THREE ESSAYS ON AUCTION THEORY AND CONTEST THEORY

by

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In the first chapter, *All-Pay Auctions with Resale*, I study equilibria of first- and second-price all-pay auctions with resale when players' signals are affiliated and symmetrically distributed. I show that existence of resale possibilities introduces an endogenous element to players' valuations and creates a signaling incentive for players. I characterize symmetric bidding equilibria for both first- and second-price all-pay auctions with resale and provide sufficient conditions for existence of symmetric equilibria. Under those conditions I show that second-price all-pay auctions generate no less expected revenue than first-price all-pay auctions with resale. The initial seller could benefit from publicly disclosing her private information which is affiliated with players' signals. Outcome in all-pay auctions is deterministic since the highest bidder wins the prize with probability one. However, many realistic contests have in-deterministic outcome and no player can guarantee winning the prize.

The second chapter, *Rent-Seeking Contest with Private Values and Resale*, studies rent-seeking contests with private values and resale possibilities. With an in-deterministic success function, the resulting possible inefficiency 1 creates a motive for aftermarket trade. Players' valuations are endogenously determined when there is an opportunity of resale. I characterize symmetric equilibria. I assume that the winner has full bargaining power; however, the results extend to other resale mechanisms. I show that resale enhances allocative efficiency ex post at the expense of more wasted social resources since players compete more aggressively with resale possibilities.

In the third chapter, *The Imperfectly Discriminating Contests with Incomplete Information*, I study the existence of monotone pure-strategy equilibria in imperfectly discriminating contests with incomplete information. Sufficient conditions under which equilibria exist are provided for

both finite-action and continuum-action cases. Using a two-bidder example, we derive some properties of equilibria and show a special case of revenue equivalence between contests with incomplete information and contests with complete information.

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PREFACE

First, I thank my advisor, Professor Andreas Blume for his continuing support and advice during the whole process of my study. Without his guide, I can not finish my dissertation with a decent job. Those regular meetings with him were always exciting and challenging since he always asks ”Anything new?” first. Moreover, I want to express my gratitude to my other committee members, Professors Alexander Matros, Utku Ünver, Esther Gal-Or for their insightful comments on my dissertation.

Second, I want to thank my parents. It is their wholehearted love and support that brought me out of my hometown to Beijing, and further to Pittsburgh. Without them, I would not have the luxury to be here and chase my intellectual dreams.

Finally, I would like to thank my wife Xi Zhang for her faith in me and priceless encouragement, which help me through the whole process of job hunting and dissertation writing. This dissertation is dedicated to her.

1.0 ALL-PAY AUCTIONS WITH RESALE

1.1 INTRODUCTION

This paper studies all-pay auctions with resale. In contrast to standard auctions where only winners are required to make payments, all-pay auctions exhibit special characteristic of unconditional payment, that is, bidders always pay their bids regardless of winning or losing. All-pay auctions or equivalent models have been widely used to model a variety of economic and social instances of conflict and competition such as lobbying, contests and tournaments, political campaigns, patent races, and so on.[1] Two formats of all-pay auctions are widely used in the existing literature: first-price all-pay auctions and second-price all-pay auctions.[2] They differ only in the winning bidders' payment. The winning bidder pays his own bid in first-price all-pay auctions, and the highest losing bid in second-price all-pay auctions. In both auctions, all losing bidders pay their own bids. Hence, they are analogous to standard first-price and second-price sealed-bid auctions.

Using the general symmetric model, Krishna and Morgan (1997) study all-pay auctions by characterizing equilibrium strategies and comparing expected revenues resulting from both first-price and second-price all-pay auctions. However, there are many instances a static all-pay auction model could not account for. Participants in such instances often face aftermarket competition, as in the following examples.

Patent Races. Patent races are frequently formalized as all-pay auctions since resources devoted by competitors are irreversible, regardless winning or losing. Winners in patent races can either retain exclusive use of the innovations or license the innovations for use by other producers.

[1] See, for example, Baye et al. (1993), Krishna and Morgan (1997), and Moldovanu and Sela (2001).

[2] Second-price all-pay auctions are better known as the war of attrition and are used to model conflicts among animals (Maynard Smith, 1982) and struggles for survival among firms (Fudenberg and Tirole, 1986).

When an innovation is to be patented, the winner who comes from a research institution is very likely to sell the patent to interested producers. On the demand side, an incumbent monopolist possessing related or substitutable technology has more incentive than a potential entrant to buy the patent.[3] Hence, very often there is a secondary market for patents. In a broader view, the resale of patented technology takes place all over the world. In particular, the transfer of patented technology from developed countries to developing ones promotes the economic performances of the latter. This could be evidenced by the development of many east Asian countries.[4]

Lobbying. Rent-seeking activities such as lobbying play an important role in the allocation of government contracts. Lobbyists make implicit payments to politicians through campaign contributions or other channels in order to influence political decisions. Lobbying is usually formalized as an all-pay auction since lobbyists' up-front payments are not refundable to those failing to win the prize. Again there is aftermarket competition in lobbying. Successful lobbyists often use subcontracting to reduce their production costs that are, in particular, strictly convex.[5]

Waiting-Line Competition. The allocation procedures based on a first-come-first-served principle could be considered as waiting-line competitions. Examples include allocating tickets of sports or concerts, foods or other necessities with scarcity, university parking lots or day-care services, discounted commodities, and so on. These waiting-line allocation procedures are often formalized as all-pay auctions, since players are involved in costly competition in some non-price dimensions for a limited number of prizes. Regardless of winning or losing, players' effort is sunk. Although resale is usually not allowed, in practice, speculative behaviors are prevalent. For instance, people with lower time cost could wait in line and profit by reselling the objects to those with higher opportunity cost of time.[6]

The above examples show that many realistic situations could be best analyzed through a model that incorporates resale possibilities into all-pay auctions. It remains open to characterize players'

[3]By doing this the monopolist could maintain market power, whereas competition results if the entrant obtains the patent. See Gilbert and Newbery (1982).

[4]I thank John Morgan for pointing out this example to me.

[5]For general results regarding subcontracting, please refer to Kamine et al. (1989) and Gale et al. (2000).

[6] For example, near the end of each year in China, a large number of migrant workers have to pay much more for train tickets than their face value in order to go back to their hometown for family reunion during the Chinese New Year. The middlemen, or *huangniu* (yellow bulls), who have lower opportunity cost of time would wait in line and make illegal profits by reselling the tickets to those who long for going home but have no time waiting in line to buy the tickets.

behaviors in all-pay auctions with resale. Intuitively existence of resale possibilities exhibits influence on bidders' bidding behaviors in the first stage. The aftermarket buyers usually have access to information revealed by the initial seller, such as submitted bids. If the submitted bids reveal private information of primary bidders, resale price will be responsive to those bids and a bidder's resale profit can depend on the bid he makes in the primary auction. Therefore, resale possibilities introduce an endogenous element to bidders' valuations upon winning the auction and creates an incentive for primary bidders to signal their private information to aftermarket buyers. This information connection between resale price and submitted bids is our primary focus.

The main objective of this paper is to investigate the effect of resale possibilities on bidders' bidding behaviors and the resulting expected revenues from both first- and second-price all-pay auctions. This paper considers a two-stage model in which an all-pay auction in the first stage is followed by resale. For the first stage, we analyze both first- and second-price all-pay auctions. For the second stage, we do not specify a resale mechanism and simply assume that resale is conducted through a competitive market. Therefore, winners of the first-stage auction have no bargaining power and can only affect their profits by signalling their private information through bids. This assumption could be relaxed if there is only one aftermarket buyer. For multiple buyers, this assumption is justified if interested buyers come to the market randomly and each proposes a take-it-or-leave-it offer to the winner.[7] In certain industries such as automobile and electronics, subcontracting is based on competitive bids.[8]

We extend Krishna and Morgan (1997)'s general symmetric setting to incorporate resale possibilities into all-pay auctions. We characterize symmetric bidding equilibria for both first- and second-price all-pay auctions with resale. Based on these equilibria, we compare the two formats from the perspective of a revenue-maximizing seller. In addition, we examine the impact of information disclosure of the initial seller.

This paper is related to the literature regarding auctions with resale. Bikhchandani and Huang (1989) present a closely related model with symmetric information applicable to treasury bill auctions, where pure common values and a competitive resale market are assumed. Resale takes place because most bidders in the first stage are speculators and bid for resale. They characterize equilib-

[7]In equilibrium, such offer will be accepted by the winner. It makes more sense if the winner discounts future payoff more than the buyer does.

[8]The point is that subcontracting is used to reduce costs if we assume production cost is strictly convex.

rium bidding strategies for both discriminatory and uniform-price auctions. Provided symmetric equilibria exist, they show that uniform-price auctions generate no less expected revenue than discriminatory auctions. We study all-pay auctions with resale using a similar model, but we allow bidders' valuations to be interdependent.

Using a model with independent private values, Haile (2003) studies auctions with resale under private uncertainties. Resale takes place because of the discrepancy between the estimated values at the time of bidding and the true values realized after the auction. He characterizes equilibrium bidding strategies for first-price, second-price and English auctions followed by resale which could be formalized as an optimal auction or an English auction. He argues that the option to resell creates endogenous valuations and induces signaling incentives that may reverse the revenue results obtained in the literature that assumes no resale.[9] Assuming positively correlated signals and interdependent valuations, we show that second-price all-pay auctions generate no less expected revenue than first-price all-pay auctions with resale possibilities.

The rest of this paper is organized as following. Section 1.2 contains the model. Section 1.3 and 1.4 study second-price and first-price all-pay auctions respectively. Section 1.5 provides ranking of expected revenues of these two auction formats. Section 1.6 examines the effect of information disclosure. Section 1.7 concludes. The appendix contains all proofs.

1.2 THE MODEL

The game proceeds as following. In the first stage, N risk-neutral bidders compete for an indivisible good in an all-pay auction–either first-price or second-price. The bidder who submits the highest bid wins the object and pays either his own bid or the highest losing bid, depending on the exogenously chosen auction format. All losing bidders pay their own bids. Due to institutional or other reasons, there are some bidders who cannot participate in the first-stage competition.[10] However, they will try to obtain the object through aftermarket bargaining with the winner. Therefore,

[9] A similar signalling incentive is also examined in Goeree (2003).

[10] Some bidders may not be eligible to participate in certain competition for a special prize, say an monopoly privilege. Some bidders may be excluded by strategic behaviors of a revenue-maximizing seller. See an example in Section 5.

after the primary auction is over, there is possibility for resale.

In the second stage, potential buyers approach the primary winner in order to obtain the object. We assume that those losers from the first stage do not participate in the aftermarket competition. Actually in our symmetric setting of the game, if the equilibrium strategies are nondecreasing, there is no potential gain for resale taking place within the same group of bidders. On the other hand, it is likely that the valuation of certain aftermarket buyer exceeds that of primary winner, therefore there may be potential gain to be realized if new entrants bargain with the first-stage winner.

After the first-stage auction is over, we assume that the initial seller announces the winning bid and the highest losing bid.[11] Based on the released information, aftermarket buyers could infer the private information held by the first-stage winner. To focus on the information transmission given resale possibilities, we assume that resale price will be the expected first-stage winner's valuation of the object conditional on all publicly available information. Therefore, the seller has no bargaining power.

For the first-stage all-pay auction, we follow the framework and notation of Krishna and Morgan (1997) to make the analysis consistent with the literature. Prior to auction each bidder i receives a private signal, X_i, that affects value of the object V_i defined as:

$$V_i = V(S, X_i, \{X_j\}_{j \neq i}) \tag{1.1}$$

where $S = (S_1, S_2, ..., S_m)$ are any other random variables that influence the valuation but are not observed by any bidder. We assume that V is non-negative, continuous, increasing in all its variables. For each i, $E[V_i] < \infty$. Moreover, all bidders' valuations depend on S in the same manner, and each bidder's valuation is a symmetric function of other bidders' signals.

Let X_0 be private information held by the initial seller who may or may not reveal it. Let $f(S, X_0, X_1, X_2, ..., X_n)$ be the joint density of random variables $S, X_0, X_1, X_2, ..., X_N$, where f is symmetric in bidders' signals. We assume that f satisfies the affiliation inequality:

$$f(z \vee z')f(z \wedge z') \geq f(z)f(z') \tag{1.2}$$

[11]This information release procedure is standard in most auction literature. Although interesting, the optimal information disclosure is not addressed in this paper. Calzolari and Pavan (2006) studies optimal information disclosure for a monopolist who cannot commit to prevent resale.

where $z \vee z'$ denotes the component-wise maximum of z and z' and $z \wedge z'$ denotes the component-wise minimum of z and z'. Roughly, this means that a high value of one of the variables, S_j or X_i, makes it more likely that the other variables also take on high values. Let $[0,\bar{s}]^m \times [0,\bar{x}_0] \times [0,\bar{x}]^n$ be the support of f, where $[0,\bar{x}]^n$ denotes the n-fold product of $[0,\bar{x}]$.

Let $f_{Y_1}(\cdot|x)$ denote the conditional density of Y_1, where $Y_1 = \max\{X_j\}_{j\neq 1}$, given $X_1 = x$. Standard results from Milgrom and Weber (1982) show that X_1 and Y_1 are also affiliated. Throughout the paper, we make use of the following facts: $F_{Y_1}(y|x)$ and $\frac{f_{Y_1}(y|x)}{1-F_{Y_1}(y|x)}$ are non-increasing in x.[12] Moreover, if H is any nondecreasing function, affiliation implies that $h(a_1,b_1;\ldots,a_n,b_n) = E[H(X_1,\ldots,X_n)|a_1 \leq X_1 \leq b_1,\ldots,a_n \leq X_n \leq b_n]$ is nondecreasing in all of its arguments. For simplicity, we also assume that if H is continuously differentiable, then $E[H(X_1,\ldots,X_n|a_1 \leq X_1 \leq b_1,\ldots,a_n \leq X_n \leq b_n]$ is also continuously differentiable in all its arguments.[13]

A *pure strategy* for bidder i is a measurable function, $\beta_i : [0,\bar{x}] \to \mathbb{R}$. Such a pure strategy is *monotone* if $x' \geq x$ implies $\beta_i(x') \geq \beta_i(x)$.

An N-tuple of pure strategies, $(\beta_1,\ldots,\beta_n)$ is an *equilibrium* if for every bidder i and every pure strategy β_i',

$$EU_i(\beta(x),x) \geq EU_i(\beta_i'(x),\beta_{-i}(x_{-i}),x) \tag{1.3}$$

where the left-hand side, is bidder i's expected utility given the joint strategy β, and the right-hand side is his expected utility when he employs β_i' and the others employ β_{-i}.

The equilibrium is *symmetric* if $\beta_1 = \cdots \beta_N = \beta$. Since bidders are ex ante identical, we are considering symmetric equilibrium bidding strategies.

1.3 SECOND-PRICE ALL-PAY AUCTIONS WITH RESALE

We first characterize the symmetric equilibrium for second-price all-pay auction with resale. Without loss of generality, we analyze the game from bidder 1's point of view. When bidder 1 submits his bid, he only observes his own private signal X_1.

[12] In the appendix, we provide the detailed proof for these facts.

[13] For more details about affiliation, please refer to Milgrom and Weber (1982).

According to our assumption, buyers on the secondary market observe only publicly announced information and resale price is the expectation of the primary winner's valuation conditional on all public information. It is useful to begin with a heuristic derivation of the first-order condition for β_s to be a symmetric Nash equilibrium in strictly increasing and differentiable strategies.[14]

Suppose bidders $j \neq 1$ follow the symmetric equilibrium strategy β_s. Suppose bidder 1 receives a private signal $X_1 = x$ and bids b. If bidder 1 wins with a bid b and the secondary market buyers believe that he is following β_s, the resale price will be

$$P(\beta_s^{-1}(b), Y_1) = E[V_1 | X_1 = \beta_s^{-1}(b), Y_1] \tag{1.4}$$

where β_s^{-1}denotes the inverse of β_s and Y_1 is the first-order statistic of $(X_2, ..., X_N)$. When bidder 1 wins the object and buyers on the secondary market believe that his private signal is equal to x', the expected resale price conditional on X_1 and Y_1 is:

$$v(x', x, y) \equiv E[P(x', Y_1) | X_1 = x, Y_1 = y] \tag{1.5}$$

By affiliation, both P and v are non-decreasing in all their arguments. With this notation, the expected payoff for bidder 1 is:

$$\Pi(b, x) = \int_0^{\beta_s^{-1}(b)} (v(\beta_s^{-1}(b), x, y) - \beta_s(y)) f_{Y_1}(y|x) dy - [1 - F_{Y_1}(\beta_s^{-1}(b)|x)]b \tag{1.6}$$

Maximizing (6) with respect to b yields the first-order condition

$$\begin{aligned} 0 \quad = \quad & \frac{1}{\beta_s'(\beta_s^{-1}(b))} v(\beta_s^{-1}(b), x, \beta_s^{-1}(b)) f_{Y_1}(\beta_s^{-1}(b)|x) \\ & + \frac{1}{\beta_s'(\beta_s^{-1}(b))} \int_0^{\beta_s^{-1}(b)} v_1(\beta_s^{-1}(b), x, y) f_{Y_1}(y|x) dy] \\ & - [1 - F_{Y_1}(\beta_s^{-1}(b)|x)] \end{aligned}$$

where v_1 is the partial derivative with respect to $\beta_s^{-1}(b)$.

