THE CARTAN-WEYL CONFORMAL GEOMETRY OF A PAIR OF SECOND-ORDER PARTIAL-DIFFERENTIAL EQUATIONS

by

Kiplin Perkins

M.S. Physics, University of Pittsburgh, 2002

Submitted to the Graduate Faculty of

the Department of Physics & Astronomy in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2006

UNIVERSITY OF PITTSBURGH

DEPARTMENT OF PHYSICS & ASTRONOMY

This dissertation was presented

by

Kiplin Perkins

It was defended on

05 July 2006

and approved by

Ezra (Ted) Newman, Emeritus Professor

Anthony Duncan, Professor

Simonetta Frittelli, Associate Professor, Department of Physics, Duquesne University

Rainer Johnsen, Professor

George Sparling, Associate Professor, Department of Mathematics

Dissertation Advisors: Ezra (Ted) Newman, Emeritus Professor,

Anthony Duncan, Professor

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Abstract: We explore the conformal geometric structures of a pair of second-order partial-differential equations. In particular, we investigate the conditions under which this geometry is conformal to the vacuum Einstein equations of general relativity. Furthermore, we introduce a new version of the conformal Einstein equations, which are used in the analysis of the conformal geometry of the PDE's.

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PREFACE

We thank the committee, especially Ezra Newman for all of his guidance and enlightening discussions. We also thank Carlos Kozameh, Al Janis, Pawel Nurowski, and Philip Tillman for their interest in this work. Lastly, we thank our fellow students, in particular Cameron McBride and Ramin Skibba for their support at all levels during the preparation of this work.

1.0 INTRODUCTION

In late 1915, Einstein introduced his theory of general relativity (GR), which described gravitation in terms of the curved geometry of space-time. Only a few months after its publication, Schwarzschild discovered the first exact solution to Einstein's field equations, the *Schwarzschild solution*, which paved the way for the theory of black holes. In the intervening 90 years, GR has had considerable success in multiple branches of physics and mathematics. Some of the many topics explored using the theory are the solutions of the field equations, gravitational waves, the formation of galaxies, the formation of stars and black holes, quantum theories of gravity, and the origin of the universe.

Another branch of research in GR has been the study the geometric structure of the field equations. Because these equations are a set of 10 complicated, non-linear partial differential equations, they are, in general, rather difficult to analyse. Thus, many different techniques and concepts have been developed to aid in their analysis. One idea in particular, conformal transformations, is used through-out this work.

Conformal transformations are transformations that preserve angles. They have many applications in physics and mathematics, *e.g.*, the study of solutions to Laplace's equation in 2 dimensions. Another common application is cartography: on a city's map, the lengths of the streets have been rescaled, but the relative angles between the streets have been preserved.

In GR, conformal transformations are obtained by multiplying the metric tensor by a positive scalar function (the *conformal factor*) to obtain a new, conformally-related metric. Since metrics are used to measure distance, the effect of a conformal transformation is a rescaling of space-time. The rescaling can either be an expansion or a contraction.

One of the major successes of conformal transformations in GR has been the study of conformal infinity [21], [3]. In this case, one is interested in the "fall-off" behaviour of physical quantities (*e.g.*, solutions to the field equations) as they are extended to infinity. In general, the limit procedures are very complicated. One can simplify the issue, however, by using a conformal transformation to rescale the infinite boundary of space-time to a finite, conformal boundary. Then any asymptotic properties of interest are more easily found at this finite boundary.

Conformal transformations of the metric raise a natural question: what are the properties of the Einstein field equations under a conformal transformation. This issue, first raised in the context of null infinity, has found many related applications, such as the numerical evolution of space-time for simulations [3]. In particular, we will be concerned with the conformal transformations of the vacuum (*i.e.*, source-free) Einstein equations, which are a set of differential equations for the metric. If the metric is conformally transformed, however, the *conformal vacuum Einstein equations* are (in the standard form) differential equations for the metric *and* the conformal factor. These conformal field equations, containing the conformal factor, have been extensively studied [3], [4].

In general, an arbitrary metric will not satisfy the Einstein equations, *i.e.*, it will not be an *Einstein metric*. One can ask, however, if there exists a conformal factor that transforms this metric into a new metric that *does* satisfy the Einstein equations. If such a conformal factor exists, then it and the original metric must satisfy the conformal Einstein equations. In other words, this metric is *conformally Einstein*.

In this work, one of our objectives will be the introduction of a new version of the conformal Einstein equations which does not explicitly contain the conformal factor. Thus, this version is a set of differential equations solely for the conformal metric; the conformal factor is contained in a set of auxiliary differential equations. Therefore, one can use these field equations to determine whether a given metric is conformally Einstein without having to consider the conformal factor. The field equations only determine whether this factor exists. If information about it is needed, one then uses the auxiliary equations that contain the factor.

As an application of this new version of the conformal field equations, we explore the Cartan-Weyl geometry of differential equations. Geometric methods have been used to study differential equations and their solutions since the 1930's [1], [2]. For most of the time since then, this analysis has essentially been mathematical in nature. Beginning in the early 1980's [16] - [17], however, and considerably expanded upon in the 1990's, Frittelli, Kozameh, Newman, *et al.* [6] - [10] demonstrated that when a certain condition, known as the *Wünschmann condition* [23], is satisfied, the 4-dimensional solution space of a pair of second-order partial-differential equations (PDE's) naturally contains a conformal Lorentzian metric. In other words, the solution space of these PDE's can be interpreted as a conformal space-time.

In this picture, the conformal metric is constructed as a functional of the inhomogeneous functions of the PDE's. Thus, all other geometric quantities (the connection and the curvatures) are also functionals of these functions. This raises two issues, which will be explored in detail in this work. First, what are the explicit forms of these quantities in terms of the inhomogeneous functions. And second, what further conditions on the inhomogeneous functions are needed in order to make the metric constructed from them conformally Einstein.

These issues have been partially addressed using the standard form of the conformal Einstein equations [8] - [9], which explicitly contains the conformal factor, and by considering the simpler case of 3-dimensional space-times [11] - [12]. Our objective here is to 1.) fill in the details of the construction the geometric quantities in 4 dimensions, namely the Weyl connection and the Cartan curvatures; and 2.) use our new version of the conformal Einstein equations to analyse this geometry.

This work is organised in the following way. In chapter 2, we briefly present the vacuum Einstein equations and some of their geometric structures. We also present an alternate, non-standard form of the vacuum Einstein equations. Next, in chapter 3, we begin by reviewing conformal transformations of the metric and the associated curvatures. We then construct three versions of the conformal Einstein equations, the last being our new version, which does not contain the conformal factor. Then, in chapter 4, we give an over-view of the null-surface formulation (NSF) of a conformal Lorentzian metric. This introduces the notion of constructing a metric in terms of the inhomogeneous functions of a pair of PDE's. Also in that chapter, we first encounter the Wünschmann condition, which is the condition for the existence of a conformal metric on the space-time (*i.e.*, the solution space of the PDE's). After reviewing some of the details of the NSF, we change the context of our discussion in chapter 5 to the pair of PDE's and their conformal geometry. In that chapter, we examine the geometry (the metric, the connections, and the curvatures) in a fairly general and technical manner. The purpose of this, as we have said, is to fill in many gaps in the development of this geometry. Accordingly, much of the work in that chapter is new. Furthermore, we will derive the Wünschmann condition from a very different point of view compared to its derivation in the NSF. Lastly, in chapter 6, we apply our new version of the conformal Einstein equations to the Cartan-Weyl geometry of the PDE's. In particular, we present an outline for how one finds the conditions on the inhomogeneous functions of the PDE's that make the metric conformally Einstein. The work discussed in chapter 6 is new, as is most of the work in chapter 5 and section 3.3.

2.0 THE VACUUM EINSTEIN EQUATIONS

In this preliminary chapter, we briefly discuss the vacuum Einstein equations and some of their features. Beginning in section 2.1, we define the necessary curvature tensors and their identities on a standard space-time manifold. Then, in section 2.2, we discuss the standard version of the vacuum Einstein equations and introduce an alternate form.

2.1 DEFINITIONS OF THE CURVATURES

We begin with a standard four-dimensional space-time M which has a metric g_{ab} and a *metric connection* ∇_c, which has the property

$$\nabla_c \, g_{ab} = 0 \quad . \tag{2.1}$$

Using this connection, the *Riemann curvature tensor* $R^a{}_{bcd}$ is given by

$$(\nabla_c \nabla_d - \nabla_d \nabla_c) V^a = R^a{}_{bcd} V^b \quad , \tag{2.2}$$

for an arbitrary vector V^a. From the definitions of the Riemann tensor and the metric connection, it follows that the Riemann tensor has the symmetries

$$R_{abcd} = R_{[ab][cd]} \quad , \tag{2.3}$$

$$R_{abcd} = R_{cdab} \quad , \tag{2.4}$$

$$R_{a[bcd]} = 0 \qquad \text{(Bianchi symmetry)} \quad , \tag{2.5}$$

where $R_{a[bcd]} = \frac{1}{3}(R_{abcd} + R_{acdb} + R_{adbc})$. R_{abcd} also satisfies the *Bianchi identity*,

$$0 = \nabla_{[a}R_{bc]de} = \frac{1}{3}(\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde}) \qquad \text{(Bianchi identity)} \quad . \tag{2.6}$$

From the Riemann tensor and the metric, the *Ricci tensor* R_{bd}, the *Ricci scalar* R, the *Schouten tensor* P_{ab}, and the *Schouten scalar* P are defined as

$$R_{bd} = g^{ac} R_{abcd} \quad , \tag{2.7}$$

$$R = g^{ab} R_{ab} \quad , \tag{2.8}$$

$$P_{ab} = \frac{1}{2} R_{ab} - \frac{1}{12} R g_{ab} \quad , \tag{2.9}$$

$$P = g^{ab} P_{ab} = \frac{1}{6} R \quad . \tag{2.10}$$

Using Eqs (2.4) and (2.7), it is clear that the Ricci tensor, and, thence, the Schouten tensor are symmetric:

$$R_{ab} = R_{(ab)} \quad , \qquad P_{ab} = P_{(ab)} \quad . \tag{2.11}$$

Remark 2.1.1. From its definition, one sees that the Schouten tensor contains the same information as the Ricci tensor. We will find this fact useful in simplifying many of our subsequent expressions (as we see in the definition of the Weyl tensor, below) and in connecting the tensor language of the Einstein equations with the p-form language of the Cartan construction found in chapters 5 and beyond.

The *Weyl curvature tensor* C_{abcd} is defined as

$$C_{abcd} = R_{abcd} - 2P_{a[c}g_{d]b} + 2P_{b[c}g_{d]a} \quad . \tag{2.12}$$

From its definition, the Weyl tensor inherits the symmetries of the Riemann tensor:

$$C_{abcd} = C_{[ab][cd]} \quad , \tag{2.13}$$

$$C_{abcd} = C_{cdab} \quad , \tag{2.14}$$

$$C_{a[bcd]} = 0 \quad . \tag{2.15}$$

In addition, it is completely traceless,

$$g^{ac}C_{abcd} = 0 \quad . \tag{2.16}$$

At this point, it will be useful for us to restrict our attention to so called *generic metrics*, which are defined by the properties:

$$\begin{aligned} V^a C_{abcd} = 0 \quad &\Rightarrow \quad V^a = 0 \quad , \\ H^{[ab]} C_{abcd} = 0 \quad &\Rightarrow \quad H^{[ab]} = 0 \quad , \\ T^{(ac),\mathrm{TF}} C_{abcd} = 0 \quad &\Rightarrow \quad T^{(ac),\mathrm{TF}} = 0 \quad , \end{aligned} \tag{2.17}$$

where V^a, $H^{[ab]}$, and $T^{(ac),\mathrm{TF}}$ are arbitrary, and the trace-free tensor $T^{(ac),\mathrm{TF}}$ is defined by

$$T^{(ac),\mathrm{TF}} = T^{ac} - \frac{1}{4}Tg^{ac} \quad . \tag{2.18}$$

Since this restriction only excludes a few members of a small class of metrics (the algebraically special metrics), we have not lost much generality. The above properties of generic metrics will be used in the proofs of several important theorems in the this chapter and the next.

2.1.1 The Contracted Bianchi Identities

Taking two consecutive traces on the Bianchi identity yields the *contracted Bianchi identities*. They are, respectively,

$$\nabla^e R_{ebcd} = \nabla_c R_{db} - \nabla_d R_{cb} \quad , \tag{2.19}$$

$$\nabla^e R_{ed} = \frac{1}{2}\nabla_d R \quad . \tag{2.20}$$

Using Eqs (2.12), (2.9), and (2.10), the contracted Bianchi identities can be re-written in terms of the Weyl and Schouten tensors:

$$\nabla^e C_{ebcd} = \nabla_c P_{db} - \nabla_d P_{cb} \quad , \tag{2.21}$$

$$\nabla^e P_{ed} = \nabla_d P \quad . \tag{2.22}$$

Clearly Eqs (2.19) and (2.21) have the same form, as do Eqs (2.20) and (2.22). Under a conformal transformation, however, the left-hand side of Eq (2.21) is much simpler than that of Eq (2.19). (See section 3.1, Eq (3.18).) Therefore, we will often use the second set of the contracted Bianchi identities, *i.e.* those that contain C_{abcd} and P_{ab}.

2.2 THE EINSTEIN EQUATION

The *(vacuum) Einstein equations* are

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 0 \quad , \tag{2.23}$$

where Λ is the cosmological constant. A metric that satisfies these equations is said to be an *Einstein metric*.

From the trace of Eq (2.23),

$$\Lambda = \frac{1}{4}R \quad , \tag{2.24}$$

and, thus, the Einstein equations can be re-written as

$$R_{ab}^{\mathrm{TF}} \equiv R_{ab} - \frac{1}{4}g_{ab}R = 0 \quad , \tag{2.25}$$

where R_{ab}^{TF} is the *trace-free* part of R_{ab}. Thus, for an Einstein metric, the Ricci tensor is given as

$$R_{ab} = \frac{1}{4}Rg_{ab} \quad . \tag{2.26}$$

Remark 2.2.1. If one takes the cosmological constant to be zero,

$$\Lambda = 0 \quad , \tag{2.27}$$

then the Ricci tensor vanishes, $R = 0$, and the Einstein equations are simply

$$R_{ab} = 0 \quad . \tag{2.28}$$

2.2.1 The Yang Equation and C-Space

According to Eq (2.24), $R = 4\Lambda =$ constant for Einstein metrics. Thus, the covariant derivative of Eq (2.26) yields

$$\nabla_c R_{ab} = \frac{1}{4}\nabla_c(g_{ab}R) = 0 \quad ; \tag{2.29}$$

or, terms of the Schouten tensor,

$$\nabla_c P_{ab} = 0 \quad . \tag{2.30}$$

Therefore, for Einstein metrics the contracted Bianchi identity, Eq (2.21), is

$$Y_{bcd} \equiv \nabla^e C_{ebcd} = 0 \quad . \tag{2.31}$$

The tensor Y_{bcd} has 16 components and is known as the *Yang tensor*. The 16 equations $Y_{bcd} = 0$ are the *Yang equations*, and from their definitions, they are clearly necessary conditions for the metric g_{ab} to be Einstein. They are not sufficient, however, since the substitution of $\nabla^e C_{ebcd} = 0$ into Eq (2.21) only implies

$$\nabla_c P_{db} = \nabla_d P_{cb} \quad . \tag{2.32}$$

This equation is not sufficient to have $R_{ab} = \frac{1}{4}Rg_{ab}$.

A metric that satisfies the Yang equations defines a *C-space* [18]. Since the Yang equations are necessary but not sufficient for a metric to be Einstein, it follows that Einstein spaces are subsets of C-spaces [18].

2.2.2 An Alternate Form of the Einstein Equations

As we have seen, the Yang equations, $Y_{abc} = \nabla^e C_{eabc} = 0$, are not sufficient for a metric to be Einstein. To gain sufficiency, metrics need to also satisfy another condition, namely the vanishing of the *Bach tensor* B_{ab},

$$B_{ab} = \nabla^c \nabla^a C_{abcd} + \frac{1}{2} R^{ca} C_{abcd} \quad . \tag{2.33}$$

Its vanishing, $B_{ab} = 0$, is called the *Bach equations*.

Theorem 2.2.1 (Kozameh, *et al.* [18]). *The Yang- and Bach equations are necessary and sufficient conditions for a generic metric to be Einstein (i.e., generically Einstein):*

$$g_{ab} \quad \textit{is generically Einstein} \quad \Leftrightarrow \quad \begin{cases} Y_{abc} = 0 \quad , \\ B_{bd} \ = 0 \quad , \end{cases} \quad \Rightarrow \quad R_{ab} = \frac{1}{4} R g_{ab} \quad . \tag{2.34}$$

Proof. The proof follows the work of [18]. We begin by showing that the pair ($Y_{abc} = 0\,, B_{ab} = 0$) is necessary for *all* Einstein metrics. We have already shown that if a metric is Einstein, then the Yang tensor must vanish. Since the first term of the Bach tensor is just a derivative of the Yang tensor, it too must vanish. Moreover, using Eq (2.26), the second term of the Bach tensor is

$$R^{ca} C_{abcd} = \frac{1}{4} R g^{ca} C_{abcd} = 0 \quad , \tag{2.35}$$

where we have used the trace-free property of the Weyl tensor, Eq (2.16). Therefore, the vanishing of the Bach tensor is necessary for a metric to be Einstein.

Before showing that the pair ($Y_{abc} = 0\,, B_{ab} = 0$) is sufficient, we demonstrate the insufficiency of each member of the pair. We have already seen that $Y_{abc} = 0$ alone is insufficient. In fact, the insufficiency of $B_{ab} = 0$ follows a similar argument:

the vanishing of B_{ab} only determines that $\nabla^c \nabla^a C_{abcd} = -\frac{1}{2} R^{ca} C_{abcd}$, which, in turn, does not imply that $R_{ab} = \frac{1}{4} R g_{ab}$. Therefore, at least both the Yang tensor and Bach tensor must vanish in order to have sufficiency.

