HAAG'S THEOREM AND THE INTERPRETATION OF QUANTUM FIELD THEORIES WITH INTERACTIONS

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ABSTRACT

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Quantum field theory (QFT) is the physical framework that integrates quantum mechanics and the special theory of relativity; it is the basis of many of our best physical theories. QFT's for interacting systems have yielded extraordinarily accurate predictions. Yet, in spite of unquestionable empirical success, the treatment of interactions in QFT raises serious issues for the foundations and interpretation of the theory. This dissertation takes Haag's theorem as a starting point for investigating these issues. It begins with a detailed exposition and analysis of different versions of Haag's theorem. The theorem is cast as a *reductio ad absurdum* of canonical QFT prior to renormalization. It is possible to adopt different strategies in response to this *reductio*: (1) renormalizing the canonical framework; (2) introducing a volume (i.e., long-distance) cutoff into the canonical framework; or (3) abandoning another assumption common to the canonical framework and Haag's theorem, which is the approach adopted by axiomatic and constructive field theorists. Haag's theorem does *not* entail that it is impossible to formulate a mathematically well-defined Hilbert space model for an interacting system on infinite, continuous space. Furthermore, Haag's theorem does not undermine the predictions of renormalized canonical QFT; canonical QFT with cutoffs and existing mathematically rigorous models for interactions are empirically equivalent to renormalized canonical QFT. The final two chapters explore the consequences of Haag's theorem for the interpretation of QFT with interactions. I argue that no mathematically rigorous model of QFT on infinite, continuous space admits an interpretation in terms of quanta (i.e., quantum particles). Furthermore, I contend that extant mathematically rigorous models for physically unrealistic interactions serve as a better guide to the ontology of QFT than either of the other two formulations of QFT. Consequently, according to QFT, quanta do not belong in our ontology of fundamental entities.

Keywords: quantum field theory, Haag's theorem, quanta, particles, interactions, philosophy of physics.

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PREFACE

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0.0 INTRODUCTION

One motivation for undertaking this project was the observation that there is a gap in the philosophical literature on relativistic quantum field theory (QFT). For the highly idealized case of free systems, philosophers have attended to a variety of formulations of QFT. For example, the first five chapters of Teller (1995) rely upon what I will call the 'canonical' formulation of QFT, the version of the theory developed by Feynman, Dyson, and their colleagues that can still be found in most introductory textbooks on the subject. More recently, philosophical analysis of free systems has focused on the more mathematically rigorous algebraic formulation of QFT that originated with Segal, Haag, and Kastler (e.g., Clifton and Halvorson 2001, Ruestsche 2002; see also Arageorgis et. al. 2003, Arageorgis et. al. 2002). However, for interacting systems, the discussion has almost exclusively been based on variants of the canonical formulation of QFT; that is, canonical QFT with infinite renormalizations and finitely renormalized QFT with volume (i.e., long-distance) cutoffs and ultraviolet (i.e., short-distance) cutoffs (e.g., Cao and Schweber 1993; Teller 1995; Huggett and Weingard 1995; Huggett 2002). The alternative axiomatic and constructive approaches are rarely mentioned. Granted, there seems to be a good reason for this omission. Teller explains that he does not discuss "formal and rigorous work in axiomatic field theory" in his book because "[a]lthough [axiomatic field theory] is a useful enterprise in the study of formal properties of quantum field theories, axiomatic quantum field theory as it exists today does not appear usefully to describe real physical phenomena" (Teller 1995, p. 146, fn. 22). He has a point: to date, no model of any set of axioms has been constructed for any remotely realistic interaction. Yet this does not necessarily mean that axiomatic and constructive QFT holds no insights into the foundations or interpretation of QFT. The theme of this dissertation is that philosophers should look to axiomatic and constructive QFT in order to deepen our understanding of QFT with interactions.

The starting point for this investigation is Haag's theorem, which was one of the earliest results of the program of specifying mathematically rigorous axioms for QFT. It predates both Wightman's proposed axiomatization (Wightman 1956) and the algebraic axiomatization proposed by Haag and Kastler (Haag and Kastler 1964), and it certainly influenced the development of these axiom systems. Haag's theorem is a significant result for the study of the foundations of relativistic quantum field theory (QFT) with interactions because it simultaneously furnishes a framework for understanding the notorious renormalization procedure in canonical QFT and reveals constraints on a rigorous mathematical formulation of QFT. Haag's theorem can be regarded as giving a *reductio* of a set of assumptions deployed in setting up the interaction picture representation in canonical QFT. If the interaction picture representation adopts this complete package of assumptions and is mathematically well-defined, then it necessarily describes a free system. Renormalization is a response to this *reductio*: once infinite renormalization counterterms are introduced, the interaction picture is not mathematically well-defined; therefore, it is no longer possible to prove Haag's theorem. As far as the possibility of erecting a rigorously mathematically well-defined framework for relativistic QFT is concerned, Haag's theorem can be regarded as a 'no-go' result: no rigorous framework can adopt all of the assumptions of (any version of) Haag's theorem.

This dissertation begins with a detailed exposition and analysis of different versions of Haag's theorem (Chapter 1). The theorem is cast as a *reductio* of canonical QFT without renormalization. It is possible to adopt different strategies in response to this *reductio*: (1) renormalizing the canonical framework; (2) introducing a volume (i.e., long-distance) cutoff into the canonical framework; or (3) abandoning another assumption common to the canonical framework and Haag's theorem, which is the approach adopted by axiomatic and constructive field theorists (Chapter 2). The good news is that Haag's theorem does *not* entail that it is impossible to formulate a mathematically well-defined Hilbert space model for an interacting system on infinite, continuous space. Furthermore, Haag's theorem does not undermine the predictions of renormalized canonical QFT; canonical QFT with cutoffs and existing mathematically rigorous models for interactions are empirically equivalent to renormalized canonical QFT (Chapter 3). The final two chapters explore the consequences for the interpretation of QFT with interactions. I argue that no mathematically rigorous model of QFT on infinite, continuous space admits an interpretation in terms of quanta (i.e., quantum particles) (Chapter 4). This chapter is intended to be self-contained; the presentation does not rely on the preceding chapters. Furthermore, I contend that extant mathematically rigorous models for physically unrealistic interactions serve as a better guide to the ontology of QFT than either of the other two formulations of QFT. Consequently, according to QFT, quanta do not belong in our ontology of fundamental entities.

Some highlights of individual chapters:

Chapter 1: Explication of Haag's theorem

- Explication of informal and rigorous versions of Haag's theorem (including the rigorous Hall-Wightman, Lopuszanksi and Streit-Emch versions). I use the term "Haag's theorem" loosely to denote this family of results. The family resemblance is that they all delimit sets of sufficient conditions for the absence of vacuum polarization in a representation for two systems governed by specified dynamics; all include conditions requiring invariance under spatial translations; and all derive the unintended consequence that the two systems described are governed by the same dynamics.

- Special relativity and Haag's theorem. In relativistic QFT, the interaction picture representation does not exist for any interaction. In contrast, in Galilean QFT, there are some non-trivial interactions for which the interaction picture exists. The upshot is that while it is possible to prove a version of Haag's theorem that applies universally to all relativistic interacting systems, no version of Haag's theorem can apply to all Galilean interacting systems.
- A suitable slogan for Haag's theorem is that "the representation determines the dynamics." (But *not* the converse: not that "the dynamics determines the representation.")

Chapter 2: Haag's theorem as a *reductio* of (unrenormalized) canonical QFT

- Prior to renormalization, the interaction picture representation employed in canonical QFT endorses all of the assumptions of the rigorous Hall-Wightman version of Haag's theorem. Thus, Haag's theorem can be regarded as giving a *reductio* of (unrenormalized) canonical QFT; Haag's theorem establishes that the interaction picture necessarily describes a free system. Different formulations of QFT respond to this *reductio* by rejecting at least one of the shared assumptions of the interaction picture and Haag's theorem.

- Haag's theorem does not result from the employment of an approximation method (i.e., a perturbative expansion) to evaluate S-matrix elements in the interaction picture.
- Renormalized canonical QFT responds to Haag's theorem by introducing an infinite vacuum self-energy counterterm into the interaction Hamiltonian. Consequently, there does not exist a unique, normalizable Poincaré invariant state and Poincaré transformations are not implemented by well-defined unitary operators. Put another way, renormalized canonical QFT evades Haag's theorem by adopting a mathematically ill-defined framework for QFT; Haag's theorem cannot be proven in this context because the proof requires formal mathematics.
- Canonical QFT with a volume cutoff responds to Haag's theorem by introducing a volume (i.e., long-distance) cutoff. Introduction of a volume cutoff is a necessary and sufficient response to Haag's theorem.
- An alternative to these approaches is to abandon the canonical framework altogether and reject the assumption that there is a time at which the representation for an interacting system is unitarily equivalent to the representation for a free system.

Chapter 3: Alternative formulations of QFT

- The $(\phi^4)_2$ model constructed by Glimm and Jaffe demonstrates that it is possible to give a Hilbert space representation for a non-trivial interaction that satisfies all premises of the Hall-Wightman theorem with the sole exception of the premise that this representation is unitarily equivalent to the representation for a free field.
- Haag-Ruelle scattering theory furnishes a rigorous mathematical framework for scattering theory that is compatible with Haag's theorem. It achieves this feat

by not entailing that at $t = \pm\infty$ there is a unitary transformation that relates the interacting field to a free field. This implication is avoided by careful formulations of the asymptotic condition. The strong limit version does not contain the interacting field operator; in some circumstances the weak limit version may contain the interacting field operator, but this is not problematic because the weak limit does not entail that there is a unitary transformation relating the interacting and asymptotic free fields.

- I argue that renormalized canonical QFT is empirically equivalent to canonical QFT with a volume cutoff; i.e., in the limit in which the volume cutoff is removed, the S-matrix elements predicted by the cutoff representation approach the S-matrix elements predicted by the renormalized representation.
- I argue that renormalized canonical QFT is empirically equivalent to extant mathematically rigorous models of interacting systems; i.e., modulo the complication that renormalized canonical QFT employs a perturbation series that likely diverges, renormalized canonical QFT yields precisely the same set of S-matrix elements as the mathematically rigorous models that have been constructed to date.

Chapter 4: The fate of 'particles' in quantum field theories with interactions

- I argue that QFT with interactions does not support the inclusion of quanta as fundamental entities in our ontology because

  (a) Haag's theorem entails that the Fock representation for a free field does not support a quanta interpretation for an interacting field.

  (b) The positive-negative frequency decomposition of a classical interacting field cannot form the basis of the construction of an analogue of the Fock representation for a free field. The most serious problem is that, in general, the decomposition is not Lorentz covariant.

(c) Adopting the same formal definitions for the annihilation and creation operators and the no-particle state as in the free Fock representation does not yield a Hilbert space representation for an interaction with the right properties. (e.g., there is no way to establish that the that the one-particle states have the correct energies for states in which one quantum is present).

- Unlike Malament's argument against the existence of particle-like entities (Malament 1996; Halvorson and Clifton 2002), this argument does not require that a particle-like entity be localizable. This argument also establishes that the quanta notion is already untenable in QFT on Minkowski spacetime, before the geometry of spacetime is generalized to (presumably) accommodate gravity (see Clifton and Halvorson (2001); Arageorgis et. al. (2003)). While most philosophers of physics would probably not be surprised by the conclusion that QFT is not fundamentally a theory of quanta, they may be surprised by my arguments for it. The obstacle to obtaining a Fock representation for an interaction is not that problems are encountered representing multi-particle states in the presence of interactions, but that special relativity does not furnish the right constraints on the vacuum and one-particle states.

Chapter 5: Implications of Haag's theorem for the interpretation of QFT

- The thesis of this chapter is that the mathematically rigorous model for the physically unrealistic $(\phi^4)_2$ interaction is a more reliable guide to the ontology of QFT than either renormalized canonical QFT or canonical QFT with cutoffs, in spite of the fact that both of these formulations of QFT can be applied to physically realistic interactions.

1.0 CHAPTER 1: EXPLICATION OF HAAG'S THEOREM

It would be nice to be able to provide a concise statement of Haag's theorem in one or two sentences, but, unfortunately, such a statement would be deficient and misleading. There is a family of related results, stated and proven with varying degrees of rigor, which can individually be regarded as versions of Haag's theorem. What these results have in common is the contention that a contradiction is generated by a set of *prima facie* innocuous assumptions about how the Hilbert space representation used to represent an interaction relates to the Hilbert space representation used for the free case. As we shall see, different sets of assumptions can be used to generate the contradiction.

1.1 PART I: VERSIONS OF HAAG'S THEOREM

1.1.1 Unitarily inequivalent representations of ETCCR's

It is useful to set up the analysis of Haag's theorem in this chapter by briefly considering Haag's motivation for stating the theorem. In a quantum field theory (QFT) of free bosons, the Fock representation[1] of the equal-time canonical commutation relations (ETCCR's) is typically employed. But this is only one among an uncountable infinity of Hilbert space representations of the ETCCR's (Gårding and Wightman 1954). This raises important questions: Is the choice among these representations a conventional matter of choosing among physically equivalent representations or a substantive matter of choosing among physically inequivalent representations? If the choice is significant, then why choose Fock space over the alternatives? Haag was motivated to formulate Haag's theorem by reflecting upon these questions in the context of interacting field theories.

In their familiar form, the ETCCR's for a canonical field $\phi(\mathbf{x},t)$ and its conjugate $\pi(\mathbf{x},t)$ are

$$[\phi(\mathbf{x},t),\phi(\mathbf{x}',t)]=0,\ [\pi(\mathbf{x},t),\pi(\mathbf{x}',t)]=0,\ [\phi(\mathbf{x},t),\pi(\mathbf{x}',t)]=i\delta^3(\mathbf{x}-\mathbf{x}') \tag{1.1}$$

$\phi(\mathbf{x},t)$, $\pi(\mathbf{x},t)$ are often 'smeared' in their spatial variables to make quantities of interest well-defined:[2] i.e., $\phi(f,t)=\int\phi(\mathbf{x},t)f(\mathbf{x})d^3x$ where $f,g\in\mathcal{T}$, the test function

[1] See Section 2 of Chapter 4 for a comprehensive description of the Fock representation for a free field. In short: the Fock representation for a free field ($m>0$) is the unique (up to unitary equivalence) representation of the ETCCR's that, when $c(\mathbf{k},t)$ is defined as follows,

$$c(\mathbf{k},t)=\int\frac{d^3x}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\mathbf{k}}}}e^{ik\cdot x}[\omega_{\mathbf{k}}\phi(\mathbf{x},t)+i\pi(\mathbf{x},t)]$$

contains a unique normalizable no-particle state $|0\rangle$ such that $c(\mathbf{k},t)|0\rangle=0$ for all $\mathbf{k}$ such that $\mathbf{k}^2=k_0^2-m^2$.