At a symmetric equilibrium, it is optimal that $\beta_s(x) = b$, then we have

$$\beta_s'(x) = v(x, x, x) \frac{f_{Y_1}(x|x)}{1 - F_{Y_1}(x|x)} + \int_0^x v_1(x, x, y) \frac{f_{Y_1}(y|x)}{1 - F_{Y_1}(x|x)} dy \tag{1.7}$$

[14]We will show later that the equilibrium strategy is indeed strictly increasing and differentiable.

The solution with the boundary condition $\beta_s(0) = 0$ is:

$$\beta_s(x) = \int_0^x v(t,t,t)\frac{f_{Y_1}(t|t)}{1-F_{Y_1}(t|t)}dt + \int_0^x k(u)du \tag{1.8}$$

where $k(u) = \int_0^u v_1(u,u,y)\frac{f_{Y_1}(y|u)}{1-F_{Y_1}(u|u)}dy$.

This is only necessary condition for a symmetric equilibrium. For sufficiency, we need the following assumption.

Assumption 1. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\psi(x',x,y) = v(x',x,y)\frac{f_{Y_1}(y|x)}{1-F_{Y_1}(y|x)}$. We assume that for all y, (i) $\psi_2 > 0$, and (ii) $\psi_{12} > 0$.*

Given the above assumption, we have the following result.

Theorem 1. *Suppose Assumption 1 hold, then a symmetric equilibrium of second-price all-pay auction with resale is given by β_s defined as*

$$\beta_s(x) = \int_0^x v(t,t,t)\frac{f_{Y_1}(t|t)}{1-F_{Y_1}(t|t)}dt + \int_0^x k(u)du$$

where $k(u) = \int_0^u v_1(u,u,y)\frac{f_{Y_1}(y|u)}{1-F_{Y_1}(u|u)}dy$.

Remark 1. *Assumption 1 is restrictive and critical to the sufficiency result. Note that affiliation implies that $\frac{f_{Y_1}(y|x)}{1-F_{Y_1}(y|x)}$ is non-increasing in x, and $v(x',x,y)$ is non-decreasing in x for every y. Therefore, part (i) ensures that the affiliation between X_1 and Y_1 is not so strong that it overwhelms the increase in the expected valuation of the object, $v(x',x,y)$, resulting from a higher signal x. Part (ii) is needed to ensure that the responsiveness of resale price and primary bidders' signalling incentive are increasing in signal.*

Example 1. *Suppose $N = 2$. Let $f(x,y) = \frac{4}{9}(2+xy)$ on $[0,1]^2$, and $v(x',x,y) = x'x + \frac{1}{2}y$. Simple manipulation yields $f_y(y|x) = \frac{4+2xy}{4+x}$, and $F_Y(y|x) = \frac{4y+xy^2}{4+x}$. Then we have*

$$\psi(x',x,y) = v(x',x,y)\frac{f_{Y_1}(y|x)}{1-F_{Y_1}(y|x)} = \frac{(2x'x+y)(2+xy)}{4+x-4y-xy^2}$$

It can be verified that all conditions in Assumption 1 is satisfied.

Using a variation of part (i) of Assumption 1, Krishna and Morgan (1997) derive sufficient condition for a symmetric equilibrium of second-price all-pay auction (the war of attrition) without resale. In their context, $v(x,y) = E[V_1|X_1 = x, Y_1 = y]$, and the symmetric equilibrium is given by:

$$\alpha_s(x) = \int_0^x v(t,t)\frac{f_{Y_1}(t|t)}{1-F_{Y_1}(t|t)}dt \tag{1.9}$$

To ensure it is indeed a symmetric equilibrium, they assume that $\phi(\cdot,y)$ is increasing for all y, where $\phi(x,y) \equiv v(x,y)\frac{f_{Y_1}(y|x)}{1-F_{Y_1}(y|x)}$.

Examining both equilibria, we could observe that $\beta_s(x)$ reduces to $\alpha_s(x)$ if there is no resale possibilities and the primary bidders have no incentive to signal. Resale possibilities introduce an endogenous element to primary bidders' valuation since bidders' resale profit depend on the bids they make in the primary auction. Hence primary bidders have incentives to signal their private information to aftermarket buyers. They signal in order to convince aftermarket buyers that the object is of high value since resale price is responsive to the announced bids. This responsiveness is measured by v_1; v_1 is nonnegative and increasing in x due to affiliation and Assumption 1.

From the analysis above, we can conclude that the information disclosing policy is crucial to our characterization. In second-price all-pay auctions with resale, if only the highest losing bid (price paid by the winner) is revealed, primary bidders' signalling incentive will be reduced since the winning bid conveys private information for the first-stage winner. On the other hand, if the initial seller releases more information, the expected resale price will not decrease, may increase based on more information. This increases the expected valuation of bidders upon winning, hence they will bid more aggressively than otherwise.

1.4 FIRST-PRICE ALL-PAY AUCTIONS WITH RESALE

The analysis is parallel to previous section. Again we begin with a heuristic derivation of equilibrium. Suppose bidders $j \neq 1$ follow the symmetric equilibrium strategy β_f. Suppose bidder 1 receives a private signal $X_1 = x$ and bids b. If bidder 1 wins with a bid b and aftermarket buyers believe that he is following β_f, slightly abusing notation yields

$$P(\beta_f^{-1}(b), Y_1) = E[V_1|X_1 = \beta_f^{-1}(b), Y_1]$$

where β_f^{-1}denotes the inverse of β_f. When bidder 1 wins the object and buyers on the secondary market believe that his private signal is equal to x', the expected resale price conditional on X_1 and Y_1 is:

$$v(x',x,y) \equiv E[P(x',Y_1)|X_1 = x, Y_1 = y]$$

By affiliation, both P and v are non-decreasing in all their arguments. With this notation, the expected payoff for bidder 1 is

$$\Pi(b,x) = \int_0^{\beta_f^{-1}(b)} v(\beta_f^{-1}(b),x,y) f_{Y_1}(y|x)dy - b \tag{1.10}$$

Maximizing (10) with respect to b yields the first-order condition:

$$\begin{aligned} 0 &= v(\beta_f^{-1}(b),x,\beta_f^{-1}(b)) f_{Y_1}(\beta_f^{-1}(b)|x)\frac{1}{\beta_f'(\beta_f^{-1}(b))} \\ &+ \frac{1}{\beta_f'(\beta_f^{-1}(b))}\int_0^{\beta_f^{-1}(b)} v_1(\beta_f^{-1}(b),x,y) f_{Y_1}(y|x)dy - 1 \end{aligned}$$

where β_f' is the first derivative of β_f, and $v_1(\beta_f^{-1}(b),x,y)$ is the partial derivative of v with respect to its first argument.

At a symmetric equilibrium, $\beta_f(x) = b$ and thus

$$\beta_f'(x) = v(x,x,x) f_{Y_1}(x|x) + \int_0^x v_1(x,x,y) f_{Y_1}(y|x)dy \tag{1.11}$$

The solution to equation (11) with the boundary condition $\beta_f(0) = 0$ is:

$$\beta_f(x) = \int_0^x v(t,t,t) f_{Y_1}(t|t)dt + \int_0^x h(u)du \tag{1.12}$$

where $h(u) = \int_0^u v_1(u,u,y) f_{Y_1}(y|u)dy$.

The derivation is heuristic since (11) is only a necessary condition. For the sufficiency, we need additional restriction like Assumption 1. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\Phi(x',x,y) = v(x',x,y) f_{Y_1}(y|x)$$

One implication of Assumption 1 leads to the following lemma. The argument makes use of the fact that $F_{Y_1}(y|x)$ is non-increasing in x, and proof is contained in appendix.

Lemma 1. *Suppose Assumption* 1 *hold. Then for all y, we have (i)* $\Phi_2 > 0$*, and (ii)* $\Phi_{12} > 0$.

With Lemma 1, we can show that the equilibrium we characterize is indeed a symmetric equilibrium.

Theorem 2. *Suppose Assumption* 1 *hold, then a symmetric equilibrium in first-price all-pay auctions with resale is given by* β_f *defined as*

$$\beta_f(x) = \int_0^x v(t,t,t) f_{Y_1}(t|t)dt + \int_0^x h(u)du$$

where $h(u) = \int_0^u v_1(u,u,y) f_{Y_1}(y|u)dy$*.*

For first-price all-pay auctions without resale, Krishna and Morgan (1997) characterize a symmetric equilibrium:

$$\alpha_f(x) = \int_0^x v(t,t) f_{Y_1}(t|t)dt \tag{1.13}$$

To ensure it is indeed a symmetric equilibrium, they assume that $\varphi(\cdot,y)$ is increasing for all y, where $\varphi(x,y) \equiv v(x,y) f_{Y_1}(y|x)$.[15]

Examining both equilibria, we find that $\beta_f(x)$ reduces to $\alpha_f(x)$ if there is no resale possibility and primary bidders have no incentive to signal. As we argue in previous section, resale possibilities introduce an endogenous element to primary bidders' valuation since bidders' resale profit depend on the bids they make in the primary auction. Hence primary bidders have incentives to signal their private information to aftermarket buyers. The implication of Assumption 1, say Lemma 1, ensures that the responsiveness of resale price to announced bids (measured by v_1) increases with a bidder's private signal. This further guarantees that each bidder's incentive to signal increases with his private signal. Without resale, only $\Phi_2 > 0$ is needed to characterize the symmetric equilibrium.

Because of responsiveness of resale price to announced bids, the information disclosing policy affects primary bidders' bidding behaviors. Affiliation implies that resale price is non-decreasing in all bidders' private signals. Therefore, the first-stage bidders may bid more aggressively if the initial seller announces more bids. If the initial seller announces less information, bidders' incentives to signal their private information will be reduced.

[15] Krishna and Morgan (1997) show that for all y, that $\phi(\cdot,y)$ is increasing implies $\varphi(\cdot,y)$ is increasing.

1.5 REVENUE COMPARISON

In this section, we investigate the performance of first- and second-price all-pay auctions with resale in terms of expected revenue accruing to the initial seller. Given the symmetric equilibria we characterize, which auction format is better from the perspective of a revenue-maximizing seller? Without resale, Krishna and Morgan (1997) derive a revenue ranking between these two auction formats: second-price all-pay auctions generate no less expected revenue than first-price all-pay auctions.[16] The following subsection examines whether this ranking remains true with resale possibilities.

1.5.1 TWO ALL-PAY AUCTION FORMATS WITH RESALE

First, let us compare the expected revenue generated by both auction formats at symmetric equilibria.

Theorem 3. *Suppose Assumption* 1 *hold. With resale possibilities, the expected revenue from second-price all-pay auction is greater than or equal to that from first-price all-pay auction at the symmetric equilibria.*

The proof is contained in the appendix. Here we provide an intuitive explanation using *linkage principle*. Milgrom and Weber (1982) originally introduce linkage principle to auction literature in order to derive the revenue ranking among first-price, second-price, and English auction when signals are affiliated and valuations are interdependent. One of the implications is that the expected revenue from second-price auctions is no less than that from first-price auctions. Krishna and Morgan (1997) further apply linkage principle to all-pay auctions and show that this ranking remains true if we require all bidders pay their bids. Theorem 3 further implies that this revenue ranking maintains when there are resale possibilities. The common thread running through is *linkage principle*.

Consider the auctions as revelation games, then the selling price (the revenue of seller) could depend only on the bids or bidders' reports, and on the seller's information. Then if the winner's

[16] They further show that second-price all-pay auctions generate no less expected revenue than second-price auctions, and first-price all-pay auctions generate no less expected revenue than first-price auctions.

payment depend on the second-highest bidder's signal, which is affiliated with the winner's private signal, the expected payment would be statistically linked to that information. As a result of affiliation, this linkage reduces the information rent the seller must leave to bidders to induce truthful revelation of private information. For a fixed bid, a higher private signal of the winner means that the second highest signal is more likely to take on high values, so the expected payment of the winner is also higher. Hence, the linkage makes the expected price paid in equilibrium by the winner increase more steeply as a function of his signal than otherwise. By our boundary condition, a bidder with the lowest type pays nothing, then a steeper expected payment function yields higher expected prices. This is the intuition behind the revenue ranking result. Clearly this linkage still works when resale is allowed.

To see how the linkage principle work, suppose bidder 1 learns his private signal as x, but bids as if it were z. Let $e^k(z,x)$ denote the expected payment made by bidder 1 in the k-price all-pay auction with resale, where $k = \{f,s\}$.

Krishna and Morgan (1997) derive a variation of linkage principle that is more useful in our model of all-pay auctions with resale.

Proposition 1. *Suppose L and M are two auction mechanisms with symmetric increasing equilibria such that the expected payment of a bidder with the lowest signal is* 0*. If for all x,* $e_2^M(x,x) \geq e_2^L(x,x)$ *then for all x,* $e^M(x,x) \geq e^L(x,x)$*.*

To see how this principle works, let $R(z,x)$ denote the expected value received by bidder 1. Then

$$R(z,x) = \int_0^z v(z,x,y) f_{Y_1}(y|x) dy \tag{1.14}$$

Indeed, this expression is the same for both all-pay auction forms at the symmetric equilibria. Then the expected payoff for bidder 1 is:

$$\Pi^k(z,x) = R(z,x) - e^k(z,x) \tag{1.15}$$

In equilibrium, it is optimal to choose $z = x$, and the first-order condition yields $e_1^f(x,x) = e_1^s(x,x)$, where $e_1^k(x,x)$ is the derivative of $e^k(z,x)$ with respect to z evaluated at $z = x$.

Note that we have

$$e^k(z,x) = \begin{cases} e^f(z) = \beta_f(z), & \text{if } k = f \\ \int_0^z \beta_s(y) f_{Y_1}(y|x)dy + [1 - F_{Y_1}(z|x)]\beta_s(z), & \text{if } k = s \end{cases}$$

Taking the derivative with respect to x, it is trivial to show that $e^f_2(z,x) = 0$ and

$$e^s_2(z,x) = -\frac{\partial}{\partial x}\int_0^z \beta'_s(y) F_{Y_1}(y|x)dy \geq 0$$

since $F_{Y_1}(y|x)$ is non-increasing in x due to affiliation. Applying Proposition 1, we have $e^s(x,x) \geq e^f(x,x)$ since $e^s(0,0) = e^f(0,0) = 0$.

Therefore, the initial seller could benefit from exogenously choosing second-price instead of first-price all-pay auctions if resale is allowed.

Recall Example 1: Let $f(x,y) = \frac{4}{9}(2+xy)$ on $[0,1]^2$, and $v(x',x,y) = x'x + \frac{1}{2}y$.

Based on the equilibrium strategies derived above, We have:

$$R_f = 2\int_0^1 \int_0^x \frac{3t^4 + t^3 + 8t^2 + 2t}{4+t} dt dx$$

and

$$R_s = 2\int_0^1 \int_0^x \frac{3t^4 + t^3 + 8t^2 + 2t}{4+t} \theta dt dx$$

where $\theta = \frac{1 - F_t(t|x)}{1 - F_t(t|t)}$. Therefore, we have $R_s \geq R_f$ since $\theta \geq 1$ when $x \geq t$.

1.5.2 DOES RESALE BENEFIT SELLER?

There are still some questions to be explored. First, if the seller can commit to allow resale or not, should he allow resale? Under what conditions does the existence of an active secondary market benefit the initial seller?[17] Generally, if the first-stage winner has access to some potential buyers to bargain, primary participants not only compete for the object, but also compete for the right to resell. By signalling their private information, they bid more aggressively than without resale possibilities.

Second, does a revenue-maximizing seller have incentive to exclude some bidders from the first stage competition, forcing them to the secondary market? This question remains open with affiliated signals and interdependent values.

Baye et al. (1993) present an interesting *exclusion principle*: a revenue maximizing politician may find it in his best interest to exclude lobbyists with valuations above a threshold from participating in the all-pay auction. Since values are public information, the exclusion makes the competition more even and bidders submit higher bids, which in turn increases the initial seller's expected revenue. Bose and Deltas (1999) study English auction with two distinct types of potential bidders: consumers who bid for their own consumption and speculators who bid for resale. They show that, if the speculators have access to a larger market of consumers than the seller, then the seller prefer to prevent the consumers from participating in the auction.

1.6 INFORMATION DISCLOSURE BY THE INITIAL SELLER

Very often the initial seller has private information that may affect bidders' valuation or private information. Suppose the initial seller has private signal X_0 that is affiliated with all bidders' signals. Now consider how equilibria in all-pay auctions are affected when the initial seller publicly reveals X_0. Conditional on $X_0 = x_0$, we could derive the symmetric equilibria for both second-price and first-price all-pay auctions with resale.

[17] Calzolari and Pavan (2006) show that a monopolist benefits from the existence of resale market when he cannot contract with all potential buyers and he can prohibit the winner from reselling to the losers. However, the monopolist will get hurt if resale cannot be banned and takes place among the same group of bidders.

