EXAMINING PRESERVICE SECONDARY MATHEMATICS TEACHERS' ABILITY TO REASON PROPORTIONALLY PRIOR TO AND UPON COMPLETION OF A PRACTICE-BASED MATHEMATICS METHODS COURSE FOCUSED ON PROPORTIONAL REASONING

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Submitted to the Graduate Faculty of

the School of Education in partial fulfillment

of the requirements for the degree of Doctor of Education

University of Pittsburgh

2005

UNIVERSITY OF PITTSBURGH

SCHOOL OF EDUCATION

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The purpose of this study was to examine preservice secondary mathematics teachers' understandings about proportional reasoning prior to and upon completion of a practice-based methods course focused on proportional reasoning, their opportunities to learn the intended content, and their ability to apply what was learned in a new setting. Ten teachers completed a pre/posttest and pre/post interview that was designed to explore their ability to reason proportionally. All classes were videotaped so as to examine teachers' opportunities to learn to reason proportionally and to utilize their understandings in a new setting. In addition, six teachers who were not enrolled in the course served as a contrast group and completed the pre/post instruments.

The analysis of the data suggests that teachers learned important aspects of proportional reasoning from the course. Prior to the course, there were no differences between the understandings of the teachers enrolled in the course and those who were not. However, by the end of the course, teachers enrolled in the course utilized a broader range of solution strategies, significantly improved their capacity to distinguish between proportional and nonproportional

relationships, and significantly enhanced their understanding of the nature of proportional relationships, while those in the contrast group did not.

In addition, the analysis of the class sessions made salient that all of the mathematics that teachers learned during the course was made public during multiple classes and by multiple teachers. The analysis also revealed that even teachers who remained mostly silent during class discussions still learned the same mathematics that more the vocal teachers learned.

The results of the analysis of class sessions from a subsequent course on algebra revealed that the teachers who participated in the proportional reasoning course drew upon their enhanced understandings of proportional relationships when appropriate. This result suggests that teachers had not merely memorized discrete facts about proportional relationships, but had developed flexible understandings that allowed them to access their knowledge as they explored different mathematical ideas. Finally, the results of the study suggest that practice-based teacher education courses can be fruitful sites for helping teachers develop mathematical knowledge needed for teaching.

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Many people have provided invaluable support and guidance that has enabled me to complete this dissertation. First I would like to thank Margaret S. Smith – my advisor, mentor, and boss – for her ongoing support and encouragement throughout this journey. I am especially grateful for her insightful comments and suggestions and for her constant presses for me to "say more." Her feedback helped make this a better dissertation study. In addition, throughout my graduate studies, she has provided me with numerous learning opportunities that have helped develop my own mathematical knowledge for teaching. I am also indebted to the members of my committee, Ellen Ansell, Ellice Forman, and Gaea Leinhardt, whose thoughtful comments and suggestions helped shape this research and made this study much stronger.

I am also grateful for the support of many colleagues with whom I have had an opportunity to work during my graduate studies. First, I would like to thank my colleagues in the doctoral program, Melissa Boston, Elizabeth Hughes, Mary Lou Metz, Jennifer Mossgrove, and Michael Steele, who have provided friendship and thoughtful comments and suggestions on my work. I am especially indebted to my friend and "office mate," Elizabeth Hughes, who provided assistance in data collection and analysis. I would also like to thank Kathy Day and Joyce Fienberg for their friendship and support throughout this process; Wade Emrick for producing flawless transcripts of the videotapes of the algebra course; and Elaine Rubinstein for her helpful suggestions for data analysis.

I would also like to thank my parents, Nancy and David Fleeger, who have provided unwavering support both during the process of writing this dissertation and in my pursuit of a career that combines my interests in mathematics and teaching. In addition, my sister, Lauren Fleeger, provided encouragement and pep talks at crucial moments.

Finally, I would like to thank my husband and best friend, Kevin Hillen. He provided twenty-four-hour-a-day technical assistance, which allowed me to work through more substantive issues on this dissertation. More importantly, he was a constant source of patience, unconditional love, and emotional support, without which I could have never completed this journey.

1. CHAPTER ONE: STATEMENT OF THE PROBLEM

1.1. Introduction

Teachers need a deep and broad understanding of the mathematics they teach (Ball & Cohen, 1999; Conference Board of the Mathematical Sciences [CBMS], 2001; Ma, 1999). Their content knowledge affects the mathematics that is discussed and produced in the classroom (Ma, 1999; Sowder, Philipp, Armstrong, & Schappelle, 1998). In fact, Fennema and Franke (1992) note, “what a teacher knows is one of the most important influences on what is done in classrooms and ultimately on what students learn” (p. 147). Teachers make use of their content knowledge in ways that others who use mathematics do not. For example, teachers draw upon their knowledge of mathematics on a daily basis through the questions they ask and the feedback they provide their students (National Research Council [NRC], 2001). Teachers need to know how mathematics across grade levels is connected and consider how they might select and sequence tasks in ways that foster the development of key mathematical ideas (Hiebert et al., 1997; NRC, 2001).

Ma (1999) describes the kind of mathematics knowledge needed for teaching as a “profound understanding of fundamental mathematics (PUFM)” (p. 120). She notes that teachers who have a PUFM see connections among mathematical ideas, appreciate multiple perspectives of ideas and problems, and have a deep and broad understanding of the entire mathematics curriculum. In addition, teachers with a PUFM are not only able to efficiently carry out mathematical procedures such as subtraction with regrouping, but also are able to explain the underlying rationale for procedures. By contrast, teachers who do not have a PUFM can carry out procedures fluently, but are unable to explain why the procedures work. Ma argues that these differences are important in terms of the potential for student learning – teachers with a PUFM create different plans for interacting with students than teachers without a PUFM (e.g., in confronting a student who exhibits a common misconception, teachers with a PUFM ask questions to help the student understand or uncover the mathematical ideas, whereas teachers without a PUFM help the student develop methods for carrying out the procedure).

Recent recommendations made by the mathematics education community call for teachers to create classrooms in which students make sense of mathematics through solving challenging problems, using multiple strategies to solve problems, explaining and justifying their strategies, and engaging in discussions about how those strategies are mathematically related (National Council of Teachers of Mathematics [NCTM], 2000; NRC, 2001). In fact, many school districts are adopting curricula that embody these recommendations (CBMS, 2001). In order to meet the demands of these curricula and the community's recommendations, teachers will need an extensive knowledge of mathematics.

1.2. Limitations in Teachers' Knowledge of Mathematics

Although there is consensus that mathematics content knowledge is needed for teaching, research has documented that preservice and inservice elementary teachers have limited understanding of ideas central to the elementary curriculum, such as place value (Ball, 1988; Ma, 1999), division (Ball, 1988; Borko et al., 1992; Eisenhart et al., 1993; Ma, 1999), and rational number (Cramer & Lesh, 1988). Although secondary teachers must complete far more mathematics courses for certification, they too appear to have limited understandings of the mathematical ideas that span the K-12 curriculum. Secondary teachers' knowledge of elementary mathematics topics such as place value and division are frequently characterized by a focus on procedures (Ball, 1988; Post, Harel, Behr, & Lesh, 1991) and their conceptions of more sophisticated mathematical ideas, such as slope and function, also appear to be fragile (Ball, 1988; Even, 1993; Even & Tirosh, 1995; Wilson, 1994) and tend to mirror students' limited conceptions (Even, 1993).

One area of mathematics that appears to be particularly problematic for teachers at all levels (elementary, middle, and secondary) is proportional reasoning (Cramer, Post, & Currier, 1993; Heinz, 2000; Post et al., 1991; Simon & Blume, 1994; Smith, Silver, Leinhardt, & Hillen, 2003; Sowder, Armstrong, et al., 1998). Teachers who are unable to reason proportionally often exhibit the same misconceptions as students (Cramer et al., 1993; Simon & Blume, 1994). Even teachers who can successfully reason proportionally in many situations may have limited understandings that surface only when they encounter complex or unfamiliar proportionality problems (Post et al., 1991; Simon & Blume, 1994; Smith, Stein, Silver, Hillen, & Heffernan, 2001).

1.3. Mathematics Crucial to the Middle Grades: Proportional Reasoning

The ability to reason proportionally is crucial to students' mathematical development. In fact, the NCTM (1989) argues that proportional reasoning "is of such great importance that it merits whatever time and effort must be expended to assure its careful development" (p. 82). Proportional reasoning is so important because it brings together the mathematics explored in the elementary grades, and opens the door to high school mathematics and beyond. That is, proportional reasoning "is the capstone of children's elementary school arithmetic; on the other hand, it is the cornerstone of all that is to follow" (Lesh, Post, & Behr, 1988, pp. 93-94). Proportional reasoning is the capstone of work in the elementary grades because students shift from making additive comparisons (e.g., how *many* more…?) to making multiplicative comparisons (e.g., how *much* more…?) (Lamon, 1999; Sowder, Armstrong, et al., 1998). Proportional reasoning can also be viewed as the cornerstone of students' future work because it lays the groundwork for the study of advanced mathematical ideas such as slope, probability and statistics, trigonometry, and calculus.

A proportional relationship is just one type of relationship that can exist between two sets of quantities. It is a special class in which multiplication defines the relationship (Cramer et al., 1993). For example, the relationship between the quantities in the situation "Four tents will house 12 scouts" (Carpenter et al., 1999, p. 25) is proportional because the quantities, tents and scouts, are related multiplicatively. That is, the number of scouts that can be housed by a number of tents will always be related by a factor of three. By contrast, in the situation "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps" (Cramer et al., 1993, p. 159), the relationship between the quantities is nonproportional. That is, the relationship between the number of laps that Sue has run and the number of laps that Julie has run is related by addition, or a constant difference – Sue is always 6 laps ahead of Julie.

Proportional reasoning is often considered to be an integrative theme in middle grades mathematics (CBMS, 2001; NCTM, 2000). The NCTM (2000) recommends that students develop their capacity to reason proportionally in a variety of contexts that span the curriculum. Towards that end, proportionality problems permeate all five middle grades' content strands (Number & Operations, Algebra, Geometry, Measurement, Data Analysis & Probability) in the NCTM's (2000) *Principles and Standards for School Mathematics*. For example, students might solve problems in which they determine the “better buy”; work with linear functions and notice that linear functions that are proportional and those that are not have different properties; determine if two figures are similar; build scale drawings; or determine the likelihood of an event based on outcomes of a sample of the population.

In a review of research on ratio and proportion, Cramer et al. (1993) suggest that a “proportional reasoner” must be able to do the following: (1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations. These abilities are briefly described in the following sections and are discussed in more detail in Chapter Two.

1.3.1. Solve a Variety of Problem Types

Cramer et al. (1993) identify three types of proportionality problems: missing value, numerical comparison, and qualitative. A missing value problem is one in which three of the four values in the proportion a/b = c/d are provided and the solver must determine the fourth, or missing, quantity (Lamon, 1989). For example, the problem “3 U.S. dollars can be exchanged for 2 British pounds. How many pounds for 21 U.S. dollars?” is a typical missing value problem (Cramer et al., 1993, p. 159). Although cross multiplication is frequently used to solve this class of problems, Lamon (1999) argues that the ability to implement the cross multiplication procedure does not constitute proportional reasoning and that the use of alternative strategies might reveal a more sophisticated understanding of the relationships between the quantities.

In numerical comparison problems, all four values in the proportion a/b = c/d are provided and it must be determined whether a/b is greater than, less than, or equal to c/d (Lamon, 1989). For example, the following problem is a typical numerical comparison problem: “Richard bought 3 pieces of gum for 12 cents. Susan bought 5 pieces of gum for 20 cents. Who bought the cheaper gum?" (Karplus, Pulos, & Stage, 1983a, p. 222). Numerical comparison problems are more difficult than missing value problems because the cross multiplication strategy (the strategy most likely taught to teachers during their own middle grades experience [CBMS, 2001]) is not helpful (Lamon, 1999).

Qualitative problems "contain no numerical values but require the counterbalancing of variables in measure spaces" (Cramer et al., 1993, p. 166). For example, the following is a typical qualitative problem: "Mary ran more laps than Greg. Mary ran for less time than Greg. Who was the faster runner?" (Cramer et al., 1993, p. 166). These problems are difficult because neither cross multiplication nor alternative quantitative strategies are useful.

1.3.2. Discriminate Proportional From Nonproportional Situations

Cramer et al. (1993) argue that a proportional reasoner is able to distinguish between proportional and nonproportional situations. That is, a proportional reasoner would recognize whether the quantities in a problem situation were related additively, multiplicatively, or in some other way, and then apply an appropriate strategy. For example, consider a problem situated in the running laps context described earlier: "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?" (Cramer et al., p. 159). A proportional reasoner would recognize that the quantities in this problem are not related multiplicatively, and would therefore not apply a proportional strategy.

1.3.3. Understand the Mathematical Relationships Embedded in Proportional Situations

A proportional reasoner also understands that proportional relationships are multiplicative in nature and can be expressed algebraically in the form $y = mx$. Graphically, these relationships are depicted by a straight line that passes through the origin. The m in the equation represents the slope of the line and is the unit rate, or constant factor that multiplicatively relates the quantities. In addition, all rate pairs in proportional situations are equivalent and appear on the line $y = mx$ (Cramer et al., 1993). For example, the situation "3 U. S. dollars can be exchanged for 2 British pounds" is proportional because the relationship can be expressed algebraically as $y = 2/3x$. By contrast, the situation "A taxicab charges $1.00 plus 50 cents per kilometer", is not proportional because the relationship is defined by both multiplication and addition: $y = .50x + 1$ (Cramer et al., p. 162).

1.4. Teachers' Opportunities to Become Proportional Reasoners

In order for students to develop an ability to reason proportionally, teachers need to have an extensive knowledge of proportionality and be proportional reasoners themselves. By considering teachers' opportunities to develop their capacity as proportional reasoners, we begin to understand why limitations in their knowledge exist. For example, their own school experience was often limited to a few days of instruction, and success was defined by correctly solving missing value problems using abstract procedures (e.g., cross multiplication). Teachers were not expected to make sense of or understand these strategies; but rather, to be able to use them fluently (CBMS, 2001). Teachers' university experiences are typically no different. Although secondary teachers complete numerous mathematics courses, none of them focus specifically on proportional reasoning. In addition, preparation for teachers of the middle grades is often overlooked. Few teacher preparation programs offer courses specifically targeted for middle grades teachers (CBMS, 2001). If teachers are to meet the demands of new curricula and develop a PUFM, they need different opportunities to develop their capacity as proportional reasoners.

1.5. Recommendations for Teacher Education

Howe (1999) argues that university mathematics courses serve the needs of professional mathematicians, not teachers of mathematics. Others (e.g., CBMS, 2001; Noddings, 1998; Schifter, 1993) suggest that mathematics majors who plan to teach do not need all of the mathematics courses that future mathematicians need, but rather they need alternative, rigorous courses designed to meet their specific needs. The CBMS (2001) calls for university mathematics courses that allow teachers to revisit the mathematics of their past and to make sense of mathematics. Others (e.g., Ball, 1990; Ma, 1999; Sowder, Armstrong, et al., 1998) argue that mathematics methods courses could be sites in which teachers strengthen their understanding of mathematics.

Mathematics teacher educators (e.g., Sowder, Armstrong, et al., 1998; Schifter, 1993) agree that in order for teachers to come to know and understand mathematics in meaningful ways and be able to use this knowledge in ways that impact student learning, they must have opportunities to solve challenging mathematical tasks. Sowder and her colleagues make specific recommendations for middle grades teachers. In particular, they call for teachers to solve problems that move beyond missing value problems that are solved with cross multiplication, and instead require reasoning about and making sense of quantities and their relationships. In addition, they recommend that teachers should encounter both proportional and nonproportional problems and compare and contrast them.

Two recent teacher education experiences provide insights into the potential of such teacher education opportunities. Sowder, Philipp, et al. (1998) engaged five inservice middle school teachers in a two-year professional development program that focused on rational number and proportionality. Teachers participated in seminars in which they solved mathematics tasks, examined student work, and attended presentations given by mathematics teacher educators (e.g., Susan Lamon, Patrick Thompson). Sowder and her colleagues examined teachers' content knowledge and the impact this knowledge appeared to have on their teaching and, ultimately, students' learning. At the end of the professional development program, teachers understood the difference between proportional and nonproportional situations and had a conceptual understanding of the mathematics discussed during the seminars. In addition, as teachers' content knowledge grew broader and deeper, their instructional practice also changed. Specifically, teachers' lessons featured an increased focus on promoting conceptual understanding, a decreased emphasis on carrying out procedures, and a shift in classroom discourse (e.g., as teachers asked questions to probe students' understanding and press students for explanations, students began to ask such questions of each other). Finally, teachers' broadened content knowledge appeared to impact students' understanding of mathematics. For example, student performance on a posttest measuring understanding of fractions and proportionality was considerably better than their performance on the pretest.

In another teacher education experience, Smith et al. (2003) examined the learning that occurred as fourteen preservice elementary teachers and three secondary teachers (one preservice, two inservice) participated in a six-week mathematics methods course that focused on proportional reasoning in the middle grades. During the course, teachers solved mathematical tasks and discussed alternative solution paths, examined student work on these tasks, and read and analyzed written narrative cases of teaching. Smith et al. examined teachers' ability to distinguish between proportional and nonproportional situations and their understandings of the mathematical relationships embedded in proportional situations before and after their participation in the course. Prior to the course, teachers struggled to distinguish between proportional and nonproportional situations and had a limited understanding of the mathematical relationships embedded in proportional situations. By the end of the course, teachers were able to classify situations as either proportional or nonproportional and also had a deeper and more flexible understanding of the relationships in proportional situations.

These two studies of teacher education experiences also provide evidence regarding the viability of what Ball and Cohen (1999) call a "practice-based" approach to teacher education. In particular, Ball and Cohen argue:

> [i]f teachers' professional learning could be situated in the sorts of practice that reformers wish to encourage, it could become a key element in a curriculum….A practice-based curriculum could be compelling for teachers and would help them to improve students' learning. (p. 6)

In a practice-based approach, teacher educators identify the activities central to teaching and select or create materials that usefully depict that work (e.g., student work, mathematical tasks, written or video cases of instructional episodes). Teachers then use these practice-based materials to engage in tasks grounded in the activities of practice. For example, teachers might examine samples of student work to explore what students appear to understand (or not understand), or analyze a mathematical task to understand the mathematical territory it makes possible.

1.6. The Study

The purpose of this study was to examine preservice secondary mathematics teachers' understanding of proportionality prior to and upon completion of a practice-based methods course focused on proportional reasoning in the middle grades, their opportunities to learn the intended content, and their ability to apply what was learned in a new setting. In particular, the study sought to explore the following five research questions:

1. What do preservice secondary mathematics teachers know and understand about proportional reasoning prior to participation in a course specifically focused on proportional reasoning?
2. What do preservice secondary mathematics teachers know and understand about proportional reasoning immediately after participation in a course specifically focused on proportional reasoning?
3. How do preservice secondary mathematics teachers who participated in a course specifically focused on proportional reasoning differ from preservice secondary mathematics teachers who did not participate in the course in their understandings about proportional reasoning?
4. To what extent can teacher learning be accounted for by participation in a course specifically focused on proportional reasoning?
5. To what extent do preservice secondary mathematics teachers who participated in a course specifically focused on proportional reasoning draw upon their understandings of proportional reasoning in a subsequent course?

The first research question sought to document preservice secondary teachers' current mathematical understandings. The second research question sought to document the impact of a practice-based course that interweaves content, pedagogy, and students' understanding of mathematics on teachers' mathematical understandings. The third research question sought to document the impact of the practice-based course focused on proportional reasoning by comparing the understandings of two groups of teachers before and after the course was implemented. The fourth research question sought to account for the learning that occurred (as documented by research questions 1 and 2) by connecting particular learning outcomes to course activities. Finally, the fifth research question sought to examine whether and how teachers drew upon their understandings about proportional reasoning during participation in a practice-based mathematics methods course that focused on algebra as the study of patterns and functions.

Through an analysis of two groups of preservice secondary teachers' (the *treatment* group, who was enrolled in the course, and the *contrast* group, who was not enrolled in the course) responses to various mathematics tasks completed at the beginning and end of the semester in which the course was offered, the study examined teachers' ability to reason proportionally. A pre/posttest and a pre/post interview that contained tasks drawn from the literature whose purpose was to examine teachers' ability to: (1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations, served as data sources to explore research questions 1, 2, and 3.

In addition, the study sought to provide an explanation for the learning that occurred for teachers in the treatment group by examining the enactment of the course. Teachers' work during the course - specifically, their public contributions during class discussions - was examined so as to illustrate that teachers had the opportunity to explore the mathematical ideas that they learned during the course.

Finally, the study examined how the teachers in the treatment group made use of their understandings about proportional reasoning in a subsequent course. Upon completion of the proportional reasoning course, these teachers began a practice-based mathematics methods course that focused on algebra as the study of patterns and functions. Since many of the functions that teachers examined in this course were linear functions, it was expected that teachers might draw upon their understandings about proportional reasoning in describing linear functions that were also proportional during the algebra course.

1.6.1. Significance of the Study

The study contributed to the literature base in several important ways. First, the study examined teachers' ability to reason proportionally via a more comprehensive measure of proportional reasoning knowledge than previous studies have employed. Although studies have examined teachers' ability to solve proportionality problems (Post et al., 1991), discriminate proportional from nonproportional situations (e.g., Simon & Blume, 1994), and understand the mathematical relationships embedded in proportional situations (Smith et al., 2003), no studies have described teachers' understandings of all of these components. The study therefore provided a more complete picture of the extent to which preservice secondary teachers can reason proportionally prior to and upon completion of a practice-based course. The findings also contributed to the work done by Ball (1988), Ma (1999), and Smith et al. (2003) in describing what preservice teachers know and can do mathematically.

In addition, although several studies have examined preservice elementary teachers' understandings about proportional reasoning (e.g., Heinz, 2000; Simon & Blume, 1994), few studies have examined what preservice secondary teachers know and understand in this domain.

Although the study sought to examine teacher learning in a course similar to the one analyzed in Smith et al. (2003), the participants of this study were all preservice secondary teachers. By contrast, the Smith et al. study included only one preservice secondary teacher; the majority of the participants were preservice elementary teachers. Since secondary-certified teachers can also teach in the middle grades, it is important to examine their current understandings and the extent to which practice-based courses impact their learning.

Finally, the results from the fifth research question served to explore the extent to which teachers were able to transfer knowledge learned in one setting to a novel setting. Since the goal of teacher preparation is for teachers to construct knowledge about mathematics, students as learners of mathematics, and mathematics pedagogy that can inform what they actually do in the classroom, it is important to understand whether and how teachers appear to draw upon understandings developed in one setting as they work in a new setting.

1.6.2. Limitations of the Study

There were a number of limitations to the study. First, all participants in this study were enrolled in one of two teacher education programs at a large, urban university that culminated in certification in 7-12 mathematics. In order to be accepted into these master's-level programs, applicants needed to have a bachelor's degree in mathematics (or the equivalent) and a minimum QPA of 3.0. Thus, these preservice secondary teachers may not have been typical middle school teachers, less than half of which hold a bachelor's degree in mathematics (Grouws & Smith, 2000).

In addition, the study did not utilize a true experimental design (Campbell & Stanley, 1963). Since teachers were already enrolled in the course (or not) depending on the teacher education program in which they were enrolled, it was not possible to randomly assign teachers to the treatment in this study, enrollment in the proportional reasoning course. It is also important to note that while all teachers eligible to participate in the treatment group elected to participate in the study, only a subset of teachers eligible to participate in the contrast group elected to participate. As such, the contrast group may not have been representative of the population it aimed to represent. In addition, the size of both the treatment and the contrast group were small, with the treatment group consisting of ten teachers and the contrast group consisting of six teachers.

Finally, only a subset of the interview items was asked on both the pre- and post-interview. Several items were asked only on the pre-interview, and one item was asked only on the post-interview. Therefore, teachers' work on the interview items was used to supplement information gathered from the pre/posttests regarding teachers' ability to: (1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations.

1.7. Organization of the Document

In the next chapter, three bodies of literature that were pertinent to the study are reviewed: (1) literature that describes what proportional reasoning is and what proportional reasoners should be able to do mathematically; (2) studies that examine students' and teachers' ability to reason proportionally; and (3) literature that describes experiences teachers should encounter in order to develop their ability to reason proportionally. In Chapter Three, the methodology for the study is described. Results of data analysis are reported in Chapter Four. Finally, the results of the study are briefly summarized and implications and recommendations for future study are discussed in Chapter Five.

2. CHAPTER TWO: REVIEW OF RELATED LITERATURE

2.1. Introduction

The purpose of this study was to examine preservice secondary mathematics teachers' understanding about proportional reasoning prior to and upon completion of a practice-based methods course focused on proportional reasoning in the middle grades, their opportunities to learn the intended content, and their ability to apply what was learned in a new setting. In this chapter, three bodies of literature that were relevant to the study are reviewed. First, literature on proportional reasoning is used to define the domain and describe what it means to reason proportionally. Next, literature that examines students' and teachers' knowledge of proportionality is reviewed. Although this study examined teachers' knowledge, reviewing the literature on students' knowledge was necessary because much more research related to proportionality has been conducted with students. In addition, in recent years, students' performance on assessments such as the National Assessment of Educational Progress (NAEP) and the Third International Mathematics and Science Study (TIMSS) indicate that U.S. students have a limited understanding of mathematics and of proportionality, in particular (Beaton et al., 1996; Martin & Strutchens, 2000; Wearne & Kouba, 2000). These results suggest that teachers' understandings may not be any more robust than their students' knowledge. Thus, the literature on students' knowledge of proportionally may have implications for examining teachers' knowledge. Finally, literature that describes the mathematical experiences that teachers need to encounter in order to develop their capacity as proportional reasoners is reviewed. The chapter concludes by summarizing the literature and highlighting implications for the study.

2.2. What is Proportional Reasoning? A Framework for Examining Teachers' Knowledge

A proportional relationship is a special type of relationship in which multiplication defines the relationship between quantities (Cramer et al., 1993). Making the transition from additive to multiplicative reasoning is considered to be the hallmark of proportional reasoning development (CBMS, 2001; Inhelder & Piaget, 1958; Sowder, Armstrong, et al., 1998). However, the development of proportional reasoning comes slowly and some never fully develop this ability (Carpenter et al., 1999; Hoffer, 1992; Resnick & Singer, 1993; Steinthorsdottir, 2003; Tournaire & Pulos, 1985).

Middle grades curricula typically devote only a few pages to proportional reasoning, and focus on helping students develop facility with strategies such as cross multiplication to solve missing value problems (CBMS, 2001). However, researchers (e.g., Lamon, 1999; Sowder, Armstrong, et al., 1998) agree that proportional reasoning entails much more than setting up a proportion and cross-multiplying. In particular, a proportional reasoner should be able to do the following: (1) solve a variety of problem types (Carpenter et al., 1999; Cramer et al., 1993; Heller, Ahlgren, Post, Behr, & Lesh, 1989; Karplus et al., 1983b; Lamon, 1993b; Noelting, 1980; Post, Behr, & Lesh, 1988; Steinthorsdottir, 2003); (2) discriminate proportional from nonproportional situations (CBMS, 2001; Cramer et al., 1993; Lamon, 1995; Sowder, Armstrong, et al., 1998); and (3) understand the mathematical relationships embedded in proportional situations (Cramer et al., 1993; Post et al., 1988). These three abilities served as the framework through which the instruments measuring teacher knowledge and learning were developed and the data was analyzed. In this section, these three abilities are described both generally and in the context of examples.

2.2.1. Solve a Variety of Problem Types

Missing value and numerical comparison problems are the two most common types of proportionality problems, both in the literature and in middle grades curricula (Karplus et al., 1983a; 1983b; Lamon, 1999; Noelting, 1980; Vergnaud, 1988). Others (e.g., Heller et al., 1989; Post et al., 1991) identify an additional type of proportionality problem, qualitative problems. In the first part of this section, the strategies that can be used to solve missing value and numerical comparison problems and the ways in which researchers classify these strategies are discussed. In the latter part of this section, each of the three problem types is described in the context of an example.

In any proportion between two measure spaces, there exist two multiplicative relationships: *within*-ratio and *between*-ratio (Vergnaud, 1983). Vergnaud defines measure spaces as the two quantities that are related proportionally. For example, in the problem, “4 tents house 12 scouts. How many scouts will 40 tents house?” (Carpenter et al., 1999, p. 25), the measure spaces are tents and scouts. In this example, the multiplicative relationship *within* 4 tents and 12 scouts is three, and the multiplicative relationship *between* 4 and 40 tents is ten (Carpenter et al., 1999; Karplus et al., 1983a; 1983b; Steinthorsdottir, 2003; Vergnaud, 1988)[1], as shown in Figure 1.

[1] There is some disagreement in the use of the terms *within* and *between* in the literature. In this study, the terms were used so as to be consistent with the majority of the literature. Lamon (1993a), however, uses the terms in the opposite way. That is, Lamon would say that the multiplicative relationship *between* 4 tents and 12 scouts is three, and the multiplicative relationship *within* 4 and 40 tents is ten.

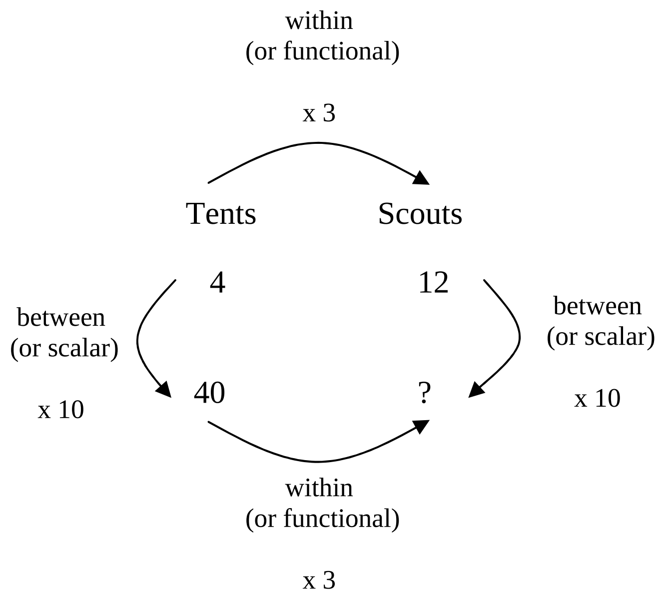


Figure 1. Illustrating within-ratio and between-ratio relationships.

Note that in the tent/scout problem, both the multiplicative relationship within and between the quantities are integer. Note also that in any proportional situation, the multiplicative relationship within quantities remains constant (i.e., the multiplicative relationship between any number of tents and scouts is three). The within-ratio then also defines a functional relationship (i.e., the number of tents multiplied by three will determine the number of scouts that can be housed). The within-ratio is therefore sometimes referred to as the functional-ratio (Vergnaud, 1988). The between-ratio is sometimes referred to as the scalar ratio (Vergnaud, 1983) or the factor-of-change (Post et al., 1988).

Early research on students' solution strategies to missing value and numerical comparison problems focused on identifying the relationship students used to solve a problem - within-ratio or between-ratio (Karplus et al., 1983a; 1983b; Vergnaud, 1983). Studies on students' preference of using within or between strategies have been inconclusive. Karplus et al.

(1983a) argue that neither strategies based on within or between relationships are more "natural" than the other, but rather, that students will use whichever strategy makes use of an integer ratio, if one is present in a problem. Other studies (e.g., Vergnaud, 1983) found that between strategies are used more frequently by students; while still others (e.g., Karplus et al., 1983b) found that within strategies are preferred by students.

Lamon (1993a; 1993b; 1994; 1995) argues that examining students' thinking through a lens of *unitizing* and *norming* may paint a more complete picture of their understanding of proportional relationships. Lamon (1993a) uses the term *unitizing* to describe the construction of a reference unit or unit whole, and argues that the ability to unitize "appears critical to the development of increasingly sophisticated mathematical ideas" (p. 133). Thus, Lamon (1995) contends that students who can only conceptualize a case of 24 cans of soda as 24 individual cans have less mathematical power than students who are able to conceptualize that case of soda as two 12-packs, four 6-packs, and other useful groupings of cans of soda.

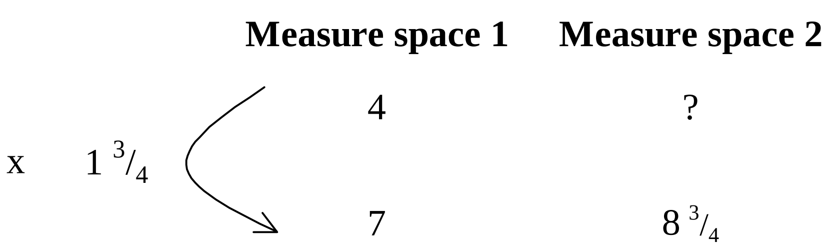
Similarly, Lamon (1993a; 1994) argues that students have more mathematical power when they conceptualize ratios as units. For example, consider the following problem:

> Ellen, Jim, and Steve bought 3 helium-filled balloons and paid $2.00 for all three. They decided to go back and get enough balloons for all of the students in their class. How much did they have to pay for 24 balloons? (Lamon, 1993a, p. 145)

A variety of strategies can be used to solve Lamon's (1993a) balloons problem. Lamon (1993a) suggests that the various ways in which students use ratio as a unit can provide a framework for analyzing students' thinking. In particular, she argues that strategies in which the ratio is conceptualized as a composite unit – that is, viewing the cost for 24 balloons as 8 groups of the ratio 3:2, is a highly sophisticated strategy. By contrast, strategies that "build up" to the desired ratio (e.g., pairs of 3 balloons and $2 are added to the initial ratio until the cost for 24 balloons is determined) are less sophisticated, even though the 3:2 ratio is viewed as a composite unit. Even less sophisticated are strategies that make use of the unit rate (i.e., determining the cost for one balloon, about 66 cents, and then multiplying the cost for one balloon by the desired number of balloons, 24) because they focus on only single units. Finally, the least sophisticated strategies are incorrect ones based on constant differences.

Lamon (1993a; 1994) uses the term *norming* to describe the reinterpretation of a given situation in terms of a composite ratio unit. For example, Lamon (1993a) argues that norming can be used to determine the scalar- or functional-ratio in solving missing value proportion problems, as shown in Figures 2 and 3.

Suppose the pharmacist gave you 7 ounces of medicine for $8.75.
What would you expect to pay for a bottle containing 4 ounces?



$7 = 1\,(4) + {}^{3}/_{4}\,(4)$

Figure 2. Using norming to determine the scalar ratio.
Adapted from Lamon, S. J. (1993a). Ratio and proportion: Children's cognitive and metacognitive processes. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research,* p. 137.

Thus, in the solution shown in Figure 2, the scalar ratio is determined by reinterpreting, or norming, 7 in terms of 4. That is, 7 is the composition of one 4 and three-fourths of 4. The solution shown in Figure 3 illustrates how norming could be used to determine the functional ratio:

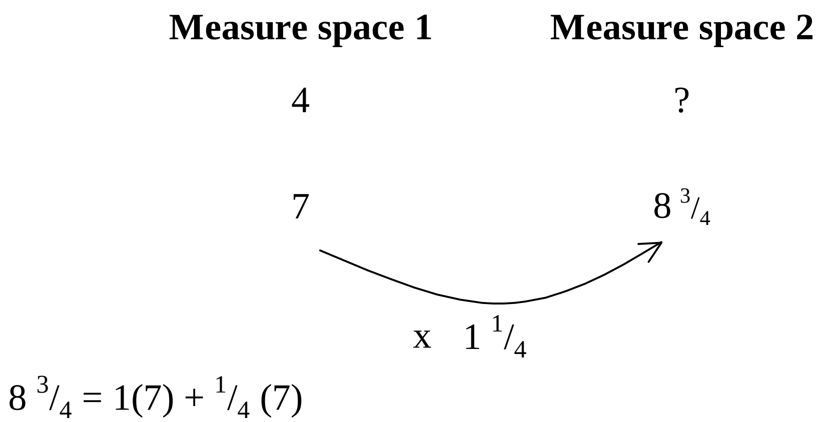


Figure 3. Using norming to determine the functional ratio.
Adapted from Lamon, S. J. (1993a). Ratio and proportion: Children's cognitive and metacognitive processes. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research*, p. 137.

Therefore, Lamon (1993a; 1995) argues that examining students' thinking through a lens of unitizing and norming is important because unitizing and norming encompass some of the critical relationships about ratios that students should understand. In addition, she argues that:

> [a]t the middle school level, one of the most salient differences between the students identified by researchers as proportional reasoners and nonproportional reasoners, is that proportional reasoners are adept at building and using composite units and that they make decisions about which unit to use when choices are available, choosing more composite units when they are more efficient than using singleton units. (Lamon, 1995, p. 169)

Carpenter et al. (1999) and Steinthorsdottir (2003) refined and expanded upon Lamon's (1993a; 1993b; 1994; 1995) work on unitizing and norming to develop a four-level "hypothetical learning trajectory" that attempts to account for the degree to which students conceptualize the ratio as a composite unit, and therefore develop more advanced levels of proportional reasoning. This hypothetical learning trajectory is based on their analysis of fourth and fifth grade students' solution strategies to missing value (Carpenter et al., 1999; Steinthorsdottir, 2003) and numerical comparison (Steinthorsdottir, 2003) problems. They argue that as students move along the trajectory, they are able to solve increasingly more complex proportionality problems, and their solution strategies become more mathematically sophisticated. The four levels of the trajectory identified in Carpenter et al. and Steinthorsdottir are next described in the context of Lamon's (1993a) balloons problem:

> Ellen, Jim, and Steve bought 3 helium-filled balloons and paid \$2.00 for all three. They decided to go back and get enough balloons for all of the students in their class. How much did they have to pay for 24 balloons? (Lamon, 1993a, p. 145)

Level 1. Students at level 1 cannot correctly solve proportionality problems. Their strategies are either based upon random calculations or constant differences. For example, a student at level 1 might solve the balloons problem by maintaining the constant difference of one between 3 balloons and \$2, and conclude that 24 balloons will cost \$23.

Level 2. Students at level 2 view the ratio as an indivisible unit. Therefore, they are able to combine ratio units either by adding (e.g., as shown in Figure 4) or multiplying (e.g., scaling the 3:2 ratio up by 8), but cannot solve problems in which the ratio must be partitioned. Thus, students at level 2 are unable to solve problems in which the between-ratio is noninteger. In addition, students at level 2 are only able to solve problems that involve enlarging, or scaling up, the original ratio.

+3 +3 +3 +3 +3 +3 +3

Balloons	3	6	9	12	15	18	21	24
Cost in dollars	2	4	6	8	10	12	14	16

+2 +2 +2 +2 +2 +2 +2

Figure 4. A solution to the balloons problem that uses a level 2 strategy based on addition.

Level 3. Students at level 3 view the ratio as a reducible unit. This conceptualization allows students to solve a broader set of problems – including ones that have noninteger between-ratios. For example, consider if the balloons problem asked how much it would cost for 25 balloons. A student at level 3 might use a ratio table (Lamon, 1999) to determine the cost, as shown in Figure 5.

x8 ÷ 24 x25

Balloons	3	24	1	25
Cost in dollars	2	16	~0.67	~16.67

x8 ÷ 24 x25

Figure 5. A solution to a modified balloons problem that uses a ratio table to determine the cost for 25 balloons.

Alternatively, a student might reduce the 3:2 ratio to 1:0.67, and then multiply by 25 to determine the cost for 25 balloons. Although this strategy may appear to resemble the within-ratio strategy, Carpenter et al. (1999) and Steinthorsdottir (2003) argue there exists a subtle but important difference between the two strategies. In the level 3 strategy, the student treats the 3:2 ratio and the new 1:0.67 ratio as *units*, and multiplies the 1:0.67 *unit* by 25. By contrast, in the within-ratio strategy, 0.67 is the number that defines the relationship within 3 balloons and 2 dollars, and 0.67 is then multiplied by 25 to determine the cost of 25 balloons.

In addition, students at level 3 are able to solve problems that involve scaling down the original ratio. For example, a student at level 3 could solve a missing value problem of the form 8/24 = 2/x, while a student at level 2 could not (because it involves reducing, or scaling down, the original ratio).

Based on her work with fifth grade female students, Steinthorsdottir (2003) refined level 3 to include an "emerging level 3" (p. 29). Students at emerging level 3 can only scale down ratios by whole numbers. By contrast, students at level 3 can solve problems that must be scaled down by numbers other than whole numbers (e.g., 15/10 = 6/x).

<u>Level 4</u>. Finally, students at level 4 are able to solve the largest class of problems because they recognize both within- and between-ratios. Therefore, they frequently use the most efficient strategy – that is, the strategy that makes use of the integer ratio. Students at level 4 have a flexible set of solution strategies at their disposal.

Steinthorsdottir (2003) tested the viability of the hypothetical learning trajectory proposed in Carpenter et al. (1999) by examining 26 fifth grade Icelandic girls' solution strategies to missing value and numerical comparison problems prior to, during, and upon completion of instruction that embodied reform practices (e.g., students shared their solution strategies with the class; the teacher scaffolded students' learning through questioning). Similar to Carpenter et al.'s results with fourth and fifth grade students, Steinthorsdottir found that prior to instruction, about one-third of the students were at level 1 and used mainly additive, constant difference strategies to solve the problems. Another one-third of the students were at level 2, and the remaining one-third was at either emerging level 3 or level 3.

In addition, Steinthorsdottir (2003) suggests that the transition from level 1 to level 2 happens quickly for students. For example, two-thirds of the girls who were at level 1 prior to instruction were able to correctly solve the first problem explored during instruction. She attributes this movement to teacher scaffolding as the students worked on the problem and class discussions. Similarly, Steinthorsdottir found that students made the transition from level 2 to level 3 easily. However, the transition from level 3 to level 4 was slow and infrequent. Asking students to find multiple ways of solving problems appeared to facilitate students' transition from level 3 to level 4.

In the remainder of this section, an example of each type of proportionality problem (missing value, numerical comparison, and qualitative) is provided. In addition, appropriate solution strategies for solving these problems are described.

2.2.1.1. Missing Value Problems

A missing value problem is one in which three of the four values in the proportion a/b = c/d are provided and the solver must determine the fourth, or missing, quantity (Lamon, 1989). For example, in the problem "3 U.S. dollars can be exchanged for 2 British pounds. How many pounds for 21 U.S. dollars?" the fourth, or missing, quantity is 14 pounds (Cramer et al., 1993, p. 159). Traditionally, students have been shown the cross multiplication procedure (i.e., setting up a proportion, cross multiplying, and solving for the missing value) to solve this class of problems

(CBMS, 2001). Although cross multiplication is the most efficient strategy for solving this class of problems, Lamon (1999) argues that the ability to implement this procedure does not necessarily constitute proportional reasoning. Alternative strategies, such as: building up (i.e., increasing the amount of dollars by 3 and pounds by 2 until 21 dollars and 14 pound is reached) (Hart, 1981); using a factor-of-change (i.e., noticing that there are seven times as many U.S. dollars, so there should be seven times as many British pounds) (Cramer et al., 1993); scaling up the 3:2 ratio by a factor of seven (Steinthorsdottir, 2003); or determining the unit rate (i.e., determining the number of British pounds that can be exchanged for every *one* U.S. dollar, and then multiplying that value by 21) call upon a deeper understanding of the relationships between the quantities.

2.2.1.2. Numerical Comparison Problems

A numerical comparison problem is one in which all four values in the proportion a/b = c/d are provided and it must be determined whether a/b is greater than, less than, or equal to c/d (Lamon, 1989). For example, numerical comparison problems in middle grades curricula are often situated in "better buy" contexts: "Richard bought 6 pieces of gum for 12 cents. Susan bought 8 pieces of gum for 15 cents. Who bought the cheaper gum?" (Karplus et al., 1983a, pp. 222-223). For students and teachers for whom the cross multiplication strategy is their only tool, these problems are more difficult than missing value problems because the cross multiplication strategy is not helpful (Lamon, 1999). Alternative strategies, such as scaling up to a common amount of one of the quantities (i.e., scale Richard's gum up by a factor of 4 -- Richard can buy 24 pieces of gum for 48 cents, and scale Susan's gum up by a factor of 3 -- Susan can buy 24 pieces of gum for 45 cents, so Susan's is cheaper) or determining the unit rate (i.e., Richard pays 2 cents for every piece of gum and Susan pays slightly less than 2 cents for every piece of gum, so Susan's is cheaper) (Cramer et al., 1993), must be called upon in order to correctly solve the problem.

Karplus et al. (1983a) report that students also use between-ratio strategies to solve numerical comparison problems. For example, to solve the problem, "Jane and Phyllis were running around the track after school. Jane finished 3 laps in 9 minutes. Phyllis finished 6 laps in 15 minutes. Which girl was running faster, or were their speeds equal?" (pp. 222-223), the authors report that students use strategies such as the following, "3 times 2 is 6, 9 times 2 is 18, and Phyllis takes only 15 minutes, so she is faster" (p. 225). Karplus et al. note that this strategy:

> made use of the scale ratio of 2 to compare the numbers of laps and then the times. This procedure illustrates the between approach to proportional reasoning…and led to the time for Phyllis to complete her nine laps at Jane's speed. This approach to comparison problems is also applicable to missing value problems and yield immediate closure for problems with equal ratios. With unequal ratio problems, however, it requires the additional inference that Phyllis's shorter time implies a greater speed than Jane's. (p. 226)

Thus Karplus et al. (1983a) argue that in order to solve numerical comparison problems involving unequal ratios, students must be able to make qualitative comparisons - that is, comparisons that do not use numeric values.