[2] For example, in the Fock representation for a free field,

$$\phi(\mathbf{x},t)=\int\frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\mathbf{k}}}}[c^\dagger(\mathbf{k},t)e^{ik\cdot x}+c(\mathbf{k},t)e^{-ik\cdot x}]$$

This expression is only well-defined for $\phi(f,t)$.

space, a real vector space with scalar product (f, g) and norm $\|f\| = (f, f)^{\frac{1}{2}}$. The ETCCR's for $\phi(f, t)$, $\pi(g, t)$ are

$$[\phi(f,t), \phi(g,t)] = [\pi(f,t), \pi(g,t)] = 0, [\phi(f,t), \pi(g,t)] = i(f,g) \tag{1.2}$$

It is inconvenient to employ this form of the ETCCR's with unbounded operators $\phi(f, t)$, $\pi(g, t)$ because $\phi(f, t), \pi(g, t)$ are not defined on the entire Hilbert space and care would have to be taken to keep track of their domains. The Weyl form of the ETCCR's does not have these drawbacks. A representation of the Weyl form of the ETCCR's is given by the pair of families of unitary operators $U(f, t) = e^{i\phi(f,t)}$, $V(g, t) = e^{i\pi(g,t)}$ where f,$g \in \mathcal{T}$ and $U(f, t)$ and $U(g, t)$ satisfy the conditions

$$\begin{aligned} U(f,t)U(g,t) &= U(f+g,t) \\ V(f,t)V(g,t) &= V(f+g,t) \\ U(f,t)V(g,t) &= e^{i(f,g)}V(g,t)U(f,t) \end{aligned} \tag{1.3}$$

and $U(\alpha f, t)$, $V(\alpha g, t)$ are continuous in the real number α (Wightman 1967b, p. 189).

When can representations of the ETCCR's be considered physically equivalent? In ordinary non-relativistic quantum mechanics (NRQM), Heisenberg and Schrödinger representations of the ETCCR's the canonical position operators $\hat{q}_i$ and momentum operators $\hat{p}_j$

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, [\hat{q}_i, \hat{p}_j] = i\delta_{ij} \tag{1.4}$$

are often employed. The choice between the Heisenberg and Schrödinger representations is regarded as a conventional choice between physically equivalent representations. This attitude is warranted by the fact that these representations are unitarily equivalent. Two representations of the ETCCR's $(\mathcal{H}_1, \{O_1^i\})$ and $(\mathcal{H}_2, \{O_2^i\})$ (where $\{O_n^i\}$ is the collection of operators appearing in the ETCCR's) are unitarily equivalent if and only if there exists some unitary mapping U from Hilbert space $\mathcal{H}_1$ to Hilbert space $\mathcal{H}_2$ that such that for each operator $O_1^j \in \{O_1^i\}$ there exists an operator $O_2^j = UO_1^jU^{-1} \in \{O_2^i\}$ (Wald 1994, p. 19). If two representations are unitarily equivalent, then each produces the same expectation values for its respective set of operators, $\{O_1^i\}$ or $\{O_2^i\}$: (using $U^\dagger = U^{-1}$)

$$
{}_2\langle\alpha|O_2^j|\alpha\rangle_2 =_1 \langle U^\dagger\alpha|UO_1^jU^{-1}|U\alpha\rangle_1 =_1 \langle\alpha|(U^{-1}U)O_1(U^{-1}U)|\alpha\rangle_1 =_1 \langle\alpha|O_1|\alpha\rangle_1 \tag{1.5}
$$

$$
\text{For all } O_2^j \in \{O_2^i\},\ |\alpha\rangle_2 \in \mathcal{H}_2 \text{ and for all } O_1^j \in \{O_1^i\},\ |\alpha\rangle_1 \in \mathcal{H}_1
$$

When a representation of the ETCCR's $(\mathcal{H}_n, \{O_n^i\})$ is irreducible–a standard assumption of QFT–the only operator that commutes with all $\{O_n^i\}$ is the identity operator (Streater and Wightman 2000, p. 101). Consequently, unitarily equivalent irreducible representations of the ETCCR's are physically equivalent in the sense that both representations generate the same expectation values.

For NRQM with a finite number of degrees of freedom, the Stone-von Neumann uniqueness theorem guarantees that not only are the Schrödinger and Heisenberg representations of the Weyl form of the ETCCR's[3] unitarily equivalent, but that all irreducible representations of the Weyl ETCCR's are unitarily equivalent. In general,

[3] In analogy to (1.3), the Weyl form of the ETCCR's for NRQM is, where $\bar{U}(\alpha) = \exp(i\sum_{j=1}^n \alpha_j q_j)$, $\bar{V}(\beta) = \exp(i\sum_{j=1}^n \beta_j p_j)$

$$
\begin{aligned}
U(\alpha)U(\alpha') &= U(\alpha+\alpha') \\
V(\beta)V(\beta') &= V(\beta+\beta') \\
U(\alpha)V(\beta) &= \exp\left(i\sum_{j=1}^n \alpha_j\beta_j\right) V(\beta)U(\alpha)
\end{aligned}
$$

the Stone-von Neumann theorem is not applicable in QFT because QFT's typically have an infinite number of degrees of freedom. For an infinite number of degrees of freedom, not only are unitarily inequivalent representations possible; for an infinite number of degrees of freedom, there exist uncountably many unitarily inequivalent representations of the ETCCR's (Gårding and Wightman 1954). Accordingly, there do not exist unitary transformations that could be used to show that the expectation values generated by these representations are equal. The failure of the Stone-von Neumann theorem raises a "choice problem": on what grounds are we to choose among the unitarily inequivalent representations of the ETCCR's?

The choice problem was very much on Haag's mind when he formulated Haag's theorem. In the abstract of Haag (1955), which contains the first published statement of Haag's theorem, Haag gives the following summary of the paper:

> Some difficulties connected with the infinite number of degrees of freedom are pointed out. Especially the fact that the canonical commutation relations no longer have unique solutions must be taken into account in all discussions of field theory. (p. 3)

In particular, Haag was concerned about the repercussions of the failure of the Stone-von Neumann theorem for the representation of dynamics. A Hilbert space representation of the ETCCR's–the *equal time* CCR's–gives a representation of the canonical field operators at one time (i.e., the time specified in the time argument of the operators). The dynamics is usually taken to be implemented by unitary operators. This introduces the assumption that representations of the ETCCR's at different times are unitarily equivalent. In light of the failure of the Stone-von Neumann theorem, there are circumstances in which this assumption may not be justified.

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where U, V are weakly continuous and $\alpha_1 \dots \alpha_n, \beta_1 \dots \beta_n \in \mathbb{R}$ (Wightman 1967a, p. 245). The conditions that U, V be weakly continuous and $\alpha_1 \dots \alpha_n, \beta_1 \dots \beta_n \in \mathbb{R}$ are required to prove the Stone-von Neumann theorem (Summers 2001).

In the early 1950's, when Haag was pondering this issue, the canonical approach to treating interactions in QFT had recently been developed by Feynman, Dyson and their colleagues. All frameworks for scattering theory incorporate the assumption that an interacting system approaches a free system in the limit of infinitely early time before the interaction. In the canonical framework, this assumption is implemented by postulating that in the limit $t \to -\infty$ there exists a unitary transformation $U(0,t)$ ("Dyson's matrix") that relates the interacting canonical field operators $\phi_I(\mathbf{x},0), \pi_I(\mathbf{x},0)$ and the free canonical field operators $\phi_F(\mathbf{x},t), \pi_F(\mathbf{x},t)$:

$$\phi_I(\mathbf{x},0) = U^\dagger(0,-\infty)\phi_F(\mathbf{x},0)U(0,-\infty) \tag{1.6}$$

$$\pi_I(\mathbf{x},0) = U^\dagger(0,-\infty)\pi_F(\mathbf{x},0)U(0,-\infty) \tag{1.7}$$

Even though the pioneers of canonical QFT were apparently unaware of the "choice problem," their formulation of QFT implicitly contains a response to it: as Haag puts it, that "[$\phi_I(\mathbf{x},0), \pi_I(\mathbf{x},0)$] belong to the same representation of the canonical commutator ring as [$\phi_F(\mathbf{x},t), \pi_F(\mathbf{x},t)$]" (p. 30). Put another way, the assumption is that the only representation of the ETCCR's that is physically relevant is the Fock representation for the corresponding free field. (See Chapters 2 and 3 for further discussion of scattering theory in the context of Haag's theorem.)

Haag worried that this resolution of the choice problem is untenable. In light of the failure of the Stone-von Neumann theorem, the existence of such a unitary transformation $U(0,-\infty)$ is no longer automatic, as it would be in NRQM. Haag's theorem was formulated to answer the question of whether $U(0,-\infty)$ exists. The interesting answer supplied by the theorem is that it does not: Haag's theorem shows that the assumption that $U(0,-\infty)$ exists is not compatible with the other assumptions of canonical QFT. In Haag's words, the significance of this result is that the existence of unitarily inequivalent representations of the ETCCR's is not "a matter of mathematical sophistication without relevance to field theory" (p. 18).

1.1.2 Informal versions of Haag's theorem

Before considering the rigorously stated and proven versions of Haag's theorem, it is helpful to consider some less formal but more intuitive versions of the result. The original statement of the theorem published in Haag (1955) falls into this category.

Heuristic versions of Haag's theorem concern the phenomenon of vacuum polarization. The term "vacuum polarization" describes the state of affairs that $H_I|0\rangle_F \neq 0$, where H_I is a full interaction Hamiltonian ($H_I = H_F + H'$) and $|0\rangle_F$ is the physical vacuum for the corresponding free field theory with Hamiltonian H_F.[4] The physical vacuum $|0\rangle_F$ is, by definition, the ground state of H_F with eigenvalue zero: $H_F|0\rangle_F = 0$. Vacuum polarization occurs for typical interaction Hamiltonians. For example, consider the $(\phi^4)_2$ interaction (i.e., an interaction in two-dimensional spacetime with a scalar ϕ^4 term in the Lagrangian) which has the formal Hamiltonian $H^{(\phi^4)_2}$(Bogolubov, Logunov, and Todorov 1975, p. 613)

$$\begin{aligned} H^{(\phi^4)_2} &= H_F + H' - E \\ &= \frac{1}{2}\int d^1x\{m^2 : \phi^2(\mathbf{x},t) : + : (\frac{d\phi(\mathbf{x},t)}{dx})^2 : + : \pi^2(\mathbf{x},t) :\} \\ +\lambda \int d^1x &: \phi^4(\mathbf{x},t) : -E \end{aligned} \tag{1.8}$$

[4]Renormalizing the mass of H_F (to yield $H_F'|0\rangle_F'$) does not undo vacuum polarization. This follows from the Hall and Wightman theorem (see Section 3), which is insensitive to the form taken by the free and interaction Hamiltonians. Wave function renormalization renders the Hilbert space representation for the renormalized interacting field unitarily inequivalent to the Fock representation for the free field, so vacuum polarization cannot be expressed (see Chapters 4 and 5).

where $\phi(\mathbf{x}, t)$ satisfies the free field equation with the same mass m[5] at time t. E is a constant that will be set to make the lowest eigenvalue of H zero. Substituting creation and annihilation operators using the expression in footnote 1,

$$\begin{aligned} &: \phi^4(x,t) :=: (\int \frac{d^1k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}}[c^\dagger(k,t)e^{ik\cdot x} + c(k,t)e^{-ik\cdot x}])^4 : \\ &= (\int \frac{d^1k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k}}c^\dagger(k,t)e^{ik\cdot x})(\int \frac{d^1k'}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{k'}}}c^\dagger(k',t)e^{ik'\cdot x}) \\ &\quad (\int \frac{d^1k''}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{k''}}}c^\dagger(k'',t)e^{ik\cdot x})(\int \frac{d^1k'''}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{k'''}}}c^\dagger(k''',t)e^{ik'''\cdot x}) + \ldots \end{aligned} \tag{1.9}$$

Apply the full Hamiltonian $H^{(\phi^4)_2}$ to $|0\rangle_F$. As Bogolubov, Logunov, and Todorov put it, $H^{(\phi^4)_2}|0\rangle_F$ "is not proportional to $|0\rangle_F$ since $: \phi^4(x) :$ contains a term with four creation operators" (p. 613). The most creation operators occurring in any term of the free part of the Hamiltonian, H_F, is two. The most creation operators occurring in any term of the interaction part of $H^{(\phi^4)_2}$, H', is four, and only one term contains this many creation operators. This term of $H^{(\phi^4)_2}|0\rangle_F$ will not be canceled because it is orthogonal to all other terms of $H^{(\phi^4)_2}|0\rangle_F$, including $E|0\rangle_F$; therefore, $H^{(\phi^4)_2}|0\rangle_F \neq 0$ and, in fact, $|0\rangle_F$ is not an eigenstate of $H^{(\phi^4)_2}$. Thus, the $(\phi^4)_2$-Hamiltonian exhibits vacuum polarization.

Informal versions of Haag's theorem derive the absence of vacuum polarization from a set of generally accepted assumptions. It is then observed that this contradicts the occurrence of vacuum polarization for typical Hamiltonians, which can be verified independently, as in the $(\phi^4)_2$ example. An extended argument of this type can be found in Wightman (1967a) and (1967b). Ten years earlier Wightman had, with

[5]Mass and charge renormalization terms should also be included, but since these are finite for a $(\phi^4)_2$ interaction they will be left out (Bogolubov, Logunov, and Todorov 1975, p. 612). Finite mass and charge renormalization does not undo vacuum polarization, as the Hall-Wightman theorem demonstrates (see Section 1.3). The relationship between Haag's theorem and infinite renormalization will be discussed in the next chapter.

Hall, proven a more rigorous and general version of Haag's theorem (see section 1.3), so he is not resorting to a heuristic argument because a more rigorous one was not known to him. Rather, his motivation in presenting a less formal version of the theorem was to communicate the result to a wider audience. He explains that this is necessary because the "treatment [of this topic] in the textbook literature is execrable" (Wightman 1967a, p. 255).

Wightman expresses his conclusion as being that "[a] *necessary* condition that a[n interacting] theory make physical sense is therefore that one use a strange representation of the commutation relations" (1967a, p. 250). Like many of the other commentators on this issue, Wightman employs the term "strange representation." The first use of this term that I have found is in Haag (1955, p. 21), the article which contains the first version of what has come to be known as Haag's theorem. Wightman does not explicitly define "strange representation," but a definition can be found in Bogolubov et. al. (1975, p. 560): a "strange representation" is any representation of the ETCCR's for a free field that is unitarily inequivalent to the Fock representation. It is clear from the context[6] that Wightman's use of "strange representation" is consistent with Bogolubov's definition.[7]

[6]In particular, it is clear from the context that Wightman does not interpret "strange representation" as referring to any non-ΦOK_2 representation of the ETCCR's for an *interacting* field (see Section 5 of Chapter 4 for a definition). For example, in (1967a), Wightman introduces his argument by posing the problem that a physicist "is given some Hamiltonian expressed as function of [free annihilation and creation operators] b's and b^*'s, he has to decide which irreducible representation of the commutation relations to put in" (p. 249). In the same presentation of his argument, Wightman gives an argument for premise 1.–that in a Euclidean invariant theory using the ΦOK_2 representation the no-particle state is Euclidean invariant–that only applies to a field that satisfies the free Klein-Gordon equation (1967a, p. 249). Also, the appeal to Haag's theorem in the context of vacuum polarization (1967b, p. 193) only makes sense if the ΦOK_2 representation in question is a ΦOK_2 representation for a free field. The reasons that Haag's theorem is inapplicable in the context of ΦOK_2 representations for interacting fields will be discussed in Chapter 4.