Before stating the results, we need more notations. Let $\tilde{\beta}_k(\cdot,x_0)$ be a symmetric equilibrium bidding strategy conditional on $X_0 = x_0$, $k = \{f,s\}$. Define

$$P(\tilde{\beta}_k^{-1}(b,x_0),Y_1,X_0) = E[V_1|X_1 = \tilde{\beta}_k^{-1}(b,x_0),Y_1,X_0 = x_0] \tag{1.16}$$

as the resale price if bidder 1 wins the auction with a bid b, and aftermarket buyers believe that he is following $\tilde{\beta}_k(\cdot,x_0)$. Similarly, define

$$\tilde{v}(x',x,y,x_0) = E[P(x',Y_1,X_0)|X_1 = x,Y_1 = y,X_0 = x_0] \tag{1.17}$$

as the expected resale price conditional on X_1,Y_1 and X_0. To derive the symmetric equilibria given $X_0 = x_0$, a modification of Assumption 1 is needed.

Assumption 2. *Let $\tilde{\psi} : \mathbb{R}^4 \to \mathbb{R}$ be defined by $\tilde{\psi}(x',x,y,x_0) = \tilde{v}(x',x,y,x_0)\frac{f_{Y_1}(y|x,x_0)}{1-F_{Y_1}(y|x,x_0)}$. We assume that for all y, (i) $\tilde{\psi}_2 > 0, \tilde{\psi}_4 > 0$, and (ii) $\tilde{\psi}_{12} > 0, \tilde{\psi}_{14} > 0$.*

By the same argument as Lemma 1, we have

Lemma 2. *Let $\tilde{\Phi} : \mathbb{R}^4 \to \mathbb{R}$ be defined by $\tilde{\Phi}(x',x,y,x_0) = \tilde{v}(x',x,y,x_0)f_{Y_1}(y|x,x_0)$. Suppose Assumption 2 hold, then for all y, (i) $\tilde{\Phi}_2 > 0, \tilde{\Phi}_4 > 0$, and (ii) $\tilde{\Phi}_{12} > 0$, $\tilde{\Phi}_{14} > 0$.*

As we characterize the symmetric equilibria without information about the seller's private signal, we could derive the symmetric equilibria conditional on seller's private information for both auction formats.

Proposition 2. *Suppose Assumption 2 hold. Conditional on the seller's private signal X_0, a symmetric equilibrium in second-price all-pay auctions with resale is given by $\tilde{\beta}_s$ defined as*

$$\tilde{\beta}_s(x,x_0) = \int_0^x \tilde{v}(t,t,t,x_0)\frac{f_{Y_1}(t|t,x_0)}{1-F_{Y_1}(t|t,x_0)}dt + \int_0^x \tilde{k}(u)du \tag{1.18}$$

where $\tilde{k}(u) = \int_0^u \tilde{v}_1(u,u,y,x_0)\frac{f_{Y_1}(y|u,x_0)}{1-F_{Y_1}(u|u,x_0)}dy$.

Proposition 3. *Suppose Assumption 2 hold. Conditional on the seller's private signal X_0, a symmetric equilibrium in first-price all-pay auctions with resale is given by $\tilde{\beta}_f$ defined as*

$$\tilde{\beta}_f(x,x_0) = \int_0^x \tilde{v}(t,t,t,x_0)f_{Y_1}(t|t,x_0)dt + \int_0^x \tilde{h}(u)du \tag{1.19}$$

where $\tilde{h}(u) = \int_0^u \tilde{v}_1(u,u,y,x_0)f_{Y_1}(y|u,x_0)dy$.

Remark 2. *Note that the equilibrium bidding function* $\tilde{\beta}$ *now maps two variables into a bid. For any fixed value of* X_0*, the equilibrium bidding strategy is a function of bidder's private signal only and is similar to* β*. Affiliation between* X_0 *and* $(X_1,\ldots,X_N)$*, Assumption* 2 *and Lemma* 2 *guarantee that the equilibrium bidding function* $\tilde{\beta}$ *is increasing as* X_0 *increases. Conditional on* X_0*, primary bidders signal their private information to aftermarket buyers through their bids. The responsiveness of resale price to announced bids and information increases as the realization of a bidder's private signal.*

An immediate implication of affiliation and Assumption 2 is that the initial seller could benefit from publicly releasing his private signal.

Proposition 4. *Suppose that Assumption* 2 *hold. A policy of publicly revealing the initial seller's private information cannot lower, and may raise the expected revenue for the seller in all-pay auctions with resale.*

The intuition underlying this result can be best understood through linkage principle. Publicly releasing his private signal, the initial seller establishes a link between the bids submitted and that signal. This additional link reduces the information rent enjoyed by the bidders possessing private information. Hence, the revenue-enhancing result follows as a consequence of linkage principle. Therefore, releasing the seller's private signal has similar effect as releasing more bids.

Obviously Proposition 4 relies crucially on Assumption 2. If Assumption 2 fails to hold, $\tilde{\beta}(x,x_0)$ may not be an increasing function of x_0 because the marginal effect of the bid on the resale price may be reduced and revealing X_0 may reduce the bidders' incentive to signal, and then lower the expected revenue for the seller even though $X_0,X_1,\ldots,X_N$ are affiliated. Extremely, if X_0 contains all the relevant information in $X_1,X_2,\ldots,X_N$, there will be no signaling incentive for the bidders, and the expected price in all-pay auctions will be lower than otherwise. The various information structure and corresponding optimal information disclosure policy is technically complex and remains open. Intuitively the optimal information disclosure policy depends on the specific resale mechanism and the distribution of bargaining power between the winner and aftermarket buyers.[18]

[18] Calzolari and Pavan (2006) studies optimal information disclosure for a monopolist who cannot commit to prevent resale.

1.7 CONCLUSIONS

This paper studies all-pay auctions with resale. Costly competitions over a limited number of prizes are often followed by aftermarket interaction, as winners of patent races sell or license patents to other producers. We find that introducing resale possibilities changes bidders' behaviors in all-pay auctions. The information connection between the resale prices and the bids submitted by the first-stage bidders creates signalling incentive for primary bidders. We provide sufficient conditions under which symmetric equilibria exist and characterize equilibria strategies. Provided the existence of symmetric equilibria, we show that second-price all-pay auctions generate no less expected revenue than first-price all-pay auctions with resale. Furthermore, if the bidders' signals are affiliated to the initial seller's private signal, the seller could enhance his expected revenue by publicly disclosing that information.

Several extensions of this model are as following. First, we do not explicitly formalize the resale mechanism. We simply assume that the resale price equals to the expected valuation of the winner in the first stage. In practice, the resale mechanism could be another auction, or a multilateral bargaining. Intuitively different resale mechanisms and the distribution of bargaining power will affect the split of resale surplus between resale seller and buyers, and in turn affect the bidding strategies adopted by primary bidders.[19]

Second, we assume that the initial seller announce the winning bid and the highest losing bid after the primary auction. Obviously different information disclosing policies have different impacts on the significance of information linkage between resale price and submitted bids. The signalling behavior relies on the announcement of winning bids. Therefore, it remains a challenging question to investigate the optimal information disclosing policy from the seller's perspective. From the standpoint of mechanism design, the optimal auction with resale may also depend on the information disclosing policy. The characterization of optimal selling mechanism with resale seems to be another challenging exercise.[20]

[19]Using an asymmetric two-bidder independent private value model, Hafalir and Krishna (2006) characterize the equilibrium bidding strategies when resale takes place via monopoly pricing. They also show that the results could easily extend to other resale mechanisms such as monopsony pricing and a probabilistic k-double auction.

[20] Ausubel and Cramton (1999) consider an optimal multi-unit auction with efficient secondary market. Zheng (2002) extends Myerson (1981)'s optimal auction design to the case in which resale cannot be prevented. Lebrun (2005) shows that the second-price auction with resale implements Myerson's optimal auction.

Third, we assume that the first-stage losers do not participate in aftermarket competition. Resale takes place when new entrants come to the market. If there are a fixed number of competitors, it will be interesting to examine the optimal excluding policy from the seller's perspective. We provide a simple example to illustrate that the seller may find in his best interest to exclude one bidder randomly. A more general analysis is worth exploring.

A special characteristic of all-pay auctions lies in the deterministic winning probabilities. Many interesting instances, however, have stochastic allocation of the objects. If the winning probability is stochastic, the more a bidder bids, the higher the probability he wins. But he never guarantees winning. Then the allocation will be ex post inefficient with positive probability. It will be worth investigating whether resale enhances allocative efficiency and compare expected revenue resulting from the deterministic model with that resulting from the stochastic model.[21]

[21] Using a simple two-bidder-two-value stochastic model, Sui (2006) shows that resale enhances allocative efficiency and increases expected revenue for the seller as long as the winner has more bargaining power than the loser.

2.0 CONTESTS WITH PRIVATE VALUES AND RESALE

2.1 INTRODUCTION

Situations in which competitors expend costly and irreversible resources to win a limited number of prizes are ubiquitous. Since Tullock's (1975, 1980) seminal contribution, the theory of rent-seeking contests has advanced considerably.[1] Most literature studying Tullock's rent-seeking contest assumes stochastic success function, in the sense that the winning probability of a player is proportional to her expenditure relative to the total expenditure. The more one spends, the more likely he will win the prize. But he can never guarantee winning even if he spends the most. Therefore, the allocation of the prize is stochastic and thus inefficient ex post with positive probability. This is true even if competitors are ex ante identical and follow symmetric strategies in equilibrium.

The possibility of ex post inefficiency lies in the stochastic winning probability and it leaves space for potential gain through aftermarket trade–resale. Indeed, many realistic characteristics of rent-seeking contests cannot be captured by a static Tullock model. For instance, the prize in rent-seeking contests could be a patented innovation. If a cost-reducing innovation is to be patented, the incumbent monopolist holding related or substitutable technology has more incentive to acquire the patent than a potential entrant. Then the winner could benefit from selling the patent to the incumbent monopolist. If the rent is a certain operating license that is usually not allowed to sell, the loser could obtain the license by taking over the whole company holding it. If the rent relates to government contracts, like defense contracts, the winner could benefit through subcontracting with the losing rival. In all above situations, the winner may not have the highest valuation due to the stochastic success function. Hence, the winner has incentive to resell the prize to those having higher valuations in order to seek additional profit.

[1] See Nitzan (1994) for an excellent survey.

The purpose of this paper is to investigate the effect of allowing resale on players' strategic behaviors and seller's expected revenue. Malueg and Yates (2004) were the first to study rent-seeking contests with two-sided private information. Using a two-player-two-value model, they characterize the equilibrium strategies and provide sufficient conditions for the existence of a symmetric equilibrium. This paper follows the same model and introduce resale possibility into the standard rent-seeking contests with private information.

Resale possibilities introduce an endogenous element to players' valuations. Upon winning, a low-value player could resell the prize to his rival who may have higher valuation with positive probability. Similarly, a high-value loser may benefit from trading with a low-value winner, depending on the distribution of bargaining power. This changes both players' strategic behaviors since they will incorporate this additional opportunities of buying and selling when they formulate their outlay of effort. We characterize the symmetric equilibrium and find an interesting proportional difference property. At symmetric equilibrium, both players compete more aggressively and thus increase expected revenue for the seller. Moreover, if the seller could commit to publicly disclosing players' private values, s/he could further increase his or her expected revenue.

Introducing resale possibility to the standard rent-seeking contests will improve allocative efficiency ex post. This is straightforward if both players have two valuations. If both players have the same valuation about the prize, the allocation is always efficient. Whenever they value the prize differently, there will be positive probability that the ex post allocation is inefficient. Indeed, if a low-value player competes with a high-value rival and wins the prize, there is potential gain if he resells the prize to the high-value rival. The resale price will be determined by their relative bargaining power. For analytical simplicity, we assume that the winner has full bargaining power. However, we can show that our results are robust to other resale mechanisms, like monopsony pricing resale or probabilistic k-double auctions that will be defined in Section 6.

More and more theoretical literature studies resale in standard auctions, such as sealed-bid and English auctions.[2] This paper is the first one that studies resale in rent-seeking contest environment.[3] The main focus of this paper is to investigate how resale changes players' strategic behaviors in rent-seeking contests and its effect on expected revenue and allocative efficiency. We

[2] See Haile (2003) for a thorough analysis.

[3] Sui (2006) studies resale through a different setting: all-pay auctions. The resale results from new entrants to the market, instead of possible inefficient allocation.

hope that our study sheds some light on this interesting question.

The rest of the paper is organized as following. Section 2 presents the model. Section 3 studies rent-seeking contests with private values and resale possibility and Section 4 contains parallel study when values become public. Section 5 derives a general revenue ranking result. Section 6 examines two other resale mechanisms and Section 7 concludes. All proofs are contained in the appendix.

2.2 THE MODEL

The rent-seeking contest proceeds as following. Two risk-neutral players, 1 and 2, compete for an indivisible prize to be awarded by a seller. Each player privately learns her valuation of the prize, v_1 and v_2. We assume that the possible realizations of valuation could be either low (v_L) or high (v_H). The prior probability distribution of (v_1, v_2) is given by

$$f(v_1, v_2) = \begin{cases} \frac{1}{2}\sigma, & \text{if } v_1 = v_2 \\ \frac{1}{2}(1-\sigma), & \text{if } v_1 \neq v_2 \end{cases} \tag{2.1}$$

The distribution is symmetric, but players' values could be different and correlated. For instance, $\sigma = 0$ refers to perfect negative correlation, $\sigma = 1$ to perfect positive correlation, and $\sigma = \frac{1}{2}$ to independence. From (1), we have the following conditional probabilities.

$$Pr(v_2 = v_L | v_1 = v_L) = Pr(v_2 = v_H | v_1 = v_H) = \sigma \tag{2.2}$$

and

$$Pr(v_2 = v_L | v_1 = v_H) = Pr(v_2 = v_H | v_1 = v_L) = 1 - \sigma \tag{2.3}$$

After learning their private valuations, both players simultaneously submit nonnegative bids, b_1 and b_2, which could also be considered as effort levels. Since both players are ex ante identical, without loss of generality, we analyze the game from the standpoint of player 1. The probability of player 1 wins the prize is given by $p(b_1, b_2)$ defined as

$$p_1(b_1, b_2) = \frac{b_1}{b_1 + b_2} \tag{2.4}$$

The winning probability of player 2 is $1 - p_1(b_1, b_2)$. For given values and bids, the expected utility for player 1 is

$$U_1(b_1, b_2) = \frac{b_1}{b_1 + b_2} v_1 - b_1 \tag{2.5}$$

Similarly, we could define $U_2(b_1, b_2)$. The utility functions and the probability distribution of values are common knowledge.

After the prize is awarded, there is possibility of resale if the low-value player wins the prize. With positive probability, the loser has high value. Thus there is potential gain resulting from resale for the low-value winner. If the winner has high value, there is no resale since no potential gain exists. For resale mechanism, we assume the winner possess full bargaining power, so he will propose a take-it-or-leave-it offer to the loser. The winner tries to extract as much surplus as possible, so he will ask for v_H and the high-value loser will accept the offer in equilibrium.

A *pure strategy* β_1 for player 1 specifies a bid contingent on the realization of his private value. Formally, $\beta_1 = (\beta_L, \beta_H)$ specifies bids β_L if $v_1 = v_L$, β_H if $v_1 = v_H$. A *Bayesian equilibrium* is a pair of strategies (β_1, β_2) such that β_1 maximizes player 1's expected payoff conditional on his realizations of value and player 2 using β_2; and β_2 maximizes player 2's expected payoff conditional on his realizations of value and player 1 using β_1. The Bayesian equilibrium is *symmetric* if $\beta_1 = \beta_2 = \beta$.

2.3 EQUILIBRIUM WITH PRIVATE VALUES AND RESALE

First we characterize a symmetric equilibrium and then provide the sufficient conditions under which this equilibrium exists.

Suppose player 2 follow strategy β, player 1 learns his private value as v_L and submits b_L, then his expected utility is

$$EU_1(b_L, \beta) = \sigma \frac{b_L}{b_L + \beta_L} v_L + (1 - \sigma) \frac{b_L}{b_L + \beta_H} v_H - b_L \tag{2.6}$$

With probability $1 - \sigma$, he competes with a high-value player and resells the prize at price equal to player 2's valuation v_H; with probability σ, the competitor has low value and there will be no resale. Maximizing (6) with respect to b_L yields

$$\frac{\sigma\beta_L}{(b_L+\beta_L)^2}v_L+\frac{(1-\sigma)\beta_H}{(b_L+\beta_H)^2}v_H=1 \tag{2.7}$$

At the symmetric equilibrium, $b_L=\beta_L$, then

$$\frac{\sigma}{4\beta_L}v_L+\frac{(1-\sigma)\beta_H}{(\beta_L+\beta_H)^2}v_H=1 \tag{2.8}$$

Multiplying both sides of (8) by β_L, we have

$$\frac{\sigma}{4}v_L+\frac{(1-\sigma)\beta_L\beta_H}{(\beta_L+\beta_H)^2}v_H=\beta_L \tag{2.9}$$

By symmetry, we also have

$$\frac{\sigma}{4}v_H+\frac{(1-\sigma)\beta_L\beta_H}{(\beta_L+\beta_H)^2}v_H=\beta_H \tag{2.10}$$

(9) and (10) imply that

$$\beta_H-\beta_L=\frac{\sigma}{4}(v_H-v_L) \tag{2.11}$$

Since the objective functions are globally concave, solutions to (9) and (10) determine a unique symmetric equilibrium. Due to analytical complexity, we cannot derive the closed-form equilibrium strategies. However, we do know that there exists one symmetric equilibrium determined by (9) and (10).