To show sufficiency, begin with the pair

$$0 = Y_{abc} = \nabla^a C_{abcd} \quad , \tag{2.36}$$

$$0 = B_{ab} = \nabla^c \nabla^a C_{abcd} + \frac{1}{2} R^{ca} C_{abcd} \quad . \tag{2.37}$$

Since the Yang tensor vanishes, the first term of the Bach tensor vanishes identically. Thus, the vanishing of the Bach tensor becomes

$$0 = R^{ca} C_{abcd} = (R^{ca,\mathrm{TF}} + \frac{1}{4} R g^{ca}) C_{abcd} = R^{ca,\mathrm{TF}} C_{abcd} \quad , \tag{2.38}$$

where we have again used the trace-free property of the Weyl tensor and the definition of R_{ab}^{TF} given in Eq (2.25). Restricting our attention to generic metrics,

$$R^{ca,\mathrm{TF}} C_{abcd} = 0 \quad \Rightarrow \quad R^{ca,\mathrm{TF}} = 0 \quad \Rightarrow \quad g_{ab} \quad \text{is generically Einstein} \quad . \tag{2.39}$$

□

Therefore, presuming that we have generic metrics, as defined in Eq (2.17), the set of conditions

$$\begin{cases} Y_{abc} = 0 \quad , \\ B_{bd} = 0 \quad , \end{cases} \tag{2.40}$$

is an alternate, though non-standard, form of the Einstein equations. These conditions are actually the integrability conditions of Eq (2.26),

$$R_{ab} = \frac{1}{4} R g_{ab} \quad , \tag{2.26}$$

which ensure the existence of solutions, namely generic Einstein metrics. Although these integrability conditions are useful in the study of differential equations, they are not of much interest to physicists. One normally either begins with a specific metric (or class of metrics) and then imposes the Einstein equations on that metric, or one begins with the Einstein equations for a specific (class of) Ricci tensor and then attempts to solve for the appropriate metric solutions. In either case, the conditions (2.40) are not generally needed for these procedures.

The conformal transformations of Eq (2.26), however, are particularly interesting, as they arise naturally from physical considerations. We will demonstrate this physical perspective in a later chapter. First, we define and briefly explore conformal transformations in the following chapter.

3.0 CONFORMAL TRANSFORMATIONS

In this chapter, we introduce and discuss the idea associated with conformal transformations. In section 3.1, we define conformal transformations applied to metrics on the space-time and then apply it to the curvature tensors of the previous chapter. Next, in section 3.2, we transform the Einstein equations and the pair of Yang- and Bach equations, from which we obtain two versions of the conformal Einstein equations. Finally, in section 3.3, we introduce a new version of the conformal Einstein equations.

3.1 THE TRANSFORMATION OF THE METRIC AND THE CURVATURES

Basically, the idea of a conformal transformation is a transformation that preserves angles, *i.e.*, a metric $\hat{g}_{ab}$ is the conformal transformation of g_{ab} if

$$\hat{g}_{ab} = e^{2\phi} g_{ab} \tag{3.1}$$

for some arbitrary smooth scalar function ϕ, called the *conformal parameter*. Furthermore, by requiring that

$$\hat{g}^{ae}\hat{g}_{eb} = \delta^a_b = g^{ae}g_{eb} \quad , \tag{3.2}$$

one also has that

$$\hat{g}^{ab} = e^{-2\phi} g^{ab} \quad . \tag{3.3}$$

Remark 3.1.1. One can also give a conformal transformation as

$$\hat{g}_{ab} = \Omega^2 g_{ab} \quad , \tag{3.4}$$

where Ω is called the *conformal factor*. By comparing the above equation with Eq (3.1), the conformal-parameter and factor are clearly related by $\phi = \ln \Omega$. Throughout this work, we will actually use both versions of conformal transformations. After the following comment, however, we will only use the version given in Eq (3.1) for the remainder of this chapter

Remark 3.1.2. In addition to depending on the the space-time coordinates x^a, the function Ω (or, equivalently, ϕ) can, in general, depend on other parameters. In particular, in chapters 4 and 5, we will be investigating the conformal transformations of the form

$$\hat{g}_{ab}\left(x^a\right) = \left[\Omega\left(x^a,\, s,\, s^*\right)\right]^2 g_{ab}\left(x^a,\, s,\, s^*\right) \quad , \tag{3.5}$$

where the parameters $(s,\, s^*)$ are the complex stereographic coordinates on the 2-sphere.

For an arbitrary vector V^a, the two metric connections of Eq (3.1) are related by

$$\hat{\nabla}_c V^a = \nabla_c V^a + \chi^a{}_{bc} V^b \quad , \tag{3.6}$$

where

$$\chi_{abc} = g_{ab}\phi_c - g_{bc}\phi_a + g_{ca}\phi_b \tag{3.7}$$

and

$$\phi_a \equiv \phi_{,a} = \nabla_a \phi = \hat{\nabla}_a \phi = \partial_a \phi \quad . \tag{3.8}$$

In particular,

$$\hat{\nabla}_c \hat{g}_{ab} = 0 \quad , \quad \nabla_c g_{ab} = 0 \quad , \tag{3.9}$$

and

$$\hat{\nabla}_c g_{ab} = -2\phi_c\, g_{ab} \quad . \tag{3.10}$$

From Eq (3.6) and the definition of the Riemann tensor, Eq (2.2), one can conformally transform $R^a{}_{bcd}$ to obtain

$$\begin{aligned}\hat{R}^a{}_{bcd} &= R^a{}_{bcd} \\ &\quad + 2\left(\delta^a_{[c}\phi_{d]}\phi_b - g_{b[c}\phi_{d]}\phi^a + \nabla_b\phi_{[c}\delta^a_{d]} - \nabla^a\phi_{[c}g_{d]b} - \phi_e\phi^e\delta^a_{[c}g_{d]b}\right) \quad .\end{aligned} \tag{3.11}$$

Then, from Eqs (3.3), (2.7), (2.8), (2.9), and (2.10), the conformal transformations of the Ricci tensor, etc. are

$$\hat{R}_{ab} = R_{ab} + 2\phi_a\phi_b - 2\nabla_a\phi_b - g_{ab}\left(2\phi^e\phi_e + \nabla^e\phi_e\right) \quad , \tag{3.12}$$

$$\hat{R} = e^{-2\phi}\{R - 6\left(\nabla^e\phi_e + \phi^e\phi_e\right)\} \quad , \tag{3.13}$$

$$\hat{P}_{ab} = P_{ab} + \phi_a\phi_b - \nabla_a\phi_b - \frac{1}{2}g_{ab}\phi^e\phi_e \quad , \tag{3.14}$$

$$\hat{P} = e^{-2\phi}\{P - \nabla^e\phi_e - \phi^e\phi_e\} \quad . \tag{3.15}$$

Since the connections $\hat{\nabla}_a$ and ∇_a are torsion-free,

$$\nabla_a f_b = \nabla_b f_a \quad , \tag{3.16}$$

for some arbitrary smooth function f, the conformal Ricci- and Schouten tensors are symmetric.

From Eq (3.11) and the definition of the Weyl tensor, Eq (2.12), it is straightforward to show that, given the appropriate index position, the Weyl tensor is conformally invariant,

$$\hat{C}^a{}_{bcd} = C^a{}_{bcd} \quad . \tag{3.17}$$

Thus, using Eqs (3.6) and (3.7), it is easy to compute the conformal Yang tensor,

$$\hat{Y}_{bcd} = \hat{\nabla}_a \hat{C}^a{}_{bcd} = \nabla_a C^a{}_{bcd} + \phi_a C^a{}_{bcd} = Y_{bcd} + \phi_a C^a{}_{bcd} \quad . \tag{3.18}$$

The vanishing of $\hat{Y}_{bcd}$ is important for obtaining one version of the conformal Einstein equations. In fact, as we will see in Section 3.3, it plays a central role in the development of the third, new version.

3.2 PREVIOUSLY KNOWN VERSIONS OF THE CONFORMAL EINSTEIN EQUATIONS

In this section, we encounter and briefly discuss two known versions of the conformal Einstein equations. The first version is very well known, while the second is less known. Each of the two versions is a set of equations containing the metric g_{ab} and the conformal factor ϕ. We will expound upon this important point below. For comparison, in Section 3.3 we will discuss a new version of the conformal Einstein equations that does not explicitly contain ϕ.

3.2.1 The First Version of the Conformal Einstein Equations

Suppose that a metric $\hat{g}_{ab}$ is Einstein:

$$\hat{R}_{ab}^{\mathrm{TF}} = \hat{R}_{ab} - \frac{1}{4}\hat{R}\hat{g}_{ab} = 0 \quad . \tag{3.19}$$

Now, using Eqs (3.12) and (3.13), the inverse conformal transformation of $\hat{R}_{ab}^{\mathrm{TF}}$ is

$$\hat{R}_{ab}^{\mathrm{TF}} = R_{ab}^{\mathrm{TF}} - 2\nabla_a\phi_b + 2\phi_a\phi_b + \frac{1}{2}g_{ab}\left(\nabla^c\phi_c - \phi^c\phi_c\right) \quad ; \tag{3.20}$$

thus, we have our first version of the conformal Einstein equations,

$$0 = R_{ab}^{\mathrm{TF}} - 2\nabla_a\phi_b + 2\phi_a\phi_b + \frac{1}{2}g_{ab}\left(\nabla^c\phi_c - \phi^c\phi_c\right) \quad . \tag{3.21}$$

The above equation is the first, and most well known, version of the conformal Einstein equations. It has been extensively used to study null-infinity [20], [3] and gravitational radiation [3], [4].

It is important to point out that this version of the conformal Einstein equations is a set of equations for both the metric g_{ab} and the conformal factor ϕ. If one is given an arbitrary metric g_{ab} *a priori*, then, in general, it will not be Einstein. One may wonder, however, if there exists a conformal factor ϕ so that the conformal transformation of g_{ab} is Einstein. In general, such a ϕ will not exist. If it does exist, then the pair $(g_{ab}\,, \phi)$ must satisfy (3.21). In this case, the metric g_{ab} is *conformally Einstein*, and, in general, the equations (3.21) can be solved for both ϕ and the g^{ab}.

3.2.2 The Second Version of the Conformal Einstein Equations

To obtain the second version of the conformal Einstein [18], one simply conformally transforms the set of conditions $(Y_{abc} = 0\,,\ B_{ab} = 0)$, which are equivalent to the vacuum Einstein equations. We begin by transforming the Yang- and Bach tensors, and, then, we impose the vanishing of these conformally transformed equations.

In terms of the "unhatted" ∇_a and $C^a{}_{bcd}$, the conformal transformations of the Yang tensor Y_{abc} and the Bach tensor B_{ab} are

$$\hat{Y}_{bcd} = Y_{bcd} + \phi_a C^a{}_{bcd} = \nabla_a C^a{}_{bcd} + \phi_a C^a{}_{bcd} \quad , \tag{3.22}$$
$$\hat{B}_{ab} = e^{-2\phi}(B_{ab}) = e^{-2\phi}(\nabla^c \nabla_a C^a{}_{bcd} + \frac{1}{2} R^c{}_a C^a{}_{bcd}) \quad . \tag{3.23}$$

From theorem (2.2.1), we have that if $\hat{g}_{ab}$ is generically Einstein, then

$$\begin{cases} \hat{Y}_{abc} = 0 \\ \hat{B}_{bd} = 0 \quad . \end{cases} \tag{3.24}$$

Therefore, the second version of the conformal Einstein equations is obtained by setting the right-hand sides of Eqs (3.22) and (3.23) equal to zero:

$$\begin{cases} 0 = \nabla_a C^a{}_{bcd} + \phi_a C^a{}_{bcd} \\ 0 = \nabla^c \nabla_a C^a{}_{bcd} + \frac{1}{2} R^c{}_a C^a{}_{bcd} \quad . \end{cases} \tag{3.25}$$

3.3 THE NEW VERSION OF THE CONFORMAL EINSTEIN EQUATIONS

Recently, it was observed that one can construct the conformal Einstein equations without explicit use of the conformal factor ϕ [14]. This is done by combining the first version of the conformal Einstein equations with the conformal Yang equations from the second version, namely

$$0 = R_{ab}^{\mathrm{TF}} - 2\nabla_a \phi_b + 2\phi_a \phi_b + \frac{1}{2} g_{ab} \left(\nabla^c \phi_c - \phi^c \phi_c \right) \quad , \tag{3.26}$$

$$0 = \nabla_a C^a{}_{bcd} + \phi_a C^a{}_{bcd} \quad . \tag{3.27}$$

The idea is as follows. First, one somehow solves the conformal Yang equation for the components of the ϕ_a gradient, which determines them as functions of the Weyl tensor. Since the Weyl tensor is itself a function of the metric, the ϕ_a are then functions of the metric, $\phi_a = K_a[g_{ab}]$. The next step is to replace the ϕ_a of Eq (3.26) with the K_a, thereby obtaining equations involving only g_{ab} and its derivatives.

The difficulty with this method lies in how one actually solves *all* the conformal Yang equations for ϕ_a. To illustrate this point, we present two different methods for doing so. The first method seems elegant at first, but it then becomes fairly messy. Worse, it has been incorrectly used by a few authors [18], [14] to obtain a set of ϕ_a that did not satisfy all of the conformal Yang equations and was not a gradient. The second approach is simply straight-forward algebra and is manifestly correct by construction. Both of the approaches will depend on certain properties of the conformal Yang equations, which we now discuss.

3.3.1 Some Properties of the Conformal Yang Equations

In this subsection, we show how one counts the components of the Yang tensor. We then follow with two theorems about the uniqueness- and gradient properties of any vector V_a that satisfies the equation

$$\nabla_a C^a{}_{bcd} + V_a C^a{}_{bcd} = 0 \quad . \tag{3.28}$$

A. Components of the Yang Tensor

The Yang tensor $Y_{bcd} = \nabla_a C^a{}_{bcd}$ has 16 independent components, as the following demonstrates:

First, from the skew-symmetries of the Weyl tensor, it follows that

$$Y_{bcd} = Y_{b[cd]} \quad . \tag{3.29}$$

Since we are in four dimensions, the skew-index pair $[cd]$ has six components, and the remaining index b has four. Thus, at most the Yang tensor has $6 \times 4 = 24$ components. Next, from the symmetry $C_{a[bcd]} = 0$, we have

$$Y_{[bcd]} = 0 \quad . \tag{3.30}$$

This gives rise to four equations – from the non-trivial skew-index sets [012], [013], [023], and [123] – which constrain four components of Y_{bcd} to be linear combinations of the other 20. Finally, the Yang tensor also inherits the trace-free property of the Weyl tensor,

$$g^{bd} Y_{bcd} = 0 \quad , \tag{3.31}$$

which, from the four choices of c, yields four additional constraint equations for four components. Therefore, the Yang tensor has 16 independent components and eight dependent components, which are linear combinations of the other 16.

Using the Eqs (3.30) and (3.31) we can determine, in a non-canonical fashion, the eight dependent components of the Yang tensor. The following table represents one possible choice of these dependent components. In the table, we list the 24 values of the Yang tensor's index set $\{bcd\}$. The index values that represent the eight dependent components of Y_{bcd} are crossed-out.

001	101	201	301
002	~~102~~	202	~~302~~
003	~~103~~	~~203~~	303
~~012~~	112	212	~~312~~
~~013~~	113	~~213~~	313
023	123	223	323

Table 3.1: Index values of Y_{bcd}. Slashed entries represent dependent components.

B. Theorems on the Conformal Yang Equations

We now present two important theorems about vectors V_a that satisfy the 16 equations

$$\nabla_a C^a{}_{bcd} + V_a C^a{}_{bcd} = 0 \quad . \tag{3.32}$$

These equations have the same form as the conformal Yang equations, but there is an important difference: the conformal Yang equations are defined with a gradient $\phi_a = \nabla_a \phi$, while the above equations are defined with an arbitrary vector V_a. The following theorems, however, prove that if there exists a vector V_a that satisfies the above equations, then, with the assumption that the metric is generic, the vector is unique and is a gradient. Therefore, given a generic metric, the above equations *are* *identical* to the conformal Yang equations.

Theorem 3.3.1 (Kozameh, *et al.* [18]). *Given a generic metric, a vector V_a that satisfies the 16 equations*

$$\nabla_a C^a{}_{bcd} + V_a C^a{}_{bcd} = 0 \tag{3.33}$$

is necessarily a gradient.

Proof. From the contracted Bianchi identities and the properties of the Riemann tensor, it is straight-forward to show that

$$\nabla^b \nabla_a C^a{}_{bcd} = 0 \quad , \tag{3.34}$$

where ∇_a is a torsion-free connection. Thus, taking ∇^b of Eq (3.33) and rearranging indices yields

$$\nabla^{[b} V^{a]} C_{abcd} = 0 \quad , \tag{3.35}$$

where we have used the fact that $C_{abcd} = C_{[ab][cd]}$. Since we have assumed a generic metric, we have, from Eq (2.17),

$$\nabla^{[b} V^{a]} = 0 \quad . \tag{3.36}$$

Therefore, because ∇_a is torsion-free, we have that V_a must be the gradient $\nabla_a f$ of some smooth function f. □

Theorem 3.3.2 (Gover, *et al.* [14]). *The vector V_a that satisfies Eq (3.33) is unique.*

Proof. The proof is by contradiction. Recall that for a generic metric,

$$V_a C^a{}_{bcd} = 0 \quad \Rightarrow \quad V_a = 0 \quad . \tag{3.37}$$

Now, suppose that we have two different vectors V_a and $\hat{V}_a$ that satisfy Eq (3.33):

$$0 = \nabla_a C^a{}_{bcd} + V_a C^a{}_{bcd} \quad , \tag{3.38}$$

$$0 = \nabla_a C^a{}_{bcd} + \hat{V}_a C^a{}_{bcd} \quad . \tag{3.39}$$

Subtracting the bottom equation from the top yields

$$(V_a - \hat{V}_a) C^a{}_{bcd} \quad . \tag{3.40}$$

Since the metric is generic,

$$V_a - \hat{V}_a = 0 \quad , \tag{3.41}$$

and, therefore, the vectors V_a and $\hat{V}_a$ are identical.