[7]Another term that is sometimes employed in this context is "myriotic" (see, e.g., Friedrichs (1953)). Note that these terms are not synonymous. A strange representation is any representation of the ETCCR's for a given free field of mass m that is unitarily inequivalent to the Fock repre-

Wightman's argument that the representation of an interaction Hamiltonian H_I on the Fock representation $\mathcal{H}_F$ of the ETCCR's for the corresponding free fields $\phi_F(f,t), \pi_F(f,t)$ does not make physical sense has the following structure:

1. "In a euclidean-invariant theory which uses the [Fock] representation of the commutation relations, the no-particle state is euclidean invariant" (1967a, p. 249). A proof of this statement for any Fock representation is contained in (1967b, pp. 192-3). The euclidean invariance of the theory implies that there exists a continuous unitary representation of the euclidean group $\{\mathbf{b}, R\} \to U(\mathbf{b}, R)$ such that

$$\begin{aligned} U(\mathbf{b}, R)\phi_F(f,t)U^\dagger(\mathbf{b}, R) &= \phi_F(f(R^{-1}(\mathbf{x}-\mathbf{b}),t) \\ U(\mathbf{b}, R)\pi_F(f,t)U^\dagger(\mathbf{b}, R) &= \pi_F(f(R^{-1}(\mathbf{x}-\mathbf{b}),t) \end{aligned} \tag{1.10}$$

Consequently, (where $c(f)$ is the smeared annihilation operator)

$$U(\mathbf{b}, R)c(f)U(\mathbf{b}, R)^{-1} = c(f(R^{-1}(\mathbf{x}-\mathbf{b}))) \tag{1.11}$$

In the Fock representation, which contains the no-particle state $c(f)|0\rangle = 0$ for all $f \in \mathcal{T}$, this implies

$$0 = U(\mathbf{b}, R)c(f)|0\rangle = c(f(R^{-1}(\mathbf{x}-\mathbf{b})))U(\mathbf{b}, R)|0\rangle \text{ for all } f \in \mathcal{T} \tag{1.12}$$

which shows that $U(\mathbf{b}, R)|0\rangle$ is also a no-particle state. Therefore

$$\begin{aligned} U(\mathbf{b}, R)|0\rangle &= \lambda(\mathbf{b}, R)|0\rangle \text{ where } |\lambda| = 1 \text{ and the } \lambda \text{ form a} \\ &\quad \text{one-dimensional representation of the euclidean group} \end{aligned} \tag{1.13}$$

Since the only one-dimensional representation of the euclidean group is the trivial one $\lambda = 1$, $U(\mathbf{b}, R)|0\rangle = |0\rangle$, i.e., $|0\rangle$ is euclidean invariant.

sentation; a myriotic representation is a representation of the ETCCR's in which a total number operator is not well-defined.

2. "We have no physical interpretation in a euclidean invariant theory for a state invariant under euclidean transformations other than as the physical vacuum" (1967a, pp. 249-50).

   $U(\mathbf{b}, R)|0\rangle = |0\rangle$ implies that the vacuum looks the same to an observer who is spatially translated or (spatially) rotated. In this restricted sense, $|0\rangle$ "looks the same to all observers" (Streater and Wightman 2000, p. 21), which is an exclusive property of the physical vacuum state. In addition, the physical vacuum is the only state with zero linear and angular momentum (Streater and Wightman 2000, p. 21).[8] These properties can also be attributed to $|0\rangle$ as a consequence of $U(\mathbf{b}, R)|0\rangle = |0\rangle$; therefore, $|0\rangle$ is the physical vacuum. For example, consider linear momentum, which is represented by the operator P. The unitary operator representing spatial translations is $U(\mathbf{b}, I) = e^{i\mathbf{b}\cdot P}$, where $\mathbf{b}$ is an arbitrary spatial vector. Since $e^{i\mathbf{b}\cdot P}|0\rangle = |0\rangle$ means that $|0\rangle$ is invariant under spatial translations, $\frac{\partial}{\partial x_i}|0\rangle = 0$ $(j = 1, 2, 3)$. $\frac{\partial}{\partial x_j}(e^{i\mathbf{b}\cdot P}|0\rangle) = \mathbf{b} \cdot Pe^{i\mathbf{b}\cdot P}|0\rangle = \mathbf{b}\cdot P|0\rangle = 0$ implies that $P|0\rangle = 0$ (assuming that $|0\rangle$ is in the domain of P)[9] since $\mathbf{b}$ is arbitrary.

3. "But the physical vacuum must also be invariant under time translations (in a theory without external sources), i.e., it must be a proper vector of the Hamiltonian" (1967a, p. 250)

   The physical vacuum must be invariant under time translations (i.e., $U(a_0, I)|v\rangle = |v\rangle$) because it must "look the same to all observers" (Streater and Wightman 2000, p. 21) and time translations are a component of the Poincaré group $U(a, \Lambda)$, which is a symmetry of the action in Minkowski spacetime. The unitary operator representing time translations is $U(a_0, I) = e^{ia_0 H}$. An argument analogous

[8] All that is required is that it possess the lowest eigenvalues of any eigenstates, but constants can always be subtracted from the Hamiltonian and momentum operators to make the lowest eigenvalues zero. See the above example of the $(\phi^4)_2$-Hamiltonian.

[9] See the discussion at the end of this section.

to that given in support of premise 2 establishes that $e^{ia_0H}|v\rangle = |v\rangle$ implies that $H|v\rangle = 0$. A "proper vector of the Hamiltonian" is a vector $|z\rangle$ that satisfies $H|z\rangle = 0$. Since, from premise 2, $|0\rangle$ is the physical vacuum, $|0\rangle$ must satisfy $e^{iHt}|0\rangle = |0\rangle$ and $H|0\rangle = 0$.

4. For "the usual Hamiltonians of quantum field theory except that of a free field" (1967a, p. 250), "vacuum polarization forbids the no-particle state to be the vacuum" (1967b, p. 193).

   Wightman takes vacuum polarization to be both a verifiable fact and a consequence of Haag's theorem. In response to a question by Wigner after Wightman presented (1967b) as a lecture, Wightman argued that "[t]he kinds of Hamiltonian that you run into if you idealize and simplify the situation in quantum electrodynamics or in the Yukawa theory are in my opinion of such obvious importance that I think they ought to be studied and they have vacuum polarization" (p. 224). These are cases in which, like the $(\phi^4)_2$ Hamiltonian, it can be verified that vacuum polarization occurs. This argument that vacuum polarization occurs is not entirely rigorous and is restricted to typical Hamiltonians. As I will indicate below, the Hall-Wightman version of Haag's theorem provides a rigorous demonstration that vacuum polarization occurs for all interaction Hamiltonians, not just the typical ones.

Therefore

5. "one must use a strange representation to make sense of the Hamiltonian in such a theory" (1967b, p.193)

In summary, Wightman argues that since the no-particle state in a free Fock representation can only be interpreted as the physical vacuum yet the no-particle state is *not* the physical vacuum for our typical interaction Hamiltonian H_I, one must use a representation other than $\mathcal{H}_F$ to represent H_I. This argument seems plausible, as far as it goes. However, there is a conceivable circumstance in which it does not seem problematic for $|0\rangle$ to be invariant under euclidean transformations but not time translations: if H is an unbounded operator defined on a dense domain of the Hilbert space that does not include $|0\rangle$. Intuitively, in this case $|0\rangle$ is not a dynamically possible state; consequently, in this case we should not feel the need to give $|0\rangle$ any physical interpretation. (More will be said about this possibility below.) Excepting this reservation, Wightman's argument seems sound. The assumptions that Wightman needs to generate his conclusion seem unobjectionable, at least for the usual Hamiltonians: Euclidean invariance of the theory, time translation invariance of the physical vacuum, and vacuum polarization.

Insight into Wightman's reasons for dismissing the possibility that $|0\rangle$ is outside the domain of H_I can be gained by examining a related argument against the use of free Fock representations for interacting fields in Haag (1996, pp. 54-55). A difference between the two arguments is that Haag frames his conclusion in more general terms. While Wightman argues that the representation of an interaction Hamiltonian on the Fock space for the corresponding free field does not make physical sense, Haag's argument applies to representations with physical vacuum states for any two Hamiltonians of a given form with the same mass[10] and different coupling constants. For example

$$\begin{aligned} H_I^1 &= H_F + \lambda_1 \int dx : \phi^4(\mathbf{x}, t) : \\ H_I^2 &= H_F + \lambda_2 \int dx : \phi^4(\mathbf{x}, t) : \end{aligned} \tag{1.14}$$

[10]i.e., the same bare mass. This argument does not take into account renormalization. The relationship between Haag's theorem and renormalization will be discussed in Chapter 2.

The free Fock representation is the special case in which the coupling constant is set to zero. Haag's conclusion is that any representation of the ETCCR's for fields governed by a given Hamiltonian that contains a physical vacuum state is unitarily inequivalent to any representation of the ETCCR's for fields governed by a Hamiltonian derived from a Lagrangian with the same mass but different coupling constant that contains a physical vacuum state. Haag also uses the occurrence of vacuum polarization as a premise in his argument. However, in contrast to Wightman, Haag justifies this assumption by arguing that because a physical vacuum is the lowest eigenstate of the Hamiltonian it is dependent on the coupling constant; therefore, Hamiltonians with different coupling constants cannot have the same vacuum vector.[11] The rest of the argument that Hamiltonians with different coupling constants must use unitarily inequivalent representations of the ETCCR's is that in an irreducible representation of the ETCCR's for a given coupling constant there is a unique vector invariant under spatial translations; since this makes this vector the only candidate for the physical vacuum in this representation and we know that the vacua for Hamiltonians with different values of the coupling constant are different, this representation does not contain a physical vacuum for a Hamiltonian with any other value assigned to the coupling constant. Haag's own gloss on his argument is that "the determination of the representation class of [the ETCCR's] is a dynamical problem" (p. 55). The longer-winded version is that this argument traces the origins of the need for unitarily inequivalent representations for different Hamiltonians to the fact that the spatial translation operators–which are independent of the form taken by the Hamiltonian–uniquely determine a candidate for a physical vacuum state in a given irreducible representation of the ETCCR's and the fact that the time translation operators are

[11] Again, a rigorous argument that this is actually the case is given by the Hall-Wightman theorem, discussed in Section 1.3 below.

dependent on the form taken by the Hamiltonian; the problem is that the physical vacuum must be invariant under both space and time translations. An appropriate slogan for this conclusion is "the representation determines the dynamics"; i.e., if a vacuum state is demanded, then a given representation of the ETCCR's can only be used to represent one dynamics. (See Section 3.2)

Haag's argument demonstrates that, if $|0\rangle$ is not the physical vacuum for H, then there is no physical vacuum for H in $\mathcal{H}_F$. Wightman dismisses the possibility that $\mathcal{H}_F$ furnishes a 'vacuumless' representation for H when $|0\rangle$ is outside the domain of H because he believes that any physically sensible representation must contain a physical vacuum (Streater and Wightman 2000, p. 97). However, this belief is not shared by everyone; Haag and Kastler are perfectly happy to accept representations which do not contain exact physical vacua (Haag and Kastler 1964).[12]

The first published statement and proof of a version of Haag's theorem is in Haag (1955). In this article, Haag gives two proofs of his theorem: an informal one in which he observes that for typical interaction Hamiltonians vacuum polarization occurs and a second proof which is intended to go through "without reference to any particular form for $[H_I]$" (op. cit., pp. 30-31). This second proof is difficult to decipher. Hall and Wightman's assessment is that "[i]n the opinion of the present authors, Haag's proof is, at least in part, inconclusive..." (Hall and Wightman 1957, p. 41, note 10). Consequently, Hall and Wightman should be credited with the first proof of Haag's theorem that does not depend on the observation that vacuum polarization occurs for typical interaction Hamiltonians.

The informal proof in Haag (1955) relies on the following assumptions:

[12]In any case, it turns out that it is not necessary to enter into this debate in this context because–as will be discussed in future chapters–subsequent to the publication of Wightman's informal argument, Glimm showed that in a free Fock representation the only vector in the domain of an interaction Hamiltonian is the zero vector (Glimm 1969b).

1. The interacting field $\phi_I(\mathbf{x})$ and its canonical conjugate $\pi_I(\mathbf{x})$ stand in the following relations to the asymptotic incoming ($t = -\infty$) free field $\phi_F(\mathbf{x})$ and its time derivative $\dot{\phi}_F(\mathbf{x})$, where V is a unitary operator

$$\begin{aligned} \phi_I(\mathbf{x}) &= V\phi_F(\mathbf{x})V^\dagger \\ \pi_I(\mathbf{x}) &= V\dot{\phi}_F(\mathbf{x})V^\dagger \end{aligned} \tag{1.15}$$

(ϕ_I, π_I), $(\phi_F, \dot{\phi}_F)$ each give representations of the ETCCR's.

2. Where $D(\mathbf{b})$ represents spatial translations, which are generated by $\mathbf{P}$,

$$\begin{aligned} D(\mathbf{b})\phi_I(\mathbf{x})D^\dagger(\mathbf{b}) &= \phi_I(\mathbf{x} - \mathbf{b}) \\ D(\mathbf{b})\pi_I(\mathbf{x})D^\dagger(\mathbf{b}) &= \pi_I(\mathbf{x} - \mathbf{b}) \end{aligned} \tag{1.16}$$

3. $|\Phi_0\rangle$ such that $\mathbf{P}|\Phi_0\rangle = 0$ is the only eigenstate of $\mathbf{P}$ with a discrete eigenvalue.

The proof that $V|\Phi_0\rangle = |\Phi_0\rangle$ is straightforward. Assumption 2 implies that $[V, \mathbf{P}] = 0$. Then $V\mathbf{P}|\Phi_0\rangle = \mathbf{P}V|\Phi_0\rangle$. From Assumption 3, $V\mathbf{P}|\Phi_0\rangle = 0$, so $\mathbf{P}V|\Phi_0\rangle = 0$, which (applying Assumption 3 again) implies that $V|\Phi_0\rangle = |\Phi_0\rangle$. Haag then observes that "[i]n all theories considered so far, [this result] is contradicted immediately by the form of the Hamiltonian" (p. 31); that is, since V is a time evolution operator, $V|\Phi_0\rangle = |\Phi_0\rangle$ implies that vacuum polarization does not occur, which is false for typical interaction Hamiltonians.

The essential difference between these informal versions of Haag's theorem and the more rigorous versions that will be set out in the following sections is that, to generate the contradiction, the informal ones rely on the observation that vacuum polarization occurs for typical interaction Hamiltonians; the rigorous versions prove that vacuum polarization occurs for interaction Hamiltonians in general. The essential point of similarity between the informal and formal versions of the theorem is that vacuum polarization is the key consideration in the argument that representations of a specified sort can only be used to describe free systems.

1.1.3 The Hall-Wightman theorem

As mentioned in the previous section, Hall and Wightman (1957) contains the first published version of Haag's theorem that does not rely on the observation that vacuum polarization occurs for typical interaction Hamiltonians. Hereafter, this result will be referred to as the "Hall-Wightman theorem." In a footnote, Hall and Wightman acknowledge their debt to Haag: "[i]t will not have escaped the discerning reader of Haag's paper that, while we have generalized his results, eliminated one of his assumptions (the asymptotic condition), completed his proofs, and sharpened his conclusions, the essential physical points are Haag's" (p. 41, note 10). Hall and Wightman characterize their theorem as a generalization of Haag's informal result because it applies to any two fields governed by arbitrary dynamics, not only to a free and to an interacting field.

The Hall-Wightman theorem is stated and proved in two parts (Hall and Wightman 1957, pp. 122-124):

> *Hall-Wightman Theorem, Part I.* Consider two neutral scalar[13] fields ϕ_j, $j = 1, 2$, and their conjugate momentum fields π_j such that
> (i) Each pair (ϕ_j, π_j) gives an irreducible representation of the equal time CCR (ETCCR)

$$\begin{aligned} [\phi_j(\mathbf{x},t), \pi_j(\mathbf{x}',t)] &= i\delta(\mathbf{x}-\mathbf{x}') \qquad j = 1,2 \\ [\phi_j(\mathbf{x},t), \phi_j(\mathbf{x}',t)] &= [\ \pi_j(\mathbf{x},t), \pi_j(\mathbf{x}',t)] = 0. \end{aligned} \tag{1.17}$$

[13] The case of neutral scalar fields is treated for notational convenience. The theorem also holds in the more general case of fields with spin indices (Streater and Wightman 2000, p. 166). The commutation relations below are then CCR's or CAR's, as dictated by the Spin-Statistics Theorem.