2.2.1.3. Qualitative Problems

Qualitative problems contain no numerical values and ask in what direction a ratio will change when either one or both quantities of the ratio changes. For example, consider the following problem: "If Cathy ran less laps in more time than she did yesterday, her running speed would be: (a) faster; (b) slower; (c) exactly the same (d) there is not enough information to tell" (Heller et al., 1989, p. 211). In order to solve this problem, a student could not rely on strategies such as cross multiplication, but rather would need to interpret the meaning of two ratios and compare them (Post et al., 1988). In this example, a student would need to understand that Cathy's running speed can be represented by the ratio of the number of laps Cathy ran compared to the amount of time Cathy ran. If Cathy ran fewer laps in a longer amount of time, she must be running at a slower speed. This kind of thinking is quite different from the understandings required to carry out the cross multiplication procedure.

Although qualitative problems are rarely present in middle grades curricula (Heller, Post, Behr, & Lesh, 1990), researchers argue that qualitative thinking is an important precursor to (Resnick & Singer, 1993) and component of (Heller et al., 1989; Post et al., 1988) proportional reasoning. For example, Post et al. argue that qualitative reasoning is an important means to check the feasibility of solutions and that students reason qualitatively by considering questions such as "Does this answer make sense? Should it be larger or smaller?" (p. 79). Through the consideration of such questions, students engage in important qualitative thinking that can facilitate the successful solving of proportionality problems.

2.2.2. Discriminate Proportional From Nonproportional Situations

A proportional reasoner must be able to distinguish between proportional and nonproportional situations. That is, a proportional reasoner must be able to determine if the relationship between quantities is multiplicative or not. For example, a proportional reasoner would recognize that there is a multiplicative relationship between tents and scouts in the situation "Four tents will house 12 scouts" (Carpenter et al., 1999, p. 25) and understand that there will always be three times as many scouts that can be housed by a number of tents. A proportional reasoner also recognizes that the relationship between the quantities in the situation, "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps" (Cramer et al., 1993, p. 159) is not multiplicative. That is, the number of laps that the girls ran is related by a constant difference of six - Sue will always be six laps ahead of Julie.

2.2.3. Understand the Mathematical Relationships Embedded in Proportional Situations

Finally, a proportional reasoner must understand the mathematical relationships embedded in proportional situations. Specifically, one must understand four key ideas[2]: (1) proportional relationships are multiplicative in nature; (2) proportional relationships are depicted graphically by a line that contains the origin; (3) the rate pairs (i.e., x, y pairs) in proportional relationships are equivalent; and (4) proportional relationships can be represented symbolically by the equation $y = mx$, where the m is the slope, unit rate, and constant of proportionality (Cramer et al., 1993; Post et al., 1988). Understanding these relationships can help one distinguish proportional from nonproportional situations.

For example, comparing the State Park and Zoo situations (shown in Figure 6) described in Smith et al. (2003) makes the mathematical relationships embedded in proportional situations salient. The four key understandings identified in Cramer et al. (1993) and Post et al. (1988) are next described in the context of Smith et al.'s park and zoo situations.

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[2] Smith, Silver, Leinhardt, and Hillen (2003) refer to these ideas as the four "key understandings." This language is also used in this study.

State Park	Zoo
The cost of admission to the state park is $1.00 for each person in a vehicle plus $3.00 for parking the vehicle.	The cost of admission to the zoo is $5.00 per person. (There is no charge for parking.)

Figure 6. The park and zoo situations.
Taken from Smith, M. S., Silver, E. A., Leinhardt, G., & Hillen, A. F. (2003). *Tracing the development of teachers' understanding of proportionality in a practice-based course*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL, p. 42.

2.2.3.1. Proportional Relationships Are Multiplicative in Nature

In the zoo situation, the total cost for any number of people seeking admission can be determined by multiplying the number of people by 5. By contrast, the relationship between the total cost and the number of people seeking admission to the state park is not multiplicative. That is, there is no factor that can be multiplied by the number of people to predict the total cost of admission.

2.2.3.2. Proportional Relationships Are Depicted Graphically By a Line That Passes Through the Origin

Depicted graphically, the total cost for any number of people seeking admission to the zoo is a line that goes through the origin. By contrast, the solution to the state park problem, depicted graphically, is a line that intersects the y-axis at the point (0, 3). The line that depicts the zoo situation contains the origin because if no people visit the zoo, then there will be no cost. By contrast, in the state park situation, even if no people visit the park, the parking fee of $3.00 must still be paid[3].

2.2.3.3. All Rate Pairs for a Proportional Situation Reduce to the Same Ratio, the Unit Rate

In the zoo situation both quantities (the number of people and the total cost) are increasing and maintaining a constant ratio of 5 to 1 (e.g., $5 for 1 person; $10 for 2 people; $50 for 10 people). In the state park problem, the $3.00 parking charge that is added on to the cost of admission for each group results in ratios that are not constant (e.g., the ratio of $4 for 1 person is not equivalent to the ratio of $5 for 2 people or to the ratio of $13 for 10 people).

2.2.3.4. The *m* in the Equation y = *m*x Represents the Slope of the Line, the Unit Rate, and the Constant of Proportionality

In the zoo situation, the total cost for any number of people can be predicted using the equation $y = 5x$. Five is the slope of the line, the unit rate (i.e., the cost per one person), and the constant of proportionality. The equation that depicts the relationship between quantities in the state park situation is $y = x + 3$. Although the quantities depicted in the state park situation are changing (increasing) at a constant rate, there is not a constant ratio that defines the relationship between quantities.

In this section the understandings and abilities that comprise proportional reasoning have been briefly described. In the next section, the literature that examines students' and teachers' knowledge of proportionality is reviewed.

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[3] This particular example may not be realistic (because if no people go to the state park, why would they have to pay the parking fee?) However, consider an analogous situation: A cable company charges $25 a month for service plus a $50 installation fee (Hillen, 2004). In this example, one would still need to pay the installation fee, even if no television was watched.

2.3. Students' and Teachers' Knowledge of Proportionality

Research has documented that students' abilities to reason proportionally are typically limited. For example, proportionality has been documented as difficult for all students on the 1995 TIMSS assessment and was the only content area in which the international average percent correct was below 50% for both seventh and eighth graders (Beaton et al., 1996). Similar results on the 1996 NAEP are reported for U.S. students. In particular, eighth and twelfth grade students struggled to solve all but the most routine proportionality problems (Martin & Strutchens, 2000; Wearne & Kouba, 2000).

Research has documented similar limitations in teachers' understanding of proportionality. For example, Post et al. (1991) found that "intermediate grades" teachers (i.e., grades 4, 5, and 6) correctly solved less than 70% of the problems in a set that included missing value, numerical comparison, and qualitative problems. Furthermore, their rationales or explanations for their solutions were typically limited to a description of the steps they had carried out to solve the problem (e.g., setting up a proportion, cross multiplying, and dividing).

In this section, the literature that documents students' and teachers' ability to: (1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations, is reviewed. The literature provides a detailed view of students' ability to solve the two most common types of proportionality problems, missing value and numerical comparison, and the factors that influence students' thinking on these two types of problems. Of particular interest is the phenomenon in which students successfully solve certain problems but resort to using faulty strategies, even if the problems vary only slightly. Although fewer studies have been conducted with teachers, the literature reveals that teachers encounter difficulties similar to those of their students.

2.3.1. Solve a Variety of Problem Types

Numerous studies have investigated students' ability to solve missing value and numerical comparison problems and the solution strategies that students use to solve these problems.

2.3.1.1. Missing Value Problems

Missing value problems are the most common, and often only, type of proportionality problem found in textbooks (CBMS, 2001). Students' success in solving missing value problems is frequently dependent upon features of the problems, such as the numeric values and context. These features often increase (or decrease) the likelihood that students use incorrect additive, or constant difference, strategies (Hart, 1981; Kaput & West, 1994; Lawton, 1993; Rupley, 1981; Singh, 2000; Tournaire & Pulos, 1985). This section reviews studies that have documented how two problem features, numeric values and problem context, influence students' ability to successfully solve proportionality problems.

Numeric features. One numeric feature that influences whether students use multiplicative strategies is the presence of integer within-ratios and/or between-ratios. For example, in the problem "3 U.S. dollars can be exchanged for 2 British pounds. How many pounds for 21 U.S. dollars?" (Cramer et al., 1993, p. 159), the multiplicative relationship within quantities (i.e., 3 U.S. dollars and 2 British pounds) is not integer but the relationship between quantities (i.e., 3 U.S. dollars and 21 U.S. dollars) is integer. Middle grades students are most successful in solving proportionality problems in which at least one of these relationships is integer (Hart, 1981; Kaput & West, 1994; Rupley, 1981; Tournaire & Pulos, 1985). For example, Hart (1981) analyzed over 2,000 middle school students' written responses to a variety of missing value problems. Students were far more successful in solving problems in which the factor relating the quantities was integer (80% of students were successful), rather than noninteger (20% of students were successful). Hart noted that many of the same students who successfully solved problems with integer ratios resorted to implementing additive strategies in solving problems with noninteger ratios. In particular, students were most successful in solving problems in which the factor relating the quantities was two. Such problems seemed to invoke a "doubling" strategy. Few students used a unit rate approach, and no students made use of the cross multiplication procedure.

Kaput and West (1994) identify another numeric feature that affects students' ability to successfully solve missing value problems. They found that problems in which there was a relatively small difference within the quantities in the original ratio invoked sixth grade students to use additive strategies. For example, in the problem, "Joan used exactly 15 cans of paint to paint 18 chairs. How many chairs can she paint with 25 cans?" (p. 268), there is a small difference between the numeric values of 15 and 18 (i.e., 15 x 1.2 = 18) . By contrast, students were more likely to successfully solve problems in which there was a relatively large difference in the quantities in the original ratio of the problem. For example, in the problem, "Judy earns $63 in 6 weeks. If she earns the same amount of money each week, how much does she earn in 4 weeks?" (p. 267), the difference between the quantities in the original ratio, 63:6, is larger than the previous example (6 x 10.5 = 63).

Contextual features. A problem's context also influences the likelihood that students use multiplicative strategies (Hart, 1981; Heller et al., 1989; Kaput & West, 1994; Lawton, 1993; Singh, 2000; Tournaire & Pulos, 1985; Vergnaud, 1988). For example, Heller et al. (1989) found that seventh grade students were more successful in solving missing value problems with familiar contexts (e.g., purchasing items) and less successful in solving those whose contexts were unfamiliar (e.g., speed). Even slight variations of the context (e.g., purchasing gum versus purchasing records) impacted students' success in solving the problems, with a greater number of students successfully solving problems set in more familiar contexts.

Even older students' success in using multiplicative strategies can be influenced by contextual features of a problem. For example, Lawton (1993) asked college students to solve a missing value problem of the form 4/6 = 6/x, but varied the context in which the problem was situated. Students were more likely to use proportional strategies in problems in which the quantities were distinctly different from one another (water being transferred from four melted ice cubes to a cylinder) than ones in which the quantities were less distinct (water that rises from four to six marks when transferred from a wide to a narrow cylinder). In solving the latter problem, nearly half the students used an additive strategy.

One problem context that has been documented as particularly difficult is similarity (Hart, 1981, 1988; Kaput & West, 1994; Lamon, 1993b; Singh, 2000). Typically, missing value problems involving similarity present a figure and its enlargement, and ask the solver to determine the length of one of the sides of the enlargement. For example, Kaput and West (1994) found that the majority of sixth grade students solved the problem shown in Figure 7 incorrectly by making use of an additive strategy.

The two sides of Figure A are 9 cm high and 15 cm long. Figure B is the same shape but bigger. If one side of Figure B is 24 cm high, how long is the other side?

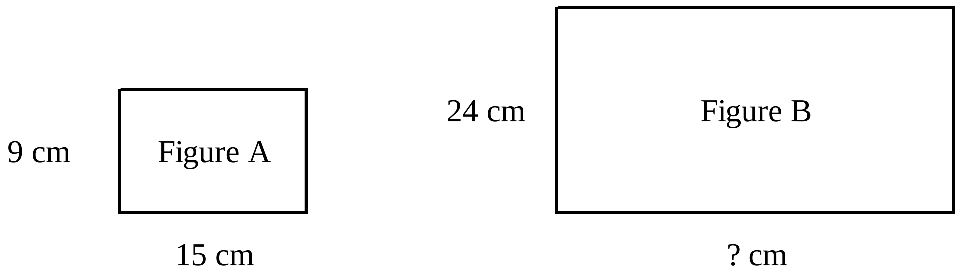


Figure 7. A typical similarity problem.
Taken from Kaput, J. J., & West, M. M. (1994). Missing value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*. Albany: State University of New York Press. p. 269.

Although young children intuitively recognize when something is “out of proportion” (van den Brink & Streefland, 1979), older children often cannot identify a strategy other than an additive approach for solving such problems (Hart, 1988). The only item that Kaput and West (1994) found to be more difficult for students than the problem shown in Figure 7 was another similarity problem in which the figure with the missing value was smaller than the original figure. That is, instead of enlarging the original figure, the original figure was shrunk, or scaled down. Hart (1981) noted similar results in her study of over 2,000 middle grades students’ written solutions to missing value problems. Between twenty-five and fifty percent of these students used additive strategies on the four most difficult items on her written instrument -- three of which were similarity problems. In a more recent study, Singh (2000) interviewed Alice, a top sixth grade student (as identified by her teacher). Although Alice could solve missing value problems situated in a range of contexts (e.g., recipes, earning money) and containing both integer and noninteger ratios, she could not solve similarity problems correctly and instead implemented additive strategies.

Thus it appears that students' ability to successfully solve missing value problems is often affected by various features of the problems. These features often affect their ability to solve other types of proportionality problems, such as numerical comparison problems.

2.3.1.2. Numerical Comparison Problems

Similar to students' ability to solve missing value problems, research has documented that the numeric features of numerical comparison problems frequently influence students' use of multiplicative strategies (Karplus et al., 1983a; 1983b; Noelting, 1980). In order to solve numerical comparison problems, one not only needs to determine the correct answer but also make sense of the quantities used in the comparison.

Numeric features. In solving numerical comparison problems, students frequently call upon additive strategies in situations in which the multiplicative relationship between the quantities is noninteger. For example, Noelting (1980) asked 321 students between the ages of 6 and 16 to compare two orange juice mixtures made from different amounts of orange juice and water and to determine the mixture that would have the stronger orange taste. Certain pairs of mixtures were easier for students to determine the stronger orange taste than others (as defined by greater frequency of success). In particular, comparing recipes that had the same amount of orange juice (and different amounts of water) was easier than comparing recipes that had different amounts of both quantities. The most difficult pairs were ones in which both recipes were related by the same additive difference (e.g., 2 cups juice and 1 cup water versus 4 cups juice and 3 cups water – both recipes have one more cup of water than juice). Of the examples in which the quantities in both recipes were related by the same additive difference, the previous example was one of the easiest, because the number of cups of juice in the recipes is related by an integer scale factor of 2 (there is twice as much juice in the second recipe, but more than twice as much water, so the first recipe is stronger). By contrast, the most difficult example in this group was comparing 5 cups juice and 7 cups water versus 3 cups juice and 5 cups water, because neither the between-ratio nor the within-ratio is an integer. Karplus et al. (1983b) had similar results when they interviewed sixth and eighth grade students about the sweetness of two lemonade mixtures made from sugar and lemon juice, finding that the presence of integer or equal ratios facilitated students' use of proportional strategies and that additive strategies replaced proportional strategies on more difficult problems.

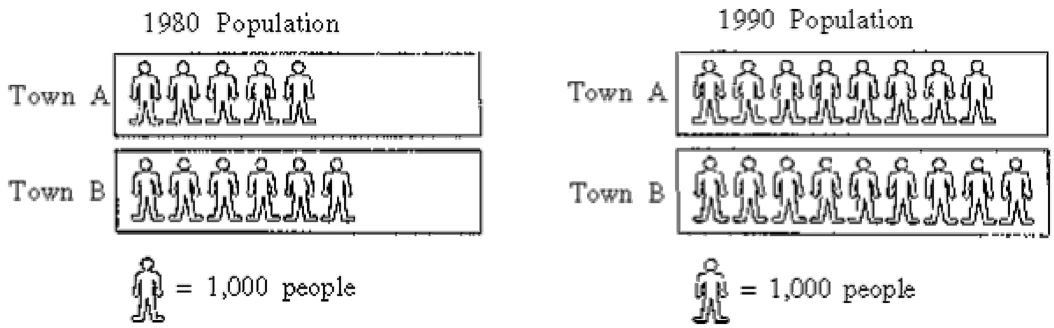
Making sense of quantities. Numerical comparison problems are also more difficult than missing value problems because as one uses ratios to make comparisons, the meaning of those ratios must be interpreted. For example, Noelting's (1980) orange juice problem can be solved by determining the unit rate. However, two unit rates can be found: the amount of juice for every one cup of water, or the amount of water for every one cup of juice. In the first case, the mixture with the *larger* value would correspond to the mixture with the stronger orange taste. Conversely, when using the unit rate of the amount of water for every one cup of juice, the mixture with the *smaller* value corresponds to the stronger mixture. In order to select the mixture with the stronger orange taste, one needs to identify what the comparisons mean in the context of the problem. This component of solving numerical comparison problems appears to be problematic for students.

For example, only 23% of twelfth graders correctly answered and explained their solution to the cherry drink problem, a numerical comparison problem similar to Noelting's (1980) orange juice tasks, that was an item on the 1996 NAEP:

> Luis mixed 6 ounces of cherry syrup with 53 ounces of water to make a cherry-flavored drink. Martin mixed 5 ounces of the same cherry syrup with 42 ounces of water. Who made the drink with the stronger cherry flavor? (Wearne & Kouba, 2000, p. 181)

The cherry drink problem could be considered difficult because of the noninteger ratios (however, students were allowed to use calculators). Another 26% of U.S. twelfth graders made appropriate comparisons using ratios, but either did not identify what their comparisons meant in the context of the problem or interpreted their comparisons incorrectly. For example, one viable strategy is to divide the water by the concentrate, calculating 8.8 for Luis' mixture and 8.4 for Martin's. However, without identifying that the 8.8 and 8.4 are the number of ounces of water for every one ounce of cherry concentrate, one cannot determine the stronger-tasting mixture. Other students made valid comparisons and attempted to interpret their comparisons, but did not do so correctly. For example, a valid strategy is to compare the amount of cherry concentrate to the amount of water for each mixture, yielding .113 and .119 for Luis and Martin, respectively. Although these values have several valid interpretations (e.g., the amount of cherry concentrate for every one ounce of water), an incorrect, but common, interpretation was that these values reflected the percent of cherry concentrate in the mixture. (This is incorrect because the concentrate was being compared to the water, rather than to the entire mixture.)

In another problem on the 1996 NAEP, both eighth and twelfth graders had considerable difficulty solving a numerical comparison problem (shown in Figure 8) in which they were asked to defend both an additive and a multiplicative position. Only 1% of eighth graders and 3% of twelfth graders were able to create arguments to support both positions, and less than 25% of students at both grade levels could produce an argument for only one of the positions. Similar to their work on the cherry drink problem, many students were able to make comparisons using ratios but incorrectly interpreted the meaning of their comparisons (Wearne & Kouba, 2000).



In 1980, the populations of Town A and Town B were 5,000 and 6,000, respectively.
In 1990 the populations of Town A and Town B were 8,000 and 9,000, respectively.

Brian claims that from 1980 to 1990 the populations of the two towns grew by the same amount.
Use mathematics to explain how Brian might have justified his claim.

Darlene claims that from 1980 to 1990 the population of Town A had grown more.
Use mathematics to explain how Darlene might have justified her claim.

Figure 8. A problem in which students are asked to make both an additive and a multiplicative comparison. Taken from Wearne, D., & Kouba, V. L. (2000). Rational numbers. In E. A. Silver & P. A. Kenney (Eds.), *Results from the seventh mathematics assessment of the national assessment of educational progress*. Reston, VA: National Council of Teachers of Mathematics. p. 186.

2.3.1.3. Range of Strategies

One final, but important, component in solving missing value and numerical comparison problems involves having a range of strategies available for solving problems. Singh's (2000) interview with sixth grader Alice illustrates what happens when students do not have a range of strategies available to them. Alice had been taught the unit rate strategy (i.e., determining how many or how much for one) and consistently tried to use this method even when solving problems in which the unit rate was noninteger, as in the problem, "To bake donuts Mariah needs 8 cups of flour to bake 14 donuts. Using the same recipe, how many donuts can she bake with 12 cups of flour?" (p. 284). Even when pressed to solve the problem in a different way, Alice could not do so. When asked to solve additional problems without using her school-taught unit rate method, she reasoned additively. She then correctly solved the same problems using her unit rate strategy, but was unable to explain why her first, additive response was incorrect.

By contrast, Karen, another sixth grader who had not been taught the unit rate strategy, was able to solve a range of missing value problems correctly and with considerable flexibility (Singh, 2000). Karen not only used different strategies, but also used different strategies depending on the particular problem on which she was working. For example, in solving the donut problem (described previously), Karen unitized the composite ratio unit of 8 cups to 14 donuts to 4 cups to 7 donuts. She then “built up” this ratio to preserve the relationship between cups and donuts (i.e., 4 to 7; 8 to 14; 12 to 21). When presented with a problem in which building up would be cumbersome (e.g., How many donuts can Mariah bake with 160 cups of flour?), Karen utilized a “scaling up” strategy (i.e., there is 20 times more flour so she can bake 20 times as many donuts). Karen was also successful in solving similarity problems involving both integer and noninteger ratios. Thus Karen was able to solve problems flexibly – that is, her strategy changed depending on the problem to one that was the most efficient.

Although far fewer studies involving proportionality have been conducted with teachers, the results of these studies indicate that many teachers encounter the same struggles that students experience. For example, Post et al. (1991) administered a written instrument with approximately 80 rational number and proportionality items (including missing value, numerical comparison, and qualitative problems) to over 200 inservice teachers at grades 4, 5, and 6. Nearly one-third of the teachers were not able to solve even half the items and the overall mean for the instrument was below 70 percent. In follow-up interviews with a subset of the teachers, Post et al. (1991) found that the teachers who correctly solved missing value problems used primarily procedural approaches, such as cross multiplication. In addition, teachers were frequently unable to provide an explanation for their solution that went beyond a description of the steps they had taken to determine the solution (e.g., setting up a proportion and cross multiplying to determine the missing quantity). Finally, only 5% of teachers used estimation to determine if their solution made sense in the context of the problem.

Post et al. (1991) also found teachers' success with numerical comparison problems to be similar to that of students' success. In general, teachers were more successful with comparing two ratios that were equivalent than those that were not. In addition, similar to Noelting's (1980) findings with students, Post and his colleagues found that teachers were less successful with comparing noninteger ratios.

Finally, familiarity with the problem context appears to positively influence teachers' ability to successfully solve proportionality problems. During problem-centered interviews, Perrine (2001) found that teachers were more successful in solving both missing value and numerical comparison problems situated in familiar contexts. These findings echo those found by Heller et al. (1989) and Lawton (1993) in studies conducted with students.

2.3.1.4. Qualitative Problems

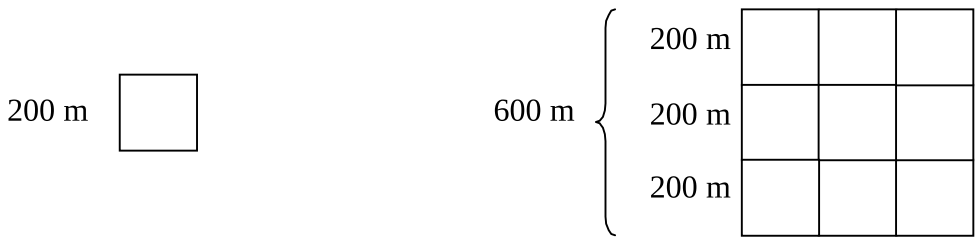
Few studies have examined students' ability to solve qualitative problems and, to date, no studies have been conducted with teachers. The studies that have been conducted with students suggest that their ability to solve qualitative problems is limited (Heller et al., 1989; Heller et al., 1990; Post et al., 1988). For example, Heller et al. (1989) found that 254 seventh grade students (who had received no instruction on proportional reasoning during the school year) solved an average of 75% of a set of qualitative problems correctly. In a similar study, Heller et al. (1990) found that over 800 seventh grade students (who had received no instruction on proportional reasoning during the school year) and eighth grade students (who had received instruction on the cross multiplication procedure prior to the study) could correctly solve only about 60% of a set of qualitative problems.

2.3.2. Discriminate Proportional From Nonproportional Situations

Several studies suggest that students have a strong tendency to apply proportional strategies to problems that do not call for it. Van Dooren, De Bock, Depaepe, Janssens, and Verschaffel (2003) refer to this phenomenon as the "illusion of linearity" (p. 113). For example, De Bock, Verschaffel, and Janssens (1998) examined seventh grade students' solutions to problems involving the relationships between the length and area of similar figures. Problems involving the relationship between the lengths of similar figures were proportional, while problems involving the relationship between the areas of similar figures were not proportional. For example, consider the following two problems:

> 1. Farmer Gus needs approximately 4 days to dig a ditch around a square pasture with a side of 100 m. How many days would he need to dig a ditch around a square pasture with a side of 300 m?
> 2. Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?
>
> (De Bock, Verschaffel, & Janssens, 1998, p. 68)

Problem 1 is proportional because it focuses on length and perimeter, and the lengths and perimeters of similar figures are proportional. Therefore, since one side of the square has tripled, it will take the farmer three times as long to dig the ditch. By contrast, problem 2 is not proportional because it focuses on area, and the areas of similar figures are not proportional. That is, even though the length of the sides has tripled, the area has increased by a factor of nine, as shown in Figure 9.



Original square piece of land with a side of 200 m.

Enlarged square piece of land with a side of 600 m.

Figure 9. Diagram that shows that when the length of the side of a square triples, the area of the square increases by a factor of nine.

De Bock et al. (1998) found that students had difficulty distinguishing between problems such as the ones shown above. Students scored an average of 92% correct on the proportional problems and an average of only 2% correct on the nonproportional problems. Nearly all of the students' incorrect strategies to the nonproportional problems involved incorrectly applying proportional strategies. Of particular interest is that even when pressed to sketch a figure of the problem situation, students still applied proportional strategies to the nonproportional situations.

In a similar study, De Bock, Van Dooren, Janssens, and Verschaffel (2002) interviewed seventh and tenth grade students as they solved a problem involving similar figures. The problem presented students with the enlargement of an irregular figure (a picture of Father Christmas, whose height and width were now three times as large as the original figure). Students were told the amount of paint needed to paint the original picture and asked how much paint was needed to paint the enlargement. Ninety-five percent of the students initially solved the problem incorrectly by applying a proportional strategy (e.g., three times as much paint is required, because the height and width are three times as large). Van Dooren et al. (2003) note that this phenomenon of students "over-applying" proportional strategies to nonproportional situations is not limited to geometric contexts such as similarity. They report similar findings in examining students' strategies in solving probability problems.

Studies that examine teachers' ability to discriminate between proportional and nonproportional situations indicate that teachers struggle to do so in situations that span a variety of contexts. For example, Cramer et al. (1993) found that thirty-two out of thirty-three preservice elementary teachers incorrectly solved the following problem: "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie had completed 15 laps, how many laps had Sue run?" (p. 159). Instead of recognizing that the number of laps that Sue and Julie ran is related by addition (Sue will always be 6 laps ahead of Julie), teachers set up a proportion, cross multiplied, and concluded that Sue had run 45 laps. Thus, teachers assumed that because the problem presented three quantities and required them to determine a fourth, that the solution required a proportional strategy. This phenomenon of "over-proportional reasoning" is similar to De Bock et al.'s (1998; 2002) observations during interviews with students. Interestingly, this same group of teachers correctly solved another problem: "3 U.S. dollars can be exchanged for 2 British pounds. How many pounds for 21 U.S. dollars?" (p. 159). However, the teachers were unable to explain why this situation was proportional while the laps problem was not.

2.3.2.1. Ratio as Measure

There is evidence that teachers do not see ratio, a multiplicative comparison, as an appropriate method to measure a particular attribute (e.g., oranginess of a drink; steepness of a hill) and instead measure the attribute using an additive comparison. For example, Simon and Blume (1994) observed that few preservice elementary teachers identified ratio as an appropriate measure of the steepness of ski ramps. During class discussions, teachers developed two ways to represent steepness: by the ratio of the height to the length of the base (a correct multiplicative approach), and by the difference between the height and length of the base (an incorrect additive approach). However, teachers had difficulty recognizing that the multiplicative approach was appropriate in this situation.

Heinz (2000) observed a similar phenomenon in a study of preservice and practicing elementary teachers. In this study, teachers were asked to determine which was more lemony -- jar A which contained 3 lemon cubes and 2 lime cubes, or jar B which contained 4 lemon cubes and 3 lime cubes. In their initial work on the task, 58% of the teachers used an incorrect additive strategy to conclude that the mixtures were equally lemony.

Smith et al. (2001) also found that at the beginning of a mathematics methods course, over 50% of preservice elementary and secondary teachers did not recognize ratio as an appropriate measure of the degree to which a rectangle is "square" and instead used additive strategies. For example, a 35 by 39 rectangle is more square than a 21 by 25 rectangle, even though in both figures the length is four units larger than the width. A ratio can be used to determine the "squareness" of a rectangle. Because the lengths of all sides of a square are equivalent, the ratio of the width to the length of a square is 1. Therefore, the rectangle whose ratio of width to the length (or length to the width) is closest to 1 would be the most square. These results are consistent with Simon and Blume's (1994) pretest results on a similar problem, in which the majority of preservice elementary teachers (19 of 26 teachers) made use of an additive strategy; Heinz's (2000) results in which nearly half of the preservice and practicing elementary teachers used a difference to measure squareness; and Perrine's (2001) results in which six preservice elementary teachers all used a difference to determine the rectangle that was the most square. Recognizing that the quantities are related multiplicatively and that comparing them additively is inappropriate is a crucial component of proportional reasoning, yet is an aspect of proportional reasoning that preservice teachers often lack.

2.3.3. Understand the Mathematical Relationships Embedded in Proportional Situations

To date, no studies have examined students' understandings of the mathematical relationships embedded in proportional situations. There is, however, evidence that teachers do not understand these relationships. For example, prior to and upon completion of a methods course focused on proportional reasoning in the middle grades, Smith et al. (2003) presented pre- and inservice elementary and secondary teachers with six relationships (two of which were depicted as graphs, two as equations, and two as tables) and asked teachers to classify them as either proportional or nonproportional and to explain how they knew. Prior to the course, nearly 70% (12 of 17) of the teachers could not correctly characterize any of the relationships. The most common misconception exhibited by this group of teachers was that all linear relationships are proportional. The remaining 30% (5 of 17) of the teachers were able to correctly classify some of the relationships. Most of these responses were supported by an appropriate rationale, the most common of which was that proportional relationships are depicted graphically by lines that contain the origin (key understanding 2).

Upon completion of the course, nearly all the teachers were able to distinguish proportional situations from nonproportional ones. In addition, they had come to understand the nature of proportional relationships, as evidenced in their use of the four key understandings to justify their distinctions. Teachers' use of the four key understandings also appeared to be flexible, as evidenced by their use of key understandings that took into account the nature of the representation. For example, an explanation involving the understanding that proportional situations are depicted graphically by a line that contains the origin (key understanding 2) was frequently used to characterize the graphical relationships, and to a lesser extent to characterize the relationships presented as equations and tables. The argument used by the least number of teachers was that a proportional relationship can be represented symbolically as $y = mx$, where m is the slope, the unit rate, and the constant of proportionality (key understanding 4).

In this section, the literature that examines students' and teachers' ability to reason proportionally has been reviewed. In the next section, the types of experiences that teachers need in order to develop their capacity as proportional reasoners are discussed.

2.4. The Experiences Teachers Need in Order to Develop Their Ability to Reason Proportionally

As noted in Chapter One, teachers' opportunities to develop their ability to reason proportionally are typically quite limited, and emphasis is placed on achieving facility in implementing procedures such as cross multiplication (CBMS, 2001). In order to help their students develop a capacity to reason proportionally, teachers need more meaningful opportunities to explore the mathematical ideas they will teach (CBMS, 2001; Sowder, Armstrong, et al., 1998). In this section, two specific types of mathematical experiences that are recommended for teachers are described.

First, teachers should have opportunities to explore and compare both additive and multiplicative situations (Lamon, 1995; Sowder, Armstrong, et al., 1998). In addition, Lamon and Sowder, Armstrong, et al. argue that teachers need opportunities to explore situations that can be viewed from both absolute (i.e., additive) and relative (i.e., multiplicative) perspectives. Lamon (1995; 1999) argues that the ability to make relative comparisons is an important type of thinking required for proportional reasoning. Therefore, teachers should explore problems such as the following, which can be viewed appropriately from both an absolute and a relative perspective:

> Jo has two snakes, String Bean and Slim. Right now, String Bean is 4 feet long and Slim is 5 feet long. Jo knows that two years from now both snakes will be fully grown. At her full length, String Bean will be 7 feet long, while Slim's length when he is fully grown will be 8 feet. Over the next two years, will both snakes grow the same amount? (Lamon, 1995, p. 174)

One way to measure the change in the snakes' growth is to consider their *absolute*, or actual, growth - they both grew three feet. Determining absolute change makes use of a constant, additive difference. Alternatively, the amount the snakes will grow (three feet) could be considered *relative* to their current length. Using a relative perspective, String Bean will grow 3/4 of her current length, while Slim will grow 3/5 of his current length. Therefore, when making a relative comparison, the snakes do not grow the same amount. Measuring change relatively, as opposed to absolutely, requires using a ratio, which is a multiplicative comparison.

Second, teachers should encounter a variety of situations in which ratio is involved because the understanding of ratio is critical in making the transition from additive to multiplicative reasoning (CBMS, 2001; Sowder, Armstrong, et al., 1998). In particular, teachers need to understand that ratio can be used to measure a particular attribute. This view of ratio is quite different from the typical ways that teachers encounter ratio (e.g., comparing two sets, such as comparing the number of boys to the number of girls in a classroom) (Sowder, Armstrong, et al., 1998). For example, a correct solution to Simon and Blume's (1994) ski ramp problem (shown in Figure 10) makes use of a ratio to measure the steepness of the ramp. In this problem, teachers must identify the relevant quantities (height of the hill and length of the base) and determine how these quantities should be related in order to describe steepness.

In Kansas, there are no mountains for skiing. An enterprising group built a series of ski ramps and covered them with a plastic fiber that permitted downhill skiing. It is your job to rate them in terms of most steep to least steep. You have available to you the following measurements for each hill: the length and width of the base (measured along the ground) and the height. How would you determine the relative steepness of the hills using the information you have?

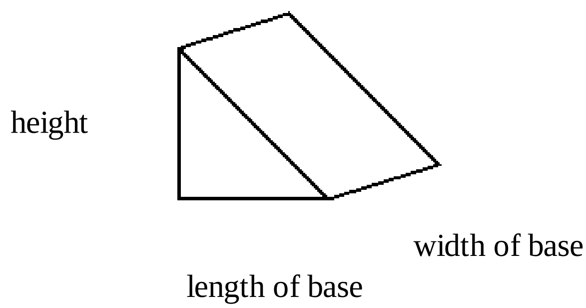


Figure 10. The ski ramp problem.
Taken from Simon, M. A., & Blume, G. W. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 13*, p. 187.

2.5. Summary

Proportional reasoning is an important topic in the middle grades curriculum, yet presents challenges both to students and to the teachers who work with them. Reasoning proportionally requires one to make a shift from additive to multiplicative thinking. In particular, reasoning proportionally includes: (1) solving a variety of problem types (missing value, numerical comparison, qualitative); (2) discriminating proportional from nonproportional situations; and (3) understanding the mathematical relationships embedded in proportional situations.

A proportional reasoner should not only be able to solve a variety of problems but also have “the mental flexibility to approach problems from multiple perspectives and at the same time [have] understandings that are stable enough not to be radically affected by large or 'awkward' numbers or by the context within which a problem is posed" (Post et al., 1988, p. 80). However, the literature suggests that students rarely use proportional reasoning consistently across sets of problems that call for multiplicative strategies. In particular, studies illustrate that even slight variations of problems dramatically influence students' (in the middle grades and beyond) strategy selection. Although fewer studies have been conducted with teachers, the literature suggests that teachers experience similar difficulties. In addition, teachers tend to rely on procedural strategies such as cross multiplication that do not necessarily call upon a deep understanding of the relationships between the quantities in proportional situations.

A proportional reasoner should also be able to distinguish between proportional and nonproportional situations. The literature suggests that both students and teachers have difficulty making this distinction. In particular, both students and teachers apply constant difference strategies to problems that require proportional strategies. Furthermore, students and teachers over-apply proportional strategies to situations in which the quantities are not related proportionally.

Finally, teachers do not appear to recognize the mathematical relationships embedded in proportional relationships. When teachers are unable to recognize these relationships, they have difficulty identifying situations presented in multiple representations as proportional or not, as shown in Smith et al. (2003).

In order to improve students' ability to reason proportionally, teachers' knowledge of proportionality must also be developed and refined. The purpose of this study was to examine preservice secondary teachers' ability to: (1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations before and after participation in a course focused on proportional reasoning. In addition, this study sought to document teachers' opportunities to learn the intended content and to examine the extent to which teachers drew upon their understandings in a new setting.

3. CHAPTER THREE: METHODOLOGY

3.1. Introduction

The purpose of this study was to describe preservice secondary mathematics teachers' understandings about proportional reasoning (specifically, their ability to: (1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations) prior to and upon completion of a practice-based methods course specifically focused on proportional reasoning. In addition, the study sought to provide an explanation for the learning that occurred during the course by examining the learning opportunities presented to teachers and their public work (in the form of oral contributions) within those opportunities. Finally, the study sought to examine the extent to which teachers applied their understandings about proportional reasoning during a subsequent course that focused on algebra as the study of patterns and functions.

The study utilized a quasi-experimental design, in which data was collected on two groups of teachers: (1) the treatment group, who underwent the treatment, enrollment in the course; and (2) the contrast group, who was not enrolled in the course (Campbell & Stanley, 1963)[4]. A pre- and posttest design approach was applied to this quasi-experiment, with both groups of teachers completing a pre/post written test and a pre/post interview. The sections that

[4] A quasi-experimental design, rather than a "true experimental design," was used because the teachers could not be randomly assigned to the treatment and contrast groups (Campbell & Stanley, 1963). The teachers were already enrolled in particular teacher certification programs that either required them to enroll in the course or not.

follow present the methodology for the study, beginning with a detailed description of the methods course that was the treatment in this quasi-experiment.

3.2. Treatment

The treatment in this quasi-experiment was a master's level advanced mathematics methods course[5] offered to students enrolled in Master of Arts in Teaching (MAT) and Master in Education (M.Ed.) programs[6] in a school of education at a large, urban university. The course was taught by an experienced mathematics teacher educator and researcher. The course was offered during a fifteen-week semester in the spring of 2003 and met once per week for two and a half hours. The goal of the course was to help teachers construct (or reconstruct) their own understanding about proportional reasoning and proportional relationships and to develop their capacity for providing meaningful learning opportunities for the students with whom they work. From a mathematical perspective, the course was intended to help teachers: (1) recognize and gain fluency using an array of mathematical and linguistic representations related to proportionality; (2) distinguish between situations in which quantities have a multiplicative relationship and those that do not, and become proficient making or using comparisons between and among quantities when there is an underlying multiplicative relationship; (3) develop and become proficient using a repertoire of strategies to solve both commonly encountered problems and non-routine problems involving proportionality; and (4) characterize and describe the occurrence of proportionality in a range of topics in the middle grades mathematics curriculum.

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[5] The course was developed under the auspices of the ASTEROID (A Study in Teacher Education: Research on Instructional Design) project (NSF Award #0101799), principal investigator, Margaret S. Smith.

[6] The course was offered to both preservice teachers (enrolled in the MAT program) and inservice teachers (enrolled in the M.Ed. Program). However, only the preservice teachers participated in the study since the purpose of the study was to examine preservice teachers' understandings.

3.2.1. Content of the Course

Teachers enrolled in the course engaged in a variety of activities that depict the work of teaching, including solving and analyzing mathematical tasks drawn from middle-grades curricula, analyzing students' mathematical understandings as shown in their written work, and analyzing and reflecting on teaching depicted in narrative and video cases. The course centered around a set of four narrative cases, each of which depicts an episode of instruction on rational numbers and proportionality in the middle grades (Smith, Silver, & Stein, 2005b)[7]. Additional tasks were drawn from a variety of sources, including a unit from The Connected Mathematics Project which focused on ratio, proportion, and percent (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), the NCTM's (2000) *Principles and Standards for School Mathematics*, and the required textbook for the course, Lamon's (1999) *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers*.

The sequence of activities in which teachers engaged during the course is shown in Figure 11. The columns denote the activities that occurred during each class or were assigned for homework (activities that are shown above the line occurred during class; activities below the line were assigned for homework)[8]. Five types of activities (denoted by the different shapes), were used in the course: (1) solving and discussing mathematical tasks (rectangles); (2) analyzing and discussing samples of student work (hexagons); (3) analyzing and discussing cases of mathematics teaching (ovals); (4) reading about and discussing issues related to mathematics teaching (triangles); and (5) discussing mathematical ideas that did not directly stem from a mathematical task that teachers solved (diamonds).

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[7] The set of cases was developed under the auspices of the NSF-funded COMET (Cases of Mathematics Instruction to Enhance Teaching) project, whose purpose was to develop materials for teacher professional development in mathematics. The project was co-directed by Margaret S. Smith, Edward A. Silver, and Mary Kay Stein.

[8] During the last class (Class 15), teachers completed the posttest. No other course activities occurred during this class.

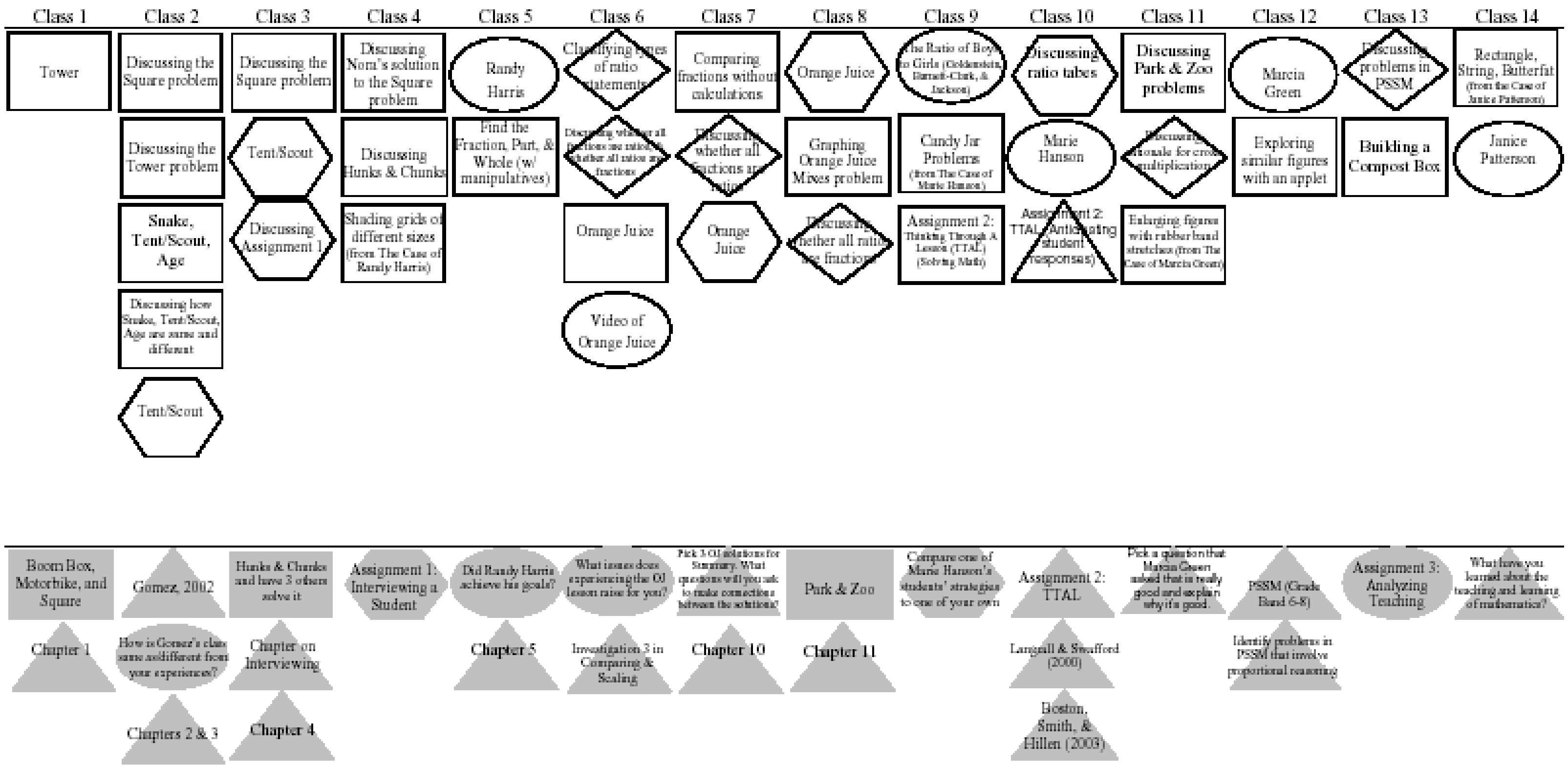


Figure 11. The course map that describes the activities in which teachers engaged during the course.
Adapted from Smith, M. S., Silver, E. A., Leinhardt, G., & Hillen, A. F. (2003). *Tracing the development of teachers' understanding of proportionality in a practice-based course*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL, p. 10.