□

Therefore, given a generic metric, the equations

$$\nabla_a C^a{}_{bcd} + V_a C^a{}_{bcd} = 0 \tag{3.42}$$

are identical to the conformal Yang equations,

$$\hat{Y}_{bcd} = \nabla_a C^a{}_{bcd} + \phi_a C^a{}_{bcd} = 0 \quad , \tag{3.43}$$

where the vector $V_a = \nabla_a \phi$ is unique.

Equipped with these properties, we are now in a position to solve the conformal Yang equations for the gradient ϕ_a using our two approaches.

3.3.2 The First Approach for Obtaining the Gradient ϕ_a

The first method of obtaining ϕ_a from the conformal Yang equations,

$$\nabla_a C^a{}_{bcd} + \phi_a C^a{}_{bcd} = 0 \quad , \tag{3.44}$$

uses the following identities [18]:

$$C^a{}_{bcd} C_e{}^{bcd} = \frac{1}{4}\delta^a_e C^2 \quad , \tag{3.45}$$

$$C^a{}_{bcd} C^{*\,bcd}_e = \frac{1}{4}\delta^a_e C^* C \quad , \tag{3.46}$$

$$C^a{}_{bcd} C^{cd}{}_{ij} C_e{}^{bij} = \frac{1}{4}\delta^a_e C^3 \quad , \tag{3.47}$$

$$-C^a{}_{bcd} C^{cd}{}_{ij} C^{*\,bij}_e = \frac{1}{4}\delta^a_e C^{*\,3} \quad , \tag{3.48}$$

where the scalars

$$C^2 = C^a{}_{bcd} C_a{}^{bcd} \quad , \tag{3.49}$$

$$C^* C = C^{*\,a}{}_{bcd} C_a{}^{bcd} \quad , \tag{3.50}$$

$$C^3 = C^a{}_{bcd} C^{cd}{}_{ij} C_a{}^{bij} \quad , \tag{3.51}$$

$$C^{*\,3} = C^{*\,a}{}_{bcd} C^{*\,cd}{}_{ij} C^{*\,bij}_e \quad , \tag{3.52}$$

are the real invariants of the Weyl tensor [19] and C^*_{abcd} is the dual to the Weyl tensor,

$$C^*_{abcd} = \frac{1}{2}\epsilon_{ab}{}^{ij} C_{ijcd} \quad . \tag{3.53}$$

To solve for ϕ_a, one can manipulate Eq (3.44) so that ϕ_a is not contracted. As an example, one can multiply by $C_e{}^{bcd}$ and use Eq (3.45) to obtain

$$0 = C_e{}^{bcd} \nabla_a C^a{}_{bcd} + \frac{1}{4}\phi_e C^2 \quad . \tag{3.54}$$

Assuming $C^2 \neq 0$ one can now solve for ϕ_e:

$$\phi_e = K_e = -\frac{4}{C^2} C_e{}^{bcd} \nabla_a C^a{}_{bcd} \quad . \tag{3.55}$$

Similarly, one can use the other three identities to obtain

$$K_e = -\frac{4}{C^2} C^{*\,bcd}_e \nabla_a C^a{}_{bcd} \quad , \tag{3.56}$$

$$K_e = -\frac{4}{C^2} C^{cd}{}_{ij} C_e{}^{bij} \nabla_a C^a{}_{bcd} \quad , \tag{3.57}$$

$$K_e = \frac{4}{C^2} C^{cd}{}_{ij} C^{*\,bij}_e \nabla_a C^a{}_{bcd} \quad . \tag{3.58}$$

At this point, we have four versions of solutions for K_a. So the first complication we encounter is the lack of uniqueness. Furthermore, none of these versions actually solve all 16 of the conformal Yang equations! Instead, each K_a is a solution to linear combinations of equations. One can overcome these complications by demanding that the four sets of K_a were all the same, which puts conditions the Weyl tensor and its divergence. In principle, this is a valid approach for finding a unique K_a, but in practice it is unnecessarily tedious. The explicit expressions of the scalars C^2, *etc.* are very long to calculate, and then many of the terms associated with these scalars cancel anyway. The end result can be obtained in a more direct manner, as we see below.

Before we move to the next approach for obtaining K_a, however, we note that previous authors ([18], [14]), missed the fact that each of the above versions of K_a by itself does not satisfy all 16 of the conformal Yang equations unless one further imposes that each of the versions are equal to one another. This observation is new and was recently made by us.

3.3.3 The Second Approach for Obtaining the Gradient ϕ_a

The following work is new.

In the second method of obtaining the gradients ϕ_a, we do not use the scalar invariants C^2, *etc.* Instead, we isolate four of the 16 equations $\nabla_a C^a{}_{bcd} + K_a C^a{}_{bcd} = 0$ and solve them for the four components of K_a. Then, we impose the 12 conditions that these components of K_a satisfy the remaining 12 equations. In this way, the vector $K_a = \phi_a$ is uniquely determined.

Note that there is some arbitrariness in how one chooses the four starting equations. The final result, however, is independent of this choice. An example of a particular choice is given now.

Using table 3.1, which lists the independent components of $\hat{Y}_{bcd} = 0$, we choose the four equations

$$0 = \nabla_a C^a{}_{001} + K_a C^a{}_{001} \quad , \tag{3.59}$$

$$0 = \nabla_a C^a{}_{101} + K_a C^a{}_{101} \quad , \tag{3.60}$$

$$0 = \nabla_a C^a{}_{201} + K_a C^a{}_{201} \quad , \tag{3.61}$$

$$0 = \nabla_a C^a{}_{301} + K_a C^a{}_{301} \quad , \tag{3.62}$$

which correspond to the top row of the table. From these equations, the four components of K_a are determined as

$$\begin{aligned}
HK_0 &= +C_{0102}\nabla_a C^a{}_{301} - C_{0103}\nabla_a C^a{}_{201} + C_{0123}\nabla_a C^a{}_{001} \quad , \\
HK_1 &= -C_{0112}\nabla_a C^a{}_{301} + C_{0113}\nabla_a C^a{}_{201} - C_{0123}\nabla_a C^a{}_{101} \quad , \\
HK_2 &= -C_{0101}\nabla_a C^a{}_{201} + C_{0102}\nabla_a C^a{}_{101} - C_{0112}\nabla_a C^a{}_{001} \quad , \\
HK_3 &= +C_{0101}\nabla_a C^a{}_{301} - C_{0103}\nabla_a C^a{}_{101} + C_{0113}\nabla_a C^a{}_{001} \quad ,
\end{aligned} \tag{3.63}$$

where the common scalar H is defined as

$$H \equiv C_{0101}C_{0123} - C_{0102}C_{0113} + C_{0103}C_{0112} \quad . \tag{3.64}$$

From their construction, there is no indication that these components of K_a necessarily satisfy the other 12 conformal Yang equations. In fact, these components are not necessarily even gradients! Thus, we must impose the conditions that these components satisfy the other 12 equations. We refer to these 12 conditions as $J_\sigma = 0$, where the label σ runs from 1 to 12. We choose the J_σ by simply reading along the columns of table 3.3.1 while ignoring the dependent entries and the top row, which was used to choose our starting equations (3.59) - (3.62):

$$\begin{aligned}
J_1 &\equiv \nabla_a C^a{}_{002} + K_a C^a{}_{002} \quad , \\
J_2 &\equiv \nabla_a C^a{}_{003} + K_a C^a{}_{003} \quad , \\
J_3 &\equiv \nabla_a C^a{}_{023} + K_a C^a{}_{023} \quad , \\
J_4 &\equiv \nabla_a C^a{}_{112} + K_a C^a{}_{112} \quad , \\
&. \\
&. \\
&.
\end{aligned} \tag{3.65}$$

With the requirement that the 12 J_σ vanish, the components of K_a given above satisfy all 16 of the conformal Yang equations. Thus, by the previous theorems, the vector K_a is a unique gradient vector for some smooth function ϕ, which we take to be our conformal factor. We can now state the last version of the conformal Einstein equations.

The third – and new – version of the conformal Einstein equations is

$$\begin{cases} 0 = R_{ab}^{\mathrm{TF}} - 2\nabla_a K_b + 2K_a K_b + \frac{1}{2} g_{ab} \left(\nabla^c K_c - K^c K_c\right) \\ 0 = J_\sigma \quad , \end{cases} \tag{3.66}$$

where the gradient vector K_a is defined as

$$\begin{aligned} HK_0 &= +C_{0102}\nabla_a C^a{}_{301} - C_{0103}\nabla_a C^a{}_{201} + C_{0123}\nabla_a C^a{}_{001} \quad , \\ HK_1 &= -C_{0112}\nabla_a C^a{}_{301} + C_{0113}\nabla_a C^a{}_{201} - C_{0123}\nabla_a C^a{}_{101} \quad , \\ HK_2 &= -C_{0101}\nabla_a C^a{}_{201} + C_{0102}\nabla_a C^a{}_{101} - C_{0112}\nabla_a C^a{}_{001} \quad , \\ HK_3 &= +C_{0101}\nabla_a C^a{}_{301} - C_{0103}\nabla_a C^a{}_{101} + C_{0113}\nabla_a C^a{}_{001} \quad , \\ H &= C_{0101}C_{0123} - C_{0102}C_{0113} + C_{0103}C_{0112} \quad . \end{aligned} \tag{3.67}$$

3.3.4 The Next Step

We will now change topics to discuss the conformal geometry of a pair of PDE's. For the purpose of clarity, we will begin with a review of the null-surface formulation of GR, in which the relevant PDE's were first discovered. After fully analysing the geometry of the PDE's in chapter 5, we will return to the new version of the conformal Einstein equations in chapter 6 in order to apply them to the PDE-geometry.

4.0 THE NULL-SURFACE FORMULATION

The purpose of this chapter is to briefly review the null-surface formulation of general relativity (NSF). In particular, we review the construction of a conformal metric in terms of null surfaces. Since this construction has been extensively discussed elsewhere [16] - [17], [6] - [10], we will not give the full details of the procedure. Instead, we are interested in highlighting the properties of this construction. One important property is the existence of a special class of differential equations, the *Wünschmann class* [23], which plays a fundamental role in the following chapters. Another purpose of the chapter is to introduce notation that will be used throughout the rest of this work.

In the standard formulation of general relativity, one treats the components of the metric as the basic variables of the theory. In the NSF, however, the metric is a derived concept, and the basic variables are families of null surfaces and a scalar function:

$$u = \text{constant} = Z\,(x^a\,,s\,,s^*) \quad , \quad \Omega\,(x^a\,,s\,,s^*) \quad . \tag{4.1}$$

The properties of these variables are as follows:

i.) The scalar function Ω is a conformal factor such that

$$g^{ab} = \Omega^2 \hat{g}^{ab} \quad , \tag{4.2}$$

for a pair of Lorentzian metrics g^{ab} and $\hat{g}^{ab}$ on the space-time M.

ii.) For each fixed value of the pair $(s\,,s^*)$, the level surfaces of the function Z in M,

$$u = \text{constant} = Z\,(x^a\,,s\,,s^*) \quad , \tag{4.3}$$

form, for different values of u, a 1-parameter family of surfaces which foliates a local region of M. The geometric meaning of the pair $(s\,,s^*)$ is, in general, arbitrary. Here, we take it to be the complex-conjugate pair of stereographic coordinates on the 2-sphere. Thus, by letting $(s\,,s^*)$ vary for a given Z, we have a collection of a sphere's worth of families of surfaces through every point in the space-time M.

The requirement that the collection of surfaces $Z\,(x^a\,,s\,,s^*) = \text{constant}$ are null for some conformal metric $g^{ab}(x^a)$ is that they satisfy the *eikonal equation*

$$g^{ab}\,(x^c)\,[\nabla_a Z][\nabla_b Z] = 0 \tag{4.4}$$

for *all* values of $(s\,,s^*)$. In the standard treatment of general relativity, one assumes the metric is known and then attempts to find the null surfaces from the eikonal equation. As an example, consider null plane waves in Minkowski space-time. For fixed values of $(s\,,s^*)$, the parameter u describes all plane waves in a particular direction. Varying $(s\,,s^*)$ yields all plane waves in all directions.

The NSF poses the converse problem: given a function $Z\,(x^a\,,s\,,s^*)$ that describes null surfaces for some *unknown* conformal metric, what conditions does the eikonal equation impose on Z, and how does one use the eikonal equation and these conditions to determine the unknown conformal metric. The procedure for solving this problem essentially relies on taking successive derivatives of the eikonal equation with respect to s and s^*. One important result of this procedure is that the functions Z must satisfy a pair of second-order partial-differential equations (PDE's), where the equations belong to a what is known as the Wünschmann class. These equations and their solutions give rise to a rich geometry, which we will partially explore in the next chapter.

This chapter will have the following organisation: In section 4.1, we explain our notation and define several important functions. Some of these functions are taken as a new set of coordinates on the space-time. Another set of functions will define the pair of PDE's that belong to the Wünschmann class. Section 4.2 will have two parts: In the first, we review the important properties that are obtained from constructing a conformal metric from the eikonal equation. In the second, we outline the actual procedure for constructing this metric.

4.1 A NEW COORDINATE SYSTEM

Before defining our new coordinate system, it is necessary to first describe our notation. We begin with a 6-dimensional fibre-bundle $M \times S^2$ with coordinates $(x^a \,, s \,, s^*)$. The x^a are the coordinates of the base space M (space-time), and the parameters $(s \,, s^*)$ are coordinates on the fibres. For an arbitrary function $f\,(x^a \,, s \,, s^*)$, we denote the x^a-derivatives by

$$\frac{\partial f}{\partial x^a} \equiv \partial_a f \equiv f_{,a} = f_a \tag{4.5}$$

and the s-derivative by

$$\frac{\partial f}{\partial s} \equiv Df \equiv f_{,s} = f_s \quad . \tag{4.6}$$

The s^*-derivative is similarly defined. When an object is clearly a derivative, we will omit the comma before the derivative label.

Next, from the function $Z\,(x^a\,,s\,,s^*)$, we define three new scalar functions,

$$W\,(x^a\,,s\,,s^*) \equiv DZ \quad , \tag{4.7}$$

$$W^*\,(x^a\,,s\,,s^*) \equiv D^*Z \quad , \tag{4.8}$$

$$R\,(x^a\,,s\,,s^*) \equiv DD^*Z = DW^* = D^*W \quad . \tag{4.9}$$

With the assumption that the Z is sufficiently generic, these four functions are used to define an $(s\,,s^*)$-dependent coordinate transformation on the space-time:

$$\begin{aligned} Z &= Z\,(x^a\,,s\,,s^*) \quad , \quad R = R\,(x^a\,,s\,,s^*) \quad , \\ W &= W\,(x^a\,,s\,,s^*) \quad , \quad W^* = W^*\,(x^a\,,s\,,s^*) \quad . \end{aligned} \tag{4.10}$$

By solving these equations, the x^a are determined as

$$x^a = X^a(Z\,,R\,,W\,,W^*\,,s\,,s^*) \quad . \tag{4.11}$$

From the gradients of the $\{Z\,,R\,,W\,,W^*\}$ coordinates, we construct the *gradient basis* $\beta^i{}_a$:

$$\{\beta^0{}_a\,,\beta^1{}_a\,,\beta^2{}_a\,,\beta^3{}_a\} \equiv \{Z_a\,,R_a\,,W_a\,,W^*{}_a\} \quad . \tag{4.12}$$

(Note that the indices $(0\,,1)$ refer to real objects, but the $(2\,,3)$ refer to a complex-conjugate pair.) The dual basis $\beta_i{}^a$ is defined by

$$\beta_j{}^a\beta^i{}_a = \delta^i_j \quad , \quad \beta_i{}^a\beta^i{}_b = \delta^a_b \quad . \tag{4.13}$$

In the next section, we will use the eikonal equation to construct a metric in terms of this gradient-basis.

From parametric derivatives of the variables in Eq (4.10), we obtain an important set of complex functions, $(S\,,S^*)$. To find them, begin with

$$\begin{aligned} D\ W\ \ (x^a\,,s\,,s^*) &= D^2\ Z\,(x^a\,,s\,,s^*) \equiv \Sigma\ \ (x^a\,,s\,,s^*) \quad , \\ D^*W^*\,(x^a\,,s\,,s^*) &= D^{*2}Z\,(x^a\,,s\,,s^*) \equiv \Sigma^*\,(x^a\,,s\,,s^*) \quad . \end{aligned} \tag{4.14}$$

Next, using the coordinate transformation $x^a = X^a(Z\,,R\,,W\,,W^*\,,s\,,s^*)$ to eliminate the x^a-dependence of the Σ's yields

$$\begin{aligned} D^2\ Z &= S\ \ (Z\,,Z_s\,,Z_{s^*}\,,Z_{ss^*}\,,s\,,s^*) \quad , \\ D^{*2}Z &= S^*\,(Z\,,Z_s\,,Z_{s^*}\,,Z_{ss^*}\,,s\,,s^*) \quad . \end{aligned} \tag{4.15}$$

We interpret these expressions as defining a pair of PDE's for the function $Z\,(s\,,s^*)$. The solutions are given by $Z = Z\,(x^a\,,s\,,s^*)$, where the constants of integration x^a become the local space-time coordinates. Furthermore, these PDE's are members of a broad but important class of equations known as the Wünschmann class. The condition for belonging to this class, the *Wünschmann condition*, is derived from the construction of the metric and will be given explicitly in section 4.2.2.