(ii) Euclidean transformations $(\mathbf{a}, \mathbf{R})$ (where $\mathbf{a}$ stands for a translation and $\mathbf{R}$ for a rotation) are induced by unitary transformations $U_j(\mathbf{a}, \mathbf{R})$.
(iii) The fields transform under $U_j(\mathbf{a}, \mathbf{R})$ as follows:

$$\begin{aligned} U_j(\mathbf{a}, \mathbf{R})\phi_j(\mathbf{x}, t)U_j^{-1}(\mathbf{a}, \mathbf{R}) &= \phi_j(\mathbf{R}\mathbf{x} + \mathbf{a}, t) \\ U_j(\mathbf{a}, \mathbf{R})\pi_j(\mathbf{x}, t)U_j^{-1}(\mathbf{a}, \mathbf{R}) &= \pi_j(\mathbf{R}\mathbf{x} + \mathbf{a}, t) \end{aligned} \tag{1.18}$$

(iv) There exist unique normalizable states $|0_j\rangle$ invariant under Euclidean transformations:

$$U_j(\mathbf{a}, \mathbf{R})|0_j\rangle = |0_j\rangle \tag{1.19}$$

(v) The fields are related at some time t by a unitary transformation V:

$$\phi_2(\mathbf{x}, t) = V\phi_1(\mathbf{x}, t)V^{-1}, \qquad \pi_2(\mathbf{x}, t) = V\pi_1(\mathbf{x}, t)V^{-1} \tag{1.20}$$

THEN

$$U_2(\mathbf{a}, \mathbf{R}) = VU_1(\mathbf{a}, \mathbf{R})V^{-1} \tag{1.21}$$

and

$$c|0_2\rangle = V|0_1\rangle \tag{1.22}$$

where $|c| = 1$.

Hall-Wightman theorem, Part II. Assume
(i) Fields satisfying the hypotheses of Part I.
(ii) Poincaré transformations (a, Λ) (where Λ stands for a Lorentz transformation) are induced by unitary transformations $T_j(a, \Lambda)$.
(iii) The fields transform under $T_j(a, \Lambda)$ as follows:

$$T_j(a, \Lambda)\phi_j(x) = \phi_j(\Lambda x + a), \quad j = 1, 2 \tag{1.23}$$

(iv) The states $|0_j\rangle$ are Poincaré invariant

$$T_j(a, \Lambda)|0_j\rangle = |0_j\rangle \tag{1.24}$$

(v) No states of negative energy exist.

THEN
The first four vacuum expectation values of the two fields (the four-point "Wightman functions") are equal:

$$\langle 0_2|\phi_2(x_1)...\phi_2(x_n)|0_2\rangle = \langle 0_1|\phi_1(x_1)...\phi_1(x_n)|0_1\rangle \tag{1.25}$$

where $n = 1, 2, 3$ or 4

In another article, Wightman remarks that "[f]rom the practical point of view of the physicist, [(1.25)] implies that the two theories are essentially identical" (Wightman 1959, p. 33; my translation). Hall and Wightman appeal to the physical interpretation of the VEV's to support this inference: "[i]t seems physically plausible that two theories in which the two-particle propagator, the vertex part, and the two-particle scattering for all energies are identical (as they must be if the first four vacuum expectation values are identical) should be completely identical" (pp. 125-6). However, Hall and Wightman naturally lament that "[f]rom both the aesthetic and the physical point of view, the version of the generalized Haag's theorem proved here is somewhat deficient because it only asserts the equality of the first four vacuum expectation values" (p. 125).

For the special case in which ϕ_1 is a free field, the desired result that all VEV's are equal can be proven. There are two ways in which this can be done. One method is to apply the Jost-Schroer theorem to the conclusion of the Hall-Wightman theorem. The Jost-Schroer theorem states that if ϕ is a Hermitian scalar field with the vacuum as a cyclic vector and if its two point vacuum expectation values coincide with those of a free field of mass $m > 0$, then ϕ is a free field of mass m (see Jost (1961)). The result was extended to the zero-mass case by Pohlmeyer (1969). A second approach was taken by Greenberg, who strengthened Part II of the Hall-Wightman theorem by proving, under the same hypotheses, that if ϕ_1 is a free field, then all of the n-point VEV's of the two fields are equal (Greenberg 1959). Thus, by either result, when ϕ_1 is a free field, all of the VEV's of ϕ_1 and ϕ_2 are equal. This licenses an expanded version of the argument based on the physical interpretation of the VEV's that was given by Hall and Wightman. The set of all VEV's for a field contains the set of all its S-matrix elements (i.e., VEV's for which the fields have time arguments $t \to \pm\infty$). Since the S-matrix elements represent the inputs and outcomes of scattering experiments, it seems reasonable to conclude with Hall and Wightman that any two fields which share a set of VEV's share the same dynamics.[14]

The two-part structure of the theorem is familiar from informal versions of Haag's theorem. Part I establishes that a certain set of assumptions implies that vacuum polarization does not occur. The conclusion that $c|0_2\rangle = V|0_1\rangle$ ($c = |1|$) is a statement of the absence of vacuum polarization in the context in which the representations of the ETCCR's are not identified, but are held to be unitarily equivalent: that is, $H_2 c|0_2\rangle = 0 = H_2(V|0_1\rangle)$, where $V|0_1\rangle$ can be regarded as the correlate of $|0_1\rangle$ in $\mathcal{H}_2$. Part II of the theorem establishes that ϕ_1 and ϕ_2 are both free fields; by modus tollens, when ϕ_2 is an interacting field, vacuum polarization must occur. The theorem takes the form of a reductio argument: vacuum polarization occurs; therefore, one of the assumptions of the theorem must be rejected.

For our purposes, what is of interest is the assumptions that go into the theorem, rather than the method of proof, so there is no need to go into the details of the proofs. Wightman explains that for Part I of the theorem, "[t]he idea of the proof is to show that the unitary operator that makes the representations of the CCR in the two theories equivalent necessarily makes the representations of the Euclidean group equivalent and therefore maps the vacuum state of one theory on that of the other" (Wightman 1989, p. 610). The proof of Part II is much more involved. It relies on the fact that the VEV's are boundary values of analytic functions. The transformation properties of the fields and vacua under the Poincaré group and the nonexistence of negative energies are both necessary for the VEV's to have this property (Hall and Wightman 1957, p. 124; Streit 1969, p. 673).

[14] Assurance that theories that share a set of VEV's share the same dynamics in the sense of having the same Hamiltonian is provided by Wightman's reconstruction theorem (see Streater and Wightman (2000, pp. 117-26)).

Part I of the theorem assumes only Euclidean invariance; the assumption of Poincaré invariance is only needed in Part II. The issue of whether relativistic assumptions are an essential ingredient in any rigorous version of Haag's theorem will be discussed in Section 2.1. The Hall-Wightman theorem employs many of the same assumptions as the informal versions of Haag's theorem in Wightman (1967a) and (1967b) and in Haag (1996). The only difference is that the assumptions employed in the Hall-Wightman theorem are slightly stronger. In contrast to the version in Wightman (1967a) and (1967b), the full set of Poincaré transformations rather than merely time translations are invoked. While the Haag (1996) version assumes that only that the physical vacuum is invariant under spacetime translations, the Hall-Wightman theorem assumes that the physical vacuum is fully Poincaré invariant.

1.1.4 The Lopuszanski theorem

In 1961, Lopuszanski proved an alternative rigorous version of Haag's theorem. Lopuszanski's theorem is essentially a variation upon the Hall-Wightman theorem. Instead of presenting the result within the framework of Wightman's axiomatization of QFT, Lopuszanski uses the Yang-Feldman representation for an interacting field, $A(x)$:

$$A(x) = A_{in}(x) - \int \Delta^{ret}(x-y)j(y)dy \tag{1.26}$$

where $\Delta^{ret}(x)$ is the retarded Green's function of the Klein-Gordon equation (Roman 1969, p. 28)

$$\Delta^{ret}(x) = \frac{1}{(2\pi)^4}\int \frac{e^{ikx}}{m^2-k^2+i\eta k_0}dk \tag{1.27}$$

η is a small positive real constant

The Yang-Feldman representation is used in the LSZ approach to QFT. Lopuszanski also restricts attention to the special case in which the representation for the free field $A_{in}(x)$ is the Fock representation and the interacting field is given an analogous representation:

$$A_{in}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{(2k_0)^{\frac{1}{2}}} \{e^{ikx} a_{in}(\mathbf{k}) + e^{-ikx} a_{in}^{+}(\mathbf{k})\} \tag{1.28}$$

$$A(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{(2k_0)^{\frac{1}{2}}} \{e^{ikx} a(\mathbf{k}, x_0) + e^{-ikx} a^{+}(\mathbf{k}, x_0)\} \tag{1.29}$$

where $k_0 = +\sqrt{\mathbf{k}^2 + m^2}$

There exist unique, normalizable states such that

$$a_{in}(\mathbf{k})|0\rangle = 0 \text{ for all } k \tag{1.30}$$

$$a(\mathbf{k}, x_0)|0(t)\rangle = 0 \text{ for all } k, \text{ for each } t \tag{1.31}$$

Lopuszanski's theorem can be stated as follows (p. 746):

IF
(a) (1.26)-(1.31)
(b) $|0(t)\rangle$ is invariant with respect to 3-dimensional space translations
(c) $\langle \mathbf{k}_1 \ldots \mathbf{k}_n | 0(t) \rangle \neq 0 \quad n = 0, 1, 2...$
where $|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle \equiv a_{in}^{+}(\mathbf{k}_1) \cdots a_{in}^{+}(\mathbf{k}_n)|0\rangle$
(d) $|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle$, $n = 0, 1, 2...$ form a complete set of eigenfunctions of energy-momentum $(\mathbf{P}, H)$
$|0\rangle$ and the one particle states are not degenerate
(e) $[j(y,t), A(x,t)] = 0$
(f) $[a_{in}(\mathbf{k}), a_{in}^{+}(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}')$; $[a_{in}(\mathbf{k}), a_{in}(\mathbf{k}')] = [a_{in}^{+}(\mathbf{k}), a_{in}^{+}(\mathbf{k}')] = 0$
THEN
$A(x) = A_{in}(x)$ (i.e., $A(x)$ is a free field).

Lopuszanski claims that an advantage of his theorem is that it dispenses with some of the assumptions of the Hall-Wightman theorem (pp. 746, 747). However, since these differences are minor and can be attributed to the different frameworks employed, Lopuszanski's theorem can be viewed as a variation upon the Hall-Wightman theorem. Lopuszanski points out that he does not assume that the free and (renormalized) interacting fields are related by a unitary transformation nor that the interacting field satisfies the ETCCR's (p. 746). In place of these assumptions, Lopuszanski's theorem employs assumptions that are not needed to prove the Hall-Wightman theorem: that free and interacting fields are related as in (1.26) and that the interacting field is decomposable into 'interacting' annihilation and creation operators according to (1.29).

Lopuszanski remarks that "we do not need as much relativistic invariance of the theory" (p. 747). However, while the explicit assumption that the interacting field is invariant under Poincaré transformations is not invoked in the proof of the theorem, relativistic assumptions do play an ineliminable role. The fact that $|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle$, $n = 0, 1, 2...$ form a complete set of eigenfunctions of energy-momentum ($\mathbf{P}$,H) (assumption (d)) is important; the relativistic assumption that $k_0 = +\sqrt{\mathbf{k}^2 + m^2}$ in expressions for both $A_{in}(x)$ and $A(x)$ plays a crucial role in an alternative version of the proof that Lopuszanski attributes to Haag (p. 747). Consequently, Lopuszanski's theorem is best viewed as a variation upon the Hall-Wightman theorem which presents essentially the same result in a different framework.

1.1.5 The Streit-Emch theorem

Another rigorous version of Haag's theorem has been formulated and proven by Streit and Emch. Unlike the Lopuszanski theorem, however, it is significantly different from the Hall-Wightman theorem. The key assumptions of the Streit-Emch theorem are different from those of the Hall-Wightman theorem.

A version of the Streit-Emch theorem was first published in Streit (1969). Emch acknowledges that his presentation of Haag's theorem in Emch (1972) "rests mainly" on a preprint of Streit (1969). Emch's contribution is the reformulation of Streit's theorem in the algebraic framework. Emch's version of the theorem will be presented here.

Streit formulated an alternative rigorous version of Haag's theorem with the aim of generalizing the Hall-Wightman theorem. He claims that his "generalization essentially consists in dropping not only locality but relativistic covariance altogether (as suggested by Haag's original argument); in fact the notion of space (Euclidean vs. e.g. lattice) need not be specified" (Streit 1969, p. 674). Recall Part II of the Hall-Wightman theorem assumes that the fields and vacua transform appropriately under Poincaré transformations. Emch echoes Streit's assessment in describing the significance of this version of the theorem: "standard proofs are available in the textbook literature (see, for instance, Chapter IV, Section 5 in Streater and Wightman [2000]...); most of these proofs, however, rely rather heavily on the analytic properties of the Wightman functions, which themselves reflect the locality and spectrum conditions, and tend to obscure the simple algebraic and group-theoretical facts actually responsible for the results obtained" (p. 247). The question of whether the Streit-Emch theorem is indeed independent of relativistic assumptions will be addressed in Section 2.1.

As presented in Emch (1972, pp. 247-53), the Streit-Emch theorem contains two parts and a lemma. Part I differs from Part I of the Hall-Wightman theorem only in minor respects; the major differences are in the respective Part II's, where the Hall-Wightman theorem invokes Poincaré transformations the Streit-Emch theorem employs alternative assumptions. As before, for the most part, the details of the proofs of these theorems are not important for our purposes. Definitions of special terms will be cited in footnotes, but for an overview of the algebraic approach to QFT see Halvorson (2006).

> *Streit-Emch theorem, Part I* Let $\mathcal{B}_H(\mathcal{C}_\mathbb{C})$ be a Weyl representation of the canonical commutation relations [defined above], with cyclic vector Φ. Suppose that the algebraic state $\hat{\phi}$ on $\mathcal{B}_H(\mathcal{C}_\mathbb{C})$ corresponding to Φ is G-invariant and η-clustering.[15] Suppose also that there exists a normalized vector Ω in $\mathcal{H}$ such that $c(f)\Omega = 0$ for all f in $\mathcal{C}$, where $c(f) = (F(f) + iP(f))/\sqrt{2}$, $F(f)$ and $P(f)$ are the generators of $U(f)$, $V(f)$, respectively, in the Weyl form of the ETCCR's (Emch 1972, p. 243).[16] Then $\Phi = \lambda\Omega$ with $\lambda \in \mathbb{C}$ and $|\lambda| = 1$.