Proposition 5. *The symmetric equilibrium* β *of the rent-seeking contest with resale is given by the solutions to* (9) *and* (10). *In addition, the equilibrium efforts satisfy* $\frac{\beta_H-\beta_L}{v_H-v_L}=\frac{\sigma}{4}$.

Malueg and Yates (2004) fully characterize equilibrium strategies for rent-seeking contests with private values but without resale. Indeed, the equilibrium strategies have proportionality property. For the convenience of comparative study, we summarize their result in the following proposition as a reference.

Proposition 6. *Let* $\rho=\frac{v_L}{v_H}$*, then the symmetric equilibrium of the rent-seeking contest without resale is given by* $\tilde{\beta}_L=\tilde{\theta}v_L$, $\tilde{\beta}_H=\tilde{\theta}v_H$ *where* $\tilde{\theta}=\frac{\sigma}{4}+\frac{1-\sigma}{(\rho^{-1/2}+\rho^{1/2})^2}$.

From (11), we have

$$\frac{\beta_H - \beta_L}{v_H - v_L} = \frac{\sigma}{4} \tag{2.12}$$

By Proposition 2, we have

$$\frac{\tilde{\beta}_H - \tilde{\beta}_L}{v_H - v_L} = \frac{\sigma}{4} + \frac{1-\sigma}{(\rho^{-1/2} + \rho^{1/2})^2} \geq \frac{\sigma}{4} = \frac{\beta_H - \beta_L}{v_H - v_L} \tag{2.13}$$

Remark 3. *The introduction of resale possibility increases low-value player's valuation upon winning the contest since he has full bargaining power. Hence the low-value player has incentive to bid more aggressively to increase the winning probability and his expected payoff upon winning. As for the high-value player, the situation is as if she is now competing with a rival possessing higher valuation than before. Standard wisdom in contest literature suggests that a player with high valuation bids more aggressively if he learns his rival's valuation is high rather than low. However, the high-value player does not bid as more aggressively as the low-value player does. This is the intuition behind* (13).

The same intuition could explain the comparative statics of σ. Let $\delta = \beta_H - \beta_L$, and $\tilde{\delta} = \tilde{\beta}_H - \tilde{\beta}_L$, then we have

$$\frac{\partial \delta}{\partial \sigma} = \frac{1}{4}(v_H - v_L) \tag{2.14}$$

and

$$\frac{\partial \tilde{\delta}}{\partial \sigma} = (\frac{1}{4} - \frac{1}{\rho + \rho^{-1} + 2})(v_H - v_L) \leq \frac{1}{4}(v_H - v_L) \tag{2.15}$$

Increases in σ implies greater ex post similarity in realized values, thus evens out the contest. But the effects of this evening out on equilibrium effort levels are different with or without resale. Without resale, evening out the contest leads to closer competition; however, resale opportunities lead to more aggressive competition for both players, thus distort the effect of evening out.

In rent-seeking contests, the overall welfare with resale possibilities is hard to tell. It is good that resale promotes allocative efficiency ex post, however, more aggressive bids means more social waste since more resources are devoted to unproductive activities. Our analysis may be justified if revenue is one of goals when politicians allocate the rent.

2.4 EQUILIBRIUM WITH PUBLIC VALUES AND RESALE

Consider the rent-seeking contests with public values and resale possibilities. We assume that the realizations of (v_1, v_2) are revealed publicly to both players before they submit their bids. We first characterize the pure-strategy Nash equilibrium and then compare it with the symmetric equilibrium we characterize in previous section.

According to the combination of values, we have four cases: (v_L, v_L), (v_L, v_H), (v_H, v_L), and (v_H, v_H). For the first and last cases, there is no potential gain of resale, so resale indeed does not take place for those two cases. Resale only happens when a low-value player competes with a high-value rival.

Using similar arguments as in Section 3, we have the following result.

Proposition 7. *The Nash equilibrium of rent-seeking contest with public values and resale possibilities is given by* β *defined as*

$$\beta_{LL} = \frac{v_L}{4},\ \beta_{LH} = \beta_{HL} = \beta_{HH} = \frac{v_H}{4}$$

Nti (1999) characterizes the equilibrium for contests with public values without considering resale possibilities. We summarize his result in the following proposition:

Proposition 8. *The unique pure-strategy Nash equilibrium of the rent-seeking contest with public values and without resale is given by* $(\tilde{\beta}_1, \tilde{\beta}_2)$ *defined as*

$$\tilde{\beta}_i(v_i, v_j) = \frac{v_i^2 v_j}{(v_i + v_j)^2} \tag{2.16}$$

From Proposition 4, we have $\tilde{\beta}_{LL} = \frac{v_L}{4}$, $\tilde{\beta}_{HH} = \frac{v_H}{4}$, $\tilde{\beta}_{LH} = \frac{v_L^2 v_H}{(v_L+v_H)^2}$, and $\tilde{\beta}_{HL} = \frac{v_H^2 v_L}{(v_L+v_H)^2}$.

Again resale opportunities make the low-value player compete more aggressively than otherwise, in turn makes the high-value player compete more aggressively as well. This is clearly seen from that

$$\tilde{\beta}_{LH} \leq \tilde{\beta}_{HL} \leq \beta_{LH} = \beta_{HL} \tag{2.17}$$

2.5 REVENUE AND WELFARE

Now let us compare expected revenues resulting from rent-seeking contests with or without resale.

Let R_C denote ex ante expected revenue resulting from contests with public values and resale, then we have

$$R_C = \sigma(\beta_{LL} + \beta_{HH}) + (1-\sigma)(\beta_{LH} + \beta_{HL}) = \frac{1}{4}(\sigma v_L + (2-\sigma)v_H) \tag{2.18}$$

Let $\tilde{R}_C$ denote ex ante expected revenue resulting from contests with public values and no resale, then we have

$$\tilde{R}_C = \sigma(\tilde{\beta}_{LL} + \tilde{\beta}_{HH}) + (1-\sigma)(\tilde{\beta}_{LH} + \tilde{\beta}_{HL}) = \frac{1}{4}[\sigma(v_L + v_H) + 4(1-\sigma)\frac{v_L v_H}{v_L + v_H}] \tag{2.19}$$

Therefore, we have

$$R_C - \tilde{R}_C = \frac{(1-\sigma)v_H(v_H - v_L)}{2(v_L + v_H)} \geq 0 \tag{2.20}$$

Proposition 9. *For rent-seeking contests with public values, the expected revenue with resale exceeds that without resale.*

Malueg and Yates (2004) documents a revenue equivalent result for rent-seeking contest without resale. They show that, ex ante expected revenue in rent-seeking contests with private information equals that in rent-seeking contests with public information. This could be verified as following.

Let $\tilde{R}_I$ denote ex ante expected revenue resulting from contests with incomplete information and no resale, and $\tilde{R}_C$ with complete information and no resale. Then we have

$$\tilde{R}_I = \sigma(\tilde{\beta}_L + \tilde{\beta}_H) + (1-\sigma)(\tilde{\beta}_L + \tilde{\beta}_H) = \tilde{\theta}(v_L + v_H) \tag{2.21}$$

and

$$\tilde{R}_C = \sigma(\tilde{\beta}_{LL} + \tilde{\beta}_{HH}) + (1-\sigma)(\tilde{\beta}_{LH} + \tilde{\beta}_{HL}) = \tilde{\theta}(v_L + v_H) \tag{2.22}$$

As we know, the existence of resale possibilities introduces an endogenous element for low-value player's valuation. Therefore, both players will compete more aggressively, which implies that $\tilde{R}_I \leq R_I$. But the extent for this upward change of valuation is different under different informational regimes. With public information, the low-value player's valuation upon winning is v_H if competing with a high-value rival. With private information, however, it only becomes $\sigma v_L + (1-\sigma)v_H$ since he does not know whether he competes with low-value or high-value rival. It is this uncertainty that decreases low-value player's incentive to bid more aggressively, hence reduces seller's expected revenue. Therefore, $R_I \leq R_C$. This argument shows that the endogenous element for low-value player's valuation upon winning explains why rent-seeking contest with public information is revenue superior. Without resale possibility, there is no such element. Therefore, revenue equivalence follows.

Given the above results, we could derive a general revenue ranking for rent-seeking contests with or without resale and with private values or public values. Therefore, the general revenue ranking result follows.

Proposition 10. *The ex ante expected revenues resulting from rent-seeking contests could be ranked as*

$$\tilde{R}_C = \tilde{R}_I \leq R_I \leq R_C \tag{2.23}$$

As for the efficiency of allocation, with resale possibility the ex post allocation of the prize is always efficient by construction of the equilibria. This is true regardless of informational regimes.

Proposition 11. *For a two-player-two-value rent-seeking contest with resale possibilities, the ex post allocation is always efficient regardless of informational regimes. Moreover, the ex post allocation is inefficient with positive probability without resale. Hence, introducing resale possibility enhances allocative efficiency ex post.*

Without resale possibilities, Malueg and Yates (2004) shows that private-information and public-information contests are identical in terms of allocative efficiency. Indeed, for each possible realization of players' values (v_1, v_2), the prize is awarded to a player with the highest value with the same probability in private-information contests as in public-information contests. However, in both informational regimes, ex post allocation is not efficient with positive probability. This inefficiency will disappear if resale possibilities are introduced.

We must admit the limitation of Proposition 7. The ex post efficiency is only restored for the simple case with two players and each player has two possible values. Once we deviate from this simple model, Proposition 7 no longer holds. Consider the following example.

Example 2. *Two players compete for one indivisible prize. Each player's valuation is independent* *draw from* $\{v_L, v_M, v_H\}$ *with equal probability, where* $v_L \leq v_M \leq v_H$.

If the realized values are the same, the ex post allocation is efficient no matter who wins the prize. If the realized values are different, there is no scheme ensuring ex post efficiency. If the winner has high value, there will no resale. If the winner has medium value, he will ask v_H, and the offer will be accepted in equilibrium if his rival has high value. If the winner has low value, there is no optimal bargaining scheme that ensures efficiency ex post. If the low-value asks v_M, his expected valuation will be $\frac{v_L+2v_M}{3}$; if the low-value asks v_H, his expected valuation will be $\frac{2v_L+v_H}{3}$. Depending on different parameter values, the low-value winner's optimal asking price may be different. For instance, if $v_M < \frac{v_L+v_H}{2}$, he will ask v_H. However, with positive probability his rival may have medium valuation. Hence, the final allocation may not be efficient with positive probability.

Intuitively, the more values both players have, the more difficult to ensure ex post efficiency through resale. If both players have continuous private valuation, Myerson and Satterthwaite (1983) show that there is no incentive-compatible individually rational bargaining mechanism can be ex post efficient.

2.6 OTHER RESALE MECHANISMS

Until now we just consider only one possibility of resale mechanism: the winner makes a take-it-or-leave-it offer to the loser. We refer to this as monopoly resale since the winner has full bargaining power. In practice the resale buyer may share the bargaining power with the seller or even have full bargaining power. In this Section, we consider other possible resale mechanisms: monopsony pricing and probabilistic k-double auctions. Specifically, monopsony pricing means that the buyer has full bargaining power and makes a take-it-or-leave-it offer to the seller. For probabilistic k-double auctions, we refer to the case in which with probability k, the seller makes a take-it-or-leave-it offer to the buyer and with probability $1-k$, the buyer makes a take-it-or-leave-it offer to the seller.[4]

[4] As Hafalir and Krishna (2006) points out, the term k-double auction usually refers to a situation in which the price is weighted average of the price demanded by the seller and that offered by the buyer.

2.6.1 MONOPSONY RESALE

For monopsony resale, the loser of contest has full bargaining power and can make a take-it-or-leave-it offer to and buy the prize from the winner. As before, resale only takes place when the realized valuations of both players are different. Then the high-value loser exerts his bargaining power by offering a price equals to the low-value winner's valuation, v_L, and extracts all the surplus.

Suppose player 2 follow strategy μ, player 1 learns his private value as v_L and submits b_L, the expected payoff for him is

$$EU_1(b_L,\mu) = [\sigma\frac{b_L}{b_L+\mu_L} + (1-\sigma)\frac{b_L}{b_L+\mu_H}]v_L - b_L \tag{2.24}$$

Similarly, the expected payoff for player 1 if his private value is v_H and he bids b_H:

$$EU_1(b_H,\mu) = \sigma\frac{b_H}{b_H+\mu_H}v_H + (1-\sigma)[\frac{b_H}{b_H+\mu_L}v_H + \frac{\mu_L}{b_H+\mu_H}(v_H - v_L)] - b_H \tag{2.25}$$

By manipulating the first-order conditions, we have

$$\mu_H - \mu_L = \frac{\sigma}{4}(v_H - v_L) \tag{2.26}$$

The above result is quite interesting since we have exactly the same relationship for monopoly resale mechanism. In other words, no matter who has the full bargaining power, the ratio between difference in equilibrium efforts and difference in valuation remains the same: $\frac{\sigma}{4}$.

Similarly, as before we can derive the equilibrium strategies with public values and monopsony resale:

$$\mu_{LL} = \mu_{LH} = \mu_{HL} = \frac{v_L}{4},\ \mu_{HH} = \frac{v_H}{4} \tag{2.27}$$

Hence the ex ante expected revenue is

$$\hat{R}_C = \frac{1}{4}(\sigma v_H + (2-\sigma)v_L) \tag{2.28}$$

Recall that $\tilde{R}_C = \frac{1}{4}[\sigma(v_L+v_H) + 4(1-\sigma)\frac{v_L v_H}{v_L+v_H}]$, then we have

$$\hat{R}_C - \tilde{R}_C = 2(1-\sigma)\frac{v_L(v_L - v_H)}{v_L+v_H} \leq 0 \tag{2.29}$$

Proposition 12. *For rent-seeking contests with public values, if the losing player possesses full bargaining power, resale opportunities decrease expected revenue for the seller.*

Remark 4. *The intuition underlying Proposition* 8 *is as follows. Assigning full bargaining power to the loser only benefits the high-value player. This makes high-value player's expected payoff upon losing the contest positive. This decreases his incentive to compete as aggressively as what he does without resale opportunities. Moreover, the low-value player has incentive to bid less aggressively since he has no bargaining power upon winning, no additional benefit upon losing. Therefore, the overall equilibrium outlay is less than that without resale.*

2.6.2 PROBABILISTIC *K*-DOUBLE AUCTIONS

In this mechanism, resale takes place as follows. With probability k, the winner of contest makes a take-it-or-leave-it offer to the loser and with probability $1-k$ the loser makes a take-it-or-leave-it offer to the winner. Again resale takes place ex post only if players have different valuations. Obviously if $k=1$, this reduces to the monopoly resale mechanism considered earlier. If $k=0$, it reduces to the monopsony resale mechanism. In this subsection, we consider the case $0<k<1$.

Let us first characterize the symmetric equilibrium strategies. Suppose player 2 follow τ, player 1 learns his valuation as v_L and submits b_L, his expected payoff will be

$$EU_1(b_L,\tau) = \sigma \frac{b_L}{b_L+\tau_L} v_L + (1-\sigma)\frac{b_L}{b_L+\tau_H}\tilde{v} - b_L \tag{2.30}$$

where $\tilde{v} = kv_H + (1-k)v_L$ is the expected valuation upon winning when player 1 competes with a high-value rival.

Similarly, the expected payoff for player 1 if his private value is v_H and he bids b_H:

$$EU_1(b_H,\tau) = \sigma \frac{b_H}{b_H+\tau_H} v_H + (1-\sigma)[\frac{b_H}{b_H+\tau_L} v_H + \frac{\tau_L}{b_H+\tau_L}\check{v}] - b_H \tag{2.31}$$

where $\check{v} = (1-k)(v_H - v_L)$ is the expected valuation upon losing when player 1 competes with a low-value rival.

By manipulating the first-order conditions, we have

$$\tau_H - \tau_L = \frac{\sigma}{4}(v_H - v_L) \tag{2.32}$$

From previous analysis, we already know that such relationship holds in cases $k=1$ and $k=0$. It is not surprising to observe the same relationship when $0<k<1$. Let us summarize this interesting finding as follows.

Proposition 13. *For rent-seeking contests with resale, the ratio between difference in equilibrium effort and difference in valuation remains a constant* ($\frac{\sigma}{4}$) *independent of the distribution of bargaining power. Furthermore, this ratio is less than that without resale* ($\frac{\sigma}{4}+\frac{1-\sigma}{\rho+\rho^{-1}+2}$).

To figure out the expected revenue if values are public information, we need to derive the equilibrium strategies first. It is trivial to show that

$$\tau_{LL}=\frac{v_L}{4},\ \tau_{HH}=\frac{v_H}{4},\ \tau_{LH}=\tau_{HL}=\frac{(kv_H+(1-k)v_L)}{4} \tag{2.33}$$

Therefore, the ex ante expected revenue with public information ($\bar{R}_C$) is

$$\bar{R}_C=\frac{1}{4}(mv_H+(2-m)v_L) \tag{2.34}$$

where $m=\sigma+2(1-\sigma)k$.