Other useful functions are the complex-conjugate pair $(T\,,T^*)$ and the real function U, which are obtained by

$$T \equiv D^*S = DR \quad , \quad T^* \equiv DS^* = D^*R \quad , \tag{4.16}$$

$$U \equiv D^*T = DT^* \quad . \tag{4.17}$$

Another useful set of derivatives are the directional derivatives

$$\partial_i = \beta_i{}^a\partial_a \quad , \tag{4.18}$$

which are the $\{Z\,, R\,, W\,, W^*\}$-coordinate derivatives,

$$\{\partial_i\} = \{\partial_0\,, \partial_1\,, \partial_2\,, \partial_3\} \equiv \left\{\frac{\partial}{\partial Z}\,, \frac{\partial}{\partial R}\,, \frac{\partial}{\partial W}\,, \frac{\partial}{\partial W^*}\right\} \quad . \tag{4.19}$$

The derivatives ∂_a and ∂_i satisfy

$$\partial_a f = \beta^i{}_a \partial_i f \quad , \quad \partial_i f = \beta_i{}^a \partial_a \quad . \tag{4.20}$$

In particular, for derivatives of the function S defined in Eq (4.15),

$$S_a = \beta^i{}_a S_i \quad , \quad S_i = \beta_i{}^a S_a \quad . \tag{4.21}$$

Since the coordinates $\{Z\,, R\,, W\,, W^*\}$ have $(s\,, s^*)$-dependence, the ∂_i will not, in general, commute with D and D^*. For any scalar function f, the commutator of these derivatives is given by

$$D(f_{,j}) = (Df)_{,j} - (S_{,j}f_{,2} + T_{,j}f_{,1} + \delta_{j2}f_{,0} + \delta_{j1}f_{,3}) \quad , \tag{4.22}$$

where the commutator for $D^*(f_{,j})$ is similarly defined.

4.2 THE CONSTRUCTION OF A CONFORMAL METRIC

For the construction of the conformal metric, we note that it is easier to work with the contravariant components in the gradient basis (*i.e.*, g^{ij}) rather than the coordinate basis (*i.e.*, g^{ab}). The components of the metric are then initially found in terms of Z and its derivatives.

4.2.1 Properties of the Construction

In the construction, we use successive $(s\,,s^*)$-derivatives of the eikonal equation to determine the components of the metric in the gradient basis. Given a metric $g^{ab}(x^c)$ in a space-time coordinate-basis, it is written in the gradient basis as

$$g^{ij}(Z\,,R\,,W\,,W^*\,,s\,,s^*) = g^{ab}(x^c)\beta^i{}_a\beta^j{}_b \quad , \tag{4.23}$$

where the $\beta^i{}_a$ are defined in (4.12). Letting the indices $(i\,,j)$ take all possible values yields the components of g^{ij} as

$$\begin{gathered} g^{00} = g^{ab}Z_aZ_b \quad , \quad g^{11} = g^{ab}R_aR_b \quad , \\ g^{22} = g^{ab}W_aW_b \quad , \quad g^{33} = g^{ab}W^*_aW^*_b \quad , \\ g^{01} = g^{ab}Z_aR_b \quad , \quad g^{02} = g^{ab}Z_aW_b \quad , \quad g^{03} = g^{ab}Z_aW^*_b \quad , \\ g^{12} = g^{ab}R_aW_b \quad , \quad g^{13} = g^{ab}R_aW^*_b \quad , \\ g^{23} = g^{ab}W_aW^*_b \quad . \end{gathered} \tag{4.24}$$

In the next section, we show how the g^{ij} can be constructed in the gradient basis (up to a conformal scale) from the eikonal equation. Assuming for the moment that the g^{ij} are known, the $g^{ab}(x^c)$ can be determined by the inverse transformation

$$g^{ab}(x^c) = g^{ij}\beta_i{}^a\beta_j{}^b \quad . \tag{4.25}$$

It is important to note that both g^{ij} and $\beta^i{}_a$ depend on the parameters $(s\,,s^*)$, but the g^{ab} depends only on x^a.

From the construction of the g^{ij}, one can only determine the ten components up to an overall scale. One can take the component g^{01} as undetermined, and the other components are scaled by it. Letting $g^{01} \equiv \Omega^2(x^a\,,s\,,s^*)$, we have that

$$g^{ij} = \Omega^2(x^a\,,s\,,s^*)\hat{g}^{ij} \quad , \tag{4.26}$$

where $\hat{g}^{ij}$ is uniquely determined by setting $g^{01} = 1$. The construction also places differential conditions on the $\Omega^2(x^a\,, s\,, s^*)$ which determine its $(s\,, s^*)$-dependence with an arbitrary multiplicative factor depending only on x^a, $i.e$,

$$\Omega^2(x^a\,, s\,, s^*) = \omega^2(x^a) F[S\,, S^*] \quad , \tag{4.27}$$

where $\omega(x^a)$ depends only on x^a, and the functional $F[S\,, S^*]$ is the determined $(s\,, s^*)$-dependent part of Ω^2. Therefore, using Eqs (4.25) - (4.27), we see that we have a pair of conformally-related metrics,

$$g^{ab}(x^c) = \omega^2(x^a) \hat{g}^{ab}(x^c) \quad , \tag{4.28}$$

where

$$\hat{g}^{ab}(x^c) \equiv F[S\,, S^*]\, \hat{g}^{ij} \beta_i{}^a \beta_j{}^b \quad . \tag{4.29}$$

Since the $F[S\,, S^*]$ and the components of $\hat{g}^{ij}$ are determined by the construction, all of the components of the $\hat{g}^{ab}(x^a)$ can be determined by the above equation. The $\omega(x^a)$ remains undetermined.

As was mentioned earlier, the functions S and S^* of Eq (4.15) are forced by the construction to satisfy the Wünschmann condition. We thus have the following important result: For every choice of the function $Z = Z(x^a\,, s\,, s^*)$ that satisfies a pair of PDE's in the Wünschmann class,

$$D^2 Z = S \quad , \quad D^{*2} Z = S^* \quad , \tag{4.30}$$

we can construct, via the eikonal equation and Eq (4.29), a conformal metric $\hat{g}^{ab}(x^c)$ that is solely a function of the x^a. Once the $\hat{g}^{ab}(x^c)$ is determined, we have a conformal class of metrics $g^{ab} = \omega^2 \hat{g}^{ab}$ that is characterised by the arbitrary scalar function $\omega(x^a)$. Thus, we can calculate a conformally-related pair of Ricci tensors, namely

$\hat{R}_{ab}$ and R_{ab} (see chapter 3). In general, $\hat{R}_{ab}$ will not satisfy the (vacuum) Einstein equations. Therefore, we are interested in finding a conformal transformation $\hat{g}^{ab} \rightarrow g^{ab}$ such that the Ricci tensor R_{ab} does satisfy the Einstein equations. Said another way, we want to find a metric $\hat{g}^{ab}$ that is conformally Einstein. Since this metric is constructed from $Z = Z(x^a\,, s\,, s^*)$, which is determined by Eq (4.30), it follows that the conditions for $\hat{g}^{ab}$ to be conformally Einstein can be interpreted as further conditions on S and S^*. We will discuss the problem of determining these conditions in chapter 6. We conclude with a brief review of the procedure for constructing g^{ij} from the eikonal equation.

4.2.2 The Procedure of Constructing g^{ij}

From the eikonal equation, the first component in Eq (4.24) becomes

$$g^{00} = g^{ab} Z_a Z_b = 0 \quad . \tag{4.31}$$

Thus, we have already determined one component of g^{ij} to be zero. To determine the other eight components of g^{ij}, we take successive D and D^* derivatives of the eikonal equation, where we assume that $g^{ab}\,(x^c)$ is independent of the parameters $(s\,, s^*)$:

$$D[g^{ab}(x^c)] = 0 \quad . \tag{4.32}$$

Below, we explicitly construct the components $\{g^{02}\,, g^{03}\,, g^{22}\,, g^{33}\,, g^{23}\}$, which sufficiently clarifies how the procedure works. The remaining components, namely $\{g^{12}\,, g^{13}\,, g^{11}\}$, are then stated without the supporting details. All of the components will be proportional to g^{01}, which we take to be a conformal factor:

$$g^{01} = \Omega^2 \quad . \tag{4.33}$$

Taking D and D^* of Eq (4.31) and using Eq (4.24) determines g^{02} and g^{03} to both be zero:

$$D\ g^{00} = 2g^{ab}Z_aW_b\ = 2g^{02} = 0 \quad \Rightarrow \quad g^{02} = 0 \quad , \tag{4.34}$$

$$D^*g^{00} = 2g^{ab}Z_aW_b^* = 2g^{03} = 0 \quad \Rightarrow \quad g^{03} = 0 \quad . \tag{4.35}$$

From D of $g^{02} = 0$, we obtain g^{22}:

$$0 = Dg^{02} = g^{ab}(W_aW_b + Z_aS_b) = g^{22} + S_1g^{01} \tag{4.36}$$

$$\Rightarrow \quad g^{22} = -S_1g^{01} \quad , \tag{4.37}$$

where we have used Eqs (4.21) and (4.24) along with $g^{00} = g^{02} = g^{03} = 0$. Similarly, $D^*g^{03} = 0$ yields

$$g^{33} = -S_1^*g^{01} \quad . \tag{4.38}$$

In an analogous fashion, we obtain g^{23} from $D^*g^{02} = 0$:

$$0 = D^*g^{02} = g^{ab}(W_aW_b^* + Z_aR_b) = g^{23} + g^{01} \tag{4.39}$$

$$\Rightarrow \quad g^{23} = -g^{01} \quad . \tag{4.40}$$

Taking $Dg^{03} = 0$ yields the same result.

Another order of parametric derivatives, along with the commutators of Eq (4.22), yields the expressions for g^{12} and g^{13} as

$$g^{12}\,[4 - S_1S_1^*] = g^{01}[S_1T_1^* - 2T_1] \quad , \tag{4.41}$$

$$g^{13}\,[4 - S_1S_1^*] = g^{01}[S_1^*T_1 - 2T_1^*] \quad . \tag{4.42}$$

From further derivatives and an application of 4.22, the g^{11} is found to satisfy

$$0 = 2g^{11}[2 + S_1S_1^*] + 4(T_1^*g^{12} + T_1g^{13}) + g^{01}[2U_1 - (S_1T_2^* + S_1^*T_3 + T_2 + T_3^*)] \quad , \tag{4.43}$$

where the g^{12} and g^{13} are given in Eqs (4.41) and (4.42).

In determining the g^{12} and g^{13}, one finds two sets of complex conditions associated with the functions S and S^*. One of them is a set of differential conditions on the conformal factor g^{01},

$$(4 - S_1 S_1^*)[D\ g^{01}] = g^{01}[2T_1 + S_1(T_1^* - S_1^* T_1)] \quad , \tag{4.44}$$

$$(4 - S_1 S_1^*)[D^* g^{01}] = g^{01}[2T_1^* + S_1^*(T_1 - S_1 T_1^*)] \quad . \tag{4.45}$$

By integrating this pair, one can determine the $(s\,, s^*)$-dependence of the g^{01}. Thus, referring to Eq (4.27), we have

$$g^{01} = \Omega^2(x^a\,, s\,, s^*) = \omega^2(x^a)F[S\,, S^*] \quad , \tag{4.46}$$

where, using the definition

$$H \equiv \frac{2T_1 + S_1(T_1^* - S_1^* T_1)}{4 - S_1 S_1^*} \quad , \tag{4.47}$$

the functional F is given as

$$F[S\,, S^*] = \exp\left[\int H ds + H^* ds^*\right] \quad . \tag{4.48}$$

(The integrability conditions of Eq (4.46) are rather complicated and are discussed in [7].)

The other set of conditions is the *Wünschmann condition*,

$$D^*(S_1) = T_1 - S_1 T_1^* \quad , \tag{4.49}$$

$$D\ (S_1^*) = T_1^* - S_1^* T_1 \quad , \tag{4.50}$$

which restricts the functions S and S^* to the Wünschmann class. Thus, we see that in order to construct a conformal metric from the eikonal equation, the function Z must satisfy a pair of PDE's, Eq (4.15), which, in turn, must satisfy the Wünschmann condition.

5.0 THE GEOMETRY OF A PAIR OF 2^{ND}-ORDER PDE'S

In the previous chapter, we saw how one can use families of surfaces,

$$u = Z\,(x^a\,, s\,, s^*) \quad , \tag{5.1}$$

to construct a conformal metric from the $(s\,, s^*)$-derivatives of the eikonal equation (which makes the surfaces null). One of the main features of that construction was that the function Z had to satisfy a pair of PDE's,

$$\begin{aligned} D^2 Z &= S\,(Z\,, Z_s\,, Z_{s^*}\,, Z_{ss^*}\,, s\,, s^*) \quad , \\ D^{*2} Z &= S^*(Z\,, Z_s\,, Z_{s^*}\,, Z_{ss^*}\,, s\,, s^*) \quad , \end{aligned} \tag{5.2}$$

where the functions S and S^*, in turn, satisfied the Wünschmann condition.

The purpose of the present chapter is to explore the conformal geometry associated with these PDE's. In particular, we construct a conformal connection and its associated curvatures as functionals of S and S^*. The overall goal of this approach, as we will discuss further in the next chapter, is to attempt to construct the conformal Einstein equations directly from the PDE's. In other words, we are seeking conditions on the S and S^*, in addition to the Wünschmann condition, that guarantee the existence of a conformally Einstein metric on the solution space of the PDE's.

In section 5.1, we begin with a pair of PDE's of the form of (5.2) and *without* the initial assumption of a space-time. The solutions, in general, depend on four constants of integration (the solution space) which is interpreted as space-time. Also, from the solution of these PDE's, we construct a tetrad, which will be taken as null. We then introduce an unknown but to-be-determined connection that is a functional of the S's. By requiring that this connection is torsion-free, we partially determine the connection and simultaneously derive the Wünschmann condition. Thus, the vanishing of the torsion guarantees the existence of a conformal metric on the solution space of the PDE's.

Next, in section 5.2, we use the formalism of Cartan [1], [2], [11], [12] to construct several curvatures that depend on S and S^*. By construction, the connection and its curvatures represent a Cartan-Weyl conformal geometry that follows directly from the PDE's (5.2). Our goal in the next chapter will be to then analyse the conditions for these curvatures to satisfy the conformal Einstein equation.

Since many of the details of the calculations in sections 5.1 and 5.2 are lengthy, we will put them at the end of the chapter in section 5.3.

Many of the results presented below were recently found by us and have been published [13]. We note, however, that some of the results of section 5.1 were originally found earlier using the NSF [16] - [17], [6] - [10], but were re-obtained here using different techniques (namely the Cartan construction) and in a totally new context. The results of section 5.2 are new.

5.1 THE NULL BASIS AND ITS CONNECTION

On a 2-dimensional space with coordinates $(s\,,s^*)$, we consider the following pair of PDE's

$$Z_{s\,s} \;= S\,(Z\,,Z_s\,,Z_{s^*}\,,Z_{ss^*}\,,s\,,s^*) \quad , \tag{5.3}$$

$$Z_{s^*s^*} = S^*(Z\,,Z_s\,,Z_{s^*}\,,Z_{ss^*}\,,s\,,s^*) \quad , \tag{5.3*}$$

where the subscripts denote partial derivatives, and $Z = Z(s,\, s^*)$ is a real function. We assume that these PDE's obey the necessary integrability conditions. Though it is possible to treat $(s\,,s^*)$ as a pair of real variables, it is more useful to consider them as a complex-conjugate pair, namely the complex stereographic coordinates. In this case, the second PDE is simply the complex-conjugate of the first equation.

Remark 5.1.1. In the following, $(^*)$ will denote the complex-conjugate. When discussing a conjugate-pair of equations, we will label each equation with the same number, but the number of the second equation will contain a $(^*)$. Occasionally, we will explicitly write only one equation of a conjugate-pair and imply the other.

To simplify notation, we define the functions W, W^*, and R as

$$W \equiv Z_s \quad , \quad W^* \equiv Z_{s^*} \quad , \quad R \equiv Z_{ss^*} \quad . \tag{5.4}$$

For an arbitrary function $H = H(Z\,,W\,,W^*\,,R\,,s\,,s^*)$, the total derivatives in s and s^* are

$$\frac{dH}{ds} \equiv D\;\,H \equiv H_s\;\, + W\;\,H_Z + S\,H_W + R\;H_{W^*} + T\;\,H_R \quad , \tag{5.5}$$

$$\frac{dH}{ds^*} \equiv D^*H \equiv H_{s^*} + W^*H_Z + RH_W + S^*H_{W^*} + T^*H_R \quad , \tag{5.5*}$$

where

$$T \;= D^* S \quad , \tag{5.6}$$

$$T^* = D \;\; S^* \quad . \tag{5.6*}$$

The T and T^* can be expressed as explicit functionals of the S and S^*. Letting $H = S^*$ in Eq (5.5) and $H = S$ in Eq (5.5*), we get two equations that each contain T and T^*. From them, we find algebraically independent expressions for T and T^*:

$$T = \frac{S_{s^*} + W^* S_Z + R S_W + S^* S_{W^*} + S_R (S^*_s + W S^*_Z + S S^*_W + R S^*_{W^*})}{1 - S_R S^*_R} \quad . \tag{5.7}$$

With the above definitions of D and D^*, the integrability condition of (5.3) is

$$D^2 S^* = D^{*2} S \quad . \tag{5.8}$$

Note that D and D^* commute with each other but not with the Z-, W-, W^*-, and R-derivatives. Instead, for $H = H(Z\,, W\,, W^*\,, R\,, s\,, s^*)$ and $y \in \{Z\,, W\,, W^*\,, R\}$,

$$D\;(H_y) = (D\;\, H)_{,y} - (S_y H_W \;\; + T_y H_R + \delta_{W,y} \;\; H_Z + \delta_{R,y} H_{W^*}) \quad , \tag{5.9}$$

$$D^*(H_y) = (D^* H)_{,y} - (S^*_y H_{W^*} + T^*_y H_R + \delta_{W^*,y} H_Z + \delta_{R,y} H_W \;\;) \quad , \tag{5.9*}$$

where $\delta_{y',y}$ is the Kronecker symbol.