Note. Columns denote activities that occurred during each class (or were assigned for homework)
Activities above the line occurred during class; activities below the line and shaded in gray were assigned for homework

The shapes indicate the *type* of activity in which teachers engaged, as shown below:

Rectangles: Solving and discussing mathematical tasks
Hexagons: Analyzing and discussing samples of student work
Ovals: Analyzing and discussing cases of mathematics teaching
Triangles: Reading about and discussing issues related to mathematics teaching
Diamonds: Discussing mathematical ideas that did not directly stem from a mathematical task that teachers solved

Figure 12 uses color-coding to illuminate the particular activities in which teachers had opportunities to: (1) solve a variety of problem types (missing value problems are shaded in green; numerical comparison problems are shaded in purple)[9]; (2) distinguish proportional from nonproportional situations (shaded in yellow); and (3) understand the mathematical relationships embedded in proportional situations (shaded in pink). As shown in Figure 12, teachers had an opportunity to solve (or examine solutions to) missing value or numerical comparison problems in 9 of 14 classes. However, teachers had no opportunities to solve qualitative problems. In addition, teachers had opportunities throughout the course to discriminate between proportional and nonproportional situations. Finally, teachers had few opportunities to examine the mathematical relationships embedded in proportional situations.

[9] Note that in Figure 12, several *hexagons* (which denote activities in which teachers analyzed and discussed student work) are shaded green or purple. This is due to the fact that by examining students' solutions, teachers had opportunities to consider alternative ways of solving missing value and numerical comparison problems.

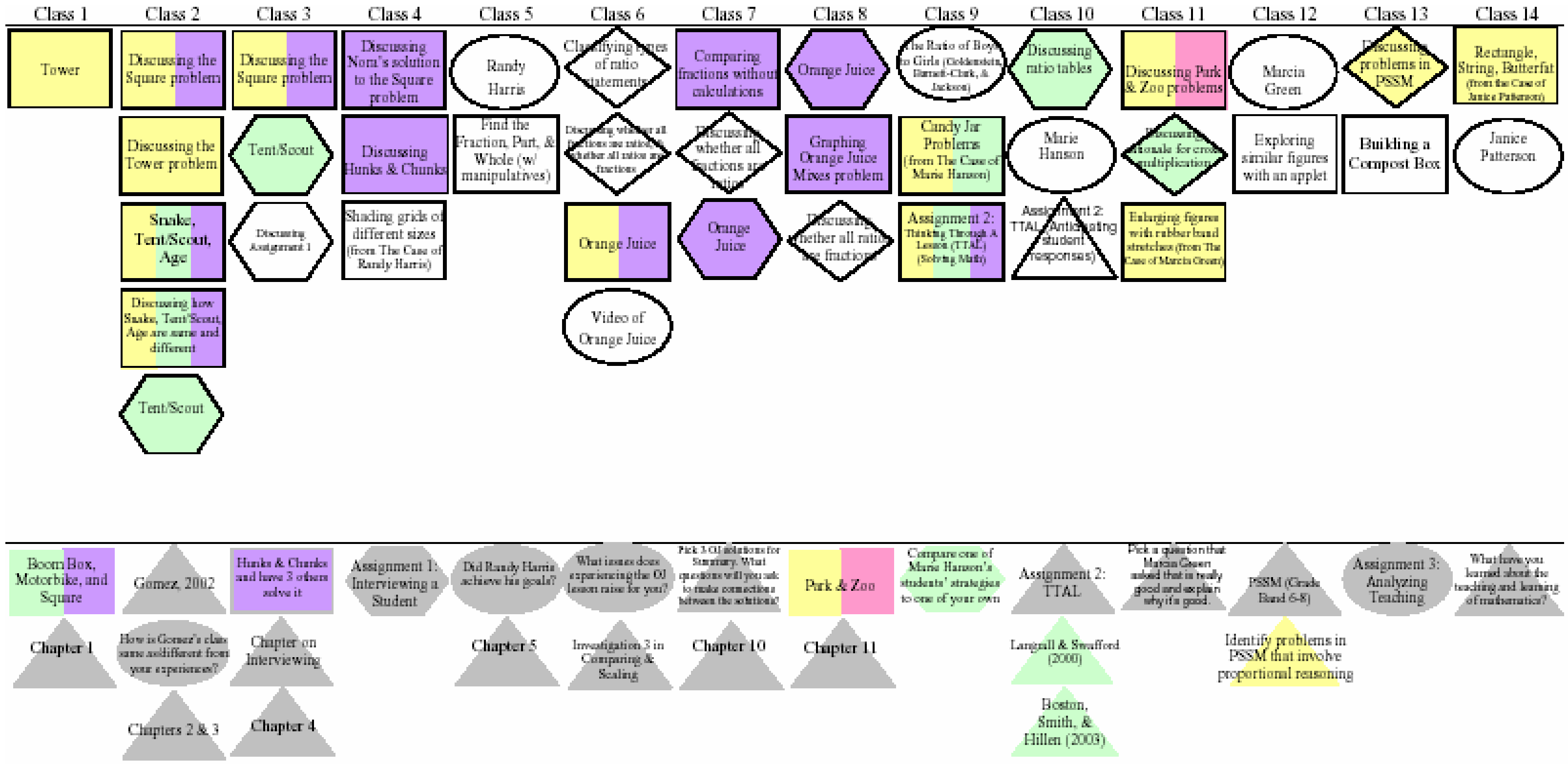


Figure 12. The color-coded course map that indicates teachers' opportunities to solve a variety of problem types, discriminate proportional from nonproportional situations, and understand the mathematical relationships embedded in proportional situations.

Note. Columns denote activities that occurred during each class (or were assigned for homework)
Activities above the line occurred during class; activities below the line and shaded in gray were assigned for homework

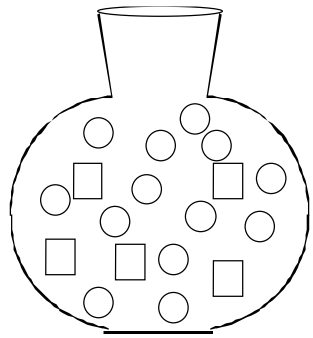
The shapes indicate the *type* of activity in which teachers engaged, as shown below:

Rectangles:	Solving and discussing mathematical tasks
Hexagons:	Analyzing and discussing samples of student work
Ovals:	Analyzing and discussing cases of mathematics teaching
Triangles:	Reading about and discussing issues related to mathematics teaching
Diamonds:	Discussing mathematical ideas that did not directly stem from a mathematical task that teachers solved

The color-coding indicates activities in which teachers had opportunities to explore the abilities described in the framework, as shown below:

Green:	Solve a variety of problems types (specifically, missing value problems)
Purple:	Solve a variety of problems types (specifically, numerical comparison problems)
Yellow:	Distinguish proportional from nonproportional situations
Pink:	Understand the mathematical relationships embedded in proportional situations

The activities were interwoven throughout the course so that teachers had multiple opportunities to explore the mathematical ideas in a variety of contexts. For example, teachers had multiple opportunities to develop a range of approaches to solve missing value problems. For instance, during the second class, teachers solved and shared their solution strategies to the tent/scout problem (“Four tents house 12 scouts. How many scouts will 40 tents house?” [Carpenter et al., 1999, p. 25]) and examined student responses to the tent/scout problem. These student responses varied with respect to correctness, strategy, and representation, and many differed from the strategies produced by the teachers. During Class 9, teachers solved and shared their solutions to the candy jar problems (shown in Figure 13) and read *The Case of Marie Hanson* (Smith, Silver, & Stein, 2005b), which highlighted a range of correct and incorrect solutions to the candy jar problems. Thus, through the exploration of student work depicted in written responses and cases, teachers were not only exposed to alternative ways of solving missing value problems but also had opportunities to analyze solution strategies for these problems.



<u>Solve</u>

1. Suppose you have a larger candy jar with the same ratio of Jolly Ranchers to Jawbreakers as shown in the candy jar above. If the jar contains 100 Jolly Ranchers, how many Jawbreakers are in the jar?

2. Suppose you have an even larger candy jar with the same ratio of Jolly Ranchers to Jawbreakers as shown in the candy jar above. If the jar contains 720 candies, how many of each kind of candy are in the jar?

3. Suppose you are making treats to hand out to trick-or-treaters on Halloween. Each treat is a small bag that contains 5 Jolly Ranchers and 13 Jawbreakers. If you have 50 Jolly Ranchers and 125 Jawbreakers, how many complete small bags could you make?

Figure 13. The candy jar problems that teachers solved during Class 9.
Taken from Smith, M. S., Silver, E. A., & Stein, M. K. (2005b). *Improving instruction in rational numbers and proportionality: Using cases to transform mathematics teaching and learning*. New York: Teachers College Press, p. 26.

3.2.2. Norms and Practices of the Course

An additional purpose of the course was to provide teachers with an opportunity to engage in mathematical activities, as students, in the very same ways in which they were being asked to teach. Therefore, the instructor posed challenging tasks that could be viewed from multiple perspectives, engaged teachers in discourse, and created a learning environment that supported and encouraged reasoning. In contrast to teachers' typical experiences as students of mathematics, the instructor rarely "told" teachers anything; rather, she created opportunities for them to engage in an activity in which the idea she wanted them to explore was embedded. For example, when the instructor wanted teachers to come to understand the mathematical relationships embedded in proportional situations, she presented them with the park and zoo situations (shown in Figure 6 in Chapter Two) and asked them to consider whether either situation was proportional and why. The instructor then compiled the teachers' individual responses to the task and created a list of teacher-generated rationales for why the zoo situation is proportional but the park situation is not. In a subsequent class, the instructor asked teachers to consider the rationales on the list. Thus individual teachers had an opportunity to consider rationales other than the ones they had generated themselves and develop explanations regarding the appropriateness of each rationale.

It is also important to note that the bulk of the instructor's conversational turns functioned to structure, support, or continue the class discussions (e.g. launching the activity, calling on teachers who wished to contribute, acknowledging teacher contributions) or to revoice ideas stated by the teachers in the classroom for the purpose of clarification, refinement, or reiteration in the public conversational space (O'Connor & Michaels, 1996). As a general rule, the instructor did not introduce new ideas into the public space.

Although teachers engaged in different types of activities during each class, activities were typically enacted in three phases: (1) teachers worked individually on the activity for about five to ten minutes; (2) teachers then discussed their individual ideas with a small group (which ranged in size between two and four teachers) to arrive at a consensus opinion or to decide that they had a fundamental disagreement that could not be resolved; and (3) teachers engaged in a public sharing and whole group discussion of ideas generated by the groups. During individual and small group work the instructor monitored teachers as they worked, asking questions to determine what they understood about the activity, providing support when they were experiencing difficulty (e.g., by helping teachers focus on salient features of the activity; by suggesting a starting point for teachers' work), and making note of key aspects of teachers' work that would serve as a resource in the sharing phase of the lesson. Sharing took various forms including posting of problem solutions on newsprint for public viewing and future reference, individual or group presentations at the overhead projector stationed in the front of the room, or individual and group contributions offered in an ongoing discussion in which the instructor acted as moderator and recorder. During the whole group discussion the instructor pressed teachers to provide justification for their solutions, strategies, or claims; invited teachers to question the ideas being discussed; pressed teachers to make connections between different solutions and ideas; and ensured that the important mathematical and pedagogical ideas that were driving the lesson were brought to the fore.

Thus, teachers in the course had an opportunity to experience firsthand a way of teaching whose focus was thinking and reasoning. Multiple strategies and perspectives were encouraged and teachers' ideas, even if they were incorrect or controversial, were respected.

3.3. Participants

All participants in this study were enrolled in one of two fifth-year teacher education programs at a large, urban university that culminated in an Instructional 1 certification in secondary (7-12) mathematics. In order to be accepted into these master's-level programs, applicants needed to have a bachelor's degree in mathematics (or the equivalent) and a minimum QPA of 3.0. During the summer or fall semesters prior to data collection, all participants completed four core courses: a teaching lab in which they learned to plan mathematics lessons and three mathematics methods courses (one whose focus was curriculum, one whose focus was instruction, and one whose purpose was to help teachers learn how and when to use technology appropriately in their mathematics lessons).

The treatment group consisted of ten preservice secondary teachers (five females, five males) who were enrolled in a Master of Arts in Teaching (MAT) program. The treatment group included all ten teachers enrolled in the MAT program. In this program, teachers earn a teaching certificate and a master's degree by completing a full-time, yearlong field placement (i.e., internship) in a public school classroom and by completing university coursework in the evenings. The advanced mathematics methods course focused on proportional reasoning described in the previous section was a required course in the MAT program. At the time the teachers in the treatment group began this course, they had completed summer courses as well as one semester of coursework and one semester of their teaching internship.

The contrast group consisted of six preservice secondary teachers (all female) who were enrolled in a Professional-Year (PY) certification program. In this program, teachers earn a teaching certificate by completing university coursework during the fall semester and by completing a one-semester field placement (i.e., student teaching) in a public school classroom during the spring semester. The advanced mathematics methods course described in the previous section was not a required course for the PY program. At the time the study was conducted, teachers in the PY program were doing their student teaching. It is important to note that the contrast group consisted of only a subset of teachers enrolled in the PY program who elected to participate in the study. In addition, over half of the teachers in the contrast group were pursuing a teaching career as a second career (by contrast, only one teacher in the treatment group was pursuing teaching as a second career). As such, the contrast group was comprised of volunteers who may not have been representative of the population it aimed to represent.

3.4. Data Sources

In this section, the data sources are described. A variety of data sources were collected in order to explore the five research questions [(1) What do preservice secondary mathematics teachers know and understand about proportional reasoning prior to participation in a course specifically focused on proportional reasoning?; (2) What do preservice secondary mathematics teachers know and understand about proportional reasoning immediately after participation in a course specifically focused on proportional reasoning?; (3) How do preservice secondary mathematics teachers who participated in a course specifically focused on proportional reasoning differ from preservice secondary mathematics teachers who did not participate in the course in their understandings about proportional reasoning?; (4) To what extent can teacher learning be accounted for by participation in a course specifically focused on proportional reasoning?; and (5) To what extent do preservice secondary mathematics teachers who participated in a course specifically focused on proportional reasoning draw upon their understandings about proportional reasoning in a subsequent course?]. Table 1 illustrates the match between the data sources and research questions. In the following sections, the data used to explore the research questions is described.

Table 1.

The Match Between the Data Sources and the Research Questions

<table>
<tr><td colspan="3" rowspan="2"></td><td colspan="5">Research question</td></tr>
<tr><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><td rowspan="6">Data sources</td><td rowspan="4">Proportional reasoning course</td><td>Pre/posttest</td><td>X</td><td>X</td><td>X</td><td></td><td></td></tr>
<tr><td>Pre/post interview</td><td>X</td><td>X</td><td>X</td><td></td><td></td></tr>
<tr><td>Written artifacts (i.e., overhead transparencies, posters) from class sessions</td><td></td><td></td><td></td><td>X</td><td></td></tr>
<tr><td>Videotapes of whole class discussions in which teachers explored the three ideas in the framework</td><td></td><td></td><td></td><td>X</td><td></td></tr>
<tr><td rowspan="2">Algebra course</td><td>Pre/posttest, question 2</td><td></td><td></td><td></td><td></td><td>X</td></tr>
<tr><td>Videotapes of whole class discussions in which teachers accessed understandings of proportionality (Classes 7 and 8)</td><td></td><td></td><td></td><td></td><td>X</td></tr>
</table>

Note. Shading indicates data sources that were collected from both the treatment and contrast groups.

3.4.1. Pre/Post Instruments

The purpose of this study was to examine changes in teacher knowledge as a result of participation in a practice-based mathematics methods course focused on proportional reasoning. Pre/post measures are particularly useful to explore changes over time and differences between the treatment group and the contrast group (Creswell, 2002). This study utilized two pre/post measures: a pre/post written test and a pre/post interview.

3.4.1.1. Pre/Posttest

The pre/posttest (shown in Appendix A) included twenty-four constructed-response mathematics tasks whose purpose was to examine teachers' ability to: (1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations. The tasks also prompted teachers to explain their thinking or justify their answers. The tasks were selected and/or adapted from the literature so as to represent a range of problem types, contexts, and numeric features and, as a collection, assess teachers' ability to reason proportionally. Table 2 illustrates what each task was intended to illuminate.

Table 2.

The Match Between the Pre/Posttest Items and the Three Abilities

Task Number	Solve a Variety of Problem Types			Discriminate proportional from nonproportional situations	Understand the mathematical relationships embedded in proportional situations
	Missing value	Numerical Comparison	Qualitative		
1-4	**X**				
5				**X**	
6		**X**		**X**	
7-8			**X**		
9				**X**	
10				**X**	
11-22				**X**	**X**
23	**X**				
24		**X**		**X**	

For example, five tasks (1, 2, 3, 4 and 23) assessed teachers' ability to solve missing value problems. In tasks 1-4, teachers were asked to solve missing value problems that were devoid of context. It was expected that teachers would correctly solve these proportions because they can be easily solved using cross multiplication. However, the tasks also prompted teachers to solve each problem in two different ways. This data served to document whether teachers had additional solution strategies available to them, and if so, the types of strategies. The remaining missing value problem on the instrument (task 23) was situated in a context that has been well documented as difficult, similarity. This problem was also expected to be difficult for teachers since neither the within- nor between-ratios were integer in this problem.

Both the treatment and contrast groups completed the pretest in the beginning of the spring semester and the posttest at the end of the same semester. The teachers were allowed as much time as they needed to complete the instrument, and most completed it in less than one hour.

3.4.1.2. Pre/Post Interview

The semi-structured interviews (see Appendices B and C for copies of the interview protocols) each contained three items whose purpose was to examine preservice secondary teachers' ability to: (1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations. The interview items supplemented the data collected from the pre/posttest. For example, only one type of task on the pre/posttest captured teachers' ability to understand the mathematical relationships embedded in proportional situations (see tasks 11-22 in Appendix A). Therefore, a focus of the interviews was to explore teachers' use of these relationships in a variety of tasks – specifically, as they defined a proportional relationship, created an example and nonexample of a proportional relationship, and examined a real-world situation. Table 3 illustrates what each question was intended to illuminate.

Table 3.

The Match Between the Interview Items and the Three Abilities

Interview item	Solve a Variety of Problem Types			Discriminate proportional from nonproportional situations	Understand the mathematical relationships embedded in proportional situations
	Missing value	Numerical Comparison	Qualitative		
Pre					
1a. Describe a proportional relationship				**X**	**X**
1b-c. Give an example of a proportional and nonproportional situation				**X**	**X**
2. Examining student work		**X**			
3. Problem Sort				**X**	
Post					
1a-c. Reflecting on learning[a]					
2. Snowfall				**X**	**X**
3a. Describe a proportional relationship				**X**	**X**
3b-c. Give an example of a proportional and nonproportional situation				**X**	**X**

[a] Only teachers in the treatment group (i.e., enrolled in the course) were asked this item during the post-interview. This item was included on the interview as part of a larger research endeavor of which this study was a part.

For example, item 2 on the post-interview (snowfall) examined teachers' ability to recognize the mathematical relationships embedded in proportional situations. In this item, teachers were asked to examine data (presented in a paragraph, a table, and a graph) on two Iowa cities' snowfall during a snowstorm. The relationship between the hours it snowed and the inches of snow on the ground was proportional for Cedar Rapids but was not proportional for Mason City since there were six inches of snow on the ground prior to the storm. Teachers were asked what they could determine from the table and the graph about each of these situations. As such, the question also provided teachers with an opportunity to spontaneously characterize the relationships depicted in the situations as proportional or not. If teachers did not spontaneously comment on the proportionality of the relationships, they were explicitly asked if either of the situations represented a proportional relationship. When teachers commented on the proportionality of the relationships, they were pressed to explain and clarify any aspect of their response that was not clear or that lacked sufficient detail.

Each teacher participated in two interviews, one early in the semester (between Class 3 and Class 5), and one at the end of the semester, after the last class (Class 15). Teachers were permitted to write on the materials that accompanied each question. These materials were kept for analysis. Each interview was audio taped and was approximately one hour in length. The author and a doctoral student in mathematics education conducted the interviews. The doctoral student underwent two hours of training in using the interview protocols and also had previous experience in interviewing teachers about their understandings of proportionality. As shown in the interview protocols, the interviews were semi-structured, and allowed the interviewer to probe the teachers' thinking. Probing was intended to press teachers to explain their thinking and clarify their responses. For example, questions such as, "What do you mean by…?" or "Can you say more about…?" were used to understand what teachers meant by particular phrases that were taken-as-shared during the course (e.g., "scaled up"; "constant of proportionality").

3.4.2. Data Related to the Proportional Reasoning Course

In addition to the pre/post data, additional data related to the course was collected. First, the author attended and videotaped each class. Brief field notes were also taken, which noted the assignment of teachers to small groups and the amount of time spent on each activity. The purpose of the field notes was to aid in locating particular discussions on the videotapes. In addition, any written work that was made public during class (i.e., posters produced by the small groups; overhead transparencies) was collected. Finally, each teacher's notebook (which contained written assignments, handouts, and classwork) was photocopied. The purpose of collecting this data was to aid in exploring research question 4 (To what extent can teacher learning be accounted for by participation in a course specifically focused on proportional reasoning?).

3.4.3. Data Related to the Algebra Course

As noted previously, the ten teachers in the treatment group were also required to complete a subsequent course whose focus was algebra as the study of patterns and functions in the middle grades. The instructor of the proportional reasoning course also taught this course. The algebra course was similar to the proportional reasoning course with respect to the types of activities, assignments, discourse, and class norms. The algebra course was offered during a six-week term during the summer of 2003, and met twice a week for three hours. The algebra course began two weeks after the end of the proportional reasoning course. The twenty-one teachers enrolled in the algebra course varied with respect to certification and subject matter preparation as shown in Table 4.

Table 4.

The Programs and Certification of the Algebra Course Participants

Certification	MAT	M.Ed.	Doctoral	Total
Elementary (K-6 all subjects)	3	1		4
Secondary (7-12 mathematics)	10	5		15
Deaf Education		1	1	2
Total	13	7	1	21

Note. Shading indicates the ten teachers in the treatment group.

A variety of data were collected during the enactment of the algebra course in order to examine teachers' understandings before, during, and after the course - pre/posttests, pre/post interviews, videotapes of each class session, and all written artifacts (i.e., overhead transparencies, posters). A small subset of this data was relevant to exploring the fifth research question, To what extent do preservice secondary mathematics teachers who participated in a course specifically focused on proportional reasoning draw upon their understandings about proportional reasoning in a subsequent course? In particular, teachers' responses to question 2 from the pre/posttest and transcripts[10] from two class discussions[11] served as data sources in examining research question 5. These sources are described in more detail in the following sections.

[10] The video of each class session of the algebra course was transcribed as part of another research study.
[11] Teachers in the algebra course spontaneously introduced proportionality during three classes, as evidenced by the transcripts of each class. However, the instructor invited further discussion about proportionality during only two of these classes. Therefore these two classes were included in the analysis.

3.4.3.1. Pre/Posttest, Question 2

During the first and last class session of the algebra course, teachers completed a paper and pencil pre/posttest. Of particular interest to this study is question 2 (shown in Figure 14) in which teachers were presented with three different situations, asked to express the relationship between the quantities in each situation using various representations (e.g., verbal description, graph, equation, table), and asked to classify each of the relationships as: (1) a function or nonfunction, (2) linear or nonlinear, and (3) proportional or nonproportional. All three relationships described functions; two of which were linear (relationships a and c), one of which was also proportional (relationship a).

Question 2.

For each of the situations below, specify the relationship between the quantities using a verbal description, numeric table, symbolic equation, and graph.

a. You are buying a number of apples. The apples cost 30 cents each. Specify the relationship between number of apples and total cost.

Check off all the terms that apply to this relationship using the list below:

☐ Function	☐ Linear	☐ Proportional
☐ Nonfunction	☐ Nonlinear	☐ Nonproportional

b. A video store charges $25 per month for unlimited rentals. Specify the relationship between the number of videos you rent in a month and the **cost per video**.

Check off all the terms that apply to this relationship using the list below:

☐ Function	☐ Linear	☐ Proportional
☐ Nonfunction	☐ Nonlinear	☐ Nonproportional

c. A cable company charges $25 a month for service plus a $50 installation fee. Specify the relationship between the number of months you subscribe and total amount paid for installation and monthly service.

Check off all the terms that apply to this relationship using the list below:

☐ Function	☐ Linear	☐ Proportional
☐ Nonfunction	☐ Nonlinear	☐ Nonproportional

Figure 14. A pre/posttest item from the algebra course that asked teachers to characterize three relationships as proportional or not.
Taken from Hillen, A. F. (2004). *Accessing proportionality in the service of developing the concept of function*. Poster presented at the annual meeting of the American Educational Research Association, San Diego, CA.

3.4.3.2. Whole Class Discussion, Class 7

During Class 7, teachers were asked to work in their small groups and create a real world situation that defined a functional relationship, as shown in Figure 15. The whole class discussion of this activity began with the instructor inviting groups to share their examples.

Ursula, one of the teachers in the treatment group, was in the first group to share their example. She spontaneously introduced the idea of proportionality to the class, and the instructor subsequently raised the idea of proportionality for further discussion (specifically, the differences between proportional and nonproportional relationships that can be seen in various representations, such as graphs, tables, equations, and verbal descriptions). This discussion lasted approximately 15 minutes.

Make up a real-world situation that defines a functional relationship. Use your example to do the following: • Sketch a graph illustrating the relationship; • State the relationship using the language of functions; • Build a chart with numbers that might go with your relationship; • Explain the graph and chart as ways of presenting the same information in different forms; • Explain how your example meets the formal definition of a function.

Figure 15. The activity that teachers worked on during Class 7 of the algebra course in which they spontaneously introduced proportionality.

3.4.3.3. Whole Class Discussion, Class 8

Teachers spontaneously revisited the idea of proportionality in Class 8 during a discussion about a task involving linear relationships situated in a "meal plan" context. Prior to the discussion during Class 8, teachers had determined "the best" of the following three meal plans: Regular Price (depicted by the equation $y = 10x$), Plan A (depicted by the equation $y = 8x + 4$), and Plan B (depicted by the equation $y = 6x + 12$)[12]. During Class 8, teachers were asked to

[12] Note that of the three meal plans, only the Regular Price Plan depicts a proportional relationship.

consider the average cost per meal (i.e., the total cost divided by the number of meals) for each of the three plans. Teachers graphed the average cost per meal as a function of the number of meals for each plan, as shown in Figure 16 (Smith, Silver, & Stein, 2005a). During the part of the whole class discussion in which proportionality was raised, teachers were discussing why the graph of the cost per meal for the Regular Plan was a horizontal line. This discussion of proportionality lasted approximately 7 minutes.

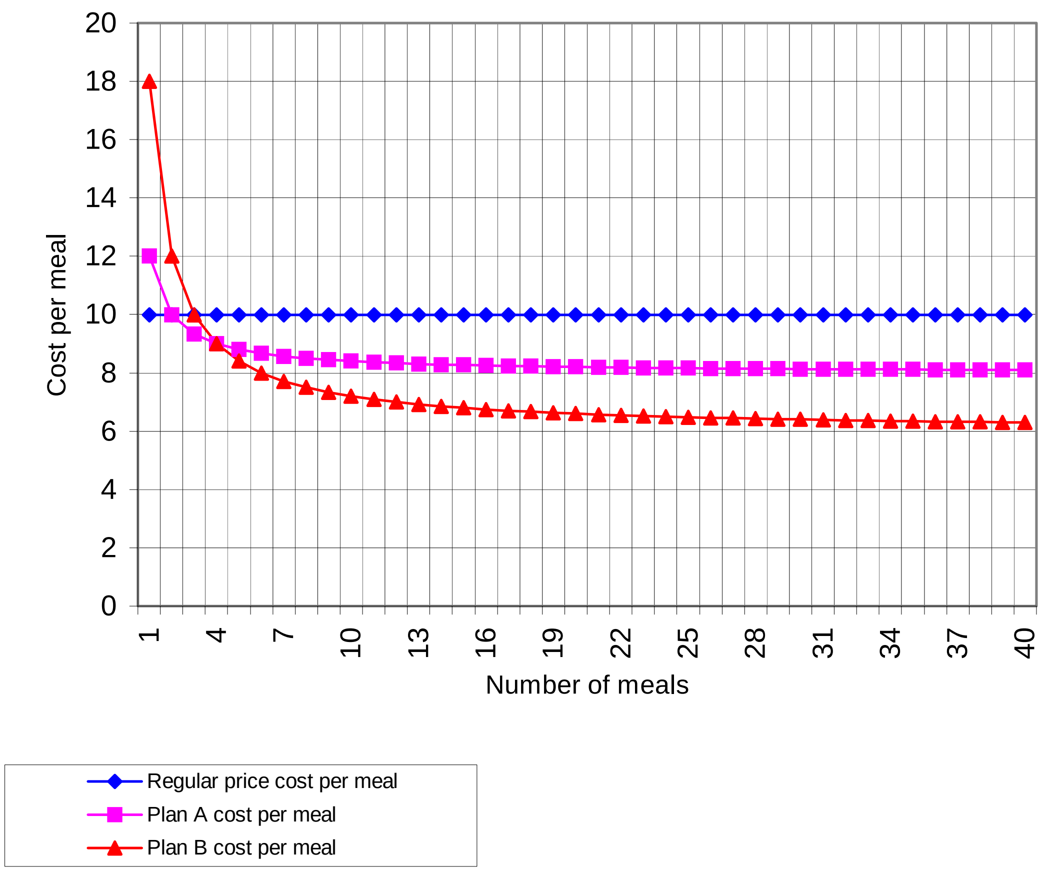


Figure 16. The cost per meal graph that teachers were discussing during Class 8 of the algebra course when they spontaneously revisited proportionality.
Taken from Smith, M. S., Silver, E. A., & Stein, M. K. (2005a). ***Improving instruction in algebra: Using cases to transform mathematics teaching and learning*****. New York: Teachers College Press, p. 113.**

3.5. Data Analysis

In this section, the ways in which the data was analyzed is described. First, the analysis of the pre- and post data, which was used to examine the first three research questions [(1) What do preservice secondary mathematics teachers know and understand about proportional reasoning prior to participation in a course specifically focused on proportional reasoning?; (2) What do preservice secondary mathematics teachers know and understand about proportional reasoning immediately after participation in a course specifically focused on proportional reasoning?; and (3) How do preservice secondary mathematics teachers who participated in a course specifically focused on proportional reasoning differ from preservice secondary mathematics teachers who did not participate in the course in their understandings about proportional reasoning?] is described. Next the ways in which the data related to the proportional reasoning course, which was used to explore research question 4 (To what extent can teacher learning be accounted for by participation in a course specifically focused on proportional reasoning?) was analyzed is described. Finally, the ways in which the data related to the algebra course, which was used to explore research question 5 (To what extent do preservice secondary mathematics teachers who participated in a course specifically focused on proportional reasoning draw upon their understandings about proportional reasoning in a subsequent course?) was analyzed is described.

3.5.1. Pre/Post Instruments

3.5.1.1. Pre/Posttest

The twenty-four mathematics tasks on the pre/posttest were coded along various dimensions, as shown in Table 5.

Table 5.

Analysis of the Pre/Posttest Items

Task	Rubric score that captured correctness and quality of explanation	Strategy used to solve problem	Correctly identified situation as proportional or nonproportional	Key understandings used
1-4		X		
5	X			
6	X	X		
7-8	X			
9	X			
10	X			
11-22			X	X
23	X	X		
24	X	X		

A scoring rubric was developed for the tasks in which teachers were asked to provide explanations. The rubrics scored teachers' responses on a scale of 0 – 4 and sought to capture various related features of teachers' responses such as correctness and quality of explanations (Lane, 1993; Silver & Lane, 1993). With respect to correctness, the rubrics described the extent to which teachers used a viable solution strategy and carried out that strategy. With respect to quality of explanation, the rubrics described the extent to which teachers justified their responses and made sense of quantities. For example, the rubric shown in Figure 17 was used to score responses to task 24. This rubric considers two characteristics of teachers' responses: (1) whether a multiplicative strategy was used; and (2) if a multiplicative strategy was used, the extent to which the values that were calculated were interpreted correctly. The rubric scores are considered ordinal data because although they are ranked (i.e., a 4 is "better" than a 3, which is better than a 2, etc.), the intervals between each score are not necessarily equal (i.e., the difference between a 4 and a 3 is not necessarily the same as the difference between a 3 and a 2).

In addition to the scoring rubric, teachers' responses to the missing value and numerical comparison problems (see tasks 1-4, 6, 23, & 24 in Appendix A) were coded by strategy (e.g., cross multiplication, within-ratio, between-ratio, building up).

4 Makes use of a correct multiplicative strategy and writes an explanation that correctly interprets the values that were calculated.

For example, the 455 feet by 508 feet size lot is the most square because the ratio of the width:length is .89 (the ratio of the width:length of the 75 feet by 114 feet sized lot is .65; the ratio of the width:length of the 185 feet by 245 feet sized lot is .75). A square's width and length are the same, so the ratio of width:length of a square is always 1. Therefore the rectangle whose ratio of width:length is closest to 1 is the most square.

3 Makes use of a correct multiplicative strategy and writes an explanation that vaguely interprets the values that were calculated.

For example, the 455 feet by 508 feet size lot is the most square because the ratio of the width:length is .89 (the ratio of the width:length of the 75 feet by 114 feet sized lot is .65; the ratio of the width:length of the 185 feet by 245 feet sized lot is .75). The rectangle with the ratio closest to 1 is the most square.

2 Makes use of a correct multiplicative strategy and either writes no explanation or writes an explanation that incorrectly interprets the values that were calculated.

For example, the 455 feet by 508 feet size lot is the most square because the ratio of the width:length is .89 (the ratio of the width:length of the 75 feet by 114 feet sized lot is .65; the ratio of the width:length of the 185 feet by 245 feet sized lot is .75).

1 Makes use of an incorrect strategy (most likely an additive strategy).

For example, argues that the 75 feet by 114 feet sized lot is the most square because the difference between the sides is only 39, and the differences for the other lots are 53 and 60, respectively.

0 No response

Figure 17. Scoring rubric for task 24 on the pre/posttest.
Adapted from work developed under the auspices of the ASTEROID project (NSF Award #0101799), principal investigator Margaret S. Smith.

Finally, teachers' responses to tasks 11-22 were coded in the same manner that Smith et al. (2003) coded similar pre/post tasks. That is, teachers' work on tasks 11-22 was coded so as to indicate whether teachers correctly identified each situation as proportional or nonproportional and the nature of their rationales for each situation. In particular, teachers' responses were coded so as to indicate which, if any, of the four key understandings identified by Cramer et al. (1993) and Post et al. (1988) and described in Chapter Two [(1) proportional relationships are multiplicative in nature; (2) proportional relationships are depicted graphically by a line that contains the origin; (3) the rate pairs are equivalent in proportional relationships; and (4) proportional relationships can be represented symbolically by the equation $y = mx$, where the m is the slope, unit rate, and constant of proportionality] teachers used to justify their classifications.

Teachers' work was fully blinded so that the teacher who produced the response (and whether the response was produced at the beginning or end of the course) could not be identified. Two raters (the author and a doctoral student in mathematics education who had previous experience in coding teachers' responses to items that measured their understandings about proportional reasoning) independently scored five percent of the responses for each item on the pre/post written test and the interviews to determine interrater reliability. The doctoral student underwent two hours of training in which the coding schema for each item (e.g., scoring rubrics) were discussed and examples and nonexamples of codes for each item were presented. For each item, interrater reliability ranged from 80% - 100%. For example, interrater reliability for items that were coded for use of the four key understandings (tasks 11-22 on the pre/post written test; defining a proportional relationship, creating an example and nonexample of a proportional relationship, and the snowfall item on the interviews) was 100%. Interrater reliability for several items that were scored using rubrics that captured the extent to which teachers' responses were correct and complete (tasks 6 and 24 on the pre/post written test) was 80%.

Two types of analyses were conducted in order to explore the first three research questions. First, the pre/post data on the treatment and contrast groups was analyzed in order to explore changes in the teachers' knowledge over time. Second, the pre/post data on the treatment group and contrast group was compared in order to examine similarities and differences between the two groups. Differences between the two groups of teachers at the conclusion of the course were of particular interest in gaining insight about the impact of the course on teacher learning.

Therefore, to analyze change in both the treatment and contrast groups over time, the matched-pair Wilcoxon test, which is used with ordinal data, was used to determine if there was a significant difference in teachers' rubric scores. The extent to which teachers' repertoire of strategies increased for particular tasks was described qualitatively by counting the number of strategies they used on the pretest and posttest, and noting whether they made use of any new strategies on the posttest. A dependent samples t-test was used to determine whether teachers used particular strategies significantly more frequently at the end of the course.

In order to determine whether there was a significant change in teachers' ability to characterize the relationships as proportional or not (measured by tasks 11-22), a dependent samples t-test was used. The extent to which teachers' use of the four key understandings in making the distinction between proportional and nonproportional changed over time was described qualitatively, as reported in Smith et al. (2003). A dependent samples t-test was used to determine whether teachers drew upon more key understandings by the end of the course than at the beginning of the course.

In order to examine differences between the groups both at the beginning and end of the course, quantitative analyses were used. In particular, the Mann Whitney U-Test was used to analyze teachers' rubric scores. Fisher's exact test was used to determine whether there was a significant difference between the two groups' ability to characterize the relationships as proportional or not (measured by tasks 11-22) and other differences in the categorical data.

Independent samples t-tests were used to examine differences involving continuous data, such as the number of key understandings that teachers drew upon in explaining their classifications in tasks 11-22. In comparing the two groups at the beginning of the course, two-tailed tests were used. However, since it was expected that the treatment in this study would positively impact teachers' mathematical understandings, one-tailed tests were used in comparing the two groups at the end of the course. The expectation that participation in the course would have a positive impact on teachers' mathematical understandings was reasonable since a main goal of the course was to help teachers construct (or reconstruct) their mathematical knowledge, and as such, teachers were provided with multiple opportunities throughout the course to explore ideas central to proportional reasoning (as shown in Figure 12).

3.5.1.2. Pre/Post Interview

The interviews were transcribed and teachers' responses were coded along various dimensions, as shown in Table 6. Rubrics were developed to capture various related features of teachers' responses such as correctness and quality of explanations. For example, item 2 on the pre-interview (examining student work) asked teachers to make sense of five different student responses to a numerical comparison problem set in a mixture context (similar to Noelting's [1980] orange juice problems). Teachers' work on this item was coded so as to indicate the extent to which they were able to make sense of each student's strategy.

Table 6.

Analysis of the Interview Items

Interview item	Rubric score that captured correctness and quality of explanation	Distinguished between proportional and nonproportional situations	Correctly identified situation as proportional or nonproportional	Key understandings used
Pre				
1a. Describe a proportional relationship	**X**			**X**
1b-c. Give an example of a proportional and nonproportional situation	**X**			**X**
2. Examining student work	**X**			
3. Problem Sort		**X**		
Post				
1a-c. Reflecting on learning				
2. Snowfall			**X**	**X**
3a. Describe a proportional relationship	**X**			**X**
3b-c. Give an example of a proportional and nonproportional situation	**X**			**X**

Other items (e.g., item 3 on the pre-interview, item 2 on the post-interview) were not coded using rubrics. For example, item 2 on post-interview (snowfall) sought to examine teachers' ability to distinguish between proportional and nonproportional situations and to recognize the mathematical relationships embedded in proportional situations and was coded as described in Smith et al. (2003). That is, teachers' responses were coded so as to indicate whether they correctly identified the relationship for Cedar Rapids as proportional and Mason City as nonproportional and upon which of the four key understandings teachers drew in order to make this distinction.

It is important to note that two items from the interviews were intended to provide evidence of teachers' ability to discriminate proportional from nonproportional situations but were not included for analysis. First, an item in which teachers were asked to describe a proportional relationship (see Appendix B for item 1a on Interview 1 and Appendix C for item 3a on Interview 2) was not included because upon reviewing the data, this item did not elicit teachers' understandings of the *differences* between proportional and nonproportional relationships; rather, the item elicited teachers' understandings about proportional relationships (and as such, was still analyzed for teachers' understandings of the mathematical relationships embedded in proportional situations, as intended). In addition, an item in which teachers were asked to examine a set of problems (see Appendix B for item 3 on Interview 1) was not included for analysis since this item was asked only on Interview 1.

The interview data was also analyzed in two different ways: (1) an analysis of the pre/post data on the treatment and contrast groups to explore changes in teachers' knowledge over time; and (2) a comparison of the pre/post data on the treatment and contrast groups to examine differences between the two groups. Data from items coded using rubrics was analyzed using the same statistical tests described for the analysis of the pre/posttest. The snowfall data was analyzed in similar ways as tasks 11-22 on the pre/posttest.

3.5.2. Data Related to the Proportional Reasoning Course

As noted previously, the data related to the proportional reasoning course was used to explore research question 4, To what extent can teacher learning be accounted for by participation in a course specifically focused on proportional reasoning? In particular, the mathematics that teachers came to know and understand (as evidenced from teachers' work on the pre/posttest and pre/post interview) was used as a lens through which to code the videotapes of the class discussions.

In order to identify the class discussions to be analyzed, the author examined the entire written record of the course (i.e., all overhead transparencies, writing on the chalkboard, and posters) by watching each whole class discussion on videotape. Any whole class discussion in which the mathematics that teachers appeared to learn was evident in the written record was included in the analysis. For example, class discussions in which teachers presented solution strategies on an overhead transparency were included in the analysis. The analysis consisted of re-watching each identified class discussion and indicating the total number of "turns" spoken during each discussion (including the number of turns spoken by each of the ten teachers in the treatment group, the instructor, and the five other teachers [who were inservice teachers pursuing an M.Ed.] in the course), and the number of turns spoken that were related to the mathematics that teachers appeared to learn during the course. A turn was defined as an uninterrupted audible contribution by a speaker (either the instructor or a teacher). A new turn occurred with every new speaker (Inagaki, Hatano, & Morita, 1998; Smith et al., 2003).

3.5.3. Data Related to the Algebra Course

As noted previously, the data related to the algebra course was used to explore research question 5, To what extent do preservice secondary mathematics teachers who participated in a course specifically focused on proportional reasoning draw upon their understandings about proportional reasoning in a subsequent course? Since the purpose of research question 5 was to examine the extent to which the ten teachers in the treatment group drew upon their understandings of proportionality, only these teachers' work was analyzed (i.e., the remaining eleven teachers in the algebra course were not included in this analysis).

3.5.3.1. Pre/Posttest, Question 2

Teachers' work on this question was coded so as to indicate whether or not teachers correctly identified the three relationships as proportional by checking the appropriate box ("proportional" or "nonproportional"). The number of teachers who correctly classified each of the relationships on the pretest and the posttest was reported. In addition, a dependent samples t-test was used to determine whether there was a significant difference in the number of relationships teachers were able to correctly classify at the beginning and end of the algebra course.

3.5.3.2. Whole Class Discussions, Class 7 and 8

The whole class discussions in Class 7 and Class 8 were transcribed and organized into "turns" which indicated an uninterrupted contribution by a speaker (either the instructor or a teacher). A new turn occurred with every new speaker (Inagaki, Hatano, & Morita, 1998; Smith et al., 2003). The portion of the Class 7 discussion related to proportionality consisted of 178 turns and lasted approximately 15 minutes. The portion of the Class 8 discussion related to proportionality consisted of 68 turns and lasted approximately 7 minutes.

The four key understandings described in Cramer et al. (1993) and Post et al. (1988) provided a lens for analyzing the portions of the two class discussions related to proportionality: (1) proportional relationships are multiplicative in nature; (2) proportional relationships are depicted graphically by a line that contains the origin; (3) the rate pairs are equivalent in proportional relationships; and (4) proportional relationships can be represented symbolically by the equation y = *m*x, where the *m* is the slope, unit rate, and constant of proportionality. In particular, the nature of the argument or justification made public by the teachers was identified.

3.6. Summary

The study sought to examine changes in teachers' ability to reason proportionally using a quasi-experimental design. Two groups of teachers, the treatment group (who was enrolled in the proportional reasoning course), and the contrast group (who was not enrolled in the course), served as participants for the study. A pre/posttest in which teachers solved 24 mathematics tasks and a pre/post interview in which both groups of teachers solved mathematics tasks, discussed mathematical ideas, and examined students' responses to mathematics tasks served as data sources for exploring research questions 1, 2, and 3. This data was analyzed to examine the extent to which the teachers' ability to: (1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations, changed over time. The data was analyzed to examine similarities and differences between the two groups at the beginning and end of the proportional reasoning course. In addition, data collected during the enactment of the proportional reasoning course (e.g., videotapes of each class session) served to examine research question 4. This data was analyzed so as to indicate the activities in which teachers had opportunities to learn the mathematics they appeared to learn (as indicated by the pre/post instruments), and who made public contributions related to those ideas. Finally, teachers' work during a subsequent course served to examine research question 5. This data was analyzed so as to indicate how teachers drew upon their understandings of proportionality in a new setting. In the next chapter, the results of these analyses are presented.

4. CHAPTER FOUR: RESULTS

The purpose of this study was to examine preservice secondary mathematics teachers' understandings about proportional reasoning prior to and upon completion of a practice-based methods course focused on proportional reasoning in the middle grades, their opportunities to learn the intended content, and their ability to apply what was learned in a new setting. In order to provide further evidence of the impact of the course on teachers' learning, the study also examined a similar group of teachers' (who were not enrolled in the course) understandings about proportional reasoning before and after the course.

In this chapter, the results of the study are presented, and are also situated in Ball and colleagues' (Ball et al., 2005; Ball, Bass, & Hill, 2004; Hill & Ball, 2004; Hill, Schilling, & Ball, 2004) recent framework of *mathematical knowledge for teaching*. In this framework, Ball and her colleagues argue that teachers need two important types of content knowledge in order to successfully teach mathematics: *common content knowledge* and *specialized content knowledge*. Common content knowledge consists of the mathematical understandings that are expected of everyday users of mathematics. For example, solving a missing value problem draws on one's common content knowledge. By contrast, specialized content knowledge consists of the mathematical understandings that are needed for *teaching*, and are beyond that of everyday users of mathematics. For example, teachers draw on their specialized content knowledge when they represent mathematical ideas or analyze students' errors (Ball et al., 2005; Hill, Schilling, & Ball, 2004).