The conclusion is exactly the same as line (1.22) of the conclusion of the Hall-Wightman theorem (with $V = I$; the generalization is proven in the lemma). However, the Streit-Emch theorem proves this conclusion within the algebraic framework. As advertised, the Streit-Emch result is more general in that it applies to any G-invariant and η-clustering algebraic state, while the Hall-Wightman theorem restricts

15

> (i) G-invariant: where G is a symmetry group for $(\Re, \mathcal{G}, \langle ; \rangle)$ [where $\Re$ is a C^*-algebra, $\mathcal{G}$ is the convex set of all states on $\Re$, $\langle ; \rangle$ is the expectation value of the algebraic state in the first entry for the operator in the second entry (e.g., $\langle \phi; R\rangle = \phi(R)$)],
>
> $$v_g\hat{\phi} = \hat{\phi} \;\; \forall g \in G$$
>
> (Emch 1972, p. 172), and
> (ii) η-clustering: with respect to the invariant mean η on G (Emch 1972, pp. 165, 175)
>
> $$\eta\langle \phi; \alpha_g[A]B\rangle = \eta\langle \phi; \alpha_g[A]\rangle\langle \phi; B\rangle$$
>
> where α_g is a C^*-automorphism of $\Re$. Intuitively, this condition is satsified in relativistic and Galilean theories when G is taken to be the group of spatial translations because observables commute when they are far enough apart, so "on average" B commutes with all the translations of A (Emch 1972, p. 176).

16 See the related discussion of the ΦOK_2 representation in Section 5 of Chapter 4.

attention to the special case in which the corresponding vector Φ is Euclidean invariant. (This is a special case because the algebraic state associated with a vector invariant under the spatial translation subgroup of the Euclidean group is G-invariant and η-clustering (Emch 1972, p. 247)). This theorem is less general than Part I of the Streater-Wightman theorem in that it applies only to a free and an interacting field, as indicated by the assumption that Ω is a no-particle state in a Fock representation (Streit 1969, pp. 674-5).

Lemma Let G be an amenable group of symmetries for $(\Re, \mathcal{G}, \langle;\rangle)$ and assume η-abelianness in G;[17] let ϕ_j $(j = 1, 2)$ be G-invariant states on $\Re$, with ϕ_1 η-clustering; let $\{\pi_j(\Re), U_j(G)\}$ be the covariant representation associated with ϕ_j,[18] $\mathcal{H}_j$ the corresponding representation space, and Φ_j the corresponding cyclic vector; let $\{U_j(t)|t \in \mathbb{R}\}$ be a weakly continuous, one-parameter group of unitary operators acting within $\mathcal{H}_j$, and such that $U_j(t)\Phi_j = \Phi_j$ $\forall t \in \mathbb{R}$; let $\{\pi_j^{(t)}(\Re), U_j^{(t)}(G)\}$ be the covariant representation defined for each t in $\mathbb{R}$ by

$$\pi_j^{(t)}(R) = U_j(t)\pi_j(R)U_j(-t) \tag{1.32}$$

$$U_j^{(t)}(g) = U_j(t)U_j(g)U_j(-t) \tag{1.33}$$

We suppose that for some $t = t_0$ there exists a unitary mapping $V(t_0)$ from $\mathcal{H}_2$ onto $\mathcal{H}_1$ such that

$$V(t_0)^*\pi_1^{(t_0)}(R)V(t_0) = \pi_2^{(t_0)}(R)\ \forall R \in \Re \tag{1.34}$$

Then, $\phi_1 = \phi_2$, and for each $t \in \mathbb{R}$ there exists a unitary mapping $U_0(t)$ from $\mathcal{H}_2$ onto $\mathcal{H}_1$ such that

$$\Phi_2 = U_0^*(t)\Phi_1 \tag{1.35}$$

$$\pi_2^{(t)}(R) = U_0^*(t)\pi_1^{(t)}(R)U_0(t) \tag{1.36}$$

$$U_2^{(t)}(g) = U_0^*(t)U_1^{(t)}(g)U_0(t) \tag{1.37}$$

$$U_0(t) = U_1(t)U_0(0)U_2(-t) \tag{1.38}$$

[17] η-abelianness in G: A system $(\Re, \mathcal{G}, \langle;\rangle)$ is η-abelian in G iff for all ϕ in $\mathcal{G}_G$ (the set of G-invariant states)

$$\eta\langle\phi; R^*(\alpha_g[A]B - B\alpha_g[A])R\rangle = 0$$

for all A, B in $\mathcal{U}$ (the Jordan algebra of all self-adjoint elements of $\Re$ (Emch 1972, p. 152)), and all R in $\Re$.

[18] The strongly continuous unitary representation $\{U_\phi(g)|g \in G\}$ of G in $\mathcal{H}_\phi$ such that $\pi_\phi(\alpha_g[R]) = U_\phi(g)\pi_\phi U_\phi(g)^*$ for all g in G and all R in $\Re$ (Emch 1972, pp. 163-4).

This lemma establishes that at each time the two representations are related by a unitary transformation $U_0(t)$. This does not entail that the systems described are governed by the same dynamics because $U_0(t)$ can evolve with time according to (1.38). Part II of the Streit-Emch theorem–which does establish that the systems have the same dynamics–is proven by showing that, given additional assumptions, $U_0(t)$ is in fact time-independent.

> *Streit-Emch theorem, Part II* Let $\{U_j(f), V_j(f) | f \in C\}$ $(j = 1, 2)$ be two Weyl representations of the canonical commutation relations satisfying the assumptions of the lemma. Suppose that the respective generators $F_j(f)$, $P_j(f)$, and H_j of $V(_j f)$, $U(_j(f)$, and $U_j(t)$ satisfy the following conditions:
> (i) there exists a linear manifold $\mathcal{D}_j$, dense in $\mathcal{H}_j$, and stable with respect to $F_j(f, 0)$, $P_j(f, 0)$, and H_j,
> (ii) on $\mathcal{D}_j$, $P_j(f, 0) = i[H_j, F_j(f, 0)]$,
> (iii) the set of vectors obtained by applying polynomials in $F_1(f, t)$ for $f \in \mathcal{C}$ to Φ_1 is dense in $\mathcal{H}_1$
> Then $H_2 = U^* H_1 U$ where U is the time-independent unitary operator which has all the properties established in the lemma.

The proof of this theorem is relevant to the discussion in the next section. The strategy is straightforward:

By assumption (iii), the set of all polynomials in $F_1(f, t)$ applied to Φ_1 is dense in $\mathcal{H}_1$. Consider the arbitrary vector $F_1(f_1, t) \cdots F_1(f_n, t)\Phi_1$. Using $F_1(f, t) = U_1(t)F_1(f, 0)U_1(-t)$ and $U_1(t - 0)\Phi_1 = \Phi_1$, $\forall t \in \mathbb{R}$

$$F_1(f_1, t) \cdots F_1(f_n, t)\Phi_1 = U_1(t - 0)F_1(f_1, 0) \cdots F_1(f_n, 0)\Phi_1 \tag{1.39}$$

Taking the time derivative of both sides, applying the product rule and then setting

$t = 0$ gives (noting that $U_1(0) = e^{iH_1 \cdot 0} = 1$)[19]

$$iH_1 F_1(f_1,0)\cdots F_1(f_n,0)\Phi_1 = \sum_{\nu=1}^{n} F_1(f_1,0)\cdots F_1(f_{\nu-1},0)P_1(f_\nu,0)F_1(f_{\nu+1},0)\cdots F_1(f_n,0)\Phi_1 \quad (1.40)$$

From (1.36) of the conclusion of the Lemma[20]

$$F_1(f_i,0) = U_0(0)F_2(f_i,t)U_0^*(0) \quad (1.41a)$$

$$P_1(f_i,0) = U_0(0)P_2(f_i,t)U_0^*(0) \quad (1.41b)$$

Substituting in the RHS of (1.40),

$$iH_1(0)F_1(f_1,0)\cdots F_1(f_n,0)\Phi_1 = U_0(0)\sum_{\nu=1}^{n} F_2(f_1,0)\cdots F_2(f_{\nu-1},0)P_2(f_\nu,0)F_2(f_{\nu+1},0)\cdots F_2(f_n,0)\Phi_1 \quad (1.42)$$

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[19] Emch's notation is potentially confusing. I take it that

$$\sum_{\nu=1}^{n} F_1(f_1,0)\cdots F_1(f_{\nu-1},0)P_1(f_\nu,0)F_1(f_{\nu-1},0)\cdots F_1(f_n,0)\Phi_1 = P_1(f_1,0)F_1(f_2,0)\cdots F_1(f_n,0)\Phi_1 + \sum_{\nu=2}^{n} F_1(f_1,0)\cdots F_1(f_{\nu-1},0)P_1(f_\nu,0)F_1(f_{\nu+1},0)\cdots F_1(f_n,0)\Phi_1$$

[20] $V_j(f,0), U_j(f,0) \in \pi_j^{t=0}(R)$ and by assumption of Part II of the theorem $F_j(f,0), P_j(f,0)$ are generators of $V_j(f,0), U_j(f,0)$; therefore

$$e^{-iF_2(f,0)} = U_0(0)e^{-iF_1(f,0)}U_0^*(0)$$

$$\sum_{n=1}^{\infty}\frac{(-iF_2(f,0))^n}{n!} = U_0(0)\left[\sum_{n=1}^{\infty}\frac{(-iF_1(f,0))^n}{n!}\right]U_0^*(0)$$

$$-iF_2(f,0) - \frac{1}{2}F_2(f,0)F_2(f,0) - \ldots = -iU_0(0)F_1(f,0)U_0^*(0) - \frac{1}{2}U_0(0)F_1(f,0)U_0^*(0)U_0(0)F_2(f,0)U_0^*(0) - \ldots$$

Therefore (?) $F_2(f,0) = U_0(0)F_1(f,0)U_0^*(0)$. Similarly, $P_2(f,0) = U_0(0)P_1(f,0)U_0^*(0)$.

Then on the RHS eliminate the P_2 in each term by applying the commutation relation from assumption (ii) of Part II with $t = 0, j = 2$:

$$P_2(f, 0) = i[H_2, F_2(f, 0)] \text{ on } \mathcal{D}_2 \tag{1.43}$$

After cancellations,

$$iH_1F_1(f_1, 0)\cdots F_1(f_n, 0)\Phi_1 = iU_0(0)H_2F_2(f_1, 0)\cdots F_2(f_n, 0)\Phi_1 \tag{1.44}$$

Then use (1.41a) to eliminate the $F_2(f_i, 0)$'s in favour of $F_1(f_i, 0)$'s

$$iH_1F_1(f_1, 0)\cdots F_1(f_n, 0)\Phi_1 = iU_0(0)H_2U_0^*(0)F_1(f_1, 0)\cdots F_1(f_n, 0)\Phi_1 \tag{1.45}$$

Thus

$$H_1 = U_0(0)H_2U_0^*(0) \tag{1.46}$$

Emch then cites Streit's proof that H_1 is essentially self-adjoint on the domain of polynomials of $F_1(f_i, t)$ applied to Φ_1(i.e., H_1 has a unique self-adjoint extension to the rest of $\mathcal{H}_1$) to establish that domain problems do not render (1.46) uninformative.

Since, by assumption of Part II, H_j is the generator of $U_j(t)$, this implies that systems 1 and 2 are governed by the same dynamics:

$$\begin{aligned}
e^{iH_it} &= e^{iU_0(0)H_2U_0^*(0)t} && (1.47)\\
&= \sum_{n=1}^{\infty} \frac{(iU_0(0)H_2U_0^*(0)t)^n}{n!} && (1.48)\\
&= 1 + U_0(0)\,[iH_2t]\,U_0^*(0) + U_0(0)\,[iH_2t]\,U_0^*(0)U_0(0)[iH_2t]U_0^*(0) + \ldots && (1.49)\\
&= U_0(0)\left[1 + iH_2t + (iH_2)^2 + \ldots\right]U_0^*(0) && (1.50)\\
&= U_0(0)e^{iH_2t}U_0^*(0) && (1.51)
\end{aligned}$$

That is, $U_1(t) = U_0(0)U_2(t)U_0^*(0)$. The crucial difference between this expression and conclusion (1.38) of the Lemma

$$U_0(t) = U_1(t)U_0(0)U_2(-t) \tag{1.38}$$

is that $U_1(t), U_2(t)$ are related by the *same* unitary operator $U_0(0)$. Since $U_1(t), U_2(t)$ are time-independent operators (the t represents translation by time t, not time-dependence), they are related by $U_0(0)$ at all times. Therefore, from (1.38), $U_0(t) = U_0(0)$ $\forall t \in \mathbb{R}$. The fact that the systems are related by the same unitary transformation at all times is another way of appreciating the fact that the systems are governed by the same dynamics.

The Streit-Emch and Hall-Wightman theorems are both variants of Haag's theorem because they share a common conclusion: a free system and a purportedly interacting system in fact share the same free dynamics because, as Emch puts it, Part I of theorem "*prevents vacuum polarization* from occurring" (p. 248). However, while the Hall-Wightman theorem relies on relativistic assumptions (e.g., Poincaré transformations) to establish this conclusion, the Streit-Emch theorem instead employs the assumption that the systems are governed by canonical dynamics, so that $P_j(f,t) = i[H_j, F_j(f,t)]$ on $\mathcal{D}_j$. Since the Streit-Emch theorem does not invoke any overtly relativistic assumptions, it raises the question of whether the unwelcome conclusion of Haag's theorem also obtains in nonrelativistic contexts. This is the subject of the next section.[21]

[21] It may also be possible to prove a rigorous version of Haag's theorem in the context of Euclidean field theory. Schrader (1974, p. 134) interprets a result as "a strong euclidean version of Haag's theorem" for polynomial scalar self-interaction terms on two-dimensional spacetime.

1.2 PART II: INGREDIENTS OF HAAG'S THEOREM

The informal result first stated by Haag in Haag (1955) can be formalized in different ways. Different versions employ different frameworks for QFT and invoke different assumptions. The next two sections investigate the role played by relativistic assumptions and by the assumption of an infinite number of degrees of freedom in all versions of Haag's theorem.

1.2.1 The role of relativity

Emch interprets the Streit-Emch theorem as showing that relativistic assumptions are not required for the proof of Haag's theorem: "standard proofs are available in the textbook literature (see, for instance, Chapter IV, Section 5 in Streater and Wightman (1963)...); most of these proofs, however, rely rather heavily on the analytic properties of the Wightman functions, which themselves reflect the locality and spectrum conditions, and tend to obscure the simple algebraic and group-theoretical facts actually responsible for the results obtained" (p. 247). Is this correct?[22]

Intuitively, there seems to be good reason to suspect that Haag's theorem holds in relativistically covariant theories but not Galilean covariant ones, contrary to Emch's view. By assumption, there is one time t_0 at which the representations are related by a unitary transformation; proving Haag's theorem involves showing that the representations are related by the same unitary transformation at all times. Intuitively, in a relativistic theory a constraint "at a time" also imposes constraints at other times because a time translation is equivalent to a Lorentz boost plus a spatial translation plus a Lorentz boost. In contrast, in the nonrelativistic setting of absolute

[22] Thanks to Hans Halvorson for a helpful correspondence on this topic.

time, constraints at a time are genuinely constraints at a time. In place of relativistic constraints on the dynamics, Streit and Emch adopt the dynamical assumption that the representations employ canonical dynamics for which $P_j(f,t) = i[H_j, F_j(f,t)]$ on $\mathcal{D}_j$. The question is whether this is a strong enough constraint on the dynamics to generate the conclusion of Haag's theorem.