Recall the expected revenue with public values and without resale $\tilde{R}_C=\frac{1}{4}[\sigma(v_L+v_H)+4(1-\sigma)\frac{v_Lv_H}{v_L+v_H}]$, then we have

$$\bar{R}_C-\tilde{R}_C=2(1-\sigma)\frac{(v_H-v_L)[k(v_L+v_H)-v_L]}{v_L+v_H} \tag{2.35}$$

Hence, $\bar{R}_C\geq\tilde{R}_C$ if and only if $k\geq\frac{v_L}{v_L+v_H}$.

It is interesting to examine how the ex ante expected revenue will change if the distribution of bargaining power varies. Given $\bar{R}_C=\frac{1}{4}(mv_H+(2-m)v_L)$, we have

$$\frac{\partial\bar{R}_C}{\partial k}=\frac{1-\sigma}{2}(v_H-v_L)\geq 0 \tag{2.36}$$

The intuition is quite straightforward. The resale possibility introduces an endogenous element to the winner's valuation upon winning. The more bargaining power the winner has, the more surplus he can extract from resale.[5] Hence, the low-value player will bid more aggressively. This means the high-value player is competing against a rival with higher valuation, thus he will also bid more aggressively. The unconditional payment rule of rent-seeking contests renders the expected revenue to become larger as more bargaining power goes to the winner. Indeed, we have[6]

$$\hat{R}_C\leq\bar{R}_C\leq R_C \tag{2.37}$$

[5] When $v_L=v_H$ or $\sigma=1$, the expected revenue is independent of the distribution of bargaining power. Under these two extremes, both players' valuations are perfectly aligned, hence there will be no resale.

[6] It is trivial to show that $R_C\geq\hat{R}_C$. Note $\bar{R}_C-\hat{R}_C=\frac{r}{4}(v_H-v_L)(m-\sigma)\geq 0$ since $m=2k-2k\sigma+\sigma\geq\sigma$. Moreover, $R_C-\bar{R}_C=\frac{r}{4}(v_H-v_L)(2-m-\sigma)\geq 0$ since $2-m-\sigma=2(1-\sigma)(1-k)\geq 0$.

It is trivial to show that $\beta_{LL} = \mu_{LL} = \tau_{LL}$, $\beta_{HH} = \mu_{HH} = \tau_{HH}$, $\beta_{LH} \geq \tau_{LH} \geq \mu_{LH}$, and $\beta_{HL} \geq \tau_{HL} \geq \mu_{HL}$.[7] Again the intuition is as before. Although the high-value player loses more bargaining power as k increases, he still bids more aggressively since the low-value player bids more aggressively as k increases. It is as if the high-value player is competing against with a rival with higher and higher valuation. Simple manipulation of equilibria strategies shows that the ex ante expected bids are getting bigger as the winner gets more bargaining power. This is true for both the low-value player and the high-value player.

A general revenue ranking among all possible situations is not available. It can be shown that the general ranking will depend on specific realization of parameter values.

2.7 DISCUSSION AND CONCLUSION

We have introduced into a standard rent-seeking contest with private values the interesting feature that resale may take place whenever there is a potential gain from implementing it. We characterize the symmetric equilibrium and find an interesting proportional difference property. We show that resale enhances both ex post allocative efficiency and seller's ex ante expected revenue. By comparing both private and public information regimes, we derive a general revenue ranking for rent-seeking contest. It turns out that the highest expected revenue goes to the case with resale and public information. Depending on different parameter values, ex ante a player may prefer values to be private, public or indifferent. Similarly, whether resale possibility benefits a player ex ante depends on different parameter values.

For analytic simplicity, we focus on a two-player-two-value model and assume that the winner has full bargaining power in resale. The two-player-two-value setting ensures that most of our results are robust to variation of resale mechanisms. Indeed, they survive for any resale mechanism in which the resale price is somewhere between both players' valuations. But they do not survive if each player has more than two values. For example, if each player has three values, resale cannot lead to efficient allocation ex post. Depending on different configurations of values, the allocation

[7] The ex ante bids submitted by the low-value player are $\frac{v_L}{4}$, $\sigma\frac{v_L}{4} + (1-\sigma)\frac{(kv_H + (1-k)v_L)}{4}$ and $\sigma\frac{v_L}{4} + (1-\sigma)\frac{v_H}{4}$ for $k = 0, 0 < k < 1, k = 1$ respectively. Similarly, the ex ante bids submitted by the high-value player are $\sigma\frac{v_H}{4} + (1-\sigma)\frac{v_L}{4}$, $\sigma\frac{v_H}{4} + (1-\sigma)\frac{(kv_H + (1-k)v_L)}{4}$, and $\frac{v_H}{4}$ for $k = 0, 0 < k < 1, k = 1$ respectively.

may not be always efficient ex post. Suppose that possible values are low, intermediate or high, the low-value player wins the contest and has full bargaining power. After updating his posterior belief, the low-value winner makes a take-it-or-leave-it offer in order to maximize his expected payoff. If values are private information, there is no general optimal pricing scheme that ensures efficiency ex post.[8]

More general analysis needs to be addressed with more than two players and continuous type space. Unfortunately there is no theoretical benchmark for rent-seeking contests with private information and more than two players or continuous type space. Actually it remains open to characterize equilibrium strategies for just two players whose valuations' supports have three points. The characterization of equilibrium strategies for these situations with or without resale seems a challenging exercise.

[8] I am indebted to Andreas Blume for pointing out to me this arguments, which lead to an interesting generalization for the current model.

3.0 THE IMPERFECTLY DISCRIMINATING CONTESTS WITH INCOMPLETE INFORMATION

3.1 INTRODUCTION

This paper studies the existence of pure-strategy equilibria for imperfectly discriminating contests with incomplete information. Many economic allocations are decided by competition over a prize where each participant exerts effort or spends resource to win the prize, and the losers' efforts or expenditures are irreversible. Rent seeking activities such as lobbying, R&D contests, arm races, political campaigns, and sports contests are a few examples.

Since Tullock's (1975, 1980) seminal contribution, two models are widely used in this literature to study players' behavior under these situations: perfectly discriminating model–all-pay auctions, and imperfectly discriminating model–lotteries. In the former situation, the player making the highest effort or spend the most resource is designated the winner, while in the latter situation, the allocation of the prize is stochastic in the following sense: the probability that a given player wins the prize is proportional to her expenditure relative to the total expenditure. Even if one player spends the most, she can never guarantee winning the prize as long as there is any positive expenditure made by the others. The exogenous choice of deterministic or stochastic success function is the essential difference between these two models.

For all-pay auctions with complete information, the equilibria are fully characterized by Baye et al. (1996) for arbitrary configurations of valuations. For incomplete information all-pay auctions, equilibrium behaviors have been studied by Hillman and Riley (1989), and Krishna and Morgan (1997). Equilibrium uniqueness in frameworks with two players is addressed in Amann and Leininger (1996) and Lizzeri and Persico (2000). For independent private values and risk averse bidders, or positive interdependent values and independent information, the existence of a non-decreasing pure strategy equilibrium is documented in Athey (2001).

For lotteries, equilibria with complete information are fully characterized by Hillman and Riley (1989). Fang (2002) further shows that the equilibrium characterized by Hillman and Riley (1989) is unique. Using a two-player-two-value model, Malueg and Yates (2004) first studies lotteries with private values.[1] However, a general theoretical benchmark for imperfectly discriminating contests remains open.

The imperfectly discriminating contests addressed in this paper could be embedded into a large class of Bayesian games whose existence of equilibria is thoroughly studied. For a large class of games of incomplete information, Athey (2001) shows the existence of pure strategy equilibria under the so-called single crossing condition.[2] Mcadams (2003) extends Athey's results to settings in which type and action spaces are multi-dimensional and partially ordered.[3] Most recently, Reny (2006) provides a generalization of the results of both Athey and Mcadams through a new route.[4]

Following their methods, we provide the sufficient conditions under which a monotonic pure strategy equilibrium exists in the imperfectly discriminating contests. We use Athey's method to show the existence for finite-action contest games, and use Reny's method for continuum-action contest games. Although Reny's method could work for both cases, Athey's method is more helpful for computational analysis of the equilibrium strategies. Actually Athey first discretizes the action space to show the existence results for finite-action games, and then invokes limiting arguments to show the existence results for continuum-action games. This naturally provides an algorithm for computational study.

Using a two-player-two-value example, we characterize the symmetric equilibrium and show the monotonicity of equilibrium strategies. The ex post allocation may be inefficient since the low-value player wins the prize with positive probability in equilibrium.[5] If each player has three values about the prize, it is too complex to derive the closed-form equilibrium strategies though

[1] They actually adopt a more general model than a pure lottery since a discriminating factor r is included.

[2] Athey's result is now a central tool for establishing the existence of pure strategy equilibria in auction theory. See Reny and Zamir (2004).

[3] This permits new existence results in auctions with multi-dimensional signals and multi-unit demands. See Mcadams (2004).

[4] Reny achieves this goal by relying on a more powerful fixed point theorem than those used by Athey and Mcadams.

[5] Sui (2006) introduce resale possibility to this example and show that the allocation is always efficient ex post and the contest designer could benefit by allowing resale.

there does exist a non-decreasing equilibrium.

The remainder of the paper proceeds as following. Section 3.2 provides a simple two-player example satisfying the hypotheses of our main result. The essential ideas behind the present technique are also provided there. Section 3.3 contains existence results for finite-action contests and another example with finite actions. Existence of equilibria for continuum-action contests is presented in Section 3.4. Section 3.5 concludes.

3.2 AN EXAMPLE

We begin with a simple example to present the main approach in order to show the existence of monotone pure strategy equilibria.

Consider a contest between two bidders competing for a single prize. Each bidder's valuation is private information, which is independent random draw from $[\underline{v}, \bar{v}]$ according to the identical distribution F. After observing her private valuation, bidder i will decide how much to bid. The allocation of the prize will depend on both bids as follows:

$$p_i = \frac{b_i}{b_i + b_j} \tag{3.1}$$

where p_i is the winning probability of bidder i, and b_i is the bid submitted by bidder i. The above success function is the main characteristic of imperfectly discriminating contest. It is a special case of Tullock's original model about rent-seeking.[6] As long as the other bidder submits a positive bid, one bidder can never win the prize with probability one, contrasting to the all-pay auctions in which the highest bid wins the object with probability one if there is no tie. In this sense, our model is like a "lottery", the more tickets you buy, the higher probability you could win the prize. However, as long as any other people buy any tickets, you can never guarantee winning the prize.[7]

A pure strategy for bidder i is a function $\beta_i : [\underline{v}, \bar{v}] \to B_i \subset \mathbb{R}_+$. Since both bidders are ex ante identical, it is natural to assume that they use symmetric bidding strategies in this example.

[6]In Tullock's original model, the probability of winning for every bidder i is $\frac{b_i^r}{b_i^r + b_j^r}$, where r is the discriminating factor. If r equals to 1, it is just our lottery model. If r goes to infinity, that will be all-pay auction model.

[7]We implicitly assume that any single bidder cannot buy all the tickets, due to her budget constraint or some other reasons.

Suppose bidder 2 follow the strategy β, bidder 1 learns her private valuation v_1 and submits b, the expected utility for bidder 1 is:

$$\pi_1 = \int_{\underline{v}}^{\bar{v}} \frac{b}{b+\beta(v)} dF(v) v_1 - b \tag{3.2}$$

then the maximizing problem facing bidder 1 is:

$$\max_{b\geq 0} \pi_1 = \int_{\underline{v}}^{\bar{v}} \frac{b}{b+\beta(v)} dF(v) v_1 - b$$

The first order condition is given by:

$$\int_{\underline{v}}^{\bar{v}} \frac{\beta(v)}{[b+\beta(v)]^2} dF(v) v_1 - 1 = 0 \tag{3.3}$$

and equilibrium requires that $b = \beta(v_1)$. Substituting the latter equality into the first order condition yields:

$$\int_{\underline{v}}^{\bar{v}} \frac{\beta(v)}{[\beta(v_1)+\beta(v)]^2} dF(v) v_1 - 1 = 0 \tag{3.4}$$

Suppose the bidding strategy for bidder 2 is strictly increasing, that is, $\beta'(v) > 0$, we could use the first order condition to characterize that the best response of bidder 1 is also strictly increasing. A simple transformation gives us:

$$\int_{\underline{v}}^{\bar{v}} \frac{\beta(v)}{[\beta(v_1)+\beta(v)]^2} dF(v) = \frac{1}{v_1} \tag{3.5}$$

Take derivative with respect to v_1 on both sides of equation (5), by simple transformation we have:

$$\int_{\underline{v}}^{\bar{v}} \frac{\beta(v)\beta'(v_1)}{[\beta(v_1)+\beta(v)]^3} dF(v) = \frac{1}{2v_1^2} \tag{3.6}$$

So long as β is strictly positive, we can conclude that the best response for bidder 1 is also using a strictly increasing bidding strategy, given the other bidder uses a strictly increasing bidding strategy. By symmetry, we know that if bidder 1 adopts a strictly increasing strategy, the best response strategy for bidder 2 will also be strictly increasing. Hence, our example satisfies the Spence-Mirlees single-crossing condition and we could use similar approach as Athey (2001) to show the existence of a pure strategy equilibrium, although we cannot get the closed form of equilibrium bidding strategies directly from the first order condition.

Let $U_i(b_i, v_i, \beta_j)$ denote the interim utility function for bidder i, given the bidding strategy of bidder j, β_j, we have:

$$U_i(b_i, v_i, \beta_j) = \int_{\underline{v}}^{\bar{v}} \frac{b_i}{b_i+\beta_j(v)} dF(v) v_i - b_i \tag{3.7}$$

Definition 1. *The function U satisfies the (Milgrom-Shannon)single crossing property of incremental returns (SCP-IR) in (b,v) if, for all $b' > b$ and all $v' > v$, $U(b',v) - U(b,v) \geq 0$ implies $U(b',v') - U(b,v') \geq 0$.*

Definition 2. *The single crossing condition for games of incomplete information (SCC) is satisfied if for each $i = 1,2$, whenever every opponent $j \neq i$ uses a strategy β_j that is nondecreasing, bidder i's objective function $U_i(b_i, v_i, \beta_j)$ satisfies single crossing of incremental returns in (b_i, v_i).*

We have the following sufficient condition for SCP-IR:

$$\frac{\partial^2}{\partial b \partial v} U(b,v) \geq 0$$

Then, consider the interim utility function given in equation (7), we have:

$$\frac{\partial^2}{\partial b_i \partial v_i} U_i(b_i, v_i, \beta_j) = \int_{\underline{v}}^{\bar{v}} \frac{\beta_j(v)}{[b_i + \beta_j(v)]^2} dF(v) \geq 0 \tag{3.8}$$

Moreover, U_i is continuous in both bidders' bids if they both submit positive bids.[8] Therefore, we could apply one of Athey (2001)'s corollary to show the existence of a pure strategy Nash equilibrium (PSNE) in nondecreasing strategies.

Before we state that corollary, we need the following assumption. Let $f(\cdot)$ be the join density over bidders' type, with the conditional density of $\mathbf{v}_{-i}$ given v_i denoted $f(\mathbf{v}_{-i}|v_i)$.[9] Then, we have:

Assumption 3. *The valuations have bounded and atomless join density with respect to Lebesgue measure, $f(\cdot)$. Further, $\int_{v_i \in S} u_i(b_i, \beta_{-i}(v_{-i}), \mathbf{v}) f(\mathbf{v}_{-i}|v_i) d\mathbf{v}_{-i}$ exists and is finite for all convex S and all nondecreasing functions $\beta_j : V_j \to B_j$, $j \neq i$.*

Theorem 4 (Athey)**.** *Given Assumption 1 and the SCC, if each i has identical action space B_i, $u_i(\mathbf{b}, \mathbf{v})$ is continuous in $\mathbf{b}$, then there exists a PSNE in nondecreasing strategies.*

Remark 5. *The key point to apply Athey's existence result is the SCC, which ensures the existence of monotone best replies for one bidder whenever the other bidder chooses monotone bidding strategies. Moreover, the special feature of this simple game–stochastic success function, simplifies the exposition by avoiding the discontinuity problem commonly observed in auction models.*

[8] Fortunately, this is due to the stochastic success function in our contest model. On the contrary, if we use the all-pay auction model, there is discontinuity of the utility function upon winning or losing the auction.

[9] Although we focus on independent types in this example, for more general analysis in next section, we state the assumption in a general setting.

We already show that the SCC is satisfied in our simple example. Since bidders's private valuations are independently and identically distributed, and each bidder's utility function is continuous in valuations and bids, the Assumption 1 is satisfied as long as the distribution F admits a bounded and atomless density. Both bidders are ex ante identical, it is natural to assume that they have identical action space, that is, the support of their bidding strategies are the same. Therefore, our simple example satisfies all the conditions and assumptions of Athey's Corollary, and hence there exists one PSNE in nondecreasing strategies in our example.