Remark 5.1.2. The D and D^* are actually the basis vectors e_s and e_{s^*}, respectively, of the 6-dimensional space $(Z\,, W\,, W^*\,, R\,, s\,, s^*)$:

$$e_s \;\equiv D \;\; = \frac{d}{ds} \;\; = \frac{\partial}{\partial s} \;\; + W \; \frac{\partial}{\partial Z} + S \frac{\partial}{\partial W} + R \; \frac{\partial}{\partial W^*} + T \; \frac{\partial}{\partial R} \quad , \tag{5.10}$$

$$e_{s^*} \equiv D^* = \frac{d}{ds^*} = \frac{\partial}{\partial s^*} + W^* \frac{\partial}{\partial Z} + R \frac{\partial}{\partial W} + S^* \frac{\partial}{\partial W^*} + T^* \frac{\partial}{\partial R} \quad . \tag{5.10*}$$

In addition to the integrability condition, Eq (5.8), we assume that the functions S and S^* satisfy the inequality

$$1 - S_R S_R^* > 0 \quad . \tag{5.11}$$

From this inequality and the Frobenius theorem, it can be shown [5] that the solution space of the PDE's M is 4-dimensional. Thus, we can write

$$\begin{aligned} Z &= Z\,(x^a\,,s\,,s^*) \quad , \quad W = W\,(x^a\,,s\,,s^*) \quad , \\ R &= R\,(x^a\,,s\,,s^*) \quad , \quad W^* = W^*(x^a\,,s\,,s^*) \quad , \end{aligned} \tag{5.12}$$

where the constants of integration x^a (the solution space) are taken as space-time coordinates. As in chapter 4, we interpret the above as defining an $(s\,,s^*)$-dependent coordinate transformation on M.

The exterior derivatives of (5.12),

$$\begin{aligned} dZ &= Z_a\; dx^a + W ds + W^* ds^* \quad , \\ dR &= R_a\; dx^a + T\; ds + T^*\; ds^* \quad , \\ dW &= W_a\, dx^a + S\; ds + R\; \; ds^* \quad , \\ dW^* &= W_a^* dx^a + R\, ds + S^*\; ds^* \quad , \end{aligned} \tag{5.13}$$

can be re-written as the *Pfaffian system* of 1-forms

$$\begin{aligned} \beta^0 &\equiv dZ \; - W ds - W^* ds^* = Z_a\; dx^a \quad , \\ \beta^1 &\equiv dR \; - T\; ds - T^*\; ds^* = R_a\, dx^a \quad , \\ \beta^2 &\equiv dW \; - S\; ds - R\; \; ds^* = W_a\, dx^a \quad , \\ \beta^3 &\equiv dW^* - R\, ds - S^*\; ds^* = W_a^* dx^a \quad . \end{aligned} \tag{5.14}$$

The vanishing of the four β^i is equivalent to the PDE's of Eqs (5.3), which motivates their definitions. (Note that β^0 and β^1 are real, but β^2 and β^3 are complex conjugates.)

The components of the Pfaffian system, $\beta^i{}_a$, are identical to the gradient basis of the previous chapter. Here, it will be useful to use another basis, θ^i, defined by

$$\begin{aligned} \theta^0 &= \beta^0 \quad , & \theta^1 &= \beta^1 + a\beta^2 + a^*\beta^3 + c\beta^0 \quad , \\ \theta^2 &= \alpha(\beta^2 + b\beta^3) \quad , & \theta^3 &= \alpha(\beta^3 + b^*\beta^2) \quad , \end{aligned} \tag{5.15}$$

along with its dual basis e_i ,

$$\begin{aligned} e_0 &= (\partial_Z - c\partial_R) \quad , & e_1 &= \partial_R \quad , \\ e_2 &= \frac{\partial_W - b^*\partial_{W^*} - (a - a^*b^*)\partial_R}{\alpha(1 - bb^*)} \quad , & e_3 &= (e_2)^* \quad . \end{aligned} \tag{5.16}$$

The set of parameters $\{\alpha\,, b\,, b^*\,, a\,, a^*\,, c\}$ are referred to as *tetrad parameters*. For now, these parameters are undetermined functionals of S and S^*. Later, after we have introduced the torsion-free connection, we will uniquely determine the tetrad parameters explicitly in terms of the S and S^*.

From the θ^i, we define a degenerate metric

$$\begin{aligned} g(Z\,, W\,, W^*\,, R\,, s\,, s^*) &= \theta^0 \otimes \theta^1 + \theta^1 \otimes \theta^0 - \theta^2 \otimes \theta^3 - \theta^3 \otimes \theta^2 \\ &= \eta_{ij}\theta^i \otimes \theta^j \quad , \end{aligned} \tag{5.17}$$

which defines the η_{ij} as

$$[\eta_{ij}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad . \tag{5.18}$$

We will use the η_{ij} to raise and lower indices.

Remark 5.1.3. At this point, we are simply defining a metric on the 6-dimensional space $\{Z\,, R\,, W\,, W^*\,, s\,, s^*\}$ that makes the θ^i basis a *null tetrad*. In the next chapter, we will see how once can use this metric, along with a particular conformal transformation, to define a conformal space-time metric on M.

Remark 5.1.4. It is possible to generalise the θ^i to include more parameters corresponding to Lorentz transformations [11], [13]. In the Cartan's equivalence problem for differential equations, these extra parameters are needed. For our purposes, however, the transformation (5.15) is sufficiently general.

By adding to the θ^i the pair of 1-forms

$$\theta^s \equiv ds \quad , \quad \theta^{s^*} \equiv ds^* \quad , \tag{5.19}$$

which are dual to the e_s and e_{s^*} of Eq (5.5), we have a basis of 1-forms on the 6-dimensional space $\{Z\,, R\,, W\,, W^*\,, s\,, s^*\}$. We will refer to θ^s and θ^{s^*} as the fibre 1-forms and the four θ^i as the space-time 1-forms. The space-time indices are denoted by lower-case *i, j, etc.,* and the general indices are denoted by upper-case *I, J, etc.*:

$$\theta^i \in \{\theta^0\,, \theta^1\,, \theta^2\,, \theta^3\} \quad , \tag{5.20}$$

$$\theta^I \in \{\theta^0\,, \theta^1\,, \theta^2\,, \theta^3\,, \theta^s\,, \theta^{s^*}\} \quad . \tag{5.21}$$

Note that, in general, a p-form with tetrad indices will have components in all six directions. For example, the 1-form $\Pi^i{}_j$ and the 2-form Υ^i will have the respective expansions

$$\Pi^i{}_j = \Pi^i{}_{jK}\theta^K = \Pi^i{}_{jk}\theta^k + \Pi^i{}_{js}\theta^s + \Pi^i{}_{js^*}\theta^{s^*} \quad , \tag{5.22}$$

and

$$
\begin{aligned}
\Upsilon^i &= \frac{1}{2}\Upsilon^i{}_{JK}\theta^J \wedge \theta^K \\
&= \frac{1}{2}\Upsilon^i{}_{jk}\theta^j \wedge \theta^k + \Upsilon^i{}_{js}\theta^j \wedge \theta^s + \Upsilon^i{}_{js^*}\theta^j \wedge \theta^{s^*} + \Upsilon^i{}_{ss^*}\theta^s \wedge \theta^{s^*} \quad .
\end{aligned} \tag{5.23}
$$

In particular, the exterior derivatives of the θ^i yield 2-forms with components $\triangle^i{}_{JK}$:

$$
d\theta^i \equiv \frac{1}{2}\triangle^i{}_{JK}\theta^J \wedge \theta^K \quad . \tag{5.24}
$$

The $\triangle^i{}_{JK}$ can be found by direct calculation and are functionals of the functions S and S^*. Their explicit expressions are given in section 5.3.

Using the $\triangle^i{}_{JK}$ defined above, we define two symmetric tensors G_{ij} and G^*_{ij} by the fibre derivatives of the metric (5.17),

$$
Dg \equiv G_{ij}\theta^i \otimes \theta^j \quad , \tag{5.25}
$$

where

$$
G_{ij} = -2\triangle_{(ij)s} \quad , \tag{5.26}
$$

and

$$
\triangle_{iJK} = \eta_{im}\triangle^m{}_{JK} \quad . \tag{5.27}
$$

5.1.1 The First Structure Equation

Following the formalism of Cartan, we will now use the 6-dimensional basis of θ^I to construct a connection and its associated curvatures. Since the θ^I depend on the functions S and S^*, the connection and its curvatures will also depend on these functions.

We begin with the construction of the connection, which will be found from Cartan's torsion-free first structure equation,

$$d\theta^i + \omega^i{}_j \wedge \theta^j = 0 \quad . \tag{5.28}$$

Our goal now is to solve this equation for the *connection* 1-*forms*, $\omega^i{}_j$. To do so, we write

$$d\theta^i = \frac{1}{2}\triangle^i{}_{JK}\theta^J \wedge \theta^K \quad , \tag{5.29}$$

$$\omega_{ij} = \omega_{ijK}\theta^K \quad , \tag{5.30}$$

$$\omega^i{}_k = \eta^{ij}\omega_{jk} \quad , \tag{5.31}$$

which defines the ω_{ijK}. The structure equation then becomes

$$\frac{1}{2}\triangle^i{}_{JK}\theta^J \wedge \theta^K + \eta^{ij}\omega_{jmL}\theta^L \wedge \theta^m = 0 \quad . \tag{5.32}$$

Since we are interested the conformal geometry contained in the structure equation, we require that the connection 1-forms be generalised Weyl connections ("generalised" because of the extra degrees of freedom in the fibre directions, s and s^*):

$$\omega_{ij} = \omega_{[ij]} + \eta_{ij}A \quad , \tag{5.33}$$

where the 1-form

$$A = A_I\theta^I = A_i\theta^i + A_s\theta^s + A_{s^*}\theta^{s^*} \quad , \tag{5.34}$$

is the (generalised) Weyl 1-form.

In Eqs (5.15), we expressed our space-time tetrad, θ^i, in terms of S, S^*, and the unspecified tetrad parameters, $\{\alpha\,, b\,, b^*\,, a\,, a^*\,, c\}$. Thus, we can explicitly compute the $\triangle^i{}_{JK}$ in terms of these functions and their derivatives. (The explicit expressions for the $\triangle^i{}_{JK}$, are given in section 5.3.) Therefore, we will use Eq (5.32) to solve for the connection coefficients, ω_{ijK}, in terms of the $\triangle^i{}_{JK}$ and the undetermined A_I. In doing so, we will find several things: i) the four space-time components of the Weyl 1-form, A_i, remain arbitrary; ii) the skew-symmetric part of the connection, $\omega_{[ij]}$, and the fibre parts of the Weyl 1-form, A_s and A_{s^*}, are uniquely determined functionals of S, S^*, and A_i ; iii) the tetrad parameters are uniquely determined functionals of S and S^*; and iv) the functions S and S^* must satisfy the Wünschmann condition, which, as we have seen, is a set of complex-conjugate differential equations in all six variables of our 6-dimensional space, $\{Z\,, R\,, W\,, W^*\,, s\,, s^*\}$.

We begin by splitting the structure equation it into its fibre-fibre-, tetrad-fibre-, and tetrad-tetrad components.

A. The fibre-fibre component contains no information. By direct calculation,

$$\triangle^i{}_{ss^*} = 0 \quad . \tag{5.35}$$

Our connection 1-form is compatible to this since it does not have fibre-fibre parts,

$$\omega_{ij} = \omega_{ijK}\theta^K \quad . \tag{5.36}$$

B. The tetrad-fibre parts of the structure equation are

$$\omega_{ijs} = \triangle_{ijs} \quad . \tag{5.37}$$

Symmetrising on $(i\,,j)$ in Eq (5.37) and using Eq (5.33) yields

$$\eta_{ij}A_s = \triangle_{(ij)s} \quad , \tag{5.38}$$

while the skew-symmetric parts gives

$$\omega_{[ij]s} = \triangle_{[ij]s} \quad . \tag{5.39}$$

Eqs (5.38) uniquely determine A_s and A_{s^*} in terms of S and S^*,

$$A_s = \frac{1}{4}\triangle^k{}_{ks} \quad . \tag{5.40}$$

In addition, the trace-free part of Eqs (5.38),

$$\triangle_{(ij)s} - \frac{1}{4}\eta_{ij}\triangle^k{}_{ks} = 0 \quad , \tag{5.41}$$

uniquely determines the Wünschmann condition and the tetrad parameters.

Alternatively, from Eq (5.26), *i.e.*, $G_{ij} = -2\triangle_{(ij)s}$, and Eq (5.41) we have

$$G^{TF}_{ij} = 0 \quad , \tag{5.42}$$

where TF denotes the trace-free part. It is from these equations that we actually find the explicit expressions of the tetrad parameters (see theorem 5.1.1). The details for analysing Eqs (5.42), however, are quite involved and will be given in section 5.3.

Remark 5.1.5. With the determined tetrad parameters and the Wünschmann condition (see theorem 5.1.1 in the following subsection, 5.1.2), we have, using Eqs (5.38) and (5.26), the result

$$Dg = \underset{e_s}{\pounds} g = -2A_s g \quad , \tag{5.43}$$

where $\underset{e_s}{\pounds}$ is the Lie derivative along the parameter s. This result will be discussed further in subsection 5.1.3 below.

C. Returning to the tetrad-tetrad parts of the structure equations, we have that

$$\triangle^i{}_{mn} + \eta^{ij}(\omega_{jnm} - \omega_{jmn}) = 0 \quad , \tag{5.44}$$

or

$$\omega_{i[jk]} = \frac{1}{2}\triangle_{ijk} \quad . \tag{5.45}$$

From the tensor identity

$$\omega_{ijk} = \omega_{(ij)k} - \omega_{(jk)i} + \omega_{(ki)j} + \omega_{i[jk]} - \omega_{k[ij]} + \omega_{j[ki]} \quad , \tag{5.46}$$

and Eqs (5.33) and (5.45), we obtain the tetrad-tetrad coefficients of the connection,

$$\omega_{ijk} = \eta_{ij}A_k - \eta_{jk}A_i + \eta_{ki}A_j + \frac{1}{2}(\eta_{mi}\triangle^m{}_{jk} - \eta_{mk}\triangle^m{}_{ij} + \eta_{mj}\triangle^m{}_{ki}) \quad . \tag{5.47}$$

This decomposes naturally into a Levi-Civita part $\gamma_{ijk} = \gamma_{[ij]k}$ (which is independent of A_i) plus a "Weyl" part, $\tilde{\omega}_{ijk}$, *i.e.*,

$$\omega_{ijk} = \gamma_{ijk} + \tilde{\omega}_{ijk} \quad , \tag{5.48}$$

$$\gamma_{ijk} = \frac{1}{2}(\eta_{mi}\triangle^m{}_{jk} - \eta_{mk}\triangle^m{}_{ij} + \eta_{mj}\triangle^m{}_{ki}) \quad , \tag{5.49}$$

$$\tilde{\omega}_{ijk} = \eta_{ij}A_k + 2\eta_{k[i}A_{j]} \quad . \tag{5.50}$$

Since the $\triangle_{ijk}$ depend only on S and S^* (once the tetrad parameters have been determined), the Levi-Civita part of the connection γ_{ijk} also only depends on these functions.

Remark 5.1.6. The Levi-Civita part of the connection, γ_{ijk}, is actually the metric connection of g of Eq (5.17). For an arbitrary vector V^i, we denote this connection by ∇:

$$\nabla_k V^i = e_k \left(V^i\right) + \gamma^i{}_{jk} V^j \quad , \tag{5.51}$$

and

$$\nabla g = 0 \quad . \tag{5.52}$$

In summary, we have shown that Eqs (5.37) and (5.47) completely determine the ω_{ijK} in terms of the $\triangle^i{}_{JK}$ and the undetermined space-time parts of the Weyl 1-form, A_i. In general, the undetermined A_i are functions of $(x^a\,, s\,, s^*)$.

5.1.2 A Theorem

To conclude this section we return to the vanishing of the trace-free part of $\triangle_{(ij)s}$. They are nine complex equations for the determination of the tetrad parameters, $\{\alpha\,, a\,, a^*\,, b\,, b^*\,, c\}$. Thus, there must be several identities and/or conditions to be imposed on the S and S^*. By explicitly solving these equations (see section 5.3) the results can be summarised in the following theorem:

Theorem 5.1.1 (**Gallo, *et al.*** [13])**.** *The torsion free condition on the connection:*
1. *Uniquely determines the connection ω_{ij}, via Eqs (5.47) and (5.37).*

2. *Uniquely determines the tetrad parameters in terms of* S *and* S^* *(see below).*

3. *Imposes a (complex) condition, the vanishing of the Wünschmann invariant,*

$$M\left[S\,,S^*\right] \equiv \frac{Db + bD^*b + S_{W^*} - bS_W + b^2 S^*_{W^*} - b^3 S^*_W}{1 - bb^*} = 0 \quad , \tag{5.53}$$

where the tetrad parameters are given by:

$$b = \frac{-1 + \sqrt{1 - S_R S^*_R}}{S^*_R} \quad , \tag{5.54}$$

$$\alpha^2 = \frac{1 + bb^*}{(1 - bb^*)^2} \quad , \tag{5.55}$$

$$\begin{aligned} a = {} & b^{-1} b^{*-1} (1 - bb^*)^{-2} (1 + bb^*) \\ & \times \left\{ b^{*2}(-Db + bS_W - S_{W^*}) + b(-D^*b^* + b^* S^*_{W^*} - S^*_W) \right\} \quad , \end{aligned} \tag{5.56}$$

and

$$\begin{aligned} c = {} & -\frac{Da + D^*a^* + T_W + T^*_{W^*}}{4} - \frac{aa^*(1 + 6bb^* + b^2 b^{*2})}{2(1 + bb^*)^2} \\ & + \frac{(1 + bb^*)(bS^*_Z + b^* S_Z)}{2(1 - bb^*)^2} + \frac{a(2ab - b^* S_{W^*}) + a^*(2a^*b^* - bS^*_W)}{2(1 + bb^*)} \quad . \end{aligned} \tag{5.57}$$

5.1.3 A Space-Time Metric

Here we point out that one can use the result of remark 5.1.5 to construct a conformal metric that is independent of the fibre coordinates $(s\,,s^*)$. For the purpose of generality, however, we will return to considering the metric (5.17), which is a function of everything, for the rest of this work, *i.e.*, the $(s\,,s^*)$-independent metric discussed below will only appear in this subsection.