With respect to proportional reasoning, the three abilities that one needs to be a proportional reasoner [(1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations] can be consider common content knowledge, since these are capacities that even students need to develop (NCTM, 2000). In addition, teachers also need to know and understand a variety of solution strategies. This can be considered an aspect of specialized content knowledge, since everyday users of mathematics do not necessarily need a broad repertoire of solution strategies[13].

In this chapter, the results of the study are presented, beginning with the results that identify the common and specialized content knowledge that teachers in the proportional reasoning course appeared to learn during the course [i.e., results from the first two research questions: (1) What do preservice secondary mathematics teachers know and understand about proportional reasoning prior to participation in a course specifically focused on proportional reasoning?; and (2) What do preservice secondary mathematics teachers know and understand about proportional reasoning immediately after participation in a course specifically focused on proportional reasoning?]. In order to provide evidence that teachers' participation in the course influenced their enhanced understandings of proportional reasoning, the similarities and differences between the understandings of teachers who completed the course and those who did not are examined. In addition, teachers' opportunities to explore proportional reasoning during the course are explored. Finally, the extent to which teachers drew upon their understandings of proportional relationships during a subsequent course focused on algebra as the study of patterns and functions is described.

[13] Ball and colleagues identify additional aspects of specialized content knowledge, such as analyzing students' errors and providing mathematically careful explanations, but these aspects were not investigated in this study -- although teachers likely had opportunities to engage in these practices during the course.

4.1. Teachers' Common and Specialized Content Knowledge Prior To and Upon Completion of the Proportional Reasoning Course

In this section, research questions 1 and 2 are explored: (1) What do preservice secondary mathematics teachers know and understand about proportional reasoning prior to participation in a course specifically focused on proportional reasoning?; and (2) What do preservice secondary mathematics teachers know and understand about proportional reasoning immediately after participation in a course specifically focused on proportional reasoning? The purpose of these two research questions was to identify the mathematics that teachers appeared to learn from participating in the course. In this section, teachers' understanding of proportional reasoning is discussed based on their work on a pre/posttest and a pre/post interview that examined their ability to: (1) solve a variety of problem types; (2) discriminate proportional from nonproportional situations; and (3) understand the mathematical relationships embedded in proportional situations.

4.1.1. Solve a Variety of Problem Types

The results indicated that even prior to the course, the ten teachers in the treatment group were able to correctly solve a variety of missing value, numerical comparison, and qualitative problems – an important aspect of common content knowledge of proportional reasoning. However, the results also indicated that teachers' specialized content knowledge was somewhat limited, as evidenced by their narrow repertoire of strategies at the beginning of the course. Teachers appeared to enhance their specialized content knowledge by the end of the course, as evidenced their use of a broader range of strategies on the posttest. In this section, teachers' solutions to missing value, numerical comparison, and qualitative problems are described in more detail.

4.1.1.1. Missing Value Problems

The pre/posttest contained five missing value problems – four of which were not situated in a context (see tasks 1-4 in Appendix A), and one of which was situated in a context involving similar figures (see task 23 in Appendix A). It was expected that teachers would experience little difficulty in solving the four problems devoid of context in at least one way, since these problems could be solved using cross multiplication (a procedure well-known to most teachers). However, teachers were asked to solve these four problems in more than one way if they could. This aspect of the problems was expected to be challenging for teachers. In addition, the missing value problem that was situated in a context was expected to be difficult because the context, similarity, has been documented as being particularly difficult for students (Hart, 1981, 1988; Kaput & West, 1994; Lamon, 1993b; Singh, 2000).

The results indicated that the ten teachers in the treatment group were able to correctly solve the five missing value problems on the pre- and the posttest[14]. However, the results also indicated that teachers tended to rely on procedural strategies such as cross multiplication at the beginning of the course. Teachers used a broader range of strategies to solve the five missing value problems on the posttest. In addition, the frequency with which teachers used cross multiplication and other algebraic strategies decreased by the end of the course. In this section, teachers' solutions to missing value problems (tasks 1-4 and 23) are described in more detail.

<u>Tasks 1-4: Missing value problems devoid of context</u>. As noted previously, all ten teachers in the treatment group correctly solved these four missing value problems on both the pre- and posttest. However, key differences are revealed upon examination of teachers' solution strategies, as shown in Table 7. First, teachers' work on the pretest is described, followed by a

[14] One teacher, Elaine, correctly determined that $x = 60/8$ for task 3, but incorrectly reduced 60/8 to 15/4. However, her response was still considered to be correct since she used an appropriate multiplicative strategy to solve for x, and her error was in reducing the fraction.

description of teachers' work on the posttest and a comparison of their work on the pre- and posttest.

As shown in Table 7, cross multiplication was the prevalent strategy in teachers' work on the pretest. In fact, all ten teachers used cross multiplication as one of their strategies to solve the four problems on the pretest. However, five teachers (Bert, Bruce, Elaine, Owen, and Ursula) were unable to solve some, or all, of the problems in a way *other than* cross multiplication. That is, if a teacher solved a problem in only one way, that way was always cross multiplication. Three of these teachers (Bruce, Elaine, and Owen) solved three of the four problems in two different ways, and one problem in one way, using cross multiplication. For Bert and Ursula, cross multiplication was the only strategy used to solve these four problems.

Table 7.

Treatment Group Teachers' Solution Strategies to Tasks 1-4

	Task 1 4/20=x/35		Task 2 2/7=6/x		Task 3 3/8=x/20		Task 4 9/15=12/x	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Bert	Cross multiplication	Cross multiplication	Cross multiplication	Between-ratio	Cross multiplication	Cross multiplication	Cross multiplication	Between-ratio
Bonnie	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio
Bruce	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio
Carl	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio
Christopher	Cross multiplication Between-ratio	Within-ratio Between-ratio	Cross multiplication Between-ratio	Within-ratio Between-ratio	Cross multiplication Between-ratio	Between-ratio Between-ratio[a]	Cross multiplication Between-ratio	Between-ratio Between-ratio[b]
Elaine	Cross multiplication Algebraic strategy	Cross multiplication Between-ratio Between-ratio[c]	Cross multiplication Algebraic strategy	Cross multiplication Between-ratio	Cross multiplication Algebraic strategy	Cross multiplication	Cross multiplication	Cross multiplication Between-ratio
Nanette	Cross multiplication Algebraic strategy	Cross multiplication	Cross multiplication Algebraic strategy	Cross multiplication	Cross multiplication Algebraic strategy	Cross multiplication	Cross multiplication Algebraic strategy	Cross multiplication
Nora	Cross multiplication Algebraic strategy	Cross multiplication Within-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Algebraic strategy	Cross multiplication Between-ratio	Cross multiplication Cross multiplication[d]	Cross multiplication Between-ratio
Owen	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio
Ursula	Cross multiplication	Cross multiplication Between-ratio	Cross multiplication	Cross multiplication Between-ratio	Cross multiplication	Cross multiplication Between-ratio	Cross multiplication	Cross multiplication Between-ratio

Note: Shading indicates tasks in which the teacher solved in only one way.

[a] Christopher's second strategy involved identifying the between-ratio, 2.5, in a different way than he had in his first strategy.

[b] Christopher's second strategy involved first reducing 9/15 to 3/5, and then identifying the between-ratio of 4.

[c] Elaine's third (and unsolicited) strategy involved reducing 4/20 to 1/5, and then identifying the between-ratio of 7.

[d] Nora's second strategy involved first reducing 9/15 to 3/5, and then using cross multiplication to solve the proportion 3/5 = 12/x.

The second-most common strategy teachers used to solve tasks 1-4 on the pretest was the between-ratio strategy (see Figure 18 for an example of the between-ratio strategy to solve task 2). Teachers also used an adaptation of the between-ratio strategy, in which they reduced one of the ratios in order to more easily identify the between-ratio, as shown in Figure 19. For the purposes of this study, the strategies illustrated in Figures 18 and 19 were both classified as between-ratio strategies.

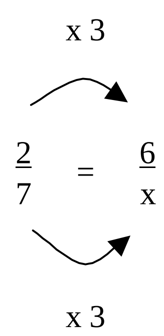


Figure 18. Example of a between-ratio strategy to solve task 2.

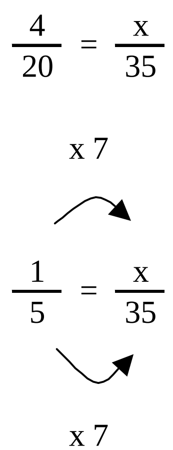


Figure 19. Example of a between-ratio strategy to solve task 1 that first makes use of reducing one of the ratios.

Six teachers (Bonnie, Bruce, Carl, Christopher, Nora, and Owen) used a between-ratio strategy to solve at least one problem on the pretest. As shown in Table 7, for some teachers (Bonnie, Carl, and Christopher), the between-ratio strategy and cross multiplication were the two strategies used at the beginning of the course. For other teachers (Bert, Elaine, Nanette, and Ursula), the between-ratio strategy did not appear to be part of their repertoire at the beginning of the course[15].

The remaining strategies teachers used to solve the problems on the pretest can be classified as algebraic in nature. For example, Nora consistently used a "solve for x" strategy to solve three of the problems, as shown in Figure 20. Nanette also used an algebraic strategy to solve each of the four problems.

$$\frac{3}{8} = \frac{x}{20}$$

$$20\left(\frac{3}{8}\right) = \left(\frac{x}{20}\right)20$$

$$\frac{60}{8} = x$$

$$7.5 = x$$

Figure 20. Example of an algebraic strategy to solve task 3.

[15] The claim that Bert, Elaine, Nanette, and Ursula did not appear to have the between-ratio strategy at their disposal at the beginning of the course is further strengthened by the fact that none of these four teachers used the between-ratio strategy to solve task 23 (the other missing value problem) on the pretest. Of course, just because these four teachers did not use the between-ratio strategy on the pretest does not necessarily mean that it was not part of their repertoire.

A subset of tasks 1-4 was specifically designed to explore whether teachers used strategies that were particularly useful for solving certain problems. For example, task 1, (4/20=x/35), contains an integer within-ratio (i.e., the multiplicative relationship between 4 and 20 is 5). Thus it was expected that teachers might use a within-ratio to solve task 1, but not use a within-ratio strategy to solve, for example, task 2 (2/7=6/x) (because the multiplicative relationship between 2 and 7 is noninteger and thus more difficult to identify). However, despite the presence of an integer within-ratio in task 1, no teacher used a within-ratio approach to solve task 1 (or tasks 2, 3, & 4) on the pretest. This suggests that the within-ratio strategy may not have been an accessible strategy for this group of teachers prior to the course.

Another problem, task 2 (2/7=6/x), was designed so as to examine whether teachers recognized the between-ratio relationship in the proportion (i.e., the multiplicative relationship between 2 and 6 is 3). Six of the ten teachers (Bonnie, Bruce, Carl, Christopher, Nora, and Owen) used a between-ratio strategy to solve task 2. However, the between-ratio strategy may have been particularly salient for these six teachers. Five of these teachers (Bonnie, Bruce, Carl, Christopher, and Owen) consistently used between-ratio strategies on three or four of the problems (as shown in Table 7) – even when the between-ratio was noninteger and therefore not particularly easy to identify. Therefore, although over half the teachers used the strategy for which task 2 was designed, it is not clear whether they used the between-ratio strategy because it was particularly useful for this problem, or because it was a strategy with which they were already familiar.

Finally, teachers were more successful in solving the "easier" problems (tasks 1 and 2) in two different ways than more difficult problems (tasks 3 and 4) (easier problems contain integer between- or within-ratios, whereas difficult problems contain only noninteger ratios), as indicated by the shading in Table 7. Eight teachers solved the easier two problems in two different ways. However, even teachers who were able to solve the easier two problems in two different ways could not solve the more difficult problems, task 3 (3/8=x/20) and task 4 (9/15=12/x), in more than one way. For example, two teachers (Bruce and Owen) could only solve task 3 in one way - cross multiplication.

Teachers' work on the posttest differed with respect to solution strategies and the number of solutions teachers produced for each problem, as shown in Table 7. Although cross multiplication was not used significantly less on the posttest than on the pretest ($t(9) = 1.41$, $p = .09$ [one-tailed]), and eight teachers still used cross multiplication as one of their ways to solve each of the four problems, it is interesting to note that one teacher, Christopher, solved each problem in two different ways – neither of which was cross multiplication. The remaining teacher, Bert, solved each problem in only one way. Bert solved two of the problems using cross multiplication and two problems using a between-ratio strategy. Although Bert solved each problem on the posttest in only one way, it is interesting to note that on the pretest, he used cross-multiplication exclusively. Therefore it appears that Bert added a strategy to his repertoire – the between-ratio strategy[16]. The other teacher who solved the pretest problems using only cross multiplication, Ursula, solved all the problems on the posttest in two different ways. On the posttest, she solved all the problems using both cross multiplication and a between-ratio strategy. Thus, it appears that Ursula's repertoire of strategies also grew to include between-ratio strategies by the end of the course.

As noted previously, task 1 (4/20=x/35), was designed to examine teachers' ability to recognize the within-ratio relationship in the proportion. On the posttest, two teachers

[16] Bert's work on all five missing value problems (tasks 1-4 and 23) reveals that he used only cross multiplication on the pretest. On the posttest, he used cross multiplication and the between-ratio strategy.

(Christopher and Nora) used a within-ratio strategy. In addition, task 2 (2/7=6/x), was designed in order to examine teachers' ability to recognize the between-ratio relationship in the proportion. On the posttest, nine of the ten teachers used a between-ratio strategy as one of their ways to solve the problem. Teachers used the between-ratio strategy significantly more frequently on the posttest than on the pretest, $t(9) = 3.36$, $p = .004$ (one-tailed).

Most teachers were able to solve tasks 1-4 in two different ways on the posttest. Seven of the teachers solved tasks 1-4 in two different ways. In addition, one teacher, Elaine, solved three of the four tasks in two different ways. Elaine solved task 3 (3/8=x/20) in one way, cross multiplication. It could be argued that task 3 is the most difficult of the four problems for two reasons: (1) task 3 contains neither an integer between-ratio (as in task 2) nor an integer within-ratio (as in task 1); and (2) task 3 does not contain a ratio that can be reduced to yield an integer within- or between-ratio (as in task 4: 9/15=12/x can be rewritten as 3/5=12/x to reveal an integer between-ratio of 4). Interestingly, Elaine solved task 1 in three different ways, even though only two solution strategies were requested. Thus Elaine's posttest work on tasks 1-4 suggests that she had access to a variety of solution strategies, but selectively called upon strategies that were particular helpful in solving certain problems.

Finally, two of the teachers, Bert and Nanette, solved all four problems in only one way. Although at first glance Bert did not appear to make any gains in the use of strategies for solving these tasks (he solved tasks 1-4 in only one way on both the pre- and posttest), it is interesting to note that he used cross multiplication exclusively on the pretest, but used cross multiplication and the between-ratio strategy on the posttest. Thus, Bert did appear to gain an additional solution strategy during the course. Nanette, on the other hand, solved tasks 1-4 in two different ways on the pretest (cross multiplication and an algebraic strategy), but on the posttest, solved the tasks in only one way (cross multiplication). However, it is not surprising that Nanette did not use her algebraic strategy on the posttest, since such strategies were made public few times during the course[17] and therefore did not appear to be valued. Of particular interest is that Ursula, who solved tasks 1-4 in only one way on the pretest, solved the tasks in two different ways on the posttest. Thus it appears that by the end of the course, the difficulty of the problem (as defined by the absence of integer within- or between-ratios) did not influence teachers' ability to solve problems in two different ways.

On the posttest, half of the ten teachers used a strategy that they had not used on the pretest -- three teachers used between-ratio strategies (Bert, Elaine, and Ursula) and two teachers used within-ratio strategies (Christopher and Nora). In addition, several teachers did not utilize algebraic strategies (that they did use on the pretest) on the posttest. As noted previously, Christopher did not use cross multiplication on the posttest. Elaine and Nora also did not use algebraic strategies. It is likely that these teachers did not forget how to implement such strategies; but rather, chose to use alternative strategies perhaps because they made more sense or were more efficient, or because the teachers were aware that the instructor valued strategies that they could explain.

Finally, several teachers used appropriate mathematical language to describe their strategies on the posttest. For example, Bruce and Owen appropriately labeled one of their between-ratio strategies as "scaling" or "scaling up" and Ursula identified the scale factor in her between-ratio strategy. No such language was present in any teachers' work on the pretest.

Task 23: Missing value problem situated in a similarity context. As noted previously, the ten teachers in the treatment group were able to correctly determine the missing length of the enlarged photograph in task 23 and provide a valid explanation both prior to and upon

[17] As evidenced in the video analysis for research question 4.

completion of the course. This is notable because this task is situated in a difficult context, similarity, and also because neither the within- nor between-ratios in the problem are integer, which also contributed to the task's difficulty.

Teachers' solution strategies changed from the pre- to the posttest, as shown in Table 8. On the pretest, eight of the ten teachers set up a proportion (either 3/4=?/14 or 3/?=4/14) and cross-multiplied to determine the missing value. One additional teacher, Bruce, set up a proportion, but provided no other written evidence of how he determined the missing quantity (his written work included only the proportion and the answer, 10.5). The remaining teacher, Bonnie, used a between-ratio strategy (i.e., identified the scale factor, 3.5, and scaled up to the missing quantity), and correctly identified the 3.5 as the "scale factor."

On the posttest, significantly fewer teachers used cross multiplication to determine the missing dimension, $t(9) = 3.00$, $p = .01$ (one-tailed). In addition, significantly more teachers used the between-ratio strategy to determine the missing dimension, $t(9) = 2.45$, $p = .02$ (one-tailed). Only three of the ten teachers (Bert, Elaine, and Nanette) set up a proportion and cross-multiplied to determine the missing length. An additional teacher set up a proportion, but provided no other written evidence of how he determined the missing quantity (Bruce, the same teacher on the pretest). The remaining six teachers used a between-ratio strategy. Three of the six teachers who used a between-ratio strategy (Carl, Christopher, and Nora) also identified the 3.5 as the "scale factor" or "factor."

Table 8.

Treatment Group Teachers' Solution Strategies to Task 23

	Pre	Post
Bert[a]	Cross multiplication Cross multiplication	Cross multiplication Cross multiplication
Bonnie	Between-ratio	Between-ratio
Bruce	Unclear	Unclear
Carl	Cross multiplication	Between-ratio
Christopher	Cross multiplication	Between-ratio
Elaine	Cross multiplication	Cross multiplication
Nanette	Cross multiplication	Cross multiplication
Nora	Cross multiplication	Between-ratio
Owen	Cross multiplication Between-ratio	Between-ratio
Ursula	Cross multiplication	Between-ratio

[a] On both the pre- and posttest, Bert solved the task in two ways – by setting up two different proportions (original width:enlarged width = original length:enlarged length, and original width:original length = enlarged width: enlarged length), and cross multiplying to determine the missing dimension.

Summary. The shift from reliance on cross multiplication to between-ratio strategies suggests that some teachers may have learned an additional strategy, between-ratio (also known as scale factor or factor-of-change) to solve missing value problems, thus developing an aspect of their specialized content knowledge. (Of course, as noted previously, an alternative explanation of the shift from cross multiplication to the between-ratio strategy is that teachers learned that the instructor valued strategies that could be explained and that made sense to students. Therefore, teachers may have chosen not to use cross multiplication because it was not something they could easily explain[18].) Finally, across the five missing value problems on the pretest, significantly more teachers used appropriate language (e.g., scale factor) in their explanations on the posttest than on the pretest, $t(9) = 2.24$, $p = .03$ (one-tailed).

4.1.1.2. Numerical Comparison Problems

The pre/posttest included two numerical comparison problems, task 6 and task 24 (see Appendix A), both of which were situated in a context. Task 6 was an adaptation of a classic numerical comparison task in which teachers were asked to compare the relative strength of two orange juice recipes. Teachers were asked to solve this problem in two different ways. Task 24 presented teachers with dimensions for three rectangular plots of land and asked teachers to determine the plot that was the "most square." Both of these tasks were expected to be difficult for teachers because neither contained integer within- or between-ratios. Task 24 was also expected to be difficult because previous studies indicate that teachers typically use incorrect additive strategies to solve it (Heinz, 2000; Perrine, 2001; Simon & Blume, 1994; Smith et al., 2001).

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[18] As evidenced in the whole class discussion during Class 11 in which teachers discussed the rationale for cross multiplication.

In addition, item 2 on Interview 1 asked teachers to make sense of five students' solutions to a numerical comparison problem situated in a context similar to the orange juice context used in task 6 on the pre/post written test (item 2 on Interview 1 involved mixing chocolate syrup with milk to make chocolate milk). This item was expected to be difficult for teachers because some of the students' strategies may not have been ones the teachers produced themselves in solving such a task.

The results indicated that the teachers had fairly well-developed common content knowledge, even prior to the course, as evidenced by most teachers' ability to correctly solve the two numerical comparison problems on the pretest. However, the results also indicated that teachers' range of strategies for solving numerical comparison problems was somewhat limited, as evidenced by teachers' reliance on part-to-part ratios and some teachers' limited capacity to make sense of alternative strategies prior to the course. Teachers appeared to enhance their specialized content knowledge by the end of the course, as evidenced their use of a broader range of strategies on the posttest. In this section, teachers' ability to solve numerical comparison problems is discussed in the context of their work on tasks 6 and 23 prior to and after the course. In addition, teachers' responses to item 2 on Interview 1 are used to provide further evidence of the extent to which teachers could make sense of quantities used to make comparisons prior to the course.

Task 6: Comparing orange juice recipes. All ten teachers in the treatment group correctly determined that Luis' mixture had the stronger orange flavor in at least one way on the pretest. In addition, eight of the ten teachers solved the problem correctly in two different ways. One teacher, Nanette, attempted to solve the problem in two different ways - one of which was correct and one of which was not correct, and one teacher, Elaine, solved the problem in only one way. On the posttest, all ten teachers solved the problem correctly in two different ways[19]. This difference was not significant, $t(9) = 1.50$, $p = .08$ (one-tailed).

Task 6 can be solved using two different types of ratios: part-to-part (i.e., comparing orange juice concentrate to water, or vice versa) and part-to-whole (i.e., comparing orange juice concentrate to the total mixture, or comparing water to the total mixture). The types of ratios that the teachers used to solve the problem on the pre- and posttest are shown in Table 9. As shown in Table 9, half of the teachers (Bruce, Elaine, Nanette, Nora, and Ursula) used part-to-part ratios exclusively on the pretest. As such, these five teachers may not have had strategies based on part-to-whole ratios in their repertoire. The difference between the types of ratios teachers used on the pretest and posttest was not significant, $X^2(2, N = 10) = 5.00$, $p = .08$.

Teachers' responses to item 2 on Interview 1 provide additional evidence for the claim that part-to-whole strategies may not have been part of the repertoire for some of these teachers. For example, Elaine was unable to make sense of Student D's response (see Figure B.2 in Appendix B), which made use of a part-to-whole ratio: "I don't know what they compared here… I don't think that [pause] it's a valid [pause] strategy" (Elaine, Interview 1, lines 141-168). In addition, even though Nanette and Ursula made sense of the quantities Student D calculated, they both were unsure if such a strategy was valid. For example, Ursula commented, "So, they took three ounces of chocolate syrup over uh chocolate syrup plus chocolate milk for each one… But I'm not positive that's right. I don't know" (Ursula, Interview 1, lines 147-155). Only Nora and Bruce made sense of Student D's solution without difficulty[20], indicating that

[19] On the posttest, one teacher, Elaine, stated that Martin had the strong mixture. However, her written work revealed that she thought that Martin used the 5 ounces of orange juice concentrate and 7 ounces of water (this was actually Luis' mixture). Thus, Elaine did correctly determine the stronger mixture; she just labeled her response incorrectly.

[20] Prior to examining the student responses during Interview 1, Bruce decided to solve the chocolate milk problem and used the same strategy that Student D used.

Elaine, Nanette, and Ursula either had difficulty making sense of solutions they had not produced themselves, or difficulty making sense of part-to-whole strategies in general. By the end of the course, four of the five teachers who had not used part-to-whole strategies on the pretest (Bruce, Nanette, Nora, and Ursula) solved the problem on the posttest in two ways: one based on a part-to-part ratio and the other based on a part-to-whole ratio. (The remaining teacher, Elaine, also made gains by the end of the course – she was able to solve the problem in two different ways, both of which were based on part-to-part ratios.)

Table 9.

Types of Ratios Used By Teachers in the Treatment Group to Solve Task 6

	Pre	Post
2 Part-to-part strategies	Bruce, Elaine[a], Nanette[b], Nora, Ursula	Elaine
2 Part-to-whole strategies		Bonnie, Carl
1 part-to-part and 1 part-to-whole strategy	Bert, Bonnie, Carl, Christopher, Owen	Bert, Bruce, Christopher, Nanette, Nora, Owen, Ursula

[a] Elaine solved the problem in only one way, which made use of a part-to-part ratio.

[b] Although one of Nanette's strategies was incorrect, both her strategies made use of a part-to-part ratio.

In addition, teachers' ability to explain the quantities they used to determine the mixture with the stronger orange flavor significantly improved over time, $T = 18$, $n_{s/r} = 6$, $p < .05$ (one-tailed). A scoring rubric (shown in Figure 21) was used to capture the extent to which teachers correctly solved the problem using a multiplicative strategy and explained the quantities used to determine the stronger mixture. Since teachers were asked to solve the problem in two different ways, each teacher received two scores, one for each of their solution strategies. All ten teachers scored a 2, 3, or 4 on both the pre- and posttest (with the exception of Elaine, who received one score of 0 on the pretest because she solved the problem in only one way), meaning that no teacher used an incorrect additive strategy to solve task 6. As shown in Figure 21, the differences between scores of 2, 3, and 4 are related to the extent to which teachers correctly explained the meaning of the quantities they calculated. Table 10 illustrates the rubric scores earned by teachers on the pre- and posttest for task 6.

Rubric score	Criteria for rubric score
4	Solves problem correctly, chooses Luis. Uses a multiplicative strategy Explains the quantities that were calculated correctly, clearly, etc.
3	Solves problem correctly, chooses Luis. Uses a multiplicative strategy Explains the quantities that were calculated vaguely
2	Solves problem correctly, chooses Luis. Uses a multiplicative strategy Does not explain the quantities that were calculated or explains them incorrectly
1	Solves problem additively and says that the drinks are the same strength (since they both have 2 more ounces of water than concentrate)
0	No response, or does not know how to explain

Figure 21. Scoring rubric for task 6.

Table 10.

Quality of Treatment Group Teachers' Explanations of the Quantities They Used to Determine the Mixture With the Stronger Orange Flavor on Task 6

	Pre		Post	
	Strategy 1	Strategy 2	Strategy 1	Strategy 2
Bert	4	4	4	4
Bonnie	4	3	4	4
Bruce	4	2	4	4
Carl	4	4	4	4
Christopher	4	4	4	4
Elaine	4	0	2	2
Nanette	2	2	4	4
Nora	2	4	4	4
Owen	3	4	2	4
Ursula	2	3	4	4

The most common misconception on the pretest was interpreting a part-to-part ratio of orange juice concentrate to water as the percent orange juice (i.e., incorrectly stating that Luis's mixture is 71% orange juice. The mixture is actually about 42% orange juice). One correct interpretation of the 71% would be that the orange juice concentrate is 71% of the amount of water in the mixture. The two teachers who used part-to-part ratios and calculated a percentage, Bruce and Nora, incorrectly interpreted the percents on the pretest. This misconception was also evident in teachers' work on the pre-interview item. For example, Nora initially displayed the misconception as she examined Student A's solution (shown in Figure B.2 in Appendix B), which made use of a part-to-part ratio. When pressed by the interviewer about what the 0.625 meant, Nora replied, "It's the percent of the whole mixture, isn't it?" (Nora, Interview 1, line 107). During the course of the interview, Nora eventually made a correct interpretation of the 0.625 in Student A's solution. Bruce also struggled to make sense of Student A's solution, and stated that he did not know what the values that Student A calculated meant. Nanette and Ursula also displayed the misconception on the interview. Both Nanette and Ursula stated that the 0.625 meant that the mixture was 62.5% chocolate.

This misconception appeared to fade by the end of the course. Only one teacher used a part-to-part ratio and converted to a percent on the posttest, Nanette. She correctly interpreted the meaning of the percents on the posttest. By contrast, on the pretest, Nanette made no attempt to interpret the meaning of any of the quantities she calculated, and on the interview, Nanette incorrectly interpreted the meaning of Student A's part-to-part strategy.

Item 2 on Interview 1 reveals additional difficulties teachers encountered as they attempted to make sense of quantities prior to the course[21]. Teachers' interpretations of the five student solutions were scored using a rubric (similar to the one used for task 6, the orange juice task, shown in Figure 21) that captured the extent to which teachers described the quantities calculated by each student. This rubric was consistent with other rubrics used in the study; for example, a score of 4 reflected an explanation that was clear and complete, a score of 3 reflected explanations that were vague, a score of 2 reflected explanations that were incorrect, and a score of 0 was awarded when the teachers did not respond or said she didn't know (a score of 1 was not used in this rubric because this score corresponds to the use of additive strategies, which was not possible in this item since the strategies were already provided). Table 11 shows the rubric scores earned by teachers on this interview item.

As shown in Table 11, only two teachers (Bert and Christopher) correctly made sense of each of the five student solutions to the chocolate milk problem, thus earning a rubric score of 4

[21] The purpose of this interview item was to examine teachers' ability to make sense of quantities. Unfortunately this interview item was not included on Interview 2. However, teachers did complete two tasks at the end of the course (tasks 6 and 24 on the posttest) that indicated that most teachers were able to explain the meaning of quantities that they constructed by the end of the course.

for each student response. The remaining eight teachers were unable to correctly and completely make sense of at least two student solutions.

Table 11.

Quality of Treatment Group Teachers' Explanations of the Quantities Students Used to Determine the Mixture With the Stronger Chocolate Flavor on Interview 1

	Student Responses				
	Student A	Student B	Student C	Student D	Student E
Bert	4	4	4	4	4
Bonnie	4[a]	2	4	4	4
Bruce	0	4	3	4	4
Carl	3	3	3	3	4
Christopher	4	4	4	4	4
Elaine	4	4	4	0	3
Nanette	2	4	0	4[b]	0
Nora	4[c]	0	4	4	0
Owen	4	4	3	3	4
Ursula	2	3	3	4[b]	3

[a] Bonnie initially stated that Student A's work could be thought of as a percentage of chocolate (which is incorrect) and as the number of ounces of chocolate per one ounce of milk (which is correct). Later, when she examined Student D's work, she realized that Student D is a percentage of chocolate, and decided that Student A's work cannot be thought of as percentage of chocolate.

[b] Nanette and Ursula both struggled to make sense of Student D's solution. In particular, they had difficulty determining where the 8 and 24 came from. Ultimately, they both correctly made sense of the solution, but they also both commented several times that they still were not sure if this solution was valid.

[c] Nora was initially unable to make sense of Student A's solution. Later during her work on this interview item, she correctly made sense of the solution.

Task 24: Comparing plots of land. On the pretest, nine of the ten teachers used multiplicative strategies and correctly identified the plot of land that would be the "most square." The remaining teacher, Nanette, used an additive strategy and incorrectly solved the problem, therefore earning a 1 on the scoring rubric, as shown in Table 12. These results are quite different from previous findings, which indicate that at least fifty percent of secondary teachers solve this problem additively (Smith et al., 2001). The nine teachers who solved the problem correctly all used the same strategy to determine the rectangle that is the most square – calculating the ratio of the sides (either width:length or length:width) and selecting the ratio that was the closest to one. Six of these nine teachers (Bert, Bonnie, Christopher, Nora, Owen, and Ursula) also explained why they selected the rectangle whose ratio was closest to one – that the ratio of the sides of a square is always one. These six teachers earned the highest rubric score – a 4. The remaining three teachers (Bruce, Elaine, and Carl) earned 3s on the rubric because they did not explain why they selected the rectangle whose ratio was closest to one.

Table 12.

Treatment Group Teachers' Rubric Scores on Task 24

	Pre	Post
Bert	4	4
Bonnie	4	4
Bruce	3	4
Carl	3	4
Christopher	4	4
Elaine	3	4
Nanette	1	3
Nora	4	4
Owen	4	4
Ursula	4	4

On the posttest, all ten teachers solved the problem correctly using a multiplicative strategy. Nine teachers (Bert, Bonnie, Bruce, Carl, Christopher, Elaine, Nanette, Owen, and Ursula) calculated the ratio of the sides and selected the ratio that was the closest to one. All nine of these teachers earned a 4 on the scoring rubric, except Nanette, who earned a 3. The remaining teacher, Nora, determined the rectangle that was the most square by considering the amount that must be added if each rectangle was a square relative to the side of the square, as shown in Figure 22. Nora also earned a 4 on the scoring rubric. There was not a significant difference in the teachers' rubric scores on the posttest versus the pretest, $T = 10$, $n_{s/r} = 4$, $p > .05$ (one-tailed).

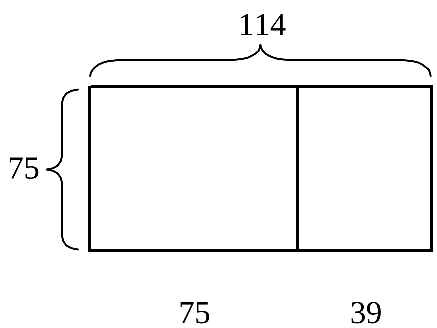

Figure 22. Considering a rectangle as a square plus an amount added on.

4.1.1.3. Qualitative Problems

Two qualitative problems (see tasks 7 and 8 in Appendix A) were included on the pre/posttest. Although teachers had no opportunities to explore qualitative problems during the course, teachers had numerous opportunities to explore the characteristics of multiplicative relationships, which may have helped them in making judgments about the relationships between quantities in these problems.

On the pretest, eight of the ten teachers solved both qualitative problems correctly and provided valid explanations. One teacher, Bert, correctly solved only task 8, and another teacher, Ursula, did not correctly solve either task 7 or 8. There was not a significant difference between the number of qualitative problems teachers solved on the pretest and the posttest, $t(9) = 1.00$, p

= .17 (one-tailed). The only difference in teachers' work on the posttest was that Ursula correctly solved one of the qualitative problems, task 8. Bert still correctly solved only task 8 on the posttest.

4.1.2. Discriminate Proportional From Nonproportional Situations

A second aspect of common content knowledge with respect to proportional reasoning requires one to be able to discriminate proportional from nonproportional situations. Teachers' ability to make this distinction was explored by examining their capacity to: (1) recognize when ratio was an appropriate measure for quantities; (2) classify relationships in a variety of representations as proportional or not; and (3) create examples and nonexamples of proportional situations. In this section, teachers' ability to discriminate proportional from nonproportional situations is described.

4.1.2.1. Ratio as Measure: Tasks 6, 9, 10, and 24

Task 6 (see Appendix A) presented teachers with two recipes for making orange juice by mixing amounts of orange juice concentrate and water, and asked teachers to determine the recipe that would yield the mixture with the strongest orange flavor. In order to correctly solve the problem, teachers needed to recognize that a ratio was an appropriate measure of concentration. Teachers' work on the pre- and posttest indicates that they understood this (although, as previously described, they were not able to interpret the ratio). No teacher solved the problem (on either the pre- or posttest) in a way that did not make use of a ratio.

Task 9 (see Appendix A) examined teachers' ability to recognize that a ratio was the appropriate measure of the shade of paint. In the task, Pat mixed blue and white paint until he had a shade of blue he liked. He needed another quart of paint, but didn't want to change the color, so he added 1 pint each of blue and white paint. Teachers were asked to comment on the effectiveness of Pat's strategy. On both the pre- and posttest, nine of the ten teachers were successful in determining that Pat's strategy for increasing the amount of paint without changing the color was ineffective. These nine teachers provided correct explanations as well. The explanations contained at least one of the following components: (1) an argument that Pat's strategy would only work if his original ratio was 1:1; (2) a counterexample showing that Pat's strategy would change the shade of paint (or a general argument that his strategy might change the shade, depending on the original shade); or (3) a general argument that Pat would need to maintain the ratio of blue to white. Only one teacher, Ursula, said that Pat's strategy was effective both on the pre- and posttest. However, it appears that Ursula may have misunderstood or misread the problem, because she explains on the pretest, "It [Pat's strategy] was a good idea b/c…the original ratio was 1-1." Her explanation on the posttest also indicates that she believes that Pat's original mixture contained 1 quart of blue paint and 1 quart of white paint. Thus it is not clear whether Ursula would have thought differently about Pat's strategy had she understood that his original ratio was not necessarily 1 to 1. (Ursula's pretest work on tasks 10 and 24 indicate that she recognized ratio as appropriate for measuring other attributes, such as steepness and squareness.)

Task 10 (see Appendix A) asked teachers how they would determine the steepness of ski ramps if the height, length of the base, and width of the base of the ramps were provided. Using ratios of height to length of the base, one could determine the steepness of the ramps (the width of the base does not affect the steepness of the ramp). On the pretest, all ten teachers correctly solved this task by explaining that a ratio of height to length of the base would measure steepness. On the posttest, nine teachers used this ratio to measure steepness. The remaining teacher, Ursula, used the Pythagorean theorem, which would determine the length of the ramp, rather than the steepness of the ramp.

As described previously, task 24 (see Appendix A) asked teachers to determine which of three rectangular plots of land was the most square. On the pretest, nine of the ten teachers used ratios to measure the squareness of each rectangle. The remaining teacher, Nanette, used an additive strategy, rather than a strategy that made use of a ratio. On the posttest, all ten teachers made use of ratios to measure the squareness of each rectangle.

Thus, it appears that eight of the teachers saw ratio as an appropriate mechanism to measure attributes prior to and after the course. In addition, one of the teachers, Nanette, appeared to come to understand that ratio could be used to measure squareness of a rectangle by the end of the course. Finally, the remaining teacher, Ursula, appeared to understand that ratio could be used to measure steepness prior to the course; however, she did not use a ratio to measure steepness on the posttest.

4.1.2.2. Classifying Relationships as Proportional or Nonproportional: Tasks 11-22 and the Snowfall Interview Item

Tasks 11-22 (see Appendix A) presented teachers with 12 relationships (3 depicted in written language, 3 depicted as graphs, 3 depicted as equations, and 3 depicted as tables) and asked them to indicate which were proportional and explain how they knew. Three teachers (Bert, Christopher, and Owen) correctly classified all twelve of the relationships, and seven teachers incorrectly classified at least one relationship. Of the 120 relationships on the teachers' pretests (10 teachers x 12 relationships), 22.5% of the relationships (27 relationships) were incorrectly classified. In addition, 2 relationships were not classified at all[22]. Therefore, 24% of

[22] Nora did not classify two of the table relationships (tasks 20 and 21).

the relationships were incorrectly classified or not classified at all on the pretest. Table 13 indicates the number of relationships that each teacher incorrectly classified (or did not classify) on the pretest. It is also interesting to note in the table that teachers' incorrect classifications were not limited to a particular representation.

Table 13.

Number of Relationships That Each Teacher in the Treatment Group Incorrectly Classified on Tasks 11-22

Teacher	Pre		Post	
	Number of relationships incorrectly classified/not classified	Representation(s) of the incorrectly/not classified relationships	Number of relationships incorrectly classified/not classified	Representation(s) of the incorrectly/not classified relationships
Bert	0	N/A	0	N/A
Bonnie	4	Language, graphs, equations, tables	0	N/A
Bruce	4	Language, graphs, equations	0	N/A
Carl	5	Language, graphs, equations, tables	1	Language
Christopher	0	N/A	0	N/A
Elaine	7	Language, graphs, equations, tables	2	Graph, equation
Nanette	2	Graphs, equations	0	N/A
Nora	3[a]	Language, tables	0	N/A
Owen	0	N/A	0	N/A
Ursula	4	Language, graphs, equations	0	N/A
Total number of relationships incorrectly classified/not classified	29		3	

[a] Nora did not classify two of the relationships, and classified one relationship incorrectly.

In addition, 50% of the teachers (Bonnie, Bruce, Carl, Elaine, and Nanette) appeared to believe that all linear relationships are proportional prior to the course. The belief that all linear relationships are proportional is a common misconception that has been well documented in the literature (De Bock, Van Dooren, Janssens, and Verschaffel, 2002; De Bock, Verschaffel, and Janssens, 1998; Smith et al., 2003). All five of these teachers consistently classified all of the linear relationships as proportional in at least two of the four representations given on the pretest. Two of these five teachers (Carl and Elaine) consistently classified all linear relationships as proportional across all four representations, and two teachers (Bonnie and Bruce) did so across three of the four representations. The explanations these teachers produced support the claim that they believed all linear relationships were proportional. For example, Nanette supported her statement on the pretest that task 17 (the equation y = 3x + 4.5) is proportional by writing, "Because this is a linear function, x and y have a proportional relationship." Bonnie, Bruce, Carl, and Elaine also wrote similar arguments in support of at least one of their classifications.

Teachers correctly classified significantly more relationships as proportional or not on the posttest than on the pretest, $t(9) = 4.21$, $p = .001$ (one-tailed). Of the 120 relationships on the posttest, only three (2.5%) were incorrectly classified on the posttest. Eight of the ten teachers correctly classified all twelve relationships as proportional or not. One teacher, Elaine, incorrectly classified two relationships, and one teacher, Carl, incorrectly classified one relationship, as shown in Table 13.

In addition, the misconception that all linear relationships are proportional appeared to have been mostly eliminated by the posttest. Significantly fewer teachers appeared to hold the misconception that all linear relationships are proportional at the end of the course than at the beginning of the course, Fisher's exact test, $p = .02$ (one-tailed). Only one teacher, Carl, incorrectly classified one linear, nonproportional relationship (task 13) as proportional because "the rates are constant." However, it is interesting to note that Carl did correctly classify the other linear relationships presented as graphs, tables, and equations.

The two relationships Elaine incorrectly classified were both quadratic relationships that contained the origin, presented as an equation (task 18) and graph (task 16). Elaine incorrectly classified these same tasks on the pretest. Interestingly, Elaine did not incorrectly classify the other quadratic relationship containing the origin (task 22) on either the pre- or posttest, as one might expect. Perhaps this is because task 22 was presented as a table, which allowed her to utilize key understanding 3 (the rate pairs in proportional relationships are equivalent) on the posttest to determine that in this case the rate pairs were not equivalent.

Teachers' work on the snowfall item on Interview 2 provided further evidence that they were able to distinguish proportional from nonproportional relationships by the end of the course. In this item, teachers examined data (presented in a paragraph, tables, and graphs) on a snowstorm that hit two cities in Iowa and were asked to discuss what they could tell from the data (see Figure C2 in Appendix C). The relationship between the number of hours it snowed and the amount of snow on the ground was proportional for Cedar Rapids, and was not proportional for Mason City. If teachers did not spontaneously identify Cedar Rapids as proportional, they were explicitly asked whether either of the relationships was proportional. Nine of the ten teachers correctly identified the proportional relationship in their work on this item[23]. Eight of these teachers did so spontaneously. Ursula did not spontaneously comment on the proportionality of the relationships, but was able to correctly identify Cedar Rapids as proportional when prompted by the interviewer.

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[23] One of the ten teachers, Nora, did not identify Cedar Rapids as proportional. Her work is discussed at the end of the next section.

4.1.2.3. Creating Examples of Proportional Situations: Task 5 and Interview Item

Task 5 asked teachers to create a word problem that would require setting up and solving the proportion 3/8 = x/20. In order to write such a problem, teachers needed to create a context in which the quantities were related multiplicatively (i.e., proportionally). In addition, since the solution to the given proportion was x = 7.5, teachers needed to make sure that a noninteger solution made sense in the context (e.g., a solution of 7.5 inches would make sense, but a solution of 7.5 people would not). Table 14 presents the results for the ten teachers in the treatment group on task 5.

As shown in Table 14, all ten teachers created missing value word problems in which the quantities were related multiplicatively both prior to and upon completion of the course. For example, eight teachers created situations in which they made the multiplicative relationship explicit (e.g., “For every 8 boxes of candy sold, $3 goes to the Make-A-Wish foundation [Bert, pretest]). The remaining two teachers (Christopher and Owen) created word problems based on situations in which the multiplicative relationship between quantities is more implicit. For example, the multiplicative relationship between height and shadow length in Owen’s word problem is not explicitly stated:

> Jamal, a basketball player, is 8 feet tall. He is standing next to a flagpole, which he knows to be 20 feet tall. He is wondering, “How long is the shadow made by the flagpole?” He asks his friend Susie to measure his shadow with the yardstick – it is 3 feet long. What is the answer to his question? (Owen, pretest)

Table 14.

Rubric Scores for Word Problems Created by Teachers in the Treatment Group for Task 5

	Pre	Post
Bert	3	1
Bonnie	1	3
Bruce	1	2
Carl	1	1
Christopher	3	2
Elaine	0[a]	3
Nanette	1	1
Nora	2	1
Owen	3	3
Ursula	1	1

Note. Teacher's word problems were scored using the following rubric:

3: Creates a word problem that does require setting up and solving 3/8=x/20. The answer of x=7.5 **makes sense in the context of the problem**

2: Creates a word problem that does require setting up and solving 3/8=x/20. The answer of x=7.5 **does not** make sense in the context of the problem, **but** the teacher appears to recognize this issue, and addresses it by addending the problem using phrases such as: "Parts…are ok", "Please round your answer", etc.

1: Creates a word problem that does require setting up and solving 3/8=x/20. The answer of x=7.5 **does not** make sense in the context of the problem.

0: Word problem does not require setting up and solving 3/8=x/20.

[a] Elaine's word problem did involve two quantities that were related multiplicatively. However, the problem she created required setting up and solving 8/3 = x/20.