3.3 FINITE-ACTION BAYESIAN CONTEST

3.3.1 EXISTENCE OF A PSNE

Now consider a general Bayesian contest between N bidders competing over one single prize, where each bidder i observes her private valuation $v_i \in V \equiv [\underline{v}, \bar{v}] \subset \mathbb{R}$ and then submits a bid b_i from a compact subset $B_i \subset \mathbb{R}_+$. Let $\mathbf{B} = B_1 \times \cdots \times B_N, \mathbf{V} = V^N = [\underline{v}, \bar{v}]^N, \underline{b_i} \equiv minB_i$, and $\bar{b}_i \equiv maxB_i$. The joint density over bidders' valuations is $f(\cdot)$, with the conditional density of $\mathbf{v}_{-i}$ given v_i denoted $f(\mathbf{v}_{-i}|v_i)$. Bidder i's utility function is $u_i : \mathbf{B} \times \mathbf{V} \to \mathbb{R}$. Given any set of strategies for the opponents, $\beta_j : V_j \to B_j, j \neq i$, bidder i's interim utility is given as follows:

$$U_i(b_i, v_i; \beta_{-i}(\cdot)) \equiv \int_{v_{-i}} u_i(b_i, \beta_{-i}(\mathbf{v}_{-i}), \mathbf{v}) f(\mathbf{v}_{-i}|v_i) d\mathbf{v}_{-i}. \tag{3.9}$$

As Athey (2001), we maintain Assumption 1 and the SCC through this section.[10] One important insight of Athey (2001) is that "when the action set is finite, any nondecreasing strategy is a step function, and the strategy can be described simply by naming the values of the player's type t_i at which the player 'jumps' from one action to the next higher action".[11] For instance, let $B_i = \{b_0, b_1, ..., b_M\}$ be the set of potential bids, in ascending order, where $M+1$ is the number of potential bids. Define $V_i^M \equiv [\underline{v}, \bar{v}]^M$,

$$\Sigma_i \equiv \{\mathbf{x} \in V_i^{M+2} | x_0 = \underline{v}, x_1 \leq x_2 \leq \cdots \leq x_M, x_{M+1} = \bar{v}\} \tag{3.10}$$

[10]As in our example, the SCC implies that in response to nondecreasing strategies by all opponents, each bidder's best response is nondecreasing in the strong set order. Thus, each bidder has a best response strategy that is nondecreasing.

[11]This is the so-called Athey Map. For more details, refer to Athey (2001).

and let $\Sigma \equiv \Sigma_1 \times \cdots \times \Sigma_N$. A nondecreasing strategy for bidder i, $\beta_i : V_i \to B_i$, can be represented by a vector $\mathbf{x} \in \Sigma_i$ according to Athey (2001).

Similar to $U_1(b_1, v_1; \mathbf{X})$, we use $V_1(b_1, v_1; \mathbf{X})$ to denote the expected utility for bidder 1 with valuation v_1 when bidder 1 chooses $b_1 \in B_1$ and bidder $2, \ldots N$ use strategies consistent with $(\mathbf{x}^2, \ldots, \mathbf{x}^N)$. Since we assume that opponent strategies are nondecreasing, the SCC implies that $V_1(b_1, v_1; \mathbf{X})$ satisfies the SCP-IR in (b_i, v_i) for all $\mathbf{X} \in \Sigma$.

The existence proof used by Athey proceeds by showing that a fixed point exists for the best response correspondence which is defined as follows. Let $\Gamma = (\Gamma_1, \ldots, \Gamma_N)$, where $\Gamma_i \equiv \{\mathbf{y} \in \Sigma_i :$ $\exists \beta_i(\cdot)$ that is consistent with $\mathbf{y}$ such that $\forall v_i \in V, \beta_i(v_i) \in b_i^{BR}(v_i|X)$. A critical property required for this purpose is convexity. Let $b_i^{BR}(v_i|X) \equiv \arg max_{b_i \in B_i} V_1(b_1, v_1; \mathbf{X})$, which is nonempty by the finiteness. She shows that the best response correspondence is convex if $b_i^{BR}(v_i|X)$ is nondecreasing in the strong set order, which results from the SCC condition. With convexity established, the existence result is given by the following theorem:

Theorem 5 (Athey)**.** *Suppose Assumption* 1 *and the SCC hold. If B_i is finite for all i, this game has a PSNE where each bidder's equilibrium strategy, $\beta_i(\cdot)$, is nondecreasing.*

Proof. See Athey (2001). □

Remark 6. *The central theme for this result is the assumption of finiteness. It guarantees that an optimal action exists for every type and also simplifies the description of strategies so that they can be represented with finite-dimensional vectors. This in turn simplifies the exposition of convexity of the best response correspondence so that we could apply Kakutani's fixed point theorem to show the existence of a PSNE.*

For finite-action Bayesian games, the existence of PSNE is also reported in Milgrom and Weber (1985). More precisely they first show the existence of mixed strategy equilibria in a game where players choose probability distributions over the actions, and then provide purification theorems.[12]However, their approach is limited by requiring that players' types are independently distributed and their utility only depend on their own types, the other players' actions, and the common state variables. Moreover, they also assume "absolute continuity" of probability measure. Under

[12]They use the war of attrition as one revealing example, which is another model dealing with contest instances.

all these conditions, a pure-strategy Nash equilibrium exists.[13]

To the best of our knowledge, there is no general theoretical benchmark dealing with imperfectly discriminating contests with incomplete information. Obviously, it is analytically difficult to derive the closed-form equilibrium bidding strategies. However, given the existence result, we can use computational methods to numerically describe the optimal bidding strategies if all bidders use nondecreasing strategies. The computation of a finite-action equilibrium requires searching a fixed point to the correspondence Γ defined above, where the calculation of $\Gamma(\mathbf{X})$ is a simple exercise of calculating the best-response jump points for each bidder i. But it is not easy to solve the nonlinear set of equations $\mathbf{X} = \Gamma(\mathbf{X})$.[14]

3.3.2 AN EXAMPLE WITH FINITE ACTIONS: TWO BIDDERS

3.3.2.1 TWO VALUES Now we use a simple example with finite actions to see some properties of the equilibrium strategies. Consider a contest with two bidders. Each bidder's valuation could be either high (v_H) or low (v_L) with equal probability $\frac{1}{2}$. Each bidder knows her valuation privately, and the distribution of valuation is common knowledge. We look for a symmetric bidding equilibrium since both bidders are ex ante identical. We solve the game from bidder 1's perspective.

Suppose bidder 2 follow the strategy β, and bidder 1 learns her valuation to be v_L and submits a positive bid b, her expected payoff is:

$$\pi = [\frac{1}{2}\frac{b}{b+\beta(v_L)} + \frac{1}{2}\frac{b}{b+\beta(v_H)}]v_L - b \tag{3.11}$$

The first order condition is:

$$\frac{\beta(v_L)}{(b+\beta(v_L))^2} + \frac{\beta(v_H)}{(b+\beta(v_H))^2} = \frac{2}{v_L} \tag{3.12}$$

The symmetric equilibrium requires that $b = \beta(v_L)$. Substituting this into equation (12) yields:

$$\frac{1}{4\beta(v_L)} + \frac{\beta(v_H)}{(\beta(v_L)+\beta(v_H))^2} = \frac{2}{v_L} \tag{3.13}$$

[13] If we relax the independence assumption, this approach may fail to work. See Radner and Rosenthal (1982).

[14] Athey (1997) provides a number of computational examples that could be computed using either variations on the algorithm $X^{k+1} = \lambda \cdot \Gamma(\mathbf{X}^k) + (1-\lambda)\cdot \mathbf{X}^k$, or quasi-Newton approaches.

By symmetry, we get the following equation:

$$\frac{1}{4\beta(v_H)}+\frac{\beta(v_L)}{(\beta(v_L)+\beta(v_H))^2}=\frac{2}{v_H} \tag{3.14}$$

Solving equations (13)(14) simultaneously, we have the following candidates of equilibrium strategies:

$$\beta(v)=\begin{cases}\frac{v_L[(v_L+v_H)^2+4v_Lv_H]}{8(v_L+v_H)^2}, & \text{if } v=v_L\\ \frac{v_H[(v_L+v_H)^2+4v_Lv_H]}{8(v_L+v_H)^2}, & \text{if } v=v_H\end{cases} \tag{3.15}$$

The payoff function is globally concave, hence the first order condition is also sufficient for the optimality. The above bidding strategy is indeed a symmetric equilibrium. Obviously we have $\beta(v_L)<\beta(v_H)$, which means that the symmetric equilibrium is strictly increasing. It is easy to check that $\beta(v)<\frac{v}{4}$, as long as $v_L<v_H$. Under complete information with identical valuation v, the optimal bid is $\frac{v}{4}$. With uncertainty about other bidder's valuation, each bidder continues to shade her bid in order to secure more rents upon winning the contest, regardless of realization of her valuation. The expected bid for each bidder would be

$$\frac{1}{2}\beta(v_L)+\frac{1}{2}\beta(v_H)=\frac{(v_L+v_H)^2+4v_Lv_H}{16(v_L+v_H)}<\frac{v_L+v_H}{8} \tag{3.16}$$

which means that the expected bid for each bidder is less than one quarter of their expected valuation.

Under complete information with two bidders, Nti (1999) characterizes the equilibrium strategies:

$$\beta_i(v_i,v_j)=\begin{cases}\frac{v_i}{4}, & \text{if } v_i=v_j\\ \frac{v_i^2v_j}{(v_i+v_j)^2}, & \text{if } v_i\neq v_j\end{cases} \tag{3.17}$$

Obviously if the valuations are asymmetric, both bidders bid less than one quarter of their valuations. With symmetric valuations, they bid exactly one quarter of their valuations. Therefore, the asymmetry between these two bidders has similar effect as uncertainty. When the weak bidder (low-value) competes with the strong bidder (high-value), she would shade her bid further so that the optimal bid is below one quarter of her valuation. Given this strategy of weak bidder, the optimal bid for the strong bidder will also be below one quarter of her valuation.

The intuition underlying is quite straightforward. Facing competition of a strong bidder, the weak bidder has relative lower probability of winning the contest, hence is reluctant to bid up to one quarter of her valuation.[15] This further-shading strategy is optimal for the weak bidder even she does know that there is still positive probability for her to win the contest. In equilibrium, the strong bidder correctly predicts the weak bidder's strategy, so she does not need bid up to one quarter of her valuation to secure the rent. Therefore, the strong bidder also shades her bid in equilibrium.

Further investigation shows that the expected revenue for the contest designer is the same for both complete information case and incomplete information case. Roughly speaking, ex ante the designer cannot manipulate the information release to enhance her expected revenue. Formally, the expected payment with incomplete information is $2(\frac{1}{2}\beta(v_L)+\frac{1}{2}\beta(v_H)) = \frac{(v_L+v_H)^2+4v_Lv_H}{8(v_L+v_H)}$. The expected payment with complete information is $\frac{1}{4}(\frac{v_L}{2}+\frac{v_H}{2}+2\frac{v_Lv_H}{v_L+v_H}) = \frac{(v_L+v_H)^2+4v_Lv_H}{8(v_L+v_H)}$.[16] This also means that ex ante it makes no difference for the designer if she does not know the exact realization of both valuations.[17]

Since ex ante the expected revenue is the same for the contest designer between complete information case and incomplete information case, the equilibrium payoffs to a typical bidder are also the same ex ante. This is quite straightforward since the total social surplus is fixed. Simple calculation yields that $\pi_I = \pi_C = \frac{5v_L^2+5v_H^2-2v_Lv_H}{16(v_L+v_H)}$. [18]

3.3.3 THREE VALUES

Now we consider the same two-bidder example except that each bidder has three values with equal probability: v_L, v_M and v_H. Again we solve the game from bidder 1's perspective.

Suppose bidder 2 follow the strategy $\tilde{\beta}$, and bidder 1 learns her valuation to be v_L and submits

[15] She would like to bid one quarter of her valuation only if she is sure that she competes with a bidder having same valuation.

[16] For complete information, the expected payment is $\frac{1}{4}(\frac{1}{2}+2\frac{2}{3}+1) = \frac{17}{24}$. For incomplete information, the expected payment is $\frac{1}{4}(\frac{34}{72}+2\frac{51}{72}+\frac{68}{72}) = \frac{17}{24}$.

[17] Using a more general two-bidder, two-valuation model, Malueg and Yates (2004) derive the same revenue equivalent result for rent seeking with private values. They do not extend the results to more general cases with $N > 2$ bidders.

[18] Numerically, $\pi_I = \pi_C = \frac{7}{16}$.

a positive bid b, her expected payoff is:

$$\pi = [\frac{1}{3}\frac{b}{b+\tilde{\beta}(v_L)} + \frac{1}{3}\frac{b}{b+\tilde{\beta}(v_M)} + \frac{1}{3}\frac{b}{b+\tilde{\beta}(v_H)}]v_L - b \tag{3.18}$$

The first order condition is:

$$\frac{\tilde{\beta}(v_L)}{(b+\tilde{\beta}(v_L))^2} + \frac{\tilde{\beta}(v_M)}{(b+\tilde{\beta}(v_M))^2} + \frac{\tilde{\beta}(v_H)}{(b+\tilde{\beta}(v_H))^2} = \frac{3}{v_L} \tag{3.19}$$

The symmetric equilibrium requires that $b = \tilde{\beta}(v_L)$. Substituting this into equation (19) yields:

$$\frac{1}{4\tilde{\beta}(v_L)} + \frac{\tilde{\beta}(v_M)}{(\tilde{\beta}(v_L)+\tilde{\beta}(v_M))^2} + \frac{\tilde{\beta}(v_H)}{(\tilde{\beta}(v_L)+\tilde{\beta}(v_H))^2} = \frac{3}{v_L} \tag{3.20}$$

Multiplying both sides of (20) by $\tilde{\beta}(v_L)$ yields:

$$\frac{1}{4} + \frac{\tilde{\beta}(v_L)\tilde{\beta}(v_M)}{(\tilde{\beta}(v_L)+\tilde{\beta}(v_M))^2} + \frac{\tilde{\beta}(v_L)\tilde{\beta}(v_H)}{(\tilde{\beta}(v_L)+\tilde{\beta}(v_H))^2} = \frac{3\tilde{\beta}(v_L)}{v_L} \tag{3.21}$$

By symmetry, we get the following equation:

$$\frac{1}{4} + \frac{\tilde{\beta}(v_L)\tilde{\beta}(v_M)}{(\tilde{\beta}(v_L)+\tilde{\beta}(v_M))^2} + \frac{\tilde{\beta}(v_M)\tilde{\beta}(v_H)}{(\tilde{\beta}(v_M)+\tilde{\beta}(v_H))^2} = \frac{3\tilde{\beta}(v_M)}{v_M} \tag{3.22}$$

and

$$\frac{1}{4} + \frac{\tilde{\beta}(v_L)\tilde{\beta}(v_H)}{(\tilde{\beta}(v_L)+\tilde{\beta}(v_H))^2} + \frac{\tilde{\beta}(v_M)\tilde{\beta}(v_H)}{(\tilde{\beta}(v_M)+\tilde{\beta}(v_H))^2} = \frac{3\tilde{\beta}(v_H)}{v_H} \tag{3.23}$$

Equations (21)-(23) determine a symmetric equilibrium, but it is analytically complex to derive the closed-form equilibrium strategies. Full characterization of equilibrium strategies for two-bidder-three-value contests with incomplete information remains an open question, so does the two-bidder case with continuous private values.[19]

[19]Wärneryd (2003) studies two-bidder contests for a prize of common by uncertain value. He shows that equilibrium expenditure are lower under asymmetric information than if either both bidders are informed or neither is informed.

3.4 CONTINUUM-ACTION BAYESIAN CONTEST

3.4.1 LIMITING APPROACH

For Bayesian contests with a continuum of actions, we could use similar limiting argument as Athey (2001) to show the existence of equilibria in nondecreasing strategies. We continue to assume the SCC, which guarantees that for any finite subset of action space, there exists one PSNE in nondecreasing strategies. Then we could use limiting argument to show the existence of a PSNE for continuum-action contests since sequences of uniformly bounded, nondecreasing functions have convergent subsequences. As we show above, one special feature of our Bayesian contest games is that the imperfectly discriminating contest could avoid the discontinuity problem by assuming that the lowest possible bid $\underline{b}_i$ is positive for every bidder i. Given this assumption, the utility function for each bidder is continuous in all bids. Formally, $u_i(\mathbf{b},\mathbf{v})$ is continuous in $\mathbf{b}$. In the following, we use this continuity to show that the limit of a sequence of equilibria in finite-action games is an equilibrium of the continuum-action game. Moreover, in our setting, all bidders are ex ante identical hence it is trivial that, for all i, $B_i = [\underline{b}_i, \bar{b}_i]$. Combining all this, we have the following result.

Theorem 6. *Given Assumption 1, if for all i, $B_i = [\underline{b}_i, \bar{b}_i]$, a PSNE exists in nondecreasing strategies in the contest game where bidders choose bids from* **B**.