Recall that the metric (5.17),

$$g\,(Z\,,R\,,W\,,W^*\,,s\,,s^*) = \eta_{ij}\theta^i \otimes \theta^j \quad , \tag{5.58}$$

depends on the fibres $(s\,,s^*)$. Using Eq (5.43),

$$Dg = \underset{e_s}{\pounds} g = -2A_s g \quad , \tag{5.59}$$

and its complex conjugate, one can construct a conformal factor such that the metric $\hat{g}$ is independent of $(s\,,s^*)$ and is, therefore, a conformal metric on the space-time M. This is done by taking the conformal transformation

$$\hat{g} = \Phi^2 g \quad , \tag{5.60}$$

and requiring, via Eq (5.59), that the conformal factor Φ satisfy

$$D\Phi = \Phi A_s \quad , \quad D^*\Phi = \Phi A_{s^*} \quad , \tag{5.61}$$

where A_s is an explicit functional of the functions S and S^*, *i.e.*, Eqs (5.40) and (5.109). Clearly then, the conformal metric $\hat{g}$ satisfies

$$D\hat{g} = D^*\hat{g} = 0 \quad . \tag{5.62}$$

Furthermore, the solution of Eq (5.61) contains the conformal freedom a multiplicative function $\varpi(x^a)$ such that $D\varpi = D^*\varpi = 0$, *i.e.* $\varpi(x^a)$ is an arbitrary function on the space-time manifold M. Thus, the solution is of the form $\Phi = \varpi(x^a)\Phi_0 [S , S^*]$, where

$$\Phi_0 [S , S^*] = \exp\left(\int A_s ds + A_{s^*} ds^*\right) \quad . \tag{5.63}$$

(The integrability conditions of Eq (5.61) are rather complicated and are discussed in [7].) The function $\varpi(x^a)$ represents the standard conformal freedom discussed in chapter 3. In other words, by taking $\varpi(x^a) \to f(x^a)\varpi(x^a)$, the metric $\hat{g}(x^a)$ is conformally transformed as $\hat{g} \to f^2\hat{g}$.

5.2 THE CARTAN CURVATURES

In the previous section, we used the first structure equation, Eq (5.28), to algebraically solve for the components of a torsion-free connection,

$$\omega_{ij} = \omega_{[ij]} + \eta_{ij} A \quad , \tag{5.64}$$

uniquely in terms of S and S^* and the undetermined A_i.

Our next goal is to compute the curvature 2-forms, Θ_{ij}, defined by the *second structure equation*,

$$d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = \Theta^i{}_j = \frac{1}{2}\Theta^i{}_{jLM}\theta^L \wedge \theta^M \quad . \tag{5.65}$$

By taking the exterior derivative of the first structure equation, Eq (5.28), and using the second structure equation, Eq (5.65), we obtain the *first Bianchi identity*,

$$\Theta_{ij} \wedge \theta^j = 0 \quad , \tag{5.66}$$

or, in terms of the tetrad-tetrad-, tetrad-fibre-, and fibre-fibre components,

$$0 = \Theta_{ijkm} + \Theta_{ikmj} + \Theta_{imjk} \quad , \tag{5.67}$$

$$0 = \Theta_{i[jk]s} \quad , \tag{5.68}$$

$$0 = \Theta_{ijss^*} \quad . \tag{5.69}$$

We now calculate the Θ_{ij} as explicit functions of S and S^* and the undetermined A_i. First, note that from Eqs (5.64) and (5.65), it is straightforward to see that the Θ_{ij} inherits the symmetry of the ω_{ij}, and thus can be written as

$$\Theta_{ij} = \Theta_{[ij]} + \eta_{ij} dA \quad , \tag{5.70}$$

with

$$dA = \frac{1}{2}(dA)_{LM}\theta^L \wedge \theta^M \quad , \tag{5.71}$$

which defines the components $(dA)_{LM}$. Next, we split the components Θ_{ijLM} into the tetrad-tetrad parts, Θ_{ijkm}, and the tetrad-fibre parts, Θ_{ijks}. (The fibre-fibre parts are identically zero from the first Bianchi identity).

We begin by simply stating the tetrad-fibre part of Θ_{ij} ,

$$\Theta_{ijks} = \eta_{ij}(dA)_{ks} + \eta_{ik}(dA)_{js} - \eta_{jk}(dA)_{is} \quad . \tag{5.72}$$

One can find this by a direct calculation that is similar to the determination of the tetrad-tetrad terms below. Instead, we will justify the above result in the next subsection, in which Θ_{ijks} is determined very easily.

In order to calculate the tetrad-tetrad parts, Θ_{ijkm} , we first note that it can be split into terms arising from the Levi-Civita part of the connection and terms arising from the Weyl part of the connection. These are denoted respectively by $\mathfrak{R}_{[ij][km]}$ and $\tilde{\Theta}_{ij[km]}$, *i.e.*,

$$\begin{aligned}\Theta_{ij[km]} &= \mathfrak{R}_{[ij][km]} + \tilde{\Theta}_{ij[km]} \\ &= \mathfrak{R}_{[ij][km]} + \tilde{\Theta}_{[ij][km]} + \eta_{ij}(dA)_{[km]} \quad .\end{aligned} \tag{5.73}$$

The $\mathfrak{R}_{[ij][km]}$ are the components of the standard Riemann tensor of the Levi-Civita connection, γ_{ijk}, of Eq (5.49).

The $\tilde{\Theta}_{[ij][km]}$ depends on A and its derivatives. Using the Levi-Civita connection, ∇, we have

$$\nabla_i A_j = e_i(A_j) - \gamma_{kji} A^k \quad , \tag{5.74}$$

and

$$(dA)_{ij} = 2\nabla_{[i} A_{j]} \quad . \tag{5.75}$$

Thus, $\tilde{\Theta}_{[ij][km]}$ can be written as

$$\frac{1}{2}\tilde{\Theta}_{[ij][km]} = \eta_{j[k}\nabla_{m]}A_i - \eta_{i[k}\nabla_{m]}A_j + A^2\eta_{j[k}\eta_{m]\,i} + A_j\eta_{i[k}A_{m]} - A_i\eta_{j[k}A_{m]} \quad , \tag{5.76}$$

where $A^2 = A^m A_m$.

Now, by defining

$$R_{jm} \equiv \eta^{ik}\Theta_{ijkm} \quad , \tag{5.77}$$

and using Eqs (5.73) and (5.76), we obtain

$$R_{jm} = \mathfrak{R}_{(jm)} - \eta_{jm}\nabla_p A^p - 2\left\{\nabla_{(m}A_{j)} + \eta_{jm}A^2 - A_j A_m\right\} + 4\nabla_{[j}A_{m]} \quad , \tag{5.78}$$

where the $\mathfrak{R}_{(jm)}$ are the components of the Ricci tensor of the connection γ_{ijk}. If we also let

$$R \equiv \eta^{jm} R_{jm} \quad , \tag{5.79}$$

then from Eq (5.78), we obtain

$$R = \mathfrak{R} - 6\left\{\nabla_p A^p + A^2\right\} \quad , \tag{5.80}$$

where $\mathfrak{R}$ is the standard Ricci scalar.

Remark 5.2.1. If the undetermined A_i are taken to be gradients of an arbitrary function f,

$$A_i = \nabla_i f \quad , \tag{5.81}$$

then the vanishing of R_{ij}^{TF},

$$0 = R_{ij}^{TF} = R_{ij} - \frac{1}{4}\eta_{ij}R \quad , \tag{5.82}$$

with R_{ij} given in terms of $\mathfrak{R}_{ij}$ above, yields the conformal Einstein equations for $\mathfrak{R}_{ij}$ (see Eqs (3.19) - (3.21)). This will be discussed further in the next chapter.

5.2.1 The First Cartan Curvature

The Cartan first curvature 2-form is given by [15]

$$\Omega_{ij} = \Theta_{ij} - \Psi_i \wedge \theta^k \eta_{kj} + \Psi_j \wedge \theta^k \eta_{ki} + \eta_{ij}\Psi_k \wedge \theta^k \quad , \tag{5.83}$$

where the (Ricci) 1-forms Ψ_i are appropriately chosen so that

$$\Omega_{ij} = \frac{1}{2}\Omega_{ijLM}\theta^L \wedge \theta^M \quad , \tag{5.84}$$

satisfies the following conditions

$$\Omega_{ijkm} = \Omega_{[ij]km} \quad , \tag{5.85}$$

$$0 = \eta^{ik}\Omega_{ijkm} \quad , \tag{5.86}$$

$$0 = \Omega_{ijks} \quad . \tag{5.87}$$

Note, from its definition, that Ω_{ij} also satisfies the first Bianchi identity, Eq (5.66), *i.e.*,

$$\Omega_{ij} \wedge \theta^j = 0 \quad . \tag{5.88}$$

With some algebra, one can show that the conditions (5.85),(5.86) and (5.87) are satisfied uniquely by the 1-form

$$\Psi_i = \Psi_{iK}\theta^K = \Psi_{ij}\theta^j + \Psi_{is}\theta^s + \Psi_{is^*}\theta^{s^*} \quad , \tag{5.89}$$

with

$$\Psi_{ij} = \frac{1}{4}R_{[ij]} + \frac{1}{2}\left(R_{(ij)} - \frac{1}{6}R\eta_{ij}\right) \quad . \tag{5.90}$$

and

$$\Psi_{is} = (dA)_{is} \quad . \tag{5.91}$$

Note that from Eqs (5.91), (5.83), and (5.87), we find Eq (5.72), *i.e.*,

$$\Theta_{ijks} = \eta_{ij}(dA)_{ks} + \eta_{ik}(dA)_{js} - \eta_{jk}(dA)_{is} \quad . \tag{5.92}$$

Using Eqs (5.78) and (5.80), we obtain

$$\Psi_{ij} = \Im_{ij} + \nabla_{[i}A_{j]} - \nabla_{(i}A_{j)} + A_iA_j - \frac{1}{2}\eta_{ij}A^2 \quad , \tag{5.93}$$

where $\mathfrak{S}_{ij}$ is the Schouten tensor associated with γ_{ijk} ,

$$\mathfrak{S}_{ij} = \frac{1}{2}\left(\mathfrak{R}_{(ij)} - \frac{1}{6}\mathfrak{R}\eta_{ij}\right) \quad . \tag{5.94}$$

By using Eq (5.93), we can insert the above expression into Eq (5.83), yielding

$$\Omega_{ijkm} = \mathfrak{R}_{[ij][km]} - \mathfrak{S}_{ik}\eta_{jm} + \mathfrak{S}_{im}\eta_{jk} + \mathfrak{S}_{jk}\eta_{im} - \mathfrak{S}_{jm}\eta_{ik} \quad , \tag{5.95}$$

which is manifestly independent of the A_i. Thus, we see that the Ω_{ijkm} is actually the Weyl tensor (see Eq (2.12)),

$$\Omega_{ijkm} = C_{ijkm} \quad , \tag{5.96}$$

i.e., the Ω_{ijkm} is the totally trace-free part of $\mathfrak{R}_{[ij][km]}$.

5.2.2 Second Cartan Curvature

Finally, we define the second Cartan curvature (with the covariant exterior derivative $\mathfrak{D}$) of the 1-form Ψ_i as

$$\Omega_i = \mathfrak{D}\Psi_i = d\Psi_i + \Psi_k \wedge \omega^k{}_i = \frac{1}{2}\Omega_{iJK}\theta^J \wedge \theta^K \quad . \tag{5.97}$$

Using Eq (5.89) in the above, we obtain, after a lengthy calculation, the simple results

$$\Omega_{ijk} = \nabla_m C^m{}_{ijk} + A_m C^m{}_{ijk} \quad , \tag{5.98}$$

and

$$\Omega_{ijs} = 0 \quad , \tag{5.99}$$

where ∇^m again is the Levi-Civita covariant derivative. Clearly, the components of the second Cartan curvature resemble the conformal Yang tensor (see Eq (3.22)). In fact, if the $\Omega_{ijk} = 0$, then, via the theorems of subsection 3.3.1, the A_i would be gradients and the Ω_{ijk} would be identical to the conformal Yang tensor. We will analyse this case in the next chapter.

5.2.3 Synopsis

Since so many different quantities and their symbols have been introduced, we have added a few essentially pedagogical remarks concerning the placement of different variables.

a.) The $\triangle_{ijK}$ depends on the functions S and S^*, which are defined by the starting PDE's.

b.) Since $\omega_{ijs} = \triangle_{ijs}$, the fibre components of the connection depend only on the S's.

c.) All the quantities $\{S\,, S^*\,, A_i\}$ appear in the tetrad components of the connection 1-forms, ω_{ijk}. These components can be split into

$$\omega_{ijk} = \gamma_{ijk} + \tilde{\omega}_{ijk} \quad , \tag{5.100}$$

where the Levi-Civita part, γ_{ijk} , depends only on S and S^*, while $\tilde{\omega}_{ijk}$ depends only on the A_i .

d.) The curvature $\Theta_{ij[km]}$ splits into two parts

$$\Theta_{ij[km]} = \mathfrak{R}_{[ij][km]} + \tilde{\Theta}_{ij[km]} \quad , \tag{5.101}$$

where the (standard) Riemann curvature $\mathfrak{R}_{[ij][km]}$ depends on the S's and $\tilde{\Theta}_{ij[km]}$ depends on everything.

e.) The first Cartan curvature 2-form, Ω_{ij} , is the Weyl tensor, C_{ijmn} , and depends only on the S's.

f.) The second Cartan curvature 2-form,

$$\Omega_i = \frac{1}{2}\left\{\nabla_m C^m{}_{ijk} + A_m C^m{}_{ijk}\right\}\theta^j \wedge \theta^k, \tag{5.102}$$

depends on everything, though the A_i appears explicitly just in the second term.

g.) Though the Ricci 1-forms,

$$\Psi_i = \Psi_{ij}\theta^j + \Psi_{is}\theta^s + \Psi_{is^*}\theta^{s^*} \quad , \tag{5.103}$$

depend on everything, their separate parts do not. Ψ_{is} depends only on the A_i. From

$$\Psi_{ij} = \Im_{ij} - \nabla_{[i}A_{j]} - 2\left\{\nabla_{(i}A_{j)} + \frac{1}{2}\eta_{ij}A^2 - A_i A_i\right\} \quad , \tag{5.104}$$

we have that $\Im_{ij}$ depends only on the S's while the remaining terms depend on everything.

5.3 EXPLICIT RELATIONS

For completeness, we give relevant explicit expressions and derivations. We begin by stating the components of $\triangle$, G, and ω. Next, we explicitly derive the tetrad parameters and the Wünschmann condition. Then, using the expressions for ω and the basis vectors e_I, we give (without proof) the commutators of the basis vectors. Finally, as an example of what the curvature terms look like, we state (without the derivation) the component Ψ_{11}. All other curvature terms are, in general, much longer and more complicated.

5.3.1 The $\triangle$, G, and ω

We begin by defining some quantities to simplify our expressions:

$$\gamma \equiv 1 - bb^* \quad , \tag{5.105}$$

$$\sigma \equiv a - b^* a^* \quad , \tag{5.106}$$

$$\zeta \equiv a_R - b^* a_R^* \quad . \tag{5.107}$$

and

$$h_2 = e_2 + b^* e_3 \quad . \tag{5.108}$$

Also, when any two quantities are members of a complex-conjugate pair, we will often display only one of the objects and imply the other.

I. The $\triangle^i{}_{JK}$:

$$
\begin{aligned}
&\triangle^0{}_{01} = \triangle^0{}_{02} = \triangle^0{}_{0s} = \triangle^0{}_{12} = \triangle^0{}_{1s} = \triangle^0{}_{23} = \triangle^0{}_{ss^*} = 0 \quad , \\
&\triangle^1{}_{ss^*} = \triangle^2{}_{01} = \triangle^2{}_{ss^*} = 0 \quad , \quad \triangle^0{}_{2s} = \frac{-1}{\alpha\gamma} \quad , \quad \triangle^0{}_{2s^*} = \frac{b^*}{\alpha\gamma} \quad , \\
&\triangle^1{}_{01} = -e_1(c) \quad , \quad \triangle^1{}_{02} = -e_2(c) + \frac{e_0(a) - b^* e_0(a^*)}{\alpha\gamma} \quad , \\
&\triangle^1{}_{0s} = -Dc + a^* c - e_0(T) - a e_0(S) \quad , \quad \triangle^1{}_{12} = \frac{\zeta}{\alpha\gamma} \quad , \\
&\triangle^1{}_{1s} = -[e_1(T) + a^* + a e_1(S)] \quad , \quad \triangle^1{}_{23} = \frac{h_2(a^*) - h_3(a)}{\alpha\gamma} \quad , \\
&\triangle^1{}_{2s} = -\left[e_2(T) + a e_2(S)\right] + \frac{b^* D a^* - D a - c + a^* \sigma}{\alpha\gamma} \quad , \\
&\triangle^1{}_{2s^*} = -\left[e_2(T^*) + a^* e_2(S^*)\right] + \frac{b^*(D^* a^* + c) - D^* a + a\sigma}{\alpha\gamma} \quad , \\
&\triangle^2{}_{02} = e_0(\ln\alpha) - \frac{b^* e_0(b)}{\gamma} \quad , \quad \triangle^2{}_{03} = \frac{e_0(b)}{\gamma} \quad , \\
&\triangle^2{}_{0s} = \alpha\left[bc - e_0(S)\right] \quad , \quad \triangle^2{}_{0s^*} = \alpha\left[c - b e_0(S^*)\right] \quad , \\
&\triangle^2{}_{12} = e_1(\ln\alpha) - \frac{b^* e_1(b)}{\gamma} \quad , \quad \triangle^2{}_{13} = \frac{e_1(b)}{\gamma} \\
&\triangle^2{}_{1s} = -\alpha\left[b + e_1(S)\right] \quad , \quad \triangle^2{}_{1s^*} = -\alpha\left[1 + b e_1(S^*)\right] \quad , \\
&\triangle^2{}_{23} = \frac{h_2(b)}{\gamma} - e_3(\ln\alpha) \quad , \\
&\triangle^2{}_{2s} = -D(\ln\alpha) + \frac{b^* D b - \alpha\gamma e_2(S) + b\sigma}{\gamma} \quad , \\
&\triangle^2{}_{2s^*} = -D^*(\ln\alpha) + \frac{b^* D^* b - \alpha\gamma b e_2(S^*) + \sigma}{\gamma} \quad , \\
&\triangle^2{}_{3s} = \frac{-Db - \alpha\gamma e_3(S) + b\sigma^*}{\gamma} \quad , \\
&\triangle^2{}_{3s^*} = \frac{-D^* b - \alpha\gamma b e_3(S^*) + \sigma^*}{\gamma} \quad ,
\end{aligned}
\tag{5.109}
$$