Despite all ten teachers creating word problems that required multiplicative solution strategies on the pretest and posttest, the extent to which they created word problems in which a noninteger solution made sense in the context of the problem varied on the pretest and posttest. As shown in Table 14, half of the ten teachers created word problems in which the answer did not make sense in the context of the problem, and therefore earned a rubric score of 1 on both the pretest and posttest. For example, consider the word problem Ursula created on the pretest:

> In a typical math classroom, there are a total of 8 students. Three out of 8 students are female. If you walked into a math classroom that contained 20 students, how many female students would you expect to see? (Ursula, pretest)

In Ursula's problem, an answer of 7.5 female students does not make sense. Other teachers appeared to recognize this issue and addressed it by stating that parts of quantities were permissible, as shown in the problem Nora created on the pretest, "Harry can buy 3 frogs for 8 cents. How many frogs can he buy with 20 cents? (Parts of frogs are ok.)"

There was no significant difference between teachers' work on task 5 at the beginning and end of the course, $T = 4$, $n_{s/r} = 6$, $p > .05$ (one-tailed). All teachers created word problems in which the quantities were related multiplicatively – which is a key, and perhaps the most important, component of this task. The extent to which teachers created problems in which a noninteger solution made sense in the context was fairly limited. However, it is important to note that teachers had no opportunities to create word problems during the course.

In addition, an item on Interview 1 and Interview 2 asked teachers to provide an example of a situation in which there was a proportional relationship between the quantities, and an example in which there was not a proportional relationship between the quantities. Teachers were also asked to explain why their examples represented (or did not represent) proportional relationships. Table 15 presents the results for the ten teachers' examples and nonexamples of proportional relationships. As shown in Table 15, all ten teachers were able to create a valid example of a proportional relationship during Interview 1 and Interview 2. The most common examples that teachers created were situated in a recipe context, in which one wanted to make more than (e.g., twice as much) the amount that the recipe yields. Another common example involved heights and shadow lengths (similar to the word problem Owen created on the pretest).

Although teachers were able to create examples of proportional situations even early in the course, their ability to explain *why* their examples were valid significantly improved over time (as shown in their rubric scores), $T = 15$, $n_{s/r} = 5$, $p < .05$ (one-tailed).

Table 15.

Rubric Scores for Examples and Nonexamples of Proportional Situations Created by Teachers in the Treatment Group

	Interview 1		Interview 2	
	example	nonexample	example	nonexample
Bert	4	2	4	4
Bonnie	4	4	4	4
Bruce	3	3	4	4
Carl	3[a]	2	3	2
Christopher	3[a]	4	4	4
Elaine	3	2	4	4
Nanette	3	1	3[a]	4
Nora	3[a]	4	4	4
Owen	4	4	4	4
Ursula	3	2	4	3

Note. Teachers' examples and nonexamples were scored using the following rubric:

4: creates a proportional/nonproportional situation and explains why it is proportional (or not) (either by drawing on the 4 key understandings, or other means – e.g., saying that the ratio between the two quantities is constant, or that the unit rate stays the same)

3: creates a proportional/nonproportional situation but explanation is vague or does not clearly explain why it's proportional (or not)

2: creates a situation in which there is *not* a relationship between the quantities – proportional or otherwise

1: incorrectly creates a proportional/nonproportional situation

[a] Carl, Christopher, Nanette, and Nora were not pressed by their interviewers to explain why their examples were proportional.

As shown in Table 15, teachers were more successful in providing an example of a proportional relationship than a nonexample early in the course. For example, half of the ten teachers were unable to provide a nonexample of a proportional relationship during Interview 1. Four teachers (Bert, Carl, Elaine, and Ursula) provided a nonexample in which there was *no* relationship between the quantities (e.g., Ursula's nonexample involved a person's height and their eye color), or told the interviewer that they could not come up with a nonexample (Nanette). Teachers' ability to provide nonexamples of proportional relationships and explain why they were not proportional significantly improved over time, $T = 15$, $n_{s/r} = 5$, $p < .05$ (one-tailed). By the end of the course, most teachers were able to provide a valid nonexample of a proportional relationship. Teachers' nonexamples at the end of the course typically involved an additive component[24], such as the nonexample provided by Bert:

> …well I guess, if you- say you have [pause] if you had a job where you got a signing bonus, [pause] And somebody asked you how much you made per hour. So, say your first paycheck included your signing bonus, [pause] the more hours that you work, the less per hour it would come out to. (Bert, Interview 2, lines 483-488)

<u>Summary</u>. Even early in the course, teachers were able to create contexts and provide examples in which the quantities were related multiplicatively. However, the extent to which they could justify why their examples were proportional significantly improved by the end of the course. Teachers' capacity to provide nonexamples of proportional situations was fairly limited early in the course. By the end of the course, most teachers were able to provide nonexamples and explain why they were not proportional. In order to justify why a relationship is proportional

[24] This is not particularly surprising, since the ideas that proportional relationships are multiplicative rather than additive and can be represented by the equation y = mx are part of the four key understandings, which were made public during the course and appeared to be learned by teachers. In addition, key problems discussed during the course included an additive component (e.g., the park and zoo problems discussed during Class 11).

or not, one needs to understand the mathematical relationships embedded in proportional situations. Teachers' understandings about these relationships are described in the next section.

4.1.3. Understand the Mathematical Relationships Embedded in Proportional Situations

In this section, the last aspect of teachers' common content knowledge with respect to proportional reasoning, their understandings of the mathematical relationships embedded in proportional situations, prior to and after the course are described. Three tasks were used to explore teachers' use of the arguments based on Cramer et al.'s (1993) and Post et al.'s (1988) key understandings. First, tasks 11-22 (see Appendix A) presented teachers with 12 relationships (3 depicted in written language, 3 depicted as graphs, 3 depicted as equations, and 3 depicted as tables) and asked them to indicate which were proportional and explain how they knew. In addition, an item on Interview 1 and Interview 2 (see item 1 in Appendix B and item 3 in Appendix C) asked teachers to define a proportional relationship and to create an example and nonexample of a proportional relationship. Teachers were also pressed by the interviewer to explain why their examples reflected proportional (or nonproportional) relationships. Finally, the snowfall item on Interview 2 asked teachers to examine data on a snowstorm that hit two cities in Iowa and to discuss what they could tell from the data (see Figure C2 in Appendix C). The relationship between the number of hours it snowed and the amount of snow on the ground was proportional for Cedar Rapids, and was not proportional for Mason City. If teachers did not spontaneously identify Cedar Rapids as proportional, they were explicitly asked whether either, if any, of the relationships was proportional. Teachers were then pressed to explain how they knew either of the relationships was proportional.

4.1.3.1. Tasks 11-22

The nature of teachers' rationales for why the relationships in task 11-22 were proportional (in particular, tasks 12, 14, 19, and 20 are proportional) changed during the course. Table 16 illustrates the key understandings upon which teachers drew on the pre- and posttest to identify the proportional relationships. As shown in Table 16, prior to the course, 50% of the teachers wrote at least one explanation that was based on a misconception (e.g., the belief that all linear relationships are proportional). By the end of the course, no teacher provided an explanation based on a misconception.

In addition, the number of key understandings teachers drew upon on the posttest was significantly greater than the number of key understandings they drew upon on the pretest, $t(9) = 4.31$, $p = .0009$ (one-tailed). Teachers also drew upon understandings on the posttest that they had not used prior to the course. All teachers (except Nora) drew upon at least one understanding on the posttest that they had not utilized on the pretest, as shown in Table 17. Seven teachers drew upon at least two "new" understandings on the posttest. In addition, six teachers used an argument that the m in $y = mx$ is the slope, the unit rate, and the constant of proportionality (i.e., key understanding 4) on the posttest. Only two teachers used such an argument on the pretest.

Table 16.

Key Understandings Upon Which Teachers in the Treatment Group Drew to Identify Proportional Relationships in Tasks 11-22

	Pre				Post			
	Task 12 (language)	Task 14 (graph)	Task 19 (equation)	Task 20 (table)	Task 12 (language)	Task 14 (graph)	Task 19 (equation)	Task 20 (table)
Bert	3, 4	4	3	3	3	2, 4	4	3, 4
Bonnie	Linears are proportional	Linears are proportional	Linears are proportional	Linears are proportional	3	2, 4	4	3
Bruce	3	3	Linears are proportional	3	1	2	2	2
Carl	Linears are proportional	Linears are proportional	Linears are proportional	Linears are proportional	1	2	2	2
Christopher	3	3	3	3	4	2	4	4
Elaine	As x increases, y increases	Linears are proportional	Linears are proportional	Linears are proportional	3	2	3	3
Nanette	Incorrectly classified	As x increases, y increases	Linears are proportional	Linears are proportional	2	2, 1	1	3, 4
Nora	3	3	4	2, 3	4	2	2	3
Owen	3	2	2	3	4	2	2	3
Ursula	Incorrectly classified	-	3	-	1	2	1	1

Note. Shading indicates that the teacher's work indicates that they have a misconception about proportional relationships (i.e., either that all linear relationships are proportional, or that a relationship is proportional because as x increases, y also increases).

Key Understandings
1: multiplicative
2: line through origin
3: rate pairs equivalent
4: m in y = mx is slope, unit rate, constant of proportionality

Table 17.

Key Understandings Upon Which Teachers in the Treatment Group Drew on the Posttest but Not on the Pretest

	Pre	Post	Key understandings that were utilized on the posttest but not on the pretest
Bert	3, 4	2, 3, 4	2
Bonnie	None	2, 3, 4	2, 3, 4
Bruce	3	1, 2	1, 2
Carl	None	1, 2	1, 2
Christopher	3	2, 4	2, 4
Elaine	None	2, 3	2, 3
Nanette	None	1, 2, 3, 4	1, 2, 3, 4
Nora	2, 3, 4	2, 3, 4	None
Owen	2, 3	2, 3, 4	4
Ursula	3	1, 2	1, 2

Note.
Key Understandings
1: multiplicative
2: line through origin
3: rate pairs equivalent
4: m in y = mx is slope, unit rate, constant of proportionality

Finally, it is interesting to note that by the end of the course, teachers' rationales appeared to be more connected with particular representations. For example, all ten teachers explained that task 14, presented as a graph, was proportional because proportional relationships are depicted by graphs of lines that contain the origin (key understanding 2) on the posttest. Teachers used this argument to a lesser extent to justify why the relationships presented in language, equations, and tables were proportional. Similarly, over half the teachers explained that task 20, presented as a table, was proportional because the rate pairs are equivalent (key understanding 3). Teachers drew upon this argument to a lesser extent to justify why the relationships presented in language and equations were proportional, and no teacher used this argument to justify why the graph was proportional.

4.1.3.2. Item on Interview 1 and Interview 2: Defining a Proportional Relationship and Creating an Example and Nonexample of a Proportional Relationship

Teachers' work on an interview item in which they were asked to define a proportional relationship and create an example and nonexample of a situation in which there was a proportional relationship between the quantities provided additional evidence that they refined their understandings about the mathematical relationships embedded in proportional relationships during the course. As shown in Table 18, teachers drew upon significantly more key understandings on the posttest than the pretest in defining a proportional relationship and creating and justifying examples and nonexamples of proportional relationships, $t(9) = 3.34$, $p = .004$ (one-tailed).

Table 18

Key Understandings Upon Which Teachers in the Treatment Group Drew in Defining a Proportional Relationship and Creating an Example and Nonexample of a Proportional Relationship

	Interview 1	Interview 2
Bert	1	2, 3
Bonnie	1	1, 2, 3
Bruce	1, 2	1, 2, 3
Carl	None	None[a]
Christopher	3	3
Elaine	None	1, 2, 3
Nanette	None	1, 2, 3
Nora	1	2
Owen	2	1, 3
Ursula	None	2

Note.
Key Understandings
1: multiplicative
2: line through origin
3: rate pairs equivalent
4: m in y = mx is slope, unit rate, constant of proportionality
[a] However, Carl's interviewer did not press him to explain *why* his example (and nonexample) did (or did not) represent a proportional relationship during Interview 2.

4.1.3.3. Item 2 on Interview 2: Snowfall

For the snowfall item, teachers were asked to examine data (presented in a paragraph, tables, and graphs) on a snowstorm that hit two cities in Iowa and to discuss what they could tell from the data. The relationship between the number of hours it snowed and the amount of snow on the ground was proportional for Cedar Rapids, and was not proportional for Mason City. By engaging with this task, teachers had opportunities to use their knowledge flexibly across three representations (written language, tables, and graphs). As such, this task was quite different from other tasks on the pre/posttest and interviews.

As noted previously, nine of the ten teachers correctly identified the proportional relationship, Cedar Rapids, in the snowfall interview item. In addition, the nine teachers who correctly identified Cedar Rapids as proportional drew upon at least one of the four key understandings to explain why the relationship was proportional, as shown in Table 19. Two teachers (Bruce and Christopher) drew upon all four key understandings in their explanations, two teachers (Nanette and Owen) drew upon three of the understandings, three teachers (Bert, Bonnie, and Carl) drew upon two understandings, and three teachers (Elaine, Nora, and Ursula) drew upon one understanding in their explanations.

The understanding that proportional relationships are depicted by graphs that contain the origin (key understanding 2) was drawn upon the most frequently, by nine teachers. The understanding that proportional relationships are multiplicative in nature (key understanding 1) was used by seven teachers. The understanding that m in the equation $y = mx$ is the slope, unit rate, and constant of proportionality (key understanding 4) was used by four teachers. Finally, the understanding that the rate pairs of proportional relationships are equivalent (key understanding 3) was used by three teachers.

It is interesting to note that all four teachers who drew upon key understanding 4 (the understanding that m in the equation $y = mx$ is the slope, unit rate, and constant of proportionality), Bruce, Christopher, Nanette, and Owen, also spontaneously generated equations that represented the relationship between the number of hours it snowed and the amount of snow on the ground for Cedar Rapids (the proportional relationship). (Bruce and Owen also generated the equation for the nonproportional relationship, Mason City.) None of the remaining six teachers generated equations to represent the relationships depicted in the snowfall item, nor did they draw upon key understanding 4 during the interview. This is not surprising, since the equation is particularly helpful in using key understanding 4.

Table 19.

Key Understandings Upon Which Teachers in the Treatment Group Drew in Their Work on the Snowfall Item

Teacher	Key understandings used
Bert	2, 3
Bonnie	1, 2
Bruce	1, 2, 3, 4
Carl	1, 2[a]
Christopher	1, 2, 3, 4
Elaine	2
Nanette	1, 2, 4
Nora	2 [b]
Owen	1, 2, 4
Ursula	1

Note.
Key Understandings
1: multiplicative
2: line through origin
3: rate pairs equivalent
4: m in y = mx is slope, unit rate, constant of proportionality
[a] Carl also attempted to use key understanding 3 (rate pairs are equivalent), but did not set up the ratios correctly. (Therefore this was not counted as a key understanding that he drew upon.)
[b] Nora was the only teacher who was not able to determine that Cedar Rapids was proportional. In her work on the problem, however, it was clear that she understood that proportional relationships are depicted by lines that contain the origin (key understanding 2).

It is also interesting to note that half the teachers (Bert, Bonnie, Bruce, Christopher, and Elaine) spontaneously made mathematical connections between the understandings in their work at the end of the course. For example, Bruce makes connections among the four key understandings as he explains why Cedar Rapids is proportional and Mason City is not in the following excerpt. (“I” stands for “Interviewer”.)

Bruce: …Cedar Rapids [pause] is [pause] proportional because the- um [pause] there’s no y-intercept, there’s no starting point. So, based on the hours, um, [pause] you can predict. So, the Mason City is not proportional. Because the proportion from one hour to the next [pause] is not the same…**So 6.5 over 1, versus 7.5 over 3, versus 8.5 over 5**, [pause] **there’s no constant of proportionality**…There’s no proportional relationship. In Mason City…the hours to the snow, there’s no proportional relationship…between hours and snow…in Mason City. There is, or should be, [pause] I think, there should be, for the other one [Cedar Rapids].

I: Why do you think there should be?

Bruce: Because it should go through the- at the- I don’t if a dot’s put- **it goes through the origin. Then there’s no y-intercept there, uh value throwing off the slope, the slope is the proportion between the hours and the inches. It’s a direct varian- it’s a direct uh relationship. It’s a direct multiplicative relationship. So I can see that from the table. I can see that from the graph.**

I: You can see what?

Bruce: I can see, looking at the graph, that it’s not- that the y-intercept, but then I could look at the table, to prove that the proportionality between the two doesn’t- those…doesn’t exist.

I: Ok.

Bruce: So that’s how I used this. I look at the- **I look at the graph, I see a y-intercept of about 6**, so **I go up here** [the table]**, and prove to myself that there is no constant of proportionality between the hours it snowed and the inches on the ground for Mason City**.

(Bruce, Interview 2, lines 681-711, bold added to highlight the mathematical connections)

In this excerpt, Bruce draws upon his understandings of the nature of proportional relationships to confirm his hypothesis that Cedar Rapids is proportional. In particular, he checks several rate pairs (key understanding 3) for Mason City and notes that there is not a constant of proportionality (key understanding 4). In addition, he notes that Cedar Rapids contains the origin (key understanding 2), which causes the relationship to be multiplicative (key understanding 1). Bruce also notices that the graph for Mason City has a y-intercept of 6, and uses that information to “prove to [him]self” that there is no constant of proportionality for Mason City, thus connecting key understandings 2 and 4. Finally, Bonnie and Christopher made similar connections in their work on the snowfall item, and Bert and Elaine did so in their work to create an example and nonexample of a proportional relationship during Interview 2.

Finally, one teacher, Nora, was unable to identify the proportional relationship (Cedar Rapids) on the snowfall item. However, her work on this interview item reveals that the source of the problem may have been the snowfall context, not necessarily an impoverished understanding of proportional relationships and their characteristics. For example, when asked by the interviewer if either of the situations reflected a proportional relationship, Nora responded, “Um, for a proportional relationship, [pause] you would have to have the graph go through zero, you have to go through the origin.” Thus, Nora knew that proportional relationships are depicted by linear graphs that contain the origin (key understanding 2). In addition, Nora was able to discuss key characteristics of proportional relationships during her work on this interview item; however, she was only able to do so when she shifted the contexts to the movie tickets and taxicab contexts from the pre/posttest (see tasks 11 and 12 in Appendix A), as shown in the following excerpt. (“I” stands for “Interviewer”.)

Nora: **Um [pause] can’t- I don’t know- can I use a different situation?**

I: Sure, if- if that seems to-

Nora: Ok.

I: if the snowfall is giving you a hard time. Sure.

Nora: Cause I know with the- this is movie tickets. [pause] For the movie tickets, no matter how many people you have, [pause]

I: Ok, so you’re drawing a little graph here

Nora: number of- number of people um for the x-axis, is the number of- er- [pause] ___ cost of [pause] all the tickets, ___ the total cost. All the tickets cost the same amount, um so zero tickets- um zero people would be zero dollars, but um one person would be five dollars if these are five dollar tickets.

I: Ok. Ok. Ok.

Nora: Two people would be ten dollars, **so the graph would be linear and go through the origin. And for- it wouldn't matter how many people [pause] you had, every ticket would still cost the same amount.**

I: Ok.

Nora: So if you had um two people, it would cost ten dollars, and that would be five dollars per ticket, and if you had five people that would be twenty five um dollars, and **it would still be five dollars per ticket**. Whereas if you were in a taxicab, you have um [pause] the number of miles versus the total cost.

I: Ok.

Nora: Then, I have miles on the x-axis, um, even though you don't go anywhere, it still costs you three dollars. [pause] Or something. Say that costs three dollars. And then for every additional- so it's like, cost will be three dollars plus, I don't know, twenty cents a mile.

I: Ok.

Nora: **Um the more miles you go, the less it costs per mile**. [pause]

I: Ok. So that- and that's different than-

Nora: That's different than this.

I: The movie tickets situation.

Nora: Right. Where, **no matter how many people- no matter- no matter how far you go along the x-axis it's still the same increase** um.

I: It's still this five dollar-

Nora: **Still five dollars**.

I: Ok, and here you said it would be changing, it would-

Nora: **the cost would change** per person-

I: Ok.

Nora: er- **per mile**.

I: Per mile. Ok.

(Nora, Interview 2, lines 346-384, bold added for emphasis)

Thus, Nora is able to identify a key difference between relationships that are and are not proportional: the unit rate (in her examples, cost per person or cost per mile) is constant in proportional relationships; by contrast, the unit rate changes in nonproportional relationships. Unfortunately, Nora is unable to relate this work back to the snowstorm situations, as shown in the following excerpt.

I: …**you think you can relate this back to the snowstorm?**

Nora: Um [long pause] the difficulty I'm having is the increase is the same each hour. But I guess the level of snow is different. Ok, for Cedar Rapids. Um, it says that there was no snow on the ground as the- when the snow started, so I'm going to extend this line through the origin.

I: Ok.

Nora: Cause at time zero, there was no snow. So after two hours, um, is that right? Yeah. After two hours, um it had snowed three- there was three inches of snow on the ground. This is um Cedar Rapids and for Mason City, [pause] um for two hours- after two hours there was, where is it? Seven um inches of snow. Then after three hours, in Mason City [pause] huh. Hmm hmm hmm. And that would be- that doesn't- I don't think it makes any sense. [pause] Cause that would be for each one hour, that'd be one and a half inches of snow on the ground.

I: That's for Cedar Rapids.

Nora: For Cedar Rapids.

I: Ok.

Nora: And it wouldn't matter [pause] cause after um four hours, in Cedar Rapids there is six inches on the ground, it's the same- it's one- it's still one and a half inches um for every hour that it snowed. But since in Mason City there's already snow on the ground, after two hours, there's seven inches and so that's like one, that's three and a half inches per hour and mmm I'll (do it with) four. At four, after four hours there was eight inches on the s- on the ground, so that would mean it was snowing two inches per- per hour, for every one hour. And [pause] I don't think it makes any sense here. [pause]

I: What's not making any sense?

Nora: I don't- I don't know how to interpret the difference in the amounts of snowfall

I: Ok, this-

Nora: per hour.

I: this, per hour rate, ok.

Nora: Yeah, that doesn't- I don't think that makes any sense because I already know that it's one half. Cause I- so I know it's a constant [pause]

I: Ok.

Nora: rate, but it's a constant rate of change but it's- the ratio between the number of hours and the number of inches is not the same.

I: Ok. So it's not giving you that one half that-

Nora: Right.

I: that you know that this line is representing.

Nora: Yeah. Yeah.

I: Ok. Now in the movie- er this is the taxi cab-

Nora: Mhm.

I: that- what's happening? Or what's happening there?

Nora: Well for the taxicab I know that for the number of as- as the number of minutes increased, the total cost for each minute went down. [pause] So [pause] can I say that the number of hours increased the total amount [pause] I think there's a- I don't know how to relate to cost, inches of snow on the ground. [laughs]

I: Ok.

Nora: I think that the total height, in each- for each hour it snows the total height [pause] doesn't increase the same amount. [pause] I don't know. [pause] It's all cause the six inches. [laughs]

I: [laughs]

Nora: That was already there.

I: Ok, and that's throwing it off?

Nora: I'm not sure, yeah. I'm not sure how to interpret this. But I don't think- **I think you have to have a linear relationship that goes through the origin-**

I: for it to be

Nora: **proportion-**

I: proportional.

Nora: Yeah.

I: Ok.

(Nora, Interview 2, lines 386-444, bold added for emphasis)

4.1.3.4. Tasks 11-22 and Interview Items (Defining a Proportional Relationship, Creating an Example and Nonexample of a Proportional Relationship, and Snowfall)

By arranging all the data that examined teachers' understandings of the mathematical relationships embedded in proportional situations together, differences between teachers' use of the key understandings prior to and after the course can be noted. Table 20 shows the key understandings used by teachers on the pre-instruments (tasks 11-22 on the pretest and the Interview 1 item in which teachers defined a proportional relationship and created an example and nonexample) and the post-instruments (tasks 11-22 on the posttest, the Interview 2 item in which teachers defined a proportional relationship and created an example and nonexample, and the Snowfall item on Interview 2).

Table 20.

Key Understandings Upon Which Teachers in the Treatment Group Drew Prior to and Upon Completion of the Course

	Pre-instruments: Pretest (Tasks 11-22) & Interview 1 (Defining, Example, Nonexample)	Post-instruments (Posttest (Tasks 11-22), Interview 2 (Defining, Example, Nonexample), Interview 2 (Snowfall)	**Key understandings that were utilized on the post-instruments, but not on the pre-instruments**
Bert	1, 3, 4	2, 3, 4	**2**
Bonnie	1	1, 2, 3, 4	**2, 3, 4**
Bruce	1, 2, 3	1, 2, 3, 4	**4**
Carl	None	1, 2	**1, 2**
Christopher	3	1, 2, 3, 4	**1, 2, 4**
Elaine	None	1, 2, 3	**1, 2, 3**
Nanette	None	1, 2, 3, 4	**1, 2, 3, 4**
Nora	1, 2, 3, 4	2, 3, 4	**None**
Owen	2, 3	1, 2, 3, 4	**1, 4**
Ursula	3	1, 2	**1, 2**

Note.
Key Understandings
1: multiplicative
2: line through origin
3: rate pairs equivalent
4: m in y = mx is slope, unit rate, constant of proportionality

As shown in Table 20, three teachers (Carl, Elaine, and Nanette) did not draw upon any of the four key understandings in their work prior to the course. By contrast, all ten teachers used at least two key understandings in their work at the end of the course – five of which utilized all four key understandings at the end of the course. In addition, Table 20 notes the understandings that teachers used on the post-instruments but not on the pre-instruments – that is, understandings that teachers appeared to learn during the course. Teachers drew upon significantly more understandings in their work at the end of the course than at the beginning of the course, $t(9) = 3.67$, $p = .002$ (one-tailed). As shown in Table 20, all but one teacher, Nora, utilized at least one key understanding at the end of the course that they had not used at the beginning of the course[25]. Seven teachers appeared to learn that proportional relationships are depicted graphically by lines that contain the origin (key understanding 2) by the end of the course. Six teachers appeared to learn that proportional relationships are multiplicative in nature (key understanding 1), five teachers appeared to learn that the *m* in y = *m*x is the slope, constant of proportionality, and unit rate (key understanding 4), and three teachers appeared to learn that the rate pairs in proportional relationships are equal (key understanding 3).

4.1.4. Summary

Teachers' work on the pretest and pre-interview revealed that the ten teachers in the treatment group were able to do a lot of mathematics prior to the course. With only one exception (Nanette's pretest solution to task 24, the plots of land problem), this group of teachers did not use incorrect additive strategies to solve the problems. This finding suggests that this particular group of teachers is not typical of teachers, many of which apply additive strategies to problems that call for multiplicative ones (Heinz, 2000; Simon & Blume, 1994; Smith et al., 2001; Post et al., 1991). In general, this group of teachers was able to successfully solve a variety of proportionality problems and recognize when multiplicative strategies were appropriate. These teachers' success in solving such problems can be attributed in part to the fact that they were about to be certified in secondary mathematics and had earned a bachelor's degree or equivalent in mathematics.

However, a closer look at teachers' work on the pretest revealed that they relied heavily on the cross multiplication procedure to solve missing value problems. In addition, many teachers may have held a narrow view of the types of ratios that could be used to solve numerical

[25] However, as shown in Table 21, Nora drew upon all four key understandings at the beginning of the course, and thus had no room for improvement.

comparison problems. Teachers' work on the posttest suggests that their range of strategies for solving missing value and numerical comparison problems was broadened as a result of the course.

In addition, teachers' work on the pretest indicated that they understood that ratio is an appropriate measure of quantities such as squareness and steepness. This understanding is an important aspect of discriminating proportional from nonproportional situations. However, when specifically asked to classify relationships presented in a variety of representations as proportional or not, teachers struggled to do so on the pretest. In particular, half of the ten teachers appeared to believe that all linear relationships are proportional. By the end of the course, teachers' capacity to determine whether or not relationships were proportional had significantly improved, and the misconception that all linear relationships are proportional was not present in teachers' work at the end of the course.

Teachers' work on the pretest also revealed that they had limited understandings of the mathematical relationships embedded in proportional situations. Teachers drew upon significantly more key understandings in their work at the end of the course than they did at the beginning of the course, and all teachers (except Nora[26]) appeared to come to know or understand additional features of proportional relationships as a result of the course.

[26] Recall that Nora drew upon all four key understandings at the beginning of the course (see Table 21), and therefore had no room for improvement with respect to utilizing additional key understandings at the end of the course.

4.2. Exploring the Impact of the Proportional Reasoning Course: Similarities and Differences Between Teachers Who Participated in the Course and Teachers Who Did Not

In this section, the influence that the proportional reasoning course had on teachers' learning is explored by comparing the understandings of teachers enrolled in the course and those who were not before and after the course was enacted. In making these comparisons, research question 3 (How do preservice secondary mathematics teachers who participated in a course specifically focused on proportional reasoning differ from preservice secondary mathematics teachers who did not participate in the course in their understandings about proportional reasoning?) is explored. As described in Chapter Three, both the treatment group (i.e., ten teachers who completed the proportional reasoning course) and the contrast group (i.e., six teachers who were not enrolled in the course, but were otherwise similar to the treatment group) completed the pre/posttest and pre/post interview. Results from these instruments indicated that teachers in both groups had similar understandings prior to the course, and that teachers in the treatment group enhanced their understandings of several aspects of proportional reasoning, as shown in Table 21.

With respect to teachers' ability to solve a variety of problem types, both groups were able to do so prior to the course, as shown in Table 21. As noted previously, by the end of the course, the teachers in the treatment group relied significantly less heavily on procedural strategies such as cross multiplication (in solving task 23, a missing value problem situated in a similarity context), and used the between-ratio strategy, which highlights the multiplicative relationship between quantities, significantly more frequently in solving all five missing value problems on the pre/posttest. By contrast, there was no change in the strategies used by the teachers in the contrast group by the end of the course. In addition, both the treatment and contrast group were also similar with respect to their ability to discriminate proportional from nonproportional situations at the beginning of the course. For example, both groups recognized that ratio is an appropriate measure for attributes such as steepness and oranginess of an orange juice mixture. Both groups also had limitations in their ability to provide examples and nonexamples of proportional situations and to classify relationships as proportional or nonproportional. However, by the end of the course, the teachers in the treatment group demonstrated significant growth in their capacity to provide examples and nonexamples and to classify relationships as proportional or not. By contrast, the contrast group did not perform significantly better on these aspects. Finally, both groups used few key understandings to justify their examples and nonexamples of proportional situations and their classifications of relationships as proportional or not prior to the course. While the treatment group drew upon significantly more key understandings in their work at the end of the course, the contrast group did not. In the following sections, the similarities and differences between the treatment and contrast groups are described in more detail.

Table 21.

Similarities and Differences Between the Treatment Group and Contrast Group Prior to and Upon Completion of the Proportional Reasoning Course

	Treatment Group	Contrast Group
1. Solve a variety of problem types	***Before course:*** • Could correctly solve • Reliance on cross multiplication ***After course:*** • Growth in use of strategies that highlight multiplicative relationship (tasks 1-4; 23)* • Less reliance on cross multiplication (task 23)* • Growth in use of proportional reasoning language (e.g., "scaled up") in explanations*	***Before course:*** • Could correctly solve • Reliance on cross multiplication ***After course:*** • No growth in use of strategies that highlight multiplicative relationship** • No change in reliance on cross multiplication ** • No use of proportional reasoning language in explanations**
2. Discriminate proportional from nonproportional situations	***Before course:*** • Recognized when ratio is appropriate • Limited capacity to provide example and nonexample of proportional relationship and explain why • Limited capacity to classify relationships • Half held misconception that all linear relationships are proportional ***After course:*** • Growth in capacity to provide example and nonexample and explain why* • Growth in capacity to classify relationships* • Elimination of misconception that all linear relationships are proportional*	***Before course:*** • Recognized when ratio is appropriate • Limited capacity to provide example and nonexample of proportional relationship and explain why • Limited capacity to classify relationships • Nearly all did not hold misconception that all linear relationships are proportional ***After course:*** • No growth in capacity to provide example and nonexample and explain why • No growth in capacity to classify relationships** • No change in number of teachers who held misconception
3. Understand the mathematical relationships embedded in proportional situations	***Before course:*** • Limited use of 4 key understandings ***After course:*** • Growth in use of 4 key understandings*	***Before course:*** • Limited use of 4 key understandings ***After course:*** • No growth in use of 4 key understandings**

Note .
* indicates that there was a significant difference between the treatment group at the end of the course and the beginning of the course
** indicates that there was a significant difference between the treatment group and the contrast group

4.2.1. Solve a Variety of Problem Types

4.2.1.1. Missing Value Problems

Like the ten teachers in the treatment group, the six teachers in the contrast group correctly solved all five missing value problems on both the pretest and the posttest (tasks 1-4, which were not situated in a context, and task 23, which was situated in a similarity context [see Appendix A]). In addition, the teachers in the contrast group relied heavily on the cross multiplication procedure on both the pretest and posttest. However, unlike the teachers in the treatment group, the teachers in the contrast group did not utilize any new strategies on the posttest.

<u>Tasks 1-4: Missing value problems devoid of context</u>. As shown in Table 22, all six teachers in the contrast group used cross multiplication to solve the four missing value problems that were not situated in a context both on the pretest and posttest. These six teachers also utilized algebraic strategies (such as cross multiplication) and the between-ratio strategy on the pretest and posttest. Like the treatment group, teachers in the contrast group did not use cross multiplication with significantly less frequency on the posttest than on the pretest, $t(5) = 1$, $p = .18$ (one-tailed). However, unlike the treatment group, teachers in the contrast group did not use the between-ratio strategy significantly more on the posttest than on the pretest, $t(5) = 1.46$, $p = .10$ (one-tailed). In addition, teachers in the treatment group used the between-ratio strategy significantly more than teachers in the contrast group on the posttest, $t(14) = 2.55$, $p = .01$ (one-tailed). It is also interesting to note that no teachers in the contrast group used strategies on the posttest that they had not used on the pretest. By contrast, five of the ten teachers in the treatment group used strategies on the posttest that they had not used on the pretest. However, the difference between the number of teachers in the treatment group and the number of teachers in

the contrast group who used strategies on the posttest that they had not used on the pretest was not significant, Fisher's exact test, $p = .057$ (one-tailed).

Task 23: Missing value problem situated in a similarity context. As shown in Table 23, all six teachers in the contrast group correctly determined the missing dimension in task 23 and provided a valid explanation on the pretest and posttest. On the pretest, these teachers used cross multiplication exclusively. On the posttest, five of the teachers used cross multiplication, and one teacher used a between-ratio strategy. (Note that this teacher also used the between-ratio strategy on the pretest to solve tasks 1 and 2, as shown in Table 22.)

There was no significant difference between the teachers in the treatment group and the teachers in the contrast group in the use of cross multiplication to determine the missing dimension on the pretest, $t(14) = 1.15$, $p = .27$ (two-tailed). By contrast, on the posttest, significantly fewer teachers in the treatment group than teachers in the contrast group used cross multiplication to determine the missing dimension, $t(14) = 2.26$, $p = .02$ (one-tailed).

Table 22.

Contrast Group Teachers' Solution Strategies to Tasks 1-4

	Task 1 4/20=x/35		Task 2 2/7=6/x		Task 3 3/8=x/20		Task 4 9/15=12/x	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Carrie	Cross multiplication Cross multiplication	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Algebraic strategy	Cross multiplication Algebraic strategy	Cross multiplication Between-ratio	Cross multiplication Between-ratio
Emily	Cross multiplication Algebraic strategy	Cross multiplication Algebraic strategy	Cross multiplication Algebraic strategy	Cross multiplication Algebraic strategy	Cross multiplication Algebraic strategy	Cross multiplication Algebraic strategy	Cross multiplication Algebraic strategy	Cross multiplication Algebraic strategy
Lynn	Cross multiplication Cross multiplication	Cross multiplication Cross multiplication	Cross multiplication Cross multiplication	Cross multiplication Cross multiplication	Cross multiplication Cross multiplication	Cross multiplication Cross multiplication	Cross multiplication Cross multiplication	Cross multiplication Cross multiplication
Natalie	Cross multiplication Cross multiplication	Cross multiplication Cross multiplication	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication	Cross multiplication	Cross multiplication Cross multiplication	Cross multiplication Cross multiplication
Nicole	Cross multiplication Algebraic strategy	Cross multiplication	Cross multiplication Algebraic strategy	Cross multiplication	Cross multiplication Algebraic strategy	Cross multiplication	Cross multiplication Algebraic strategy	Cross multiplication
Paige	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication Between-ratio	Cross multiplication	Cross multiplication Between-ratio	Cross multiplication	Cross multiplication Between-ratio

Note. The dark gray shading indicates tasks in which the teacher solved in only one way. The light gray shading indicates tasks in which the teacher solved in two ways that both involved cross multiplication. In this case, the teacher's first method would be to apply the cross multiplication procedure to the problem. Their second way would involve manipulating the original proportion (e.g., reducing one of the ratios; reducing the entire proportion) and then applying the cross multiplication procedure to the new proportion.

Table 23.

Contrast Group Teachers' Solution Strategies to Task 23

	Pre	Post
Carrie	Cross multiplication	Cross multiplication
Emily	Cross multiplication	Cross multiplication
Lynn	Cross multiplication	Cross multiplication
Natalie	Cross multiplication	Cross multiplication
Nicole	Cross multiplication	Cross multiplication
Paige	Cross multiplication	Between-ratio

4.2.1.2. Numerical Comparison Problems

<u>Task 6: Comparing orange juice recipes</u>. All six teachers in the contrast group correctly determined that Luis' mixture had the stronger orange flavor in two ways on the pretest and the posttest. As noted previously, task 6 can be solved using two different types of ratios: part-to-part and part-to-whole. The types of ratios that the teachers in the contrast group used to solve the problem on the pre- and posttest are shown in Table 24. On the pretest, two teachers (Carrie and Natalie) used part-to-part ratios exclusively, and three teachers (Emily, Nicole, and Paige) used part-to-whole ratios exclusively. The remaining teacher, Lynn, solved the task using both types of ratios. Thus, the five teachers who used only one particular type of ratio on the pretest may not have held strategies based on the other type of ratio in their repertoire.

Table 24.

Types of Ratios Used By Teachers in the Contrast Group to Solve Task 6

	Pre	Post
2 Part-to-part strategies	Carrie, Natalie	Carrie
2 Part-to-whole strategies	Emily, Nicole, Paige	Emily, Nicole
1 part-to-part and 1 part-to-whole strategy	Lynn	Lynn, Paige, Natalie

The teachers' responses to item 2 on Interview 1 provided additional evidence for the claim that certain strategies were not part of their repertoire. In particular, the two teachers who used only part-to-part ratios on the pretest (Carrie and Natalie) struggled to make sense of Student D's response (see Figure B2 in Appendix B), which was based on a part-to-whole ratio, as shown in Table 25. Carrie was unable to make sense of Student D's strategy, and although Natalie eventually did make sense of this strategy, she remained unsure if it was generalizable. For example, Natalie stated, "I think they [Student D] might have just gotten lucky" (Natalie, Interview 1, line 156).

In addition, the three teachers who used part-to-whole ratios on the pretest (Emily, Nicole, and Paige) struggled to make sense of Student A and Student C's responses (see Figure B2 in Appendix B), which were based on part-to-part ratios. As shown in Table 25, none of these three teachers were able to correctly make sense of the responses produced by Student A and Student C. Interestingly, the one teacher who used both part-to-part and part-to-whole ratios on the pretest, Lynn, was able to make sense of all five student responses on the interview. The difference between the types of ratios teachers used on the pretest and posttest was not significant, $X^2(2, N = 6) = 1.53$, $p = .465$.

As shown in Table 24, two of the teachers in the contrast group, Natalie and Paige, used ratios on the posttest that they had not used on the pretest (Natalie had not used part-to-whole ratios on the pretest; Paige had not used part-to-part ratios on the pretest). These gains in strategy use might be accounted for by their work on item 2 on Interview 1, in which they examined and attempted to make sense of strategies based on both part-to-part and part-to-whole ratios. Although these two teachers struggled to make sense of strategies based on ratios that they had not themselves used in solving task 6 on the pretest (as shown in Table 25), teachers may have learned these unfamiliar strategies simply by carefully considering their validity during their work on item 2 on Interview 1.

Teachers' ability to explain the quantities they used to determine the mixture with the stronger orange flavor significantly improved over time[27], $T = 15$, $n_{s/r} = 5$, $p < 0.05$ (one-tailed). As shown in Table 26, all six teachers scored a 2, 3, or 4 on both the pre- and posttest, meaning that no teacher used an incorrect additive strategy to solve task 6. There was no significant difference in the extent to which teachers in the treatment and contrast groups could make sense of the quantities they used to solve task 6 on the pretest, $U = 25$, $p = .62$ (two-tailed). The difference in the extent to which teachers in the two groups made sense of the quantities they used to solve task 6 on the posttest was not significant, $U = 15$, $p = .058$ (one-tailed).

[27] It was certainly not expected that teachers in the contrast group would explain the quantities they used to solve task 6 significantly better on the posttest. It is interesting to note that their work on this task was the only instance in which they significantly improved on the posttest. Their experience in Interview 1, in which they were asked to make sense of students' strategies to a similar problem (see item 2 on Interview 1) provided them with an opportunity to explain the meaning of the quantities. In addition, those doing their student teaching in middle schools may have had opportunities to work on proportional reasoning during the semester. These experiences may have impacted their capacity to make sense of quantities.

Table 25.

Quality of Contrast Group Teachers' Explanations of the Quantities Students Used to Determine the Mixture With the Stronger Chocolate Flavor on Interview 1

	Student Responses				
	Student A	Student B	Student C	Student D	Student E
Carrie	2	4	3	0[a]	3
Emily	2	3	3	4	0
Lynn	4	4	4	4	4
Natalie	4	4	4	4[a]	4
Nicole	2[b]	4	3	4	4
Paige	2	4	3	3	4

Note. Scoring rubric:

Rubric score	Criteria for rubric score
4	Explains the quantities that were calculated correctly
3	Explains the quantities that were calculated vaguely
2	Does not explain the quantities that were calculated or explains them incorrectly
0	Does not know how to explain

[a] Carrie and Natalie both were unsure about Student D's strategy and struggled to make sense of the 8 and 13 (the total number of ounces in each mixture).

[b] Nicole initially did not believe that Student A's strategy would work. Eventually she decided that the strategy "is ok," but still did not make sense of the values Student A calculated.

Table 26.

Quality of Contrast Group Teachers' Explanations of the Quantities They Used to Determine the Mixture With the Stronger Orange Flavor on Task 6

	Pre		Post	
	Strategy 1	Strategy 2	Strategy 1	Strategy 2
Carrie	4	2	4	3
Emily	3	4	4	4
Lynn	3	2	3	3
Natalie	4	3	3	4
Nicole	3	3	3	4
Paige	3	2	4	2

Unlike the treatment group, no teacher in the contrast group exhibited the misconception of interpreting a part-to-part ratio of orange juice concentrate to water as the percent orange juice. However, no teachers in the contrast group converted their ratios to percents. (Instead they scaled both ratios up so that they had a common amount of either orange juice or water.) An examination of the contrast group teachers' work during the interview reveals that half of them did exhibit this misconception as they attempted to make sense of Student A's strategy (see Figure B2 in Appendix B), which was based on a part-to-part ratio. Three teachers incorrectly identified the values that Student A calculated as the percentage of chocolate in the chocolate milk. There was no significant difference in the extent to which teachers in the treatment and contrast groups were able to make sense of the strategies produced by students, $U = 32.5$, $p = .83$ (two-tailed).

<u>Task 24: Comparing plots of land</u>. On the pretest, five of the six teachers in the contrast group used multiplicative strategies and correctly identified the plot of land that would be the

"most square." The remaining teacher, Paige, used an additive strategy and incorrectly solved the problem, therefore earning a 1 on the scoring rubric. The five teachers who solved the problem correctly all used the same strategy to determine the rectangle that is the most square – calculating the ratio of the sides and selecting the ratio that was the closest to one. These five teachers also explained why they selected the rectangle whose ratio was closest to one (because the ratio of the sides of a square is always one), and therefore earned a 4 on the scoring rubric. On the posttest, all six teachers in the contrast group correctly solved the problem by calculating the ratio of the sides for each rectangle and selecting the rectangle whose ratio was closest to one. In addition, these teachers provided valid explanations for why they selected the rectangle whose ratio was closest to one, and earned 4s on the scoring rubric. There was not a significant difference between the rubric scores earned by the contrast group between the pretest and posttest, $T = 1$, $n_{s/r} = 1$, $p > .05$. In addition, there was not a significant difference between the rubric scores earned by the treatment group and the contrast group on the pretest ($U = 35.5$, $p = .59$ [two-tailed]) or the posttest ($U = 33$, $p = .39$ [one-tailed]).

Finally, no teachers in the contrast group used proportional reasoning language in their explanations on either the pre- or posttest. By contrast, over half the teachers in the treatment group used appropriate language (e.g., "scaled up") in their explanations. Significantly more teachers in the treatment group used appropriate proportional reasoning language in their explanations than teachers in the contrast group did on the posttest, $t(14) = 2.81$, $p = .007$ (one-tailed).

4.2.1.3. Qualitative Problems

All six teachers in the contrast group correctly solved and provided valid explanations for both qualitative problems on both the pretest and posttest. There was not a significant difference between the treatment group and the contrast group in their ability to correctly solve these two qualitative problems on either the pretest, $t(14) = 1.07$, $p = .30$ (two-tailed), or the posttest, $t(14) = 1.15$, $p = .13$ (one-tailed).