The ideal context for investigating this matter is Galilean QFT's. Like relativistic QFT's, Galilean QFT's have an infinite number of degrees of freedom; as discussed above, the Stone-von Neumann entails that this is a prerequisite for Haag's theorem holding, assuming standard Hilbert space representations are used (see the discussion in the next section). Galilean QFT's have been treated axiomatically by Lévy-Leblond (see Lévy-Leblond (1967) and Lévy-Leblond (1971)). Galilean fields transform appropriately under a Galilean transformation $g = (b, \mathbf{a}, \mathbf{v}, R)$ (where $g(\mathbf{x}, t) = (R\mathbf{x} + \mathbf{v}t + \mathbf{a}, t + b) = (\mathbf{x}', t')$) (Lévy-Leblond 1967, p. 163):

$$U(g)^{-1}\phi(\mathbf{x},t)U(g) = exp\{im[\frac{1}{2}v^2t + vRx]\}\phi(\mathbf{x}',t') \tag{1.52}$$

The m that appears in this identity is mass; consequently, each Galilean field is associated with a particular, determinate mass (Lévy-Leblond 1967, p. 163). The local commutativity condition also differs from its relativistic counterpart: "at equal times, two field operators either commute or anti-commute (for non-zero spatial separation)" (Lévy-Leblond 1967, p. 164). There exist models of the axioms of Galilean QFT. One example is the model obtained by "second-quantizing" the Schrödinger equation (see Lévy-Leblond (1967, pp. 166-7), Schweber (1961, Chapter 6), or Redmond and Uretsky (1960)).

There exist Galilean QFT's with non-trivial interactions in which vacuum polarization does not occur. Consider the total Hamiltonian $H = H_0 + H_I$ where H_I is invariant under Galilean transformations (i.e., commutes with $\mathbf{P}$ [momentum], $\mathbf{J}$ [angular momentum], $\mathbf{K}$ [generator of pure Galilean boosts], M [mass operator]) (Lévy-Leblond 1967, p. 161). The Hamiltonian does appear inside the Lie brackets, but, as long as H_I commutes with $\mathbf{P}, \mathbf{J}, \mathbf{K}, M$, adding H_I does not alter the Lie algebra: e.g.,

$$[K_i, H] = [K_i, H_0 + H_I] = [K_i, H_0] + [K_i, H_I] = [K_i, H_0] \tag{1.53}$$

This means that if we have a representation of the ETCCR's in which the operators $\hat{K}_i$, $\hat{H}_0$ satisfy the correct Galilean commutation relation, then $\hat{K}_i, \hat{H}$ also satisfy it. Furthermore–assuming that the vacuum $|0\rangle_F$ for the free theory is the unique vector annihilated by $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, M, H_0–H also annihilates $|0\rangle_F$.[23] That is, vacuum polarization does not occur.[24]

This is, of course, in contrast to what happens in a relativistic theory. The Hall-

[23]Consider the Galilean commutation relation $[\hat{K}_i, \hat{H}] = i\hat{P}_i$:

$$[\hat{K}_i, \hat{H}]||0\rangle_F = 0$$
$$\hat{K}_i\hat{H}|0\rangle_F = \hat{H}\hat{K}_i|0\rangle_F = 0$$

Therefore, since $\hat{H}$ satisfies analogous commutation relations for other generators of the Galilean group, either $\hat{H}|0\rangle_F = 0$ or $\hat{H}|0\rangle_F = |0\rangle_F$. If the latter, subtract the identity from $\hat{H}$ to obtain a Hamiltonian which annihilates $|0\rangle_F$: $\hat{H}' = \hat{H} - I$, $\hat{H}'|0\rangle_F = 0$.

[24]But it should be stressed that vacuum polarization does occur in some contexts in nonrelativistic QFT's, including Galilean QFT's. This is what Wightman is referring to in the following passage:

> The strange representations [i.e., representations unitarily inequivalent to the Fock representation for a free field] associated with Haag's theorem are, in fact, an entirely elementary phenomenon and appear as soon as a theory is euclidean invariant and has a Hamiltonian which does not have the no-particle state as a proper vector. This will happen whether or not the theory is relativistically invariant... (1967a, p. 255)

One such example is described in Wightman and Schweber (1955, p. 824). Note that the remark just quoted should not be taken to imply that relativistic assumptions are not an essential ingredient of Haag's theorem. In relativistic theories, vacuum polarization *necessarily* occurs; in nonrelativistic theories, vacuum polarization may or may not occur.

Wightman theorem establishes that in any Poincaré covariant theory that satisfies its assumptions, absence of vacuum polarization entails that the system described is a free system. Lévy-Leblond opines that the fact that the Hamiltonian never appears on the RHS of the commutation relations for the Lie algebra is "perhaps the most important physical difference between Galilei and Poincaré invariance" (1967, p. 161). The Lie algebra for the Poincaré group contains the commutation relation $[P_i, K_i] = -iH$ (Ryder 1996, p. 59). In this case ensuring that H_I is Poincaré invariant clearly does not guarantee that $H_0 + H_I$ has the same Lie brackets as H_0. If we have a representation of the ETCCR's in which $[\hat{P}_i, \hat{K}_i] = -i\hat{H}_0$ and we introduce $\hat{H} = \hat{H}_0 + \hat{H}_I$, then $[\hat{P}_i, \hat{K}_i] \neq -i(\hat{H}_0 + \hat{H}_I)$, so we do not have the correct Poincaré commutation relation for $(\hat{H}_0 + \hat{H}_I)$. To satisfy the Poincaré commutation relation for $(\hat{H}_0 + \hat{H}_I)$, we will have to pick different sets of self-adjoint operators to represent momentum and/or Lorentz boosts: i.e., $[\hat{P}'_i, \hat{K}'_i] = -i(\hat{H}_0 + \hat{H}_I)$. (Possibly suitable $\hat{P}'_i, \hat{K}'_i$ could be found in the original representation in which $[\hat{P}_i, \hat{K}_i] = -i\hat{H}_0$ but then there would be two distinct generators of spatial translations, which is physically nonsensical). Consequently, as Lévy-Leblond notes, the fact that the Hamiltonian never appears on the RHS of the Galilean Lie algebra "[e]xplains why it is so easy to construct nontrivial galilean theories, whereas this proves a formidable task in the relativistic case" (1967, p. 161). Physically, this difference between the Galilean and relativistic cases is attributable to the fact that there is absolute time in the Galilean case but not in the relativistic case.

The fact that there exist models of Galilean QFT with non-trivial interactions but no vacuum polarization is a counterexample that demonstrates that the Streit-Emch theorem is inapplicable to the class of Galilean QFT's. The conclusion of Part I of the Streit-Emch theorem is that vacuum polarization does not occur; Part II of the theorem establishes that the absence of vacuum polarization entails that the systems described are both free systems. As I have stressed, all rigorous versions of Haag's theorem share this form. Consequently, no variant of Haag's theorem is applicable to the class of Galilean QFT's. It is possible that the Streit-Emch theorem could apply to some models of Galilean QFT–those that exhibit vacuum polarization.[25] But this would be a much different situation than that which obtains in relativistic theories. The Hall-Wightman theorem holds universally in relativistic QFT; any representation which satisfies all of its assumptions is necessarily devoid of vacuum polarization and describes a free system. Put another way, the interaction picture never exists in relativistic QFT, but it sometimes exists in Galilean QFT.

It is not immediately obvious why the Streit-Emch theorem is inapplicable to some Galilean QFT's (i.e., which premise of the theorem fails to hold). A premise of the theorem that fails to hold can be identified by particularly intuitive models of relativistic and Galilean QFT: the models obtained by 'second-quantizing' the Schrödinger and Klein-Gordon equations, respectively. The problem is not particular to interactions, so the free versions of these equations will be considered. As we shall see, the assumption of the Streit-Emch theorem that fails in Galilean QFT is assumption (iii) of Part II of the theorem:

> (iii) the set of vectors obtained by applying polynomials in $F_1(f,t)$ for $f \in \mathcal{C}$ to Φ_1 is dense in $\mathcal{H}_1$

Consequently, manipulation of the vectors of the form $F_1(f_1,t)\cdots F_1(f_n,t)\Phi_1$ does not necessarily yield information about the relationship between the Hamiltonians in the two representations. Recall that this assumption was needed to prove that H_1 is essentially self-adjoint on this domain, and therefore that the result that $H_1 = U_0(0)H_2U_0^*(0)$ on this domain is not rendered trivial by domain considerations.

[25] See the previous footnote.

The first step in obtaining an operator field governed by the Klein-Gordon equation is to Fourier decompose the classical real scalar Klein-Gordon field $\phi(\mathbf{x},t)$ into positive and negative frequency parts:

$$\begin{aligned}
\phi(\mathbf{x},t) &= \frac{1}{\sqrt{2(2\pi)^3}}\int \frac{d^3k}{k_0}\left[e^{-i[k_0t-\mathbf{k}\cdot\mathbf{x}]}c(\mathbf{k})+e^{i[k_0t-\mathbf{k}\cdot\mathbf{x}]}c^*(\mathbf{k})\right] && (1.54)\\
&= \phi^{(+)}(\mathbf{x},t)+\phi^{(-)}(\mathbf{x},t) && (1.55)\\
\text{where } k_0 &> 0,\ k_0^2=m^2
\end{aligned}$$

This expression is quantized by interpreting the coefficients as annihilation and creation operators, yielding the quantum field $\hat{\phi}(\mathbf{x},t)$. Polynomials in $\hat{\phi}(\mathbf{x},t)$ applied to the vacuum $|0\rangle$ clearly give a set of vectors that is dense in the Hilbert space since $\hat{\phi}(\mathbf{x},t)$ is an integral over all the creation operators and Fock space is spanned by the states generated by successive application of the creation operators to $|0\rangle$. $\hat{\phi}(\mathbf{x},t)$ is self-adjoint since it is an integral over an operator ($\hat{c}(\mathbf{k})$) and its adjoint ($\hat{c}^\dagger(\mathbf{k})$).

The salient difference between the Klein-Gordon and Schrödinger equations is that (infamously) while the Klein-Gordon equation has positive and negative energy solutions, the Schrödinger equation has only positive energy solutions. Consequently, the Fourier decomposition of the classical complex scalar Schrödinger field only has one term

$$\psi(\mathbf{x},t)=\frac{1}{\sqrt{2(2\pi)^3}}\int d^3p e^{-i[\frac{\mathbf{p}^2}{2m}t-\mathbf{p}\cdot\mathbf{x}]}c(\mathbf{p}) \qquad (1.56)$$

When this is interpreted naïvely in analogy with the Klein-Gordon equation, $\hat{c}(\mathbf{p})$ is promoted to an annihilation operator, so $\hat{\psi}(\mathbf{x},t)|0\rangle=0$. Clearly, then, polynomials in $\hat{\psi}(\mathbf{x},t)$ applied to $|0\rangle$ would not give a set of vectors that is dense in the Hilbert space; in fact, they would only give vectors in the subspace of complex numbers in the Fock space! But, strictly speaking, this interpretation is not correct because the real scalar Klein-Gordon field carries charge $q=0$. The analogy, therefore, should be drawn between the real scalar Klein-Gordon field and the $m = 0$ complex scalar Schrödinger field. The reason that the relativistic $q = 0$ and Galilean $m = 0$ cases are most similar is that in Galilean QFT mass behaves as like charge in relativistic QFT insofar as it carves up the Hilbert space into superselection sectors (Lévy-Leblond 1967, p. 160). But for $m = 0$, the Schrödinger equation becomes singular:

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \frac{-1}{2m}\nabla^2\psi(\mathbf{x},t) \tag{1.57}$$

However, the relativistic and Galilean models with $\pm q$ and $\pm m$, respectively, can be compared. A complex Klein-Gordon field is associated with charge $\pm q$. Following the standard treatment, decompose the classical complex Klein-Gordon field $\Phi(\mathbf{x},t)$ and its complex conjugate $\Phi^\dagger(\mathbf{x},t)$ as follows:

$$\Phi(\mathbf{x},t) = \frac{1}{\sqrt{2(2\pi)^3}}\int\frac{d^3k}{k_0}\left[e^{-i[k_0t-\mathbf{k}\cdot\mathbf{x}]}a(\mathbf{k}) + e^{i[k_0t-\mathbf{k}\cdot\mathbf{x}]}b^*(\mathbf{k})\right] \tag{1.58a}$$

$$\Phi^\dagger(\mathbf{x},t) = \frac{1}{\sqrt{2(2\pi)^3}}\int\frac{d^3k}{k_0}\left[e^{-i[k_0t-\mathbf{k}\cdot\mathbf{x}]}b(\mathbf{k}) + e^{i[k_0t-\mathbf{k}\cdot\mathbf{x}]}a^*(\mathbf{k})\right] \tag{1.58b}$$

These expressions are quantized by promoting $\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k})$ and $\hat{b}(\mathbf{k}), \hat{b}^\dagger(\mathbf{k})$ to be distinct sets of annihilation and creation operators. $\hat{a}^\dagger(\mathbf{k})|0\rangle$ (with any $\mathbf{k}$) is a state with charge $+q$, $\hat{b}^\dagger(\mathbf{k})|0\rangle$ (with any $\mathbf{k}$) is a state with $-q$, and $\hat{a}(\mathbf{k})|0\rangle = \hat{b}(\mathbf{k})|0\rangle = 0$ for all $\mathbf{k}$. Clearly, $\hat{\Phi}(\mathbf{x},t)$ is not self-adjoint. This means that Emch's assumption must be liberalized:[26] in this case, assume that polynomials in the field and its adjoint applied to the vacuum yield a set of states that is dense in the Hilbert space. Since $\hat{\Phi}(\mathbf{x},t)$ is an integral over $\hat{b}^\dagger(\mathbf{k})$, creation operators for the $-q$ states, and $\hat{\Phi}^\dagger(\mathbf{x},t)$ is

[26]The fact that $\hat{\Phi}(\mathbf{x},t)$ is not self-adjoint also means that it cannot be taken to be the generator of the unitary Weyl operator $V(f)$, another assumption of Part II of the Streit-Emch theorem. The fact that this is the case for both the relativistic and Galilean quantum fields under consideration shows that this is not the reason that the Streit-Emch theorem is inapplicable in Galilean QFT, so this circumstance will be overlooked here.

an integral over $\hat{a}^\dagger(\mathbf{k})$, creation operators for the $+q$ states, polynomials in $\hat{\Phi}(\mathbf{x},t)$ and $\hat{\Phi}^\dagger(\mathbf{x},t)$ applied to the vacuum $|0\rangle$ do give a set of states that is dense in the Hilbert space.