Proof. See Athey (2001). □

Remark 7. *Theorem 1 could be considered as one Corollary of Theorem 3. If we impose the SCC assumption, there exists a PSNE in nondecreasing strategies for every finite-action game. Then, our contest games satisfy all the conditions to apply Athey's theorem, the result follows naturally.*

Besides providing existence results, Athey (2001) also characterizes sufficient conditions under which the SCC holds. She shows that the SCC holds if $u_i(\mathbf{b},\mathbf{v})$ is supermodular in $\mathbf{b}$ and $(b_i, v_j), j = 1, ..., N$ and types are affiliated. [20] It is not easy to show similar characterization in our Bayesian contest games. Indeed, we cannot ensure that the utility functions in our contest games

[20]If X is a lattice, the function $h : X \to \mathbb{R}$ is *supermodular* if, for all $\mathbf{x},\mathbf{y} \in X, h(\mathbf{x}\vee\mathbf{y}) + h(\mathbf{x}\wedge\mathbf{y}) \geq h(\mathbf{x}) + h(\mathbf{y})$. The operations "meet" ($\vee$) and "join" ($\wedge$) are defined for product sets as follows: $\mathbf{x}\vee\mathbf{y} = (max(x_1,y_1),...,max(x_n,y_n))$ and $\mathbf{x}\wedge\mathbf{y} = (min(x_1,y_1),...,min(x_n,y_n))$.

are supermodular in general. However, she characterizes the sufficient condition for supermodularity for all-pay auction, which is quite similar to our contest games. Let $U_i(v_i - b_i)$ denote the utility for bidder i to get the prize, $U_i(-b_i)$ denote the utility for bidder i to lose the prize, then the expected payoffs are given by $(U_i(v_i - b_i) - U_i(-b_i))Pr(b_i \text{ wins}) + U_i(-b_i)$. No matter what opponents do, this expression is supermodular so long as U_i is increasing and concave, then the SCC holds.[21]

While we are working through the existence results, Reny (2006) shows the existence of monotone pure strategy equilibria in Bayesian games using a different approach from that of Athey (2001). The following subsection summarizes the application of Reny's approach to our contest games.

3.4.2 LATTICE APPROACH

Based on a more powerful fixed-point theorem, Reny (2006) works on the contractibility of bidders' set of best response, instead of working on the property of convexity, which is crucial to apply Kakutani's fixed point theorem. Although it is not trivial to show the contractibility for a general set, Reny argues that establishing the contractibility of each bidder's set of monotone best replies, for any given monotone bidding functions of the others, is rather simple. Without referring to the Athey map and without considering jump points, we consider directly the monotone bidding functions themselves.

Roughly speaking, a set is contractible if it can be continuously shrunk, within itself, to one of its points. Formally, a subset X of a topological space is *contractible* if for some $x_0 \in X$ there is a continuous function $h : [0,1] \times X \to X$ such that for all $x \in X, h(0,x) = x$ and $h(1,x) = x_0$. We then say that h is a *contraction* for X. Actually contractibility is a strictly more permissive condition than convexity.[22]

It can be shown that, so long as each bidder possesses a monotone best reply whenever the

[21] Let π denote the objective function: $(U_i(v_i - b_i) - U_i(-b_i))Pr(b_i\text{wins}) + U_i(-b_i)$. Then we have $\frac{\partial^2 \pi}{\partial b_i \partial v_i} = \frac{\partial^2 U_i(v_i - b_i)}{\partial b_i \partial v_i} Pr(b_i\text{wins}) + \frac{\partial U_i(v_i - b_i)}{\partial v_i} \frac{\partial Pr(b_i\text{wins})}{\partial b_i}$. Since Pr($b_i$ wins) is nonnegative and nondecreasing in b_i, $U_i(v_i - b_i)$ is nondecreasing in v_i, the objective function is supermodular if and only if $U_i(v_i - b_i)$ is supermodular and in turn if and only if it is concave (the bidder is risk averse).

[22] Note that every convex set is contractible since, choosing any point x_0 in the set, the function $h(\tau, x) = (1-\tau)x + \tau x_0$ is a contraction. On the other hand, there are contractible sets that are not convex (e.g., any curved line in $\mathbb{R}^2$ that does not intersect itself).

others employ monotone bidding function, an appropriate generalization of Kakutani's theorem – relying on contractible-valuedness instead of convex-valuedness –can be employed to establish the existence of monotone PSNE. Although we don't need the SCC to show the contractibility of best reply sets, it is needed to show that the monotone best replies exist.

Before we state the result, we need to restate our contest game as follows. For the lattice terminology, please refer to Appendix. There are N risk-neutral bidders, $i = 1, 2, ..., N$. Bidder i's type space is $V_i = [\underline{v}, \bar{v}]$ endowed with the Euclidean metric, and i's action space is the topological space B_i. Bidder i's bounded and measurable utility function is $u_i : \mathbf{B} \times \mathbf{V} \to \mathbb{R}$, where $\mathbf{B} = \times_{i=1}^{N} B_i$ and $\mathbf{V} = \times_{i=1}^{N} V_i$. The common prior over the bidders' types is a probability measure μ on the Borel subset of $\mathbf{V}$.

Reny imposes the following assumptions to simplify the analysis, which are satisfied in our contest game. Let μ_i denote the marginal of μ on V_i. First, for every bidder i, and every Borel subset A of V_i, he assumes that $\mu_i(A) = 0$ if $A \cap C$ is countable for every subset C in V_i, which implies that each μ_i is atomless because we may set $A = \{v_i\}$ for any $v_i \in V_i$. In our contest games, each bidder's type space is $[\underline{v}, \bar{v}]$ with its usual total order, this assumption holds if and only if μ_i is atomless. In particular, this assumption helps ensure the compactness of the bidders' sets of monotone pure strategies in a topology in which ex-ante utilities are continuous.

Secondly, he assumes that $(B_i, \geq)$ is a compact locally-complete metrizable semilattice, which holds whenever $(B_i, \geq)$ is a compact metrizable semilattice in Euclidean space with the coordinate partial order. In our contest games, given the action space, $B_i = [\underline{b}, \bar{b}]$, this assumption is trivially satisfied.[23]

Finally, he also assumes the continuity of the utility function, which is satisfied in our contest games.

A *pure strategy* for bidder i is a measurable function, $\beta_i : V_i \to B_i$. Such a pure strategy is *monotone* if $v_i' \geq v_i$ implies $\beta_i(v_i') \geq \beta_i(v_i)$.

As before, an N-tuple of pure strategies, $(\beta_1, ..., \beta_N)$ is an *equilibrium* if for every bidder i and every pure strategy β_i',

$$\int_V u_i(\beta(v), v) d\mu(v) \geq \int_V u_i(\beta_i'(v_i), \beta_{-i}(v_{-i}), v) d\mu(v) \tag{3.24}$$

[23] Reny (2006) actually considers more general setting such as multidimensional type space and action space.

where the left-hand side, is bidder i's utility given the joint strategy β, and the right-hand side is his utility when he employs β'_i and the others employ β_{-i}.

It will be helpful to speak of the interim utility to bidder i's type v_i from the bid b_i given the strategies of the others, β_{-i},

$$U_i(b_i, v_i, \beta_{-i}) = \int_V u_i(b_i, \beta_{-i}(v_{-i}), v) d\mu_i(v_{-i}|v_i) \tag{3.25}$$

where $\mu_i(\cdot|v_i)$ is a version of the conditional probability on V_{-i} given v_i.

A set of bidder i's pure strategies is *join-closed* if for any pair of strategies, β_i, β'_i, in the set, the strategy taking the action $\beta_i(v_i) \vee \beta'_i(v_i)$ for each $v_i \in V_i$ is also in the set. Now we could state the existence result. The proof mimics Reny (2006), so is omitted here.

Theorem 7. *Suppose that for each bidder i, μ_i is atomless, and each bidder's set of monotone best replies is nonempty and join-closed whenever the others employ monotone pure strategies, then the contest defined above possesses a monotone pure strategy equilibrium.*

Remark 8. *This result is more powerful than Theorem 3. The following analysis shows that we do not need a strong version of the single crossing condition to ensure the nonemptiness of best response. Actually only a weak single crossing condition will suffice.*

In the appendix, we provide the original fixed point theorem Reny uses for his existence result and show that our contest games satisfies the conditions required in order to apply that theorem.[24]

It is well-known that within the confines of a lattice, quasisupermodularity and single-crossing conditions on interim utility functions guarantee the existence of monotone best replies and that sets of monotone best replies are lattice and hence join-closed. The sufficient conditions on interim utilities are as follows.

Definition 3. *Suppose that for each bidder i, $(B_i, \geq)$ is a lattice. The interim utility function U_i is weakly quasisupermodular if for all monotone pure strategies β_{-i} of the others, all $b_i, b'_i \in B_i$, and every $v_i \in V_i$, $U_i(b_i, v_i, \beta_{-i}) \geq U_i(b_i \wedge b'_i, v_i, \beta_{-i})$ implies $U_i(b_i \vee b'_i, v_i, \beta_{-i}) \geq U_i(b'_i, v_i, \beta_{-i})$.*

In our contest games, actions are totally ordered, interim utilities are automatically supermodular, and hence weakly quasisupermodular.[25]

[24] Similar material could be also found in Reny (2006).

[25] Complementarities in the distinct dimensions of a bidder's own actions are natural economic conditions under which weak quasisupermodularity holds. However, supermodularity usually requires complementarities between the actions of distinct bidders, which is not satisfied in many auction games.

Definition 4. *Bidder i's interim utility function U_i satisfies weak single-crossing if for all monotone pure strategies β_{-i} of the others, for all bidder i action pairs $b_i' \geq b_i$, and for all bidder i type pairs $v_i' \geq v_i$, $U_i(b_i', v_i, \beta_{-i}) \geq U_i(b_i, v_i, \beta_{-i})$ implies $U_i(b_i', v_i', \beta_{-i}) \geq U_i(b_i, v_i', \beta_{-i})$.*[26]

If the interim utility functions satisfy these two properties, the following result ensures that a pure strategy equilibrium exists for the Bayesian contest game.

Corollary 1. *Suppose that for each bidder i, μ_i is atomless, each $(B_i, \geq)$ is a lattice, and the bidders' interim utility functions are weakly quasisupermodular and satisfy weak single-crossing, then the Bayesian contest game possesses a monotone pure strategy equilibrium.*

3.5 CONCLUSION

This paper studies the imperfectly discriminating contests with incomplete information. Most existing literature on contest focus on either complete information where every player's valuation is common knowledge, or incomplete information but with two players. By studying a general contest model with incomplete information and N players, we provide the sufficient conditions under which a monotone pure strategy equilibrium exists for both finite-action and continuum-action contests. The main condition we impose is the Single Crossing Condition that is widely used in economic literature. Roughly this condition guarantees that when one player adopts a non-decreasing strategy, there exists a best response for other player that is also non-decreasing. By applying certain fixed point theorem, we can establish existence of a monotone equilibrium. However, analytically it is too complex to derive the closed form equilibrium strategies. Future research may try to use computational or numerical methods to approximate equilibrium strategies.

Using a simple two-bidder example, we confirm the monotonicity of equilibrium strategies and also derive some numerical results which may be useful for the experimental study of contests. In particular, with two bidders each having two values, we could fully characterize equilibrium strategies. We further show that contests with incomplete information is equivalent to contests with

[26] To ensure the convexity of the set of monotone best replies, Athey (2001) assumes W_i satisfies a more stringent single-crossing condition, which requires that, in addition to the above, the second inequality is strict whenever the first one is.

complete information in terms of expected revenue.[27] It remains open to characterize equilibrium strategies for contests with two players each having three or more values.[28][29]

More general results for characterizing equilibrium strategies and their properties need to be addressed in the future research. It may be also interesting to study agents' behaviors through controlled experiments in the laboratory.

[27] However, this revenue equivalence fails if the contest designer cannot forbid resale. With resale possibility, the allocation is always ex post efficient. The contest designer could benefit by allowing resale since both bidders bid more aggressively if they foresee the possibility of resale. See Sui (2006)

[28] Wärneryd (2003) studies two-player contest with one-sided incomplete information. The prize has common value that is uncertain to one player. He shows that equilibrium expenditures are lower than that with complete information or two-sided incomplete information.

[29] With three values or even continuous private values, there is no incentive-compatible bargaining scheme that could ensure ex post efficiency. See Myerson and Satterthwaite (1983), Sui (2006).

APPENDIX A

CHAPTER 1

A.1 SOME FACTS FROM AFFILIATION

In this section, we prove some useful facts due to affiliation.

Fact 1. $\frac{F_{Y_1}(y|x)}{f_{Y_1}(y|x)}$ is non-increasing in x.

Proof. Let $x < x'$ and $y < y'$. By affiliation inequality, we have

$$f(x,y)f(x',y') \geq f(x,y')f(x',y)$$

Hence

$$\frac{f(x,y)}{f(x,y')} \geq \frac{f(x',y)}{f(x',y')}$$

Then we have

$$\frac{f_{Y_1}(y|x)}{f_{Y_1}(y'|x)} \geq \frac{f_{Y_1}(y|x')}{f_{Y_1}(y'|x')}$$

Integrating with respect to y over $[0,y']$ yields

$$\frac{F_{Y_1}(y'|x)}{f_{Y_1}(y'|x)} \geq \frac{F_{Y_1}(y'|x')}{f_{Y_1}(y'|x')}$$

therefore the result follows. □

Fact 2. $\frac{f_{Y_1}(y|x)}{1-F_{Y_1}(y|x)}$ is non-increasing in x.

Proof. By *Fact 1*, $\frac{f_{Y_1}(y|x)}{F_{Y_1}(y|x)}$ is non-decreasing in x. Hence, $-\frac{f_{Y_1}(y|x)}{F_{Y_1}(y|x)}$ is non-increasing in x. Therefore, the result follows. □

Fact 3. $F_{Y_1}(y|x)$ is non-increasing in x.

Proof. Fact 1 and *Fact 2* imply that $\frac{F_{Y_1}(y|x)}{1-F_{Y_1}(y|x)}$ is non-increasing in x. Hence $\frac{1-F_{Y_1}(y|x)}{F_{Y_1}(y|x)}$ is non-decreasing in x. Therefore, $\frac{1}{F_{Y_1}(y|x)}$ is non-decreasing in x. The result follows. □

A.2 PROOF OF RESULTS

Proof of Theorem 1

Proof. The necessity is established in Section 3. For sufficiency, let $z \leq x$, and $\beta_s(z) = b$. From the first order condition, we have

$$\begin{aligned}
\frac{\partial \Pi(\beta_s(z),x)}{\partial b} &= v(z,x,z)\frac{f_{Y_1}(z|x)}{1-F_{Y_1}(z|x)}\frac{1}{\beta_s'(z)} + \frac{1}{\beta_s'(z)}\int_0^z v_1(z,x,y)\frac{f_{Y_1}(y|x)}{1-F_{Y_1}(z|x)}dy - 1 \\
&\geq v(z,z,z)\frac{f_{Y_1}(z|z)}{1-F_{Y_1}(z|z)}\frac{1}{\beta_s'(z)} + \frac{1}{\beta_s'(z)}\int_0^z v_1(z,z,y)\frac{f_{Y_1}(y|z)}{1-F_{Y_1}(z|z)} - 1 \\
&= \frac{\partial \Pi(\beta_s(z),z)}{\partial b} = 0
\end{aligned}$$

The first inequality follows from Assumption 1, and the last two equalities follow from the first-order condition. That means, when $X_1 = x$ and bidder 1 bids $b = \beta_s(z) \leq \beta_s(x)$, his expected profit could be raised by bidding higher. By similar argument, when $z \geq x$, we can show $\frac{\partial \Pi(\beta_s(z),x)}{\partial b} \leq 0$. Consequently, $\Pi(b,x)$ is maximized at $\beta_s(x) = b$. Since $\Pi(0,x) = 0$ for all x, we have $\Pi(\beta_s(x),x) \geq 0$ for all $x > 0$ by affiliation. Thus, we have shown that $\beta_s(x)$ is the best response strategy for bidder 1 when he observes $X_1 = x$ and all other bidders $j \neq i$ follow β_s, and when resale market participants believe that all bidders follow β_s.

From the above argument, the equilibrium payoff to a bidder who receives a signal of x is $\Pi(\beta_s(x),x) \geq 0$, and thus it is individually rational for each bidder to participate in the auction.

It remains to show that the equilibrium bidding strategy is strictly increasing and differentiable. Since v_1 is positive by affiliation, $\beta_s'(x)$ is strictly positive. Clearly, the equilibrium bidding strategy is differentiable. Therefore, the bidding strategy we characterize is indeed a symmetric equilibrium provided Assumption 1 holds. □

Proof of Lemma 1

Proof. Let $x < z$. Since $\Psi_2 > 0$, we have that

$$v(x', x, y)\frac{f_{Y_1}(y|x)}{1 - F_{Y_1}(y|x)} < v(x', z, y)\frac{f_{Y_1}(y|z)}{1 - F_{Y_1}(y|z)}$$

By *Fact 3*, $F_{Y_1}(y|x) \geq F_{Y_1}(y|z)$ and thus

$$v(x', x, y) f_{Y_1}(y|x) < v(x', z, y) f_{Y_1}(y|z)$$

This proves that $\Psi_2 > 0$ implies that $\Phi_2 > 0$. Similar argument could show that $\Psi_{12} > 0$ implies $\Phi_{12} > 0$. □

The proof of Lemma 2 is exactly the same as above, so is omitted.