4.2.2. Discriminate Proportional From Nonproportional Situations

4.2.2.1. Ratio as Measure: Tasks 6, 9, 10, and 24

All six teachers in the contrast group correctly used ratios to solve tasks 6, 9, and 10 on both the pre- and posttest. That is, these teachers appeared to understand that ratio is an appropriate mechanism to measure attributes such as concentration of flavor, shades of paint, and steepness. In addition, as noted previously, five of the six teachers correctly used a ratio to measure squareness in task 24 on the pretest, and all six teachers used a ratio to measure squareness on the posttest. There was no significant difference between the contrast group and the treatment group in their ability to use ratios to solve this subset of tasks on either the pretest, Fisher's exact test, $p = .50$ (two-tailed) or the posttest, Fisher's exact test, $p = .62$ (one-tailed).

4.2.2.2. Classifying Relationships as Proportional or Nonproportional: Tasks 11-22 and the Snowfall Interview Item

Two teachers in the contrast group correctly classified all twelve of the relationships presented in tasks 11-22 as proportional or nonproportional and four teachers incorrectly classified at least one relationship. Of the 72 relationships on the teachers' pretests (6 teachers x 12 relationships), 7 relationships (approximately 10%) were incorrectly classified by teachers in the contrast group on the pretest. Table 27 indicates the number of relationships that each teacher incorrectly classified on the pretest.

Table 27.

Number of Relationships That Each Teacher in the Contrast Group Incorrectly Classified on Tasks 11-22

Teacher	Pre		Post	
	Number of relationships incorrectly classified/not classified	Representation(s) of the incorrectly/not classified relationships	Number of relationships incorrectly classified/not classified	Representation(s) of the incorrectly/not classified relationships
Carrie	4	Language, graph, equation, table	4	Language, graph, equation, table
Emily	0	N/A	0	N/A
Lynn	1	Language	0	N/A
Natalie	1	Language	0	N/A
Nicole	0	N/A	0	N/A
Paige	1	Language	1	Language
Total number of relationships incorrectly classified/not classified	7		5	

There was not a significant difference between the treatment and contrast groups on the pretest with respect to their ability to correctly classify the relationships presented in tasks 11-22, $t(14) = 1.60$, $p = .13$ (two-tailed). In addition, there was not a significant difference in the number of teachers who appeared to believe that all linear relationships are proportional (as noted earlier, a common misconception) on the pretest between the two groups, Fisher's exact test, $p = .31$ (two-tailed).

On the posttest, four teachers in the contrast group correctly classified all twelve of the relationships presented in tasks 11-22 as proportional or nonproportional and two teachers incorrectly classified at least one relationship, as shown in Table 27. Of the 72 relationships on the posttest, five (7%) were incorrectly classified. The number of incorrect classifications that teachers in the contrast group made on the posttest was not significantly less than the number of incorrectly classified relationships on the pretest, $t(5) = 1.58$, $p = .08$ (one-tailed). There was also not a significant difference between the number of correct classifications made by the treatment group and the contrast group on the posttest, $t(14) = 0.94$, $p = .18$ (one-tailed). Although this difference was not significant, it is important to note that the average number of correct classifications made by the contrast group on the pretest was 10.8 (out of a total of 12), while the average number of correct classifications made by the treatment group on the pretest was 9.1. Therefore, the contrast group had less room to improve their capacity to classification relationships as proportional or not than the treatment group. On the posttest, the average number of correct classifications made by the contrast group on the posttest was 11.2, while the average number of correct classifications made by the treatment group was 11.7. In addition, comparison of the difference scores (i.e., the number of correctly classified relationships on the posttest – the number of correctly classified relationships on the pretest for each teacher) between the two groups revealed that the treatment group experienced significantly more growth than the contrast group in correctly classifying relationships $t(14) = 2.54$, $p = .01$ (one-tailed).

Teachers' work on the snowfall item on Interview 2 provides further information about their ability to distinguish proportional from nonproportional relationships. Five of the six teachers in the contrast group correctly identified the proportional relationship in their work on this item. Four of these teachers did so spontaneously. Emily did not spontaneously comment on the proportionality of the relationships, but was able to correctly identify Cedar Rapids as proportional when prompted by the interviewer. The remaining teacher, Carrie, did not spontaneously comment on the proportionality of the relationships, and when prompted by the interviewer, stated that both relationships were proportional because they were both linear.

4.2.2.3. Creating Examples of Proportional Situations: Task 5 and Interview Item

Table 28 presents the results for the six teachers in the contrast group on task 5, in which they were asked to create a word problem that would require setting up and solving the proportion 3/8 = x/20.

Table 28.

Rubric Scores for Word Problems Created by Teachers in the Contrast Group for Task 5

	Pre	Post
Carrie	3	3
Emily	1	3
Lynn	1	2
Natalie	3	3
Nicole	1	1
Paige	1	3

Note. Teacher's word problems were scored using the following rubric:

3: Creates a word problem that does require setting up and solving 3/8=x/20. The answer of x=7.5 **makes sense in the context of the problem**

2: Creates a word problem that does require setting up and solving 3/8=x/20. The answer of x=7.5 **does not** make sense in the context of the problem, **but** the teacher appears to recognize this issue, and addresses it by addending the problem using phrases such as: "Parts…are ok", "Please round your answer", etc.

1: Creates a word problem that does require setting up and solving 3/8=x/20. The answer of x=7.5 **does not** make sense in the context of the problem.

0: Word problem does not require setting up and solving 3/8=x/20.

As shown in Table 28, all six teachers created word problems in which the quantities were related multiplicatively both on the pretest and posttest. Like the treatment group, the extent to which an answer of 7.5 made sense in the context of teachers' problems varied both on the pre- and posttest. There was no significant difference in the quality of the word problems produced by the contrast group between the pre- and posttest, $T = 6$, $n_{s/r} = 3$, $p > .05$. In addition, there was no significant difference between the treatment and contrast group on either the pretest (U = 31, p = .96 [two-tailed]) or the posttest (U = 42.5, p = .09 [one-tailed]).

Table 29 presents the results for the six teachers in the contrast group on the interview item in which they were asked to create an example and nonexample of a proportional relationship. As shown in Table 29, their ability to provide both examples and nonexamples of proportional relationships and explain why on Interview 1 and Interview 2 was not significantly different (for examples, $T = 3$, $n_{s/r} = 2$, $p > .05$; for nonexamples, $T = 6$, $n_{s/r} = 3$, $p > .05$). There was also not a significant difference between the treatment and contrast groups in their ability to provide examples of proportional relationships (U = 37.5, p = .48 [two-tailed]) or nonexamples (U = 28, p = .87 [two-tailed]) on Interview 1. In addition, there was not a significant difference between the two groups in their ability to provide examples of proportional relationships (U = 31, p = .48 [one-tailed]) or nonexamples (U = 20.5, p = .16 [one-tailed]) on Interview 2. A comparison of the difference scores (i.e., the posttest score – the pretest score for each teacher) for the examples also revealed no significant difference between the two groups, $t(14) = 0.0$, p = .50 (one-tailed). However, a comparison of the difference scores (i.e., the posttest score – the pretest score for each teacher) for the nonexamples revealed that the mean improvement for teachers in the treatment group was 0.9 on the rubric, while the mean improvement for teachers in the contrast group was only 0.5 on the rubric. [However, there was not a significant difference in the difference scores between the two groups, $t(14) = 0.82$, p = 21 (one-tailed).]

Table 29.

Rubric Scores for Examples and Nonexamples of Proportional Situations Created by Teachers in the Contrast Group

	Interview 1		Interview 2	
	example	nonexample	example	nonexample
Carrie	4	3	4	3
Emily	1	1	3	2
Lynn	4	4	4	4
Natalie	4	1	4	2
Nicole	4	4	4	4
Paige	3	3	4	4

Note. Teachers' examples and nonexamples were scored using the following rubric:

4: creates a proportional/nonproportional situation and explains why it is proportional (or not) (either by drawing on the 4 key understandings, or other means – e.g., saying that the ratio between the two quantities is constant, or that the unit rate stays the same)

3: creates a proportional/nonproportional situation but explanation is vague or does not clearly explain why it's proportional (or not)

2: creates a situation in which there is *not* a relationship between the quantities – proportional or otherwise

1: incorrectly creates a proportional/nonproportional situation

4.2.3. Understand the Mathematical Relationships Embedded in Proportional Situations

4.2.3.1. Tasks 11-22

On the pretest, there was no significant difference in the number of key understandings that teachers in the treatment and contrast groups drew upon to explain why the proportional relationships in tasks 11-22 were proportional, $t(14) = 0.66$, $p = .52$ (two-tailed). However, on the posttest, teachers in the treatment group drew upon significantly more key understandings than the contrast group in their explanations of why the proportional relationships were proportional, $t(14) = 2.17$, $p = .02$ (one-tailed).

Table 30 illustrates the key understandings upon which teachers in the contrast group drew upon on the pretest and posttest to explain why the proportional relationships in tasks 11-22 (in particular, tasks 12, 14, 19, and 20 are proportional) were proportional. As shown in Table 30, on the pretest, only one teacher (Carrie) consistently explained that the proportional relationships were proportional because linear relationships are proportional. (Recall that in the treatment group, five of the ten teachers exhibited this misconception. However, there was not a significant difference between the two groups, Fisher’s exact test, $p = .30$ [two-tailed].)

In addition, as shown in Table 31, the number of key understandings teachers in the contrast group drew upon on the posttest was not significantly greater than the number of key understandings they drew upon on the pretest, $t(5) = 1.58$, $p = .08$ (one-tailed). Two teachers drew upon one understanding on the posttest that they had not used on the pretest.

Table 30.

Key Understandings Upon Which Teachers in the Contrast Group Drew to Identify Proportional Relationships in Tasks 11-22 on Pre- and Posttest

	Pre				Post			
	Task 12 (language)	Task 14 (graph)	Task 19 (equation)	Task 20 (table)	Task 12 (language)	Task 14 (graph)	Task 19 (equation)	Task 20 (table)
Carrie	Linears are proportional	Linears are proportional	Linears are proportional	Linears are proportional	Linears are proportional	Linears are proportional	Linears are proportional	Linears are proportional
Emily	3	2	3	3	3	2	3	3
Lynn	3	3	3	3	3	3	3	3
Natalie	3	2	2	3	3	2	2	3
Nicole	3	2	3	3	1	2	1	3
Paige	3	None	None	3	None	2	2	3

Note. Shading indicates that the teacher's work indicates that they have a misconception about proportional relationships (i.e., either that all linear relationships are proportional, or that a relationship is proportional because as x increases, y also increases).

Key Understandings
1: multiplicative
2: line through origin
3: rate pairs equivalent
4: m in y = mx is slope, unit rate, constant of proportionality

.

Table 31.

Key Understandings Upon Which Teachers in the Contrast Group Drew on the Posttest, but Not on the Pretest

	Pre	Post	Key understandings that were utilized on the posttest, but not on the pretest
Carrie	None	None	None
Emily	2, 3	2, 3	None
Lynn	3	3	None
Natalie	2, 3	2, 3	None
Nicole	2, 3	1, 2, 3	1
Paige	3	2, 3	2

Note.

Key Understandings
1: multiplicative
2: line through origin
3: rate pairs equivalent
4: m in y = mx is slope, unit rate, constant of proportionality

4.2.3.2. Item on Interview 1 and Interview 2: Defining a Proportional Relationship and Creating an Example and Nonexample of a Proportional Relationship

The number of key understandings that teachers in the contrast group drew upon in defining and creating examples and nonexamples of proportional relationships did not increase significantly from the pretest to the posttest, as shown in Table 32, $t(5) = 1.00$, $p = .18$ (one-tailed). There was not a significant difference between the number of key understandings used by the treatment group and the contrast group on Interview 1, $t(14) = .367$, $p = .72$ (two-tailed). However, teachers in the treatment group drew upon significantly more key understandings than teachers in the contrast group on Interview 2, $t(14) = 1.82$, $p = .04$ (one-tailed).

Table 32.

Key Understandings Upon Which Teachers in the Contrast Group Drew in Defining a Proportional Relationship and Creating an Example and Nonexample of a Proportional Relationship on Interview 1 and 2

	Interview 1	Interview 2
Carrie	None	None
Emily	3	3
Lynn	3	3
Natalie	None	3
Nicole	1	1
Paige	1, 3	1, 3

Note.

Key Understandings
1: multiplicative
2: line through origin
3: rate pairs equivalent
4: m in y = mx is slope, unit rate, constant of proportionality

4.2.3.3. Item 2 on Interview 2: Snowfall

As noted previously, five of the six teachers in the contrast group correctly identified the proportional relationship, Cedar Rapids, in the snowfall interview item. In addition, the five teachers who correctly identified Cedar Rapids as proportional drew upon key understandings to explain why the relationship was proportional, as shown in Table 33. Teachers in the treatment group drew upon significantly more key understandings than the teachers in the contrast group on the snowfall item, $t(14) = 1.78$, $p = .048$. It is also interesting to note that no teachers in the contrast group drew upon more than two key understandings in their work on the snowfall item, while four of the ten teachers in the treatment group drew upon three or four key understandings in their work (as shown in Table 19).

Table 33.

Key Understandings Upon Which Teachers in the Contrast Group Drew in Their Work on the Snowfall Item

Teacher	Key understandings used
Carrie	None
Emily	3
Lynn	3
Natalie	2, 3
Nicole	1, 3
Paige	1, 2

Note.

Key Understandings
1: multiplicative
2: line through origin
3: rate pairs equivalent
4: m in y = mx is slope, unit rate, constant of proportionality

4.2.3.4. Tasks 11-22 and Interview Items (Defining & Creating Example and Nonexample, and Snowfall)

By arranging all the data that examined the contrast group teachers' understandings of the mathematical relationships embedded in proportional situations together, comparisons between these teachers' use of the key understandings prior to and after the course can be made. Table 34 shows the key understandings used by teachers in the contrast group on the pre-instruments (tasks 11-22 on the pretest and the Interview 1 item in which teachers defined a proportional relationship and created an example and nonexample) and the post-instruments (tasks 11-22 on the posttest, the Interview 2 item in which teachers defined a proportional relationship and created an example and nonexample, and the snowfall item on Interview 2).

Table 34.

Key Understandings Upon Which Teachers in the Contrast Group Drew Prior to and Upon Completion of the Course

	Pre-instruments: Pretest (Tasks 11-22) & Interview 1 (Defining, Example, Nonexample)	Post-instruments (Posttest (Tasks 11-22), Interview 2 (Defining, Example, Nonexample), Interview 2 (Snowfall)	**Key understandings that were utilized on the post-instruments, but not on the pre-instruments**
Carrie	None	None	None
Emily	2, 3	2, 3	None
Lynn	3	3	None
Natalie	2, 3	2, 3	None
Nicole	1, 2, 3	1, 2, 3	None
Paige	1, 3	1, 2, 3	2

Note.
Key Understandings
1: multiplicative
2: line through origin
3: rate pairs equivalent
4: m in y = mx is slope, unit rate, constant of proportionality

As shown in Table 34, five of the six teachers in the contrast group did not draw upon any key understandings on the post-instruments that they had not also used on the pre-instruments. That is, most teachers in the contrast group did not appear to learn any key understandings between the time they completed the pre- and post-instruments. Teachers in the contrast group did not draw upon significantly more understandings in their work on the post-instruments than on the pre-instruments, $t(5) = 1.00$, $p = .36$ (two-tailed). Teachers in the treatment group appeared to learn significantly more key understandings during the course than teachers in the contrast group, $t(14) = 3.78$, $p = .001$ (one-tailed). It is also interesting to note that key understanding 4 was never used by the contrast group. By contrast, half of the ten teachers in the treatment group appeared to learn key understanding 4 as a result of participation in the course. Since key understanding 4 involves making sense of the *m* in the equation $y = mx$, its use provides an opportunity to make connections to algebra. In addition, key understanding 4 may help teachers make connections among the remaining key understandings. For example, the *m* in the equation $y = mx$ is the slope, and it is also the multiplicative factor that relates the quantities (key understanding 1). In addition, the equation $y = mx$ is a line that contains the origin (key understanding 2). Finally, the rate pairs of a proportional relationship all reduce to the unit rate (key understanding 3), which is also the *m* in the equation $y = mx$.

4.2.4. Summary

A comparison of the treatment group and the contrast group indicated that the groups had similar understandings about proportional reasoning prior to the course. However, by the end of the course, teachers in the treatment group had developed several aspects of common and specialized content knowledge. In particular, teachers in the treatment group used additional strategies for solving problems and relied less heavily on procedures such as cross multiplication. In addition, teachers in the treatment group improved their capacity to classify relationships as proportional or not and provide examples and nonexamples of proportional relationships. These teachers also improved their justifications of their classifications and examples and nonexamples by drawing on a greater number of key understandings of the mathematical relationships embedded in proportional situations at the end of the course. A natural question that is raised by examining the similarities and differences between the treatment and contrast groups is, Where did teachers in the treatment group have opportunities to develop their common and specialized content knowledge during the course? In the next section, this question is explored.

4.3. Examining Teachers' Opportunities to Develop Common and Specialized Content Knowledge During the Proportional Reasoning Course

In this section, research question 4, To what extent can teacher learning be accounted for by participation in a course specifically focused on proportional reasoning?, is explored. The mathematics that teachers in the treatment group appeared to come to know (as evidenced by research questions 1 and 2) was used as a lens through which to examine the videotapes of the whole class discussions that occurred during the course. Specifically, teachers' work on the pre- and posttest and two interviews indicated that teachers appeared to learn the following by the end of the course: (1) strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies; (2) to classify relationships as proportional or nonproportional; (3) that not all linear relationships are proportional; and (4) the mathematical relationships embedded in proportional situations.

The course map shown in Figure 23 highlights the 16 whole class discussions in which there was evidence in the written record (e.g., overhead transparencies, poster paper, chalkboard) that teachers had opportunities to engage with the mathematics that they appeared to learn. These class discussions were further analyzed so as to indicate the total number of turns, the number of turns related to the mathematics teachers appeared to learn, and *who* made these contributions during the discussions. The results of the analysis of the content of each discussion's turns (i.e., the number of turns that were related to the mathematics that teachers appeared to learn) made salient that the mathematics that teachers appeared to learn during the course was in fact made public during multiple class discussions, thus providing all teachers who were present for the discussions an opportunity to learn. The results of the analysis of the teacher turns showed that even teachers who were relatively silent during class discussions still appeared to learn mathematics. The results of these analyses are presented in the following sections.

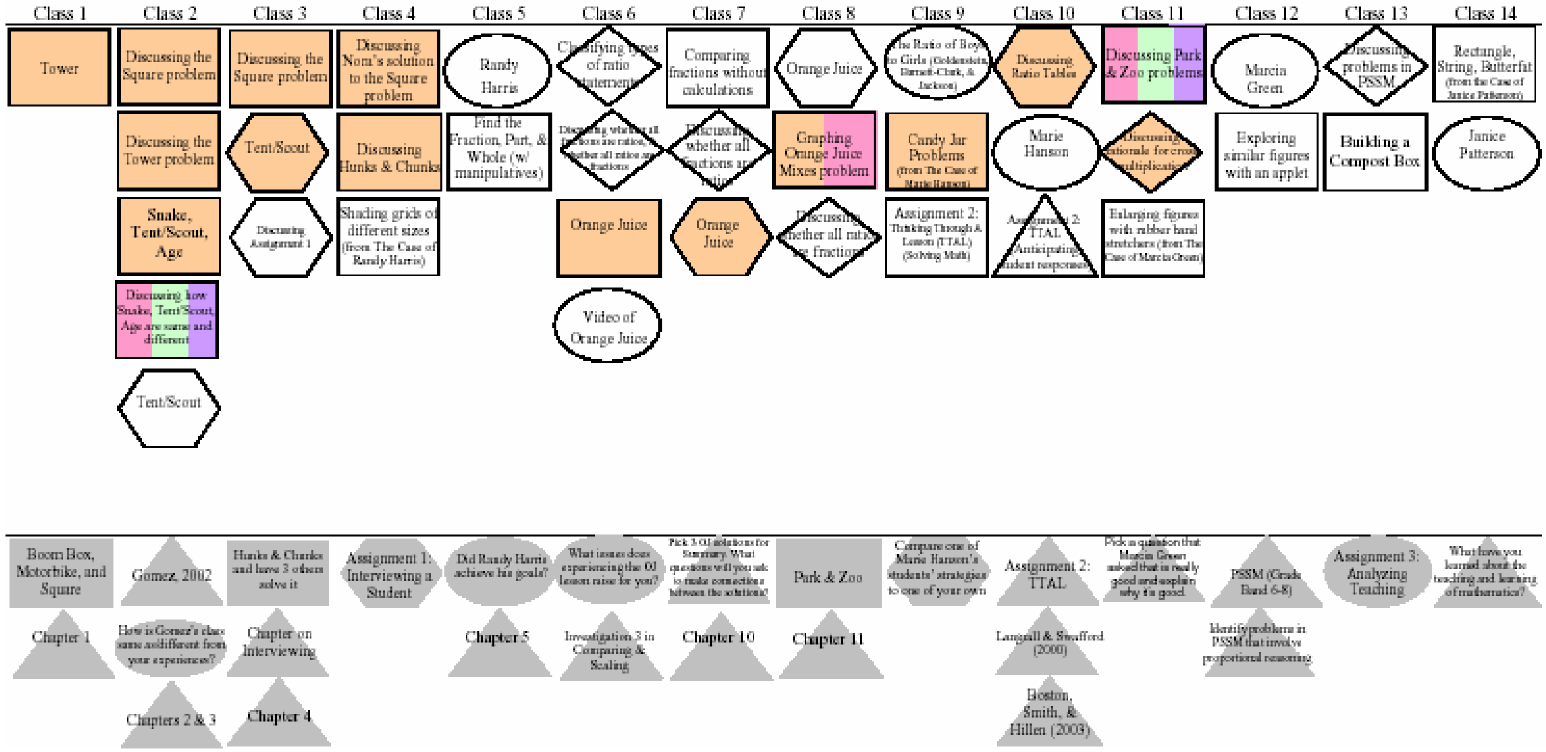

Figure 23. The color-coded course map that indicates the whole class discussions in which there was evidence in the written record that teachers had opportunities to explore the mathematics that they appeared to learn during the course.
Adapted from Smith, M. S., Silver, E. A., Leinhardt, G., & Hillen, A. F. (2003). Tracing the development of teachers' understanding of proportionality in a practice-based course. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL, p. 10.

Note. Columns denote activities that occurred during each class (or were assigned for homework)
Activities above the line occurred during class; activities below the line and shaded in gray were assigned for homework

The shapes indicate the *type* of activity in which teachers engaged, as shown below:

Rectangles: Solving and discussing mathematical tasks
Hexagons: Analyzing and discussing samples of student work
Ovals: Analyzing and discussing cases of mathematics teaching
Triangles: Reading about and discussing issues related to mathematics teaching
Diamonds: Discussing mathematical ideas that did not directly stem from a mathematical task that teachers solved

The color-coding indicates the activities in which there was evidence in the written record that teachers had opportunities to learn the mathematics, as shown below:

Orange: Strategies for solving missing value and numerical comparison problems and how to make sense of such strategies
Green: To classify relationships as proportional or nonproportional
Purple: Not all linear relationships are proportional
Pink: The mathematical relationships embedded in proportional situations

4.3.1. Strategies for Solving Missing Value and Numerical Comparison Problems and How to Make Sense of Such Strategies

As depicted by the orange shading in Figure 23, teachers had many opportunities to explore and make sense of strategies for solving missing value or numerical comparison problems. The class discussions in which strategies for solving these types of problems were made public and justified ranged in length from 9-47 minutes, and an average of 75% of the turns of these twelve discussions[28] were related to solution strategies, as shown in Table 35.

Tables 36-47 illustrate the total number of turns spoken during the twelve class discussions in which strategies for solving missing value and numerical comparison problems were presented and made sense of, who spoke these turns, and the solution strategies that were presented. Across the twelve discussions, each of the ten teachers presented at least one strategy for solving either missing value or numerical comparison problems, as shown in Table 48. Bert presented the most different[29] strategies, five, across the twelve discussions. Bruce, Carl, and Owen each presented four different strategies[30], Bonnie and Nora each presented three different strategies, Christopher and Ursula each presented two different strategies, and Elaine and Nanette each presented one strategy during the twelve discussions.

As shown in Tables 36-47, a variety of solution strategies were presented during the twelve discussions. The strategy that was presented with the most frequency was the between-

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[28] Note that there are 14 discussions highlighted in orange on the course map (shown in Figure 23), and only 12 class discussions listed in Table 35. This discrepancy is due to the fact that two of the activities, examining student work from the tent/scout problem and examining student work from the orange juice problem, included artifacts in the written record that were related to strategies – actual student responses. However, the class discussions of these activities were not related to discussing and making sense of the strategies (e.g., in the tent/scout student work activity, teachers considered which response showed the greatest and least understanding). The student work from these activities was coded for particular strategies that were made public for teachers, and is discussed at the end of this section.

[29] "Different" strategies means that the strategies that *each individual teacher* presented were different from one another. For example, Bert presented five strategies during the twelve class discussions, and each of those strategies was different from the other. However, it is important to note that multiple teachers presented the same strategy (e.g., both Ursula and Carl presented cross multiplication).

[30] Carl presented a fifth solution, but it was a strategy that he had already presented (cross multiplication).

ratio strategy, which was presented ten times. It is therefore not surprising that there was a significant increase in the frequency with which teachers used this strategy on the posttest. Strategies that made use of the unit rate were presented five times. Algebraic strategies were presented five times, and cross multiplication was presented three times. Part-to-part and part-to-whole strategies were each presented twice. A variety of solution strategies that were situated in particular representations were also presented. For example, the discussion of the graphing orange juice mixes problem in Class 8 elicited six different strategies for determining the "stronger" ratio that all made use of a graph (see Table 44). In addition, several strategies that used tables were presented during the course. Thus, a wide variety of solution strategies were made public and therefore available for those teachers present.

In addition to presenting strategies, teachers also participated in these class discussions by making contributions that were related to the solution strategies. These contributions usually involved discussing why a particular strategy made sense or adding to the explanation of a strategy. As shown in Tables 36-47, all ten teachers made contributions related to strategies that they had not presented themselves. That is, across the twelve discussions, the teachers were actively involved in making sense of and explaining both strategies that were produced by themselves and others in the course. Of the ten teachers, Bert made the most contributions, contributing 87 turns that were related to solution strategies across eleven of the twelve discussions. By contrast, Ursula and Nanette were the most silent participants across the twelve discussions. Ursula contributed six turns that were related to solution strategies across four discussions, and Nanette contributed ten turns that were related to solution strategies across two discussions.

Table 35.

Turns Related to Strategies for Solving Missing Value or Numerical Comparison Problems and How to Make Sense of Such Strategies During the Twelve Class Discussions

Whole class discussion	Length of whole class discussion (rounded to nearest minute)	Total number of turns	Percent of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies
Class 1: Tower problem	9 min	54	85%
Class 2: Square problem	15 min	112	68%
Class 2: Tower problem	10 min	80	43%
Class 2: Snake, scout/tent, and age problems	45 min	260	88%
Class 3: Square problem	19 min	139	73%
Class 4: Nora's solution to square problem	10 min	96	85%
Class 4: Hunks and Chunks problem	47 min	367	53%
Class 6: Orange juice problem	13 min	153	94%
Class 8: Graphing orange juice mixes problem	21 min	248	63%
Class 9: Candy jar problems	17 min	129	74%
Class 10: Ratio tables	11 min	76	88%
Class 11: Rationale for cross multiplication	11 min	59	88%
Across all twelve discussions	228 min = 3.8 hours	1773	75%

Table 36.

Analysis of the Discussion of the Tower Problem During Class 1 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented
Bert	**5**	**5**	
Bonnie	**0**	**0**	
Bruce	**0**	**0**	
Carl	**9**	**8**	**1. Between-ratio** **2. Informal measuring using the ladder**
Christopher	**0**	**0**	
Elaine	**0**	**0**	
Nanette	**0**	**0**	
Nora	**0**	**0**	
Owen	**0**	**0**	
Ursula	**0**	**0**	
Instructor	26	22	
Other 5 teachers in course (M.Ed. students)	11	8	3. Informal measuring using the crossbeams
Other talk (e.g., laughter, group responses, can't tell who spoke)	3	3	
Total number of turns	54	46	

Note. Bold indicates teachers in the treatment group.
Strategies are numbered according to the order in which they were presented during the whole class discussion.

Table 37.

Analysis of the Discussion of the Square Problem During Class 2 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented
Bert	**2**	**1**	
Bonnie	**2**	**2**	
Bruce	**0**	**0**	
Carl	**0**	**0**	
Christopher	**0**	**0**	
Elaine	**3**	**2**	
Nanette	**5**	**4**	
Nora	**14**	**10**	**4. Considering area added on to 35 x 35 versus 22 x 22**
Owen	**5**	**5**	**3. Counterexample: 1 x 5 versus 450 x 470**
Ursula	**0**	**0**	
Instructor	50	33	
Other 5 teachers in course (M.Ed. students)	22	13	1. Ratio of length:width 2. Additive
Other talk (e.g., laughter, group responses, can't tell who spoke)	9	6	
Total number of turns	112	76	

Note. Bold indicates teachers in the treatment group.
Strategies are numbered according to the order in which they were presented during the whole class discussion.

Table 38.

Analysis of the Discussion of the Tower Problem During Class 2 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented[a]
Bert	**5**	**3**	
Bonnie	**0**	**0**	
Bruce	**0**	**0**	
Carl	**11**	**8**	
Christopher	**0**	**0**	
Elaine	**0**	**0**	
Nanette	**9**	**0**	
Nora	**2**	**0**	
Owen	**0**	**0**	
Ursula	**0**	**0**	
Instructor	38	15	
Other 5 teachers in course (M.Ed. students)	15	8	
Other talk (e.g., laughter, group responses, can't tell who spoke)	0	0	
Total number of turns	80	34	

Note. Bold indicates teachers in the treatment group.

[a] No strategies were presented during this discussion. The purpose of this discussion was to make sense of and connect several of the strategies presented during Class 1 (shown in Table 22).

Table 39.

Analysis of the Discussion of the Snake, Tent/Scout, and Age Problems During Class 2 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented
Bert	**29**	**25**	**1. Additive (Snake)**
Bonnie	**16**	**16**	**9. Unit rate and then between-ratio (like Nanette's except not visual) (Scout/tent)**
Bruce	**17**	**13**	**2. Multiplicative (i.e., using ratios to compare lengths) (Snake)**
Carl	**3**	**3**	
Christopher	**11**	**11**	**4. Multiplicative (i.e., using ratios to compare lengths) (Snake)** **8. Visual between-ratio (Scout/tent)**
Elaine	**5**	**3**	
Nanette	**6**	**6**	**6. Visual unit rate then between-ratio to (Scout/tent)**
Nora	**5**	**4**	
Owen	**10**	**10**	**3. Multiplicative (i.e., using ratios to compare lengths) (Snake)**
Ursula	**2**	**1**	**5. Cross multiplication (Scout/tent)**
Instructor	114	97	
Other 5 teachers in course (M.Ed. students)	39	36	7. Algebraic strategy (Scout/tent) 10. Guess & check (Age) 11. Algebraic strategy (Age) 12. Table (Age)
Other talk (e.g., laughter, group responses, can't tell who spoke)	3	3	
Total number of turns	260	228	

Note. Bold indicates teachers in the treatment group.
Strategies are numbered according to the order in which they were presented during the whole class discussion.

Table 40.

Analysis of the Discussion of the Square Problem During Class 3 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented[a]
Bert	**14**	**9**	
Bonnie	**0**	**0**	
Bruce	**0**	**0**	
Carl	**6**	**4**	
Christopher	**4**	**2**	
Elaine	**0**	**0**	
Nanette	**0**	**0**	
Nora	**7**	**6**	
Owen	**0**	**0**	
Ursula	**0**	**0**	
Instructor	66	48	
Other 5 teachers in course (M.Ed. students)	34	25	
Other talk (e.g., laughter, group responses, can't tell who spoke)	8	8	
Total number of turns	139	102	

Note. Bold indicates teachers in the treatment group.

[a] No strategies were presented during this discussion. The purpose of this discussion was to make sense of quantities used in solutions that were presented during Class 2 (shown in Table 23).

Table 41.

Analysis of the Discussion of Nora's Solution to the Square Problem During Class 4 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented[a]
Bert	**1**	**1**	
Bonnie	**0**	**0**	
Bruce	**0**	**0**	
Carl	**-**	**-**	
Christopher	**-**	**-**	
Elaine	**13**	**11**	
Nanette	**0**	**0**	
Nora	**6**	**6**	
Owen	**0**	**0**	
Ursula	**3**	**1**	
Instructor	43	36	
Other 5 teachers in course (M.Ed. students)	29	26	
Other talk (e.g., laughter, group responses, can't tell who spoke)	1	1	
Total number of turns	96	82	

Note. Bold indicates teachers in the treatment group.
Shading indicates teachers who were not present for class.

[a] No strategies were presented during this discussion. The purpose of this discussion was to make sense of Nora's strategy, which had been presented during Class 2 (shown in Table 37), and revisited during Class 3.

Table 42.

Analysis of the Discussion of the Hunks and Chunks Problem During Class 4 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented
Bert	**25**	**13**	**3. Unit rate: Cost per ounce**
Bonnie	**17**	**14**	**4. Cost per 4 ounces**
Bruce	**7**	**1**	
Carl	**-**	**-**	**-**
Christopher	**-**	**-**	**-**
Elaine	**27**	**21**	**5. Cost per 4 ounces and added to 12-ounce box to compare 2 16-ounce boxes**
Nanette	**9**	**0**	
Nora	**5**	**2**	
Owen	**18**	**10**	
Ursula	**0**	**0**	
Instructor	143	70	
Other 5 teachers in course (M.Ed. students)	85	47	1. Unit rate: Cost per ounce via mental guess and check 2. Unit rate: Cost per ounce of Super Chunks and determined cost for Mighty Hunks at the SC cost per ounce 6. Cost per 4 ounces and added to 12-ounce box to compare 2 16-ounce boxes 7. Comparing the sizes of the boxes and prices using percents 8. Building up (using addition) to 24-ounce boxes 9. Between-ratio: Scale up (using multiplication) to 48-ounce boxes 10. More formal version of strategy 7
Other talk (e.g., laughter, group responses, can't tell who spoke)	31	18	
Total number of turns	367	196	

Note. Bold indicates teachers in the treatment group.
Shading indicates teachers who were not present for class.
Strategies are numbered according to the order in which they were presented during the whole class discussion.

Table 43.

Analysis of the Discussion of the Orange Juice Problem During Class 6 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented[a]
Bert	**7**	**7**	
Bonnie	**9**	**9**	
Bruce	**11**	**11**	**4. Part to part; percents**
Carl	**4**	**4**	
Christopher	**0**	**0**	
Elaine	**0**	**0**	
Nanette	**0**	**0**	
Nora	**10**	**10**	**3. Part to whole; percents**
Owen	**15**	**15**	**2. Part to part; scale up to 120 cups water**
Ursula	**0**	**0**	
Instructor	61	53	
Other 5 teachers in course (M.Ed. students)	23	22	1. Part to whole; common denominator of 40 cups of juice
Other talk (e.g., laughter, group responses, can't tell who spoke)	13	13	
Total number of turns	153	144	

Note. Bold indicates teachers in the treatment group.
Strategies are numbered according to the order in which they were presented during the whole class discussion.

[a] Note that the strategies were produced by the small groups and written on posters, which were displayed on one of the classroom walls. During the discussion, the instructor selected the order in which the posters were discussed, and the particular teachers who presented the strategies.

Table 44.

Analysis of the Discussion of the Graphing Orange Juice Mixes Problem During Class 8 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented
Bert	**11**	**11**	**4. Draw line y=x; mixtures that are above that line are less orangey**
Bonnie	**-**	**-**	**-**
Bruce	**20**	**19**	**1. Use graph to show change in y for Mix A isn't as much as change in y for Mix B, given same amount of change in x**
Carl	**10**	**5**	
Christopher	**2**	**0**	
Elaine	**0**	**0**	
Nanette	**0**	**0**	
Nora	**10**	**4**	
Owen	**12**	**7**	**3. Line that's closest to the axis for water is less orangey**
Ursula	**0**	**0**	
Instructor	100	62	
Other 5 teachers in course (M.Ed. students)	47	31	2. Fix concentrate and use graph to see which mixture has more water 5. 100% concentrate would be a horizontal line; the line becomes more vertical as the mixture becomes less orangey 6. Fix the water and use graph to see which mixture has less concentrate
Other talk (e.g., laughter, group responses, can't tell who spoke)	36	17	
Total number of turns	248	156	

Note. Bold indicates teachers in the treatment group.

Shading indicates teachers who were not present for class.

Strategies are numbered according to the order in which they were presented during the whole class discussion.

Table 45.

Analysis of the Discussion of the Candy Jar Problems During Class 9 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented
Bert	**10**	**10**	**4. Algebraic: set up and solved equations (Problem 2)**
Bonnie	**-**	**-**	**-**
Bruce	**16**	**13**	**5. Between-ratio (Problem 3)**
Carl	**8**	**5**	**2. Cross multiplication (Problem 1)** **6. Building up table (Problem 3)**
Christopher	**0**	**0**	
Elaine	**0**	**0**	
Nanette	**1**	**0**	
Nora	**4**	**3**	
Owen	**4**	**4**	
Ursula	**3**	**3**	**1. Between-ratio (Problem 1)**
Instructor	51	37	
Other 5 teachers in course (M.Ed. students)	20	16	3. Between-ratio with a table (Problem 2) 7. Dividing total candy by number of candy in 1 bag (Problem 3) (Teachers decided this strategy would not work all the time)
Other talk (e.g., laughter, group responses, can't tell who spoke)	12	5	
Total number of turns	129	96	

Note. Bold indicates teachers in the treatment group.
Shading indicates teachers who were not present for class.
Strategies are numbered according to the order in which they were presented during the whole class discussion.

Table 46.

Analysis of the Discussion of Ratio Tables During Class 10 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented
Bert	**0**	**0**	
Bonnie	**0**	**0**	
Bruce	**12**	**10**	
Carl	**5**	**2**	**1. Cross multiplication (to check the ratio table strategy)**
Christopher	**2**	**2**	
Elaine	**4**	**1**	
Nanette	**0**	**0**	
Nora	**2**	**2**	
Owen	**0**	**0**	
Ursula	**2**	**1**	
Instructor	46	31	
Other 5 teachers in course (M.Ed. students)	10	6	
Other talk (e.g., laughter, group responses, can't tell who spoke)	14	12	
Total number of turns	76	67	

Note. Bold indicates teachers in the treatment group.

Table 47.

Analysis of the Discussion of the Rationale for Cross Multiplication During Class 11 With Respect to Solution Strategies

	Number of turns	Number of turns related to strategies for solving missing value problems or numerical comparison problems and how to make sense of such strategies	Strategies presented
Bert	**3**	**2**	**1. Algebraic solving for x**
Bonnie	**3**	**2**	**4. Generalized between-ratio strategy using equivalent ratios**
Bruce	**3**	**2**	
Carl	**0**	**0**	
Christopher	**0**	**0**	
Elaine	**1**	**0**	
Nanette	**0**	**0**	
Nora	**6**	**6**	**2. Algebraic solving for x**
Owen	**0**	**0**	
Ursula	**0**	**0**	
Instructor	26	23	
Other 5 teachers in course (M.Ed. students)	4	4	3. Used balloons context to identify the between-ratio
Other talk (e.g., laughter, group responses, can't tell who spoke)	13	13	
Total number of turns	59	52	

Note. Bold indicates teachers in the treatment group.

Strategies are numbered according to the order in which they were presented during the whole class discussion.

Table 48.

Number of Different Strategies Presented by Teachers in the Treatment Group During the Twelve Discussions

	Number of different strategies presented during the twelve discussions
Bert	**5**
Bonnie	**3**
Bruce	**4**
Carl	**4**
Christopher	**2**
Elaine	**1**
Nanette	**1**
Nora	**3**
Owen	**4**
Ursula	**2**

Finally, in addition to the whole class discussions, solution strategies for solving missing value and numerical comparison problems were made public during three course activities in which teachers examined student responses (as depicted by the orange shading in the two hexagons in Figure 23[31]). The two activities represented by hexagons engaged teachers in the analysis of sets of student work and served to foster teachers' mathematics understandings as well as their understandings about students as learners of mathematics and teaching mathematics. For example, the set of ten student responses to the tent/scout problem (shown in Appendix D) varied with respect to strategy (i.e., unit rate, building up, between-ratio, incorrect additive) and representation (i.e., table, diagram, symbolic), and included some strategies (e.g., the solution produced by Student B in Appendix D) that had not been presented by teachers during the whole

[31] The third activity in which teachers examined student responses to missing value or numerical comparison problems occurred when teachers read *The Case of Marie Hanson* (Smith, Silver, & Stein, 2005b). Since teachers' reading of this case did not occur in the public space, this activity is not shaded in Figure 23.

class discussion in which they shared their own solutions. Thus, examining this set of student work might have introduced teachers to additional solution strategies that students might use. An additional purpose of the tent/scout student work activity was to help teachers come to see that not all incorrect strategies are “the same,” and to bring to light a common student misconception – applying additive strategies when multiplicative ones are needed (e.g., see the solutions produced by Students D and I in Appendix D).

In the second activity involving a set of student work, teachers examined a set of twelve student responses to the orange juice problem, as shown in Appendix E. This set of student responses varied with respect to ratios used (i.e., part-to-part; part-to-whole), strategy (i.e., unit rate, percents, building up to a common amount of juice or water), and representation (i.e., diagram, table).

One final instance in which teachers were exposed to students’ solutions to missing value problems occurred when teachers read *The Case of Marie Hanson* (Smith, Silver, & Stein, 2005b), which depicted a middle-grades mathematics lesson in which students solved the candy jar problems (see Figure 13 in Chapter Three). As teachers read this case, they had an opportunity to examine a variety of student solutions to the candy jar problems; specifically, building-up (in the form of a table), unit rate, between-ratio, and cross multiplication.

4.3.2. Classifying Relationships as Proportional or Nonproportional

As depicted by the green shading in Figure 23, teachers had opportunities to classify relationships as proportional or nonproportional during two class discussions – the discussion of the similarities and differences among the snake, tent/scout, and age problems (Class 2) and the discussion of the park and zoo problems (Class 11). As shown in Table 49, the class discussions ranged in length from 22-28 minutes, and approximately 72% of the turns of each of these two discussions were related to classifying relationships as proportional or nonproportional.

Table 49.

Turns Related to Classifying Relationships as Proportional or Nonproportional During the Two Class Discussions

Whole class discussion	Length of whole class discussion (rounded to nearest minute)	Total number of turns	Percent of turns related to classifying relationships as proportional or nonproportional
Class 2: Similarities and differences among the snake, scout/tent, and age problems	28 min	241	83%
Class 11: Park and zoo problems	22 min	247	60%
Across all three discussions	50 min	484	72%

Seven teachers made contributions related to classifying relationships as proportional or nonproportional during Class 2, as shown in Table 50. Although Bruce and Ursula made contributions, none of them were related to classifying relationships as proportional or not. Nanette remained silent during the discussion. As shown in Table 51, during Class 11, seven teachers made contributions related to classifying relationships as proportional or not. Elaine, Nanette, and Ursula remained silent during the discussion of the park and zoo problems during Class 11.

Table 50.

Analysis of the Discussion of the Similarities and Differences Among the Snake, Tent/Scout, and Age Problems During Class 2 With Respect to Classifying Relationships as Proportional or Nonproportional

	Number of turns	Number of turns related to classifying relationships as proportional or nonproportional
Bert	**32**	**25**
Bonnie	**19**	**19**
Bruce	**2**	**0**
Carl	**10**	**7**
Christopher	**12**	**12**
Elaine	**5**	**5**
Nanette	**0**	**0**
Nora	**1**	**1**
Owen	**10**	**10**
Ursula	**1**	**0**
Instructor	88	62
Other 5 teachers in course (M.Ed. students)	40	38
Other talk (e.g., laughter, group responses, can't tell who spoke)	21	21
Total number of turns	241	200

Note. Bold indicates teachers in the treatment group.

Table 51.

Analysis of the Discussion of the Park and Zoo Problems During Class 11 With Respect to Classifying Relationships as Proportional or Nonproportional

	Number of turns	Number of turns related to classifying relationships as proportional or nonproportional
Bert	**25**	**17**
Bonnie	**9**	**2**
Bruce	**15**	**10**
Carl	**8**	**3**
Christopher	**4**	**1**
Elaine	**0**	**0**
Nanette	**0**	**0**
Nora	**14**	**9**
Owen	**19**	**5**
Ursula	**0**	**0**
Instructor	83	58
Other 5 teachers in course (M.Ed. students)	34	22
Other talk (e.g., laughter, group responses, can't tell who spoke)	36	22
Total number of turns	247	149

Note. Bold indicates teachers in the treatment group.

The idea of classifying relationships as proportional or nonproportional appeared to be an important aspect of the discussions during these two class discussions – accounting for nearly 75% of the total turns. In addition, the analysis of the two class discussions shows that eight of the ten teachers made contributions related to classifying relationships as proportional or nonproportional during the two class discussions. Only Nanette and Ursula made no contributions related to classifying relationships as proportional or nonproportional.

4.3.3. Not All Linear Relationships are Proportional

As depicted by the purple shading in Figure 23, teachers had opportunities to discuss the idea that not all linear relationships are proportional during two class discussions – the discussion of the similarities and differences among the snake, tent/scout, and age problems (Class 2) and the discussion of the park and zoo problems (Class 11). As shown in Table 52, the class discussions ranged in length from 22-28 minutes, and approximately 50% of the turns of each of these two discussions were related to the idea that not all linear relationships are proportional.

Table 52.

Turns Related to the Idea That Not All Linear Relationships Are Proportional During the Two Class Discussions

Whole class discussion	Length of whole class discussion (rounded to nearest minute)	Total number of turns	Percent of turns related to the idea that not all linear relationships are proportional[a]
Class 2: Similarities and differences among the snake, scout/tent, and age problems	28 min	241	46%
Class 11: Park and zoo problems	22 min	247	52%
Across all three discussions	50 min	484	49%

[a] Note that some turns that were related to the idea that not all linear relationships are proportional were also related to classifying relationships as proportional or not (as shown in Table 49).