In a theory with $\pm m \neq 0$, the complex Schrödinger field can be second-quantized in analogy to the Klein-Gordon field. This is the way this model is treated in standard accounts (see Schweber (1961, Chapter 6), or Redmond and Uretsky (1960)). The coefficient $c(\mathbf{p})$ in the Fourier decomposition (1.56) of $\Psi(\mathbf{x},t)$ is promoted to an annihilation operator $\hat{c}(\mathbf{p})$. Like the quantized complex Klein-Gordon field $\hat{\Phi}(\mathbf{x},t)$, $\hat{\Psi}(\mathbf{x},t)$ is not self-adjoint. As Lévy-Leblond notes, this follows from the Galilean transformation properties of the field and the mass superselection rule (1967, p. 163). The field $\hat{\Psi}(\mathbf{x},t)$ transforms as (where $(\mathbf{x}',t') = (R\mathbf{x}+\mathbf{v}t+\mathbf{a}, t+b)$)

$$U(g)^{-1}\hat{\Psi}(\mathbf{x},t)U(g) = exp\{im[\frac{1}{2}v^2t + vRx]\}\hat{\Psi}(\mathbf{x}',t') \tag{1.59}$$

so is associated with mass m. The field

$$\hat{\Psi}^\dagger(\mathbf{x},t) = \frac{1}{\sqrt{2(2\pi)^3}}\int d^3p e^{i[\frac{\mathbf{p}^2}{2m}t - \mathbf{p}\cdot\mathbf{x}]}\hat{c}^\dagger(\mathbf{p}) \tag{1.60}$$

transforms as (where $(\mathbf{x}',t') = (R\mathbf{x}+\mathbf{v}t+\mathbf{a}, t+b)$)

$$U(g)\hat{\Psi}^\dagger(\mathbf{x},t)U(g) = exp\{-im[\frac{1}{2}v^2t + vRx]\}\hat{\Psi}^\dagger(\mathbf{x}',t') \tag{1.61}$$

i.e., $\hat{\Psi}^\dagger(\mathbf{x},t)$ transforms as a field with mass $-m$. In Schweber (1961) and Redmond and Uretsky (1960), $\hat{c}(\mathbf{p})$ is interpreted as an annihilation operator for $+m$ states such that $\hat{c}(\mathbf{p})|0\rangle = 0$ for all $\mathbf{p}$ and $\hat{c}^\dagger(\mathbf{p})$ as a creation operator for $+m$ states. As in the case of the quantized complex Klein-Gordon field, $\hat{\Psi}(\mathbf{x},t)$ is not self-adjoint, so the pertinent assumption of the Streit-Emch theorem will have to be liberalized in the same way. However, unlike in the Klein-Gordon case, polynomials in $\hat{\Psi}(\mathbf{x},t)$

and $\hat{\Psi}^\dagger(\mathbf{x},t)$ applied to the vacuum *do not* give a dense set of states in the Hilbert space; since successive applications of $\hat{c}^\dagger(\mathbf{p})$ (for any $\mathbf{p}$) to $|0\rangle$ yields a set of states that spans the $+m$ superselection sector, polynomials in $\hat{\Psi}(\mathbf{x},t)$ and $\hat{\Psi}^\dagger(\mathbf{x},t)$ applied to the vacuum give a dense set of states in the $+m$ superselection sector of the Hilbert space only. Contrasting this situation with that for the quantized complex Klein-Gordon field, the issue is clearly that there are not enough annihilation and creation operators. The root of this problem is that the Schrödinger equation has only positive energy solutions.

There is an intuitive connection between the reason that the Streit-Emch theorem does not apply in the Galilean case and the reason that the proof of the Hall-Wightman version of the theorem does not go through in this case. The comparison of the relevant second-quantized free Schrödinger and free Klein-Gordon theories reveals that the fact that polynomials in the Galilean fields $\hat{\Psi}(\mathbf{x},t), \hat{\Psi}^\dagger(\mathbf{x},t)$ applied to the vacuum do not yield a dense set of states is intimately related to the fact that it is not necessary that antiparticles exist in the nonrelativistic theory. One might argue that in the Galilean theory the $-m$ particle is the antiparticle of the $+m$ particle, but, as we have seen, the relationship between them is not mediated by the fields in the same way as in the relativistic case. In relativistic QFT, the existence of both particles and antiparticles ensures that the relativistic local commutativity relation holds. In a theory in which the canonical fields give an irreducible representation of the ETCCR's, the local commutativity condition is that all fields commute or anti-commute at spacelike separation. This is a relativistic condition; the Galilean local commutativity condition is that all fields commute at non-zero spatial separation. Intuitively, satisfaction of this condition does not require the existence of intimately related particles and anti-particles. This would explain the failure of assumption (iii) of the Streit-Emch theorem in the Schrödinger model. Recall that the Hall-

Wightman theorem cannot be proven in Galilean field theory because the VEV's do not have the same nice analytic properties as a result of the local commutativity condition applying at equal times (in contrast to the relativistic condition).[27] Thus, it seems plausible that the Streit-Emch and Hall-Wightman versions of Haag's theorem are ultimately inapplicable in the Galilean case for the same reason.

Returning to Emch's assessment of the Streit-Emch theorem, quoted at the outset of this section, he is probably correct that, since the theorem does not assume Poincaré invariance, it is applicable to nonrelativistic QFT's and therefore constitutes a generalization of the Hall-Wightman theorem. However, the nonrelativistic QFT of special physical interest is Galilean QFT, and the Streit-Emch theorem does not hold universally in this case. In this sense, then, Haag's theorem is tied to relativistic considerations and is not merely a consequence of "simple algebraic and group-theoretical facts [i.e., concerning the Euclidean group]," as Emch claims. A related point is that in relativistic QFT, it is not possible to evade Haag's theorem by simply abandoning the Streit-Emch assumption of canonical Hamiltonian dynamics (i.e., that $P_j(f,t) = i[H_j, F_j(f,t)]$),[28] since this is not an assumption of the Hall-Wightman version of the theorem.

1.2.2 The number of degrees of freedom

The Stone-von Neumann theorem informs us that, in principle, Haag's theorem is inapplicable to standard representations of the Weyl CCR's with a finite number of degrees of freedom. How, precisely, does Haag's theorem fail in these cases?

[27] Lévy-Leblond also cites the weakened spectral condition (1967, p. 166).

[28] In fact, it may be possible to prove a rigorous version of Haag's theorem without any explicit mention of the canonical momentum fields. Dadašev (1980) (unfortunately in Russian!) may prove a version of Haag's theorem based on Wightman's axioms without making any assumptions about the time derivatives of the fields.

The assumption that does not hold is that there exists a unique vacuum vector invariant under Euclidean transformations or, more generally, the group G that figures in the Streit-Emch theorem (Emch 1972, pp.252-3, Streit 1969, p. 680). More specifically, in the relativistic case with an infinite number of degrees of freedom and $m > 0$, the vacuum $|0\rangle$ is the only normalizable state such that $\mathbf{P}|0\rangle = 0$ because states with zero three-momentum are not normalizable (Hall and Wightman 1957, p. 38; Bogolubov et. al. 1975, p. 551). In the Hall-Wightman theorem, this is an assumption of Part I:

> Assume that there exist unique normalizable states $|0_j\rangle$ invariant under Euclidean transformations
> $$U_j(\mathbf{a}, \mathbf{R})|0_j\rangle = |0_j\rangle$$

In the Streit-Emch theorem, the relevant fact is that the algebraic state ϕ is the unique G-invariant state in the relevant vector representation. In the course of proving Part I of the theorem, Emch shows that this follows from the assumption that "the algebraic state $\hat{\phi}$ on $\mathcal{B}_H(\mathcal{C}_\mathbb{C})$ corresponding to Φ is G-invariant and η-clustering" (p. 248).

Emch gives an example that illustrates that, in his terms, there is more than one G-invariant vector state in a representation when there is a finite number of degrees of freedom (Emch 1972, pp. 252-3). Consider the following operator on a Fock representation:

$$A = \frac{\sum_{j=1}^{n} a^*(f_j)^2}{2n} \tag{1.62}$$

where $f_1 \ldots f_n$ is a basis in the test function space $\mathcal{C}$. Let $g{:}f \to g[f]$ be a unitary transformation of $\mathcal{C}$. Then

$$\frac{\sum_{j=1}^{n} a^*(g[f_j])^2}{2n} = A \tag{1.63}$$

Let Φ be a vector in the Hilbert space representation that is invariant under $U(g)$, the family of unitary operators representing the symmetry group G. Then $A\Phi$ is also invariant under $U(g)$, but $A\Phi \neq \Phi$. That is, there is more than one vector in the Hilbert space that is invariant under $U(g)$. In algebraic terms, the algebraic states ϕ, ϕ' generated by $\Phi, A\Phi$, respectively, are different, so there is more than one G-invariant algebraic state.

1.3 PART III: PREVIEW OF IMPLICATIONS OF HAAG'S THEOREM

1.3.1 The existence of Fock representations for interacting fields

The Hall-Wightman theorem dictates that if we want a representation of the ETCCR's for an interacting field that has all the desirable properties listed in the assumptions of the theorem (a unique vacuum vector, unitarily implementable Poincaré transformations, etc.), then this representation must be unitarily inequivalent to any representation of a free field which possesses the same desirable properties. The Fock representation for a free field is such a representation. Consequently, a suitable representation for an interacting field must be unitarily equivalent to all Fock representations for free fields.

Does this mean that Fock representations for interacting fields do not exist? (i.e., representations of the ETCCR's for interacting fields satisfying the conditions in footnote 1) That this conclusion is not licensed by Haag's theorem is apparent from the fact that the Hall-Wightman theorem is instantiated by Fock representations for free fields with different masses. Let (ϕ_1, π_1), (ϕ_2, π_2) be the Fock representations for free fields with masses m_1, m_2, respectively. The Hall-Wightman theorem shows that (on pain of the systems being governed by the same dynamics) these Fock representations must be unitarily inequivalent. It can be confirmed independently that the Fock representations for free fields of different masses are unitarily inequivalent (Reed and Simon 1975, Theorem X.46). Given that the Fock representations for free fields of different masses are unitarily inequivalent, it is possible that there exist Fock representations for interacting fields that are unitarily inequivalent to the Fock representation of any free field. Consequently, on its own the Hall-Wightman theorem does not entail that interacting fields do not possess Fock representations. The issue of whether there are other considerations that rule out the existence of Fock representations for interacting fields will be taken up in Chapter 4.

1.3.2 "The representation determines the dynamics" vs. "the dynamics determines the representation"

An appropriate slogan for Haag's theorem is "the representation determines the dynamics" (i.e., a given representation is suitable for representing only one dynamics). "The representation" is the Fock representation for a free field, one of the uncountably many unitarily inequivalent representations of the ETCCR's that are available. All versions of Haag's theorem show that, granting certain assumptions, the Fock representation for a specified free field can only be used to represent a system with the specified free dynamics. The choice of representation of the CCR's (up to unitary equivalence) determines the dynamics. The Hall-Wightman version of Haag's theorem attempts to extend this result from representations for free fields to representations for arbitrary fields. However, the attempt is only partly successful because it can only be proven that the first four vacuum expectation values of the two fields are equal (see Section 1.3 above).[29]

The general result that "the representation determines the dynamics" would be interesting, but it would not resolve the problem raised by the failure of the Stone-von Neumann theorem in QFT. Recall that there exist uncountably many unitarily inequivalent representations of the ETCCR's; each of these has the potential to yield different predictions because the respective sets of expectation values cannot be equated via a unitary transformation. The problem is how to choose among these alternative representations. Knowing that "the representation determines the dynamics" would not solve this problem because it leaves open the possibility that unitarily inequivalent representations determine the same dynamics, in which case we would still be confronted with the problem of choice.

Florig and Summers (2000) suggest that there is a sense in which "the dynamics determines the representation": "[r]oughly speaking, the kinematical aspects determine the choice of CCR-algebra, whereas the dynamics fix the choice of the representation of the given CCR-algebra in which to make the relevant, perturbation-free computations" (p. 452). The algebra is kinematical insofar as it contains observables and physical states can be defined as maps from this set of observables to the real numbers. However, if it is the case that "the dynamics determines the representation," this does not follow from Haag's theorem alone.[30]

[29] Araki (1960) claims to prove that, given certain assumptions, "the representation determines the dynamics" for self-interacting neutral scalar fields (p. 504). However, in his MathSciNet review of this article, Segal offers the following assessment: "[j]ust what the results mean in precise terms, and to what extent they are rigorously valid, is uncertain."

[30] See Earman and Fraser (2005, Section 9) for a discussion of what additional ingredients might be needed to support such a claim.

1.3.3 The canonical approach to QFT

The most significant consequence of Haag's theorem is that it undermines the interaction picture, which is the basis for the treatment of interactions in the canonical approach to QFT. This implication is straightforward and indisputable because all of the assumptions of the rigorous Hall-Wightman theorem are adopted in canonical QFT. A detailed analysis of precisely how Haag's theorem undermines canonical QFT will be given in Chapter 2. Alternatives to canonical QFT naturally arise out of this analysis. These alternatives will be explored further in Chapter 3. I will give a brief overview of what is to come.

The philosophical literature on Haag's theorem contains some dark speculations on its significance. For example, Teller's *An interpretative introduction to quantum field theory* (1995) tells the reader that "[a]ccording to something called *Haag's theorem* there appears to be no known consistent formalism within which interacting quantum field theory can be expressed" (p. 115), a sentiment echoed in Sklar's *Theory and Truth* (2000) where the reader is told that Haag's theorem "seemed to show the theory [QFT] incapable of describing interactions" (p. 28). Some of the more incautious statements in the physics textbook literature go further and remove the qualifiers. For example, Haag's theorem has been said to show that "the only field theory is that which leads to non-interacting particles!" (Roman 1969, p. 393). However, this statement is false. It is a good example of what Wightman had in mind when he characterized the textbook literature as "execrable" (1967a, p. 255).

The canonical approach to treating interactions is untenable, but Haag's theorem does not imply that it is not possible to treat interactions in the standard way (i.e., with a Hilbert space representation of the ETCCR's which contains a Hamiltonian, on which there exists a self-adjoint Hamiltonian and self-adjoint field operators and which gives a unitary representation of the Poincaré group). The existence of such Hilbert space models for nontrivial interactions in two and three spacetime dimensions demonstrates that the treatment of interactions within this framework is compatible with Haag's theorem. (See Chapter 3 for further discussion of the two-dimensional model.) These representations are allowed by Haag's theorem because they are unitarily inequivalent to representations for free systems.

At present, it is unknown whether or not there exist Hilbert space models for nontrivial interactions in the physically realistic case of four spacetime dimensions. However, it is important to recognize that Haag's theorem has no bearing on this issue. Haag's theorem does not give us any reason to believe that such representations do not exist; conversely, if it turns out that such representations are not possible, Haag's theorem cannot be held responsible.

2.0 CHAPTER 2: HAAG'S THEOREM AS A REDUCTIO OF (UNRENORMALIZED) CANONICAL QFT

In the paper which contains the first presentation of the theorem that bears his name, Haag made it clear that the intended target of his theorem is the canonical approach to treating interactions in QFT that was developed by Feynman, Dyson and their colleagues. In the abstract for the paper, he writes: "[i]t is shown that ... Dyson's matrix $U(t_1, t_2)$ for finite t_1 or t_2 cannot exist" (1955, p.1). As Roman puts it, "[t]he most sobering consequence of Haag's theorem is that *the interaction picture of canonical field theory cannot exist unless there are no interactions*" (Roman 1969, p. 391). As we shall see, this analysis is accurate. The application of Haag's theorem to canonical QFT is straightforward because canonical QFT endorses all of the assumptions of the Hall-Wightman theorem. The Hall-Wightman theorem can be regarded as a *reductio ad adsurbum* argument: the conclusion that a representation satisfying all of its assumptions necessarily describes free dynamics is unacceptable; therefore, at least one of the premises of the theorem must be rejected. Three responses to this *reductio* will be outlined below: (1) introducing an infinite renormalization counterterm into the interaction picture; (2) introducing a volume cutoff into the interaction picture; and (3) choosing to reject the interaction picture's assumption that there is a time at which the representation for the interaction is unitarily equivalent to the

Fock representation for a free system.[1]

2.1 THE INTERACTION PICTURE AND SCATTERING THEORY

In this section the assumptions that go into setting up the interaction picture representation employed in canonical QFT will be reviewed. As we will verify, this set of assumptions includes all of the premises of the Hall-Wightman theorem. I will refer to the classic presentation in Schweber (1961), but an account can be found in virtually any introductory textbook on QFT. Until further notice, the discussion will focus on the interaction picture *before* any renormalization procedure has been carried out. One of the main motivations behind the development of the canonical approach to QFT was the desire to derive predictions for scattering experiments. The interaction picture was instrumental in the achievement of this aim.