II. The G_{ij}:

$$
\begin{aligned}
G_{00} &= 2\left[Dc - ca^* + ae_0(S) + e_0(T)\right] \quad , \\
G_{11} &= 0 \quad , \quad G_{01} = a^* + ae_1(S) + e_1(T) \quad , \\
G_{02} &= \alpha\left[c - b^* e_0(S)\right] + ae_2(S) + e_2(T) + \frac{Da - b^* Da^* + c - a^*\sigma}{\alpha\gamma} \quad , \\
G_{03} &= \alpha\left[bc - e_0(S)\right] + ae_3(S) + e_3(T) + \frac{Da^* - bDa - bc - a^*\sigma^*}{\alpha\gamma} \quad , \\
G_{12} &= \frac{1}{\alpha\gamma} - \alpha\left[1 + b^* e_1(S)\right] \quad , \quad G_{13} = -\frac{b}{\alpha\gamma} - \alpha\left[b + e_1(S)\right] \quad , \\
G_{22} &= \frac{2}{\gamma}\left[\sigma - Db^*\right] - 2\gamma b^* e_2(S) \quad , \quad G_{33} = \frac{2}{\gamma}\left[b\sigma^* - Db\right] - 2\gamma e_3(S) \quad , \\
G_{23} &= a^* - \alpha\left[e_2(S) + b^* e_3(S)\right] + \frac{D(bb^*) - 2D(\ln\alpha)}{\gamma} \quad ,
\end{aligned} \tag{5.110}
$$

III. The ω_{ij}:

$$
\begin{aligned}
\omega_{01} &= e_1(c)\theta^0 + \left\{A_2 + \frac{\zeta}{2\alpha\gamma}\right\}\theta^2 + \left\{A_3 + \frac{\zeta^*}{2\alpha\gamma}\right\}\theta^3 + 2A_1\theta^1 \\
&\quad + 2A_s\theta^s + 2A_{s^*}\theta^{s^*} \quad , \\
\omega_{10} &= -\omega_{01} + 2A \quad , \\
\omega_{02} &= \left\{e_2(c) + \frac{b^* e_0(a^*) - e_0(a)}{\alpha\gamma}\right\}\theta^0 + \left\{A_2 - \frac{\zeta}{2\alpha\gamma}\right\}\theta^1 + \frac{e_0(b^*)}{\gamma}\theta^2 \\
&\quad + \left\{A_0 + \frac{2e_0(bb^*) + \alpha\gamma^2\left[h_2(a^*) - h_3(a)\right]}{2\gamma(1 + bb^*)}\right\}\widehat{\theta}^3 \\
&\quad + \frac{\gamma c - b^* S_Z(1 + bb^*)}{\alpha\gamma^2}\theta^s - \frac{b^*\gamma c + S_Z^*(1 + bb^*)}{\alpha\gamma^2}\theta^{s^*} \quad , \\
\omega_{03} &= (\omega_{02})^* \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\omega_{12} &= \left\{A_2 - \frac{\zeta}{2\alpha\gamma}\right\}\theta^0 + \frac{e_1(b^*)}{\gamma}\theta^2 + \left\{A_1 + \frac{e_1(bb^*)}{\gamma(1+bb^*)}\right\}\theta^3 \\
&\quad - \frac{\alpha\gamma}{1+bb^*}\theta^s + \frac{\alpha\gamma b^*}{1+bb^*}\theta^{s^*} \quad , \\
\omega_{13} &= (\omega_{12})^* \quad , \\
\omega_{23} &= \left\{-A_0 + \frac{b^* e_0(b) - b e_0(b^*)}{2\gamma} + \frac{\alpha\gamma\left[h_3(a) - h_2(a^*)\right]}{2(1+bb^*)}\right\}\theta^0 \\
&\quad - \left\{A_1 + \frac{bb^*_R - b^* b_R}{2\gamma}\right\}\theta^1 + \left\{-\frac{h_3(b^*)}{\gamma} + \frac{(3+bb^*)e_2(bb^*)}{2\gamma(1+bb^*)}\right\}\theta^2 \\
&\quad - \left\{-\frac{h_2(b)}{\gamma} + \frac{(3+bb^*)e_3(bb^*)}{2\gamma(1+bb^*)} + 2A_3\right\}\theta^3 \\
&\quad - \left\{\frac{\gamma(S_W + 2A_s) + a^*(3+bb^*)}{4} - \frac{ab(1+3bb^*)}{2(1+bb^*)}\right\}\theta^s \\
&\quad + \left\{\frac{\gamma S^*_{W^*} + (a - 2A_{s^*})(3+bb^*)}{4} - \frac{a^* b^*(1+3bb^*)}{2(1+bb^*)}\right\}\theta^{s^*} \quad , \\
\omega_{32} &= -\omega_{23} - 2A \quad .
\end{aligned}
\tag{5.111}
$$

5.3.2 The Tetrad Parameters

Here we determine the tetrad parameters $\{\alpha\,,b\,,b^*\,,a\,,a^*\,,c\}$ and the Wünschmann condition in term of the functions S and S^*. The determination of the tetrad parameters also uniquely determines the torsion-free connection above. To find the tetrad parameters, we use the vanishing of the trace of $\triangle_{ijs}$ found in Eq (5.42). From these conditions, we have

$$
\begin{aligned}
0 &= G_{01} + G_{23} = G^*_{01} + G^*_{23} \quad , \\
0 &= G_{ij} = G^*_{ij} \quad , \quad \text{for } (i\,,j) \notin \{(0\,,1)\,,(2\,,3)\} \quad .
\end{aligned}
\tag{5.112}
$$

Using the explicit expressions of the G_{ij} of Eq (5.110), along with the definitions of the e_i in Eq (5.16), we solve for *i.)* the tetrad parameters and *ii.)* the Wünschmann condition.

As before, we will often list only one member of a complex-conjugate pair and imply the other. We refer to the conjugate of a listed equation by writing the listed-equation's number with a superscript (*).

We start with the equations $G_{12} = 0$, $G_{13} = 0$, $G^*_{12} = 0$, and $G^*_{13} = 0$, which depend only on b, b^*, and α. They are four equations with three unknowns that satisfy an identity. From $G_{12} = 0$ and $G^*_{13} = 0$, we have

$$b^* S_R = b S^*_R \quad . \tag{5.113}$$

Next, using $G^*_{13} = 0$ and $G_{13} = 0$ to eliminate α^2, we obtain

$$b = \frac{-1 + \sqrt{1 - S_R S^*_R}}{S^*_R} \quad . \tag{5.114}$$

(We have chosen the positive root since we want b to vanish when S vanishes.) Using Eq (5.113), one sees that b^* is the complex conjugate of b. It useful to invert Eqs (5.114) and (5.114*), yielding

$$S_R = \frac{-2b}{1 + bb^*} \quad . \tag{5.115}$$

From Eq (5.115) and $G_{12} = 0$, we find

$$\alpha^2 = \frac{1 + bb^*}{(1 - bb^*)^2} \quad . \tag{5.116}$$

All four equations $G_{12} = 0$, $G_{13} = 0$, $G^*_{12} = 0$, and $G^*_{13} = 0$ are satisfied by Eqs (5.114), (5.114*), and (5.116).

Our next step is to determine a, a^*, and the Wünschmann condition from the equations $G_{01}+G_{23}=0$, $G_{22}=0$, $G_{33}=0$, and their conjugates. From this set of six equations we will be able to solve for a and a^*, find the Wünschmann condition, and obtain further identities on some functions.

We first state some useful relationships. Taking D of Eq (5.116), we have, after some simplification,

$$D\alpha = \frac{\alpha D(bb^*)(3+bb^*)}{2(1+bb^*)(1-bb^*)} \quad . \tag{5.117}$$

Next, we find $T_R = T_R[Db, Db^*, b, S]$ and its conjugate. By first taking D^* of Eq (5.115) and D of equation (5.115*),

$$D^*\left(S_R\right) = D^*\left(\frac{-2b}{1+bb^*}\right) \quad , \quad D^*\left(S_R^*\right) = D\left(\frac{-2b^*}{1+bb^*}\right) \quad , \tag{5.118}$$

then using, Eq (5.9) to commute the R-derivative and the fibre-derivative, we obtain two equations containing T_R and T_R^*. After simplifying with Eqs (5.115), they become

$$\begin{aligned} T_R = &\frac{4b(Db^*-b^{*2}Db)}{(1+bb^*)(1-bb^*)^2} + \frac{2(b^2D^*b^*-D^*b+2b^2S_W^*)}{(1-bb^*)^2} \\ &+ \frac{S_W(1+bb^*)^2}{(1-bb^*)^2} - \frac{2(1+bb^*)(b^*S_{W^*}+bS_{W^*}^*)}{(1-bb^*)^2} \quad . \end{aligned} \tag{5.119}$$

We are now in a position to find a, a^*, and the Wünschmann condition. First, from $G_{01}+G_{23}=0$ and $G_{01}^*+G_{23}^*=0$, we solve for a and a^*. With the aid of Eqs (5.117), (5.119) and their complex conjugates, we find

$$a = \frac{(1+bb^*)\left[b^{*2}Db+Db^*+D^*\left(bb^*\right)+(1-bb^*)\left(b^*S_W+bS_W^*\right)\right]}{\left(1-bb^*\right)^3} \quad . \tag{5.120}$$

When they are inserted into $G_{33}=0$, we find that S must obey the differential condition

$$M \equiv \frac{Db+bD^*b+S_{W^*}-bS_W+b^2S_{W^*}^*-b^3S_W^*}{1-bb^*} = 0 \quad , \tag{5.121}$$

where b is the known expression in terms of S and S^*. The expression $M = M\,[Db\,,D^*b\,,b]$ is the Wünschmann invariant. Its vanishing is the condition on the S and S^*, *i.e.*, on the original pair of PDE's, for the existence of a torsion-free connection.

This condition tells us that this invariant must vanish if we are to find a non-trivial torsion free connection. By substituting Db^* from the Wünschmann invariant and its conjugate into Eqs (5.120) and (5.120*), our expression for a becomes

$$a = \frac{\alpha^2}{bb^*}\left[b^{*2}(M - Db + bS_W - S_{W^*}) + b(M^* - D^*b^* + b^*S^*_{W^*} - S^*_W)\right] \quad . \tag{5.122}$$

or, with $M = M^* = 0$, we have

$$a = \frac{\alpha^2}{bb^*}\left[b^{*2}(-Db + bS_W - S_{W^*}) + b(-D^*b^* + b^*S^*_{W^*} - S^*_W)\right] \quad . \tag{5.123}$$

Summarising our results so far, we have obtained the five tetrad parameters, $\{b\,,b^*\,,\alpha\,,a\,,a^*\}$, as well as the Wünschmann condition in terms of S and S^*. The search for the last parameter c, is the most interesting and at the same time the most difficult part of the construction.

There are four equations for c, namely $G_{02} = 0$, $G_{03} = 0$, $G^*_{02} = 0$, and $G^*_{03} = 0$. As we will see below, three of those equations become identities once we algebraically solve for c. It is, however, instructive to keep the Wünschmann invariant different from zero when solving the equations. We then explicitly show how its vanishing yields a unique solution for c, such that the remaining identities among the G_{ij} are also satisfied. Thus, for the subsequent calculations, M is left in the equations.

By manipulating the expressions for a and M and their conjugates, we have

$$Db = M + bS_W - S_{W^*} + \frac{b(1-bb^*)(ab-a^*)}{1+bb^*} \quad , \tag{5.124}$$

$$Db^* = b^*(b^*S_{W^*} - S_W) - bM^* + \frac{(1-bb^*)(a-a^*b^*)}{1+bb^*} \quad . \tag{5.125}$$

Next, we insert the left-hand sides of Eqs (5.124), (5.125), and their conjugates into Eqs (5.119) and (5.119*) to find $T_R = T_R[a\,, b\,; M]$ and $T_R^* = T_R^*[a\,, b\,; M]$. The result is

$$T_R = (\tau + S_W) + \frac{2(3ab - b^* S_{W^*})}{1 + bb^*} - \frac{2a^*(1 + 4bb^* + b^2 b^{*2})}{(1 + bb^*)^2} \quad , \tag{5.126}$$

where $\tau = \tau[M]$ (which vanishes with M) is given below. Third, using the fact that the vectors D and D^* commute,

$$DD^* b = D^* Db \quad , \quad DD^* b^* = D^* Db^* \quad . \tag{5.127}$$

Thus by taking the appropriate fibre-derivatives of the four Eqs (5.124), (5.124*), (5.125), and (5.125*), simplifying with Eqs (5.9), (5.124), (5.125), and their conjugates, and by using the Eqs (5.127), we obtain two equations containing Da, D^*a, Da^*, and D^*a^*. They can be solved for $Da^* = Da^*[Da\,, D^*a^*]$ and $D^*a = D^*a[Da\,, D^*a^*]$ to find

$$\begin{aligned} Da^* = (\Upsilon + a^* S_W - a^{*2} - T_{W^*}) &+ \frac{S_Z(1 + b^2 b^{*2}) + 2b^2 S_Z^*}{(1 - bb^*)^2} \\ &+ \frac{b(Da - D^*a^* + T_W - T_{W^*}^* + 4aa^*) - 2a^* b^* S_{W^*}}{(1 + bb^*)} \\ &- \frac{a S_{W^*}(1 + b^2 b^{*2}) + 2b^2(2a^2 + a^* S_W^*)}{(1 + bb^*)^2} \quad . \end{aligned} \tag{5.128}$$

The term $\Upsilon = \Upsilon[DM\,, D^*M\,, M]$, which vanishes with M, is given below. Finally, in addition to Eq (5.128) above, we can use the integrability condition to derive another identity on the fibre-derivatives of a and a^*. We begin by taking D^* of Eq (5.126):

$$D^*(T_R) = D^* \left[(\tau + S_W) + \frac{2(3ab - b^* S_{W^*})}{1 + bb^*} - \frac{2a^*(1 + 4bb^* + b^2 b^{*2})}{(1 + bb^*)^2} \right] \quad . \tag{5.129}$$

On the left-hand side, we use Eq (5.9*) to commute the R-derivative and the fibre-derivative so that we obtain the term $U_R = \partial_R(D^*T)$, where

$$U \equiv D^*T = D^{*2}S = D^2S^* = DT^* \tag{5.130}$$

denotes the integrability condition. We can then solve this equation for U_R. Refer to the U_R that we obtain in this manner as $U_R^{(1)}$. In a similar manner, we can obtain $U_R^{(2)}$ by taking D of Eq (5.126*). Then equate $U_R^{(1)}$ and $U_R^{(2)}$, from which we find an identity on Da and D^*a^*. With the use of Eqs (5.9), (5.115), (5.124), (5.125), (5.128), and their conjugates this identity becomes

$$\begin{aligned} 0 = (\Gamma - \Gamma^* + Da - D^*a^* + T_W - T^*_{W^*}) + \frac{2(1+bb^*)(bS^*_Z - b^*S_Z)}{(1-bb^*)^2} \\ + \frac{4(a^{*2}b^* - a^2b) + 2(ab^*S_{W^*} - a^*bS^*_W)}{1+bb^*} \quad . \end{aligned} \tag{5.131}$$

The term $\Gamma = \Gamma[DM\,, D^*M\,, M]$ and its conjugate vanish with M and are given below. We are now in a position to find c from the four equations $G_{02} = 0$, $G_{03} = 0$, $G^*_{02} = 0$, and $G^*_{03} = 0$. We algebraically solve each of the four equations for c, calling each solution $c^{(i)}$. Next, we replace S_R , α^2, T_R, and Da^* by Eqs (5.115), (5.116), (5.126), and (5.128), and use Eq (5.131) to simplify. Finally, we separate each $c^{(i)}$ into a piece that contains all terms with the Wünschmann condition and its fibre-derivatives, namely $\xi^{(i)}$, and another piece that contains no Wünschmann terms, namely $C^{(i)}$, so that $c^{(i)}$ has the form

$$c^{(i)} = C^{(i)} + \xi^{(i)} \quad , \tag{5.132}$$

for all i. It is straightforward to verify that the four $C^{(i)}$ are real and equal. Imposing the Wünschmann condition, $M = M^* = 0$, so that the $\xi^{(i)} = 0$, then $C^{(i)} = c^{(i)} = c$, and we have our final expression for c, namely

$$\begin{aligned} c = &-\frac{Da + D^*a^* + T_W + T^*_{W^*}}{4} - \frac{aa^*(1 + 6bb^* + b^2b^{*2})}{2(1 + bb^*)^2} \\ &+ \frac{(1 + bb^*)(bS^*_Z + b^*S_Z)}{2(1 - bb^*)^2} + \frac{a(2ab - b^*S_{W^*}) + a^*(2a^*b^* - bS^*_W)}{2(1 + bb^*)} \quad . \end{aligned} \tag{5.133}$$

Had the Wünschmann invariant been non-vanishing, the whole construction obviously would have failed. Having determined all the tetrad parameters, we still have to verify that $G_{00} = 0$ and $G^*_{00} = 0$. By inspection, we see that these equations contain fibre-derivatives of c. In fact, by explicitly taking these fibre derivatives on c, we find that $G_{00} = 0$ and $G^*_{00} = 0$ are identically satisfied. We see this in the following fashion: From Eqs (5.128), (5.128*), (5.131) and (5.133), we find

$$\begin{aligned} Da^* = a^*S_W - aS_{W^*} - T_{W^*} + \frac{2a^*(2ab - b^*S_{W^*})}{1 + bb^*} \\ + \frac{S_Z(1 + bb^*)^2}{(1 - bb^*)^2} - \frac{a^{*2}(1 + 6bb^* + b^2b^{*2})}{(1 + bb^*)^2} \end{aligned} \tag{5.134}$$

and

$$\begin{aligned} Da = -(2c + T_W) + \frac{2a(2ab - b^*S_{W^*})}{1 + bb^*} \\ + \frac{2b^*S_Z(1 + bb^*)}{(1 - bb^*)^2} - \frac{aa^*(1 + 6bb^* + b^2b^{*2})}{(1 + bb^*)^2} \quad . \end{aligned} \tag{5.135}$$

By taking D^* of Eq (5.134) and D of Eq (5.135*), subtracting them and using the commutability of D and D^*, we have

$$Dc = cS_W - T_Z - aS_Z + \frac{2c(2ab - b^*S_{W^*})}{(1 + bb^*)} - \frac{ca^*(1 + 6bb^* + b^2b^{*2})}{(1 + bb^*)^2} \quad , \tag{5.136}$$

which is equivalent to $G_{00} = 0$.