As shown in Table 53, half of the teachers (Bert, Bonnie, Carl, Christopher, and Owen) made contributions related to the idea that not all linear relationships are proportional during the discussion of the similarities and differences among the snake, tent/scout, and age problems during Class 2. Four other teachers (Bruce, Elaine, Nora, and Ursula) made contributions during this discussion, but none of their contributions were related to the idea that not all linear relationships are proportional. Finally, the remaining teacher, Nanette, remained silent during this discussion.

During Class 11, seven of the ten teachers (Bert, Bonnie, Bruce, Carl, Christopher, Nora, and Owen) made contributions related to the idea that not all linear relationships are proportional, as shown in Table 54. The remaining three teachers, Elaine, Nanette, and Ursula, were silent during the discussion of the park and zoo problems during Class 11.

Table 53.

Analysis of the Discussion of the Similarities and Differences Among the Snake, Tent/Scout, and Age Problems During Class 2 With Respect to the Idea That Not All Linear Relationships Are Proportional

	Number of turns	Number of turns related to the idea that not all linear relationships are proportional
Bert	**32**	**21**
Bonnie	**19**	**19**
Bruce	**2**	**0**
Carl	**10**	**7**
Christopher	**12**	**3**
Elaine	**5**	**0**
Nanette	**0**	**0**
Nora	**1**	**0**
Owen	**10**	**3**
Ursula	**1**	**0**
Instructor	88	30
Other 5 teachers in course (M.Ed. students)	40	15
Other talk (e.g., laughter, group responses, can't tell who spoke)	21	13
Total number of turns	241	111

Note. Bold indicates teachers in the treatment group.

Table 54.

Analysis of the Discussion of the Park and Zoo Problems During Class 11 With Respect to the Idea That Not All Linear Relationships Are Proportional

	Number of turns	Number of turns related to the idea that not all linear relationships are proportional
Bert	**25**	**16**
Bonnie	**9**	**2**
Bruce	**15**	**6**
Carl	**8**	**3**
Christopher	**4**	**1**
Elaine	**0**	**0**
Nanette	**0**	**0**
Nora	**14**	**7**
Owen	**19**	**5**
Ursula	**0**	**0**
Instructor	83	49
Other 5 teachers in course (M.Ed. students)	34	16
Other talk (e.g., laughter, group responses, can't tell who spoke)	36	23
Total number of turns	247	128

Note. Bold indicates teachers in the treatment group.

The analysis of the two class discussions shows that the idea that not all linear relationships are proportional was made public by many teachers during two class discussions. In addition, this idea appeared to be an important aspect of the discussions – approximately 50% of the total turns involved discussion of the idea that not all linear relationships are proportional.

4.3.4. The Mathematical Relationships Embedded in Proportional Situations

As depicted by the pink shading in Figure 23, teachers had opportunities to discuss the mathematical relationships embedded in proportional situations during three class discussions – the discussion of the similarities and differences among the snake, tent/scout, and age problems (Class 2), the discussion of the graphing orange juice mixes problem (Class 8), and the discussion of the park and zoo problems (Class 11). Cramer et al. (1993) and Post et al. (1988) identify four relationships that are embedded in proportional situations: (1) proportional relationships are multiplicative in nature; (2) proportional relationships are depicted graphically by a line that contains the origin; (3) the rate pairs are equivalent in proportional relationships; and (4) proportional relationships can be represented symbolically by the equation $y = mx$, where the m is the slope, unit rate, and constant of proportionality.

Table 55 shows the key understandings that emerged during each of the three class discussions, and across these three class discussions. As shown in the table, the class discussions ranged in length from 21-28 minutes. In each class discussion, between 39% and 63% of the total turns were related to the four key understandings. On average, nearly half (47%) of the total turns were related to the four key understandings. Each of the four key understandings was discussed during at least two class discussions. The idea that proportional relationships are depicted by lines that contain the origin (key understanding 2) was discussed with the greatest frequency during these three discussions (19%). The idea that the rate pairs of proportional relationships are equivalent was discussed the least (2%).

Table 55.

Turns Related to the Four Key Understandings During the Three Class Discussions

				Number of turns related to the four key understandings			
Whole class discussion	Length of whole class discussion (rounded to nearest minute)	Total number of turns	Percent of turns related to the four key understandings [a]	1. Multiplicative	2. Line through origin	3. Rate pairs equivalent	4. *m* is slope, unit rate
Class 2: Similarities and differences among the snake, scout/tent, and age problems	28 min	241	39%	36	31	0	26
Class 8: Graphing orange juice mixes problem	21 min	248	40%	1	89	2	2
Class 11: Park and zoo problems	22 min	247	63%	54	22	15	65
Across all three discussions	71 min	736	47%	91	142	17	93
Percent of turns related to each key understanding				**12%**	**19%**	**2%**	**13%**

Note. Bold indicates teachers in the treatment group.

[a] Note that some turns that were related to the four key understandings were also related to the idea that not all linear relationships are proportional and/or classifying relationships as proportional or not (as shown in Tables 49 and 52).

It is also interesting to note that eight of the ten teachers made contributions related to one or more key understanding during at least one of the three class discussions, as shown in Tables 56, 57, and 58. During Class 2 (see Table 56), seven of the ten teachers made contributions related to one or more key understanding. Two of these teachers (Bonnie and Christopher) made contributions related to two key understandings. During Class 8 (see Table 57), four teachers made contributions related to one or more key understanding. Of these four teachers, Bert and Owen made contributions related to two or more key understandings. During Class 11 (see Table 58), seven teachers made contributions related to one or more key understanding. In particular, Bert, Bruce, and Nora made contributions related to at least two key understandings.

Across all three class discussions, four teachers (Bert, Christopher, Nora, and Owen) made contributions related to at least three key understandings, as shown in Tables 56, 57, and 58. In addition, three teachers (Bonnie, Bruce, and Carl) made contributions related to at least two key understandings during the three discussions. One teacher, Elaine, made contributions related to one key understanding during one of the three class discussions. Finally, two teachers (Nanette and Ursula) made no contributions related to the four key understandings during any of the three discussions[32].

[32] Ursula did make one contribution, but it was unrelated to the key understandings. Nanette was silent during the three discussions.

Table 56.

Analysis of the Discussion of the Similarities and Differences Among the Snake, Tent/Scout, and Age Problems During Class 2 With Respect to the Four Key Understandings

		Turns related to the four key understandings			
	Number of turns	1. Multiplicative	2. Line through origin	3. Rate pairs equivalent	4. *m* is slope, unit rate
Bert	**32**	**6**	**0**	**0**	**0**
Bonnie	**19**	**0**	**2**	**0**	**4**
Bruce	**2**	**0**	**0**	**0**	**0**
Carl	**10**	**0**	**0**	**0**	**7**
Christopher	**12**	**0**	**4**	**0**	**2**
Elaine	**5**	**7**	**0**	**0**	**0**
Nanette	**0**	**0**	**0**	**0**	**0**
Nora	**1**	**0**	**1**	**0**	**0**
Owen	**10**	**0**	**2**	**0**	**0**
Ursula	**1**	**0**	**0**	**0**	**0**
Instructor	88	13	12	0	7
Other 5 teachers in the course	40	6	8	0	5
Other talk (e.g., laughter, group responses, can't tell who spoke)	17	4	2	0	1
Total number of turns	241	36	31	0	26

Note. Bold indicates teachers in the treatment group.

Table 57.

Analysis of the Discussion of the Graphing Orange Juice Mixes Problem During Class 8 With Respect to the Four Key Understandings

		Turns related to the four key understandings			
	Number of turns	1. Multiplicative	2. Line through origin	3. Rate pairs equivalent	4. *m* is slope, unit rate
Bert	**11**	**1**	**4**	**0**	**0**
Bonnie	**-**	**-**	**-**	**-**	**-**
Bruce	**20**	**0**	**7**	**0**	**0**
Carl	**10**	**0**	**2**	**0**	**0**
Christopher	**2**	**0**	**0**	**0**	**0**
Elaine	**0**	**0**	**0**	**0**	**0**
Nanette	**0**	**0**	**0**	**0**	**0**
Nora	**10**	**0**	**0**	**0**	**0**
Owen	**12**	**0**	**2**	**1**	**1**
Ursula	**0**	**0**	**0**	**0**	**0**
Instructor	100	0	33	1	1
Other 5 teachers in the course	47	0	30	0	0
Other talk (e.g., laughter, group responses, can't tell who spoke)	36	0	8	0	0
Total number of turns	248	1	89	2	2

Note. Bold indicates teachers in the treatment group.
Shading indicates teachers who were not present for class.

Table 58.

Analysis of the Discussion of the Park and Zoo Problems During Class 11 With Respect to the Four Key Understandings

		Turns related to the four key understandings			
	Number of turns	1. Multiplicative	2. Line through origin	3. Rate pairs equivalent	4. *m* is slope, unit rate
Bert	**25**	**4**	**2**	**2**	**2**
Bonnie	**9**	**0**	**0**	**0**	**3**
Bruce	**15**	**6**	**1**	**0**	**0**
Carl	**8**	**0**	**0**	**0**	**7**
Christopher	**4**	**4**	**0**	**0**	**0**
Elaine	**0**	**0**	**0**	**0**	**0**
Nanette	**0**	**0**	**0**	**0**	**0**
Nora	**14**	**2**	**1**	**0**	**11**
Owen	**19**	**0**	**0**	**0**	**7**
Ursula	**0**	**0**	**0**	**0**	**0**
Instructor	83	20	7	7	22
Other 5 teachers in the course	34	13	8	5	8
Other talk (e.g., laughter, group responses, can't tell who spoke)	36	5	3	1	5
Total number of turns	247	54	22	15	65

Note. Bold indicates teachers in the treatment group.

Finally, in addition to the whole class discussions, the four key understandings were made public in a different manner during teachers' work on the park and zoo problems. Prior to the whole class discussion of these problems, teachers explored the park and zoo problems individually for homework. The instructor then compiled teachers' responses into a handout (shown in Figure 24) that was distributed to the teachers prior to the whole class discussion.

During the whole class discussion, teachers were asked whether they agreed with all of the arguments for and against proportionality on the handout. The arguments on the handout were subsequently coded so as to indicate their relationship to the four key understandings. As shown in Figure 24, each of the key understandings was also made public via the handout.

The analysis of the three class discussions shows that the four key understandings of proportional relationships were made public during multiple discussions. In addition, the key understandings were discussed frequently during these class discussions – nearly half the total turns in the three class discussions involved discussion of the key understandings. Thus, the teachers who were present for these class discussions[33] had opportunities to grapple with the four key understandings.

4.3.5. Summary

As described in this section, teachers had opportunities throughout the course to grapple with mathematics that is at the heart of proportional reasoning. Although some teachers were more vocal than others during the course, all teachers had opportunities to share their thinking publicly, and all ten teachers in the treatment group made multiple public contributions during the course and shared at least one solution strategy. In addition, the analysis presented in this section illustrated that the mathematics that teachers appeared to learn (as identified by research questions 1 and 2) was made public during the course across multiple discussions and by multiple teachers in the course. In the next section, the work of the teachers in the treatment

[33] Bonnie was not present for the discussion of the graphing orange juice mixes problem (Class 8). The idea that was discussed with the most frequency during this discussion was that proportional relationships are depicted by lines that contain the origin (key understanding 2). This understanding was made public during two other discussions, so Bonnie still had opportunities to consider this idea.

group during a subsequent course that focused on algebra as the study of patterns and functions is described.

State Park and Zoo Admission Problems: Arguments For/Against Proportionality

Zoo Admission **is** proportional because…	State Park Admission **is not** proportional because…
If you choose any point on the graph, for example $35/7 people, it is equal to the ratio $5/1 person (Key understanding 3)	If you choose two points on the graph, for example (1,4) and (7,10), they will not have the same ratio ($4/1 person ≠ $10/7 people) (Key understanding 3)
The equation, y = 5x, can be written as y/x = 5 which represents a constant of proportionality (Key understanding 4)	The equation y = x + 3 cannot be written in that way (Key understanding 4)
The cost is the same amount for each person no matter how many people	The cost is less per person as the number of people increase
There is a constant scale factor for the table (Key understanding 1)	The scale factor changes (decreases) throughout the table (Key understanding 1)
It goes through the origin; both quantities are 0 at the same time (Key understanding 2)	It does not go through the origin; when the number of people is 0 the cost is $3 (Key understanding 2)

Figure 24. Handout the instructor created based on teachers' responses to the park and zoo problems coded for the four key understandings.

4.4. How Teachers Drew Upon Their Understandings About Proportional Reasoning During a Subsequent Course

In this section, the fifth research question, To what extent do preservice secondary mathematics teachers who participated in a course specifically focused on proportional reasoning draw upon their understandings about proportional reasoning in a subsequent course?, is explored. The purpose of this research question was to examine whether and how teachers drew upon their understandings about proportional reasoning during participation in a mathematics methods course that focused on algebra as the study of patterns and functions. As described in Chapter Three, the data sources used to explore this research question included teachers' work on a pre/posttest item and transcripts from two whole class discussions in which teachers spontaneously introduced proportionality.

4.4.1. Pre/Posttest, Question 2

As noted in Chapter Three, teachers were presented with three relationships and asked to characterize them as proportional or nonproportional. Their work on this question was coded so as to indicate whether they made the correct classification. Analysis of the pre- and posttest data produced by the ten teachers in the treatment group indicated that they were all able to correctly classify the relationships as proportional or not at the beginning and end of the algebra course.

4.4.2. Whole Class Discussions, Class 7 and 8

As noted in Chapter Three, the four key understandings described in Cramer et al. (1993) and Post et al. (1988) provided a lens for analyzing the portions of the two class discussions related to proportionality: (1) the relationship is multiplicative not additive; (2) the graph goes through the origin, (3) the rate pairs are equivalent; and (4) the *m* in y = *m*x is the slope of the line, the constant of proportionality, and the constant factor that relates the quantities. In particular, the nature of the argument or justification made public by the teachers was identified.

4.4.2.1. Class 7

The mathematical idea of proportionality was spontaneously introduced during a whole class discussion around a task in which teachers were asked to think about real world situations that defined a functional relationship. In particular, teachers were asked to work with their small groups to: (1) sketch a graph of their relationship; (2) state the relationship using the language of functions; (3) build a chart or table with numbers that might go with their relationship; (4) explain the graph and the chart as ways of presenting the same information in different forms; and (5) explain how their example meets the requirements of the formal definition of a function. Teachers worked on this task individually for a few minutes and then in their small groups for approximately 10 minutes. The instructor then called the small groups together and engaged the teachers in a whole group discussion.

The portion of the Class 7 discussion related to proportionality consisted of 178 turns and lasted approximately 15 minutes[34]. The teachers in the course spoke about 58% of these turns; the instructor spoke the remaining 42% of the turns. A total of 9 of the 18 teachers present during the discussion[35] actively participated making at least one public contribution. Seven of the nine teachers who actively participated had previously completed the proportional reasoning course (either during the previous semester or a year prior to the algebra course). Four of these teachers - Bert, Bruce, Carl, and Ursula - were in the treatment group.

[34] The entire whole class discussion consisted of 193 turns and lasted approximately 18 minutes. For the purposes of this analysis, only the portion of the discussion related to proportionality was used.
[35] Bonnie, Nora, and Owen were not present.

How proportionality was introduced in this class discussion. Proportionality was introduced during the whole class discussion when the instructor asked for volunteers to share their examples of real world functions with the class. The first group to volunteer, consisting of Bruce, Ursula, and Lisa[36], nominated Ursula to go to the front of the room and present their example. The example that the group created was a car wash context, in which the relationship between the number of cars washed and the money that was earned ($5 per car) defined a functional relationship. In Ursula's first statement to the class, she introduced the idea of proportionality:

> All right, so what we did was, we came up with washing cars as our example. So if you were having a car wash, the number of cars you washed and then the money earned from that. So part a is the graph, and so we just graphed hypothetically what it would look like. And our graph is also proportional. (Class 7, turn 271)

After Ursula's statement, the instructor commented, "Oh! Oh, interesting!"[37] and many teachers in the room began laughing and speaking to one another. However, at this point, the instructor did not press Ursula for clarification or explanation of what she meant by "proportional" and how she knew this. Instead, the instructor let Ursula continue her presentation, which included a description of the equation and a numeric table that depicted the relationship, and an explanation of why their relationship was a function. When she had finished, the instructor provided the teachers with an opportunity to question Ursula's group or make comments about their example, which was a standard practice in this course. At this point, the

[36] Ursula and Bruce were in the treatment group. Lisa was pursuing a certification in deaf education and had not taken the proportional reasoning course.

[37] A standard practice of the course was for the instructor to circulate among the small groups as they worked. However, the instructor spent no time with Bruce, Ursula, and Lisa's group during the small group work during this activity. Thus, the instructor did not appear to know that proportionality would come up during this discussion.

instructor returned to Ursula's statement that her group's example reflected a proportional situation:

> I want to go back to a comment that Ursula made, uh, initially. About it being proportional...Some of you might have thought I'd go back there…Um, do you want to tell us- say a little more about that? (Class 7, turns 288-292)

<u>Key understandings that were made public during the discussion</u>. Each of the four key understandings was made public as the class responded to the instructor's request. Over 60% of the 178 turns in this discussion were related to the four key understandings. Of the turns related to the key understandings, teacher turns accounted for the majority of the turns spoken (teachers spoke 58% of the turns; the instructor spoke 42% of the turns).

Table 59 illustrates the frequency with which the four key understandings were made public during the discussion. The understanding that graphs of proportional situations are linear and contain the origin (key understanding 2) was discussed with the greatest frequency – about one-third of the discussion. The idea that the rate pairs are equivalent in proportional situations (key understanding 3) comprised about 25% of the discussion. The understanding that *m* in the equation y = *m*x is the slope of the line, the constant of proportionality, and the unit rate (key understanding 4) comprised 10% of the discussion. The understanding that proportional relationships are multiplicative in nature (key understanding 1) was discussed the least, comprising 3% of the discussion[38].

[38] Since some turns were coded as relating to more than one key understanding, the percent in the previous paragraph, 60%, does not equal the total percent in this paragraph (34+25+10+3=72%). This is because about 10% of the turns were coded as relating to two key understandings.

Table 59.

Key Understandings That Were Made Public During Class 7 of the Algebra Course

		Turns related to the four key understandings			
	Number of turns	1. Multiplicative	2. Line through origin	3. Rate pairs equivalent	4. *m* is slope, unit rate
Bert	**18**	**1**	**7**	**7**	**3**
Bonnie	**-**	**-**	**-**	**-**	**-**
Bruce	**10**	**2**	**0**	**2**	**5**
Carl	**7**	**0**	**2**	**5**	**1**
Christopher	**0**	**0**	**0**	**0**	**0**
Elaine	**0**	**0**	**0**	**0**	**0**
Nanette	**0**	**0**	**0**	**0**	**0**
Nora	**-**	**-**	**-**	**-**	**-**
Owen	**-**	**-**	**-**	**-**	**-**
Ursula	**8**	**0**	**3**	**0**	**0**
Instructor	75	2	27	16	8
Other teachers (who completed proportional reasoning course)	29	0	11	10	0
Other teachers (who had NOT completed proportional reasoning course)	5	0	1	3	0
Other talk (e.g., laughter, group responses, can't tell who spoke)	26	0	10	1	0
Total number of turns	178	5 (3% of the total turns)	61 (34% of the total turns)	44 (25% of the total turns)	17 (10% of the total turns)

Notes: Bold indicates teachers in the treatment group. Shading indicates teachers who were not present for the discussion. Thirteen turns were related to 2 key understandings, and were coded as such.

Each key understanding was discussed by at least two different teachers in the treatment group (as shown in the columns of Table 59). Three teachers (Bert, Bruce, and Carl) made contributions related to at least three of the key understandings. This suggests that their understanding about the nature of proportional relationships was not limited to just having memorized a particular characteristic (e.g., rate pairs are equivalent in proportional relationships), but was more robust. Ursula's contributions were all related to the idea that proportional relationships are depicted by a line that contains the origin.

4.4.2.2. Class 8

Proportionality was spontaneously introduced again during a whole class discussion about three linear relationships situated in a meal plan context. In particular, teachers were discussing why the graph of the actual cost per meal for the Regular Plan was a horizontal line (see Figure 16 in Chapter Three). The portion of the Class 8 discussion related to proportionality consisted of 68 turns and lasted approximately 7 minutes[39]. The teachers in the course spoke about 59% of these turns; the instructor spoke the remaining 41% of the turns. A total of 8 of the 21 teachers enrolled in the course actively participated making at least one public contribution. Seven of the eight teachers who actively participated had previously completed the proportional reasoning course (either during the previous semester or a year prior to the algebra course). Six of these teachers - Bert, Bruce, Carl, Christopher, Nora and Ursula - were in the treatment group.

<u>Key understandings that were made public during the class discussion</u>. Each of the four key understandings was made public during the portion of the Class 8 discussion that was analyzed. Over 85% of the 68 turns that made up the whole class discussion were related to the four key understandings. Of the turns related to the key understandings, teacher turns accounted

[39] The entire whole class discussion consisted of 91 turns and lasted approximately 9 minutes. As with the Class 7 discussion, only the portion of the Class 8 discussion related to proportionality was analyzed.

for the majority of the turns spoken (teachers spoke 58% of the turns; the instructor spoke 42% of the turns).

Table 60 illustrates the frequency with which the four key understandings were made public during the discussion. The understandings that graphs of proportional situations are linear and contain the origin (key understanding 2) and that rate pairs are equivalent in proportional situations (key understanding 3) were discussed with the greatest frequency – about 30% of the discussion. The understanding that proportional relationships are multiplicative in nature (key understanding 1) comprised 22% of the discussion. The understanding that m in the equation $y = mx$ is the slope of the line, the constant of proportionality, and the unit rate (key understanding 4) was discussed the least, comprising about 15% of the discussion.

As shown in Table 60, of the six teachers in the treatment group who actively participated in the Class 8 discussion, five of them (Bert, Bruce, Carl, Christopher, and Nora) made contributions related to at least one of the four key understandings. Bert made contributions related to three of the four key understandings, and Bruce made contributions related to two of the key understandings. Carl, Christopher, and Nora's contributions were each related to one key understanding.

Table 60.

Key Understandings That Were Made Public During Class 8 of the Algebra Course

		Turns related to the four key understandings			
	Number of turns	1. Multiplicative	2. Line through origin	3. Rate pairs equivalent	4. *m* is slope, unit rate
Bert	**5**	**3**	**3**	**0**	**2**
Bonnie	**0**	**0**	**0**	**0**	**0**
Bruce	**7**	**0**	**2**	**0**	**3**
Carl	**1**	**0**	**0**	**0**	**1**
Christopher	**2**	**0**	**0**	**1**	**0**
Elaine	**0**	**0**	**0**	**0**	**0**
Nanette	**0**	**0**	**0**	**0**	**0**
Nora	**7**	**0**	**7**	**0**	**0**
Owen	**0**	**0**	**0**	**0**	**0**
Ursula	**1**	**0**	**0**	**0**	**0**
Instructor	29	4	14	10	4
Other teachers (who had completed proportional reasoning course)	8	1	2	3	1
Other teachers (who had NOT completed proportional reasoning course)	7	0	0	7	0
Other talk (e.g., laughter, group responses, can't tell who spoke)	3	0	1	2	0
Total number of turns	68	15 (22% of the total turns)	22 (32% of the total turns)	21 (31% of the total turns)	11 (16% of the total turns)

Note: Six turns were coded as relating to two key understandings. In addition, two turns were coded as relating to three key understandings. Bold indicates teachers in the treatment group.

4.4.3. Summary

Approximately two weeks after they completed the proportional reasoning course, the ten teachers in the treatment group began a six-week course that focused on algebra as the study of patterns and functions in the middle grades taught by the same instructor. At the beginning and end of the algebra course, all ten teachers in the treatment group were able to use an aspect of their newly-acquired common content knowledge to classify relationships as proportional or nonproportional. In addition, the teachers spontaneously introduced the idea of proportionality during their work in the algebra course, and drew upon their understandings of the nature of proportional relationships during two class discussions. In particular, over 6 of the 10 teachers (Bert, Bruce, Carl, Christopher, Nora, and Ursula) made public contributions that were related to the four key understandings of proportional relationships during these discussions.

4.5. Summary of the Results of the Study

The analysis presented in this chapter revealed that both the ten teachers in the treatment group and the six teachers in the contrast group knew and were able to do mathematics related to proportional reasoning prior to the course, and that there was no significant difference between the two groups at the onset of the study. For example, both groups had well-developed common content knowledge, as evidenced by their ability to correctly solve missing value, numerical comparison, and qualitative problems. In addition, both groups recognized ratio as an appropriate measure of attributes such as steepness, and many teachers in each group were able to discriminate proportional from nonproportional situations.

However, a closer look at teachers' work on the pretest reveals that they relied heavily on the cross multiplication procedure to solve missing value problems. By the end of the course, half the teachers in the treatment group used strategies that they had not used at the beginning of the course to solve problems, and the teachers in the treatment group tended to rely less on cross multiplication and other algebraic strategies. By contrast, teachers in the contrast group utilized the same strategies they used on the pretest, and continued to rely on cross multiplication and algebraic strategies. These results suggest that teachers in the treatment group learned additional strategies for solving proportionality problems during the course, and thus enhanced an aspect of their specialized content knowledge. This claim is further strengthened by the analysis of the class discussions that occurred during the course, which illustrated that teachers in the course shared a variety of solution strategies, and therefore had opportunities to grapple with new strategies.

In addition, teachers in the treatment group drew upon significantly more key understandings of proportional relationships by the end of the course than teachers in the contrast group did in their work on tasks in which they classified relationships as proportional or not, defined proportional relationships, and created examples and nonexamples of proportional relationships. These key understandings were made public during several discussions during the course.

Finally, the teachers in the treatment group were still able to correctly classify relationships as proportional or not eight weeks after the end of the proportional reasoning course. In addition, in their work during a subsequent course focused on algebra, these teachers spontaneously (and appropriately) drew upon their understandings of proportional relationships and engaged in discussions that illustrated their understandings about the difference between proportional and nonproportional relationships and the mathematical relationships embedded in proportional situations. In the next chapter, the results of the study are summarized and discussed. In addition, implications of the study and suggestions for future study are discussed.

5. CHAPTER FIVE: DISCUSSION

5.1. Introduction

This study investigated teachers' understandings of proportional reasoning, a content area that is known to be difficult for both students and teachers (e.g., Heinz, 2000; Martin & Strutchens, 2000; Post, et al., 1991; Wearne & Kouba, 2000). The results of this study show that while teachers were able to solve a variety of proportionality problems prior to and upon completion of a course focused on proportional reasoning, they increased their *understanding* of proportionality as a result of their participation in the course.

Specifically, a closer look at teachers' work on the pre-instruments revealed that they did not have a particularly broad repertoire of solution strategies prior to the course. For example, teachers relied heavily on the cross multiplication procedure and other algebraic strategies to solve missing value problems. In addition, half of the ten teachers in the proportional reasoning course favored the use of part-to-part ratios to solve numerical comparison problems, and several of these teachers could not make sense of strategies that made use of an alternative type of ratio, part-to-whole ratios. By contrast, by the end of the course, teachers utilized a broader range of strategies for solving missing value and numerical comparison problems. In particular, teachers relied less heavily on the cross multiplication procedure and other algebraic strategies and instead favored within- and between-ratio strategies, which highlight the multiplicative relationship between quantities. Most teachers also used strategies based on both part-to-part and part-to-whole ratios by the end of the course.

In addition, teachers' work on the pretest revealed that they had limited ability to distinguish between proportional and nonproportional relationships, and that half the teachers appeared to believe that all linear relationships are proportional. Of the teachers who could correctly identify relationships as proportional or not, their explanations were typically limited to one feature of proportional relationships (key understanding 3, that rate pairs of proportional relationships are equivalent, was used most frequently prior to the course). By the end of the course, teachers correctly identified significantly more relationships as proportional or not, and no teacher appeared to believe that all linear relationships are proportional. In addition, teachers appeared to have a more robust understanding of the nature of proportional relationships, as evidenced by their work in describing proportional relationships, providing examples of proportional and nonproportional relationships, and classifying relationships as proportional or nonproportional, which drew upon significantly more key understandings of proportional relationships than their work prior to the course.

The results of the study also make salient that the proportional reasoning course in which teachers participated was a key factor in enhancing teachers' understandings of proportionality. For example, all of the mathematics that teachers appeared to learn during the course (as indicated by their work on the pre/post instruments) was made public during multiple classes and by multiple teachers in the course. In addition, the results of the contrast group's work on the pre/post instruments indicated that prior to the course, there was no significant difference between the understandings of the teachers enrolled in the course and those who were not. However, by the end of the course, teachers in the course used additional strategies to solve problems while the contrast group did not. In addition, teachers in the course had a deeper understanding of proportional relationships, as evidenced by their use of significantly more key understandings than the contrast group in describing proportional relationships, providing examples of proportional and nonproportional relationships, and classifying relationships as proportional or nonproportional. Thus, teachers' participation in the course appeared to be an important catalyst for the learning that appeared to occur during the course.

Finally, the study also provides evidence that the teachers who participated in the proportional reasoning course were able to draw upon their understandings about proportionality in their work during a subsequent practice-based methods course, which focused on algebra as the study of patterns and functions[40]. In particular, results indicate that all ten teachers who participated in the proportional reasoning course were able to distinguish between proportional and nonproportional relationships both at the beginning and end of the algebra course (i.e., approximately two and eight weeks after the conclusion of the proportional reasoning course). In addition, over half of the ten teachers spontaneously drew upon their understandings of proportional relationships during two whole-class discussions in the algebra course in which teachers were exploring linear functions. These results suggest that teachers had not merely memorized discrete facts about proportional relationships (e.g., proportional relationships are depicted by lines that contain the origin), but had developed flexible understandings that allowed them to access their knowledge as they explored different (albeit mathematically related) ideas such as function.

5.2. Teacher Learning From the Course: Mathematical Knowledge for Teaching

So how does the mathematics that teachers learned during the proportional reasoning course contribute to their knowledge base for teaching? As noted in Chapter One, teachers'

[40] Recall that the six-week algebra course began two weeks after the conclusion of the proportional reasoning course.

knowledge of mathematics is an important factor in their ability to help their students learn mathematics (Ball, Lubienski, & Mewborn, 2001; Fennema & Franke, 1992). For example, teachers draw upon their own mathematical understandings as they identify mathematical goals for their students, select and sequence mathematical tasks, and ask questions that assess or advance their students' understandings (Hiebert et al., 1997; NRC, 2001).

As noted in Chapter Four, one way to characterize the knowledge that teachers draw upon in their work is as *mathematical knowledge for teaching* (Ball et al., 2005; Ball, Bass, & Hill, 2004; Hill & Ball, 2004; Hill, Schilling, & Ball, 2004). Ball and her colleagues argue that teachers need two important types of content knowledge in order to successfully teach mathematics: *common content knowledge* and *specialized content knowledge*. Common content knowledge is the "mathematical knowledge and skill expected of any well-educated adult" (Ball et al., 2005, p. 13). For example, one would draw upon their common content knowledge in order to solve a missing value problem. By contrast, specialized content knowledge is the "mathematical knowledge and skill needed by teachers in their work and beyond that expected of any well-educated adult" (Ball et al., 2005, p. 14). For example, teachers draw on their specialized content knowledge when they represent mathematical ideas in multiple ways, analyze errors, and evaluate alternative ideas (Ball et al., 2005; Hill, Schilling, & Ball, 2004).

In addition, Ball and her colleagues (Ball et al., 2005; Ball, Bass, & Hill, 2004; Hill & Ball, 2004; Hill, Schilling, & Ball, 2004) argue that teachers also need *knowledge of content and students* and *knowledge of content and teaching*. Knowledge of content and students combines teachers' content knowledge with their knowledge of students. For example, teachers draw upon their knowledge of content and students when they anticipate students' solutions, errors, and common misconceptions. Knowledge of content and teaching combines teachers' content knowledge with their knowledge of teaching. For example, teachers use their knowledge of content and teaching as they sequence mathematical tasks. These two types of knowledge make up what Shulman (1986) has called *pedagogical content knowledge* (Ball et al., 2005). In reviewing the results of this study through a lens of mathematical knowledge for teaching, teachers appeared to develop the two types of content knowledge described by Ball and colleagues as a result of participating in the course: common content knowledge and specialized content knowledge.

5.2.1. Common Content Knowledge

The results of this study indicate that the preservice secondary teachers that were the focus of this study had considerable common content knowledge, even prior to the course. In general, these teachers did not make use of incorrect additive strategies to solve proportionality problems and were able to solve a variety of problem types on the pretest. By contrast, previous studies of preservice elementary teachers suggest that these teachers frequently apply additive strategies to problems that call for proportional strategies (Heinz, 2000; Simon & Blume, 1994). The findings of this study suggest that the preservice secondary teachers in this study, who hold a bachelor’s degree in mathematics (or the equivalent), have stronger common content knowledge than the preservice elementary teachers in other studies. This is not surprising, since preservice secondary mathematics teachers are typically successful in K-12 mathematics and complete numerous mathematics courses as undergraduates. By contrast, preservice elementary teachers often have minimal formal coursework in mathematics beyond their own K-12 experience.

However, the findings of this study also indicate even that though teachers had a bachelor's degree in mathematics (or the equivalent), they still appeared to learn some aspects of both common and specialized content knowledge during the course. For example, teachers in the course had limited ability to classify relationships as proportional or not prior to the course. In addition, they held a limited understanding of what it means for a relationship to be proportional. These mathematical ideas can be considered common content knowledge, since even students need to understand these ideas (NCTM, 2000). By the end of the course, teachers' understandings about proportional relationships were more robust and flexible, as shown in their capacity to classify relationships as proportional or not and explain why by drawing upon the nature of proportional relationships. The finding that teachers developed common content knowledge that their students will also need during the course is particularly important in light of Hill, Rowan, and Ball's (2005) recent findings that teachers' mathematical knowledge for teaching impacts their students' learning.

5.2.2. Specialized Content Knowledge

The results of this study also indicate that prior to the course, teachers' specialized content knowledge was fairly limited. For example, prior to the course, teachers tended to rely on solution strategies such as cross multiplication. Teachers also had difficulty making sense of solution strategies that they had not produced themselves. As shown in the analysis of the video of the course, teachers made sense of and analyzed others' strategies (i.e., strategies produced by other teachers in the course, and strategies produced by students in the middle grades) on multiple occasions throughout the course. As noted previously, the analysis of teachers' work at the end of the course indicates that teachers drew upon additional strategies and relied less on cross multiplication. The finding that teachers in the course learned additional strategies and how to make sense of strategies is particularly important because as teachers, making sense of students' strategies and ideas is part of their everyday work. In order to teach for understanding, teachers need to be able to assess what their students know, and ask questions to advance their understanding.

The results of the analysis of the whole class discussions also indicate that teachers had additional opportunities to develop their specialized content knowledge. For example, teachers need to understand why mathematical procedures such as cross multiplication work if they are to help their students develop meaning for such procedures. During the course, teachers had at least two opportunities to make sense of cross multiplication, through reading an article that highlights the relationship between a factor-of-change (i.e., scale factor or between-ratio) strategy and cross multiplication (Boston, Smith, & Hillen, 2003), and through a discussion in which teachers sought to explain why cross multiplication works.

An additional aspect of specialized content knowledge involves coordinating multiple representations of mathematical relationships. Teachers had multiple opportunities to make connections among different representations during the course, as evidenced in the analysis of the whole class discussions. For example, when teachers worked on the graphing orange juice mixes problem, they had opportunities to make connections among the context of the problem and tables and graphs that they were asked to use in solving the problem.

5.3. Implications and Recommendations for Education

5.3.1. Sites for Developing and Measuring Mathematical Knowledge for Teaching

Hill, Schilling, & Ball (2004) suggest that teachers could develop mathematical knowledge for teaching through a variety of means, including teacher preparation. The proportional reasoning course that was the treatment in this study provides one model of how mathematical knowledge for teaching could be operationalized in a learning experience for preservice teachers. The course also provides a concrete exemplar of the types of practice-based learning opportunities that Ball and Cohen (1999) argue that teachers need. In addition, this study and other related studies (e.g., Smith et al., 2003; Steele, Hillen, Engle, Smith, Leinhardt, & Greeno, 2005) are beginning to provide information about course design and teacher learning. The results of this study suggest that teachers can further develop aspects of their mathematical knowledge for teaching (specifically, their common and specialized content knowledge) by completing practice-based mathematics methods courses in which teachers explore challenging mathematical tasks from curricula, examine students' responses to such tasks, and analyze narrative and video cases of teaching. The results of this study also point to the importance of providing teachers with opportunities to revisit mathematical content that is critical to K-12 mathematics so that they can develop or refine their own understandings. By providing a mathematical focus for the course, teachers had an opportunity to not only explore proportional reasoning ideas in a deep way, but also grapple with ideas about teaching mathematics and student learning more broadly.

It is also important to note that the mathematics that teachers learned during the course was discussed during multiple classes and in various contexts over a sixteen-week semester. In addition, the instructor never "told" the teachers anything; but rather, provided them with opportunities to explore ideas central to proportional reasoning and construct their own understandings through participation in a community of practice. The nature of the course and the finding that teachers learned important aspects of proportional reasoning during the course and could use their newly-acquired knowledge appropriately up to eight weeks later suggests that having multiple and varied opportunities to engage with specific mathematics content may be critical in promoting change in teacher knowledge.

This study examined only one aspect of preservice secondary mathematics teachers' mathematical knowledge for teaching proportional reasoning – their common and specialized content knowledge. Teachers likely had opportunities to develop or refine additional aspects of their mathematical knowledge for teaching during the course that were not examined in this study. For example, the course map (shown in Figure 23) indicates that less than half of the class discussions were related to common or specialized content knowledge. Therefore, a natural extension of this study would be to examine what teachers learned (or had the opportunity to learn) from engaging in the remaining class discussions (see unshaded class discussions in Figure 23). For example, given the opportunities teachers had to examine student thinking during the course (through interviewing a student, examining sets of written responses provided by the instructor, and reading narrative cases depicting middle grades mathematics lessons), teachers may have enhanced their understandings about student errors and misconceptions and their capacity to anticipate how students will approach tasks – that is, their knowledge of content and students (Ball et al., 2005). Future studies might develop and use items that measure teachers' knowledge of content and students with respect to proportional reasoning (see Hill, Schilling, & Ball [2004] or Ball et al. [2005] for examples of similar items that focus on number concepts and operations). Such items might focus on common student misconceptions, such as the use of additive strategies when multiplicative ones are needed.

In addition, items that measure teachers' knowledge of content and teaching with respect to proportional reasoning might be developed in order to further explore teachers' learning during the course. Such items might have the potential to identify the learning that occurs from analyzing and discussing the four narrative cases that were used in the course. For example, one of the cases used in the course, *The Case of Marie Hanson* (Smith, Silver, & Stein, 2005b), makes salient the teacher's thinking as she sequenced tasks and students' responses to those tasks during the lesson (Stein, Engle, Hughes, & Smith, submitted). Analyzing and discussing this case might help teachers think deeply about the relationship between the sequence of content and the mathematical opportunities that are afforded to students, thus developing or refining their knowledge of content and teaching (Ball et al., 2005).

Finally, the items currently being developed by Ball and colleagues' Study of Instructional Improvement (Hill, Schilling, & Ball, 2004) will be helpful in describing mathematical knowledge for teaching. As additional items are developed that focus on particular content areas, such as proportional reasoning, researchers will have opportunities to use items that have proven reliability in conjunction with teacher education courses in order to examine teachers' learning.

5.3.2. The Role of Public Participation in Individual Learning

The results of the study also indicate that even teachers who were mostly "silent" throughout the course appeared to learn the same mathematics as more vocal teachers. For example, Ursula, who was the most silent of the teachers in the course (she spoke the fewest number of turns during the course and spoke in the fewest number of discussions[41]), appeared to learn the same mathematics as other teachers in the course, including those who were more vocal (e.g., Bert). For example, on the posttest, Ursula used two solution strategies that she had not used on the pretest: between-ratio strategies to solve the five missing value problems (see tasks 1-4 and 23 in Appendix A); and a part-to-whole strategy to solve the orange juice numerical comparison problem (see task 6 in Appendix A). Ursula's explanations of the quantities she used to compare the orange juice recipes in task 6 were also more clear and complete on the posttest than on the pretest. In addition, her capacity to classify relationships as proportional or not was fairly limited at the beginning of the course (she correctly classified less than 70% of the relationships in tasks 11-22 [shown in Appendix A]). By contrast, Ursula was able to correctly classify all twelve relationships as proportional or not at the end of the about proportional reasoning course (tasks 11-22 in Appendix A) and at the beginning and end of the subsequent algebra course (see Figure 14). Finally, Ursula drew upon two additional key understandings to provide a rationale for her classifications by the end of the course.

It is also interesting to note that it was Ursula who first spontaneously introduced the idea of proportionality during Class 7 of the algebra course[42], and when pressed by the instructor, drew upon the understanding that proportional relationships are depicted by lines that contain the origin (key understanding 2) to support her group's claim that the function they created was proportional. This is notable because during the proportional reasoning course, Ursula remained

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[41] Nanette and Ursula both spoke in only five of the sixteen discussions that were analyzed. However, Ursula was identified as the "most silent" because Nanette spoke nearly three times as many turns as Ursula during the sixteen discussions.

[42] Recall that Ursula had been working a small group with Bruce and another teacher (who had not taken the proportional reasoning course), and that the group nominated Ursula to present their example to the class.

silent during all three discussions related to the key understandings[43]. Thus, when pressed to explain why a relationship was proportional, Ursula could do so.

The finding that a silent teacher learned the same mathematics as the more vocal teachers in the course echoes Inagaki, Hatano, and Morita's (1998) finding that there was no significant difference between silent and vocal fourth and fifth graders' ability to add fractions of different denominators following a class discussion about this idea. Hatano and Inagaki (1991) argue that group discussions are important because:

> [a] group as a whole usually has a richer data base than any of its members for problem solving. It is likely that no individual member has acquired or has ready access to all needed pieces of information, but every piece is owned by at least one member in the group (p. 341)

Thus, during group discussions, all participants – even silent ones – have the opportunity to "collect more pieces of information about the issue of the discussion and to understand the issue more deeply" (Hatano & Inagaki, 1991, p. 346). Inagaki and colleagues suggest that silent students are in fact actively participating in discussions, even though they are not speaking out (Hatano & Inagaki, 1991; Inagaki et al., 1998). Hatano and Inagaki describe active participation:

> …silent members may be actively participating. They can learn much by observing the ongoing discussion or debate carefully. This is often characterized as a vicarious process, but it is more than that. In a sense, these students are all trying to find an agent, someone who really speaks for them. A good agent or vocal participant can articulate what a silent member has been trying without success to say… (p. 346)

[43] Ursula made one contribution during one of the three discussions related to the four key understandings. However, her contribution was not related to the four key understandings.

Thus, Ursula (and other relatively silent teachers, such as Nanette) most likely were actively engaged during the whole class discussions but nonetheless chose to remain silent during most class discussions.

The finding that a silent teacher learned the same mathematics as the more vocal teachers in the course is also important given the suggested role of public discourse in mathematics classrooms (NCTM, 2000). This finding suggests that successful class discussions (i.e., ones that have the potential to impact learning) are those in which important mathematical ideas are made public. Whether all participants make public contributions during such discussions appears to be less important. Of course, teacher educators who are orchestrating discussions with teachers (and similarly, teachers who are orchestrating discussions with students) should provide opportunities for all teachers to make public contributions. However, the findings of this study indicate that some teachers chose to remain relatively silent during class discussions, yet still learned important mathematics.

Future studies might examine the understandings of the vocal teachers compared to those of the silent teachers. For example, examining Bert's (arguably the most vocal teacher in the course) and Ursula's understandings of proportional reasoning would provide information on whether increased participation (in the form of public contributions) appears to impact their mathematical knowledge for teaching. It would also be interesting to examine the understandings of additional silent teachers. For example, comparing Ursula's understandings to those of Nanette's might provide insight into whether or not all silent teachers experience similar learning outcomes.

5.3.3. How Knowledge Influences Practice

The ultimate goal in teacher education is to improve student learning outcomes. This study provides evidence that through participation in a practice-based methods course focused on proportional reasoning, teachers can and do learn mathematical knowledge for teaching during a course in their teacher education experience. Thus, this study should be seen as a first step in improving student learning. Future studies might investigate how teachers' enhanced knowledge for teaching impacts their instructional practice, and ultimately, their students' learning. Studies that examine teachers' instructional practice could provide empirical evidence on the extent to which practice-based methods courses have an impact on teachers' practice. Further studies might investigate how teachers' mathematical knowledge for teaching influences students' learning. For example, in a recent study, Hill, Rowan, and Ball (2005) found that teachers' mathematical knowledge for teaching positively predicted student gains in mathematics achievement during the first and third grade. They note that the fact that teachers' mathematical knowledge for teaching affects student gains in the first grade "suggests that teachers' content knowledge plays a role even in the teaching of very elementary mathematics content" (p. 399). This suggests that teachers' content knowledge might be even more critical when teaching at the middle grades, where complex mathematical ideas such as proportional reasoning are key components of the curriculum.

APPENDIX A

PRE/POSTTEST

Directions: Solve the following problems. Please show all your work. If you write something that you do not wish to include in your solution, please draw a single line through it (rather than erasing any of your work). You may use a calculator if you need one.

For problems 1-4, solve for *x*. If you can, solve for *x* in two different ways.