Proof of Theorem 2

Proof. The proof mimics the proof of Theorem 1. Again the key point is to show $\frac{\partial \Pi(b,x)}{\partial b} \geq 0$ if $b = \beta_f(x') \leq \beta_f(x)$, and $\frac{\partial \Pi(b,x)}{\partial b} \leq 0$ if $b = \beta_f(x') \geq \beta_f(x)$. Assumption 2 and affiliation ensure that it is not profitable for local deviation. It is trivial to verify that β_s is strictly increasing and differentiable.

From the above argument, the equilibrium payoff to a bidder who receives a signal of x is $\Pi(\beta_f(x), x) \geq 0$, and thus it is individually rational for each bidder to participate in the auction. Similarly, it is easy to show that the equilibrium strategy is increasing and differentiable, hence it is indeed a symmetric equilibrium for the first-price all-pay auction with resale. □

Proof of Theorem 3

Proof. Let $e_f(e_s)$ denote the expected payment in equilibrium of the first-price (second-price) all-pay auction. Then we have

$$\begin{aligned}
e_s(x) &= \int_0^x \beta_s(y) f_{Y_1}(y|x)dy + [1 - F_{Y_1}(x|x)]\beta_s(x) \\
&= \beta_s(x)F_{Y_1}(x|x) - \int_0^x \beta_s'(y)F_{Y_1}(y|x)dy + [1 - F_{Y_1}(x|x)]\beta_s(x) \\
&= \beta_s(x) - \int_0^x \beta_s'(y)F_{Y_1}(y|x)dy \\
&= \int_0^x v(y,y,y)\frac{f_{Y_1}(y|y)}{1 - F_{Y_1}(y|y)}dy + \int_0^x k(y)dy \\
&\quad - \int_0^x [v(y,y,y)\frac{f_{Y_1}(y|y)}{1 - F_{Y_1}(y|y)} + k(y)]F_{Y_1}(y|x)dy \\
&= \int_0^x v(y,y,y) f_{Y_1}(y|y)[\frac{1 - F_{Y_1}(y|x)}{1 - F_{Y_1}(y|y)}]dy + \int_0^x k(y)[1 - F_{Y_1}(y|x)]dy \\
&= \int_0^x v(y,y,y) f_{Y_1}(y|y)[\frac{1 - F_{Y_1}(y|x)}{1 - F_{Y_1}(y|y)}]dy + \int_0^x h(y)[\frac{1 - F_{Y_1}(y|x)}{1 - F_{Y_1}(y|y)}]dy \\
&\geq \int_0^x v(y,y,y) f_{Y_1}(y|y)dy + \int_0^x h(y)dy \\
&= e_f(x)
\end{aligned}$$

The second equality follows from integration by parts; by *Fact 3*, we have that $F_{Y_1}(y|x)$ is non-increasing in x, so $\frac{1-F_{Y_1}(y|x)}{1-F_{Y_1}(y|y)} \geq 1$ for $x \geq y$. This gives us the last inequality, which completes the proof. □

Proof of Proposition 2

Proof. See Krishna and Morgan (1997), proof of Proposition 4. □

Proof of Proposition 2 & 3

Proof. For any given value of X_0, $\tilde{\beta}$ is similar to β, and the proofs of Proposition 1 and 2 mimic the proofs of Theorem 1 and 2, so are omitted. □

Proof of Proposition 4

Proof. Consider the first-price all-pay auction with resale. Let $\tilde{\beta}_f(\cdot, x_0)$ denote the equilibrium strategy conditional on the revealed of seller's private information $X_0 = x_0$. Lemma 2 and affiliation ensures that $\tilde{\beta}_f(\cdot, x_0)$ is increasing in x_0.

Let $e^f(x,z)$ denote the expected payment for bidder 1 if he learns his signal as z but he bids as if it were x, and $\tilde{e}^f(x,z) = E[\tilde{\beta}_f(x, X_0)|Y_1 < x, X_1 = z]$. Affiliation implies that $\tilde{e}^f_2(x,z) \geq 0$. Let $R(x,z)$ denote the expected value of winning. At the equilibrium, it is optimal to choose $z = x$, the resulting first-order condition yields

$$e^f_1(z,z) = \tilde{e}^f_1(z,z)$$

Since $e^f_2(x,z) = 0$, $\tilde{e}^f_2(x,z) \geq 0$, then according to *linkage principle*, we have $\tilde{e}^f(z,z) \geq e^f(z,z)$ since $\tilde{e}^f(0,0) = e^f(0,0) = 0$. Therefore, in the first-price all-pay auction with resale, the initial seller will benefit from publicly disclosing his private signal.

Using similar argument, we can show that the information disclosure by the seller does not decrease, may increase the expected revenue in the second-price all-pay auction with resale. □

APPENDIX B

CHAPTER 2

B.1 PROOFS OF RESULTS

Proof of Proposition 1

Proof. From (6), we have

$$\frac{\partial EU_1(b_L, \beta)}{\partial b_L} = \frac{\sigma \beta_L}{(b_L + \beta_L)^2} v_L + \frac{(1-\sigma)\beta_H}{(b_L + \beta_H)^2} v_H - 1$$

Consider the second-order condition, then

$$\frac{\partial^2 EU_1(b_L, \beta)}{\partial b_L^2} = \frac{-2\sigma v_L \beta_L}{(b_L + \beta_L)^3} + \frac{2(\sigma - 1) v_H \beta_H}{(b_L + \beta_H)^3} < 0$$

Similarly, we have $\frac{\partial^2 EU_1(b_H, \beta)}{\partial b_H^2} < 0$. Therefore, the objective function is globally concave. Hence, the first-order condition is both necessary and sufficient to characterize the symmetric equilibrium which is determined by (9) and (10). □

Proof of Proposition 2

Proof. See Malueg and Yates (2004). □

Proof of Proposition 3

Proof. If the realized values are (v_L, v_L) or $(v_H, v_H$, it is trivial to show that $\beta_{LL} = \frac{v_L}{4}$ and $\beta_{HH} = \frac{v_H}{4}$. Now let us look at the cases (v_L, v_H) and (v_H, v_L). With resale possibilities, the expected valuation upon winning is v_H for both players, since the low-value player could resell the prize to his high-value rival with price equal to v_H. Therefore, existence of resale possibilities symmetrizes valuations for both players. It remains easy to show that $\beta_{LH} = \beta_{HL} = \frac{v_H}{4}$. □

Proof of Proposition 4

Proof. See Nti (1999) or Malueg and Yates (2004). □

Proofs of Proposition 5 and 6 are contained in the text, so are omitted.

Proof of Proposition 7

Proof. This is trivial by our equilibrium construction. If the realized valuations for both players are the same, the final allocation is always efficient. If the low-value player wins the prize, he will resell it to the high-value rival and the latter will accept the offer in equilibrium. Therefore, the final allocation is always efficient. □

Proof of Proposition 8

Proof. See Section 6.1. □

Proof of Proposition 9

Proof. From (9), (10), we have

$$\frac{\beta_H - \beta_L}{v_H - v_L} = \frac{\sigma}{4}$$

From Proposition 2, we have

$$\frac{\tilde{\beta}_H - \tilde{\beta}_L}{v_H - v_L} = \frac{\sigma}{4} + \frac{1-\sigma}{(\rho^{-1/2} + \rho^{1/2})^2}$$

From (24), (25), we have

$$\frac{\mu_H - \mu_L}{v_H - v_L} = \frac{\sigma}{4}$$

From (30), (31), we have

$$\frac{\tau_H - \tau_L}{v_H - v_L} = \frac{\sigma}{4}$$

□

APPENDIX C

CHAPTER 3

C.1 A FIXED POINT THEOREM AND ITS APPLICATION

C.1.1 NOTATION AND TERMINOLOGY

Let A be a nonempty set and let $\geq$ be a partial order on A.[1] For $a, b \in A$, if the set $\{a, b\}$ has a least upper bound (l.u.b.) in A, then it is unique and will be denoted by $a \vee b$, the join of a and b. If every pair of points in A has an l.u.b in A, then we shall say that $(A, \geq)$ is a *semilattice*. Similarly, we could define the greatest lower bound (g.l.b). If every pair of points in A has both an l.u.b. in A and a g.l.b. in A, then we shall say that $(A, \geq)$ is a *lattice*. Clearly, in our contest games, the type space $[\underline{v}, \bar{v}]$ and action space $[\underline{b}, \bar{b}]$ are both lattice and hence semilattice.

A *topological semilattice* is a semilattice endowed with a topology under which the join operator, $\vee$, is continuous as a function from $A \times A$ into A. When the topology on A rendering the join operator continuous is metrizable we say that $(A, \geq)$ is a *metrizable semilattice*. When the topology on A renders A compact, we say that $(A, \geq)$ is *compact*.

A semilattice $(A, \geq)$ is *complete* if every nonempty subset S of A has a least upper bound, $\vee S$, in A. A topological semilattice $(A, \geq)$ is *locally complete* if for every $a \in A$ and every neighborhood U of a, there is a neighborhood W of a contained in U such that every nonempty subset S of W has a least upper bound, $\vee S$, contained in U. The proof of Theorem 4 relies on a corollary of Eilenberg and Montgomery's (1946) fixed point theorem, which substantially generalizes

[1] That is, $\geq$ is transitive, refexive and antisymmetric.

Kakutani's theorem.

A subset X of a metric space Y is said to be a *retract* of Y if there is a continuous function mapping Y onto X leaving every point of X fixed. A metric space (X,d) is an *absolute retract* if for every metric space (Y,δ) containing X as a closed subset and preserving its topology, X is a retract of Y. Examples of absolute retracts include closed convex subsets of Euclidean space or of any metric space, and many non convex sets as well.

Theorem 8 (Eilenberg and Montgomery (1946))**.** *Suppose that a compact metric space (X,d) is an absolute retract and that $F : X \twoheadrightarrow X$ is an upper hemicontinuous, nonempty-valued, contractible-valued correspondence. Then F has a fixed point.*

Now we use this theorem to sketch the proof of Theorem 4. Let M_i denote the set of monotone pure strategies for bidder i, and let $M = \times_{i=1}^{N} M_i$. Let $H_i : M_{-i} \twoheadrightarrow M_i$ denote bidder i's best reply correspondence when bidders are restricted to monotone pure strategies. By hypothesis, each bidder possesses a monotone best reply when the others employ monotone pure strategies (the SCC), any fixed point of $\times_{i=1}^{N} H_i : M \twoheadrightarrow M$ is a monotone pure strategy equilibrium. The following steps demonstrate that such a fixed point exists.

C.1.2 THE M_i ARE COMPACT ABSOLUTE RETRACTS

First we need to show that each bidder's space of monotone pure strategies can be metrized so that it is a compact absolute retract. Without loss, we may assume that the metriz d_{B_i} on B_i is bounded.[2] Given d_{B_i}, define a metric δ_{M_i} on M_i as follows:

$$\delta_{M_i}(\beta_i, \beta_i') = \int_{V_i} d_{B_i}(\beta_i(v_i), \beta_i'(v_i)) d\mu_i(v_i) \tag{C.1}$$

Now suppose that β_i^n is a sequence in M_i. Then, by the semilattice-extension of Helley's theorem, β_i^n has a μ_i almost everywhere pointwise convergent subsequence. That is, there exists a subsequence, $\beta_i^{n_k}$, and $\beta_i \in M_i$ such that $\beta_i^{n_k}(v_i) \to_k \beta_i(v_i)$ for μ_i almost every $v_i \in V_i$. Therefore, $d_{B_i}(\beta_i^{n_k}(v_i), \beta_i(v_i))$, a bounded function of v_i, converges to zero μ_i almost everywhere as $k \to \infty$, so that, by the dominated convergence theorem, $\delta_{M_i}(\beta_i^{n_k}, \beta_i) \to_k 0$. We hence establish the following result.

[2] As Reny (2006) points out that, for any metric, $d(\cdot,\cdot)$, an equivalent bounded metric is $min(1, d(\cdot,\cdot))$.

Lemma 3. *The metric space* (M_i, δ_{M_i}) *is compact.*

The metric δ_{M_i} also renders (M_i, δ_{M_i}) an absolute retract, as stated in the next lemma, whose proof can be found in Reny (2006).

Lemma 4. *The metric space* (M_i, δ_{M_i}) *is an absolute retract.*

C.1.3 UPPER-HEMICONTUNUITY

The second step is to show that, given the metric δ_j on M_j, each bidder i's utility function, $U_i : M \to \mathbb{R}$, is continuous under the product topology. This immediately yields upper-hemicontinuity of best reply correspondence. To see utility continuity, suppose that β^n is a sequence of join strategies in M, and that $\beta^n \to \beta \in M$. By the following lemma, this implies that for each bidder i, $\beta_i^n(v_i) \to \beta_i(v_i)$ for μ_i almost everywhere $v_i \in V_i$. Consequently, $\beta^n(v) \to \beta(v)$ for μ almost everywhere $v \in V$. Hence, since u_i is bounded, Lebesgue's dominated convergence theorem yields:

$$U_i(\beta^n) = \int_V u_i(\beta^n(v), v)d\mu(v) \to \int_V u_i(\beta(v), v)d\mu(v) = U_i(\beta) \tag{C.2}$$

establishing the continuity of U_i.

Lemma 5. *In* (M, δ)*,* β_k *converges to* β *if and only if in* $(B, \geq)$*,* $\beta_k(v)$ *converges to* $\beta(v)$ *for* μ *almost every* $v \in [\underline{v}, \bar{v}]$.

Since each bidder i's utility function, U_i, is continuous and each M_i is compact, an application of Berge's theorem of the maximum immediately yields the following result.

Lemma 6. *Each bidder i's best reply correspondence,* $H_i : M_{-i} \twoheadrightarrow M_i$*, is nonempty-valued and upper-hemicontinuous.*

C.1.4 CONTRACTIBLE-VALUEDNESS

The simple observation at the heart of this approach is that each bidder i's set of monotone best replies is contractible. As Reny (2006), we use the following contracting map.

Define $h : [0,1] \times M_i \times M_i \to M_i$ as follows: for every $v_i \in [\underline{v}, \bar{v}]$,

$$h(\tau, f, g)(v_i) = \begin{cases} f(v_i), & \text{if } v_i \le |1-2\tau|(\bar{v}-\underline{v}) + \underline{v} \text{ and } \tau < 1/2 \\ g(v_i), & \text{if } v_i \le |1-2\tau|(\bar{v}-\underline{v}) + \underline{v} \text{ and } \tau \ge 1/2 \\ f(v_i) \vee g(v_i), & \text{if } v_i \ge |1-2\tau|(\bar{v}-\underline{v}) + \underline{v} \end{cases} \tag{C.3}$$

Note that $h(\tau, f, g)$ is indeed monotone and $h(0, f, g) = f$ and $h(1, f, g) = g$. The continuity is established in Reny (2006). The following lemma shows the contractible-valuedness.

Lemma 7. *$H_i : M_{-i} \twoheadrightarrow M_i$ is contractible-valued.*

Proof. Fix $\beta_{-i} \in M_{-i}$. To establish the contractibility of $H_i(\beta_{-i})$, suppose that $f, g \in H_i(\beta_{-i})$. By hypothesis, $H_i(\beta_{-i})$ is join-closed, the monotone function, $f \vee g$, taking the action $f(v_i) \vee g(v_i)$ for each $v_i \in [\underline{v}, \bar{v}]$ is also in $H_i(\beta_{-i})$. Consequently, $[h(\tau, f, g)](v_i$, being equal to either $f(v_i)$, $g(v_i)$, or $f(v_i) \vee g(v_i)$, must maximize $U_i(b_i, v_i, \beta_{-i})$ over $b_i \in B_i$ for almost every $v_i \in [\underline{v}, \bar{v}]$, because this μ_i almost-everywhere maximization property holds, by hypothesis, for every member of $H_i(\beta_{-i})$ and so separately for each of f, g, and $f \vee g$. But this implies that for every $\tau \in [0,1], h(\tau, f, g) \in H_i(\beta_{-i})$. So, because $h(0, f, g) = f, h(1, f, g) = g$ and $(\cdot,\cdot,\cdot)$ is continuous as Reny (2006) shows, $h(\cdot,\cdot,g)$ is a contraction for $H_i(\beta_{-i})$. □

C.1.5 COMPLETING THE PROOF

The following lemma completes the proof of Theorem 4.

Lemma 8. *The product of the bidders' best reply correspondence, $\times_{i=1}^{N} H_i : M \twoheadrightarrow M$, possesses a fixed point.*

Proof. By Lemmas 1 and 2, each (M_i, δ_{M_i}) is a compact absolute retract. Consequently, under the product topology, M is both compact and, by Borsuk (1966) IV (7.1), an absolute retract. By Lemmas 4 and 5, $\times_{i=1}^{N} H_i : M \twoheadrightarrow M$ is upper-hemicontinuous, nonempty-valued, and contractible-valued. Hence, applying Theorem 5 to $\times_{i=1}^{N} H_i : M \twoheadrightarrow M$ yields the desired result. □

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