Note that in the above analysis we have used the following expressions, all of which vanish when $M = M^* = 0$:

$$\begin{aligned}
\tau &= 2(b^*\nu - b^2\nu^*) \quad , \\
\Upsilon &= 2b\rho + 2(1+bb^*)^{-1}\{\mu(1-bb^*) + \nu[a^*b^* + a(1-bb^* - b^2b^{*2})] \\
&\quad + b^2\mu^*(1-bb^*) + b^2\nu^*[ab + a^*(1-bb^* - b^2b^{*2})]\} \quad , \\
\Gamma &= -b^*[4\mu + 2\nu(2abb^* + a^*b^* + 3a)] \quad ,
\end{aligned} \tag{5.137}$$

$$\begin{aligned}
\xi_1 = \xi_4^* &= b\{\mu^*(1-bb^*) + b^*\rho + \frac{1}{2}\nu^*[a^*(3 - 2b^2b^{*2}) - ab]\} \\
&\quad + \frac{1}{2}b^*\nu(a - a^*b^*) \quad , \\
\xi_2 = \xi_3^* &= \frac{1}{b}\{\mu(1-bb^*) + b\rho + \frac{1}{2}\nu[a(2 + bb^* - 2b^2b^{*2}) - a^*bb^{*2}]\} \\
&\quad + \frac{1}{2}b\nu^*(a^* - ab) \quad ,
\end{aligned} \tag{5.138}$$

where

$$\begin{aligned}
\mu &\equiv 2^{-1}(1-bb^*)^{-3}(1+bb^*)\{(b^*DM + D^*M) \\
&\quad + M(b^{*2}S_{W^*} - 2b^*S_W - 2bS_W^* + S_{W^*}^*)\} \quad , \\
\rho &\equiv -2^{-1}(1-bb^*)^{-2}(1+bb^*)MM^* \quad , \\
\nu &\equiv (1-bb^*)^{-1}(1+bb^*)^{-1}M \quad .
\end{aligned} \tag{5.139}$$

5.3.3 The Commutators

From the definitions of the basis vectors e_i in Eq (5.16), and using the relations of the previous subsection, one can find the commutators $[e_i\,, e_j]$ and $[e_s\,, e_i]$. The derivations of these commutators is complicated and not particularly instructive. Thus, we simply state some of the results.

The commutators, $[e_i\,, e_j]$, are found either by direct computation or by using the fact that for a torsion-free connection,

$$[e_j\,, e_k] = -2\omega^i{}_{[jk]} \quad . \tag{5.140}$$

Thus, these commutators can be found from the ω_{ijk} listed above. The commutators containing e_s or its conjugate are found by direct computation:

$$\begin{aligned}
[e_s\,, e_0] &= \alpha\left\{[bc - e_0(S)]\, e_2 + [c - b^* e_0(S)]\, e_3\right\} \quad ,\\
[e_s\,, e_1] &= \left[a^* - \frac{2ab}{1+bb^*} - \frac{h_2(S)}{\alpha\gamma}\right] e_1 - \frac{e_3 - be_2}{\alpha\gamma} \quad ,\\
[e_s\,, e_2] &= -\frac{e_0}{\alpha\gamma} - \alpha\,[b^* e_0(S) - c]\, e_1\\
&\quad - \left\{\frac{\gamma ab}{2(1+bb^*)} - \frac{b^*\,[e_3(S) - be_2(S)]}{2\alpha\gamma} + \frac{e_2(S)}{\alpha\gamma}\right\} e_2 \quad ,\\
[e_s\,, e_3] &= \frac{be_0}{\alpha\gamma} + \alpha\,[bc - e_0(S)]\, e_1\\
&\quad + \left\{(a^* - ab) - \frac{\gamma ab}{2(1+bb^*)} - \frac{b^*\,[e_3(S) + be_2(S)]}{2\alpha\gamma}\right\} e_3 \quad .
\end{aligned} \tag{5.141}$$

5.3.4 The Ψ_{11}

As an example of the components of the curvatures, we give – without explicit derivation – the component Ψ_{11}:

$$\Psi_{11} = \Im_{11} - e_1(A_1) + A_1^2 \quad , \tag{5.142}$$

where

$$\begin{aligned} \Im_{11} = &- \frac{\left[b^* e_1^2(b) + b e_1^2(b^*)\right]}{\gamma(1+bb^*)} - \frac{\left[e_1(b)\right]\left[e_1(b^*)\right](3 + 4bb^* + 3b^2 b^{*2})}{\gamma^2(1+bb^*)^2} \\ &- \frac{(1+2bb^*)\left(b^{*2}\left[e_1(b)\right]^2 + b^2\left[e_1(b^*)\right]^2\right)}{\gamma^2(1+bb^*)^2} \quad . \end{aligned} \tag{5.143}$$

6.0 THE PDE'S AND THE EINSTEIN EQUATIONS

In section 5.2, we calculated several curvatures as functionals of the S and S^* and the undetermined A_i. We now discuss imposing the conformal Einstein equations on these curvatures. In other words, we explore the conditions that make the metric (5.17),

$$g\,(Z\,,R\,,W\,,W^*\,,s\,,s^*) = \eta_{ij}\theta^i\otimes\theta^j \quad , \tag{6.1}$$

conformally Einstein.

Since everything (the metric, the connection, and the curvatures) depends on the functions S and S^* which satisfy the Wünschmann condition, the conditions for the conformal Einstein equations are actually further conditions – in addition to the Wünschmann condition – on these functions. Our aim is to find these extra conditions. To do this, we apply our new version of the conformal Einstein equations, Eq (3.66). Thus, the following is the first known application of this version.

Unfortunately, this problem is very formidable. In principle, we can formulate everything in terms of S and S^*. We will not, however, give any of the explicit equations since they are *millions* of terms long, even after severe approximations are taken. Accordingly, we will only present an outline of our methods.

Finally, we note that we did not investigate the other two versions of the conformal Einstein equations using this language. The first version, Eq (3.21), has been studied elsewhere using the language of the PDE's [8], and the second version, Eq (3.25), seems to be even more complicated than the third since it contains an extra level of derivatives by way of the Bach tensor. (The study of the second version in the language of the PDE's was unsuccessfully attempted by Tod [22].)

6.1 A REVIEW

For clarity, we briefly review in this section our new version of the conformal Einstein equations of chapter 3. We begin with the conformal transformation,

$$\hat{g}_{ab} = e^{2\phi} g_{ab} \quad , \tag{6.2}$$

and the Einstein equations for $\hat{g}_{ab}$,

$$\hat{R}^{\mathrm{TF}}_{ab} = \hat{R}_{ab} - \frac{1}{4}\hat{R}\hat{g}_{ab} = 0 \quad . \tag{6.3}$$

The inverse conformal transformation of Eq (6.3) yields the first version of the conformal Einstein equations,

$$0 = R^{\mathrm{TF}}_{ab} - 2\nabla_a \phi_b + 2\phi_a\phi_b + \frac{1}{2} g_{ab} \left(\nabla_c \phi^c - \phi_c \phi^c\right) \quad , \tag{6.4}$$

where $\phi_a = \nabla_a \phi$. This form of the conformal Einstein equations is a set of conditions on the metric g_{ab} *and* the conformal parameter ϕ. The metric g_{ab} that satisfies Eq (6.4) is called conformally Einstein.

A necessary but insufficient condition for a metric g_{ab} to be conformally Einstein is that it satisfy the conformal Yang equations,

$$\nabla_a C^a{}_{bcd} + \phi_a C^a{}_{bcd} = 0 \quad . \tag{6.5}$$

It is from these 16 equations that we obtained our new version of the conformal Einstein equations. The essential idea was to solve the conformal Yang equations for the four ϕ_a, which determines them as functions of g_{ab}, via the Weyl tensor $C^a{}_{bcd}$ and its divergence $\nabla_a C^a{}_{bcd}$. Call these solutions $\phi_a[g]$. The $\phi_a[g]$ are then inserted into the first version of the conformal Einstein equations, Eq (6.4), producing a set of equations that depend *only* on g_{ab}. Note that since Eq (6.5) is 16 equations, we obtain 12 conditions ($J_\sigma[g] = 0$) on the metric g_{ab} when we solve for the four $\phi_a[g]$. The new version of the conformal Einstein equations are therefore

$$\begin{cases} 0 = R^{\mathrm{TF}}_{ab} - 2\nabla_a \phi_b[g] + 2\phi_a[g]\phi_b[g] + \frac{1}{2} g_{ab} \left(\nabla_c \phi^c[g] - \phi_c[g]\phi^c[g] \right) \\ 0 = J_\sigma[g] \quad , \end{cases} \tag{6.6}$$

where the explicit expressions for $\phi_a[g]$ and $J_\sigma[g]$ are given in Eqs (3.63) and (3.65), respectively.

6.2 THE METHOD

We now return to the Cartan-Weyl geometry of chapter 5. In particular, recall that the metric was defined as

$$g\left(Z\,,R\,,W\,,W^*\,,s\,,s^*\right) = \eta_{ij}\theta^i \otimes \theta^j \quad . \tag{6.7}$$

Our aim is to determine the conditions under which this metric is (generically) conformally Einstein. To find them, we follow the procedure reviewed in Eqs (6.5) and (6.6). (See Eq 2.17 for the definition of a generic metric.)

We begin with imposing the vanishing of the second Cartan curvature of Eq (5.98),

$$\Omega_{ijk} = \nabla_m C^m{}_{ijk} + A_m C^m{}_{ijk} = 0 \quad . \tag{6.8}$$

By theorems 3.3.1 and 3.3.2, we thus have that the vector A_i is a gradient and is unique for the choice of the metric in Eq 6.7. Thus, as before in section 6.1, we solve Eq (6.8) for the A_i, which determines them as functions of g, *i.e.*, $A_i[g]$. In doing so, we also find the conditions $J_\sigma[g] = 0$.

Now recall Eq (5.78),

$$R_{jm} = \mathfrak{R}_{(jm)} - \eta_{jm}\nabla_p A^p - 2\left\{\nabla_{(m}A_{j)} + \eta_{jm}A^p A_p - A_j A_m\right\} + 4\nabla_{[j}A_{m]} \quad , \tag{6.9}$$

which was derived from the second structure equation, Eq (5.65). To obtain the conformal Einstein equations, we first require that the trace-free part of R_{ij} vanish,

$$R^{TF}_{ij} = 0 \quad . \tag{6.10}$$

Then, we insert the four $A_i[g]$ into $R^{TF}_{jm} = 0$. Since the vector $A_i[g]$ is a gradient, the term containing $\nabla_{[j}A_{m]}$ vanishes. Thus, we have the conformal Einstein equations for our Cartan geometry,

$$\begin{cases} 0 = \mathfrak{R}^{\mathrm{TF}}_{ij} - 2\nabla_i A_j[g] + 2A_i[g]A_j[g] + \frac{1}{2}\eta_{ij}\left(\nabla_k A^k[g] - A_k[g]A^k[g]\right) \\ 0 = J_\sigma[g] \quad . \end{cases} \tag{6.11}$$

The above set of equations depend *only* on the functions S and S^* and , therefore, are the additional conditions on these functions.

Our goal, however, was to calculate the above set of equations explicitly in terms of the functions S and S^*. When we attempted to do this exactly, the length of the equations became overwhelming. The solutions for the $A_i[g]$ and the conditions

$J_\sigma[g] = 0$ contained several million terms per equation. After many months of effort, with the aid of a supercomputing cluster and the mathematics programme Maple, we were unable to successfully simplify these equations. Thus, we were forced to try a severe approximation techniques, which we discuss below. Ultimately, this approximation reduced the size of our equations by about an order of magnitude, but we still had roughly a million terms per equation. Many more months of effort to simplify the equations were unfruitful.

6.2.1 The Off-Minkowski Approximation

Recall that in Minkowski space-time, the metric η_{ab} (in the standard coordinate basis) is given by

$$[\eta_{ab}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad . \tag{6.12}$$

For null plane waves in Minkowski space-time, it can be shown [5], [8], [16] that the function $Z = Z\,(x^a\,, s\,, s^*)$ describing the collection of null planes can be written

$$Z = l_a x^a \quad , \tag{6.13}$$

where the vector of coefficients $l^a = l^a\,(s\,, s^*)$ is

$$\left[\sqrt{2}\,(1+ss^*)\right] l^a = \{(1+ss^*)\;,(s+s^*)\;,i\,(s^*-s)\;,(-1+ss^*)\} \quad . \tag{6.14}$$

Using η_{ab} from above, one can show that this expression for Z satisfies the eikonal equation, Eq (4.4), for each value of the parameters $(s\,, s^*)$. Furthermore, this Z satisfies the pair of PDE's

$$D^2\,Z = D^{*2}\,Z = 0 \quad , \tag{6.15}$$

i.e., the functions S and S^* vanish.

Alternatively, one can start with the pair of PDE's in Eq (6.15) and then follow the procedure of chapter 5. In that case, the null tetrad of Eq (5.15) becomes

$$\theta^i = \theta^i{}_a dx^a \quad , \tag{6.16}$$

where

$$\begin{aligned} &\theta^0{}_a \equiv l_a \quad , \quad \theta^1{}_a \equiv n_a = (D^* D l_a + l_a) \quad , \\ &\theta^2{}_a \equiv m_a = D l_a \quad , \quad \theta^3{}_a \equiv m^*_a = D^* l_a \quad . \end{aligned} \tag{6.17}$$

Thus, $n^a l_a = 1$ and $m^a m^*_a = -1$, and all other scalar products are zero. The null metric η_{ij} (see Eq (6.7)) is then re-constructed by

$$g = \eta_{ij} \theta^i \otimes \theta^j \quad , \tag{6.18}$$

where the components $\theta^i{}_a$ are as above.

With null plane waves in mind – in particular, Eq (6.15) – a natural approximation to take in the analysis of the conformal Einstein equations is the "off-Minkowski" approximation,

$$S \ \rightarrow \epsilon S, \tag{6.19}$$

where ϵ is small. Using this approximation, we then calculate our system of equations (6.11) to second-order in ϵ.

To do so, we first use Eq (6.19) to approximate the set of tetrad parameters $\{\alpha\,, b\,, b^*\,, a\,, a^*\,, c\}$ given in Eqs (5.54) - (5.57). For example,

$$b \approx -\epsilon \frac{e_1(S)}{2} + O\left(\epsilon^3\right) \quad , \tag{6.20}$$

and

$$\alpha \approx 1 + \frac{3}{2} b b^* \tag{6.21}$$

The approximations of the other tetrad parameters are much more complicated and will not be displayed.

The next step is to compute the connection and its curvatures in terms of these approximated tetrad parameters. Then, we again apply Eqs (6.8) - (6.11).

As we have said, even after the off-Minkowski approximation, the conformal Einstein equations of (6.11) were roughly a million terms each. Thus, although we were successful in finding these equations explicitly in terms of the functions S and S^*, the equations are much too large to be useful.

7.0 CONCLUSION

In this work, we have accomplished many things. Of primary importance, we have shown how to extend Cartan's beautiful construction of differential geometric structures to the pair of PDE's

$$D^2 Z = S\,(Z\,, Z_s\,, Z_{s^*}\,, Z_{ss^*}\,, s\,, s^*) \quad ,$$
$$D^{*2}Z = S^*(Z\,, Z_s\,, Z_{s^*}\,, Z_{ss^*}\,, s\,, s^*) \quad . \tag{7.1}$$

The resulting geometry, namely the Cartan-Weyl conformal geometry, forms a rich set of quantities which includes, as a special case, all conformal Lorentzian metrics and their space-times. Consequentially, this geometry also contains all solutions to the vacuum Einstein equations.

Specifically, the restriction of the Cartan-Weyl geometry to conformal space-times was achieved via the Wünschmann condition. This condition was obtained geometrically through the torsion-free property of the generalised Levi-Civita-Weyl connection of Eq (5.33). One of our accomplishments was to find the Wünschmann condition explicitly, Eq (5.53), in terms of the inhomogeneous functions S and S^* defined by Eq (7.1). We note that the Wünschmann condition had been previously calculated in the context of the null-surface formulation of GR (*e.g.* [11]). In this work, however, we have re-obtained it from an entirely new point of view, namely via the torsion-free connection.

In order to explore the restriction of the Cartan-Weyl geometry to a conformal Einstein geometry, we used our new version of the conformal Einstein field equations, Eq (3.66), which does not contain the conformal parameter. When written explicitly in terms of the functions S and S^*, this version of the conformal field equations becomes a set of further conditions on S and S^*. We had hoped that this set would be relatively simple to express explicitly, but, unfortunately the equations were enormous, even after making the approximation of small S. Thus we concluded that Cartan-Weyl geometry was not a particularly useful application of this new version.

Perhaps this is not surprising. On the one hand, the new version of the field equations has an appealing geometric aesthetic since it is only a set of conditions of the conformal metric. On the other hand, this version is very unappealing algebraically, as it is a set of 21 complicated equations. (Nine of these come from the vanishing of the trace-free conformal Ricci tensor, *i.e.*, the first equation of Eq (3.66); the other twelve we review now.)

In order to eliminate the gradient of the conformal parameter from the conformal field equations, we had to solve the 16 conformal Yang equations for the four components of the gradient. This left us with the twelve equations that we called $J_\sigma[g] = 0$. These equations are, in general, rather complicated, and the fact that one has to satisfy twelve such equations simultaneously makes them even more cumbersome. If one could reduce or simplify them in some way, then the new version of the field equations would certainly be more powerful.

As of now, it is not clear how to do this precisely. We note, however, that very recently we have received suggestions that may simplify this set of equations, although the analysis seems very complicated. Much more work remains to be done.

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