1. $\frac{4}{20} = \frac{x}{35}$

2. $\frac{2}{7} = \frac{6}{x}$

3. $\frac{3}{8} = \frac{x}{20}$

4. $\frac{9}{15} = \frac{12}{x}$

5. Write a word problem that would require setting up and solving $\frac{3}{8} = \frac{x}{20}$

STOP. WHEN YOU ARE FINISHED WITH PROBLEMS 1-5, PLEASE HAND THEM IN AND PICK UP THE REMAINING PROBLEMS.

6. Solve the following problem in two ways:

   Luis mixed 5 ounces of orange juice concentrate with 7 ounces of water to make orange juice. Martin mixed 3 ounces of the same orange juice concentrate with 5 ounces of water. Who made the drink with the stronger orange flavor? Explain how you know.

   Adapted from Wearne, D., & Kouba, V. L. (2000). Rational numbers. In E. A. Silver & P. A. Kenney (Eds.), *Results from the seventh mathematics assessment of the national assessment of educational progress*. Reston, VA: National Council of Teachers of Mathematics. p. 181.

For problems 7-8, answer the questions shown below and explain how you made your selection.

7. Mary ran more laps than Greg. Mary ran for less time than Greg. Who was the faster runner?

   a. Mary
   b. Greg
   c. Same
   d. Not enough information to tell

   Taken from Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom*. New York: Macmillan. p. 166.

8. Devan makes a lemon-lime drink by mixing lemonade and limeade every day for her preschool students. If Devan used less lemonade and less limeade than she did yesterday, would her lemon-lime drink taste:

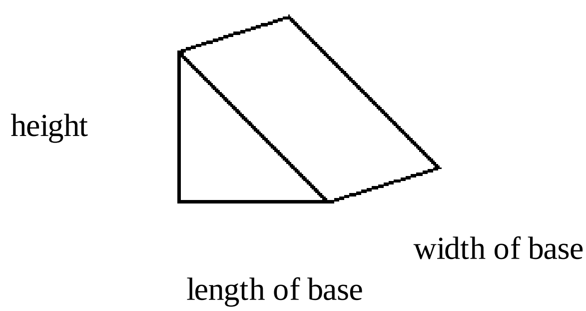
   a. More lemony than yesterday's
   b. More limey than yesterday's
   c. Exactly the same as yesterday's
   d. Less lemony than yesterday's
   e. Less limey than yesterday's
   f. Not enough information to tell

9. Pat was painting his bedroom. He mixed blue and white paint until he came up with a shade of blue that he liked. He realized however that he was probably about one quart short of the amount of the paint that he needed. He wanted to increase the amount of the paint without changing the color, so he added equal amounts of blue paint and white paint: one pint of white and one pint of blue (2 pints = 1 quart).

   Comment on the effectiveness of Pat's strategy for increasing the amount of his paint mixture without changing the color. Justify your statements.

   Taken from Heinz, K. R. (2000). *Conceptions of ratio in a class of preservice and practicing teachers*. Unpublished doctoral dissertation, The Pennsylvania State University, p. 150.

10. In Kansas, there are no mountains for skiing. An enterprising group built a series of ski ramps and covered them with a plastic fiber that permitted downhill skiing. It is your job to rate them in terms of most steep to least steep. You have available to you the following measurements for each hill: the length and width of the base (measured along the ground) and the height. How would you determine the relative steepness of the hills using the information you have?



   Taken from Simon, M. A., & Blume, G. W. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, *13,* p. 187.

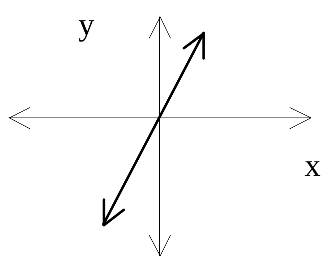
For problems 11-22, indicate (by writing either 'yes' or 'no') whether any of the relationships described below are proportional. Explain how you know.

11. _____ The relationship between the number of kilometers for a customer's taxi ride and the cost of the trip, if the customer pays a $1.00 fee, plus $0.50 per kilometer for the taxi.

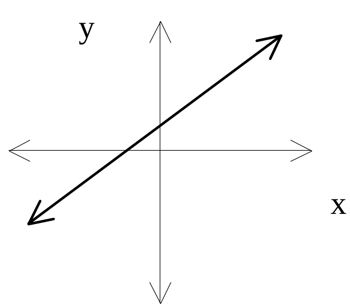
12. ______ The relationship between the number of movie tickets purchased and the total cost of the tickets, if each ticket costs $8.00.

13.______ The relationship between Jane and Sue's positions on a marathon course if they run at the same pace but Jane ran 2 miles before Sue started.

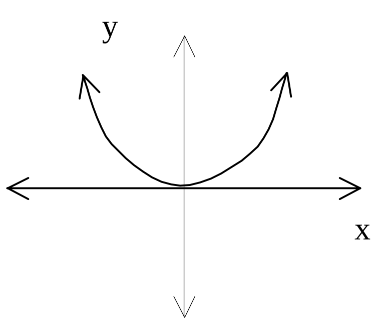
14. ______



15. ______



16. ______



17. ______ $y = 3x + 4.5$

18. ______ $y = 3x^2$

19. ______ $y = 2.5x$

20. ______

x	y
4	6
6	9
8	12
10	15
12	18

21. ______

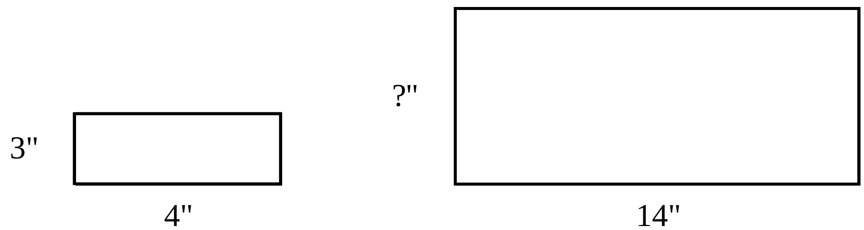
x	y
4	10
6	14
8	18
10	22
12	26

22. ______

x	y
0	0
4	8
6	18
8	32
10	50

Adapted from Smith, M. S., Silver, E. A., Leinhardt, G., & Hillen, A. F. (2003). *Tracing the development of teachers' understanding of proportionality in a practice-based course*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL, p. 54.

23. Katie and Jacob are enlarging pictures for the school yearbook and want to make sure they do not distort any of the images. They have a photograph whose width is 3” and length is 4” and they need to make an enlargement of the photograph whose length is 14”. How long will the width of the enlargement be? Explain how you know.



24. A new housing subdivision offers lots of three different sizes: 75 feet by 114 feet, 455 feet by 508 feet, and 185 feet by 245 feet. If you were to view these lots from above, which would appear most square? Which would be least square? Explain how you know.

Taken from Heinz, K. R. (2000). *Conceptions of ratio in a class of preservice and practicing teachers*. Unpublished doctoral dissertation, The Pennsylvania State University, p. 150.

APPENDIX B

PRE-INTERVIEW PROTOCOL

(Phrases in bold should be read exactly as they appear.)
Thank you for participating in this interview. I have three questions[44] for you today.

1. a. **What does it mean to say that there is a proportional relationship among quantities? If you can, I'd like to first hear your thoughts on what a proportional relationship is without your using specific examples.**

   Probes: **In your own words, what is a proportion? What do we mean when we say two quantities are related proportionally?**

   b. Now, I'd like for you to describe a situation in which there **IS a proportional relationship between the quantities**. If you can, describe a situation that was not used in class.

   Probes: If the teacher cannot think of such a situation, ask them to focus on a problem that we did in class that depicted a proportional situation. Use the class situation to see if the teacher can generalize the characteristics of the situation. For example, 'Ok, now using what you know about the tents/scouts problem, can you think of or make up a situation in which there is a proportional relationship between the quantities?

   c. Now, I'd like for you to describe a situation in which there is **NOT a proportional relationship between the quantities**. Again, describe a situation that was not used in class.

   Probes: If the teacher cannot think of such a situation, ask them to focus on a problem that we did in class that depicted a nonproportional situation (i.e., age problem). Use the class situation to see if the teacher can generalize the characteristics of the situation. For example, 'Ok, now using what you know about the Age problem, can you think of or make up a situation in which there is not a proportional relationship between the quantities?

---

[44] All questions on the pre-interview were adapted from work developed under the auspices of the ASTEROID project (NSF Award # 0101799), principal investigator Margaret S. Smith.

2. I would like to you review a set of student responses to a mathematics problem (shown in Figure B1). Here is a copy of the problem. I'll give you a few minutes to acquaint yourself with the problem.

   This is a set of five students' responses to the problem. Each of the five students picked Carla as having the strongest chocolate-flavored drink.

   For each response, **can you explain how you think each student knew that Carla had the stronger chocolate-flavored drink?**

   If the teacher doesn't explicitly talk about what the values mean – they might instead talk about how the student is using common denominators, or milk compared to syrup, etc., ask the following:

   Probe: **What do these values mean in the context of the problem?**

3. During the second class, you solved a set of three problems: Snake, Tents/Scouts, and Age (shown in Figure B2). (Give the teacher the handout with the three problems.)

   (If the teacher is in the contrast group, say: I'd like you to review this set of three problems. I'll give you some time to acquaint yourself with the problems.)

   a. How are these problems the same and how are they different?
   b. I would now like you to review a set of problems (shown in Figure B3). For each problem, A-H, you need to decide if the problem is most like Problem #1 (Snake), Problem #2 (Tents/Scouts) or Problem #3 (Age), or if it is not like any of these three problems. When you decide, I'd like you to check the appropriate box. Then I'd like you to explain what about the problem (A-H) makes it 'like' either Snake, Tent/Scout, or Age. As you examine each problem, I'd like you to think out loud – I'd like to hear how you're thinking about each one. You can revise your thinking at any time. It is okay to change your mind.

Maria mixed 3 ounces of chocolate syrup with 5 ounces of milk to make chocolate milk. Carla mixed 5 ounces of the same chocolate syrup with 8 ounces of milk. Who made the drink with the stronger chocolate flavor? Explain how you know.

A

Maria: 3/5= .6

Carla: 5/8 = .625

So Carla's drink has a stronger chocolate flavor.

B

Maria: 3/5= 24/40

Carla: 5/8 = 25/40

So Carla's drink has a stronger chocolate flavor.

C

Maria: 5/3 = 1.67

Carla: 8/5 = 1.6

So Carla's drink has a stronger chocolate flavor.

D

Maria: 3/8 = .375

Carla: 5/13 = .385

So Carla's drink has a stronger chocolate flavor.

E

24 25

$\frac{3}{5}$ $\frac{5}{8}$

So Carla's drink has a stronger chocolate flavor.

Figure B 1. The mathematics problem and set of student responses for item 2 on the pre-interview.

The snake problem

Jo has two snakes, String Bean and Slim. Right now, String Bean is 4 feet long and Slim is 5 feet long. Jo knows that two years from now, both snakes will be fully-grown. At her full length, String Bean will be 7 feet long, while Slim's length when he is fully grown will be 8 feet. Over the next two years, will both snakes grow the same amount?

> Taken from Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah: NJ: Erlbaum. p. 12.

The tent/scout problem

Four tents will house 12 scouts. If there are 40 tents, how many scouts will have a place to sleep?

> Taken from Carpenter, T. P., Gomez, C., Rousseau, C., Steinthorsdottir, O., Valentine, C., Wagner, L., & Wyles, P. (1999). *An analysis of student construction of ratio and proportion understanding*. Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada, p. 25.

The age problem

Susan and Cathy are sisters. Susan was 4 years old when Cathy was ten. Susan is now 9 years old. How old is Cathy now? When will Cathy be twice as old as Susan?

Figure B 2. The snake, tent/scout, and age problems for item 3 on the pre-interview.

A

Replace the question mark with a number to make a true statement.

$$\frac{9}{15} = \frac{12}{?}$$

This problem is most like (check one):

- ☐ The SNAKE problem
- ☐ The SCOUT/TENT problem
- ☐ The AGE problem
- ☐ NONE of these problems

B

Bob and Mary run laps together because they both run at the same pace. Today, Mary started running before Bob came out of the locker room. Mary had run 7 laps by the time Bob had run 3. How many laps had Mary run by the time Bob had run 12?

> Taken from Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah: NJ: Erlbaum. p. 223.

This problem is most like (check one):

- ☐ The SNAKE problem
- ☐ The SCOUT/TENT problem
- ☐ The AGE problem
- ☐ NONE of these problems

C

The table shows the values of x and y, where x is proportional to y.

x	3	6	P
y	7	Q	35

What are the values of P and Q?

> Taken from International Study Center (2001). *TIMSS 1999 mathematics items: Released items for eighth grade*. Chestnut Hill, MA: Boston College. p. 40.

This problem is most like (check one):

- ☐ The SNAKE problem
- ☐ The SCOUT/TENT problem
- ☐ The AGE problem
- ☐ NONE of these problems

D

Bret's family has an annual income of $30,000 and gave $400 to a charity. Barbara's family has an annual income of $300,000 and gave $4,000 to a charity. Whose family gave more money to the charity?

> Adapted from O'Daffer, P., Charles, R., Cooney, T., Dossey, J., & Schielack, J. (1998). *Mathematics for elementary school teachers*. New York: Addison Wesley. p. 402.

This problem is most like (check one):

- ☐ The SNAKE problem
- ☐ The SCOUT/TENT problem
- ☐ The AGE problem
- ☐ NONE of these problems

E

The cost of admission to the state park is $1.00 for each person in a vehicle plus $3.00 for parking the vehicle. Complete the chart below showing how much it will cost for admission based on the number of people in the vehicle. Explain how you completed the table. Examine the table. Describe at least three different patterns in the table. Write a rule to help you determine the cost given any number of people.

Number of people	Cost
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Taken from Smith, M. S., Silver, E. A., Leinhardt, G., & Hillen, A. F. (2003). *Tracing the development of teachers' understanding of proportionality in a practice-based course*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL, p. 42.

This problem is most like (check one):

- ☐ The SNAKE problem
- ☐ The SCOUT/TENT problem
- ☐ The AGE problem
- ☐ NONE of these problems

F

David and Diana are trying to lose some weight before swimsuit season. Four months ago, David weighed 250 pounds and Diana weighed 180 pounds. Today, David weighs 220 pounds, and Diana weighs 150 pounds. Who lost the most weight?

This problem is most like (check one):

- ☐ The SNAKE problem
- ☐ The SCOUT/TENT problem
- ☐ The AGE problem
- ☐ NONE of these problems

G

The cost of admission to the zoo is $5.00 per person. (There is no charge for parking.) Complete the chart below showing the cost for admission for different groups of different sizes. Explain how you completed the table. Examine the table. Describe at least three different patterns in the table. Write a rule to help you determine the cost given any number of people.

Size of group	Cost
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Taken from Smith, M. S., Silver, E. A., Leinhardt, G., & Hillen, A. F. (2003). *Tracing the development of teachers' understanding of proportionality in a practice-based course*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL, p. 42.

This problem is most like (check one):

- ☐ The SNAKE problem
- ☐ The SCOUT/TENT problem
- ☐ The AGE problem
- ☐ NONE of these problems

H

Which is more square: a rectangle that measures 35” by 39” or a rectangle that measures 22” by 25”?

> Taken from Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah: NJ: Erlbaum. p. 6.

This problem is most like (check one):

- ☐ The SNAKE problem
- ☐ The SCOUT/TENT problem
- ☐ The AGE problem
- ☐ NONE of these problems

Figure B 3. The set of eight problems for item 3 on the pre-interview.

APPENDIX C

POST-INTERVIEW PROTOCOL

(Phrases in bold should be read exactly as they appear.)
Thank you for participating in this interview. I have three questions[45] for you today.

<u>For teachers in the treatment group:</u>

1. What do you think you know or understand now that you did not know or understand before you started the course?

   a. Let's start with what you think you know or understand now about **mathematics** that you did not know or understand before you started the course?

      (After they respond to part a, introduce the course map (shown in Figure C1), explain it to them (i.e., the activities you did in each class are shown in the columns; activities above the line were completed in class; activities below line were completed outside of class). For each idea they identified, ask them to identify the activities that they think helped them come to know or understand things differently and to place a blue dot on those activities.)

   b. Now, what do you think you know or understand now about **students as learners of mathematics** that you did not know or understand before you started the course?

      (For each idea they identified, ask them to identify the activities that they think helped them come to know or understand things differently and to place a red dot on those activities.)

   c. Now, what do you think you know or understand now about **teaching mathematics** that you did not know or understand before you started the course?

      (For each idea they identified, ask them to identify the activities that they think helped them come to know or understand things differently and to place a green dot on those activities.)

[45] All questions on the post-interview were adapted from work developed under the auspices of the ASTEROID project (NSF Award #0101799), principal investigator Margaret S. Smith.

<u>For all teachers:</u>

2. Consider the data on a snowstorm that hit both Mason City and Cedar Rapids (hand teacher a copy of Figure C2).

   What can you tell me from the table and the graph about each of these situations?

   Press his/her understanding of the mathematical relationships embedded in proportional situations by asking questions such as: **How can you tell?; What does the origin have to do with proportionality?; What are you doing with the table?; etc.**

   (Let the teacher talk for as long as they want. If you're not sure they're done and they haven't spontaneously mentioned the proportionality of the situations, ask, **Anything else you want to tell me?** just to make sure you've provided them with every opportunity to spontaneously bring it up.)

   If teacher has not talked about the proportionality of either situation, ask**, Do either of these situations reflect a proportional relationship?**

   Press his/her understanding of the mathematical relationships embedded in proportional situations by asking questions such as: **How can you tell?; What does the origin have to do with proportionality?; What are you doing with the table?; etc.**

3. a. **What does it mean to say that there is a proportional relationship among quantities? If you can, I'd like to first hear your thoughts on what a proportional relationship is without your using specific examples.**

   Probes: **In your own words, what is a proportion? What do we mean when we say two quantities are related proportionally?**

   b. Now, I'd like for you to describe a situation in which there **IS a proportional relationship between the quantities**. If you can, describe a situation that was not used in class.

   Probes: If the teacher cannot think of such a situation, ask them to focus on a problem that we did in class that depicted a proportional situation. Use the class situation to see if the teacher can generalize the characteristics of the situation. For example, 'Ok, now using what you know about the tents/scouts problem, can you think of or make up a situation in which there is a proportional relationship between the quantities?

   c. Now, I'd like for you to describe a situation in which there is **NOT a proportional relationship between the quantities**. Again, describe a situation that was not used in class.

Probes: If the teacher cannot think of such a situation, ask them to focus on a problem that we did in class that depicted a nonproportional situation (i.e., age problem). Use the class situation to see if the teacher can generalize the characteristics of the situation. For example, ‘Ok, now using what you know about the Age problem, can you think of or make up a situation in which there is not a proportional relationship between the quantities?

Thank you so much for participating in the interview. Your willingness to share your thinking will help us better understand how to create learning environments that will help teachers grow and develop as teachers of mathematics.

	Class 1	Class 2	Class 3	Class 4	Class 5	Class 6	Class 7	Class 8	Class 9	Class 10	Class 11	Class 12	Class 13	Class 14
Activities completed in class	Solve the Tower problem	Discuss the Square problem	Discuss the Square problem, cont'd	Discuss the Square problem, cont'd (Nora's solution)	Discuss The Case of Randy Harris	Classify ratio statements: P:W; P:P; 2 different things	Compare fractions w/o doing calculations	Examine Orange Juice student work, cont'd	Read & discuss the Mini-case 'The Ratio of Boys to Girls'	Discuss ratio tables	Discuss the Park & Zoo problems	Discuss The Case of Marcia Green	Discuss problems identified in PSSM	Solving three problems (rectangle, string, butterfat) (from The Case of Janice Patterson)
		Discuss the Tower problem, cont'd	Examine Scout/Tent student work, cont'd	Discuss Hunks & Chunks problem	Find the Fraction; Find the Part; Find the Whole -- Manipulative activities	Discuss whether all fractions are ratios, & whether all ratios are fractions	Discuss whether all fractions are ratios	Graphing Orange Juice Mixes Problem	Solve the Candy Jar problems (from The Case of Marie Hanson)	Discuss The Case of Marie Hanson	Discuss the rationale for cross multiplication	Explore similar figures via an applet	Solve the Building a Compost Box problem	Discussing the Case of Janice Patterson
		Solve the Snake, Scout/Tent, Age problems	Discussing Assignment 1	Shade grids of different sizes (from The Case of Randy Harris)		Solve the Orange Juice problem	Examine Orange Juice student work	Discuss whether all ratios are fractions	Assignment 2: Thinking Through A Lesson (TTAL): Solving math task with group	Assignment 2: TTAL: Solving math task and anticipating student responses, cont'd	Enlarge figures with rubber band stretchers (from The Case of Marcia Green)			
		Discuss how the Snake, Scout/Tent, & Age are same and different				Discuss video of the Orange Juice lesson								
		Examine Scout/Tent student work												
Activities completed as homework	Solve boom box, motorbike & square problems	Read Gomez (2002). How is her classroom same/different from your experiences?	Solve Hunks & Chunks and have 3 people solve it	Assignment 1: Interviewing Students	Did Randy Harris achieve his goals?	What issues, questions, or concerns does experiencing OJ raise for you? What new insights do you have?	Select 3 OJ responses you want to present in Summarize phase; what questions will you ask to make connections between them?	Solve Park & Zoo problems	Pick a student in Marie Hanson's class and compare their strategy to one of mine	Assignment 2: TTAL	Pick a question that Marcia Green asks that I think is a really good question and explain why it's a good question.	Read PSSM (Grade Book 6-8) & identify problems involving proportionality	Assignment 3: Analyzing Teaching	What have I learned about the teaching and learning of mathematics?
	Read Ch.1	Read Ch. 2-3	Read Interviewing Chapter & Ch. 4		Read Ch. 5	Read Investigation 3 in Comparing & Scaling	Read Ch.10	Read Ch 11		Read Langrall & Swafford (2000) & Boston, Smith, & Hillen (2003)				

Figure C1. The course map for item 1 on the post-interview.

Mason City and Cedar Rapids have both been hit by a snowstorm. Mason City had 6 inches of snow on the ground before it started snowing. This storm brought 0.5 an inch of snow per hour to Mason City. In Cedar Rapids it snowed more heavily, 1.5 inches per hour. Fortunately, Cedar Rapids did not have any snow on the ground when the storm started.

Mason City

Hours it Snowed	Inches of Snow on the Ground
1	6.5
3	7.5
5	8.5
7	9.5

Cedar Rapids

Hours it Snowed	Inches of Snow on the Ground
1	1.5
3	4.5
5	7.5
7	10.5

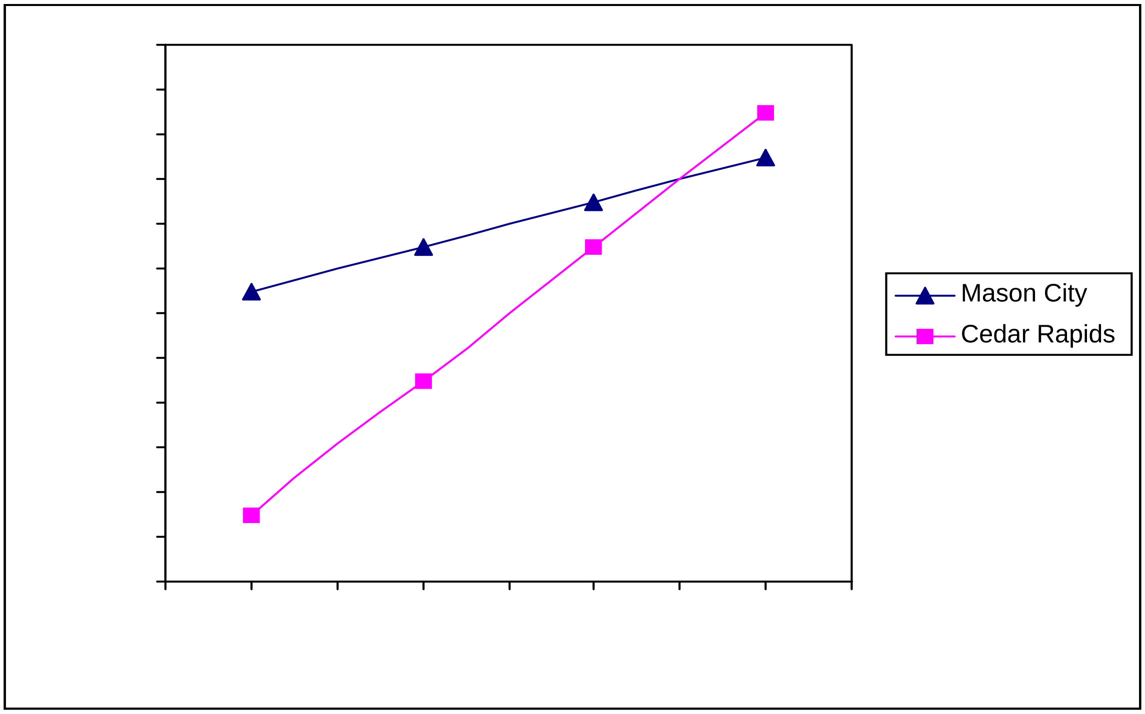


Figure C2. The snowfall data for item 2 on the post-interview.
Taken from Smith, M. S., Silver, E. A., Leinhardt, G., & Hillen, A. F. (2003). *Tracing the development of teachers' understanding of proportionality in a practice-based course*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL, p. 55.

APPENDIX D

SET OF EIGHT STUDENT RESPONSES TO THE TENT/SCOUT PROBLEM THAT TEACHERS EXAMINED DURING CLASSES 2 AND 3

A

Answer: 120 scouts

Tents	Scouts
4	12
8	24
12	36
16	48
20	60
24	72
28	84
32	96
36	108
40	120

B

Answer: 120 scouts

If 4 tents has 12 scouts, then each tent has 3 scouts.
So then you multiply 3 by 40 tents. That's 120 scouts.

C

Answer: 120 scouts

I knew the answer because I keeped adding 4 to the tents and 12 to the scouts til I got 40 tents.

# of tents	# of scouts
4	12
4	12
4	12
4	12
4	12
4	12
4	12
4	12
4	12
+ 4	+12
40	120

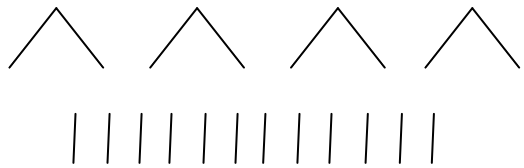
D

Answer: 48 scouts

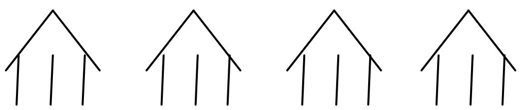
They added 36 tents and so I added 36 scouts.

E

Answer: 120 scouts
I drew 4 tents and 12 scouts.



Then I gave each tent one scout at a time and kept doing that til I had no scouts left.



If each of the tents holds 3 scouts, then you just X's. 3 X's 40 = 120.

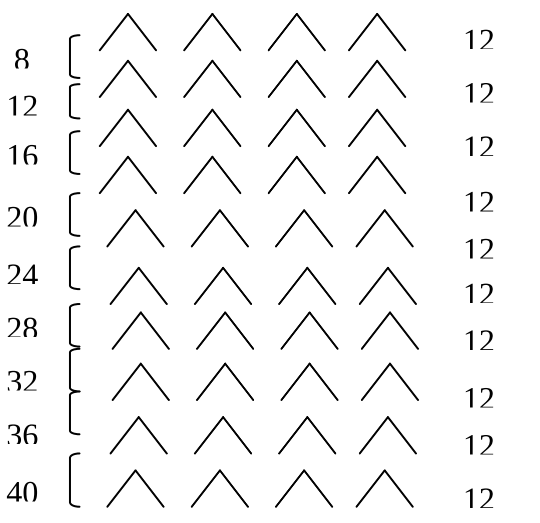
F

Answer: 56 scouts

I added 12 + 4 + 40 to get 56.

G

Answer: 120 scouts
I kept taking 4 tents and 12 scouts until I got 40 tents.



120 scouts

H

Answer: 480 scouts

  12 scouts
x 40 tents
 480 scouts for the 40 tents

I

Answer: 48 scouts

Tents	Scouts
4	12
5	13
6	14
7	15
8	16
9	17
10	18

Well there are always 8 more scouts than tents.
So 40 tents have 48 scouts.

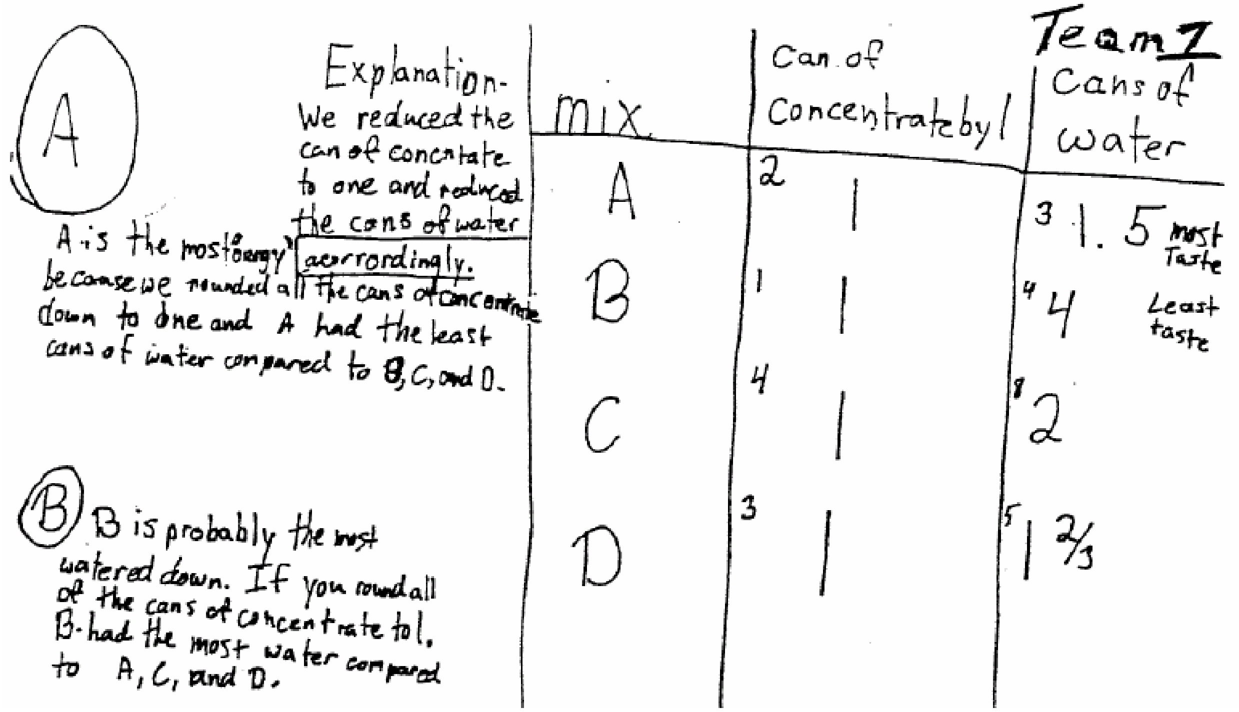
J

Answer: 120 scouts

You have 10 times more tents so you need 10 times more scouts. So you multiply 12 x 10.

APPENDIX E

SET OF TWELVE STUDENT RESPONSES TO THE ORANGE JUICE PROBLEM THAT TEACHERS EXAMINED DURING CLASSES 7 AND 8



Team 2

Mix A

$\frac{2}{5}$ 2 = 40% 5 = 60%

concentrate

$2 \div 5 = .4 = 40\%$

water

$\frac{3}{5}$ $3 \div 5 = .6 = 60\%$

Mix B

concentrate

$\frac{1}{5}$ $1 \div 5 = .2 = 20\%$

water

$\frac{4}{5}$ $4 \div 5 = .8 = 80\%$

Mix C

concentrate

$\frac{4}{12}$ $4 \div 12 = .333 = 33\%$

water

$\frac{8}{12}$ $8 \div 12 = .6667 = 67\%$

Mix D

concentrate

$\frac{3}{8}$ $3 \div 8 = .375 = 38\%$

$\frac{5}{8}$ $5 \div 8 = .625 = 62\%$

A. Mix "A" was the orangeyest because it had the most Percent Concentrate. (40%)

B. Mix "B" = the least orangest because it had the least amount of concentrate.

Mix A: 2 + 3 = 5 cups of juice × 3 = 15 cups 6 concentrate + 9 water

6/15 concentrate

Mix B: 1 + 4 = 5 cups of juice × 3 = 15 cups

3 concentrate + 12 wate

3/15 concentrate

Mix C: 4 + 8 = 12 cups of juice × 1.25 = 15 cups 4 × 1.25 = 5 concentrate

8 × 1.25 = 10 water 5/15 concentrate

Mix D: 3 + 5 = 8 cups o juice × ~~1.88~~ 1.875 = 15 c

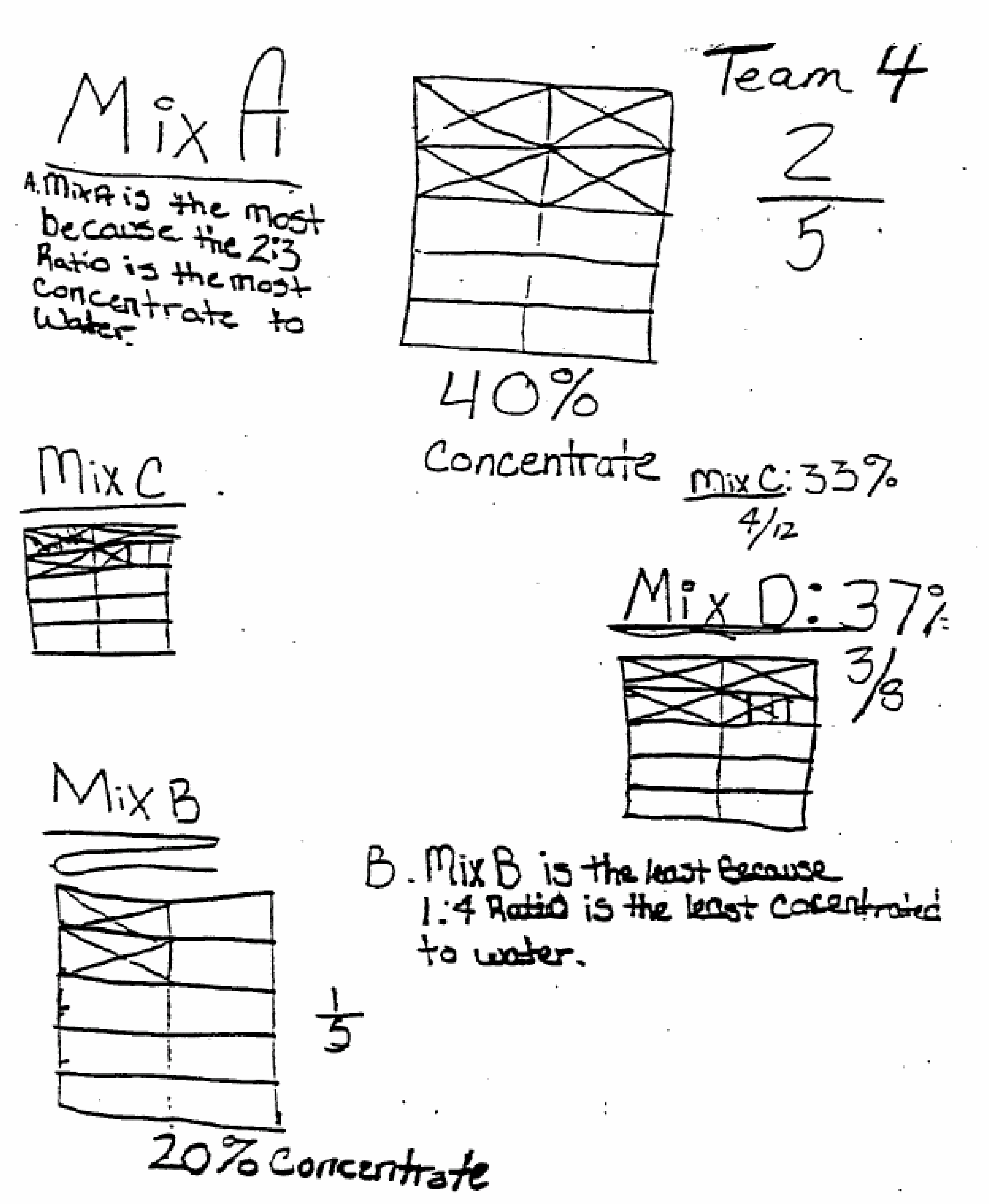
3 × ~~1.88~~ .875 = 5.6 concentrate 5 × 1.88 = 9.

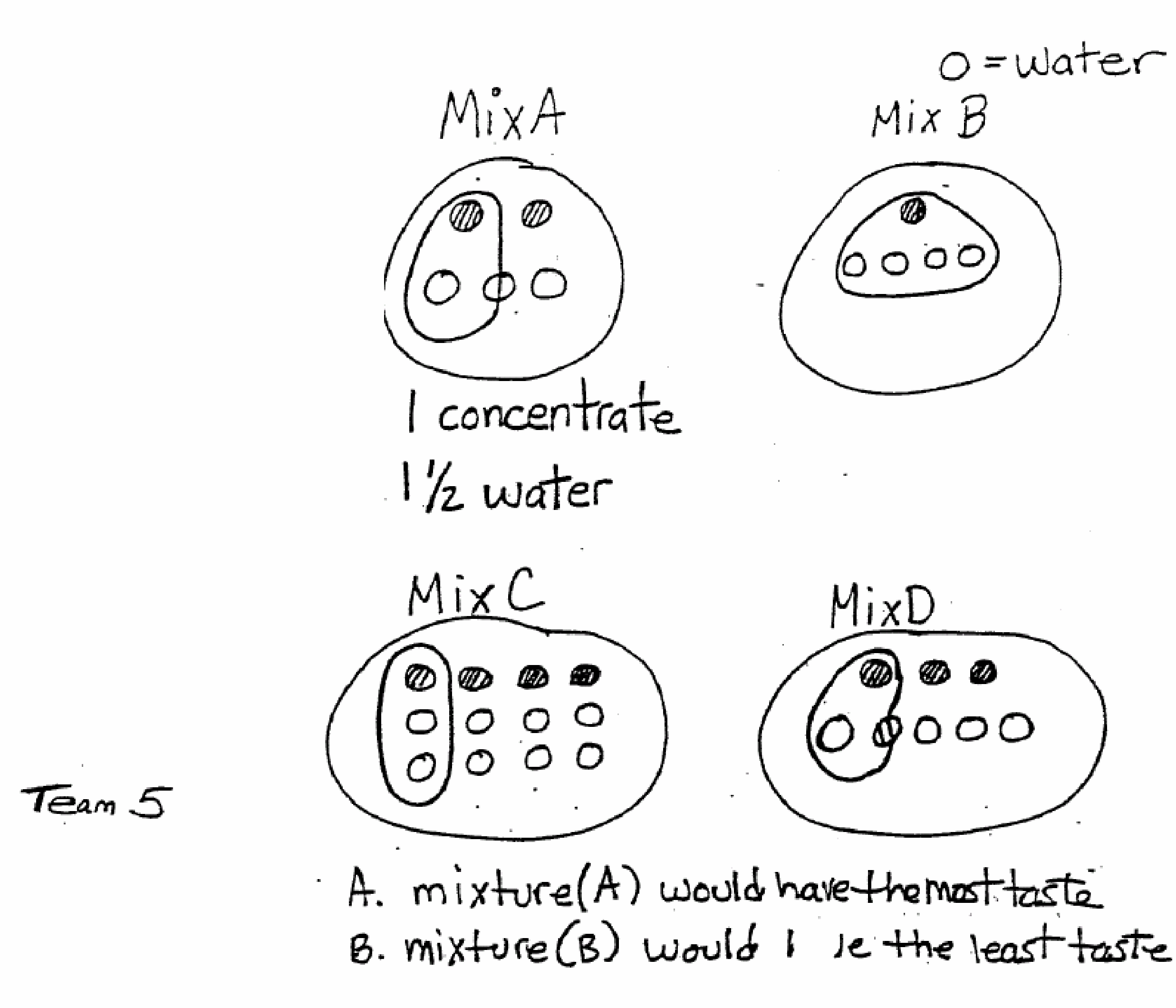
water 5.6/15 concentrat

Most Orangey (most concentrate per 15 cups of juice

Team 3

Least Orangey (least concentrate per 15 cups of ju.


Team 4
Mix A
A. Mix A is the most because the 2:3 Ratio is the most concentrate to water.
2/5
40% Concentrate
Mix C
Mix C: 33%
4/12
Mix D: 37%
3/8
Mix B
B. Mix B is the least because 1:4 Ratio is the least cocentrated to water.
1/5
20% Concentrate


O = water
Mix A
Mix B
1 concentrate
1½ water
Mix C
Mix D
Team 5
A. mixture(A) would have the most taste
B. mixture(B) would
the least taste

WATER

Team 6

A. ☆ mix A - $\frac{3}{5}$ = 60% Most orange

Mix B - $\frac{5}{..}$ → $\frac{4}{5}$ = 80%

Mix C - $\frac{8}{12}$ = 67%

Mix D - $\frac{5}{8}$ = 63%

Concentrate

B. Mix A - $\frac{2}{5}$ = 40%

☆ mix B - $\frac{5}{..}$ → $\frac{1}{4}$ = 25% least orangey

mix C - $\frac{4}{8}$ = 50%

mix D - $\frac{3}{5}$ = 60%

Team 7

A. mix "A" is the most oarngyest because. it has the least amount of water added which is $1\frac{1}{2}$ cups of water.

B. mix "B" is the least fruityest because it has 4 cups of water to 1 cup of concontrate

Mix A

$\frac{3}{2} = \frac{1}{1\frac{1}{2}}$ con. / water

Mix B

$\frac{1}{4}$ con. / water

Mix C

$\frac{4}{8} = \frac{2}{4} = \frac{1}{2}$ con. / water

Mix D

$\frac{3}{5} = \frac{1}{1\frac{2}{3}}$ con. / water

Team 8

Mix A - 3/5 water = 60% water
Mix B - 4/5 water = 80% water
Mix C - 8/12 water = 67% water
Mix D - 5/8 water = 63% water

Most orangey - Mix A
Least orangey - Mix B

Mix A 1st (most)

$\frac{2}{5} = \frac{16}{40}$

Mix B 4th (least)

$\frac{1}{5} = \frac{8}{40}$

Mix C 3rd

$4/12 = 13.3/40$

Mix D 2nd

$\frac{3}{8} = \frac{15}{40}$

Team 9.

Team 10

Mix A is most orangy, because...

100% ÷ 5 = 20% 1 cup = 20% of the mixture

20% x 2 cups = 40% concentrate
(concentrate)

Mix B is the least orangy because...

100% ÷ 5 = 20% 1 cup = 20%
(1+4=5)

1 cup x 20% = 20% concentrate

Mix C is neither because...

100% ÷ 12 = 8% 1 cup = 8%
(4+8)

4 cups x 8% = 32% concentrate

Mix D is niether because...

100% ÷ 8 = 12.5% 1 cup = 12.5%

3 x 12.5% = 37.5% concentrate

Mix A - /5 = 60% Water/40% concentrate

Mix B - 4/5 = 80% " /20% "

Mix C - 8/12 = 67% " /33% "

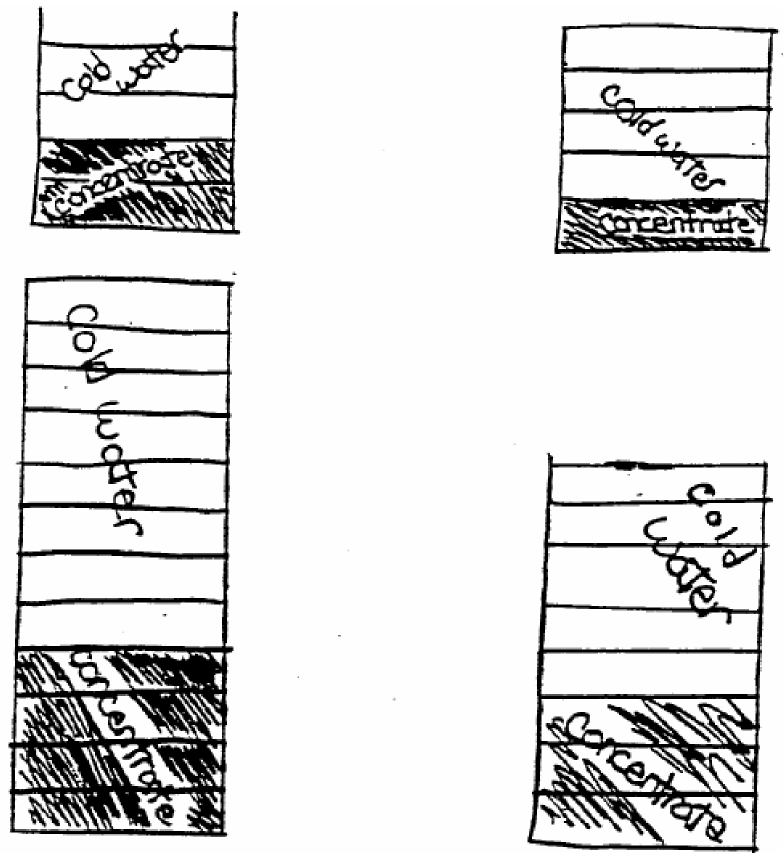
Mix D - 5/8 = 63% "/ 37% "

A.) mix A is the most orangey because it is only 60% water, the least watered down.

B.) mix B is the least orangey because it is 80% water, the most watered down

Team 11

Team 12



A) The juice that will taste most orangie is mix "A" because it does not have as much water as mixes B,c, and D.

B) The juice that will taste least orangie is mix "B" because it has more water and less cocentrate than Mixes ', C, and D

Adapted from Michigan State University, Connected Mathematics Project web site:
http://www.math.msu.edu/cmp/RREvaluation/StudentWork.html

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