To set up the interaction picture, split the total interaction Hamiltonian H into free and interacting parts, $H = H_F + H_I$. In analogy to Dirac's interaction picture formalism for ordinary NRQM, the interaction picture in canonical QFT is an irreducible representation of the ETCCR's in which the evolution of the states is governed by H_I and the evolution of the operators is governed by H_F. This picture is intermediate between the Schrödinger picture–in which the states evolve under the full Hamiltonian H and the operators are stationary–and the Heisenberg picture–in

[1] A note on my terminology: "canonical QFT" refers to the formulation of QFT developed by Feynman, Dyson and their colleagues that is presented in most introductory QFT textbooks. At points, it will be important to distinguish between canonical QFT prior to the introduction of renormalization counterterms ("unrenormalized canonical QFT") and canonical QFT post-renormalization ("renormalized canonical QFT"). "Canonical QFT with cutoffs" is canonical QFT with volume (i.e., long-distance) and/or ultraviolet (i.e., short-distance) cutoffs. (Which sorts of cutoffs are imposed will be specified on a case-by-case basis.)

which the states are stationary and the operators evolve under the full Hamiltonian H. In particular, in the interaction picture the evolution of the field operators is generated by H_F, so the fields are free.

By stipulation, the interaction picture (indicated by the subscript $_I$) coincides with the Heisenberg picture (indicated by the subscript $_H$) at time t_0:

$$\phi_I(\mathbf{x}, t_0) = \phi_H(\mathbf{x}, t_0);\ \pi_I(\mathbf{x}, t_0) = \pi_H(\mathbf{x}, t_0) \tag{2.1a}$$

$$|\psi(t_0)\rangle_I = |\psi\rangle_H \tag{2.1b}$$

Let $V(t_1, t_2)$ represent the unitary evolution operator from t_1 to t_2 for the interaction picture states: $|\psi(t_2)\rangle_I = V(t_2, t_1)|\psi(t_1)\rangle_I$. Since in the interaction picture the evolution of the states is generated by H_I,

$$V(t_1, t_2) = e^{-iH_I(t_2 - t_1)} \tag{2.2}$$

$$= e^{iH_F(t_2 - t_1)} e^{-iH(t_2 - t_1)} \tag{2.3}$$

It follows from (2.1) and the dynamics of the states and operators in the two pictures that, at any time t, the representations are related by $V(t, t_0)$:

$$\phi_I(\mathbf{x}, t) = V^{-1}(t, t_0)\phi_H(\mathbf{x}, t)V(t, t_0) \tag{2.4a}$$

$$\pi_I(\mathbf{x}, t) = V^{-1}(t, t_0)\pi_H(\mathbf{x}, t)V(t, t_0) \tag{2.4b}$$

$$|\psi(t)\rangle_I = V^{-1}(t, t_0)|\psi\rangle_H \tag{2.4c}$$

The expression that is of physical interest is

$$_H\langle\chi_{out}|\psi_{in}\rangle_H =_H \langle\psi_{in}|e^{iHt}|\psi_{in}\rangle_H \tag{2.5}$$

which represents the probability for a transition from an initial state $|\psi_{in}\rangle_H$ to a final state $|\chi_{out}\rangle_H$. This expression cannot be evaluated because we do not have a concrete Hilbert space representation for the Heisenberg picture; we do not know how to represent H as an operator on a Hilbert space of states containing $|\chi_{out}\rangle_H$ and $|\psi_{in}\rangle_H$. In the free case, on the other hand, the analogous expression

$$_{free}\langle\chi_{out}|\psi_{in}\rangle_{free} =_{free} \langle\psi_{in}|e^{iH_{free}t}|\psi_{in}\rangle_{free} \tag{2.6}$$

can be evaluated by constructing a Fock representation of the ETCCR's in the familiar way. Since finite application of the creation operators to the no-particle state generates a basis for the Fock representation, $|\psi_{in}\rangle_{free}$ can be expressed as a superposition of such vectors. H_{free} can also be written in terms of annihilation and creation operators, so (2.6) can be evaluated when a perturbative expansion is introduced for $e^{iH_{free}t}$.

For an interaction, the problem of evaluating (2.5) is solved by transforming to the interaction picture and considering the limits $t_{1,2} \to \pm\infty$. Transforming to the interaction picture using (2.4c),

$$_H\langle\chi_{out}|\psi_{in}\rangle_H =_I \langle\chi(t_2)|V(t_2,t_0)V^{-1}(t_1,t_0)|\psi(t_1)\rangle_I \tag{2.7}$$

It is convenient to consider the limits $t_{1,2} \to \pm\infty$[2] because at these times particles are assumed to be infinitely far apart and therefore not interacting; thus, the states $|\psi(-\infty)\rangle_I$, $|\chi(+\infty)\rangle_I$ are free states (Schweber 1961, p. 318). Furthermore, it is assumed that these states are superpositions of eigenstates of H_F, the free part of the Hamiltonian (ibid). Recall that the evolution of $\phi_I(\mathbf{x},t)$ is already governed by

[2]The assumption stated below that at $t = \pm\infty$ the interaction picture is a Fock representation for ϕ_I (i.e., a Hilbert space spanned by the n-particle states) implies that the limit $t_{1,2} \to \pm\infty$ should be interpreted in the sense of convergence in the strong topology:

$$\lim_{t\to\pm\infty} \left\| e^{-iH_F t}|\psi\rangle_\pm - e^{-iHt}|\psi(t)\rangle_I \right\| = 0$$

where $|\psi\rangle_\pm$ are states in the Fock representation for ϕ_I.

H_F; therefore, the Fock representation of the ETCCR's for $\phi_I(\mathbf{x},t), \pi_I(\mathbf{x},t)$ can be employed at $t = \pm\infty$. As we shall see, this is an important simplifying assumption. Taking the $t \to \pm\infty$ limits yields the familiar expression in terms of the S-matrix:

$$ {}_I\langle\chi(+\infty)|S|\psi(-\infty)\rangle_I \tag{2.8} $$

$$ \text{where } S = \lim_{t_1\to-\infty, t_2\to+\infty} V(t_2,t_0)V^{-1}(t_1,t_0) \tag{2.9} $$

The entity in (2.8) is an *S-matrix element* represented in the interaction picture. Setting $t_0 = 0$, the components of the S-matrix are the *Möller wave operators* $\Omega^{\pm}$:

$$ \Omega^{\pm} = \lim_{t\to\mp\infty} e^{iHt}e^{-iH_F t} \tag{2.10} $$

$$ = \lim_{t\to\mp\infty} V^{-1}(t,0) \tag{2.11} $$

$$ S = \Omega^{+}(\Omega^{-})^{-1} \tag{2.12} $$

S-matrix elements in the interaction picture are evaluated using perturbative methods. The assumptions of the interaction picture are also needed to set up the perturbation series. Dyson's formula gives a perturbation series for $V(t,0)V^{-1}(t',0)$

$$ V(t,0)V^{-1}(t',0) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} dt_1 \cdots \int_{t'}^{t} dt_n T\left[H_I(t_1)\cdots H_I(t_n)\right] \tag{2.13} $$

where T is the time-ordering operator and $H_I(t_n)$ is the interaction part of the Hamiltonian H_I represented in the interaction picture at time t_n (Bjorken and Drell 1965, p. 178). To arrive at this expression, the unitarity of $V(t,0)$ is exploited. Dyson's formula for $V(t,0)V^{-1}(t',0)$ can be plugged into (2.8), the expression for an S-matrix element (setting $t_0 = 0$). For the evaluation of the terms in this series, it is of critical importance that $H_I(t_n)$ is a function of $\phi_I(\mathbf{x},t_n)$. Each term of the series can be calculated because the result of applying the free field operator $\phi_I(\mathbf{x},t)$

to a state in its Fock representation is known. The Feynman rules facilitate the calculations.

All of the assumptions of the Hall-Wightman theorem are adopted in this approach to treating interactions (see Section 1.3 of Chapter 1). Some of the premises of the theorem are general presuppositions of canonical QFT while others are specific to the interaction picture. Premise I(i), that the fields give irreducible representations of the ETCCR's, is a standard assumption of this approach (see Schweber (1961, pp. 650-651)). Likewise, it is typically assumed that Poincaré (and hence Euclidean) transformations are induced by unitary operators (premises I(ii) and II(ii)) (see Schweber (1961, p. 265)). The assumption that there are no negative energies (II(v)) is justified on physical grounds. Premises I(iii) and II(iii), that the fields transform appropriately under Euclidean and Poincaré transformations, are taken to be requirements of a QFT with or without interactions (see Schweber (1961, p. 265)). The assumption (I(iv), II(iv)) that there exist unique normalizable vacuum states that are invariant under Poincaré transformations follows from Wigner's classification of representations of the inhomogeneous Lorentz group (Schweber 1961, p. 265; Streater and Wightman 2000, pp. 21-2, 27-9). This assumption holds at all times for the Heisenberg picture and for the interaction picture.[3] Premise I(v), that there is some time at which the fields in the representations are related by a unitary transformation, holds at all times in virtue of the fact that at each time t the fields in the representations are related by the unitary transformation $V(t, t_0)$. Thus, the Hall-Wightman theorem holds at all times; at times $t = \pm\infty$, all of the

[3] In the interaction picture, the unique normalizable vacuum state at $t = \pm\infty$ is the free vacuum $|0\rangle_F$ (such that $H_F|0\rangle_F = 0$) in virtue of the assumption that the representation is the Fock representation for ϕ_I. Presumably, since in the interaction picture the evolution of states is governed by the interaction part of the Hamiltonian H_I, at intervening finite times the vacuum state is $|0\rangle_I$ such that $H_I|0\rangle_I = 0$.

premises of the theorem hold for the Heisenberg picture representation, which represents an interaction, and for the interaction picture representation, which is a Fock representation for a free system. In this latter case, the special, rigorously provable case of the Hall-Wightman theorem applies (i.e., it can be proven that all VEV's of the fields are equal).

Note that Haag's theorem does not result from the interaction picture's use of an approximation method (i.e., the introduction of a perturbation series in (2.13)). The premises of the theorem make reference only to the exact expression $V(t, t_0)$. Thus, the fact that the perturbative expansion of $V(t, t_0)V^{-1}(t', t_0)$ presumably[4] diverges is unrelated to Haag's theorem.

Canonical QFT embraces all of the premises of the Hall-Wightman theorem in its treatment of interactions. Since all of the premises of the Hall-Wightman theorem hold in the interaction picture, the unwelcome conclusion must also hold: the interaction picture represents free dynamics. That is, the initial state of the system is identical to the final state; the S-matrix is trivially the identity. The Hall-Wightman theorem can be regarded as a giving a *reductio* of the interaction picture. The conclusion that the interaction picture can only represent a free system is unacceptable because we want to apply the theory to non-trivial scattering experiments. By modus tollens, one of the premises of the Hall-Wightman theorem must be rejected in order to obtain a representation for an interacting system. Different formulations of QFT reject different premises. A range of responses to Haag's theorem will be surveyed in the remainder of the chapter.

[4]Jaffe (1965) contains an argument that this is the case for non-trivial interactions in two spacetime dimensions.

2.2 RENORMALIZED CANONICAL QFT

Viewed in light of Haag's theorem, the canonical approach to QFT could be characterized as stubbornly insisting upon employing the interaction picture representation, come what may. The consequence of employing the interaction picture to describe interactions without regard for Haag's theorem is vacuum polarization:[5] $H|0\rangle_F = \infty$. Canonical QFT addresses this problem by 'renormalizing' the Hamiltonian, which involves introducing an infinite vacuum self-energy renormalization counterterm E_0:

$$H_{ren}|0\rangle_F = [H - E_0]|0\rangle_F = 0 \tag{2.14}$$

(In this context, the term "renormalization" refers to the standard procedure for introducing infinite counterterms that are evaluated to some order in the perturbative expansion.) In the following passage, Schweber explains vacuum self-energy renormalization:

> ...in relativistic field theories in general, and in the model under discussion in particular, the "bare" vacuum $|0\rangle_F$ (which is an eigenstate of H_F) is not an eigenvector of H, because $H_I(t = t_0)|0\rangle_F \neq 0$ due to the presence of the pair creation term ... in H_I. Due to this possibility of pair creation, the vacuum energy is shifted relative to the "bare" vacuum by an (infinite) amount E_0, the vacuum self-energy. To make the energy eigenvalue of the bare and physical vacuum be the same we add and

[5]Supersymmetric theories are excluded from the following discussion. Supersymmetric theories were introduced in response to the Coleman-Mandula 'no go' theorem for QFT's in four spacetime dimensions (Coleman and Mandula 1967). Supersymmetric theories generalize the framework of QFT by allowing the algebra of generators of symmetries to contain both communtation and anti-commutation relations (Wess and Bagger 1992, p. 4). In technical terms, the algebra is not a Lie algebra, but a pseudo-Lie algebra (also known as a superalgebra or a graded Lie algebra) (Haag, Łopuszański, and Sohnius 1975, pp. 257-8; Wess and Bagger 1992, p. 2). Haag's theorem is inapplicable to supersymmetric theories because all versions are predicated on the assumption that the algebra of generators of symmetries contains either commutation relations or anti-commutation relations, but not both. For example, the Hall-Wightman theorem includes *either* the premise that all field operators obey commutation relations (for bosons) *or* the premise that all field operators obey anti-commutation relations (for fermions).

subtract to H a term E_0, where E_0 is the level shift of the vacuum, and consider $H_F + E_0$ as the unperturbed Hamiltonian and $H_I - E_0$ as the perturbation. (1961, p. 423; translated into my notation)

Renormalization is evidently successful in rendering the predictions of the theory[6] finite to any finite order of the perturbation series. Since it predicts non-trivial S-matrix elements, by modus tollens it must deny at least one premise of the Hall-Wightman theorem. However, it may not be immediately obvious how this is accomplished. The fact that a constant term is added to the Hamiltonian does not in itself violate any premise of the theorem because no premise is dependent on the precise form taken by the Hamiltonian. The relevant fact is that the vacuum self-energy counterterm is infinite. Strictly speaking, H_{ren} is not a well-defined self-adjoint operator on the interaction picture because–strictly speaking–the only vector in its domain is the zero vector. "Strictly speaking" because formally it is not legitimate to subtract an infinite constant from an operator. Glimm offers an argument that it is necessarily the case that H_{ren} is only well-defined on the zero vector based on the first few terms of the perturbation series (Glimm 1969b, p. 104; see also Glimm and Jaffe 1969). (Glimm remarks that "[t]his observation is merely a proof in perturbation theory of Haag's theorem" (ibid), but this is actually a stronger result since Haag's theorem implies only that the free vacuum is not in the domain of the interaction Hamiltonian). As a consequence of the fact that H_{ren} has only the zero vector in its domain, a number of the premises of the Hall-Wightman theorem fail. By Stone's theorem,[7] since H_{ren} is not a well-defined self-adjoint operator, the operator $e^{-iH_{ren}t}$ is not unitary, violating the assumption that Poincaré transformations are induced by unitary transformations (II(i)). Also, there is no Poincaré invariant

[6] At least for a renormalizable or superrenormalizable theory!

[7] See Reed and Simon (1980, p. 266). Physical considerations rule out the possibility that time translations are generated by some self-adjoint operator other than the Hamiltonian.

vacuum state (II(ii)).

Renormalization can also be construed as evading the Hall-Wightman theorem by adopting an informal mathematical framework (i.e., a framework in which infinite subtractions are permissible). The proof of the Hall-Wightman theorem relies heavily on formal mathematics (e.g., the theory of analytic functions). Thus, a tacit premise of the theorem is that the representation picked out by the premises of the theorem be formally mathematically well-defined. Infinite renormalization violates this tacit premise.

2.3 CANONICAL QFT WITH A VOLUME CUTOFF

A second response to Haag's theorem is to introduce a volume cutoff into the interaction picture representation. (As I explain below, a volume cutoff can be implemented in one of two ways: either by introducing a spatial cutoff into the Hamiltonian or by compactifying space.) Introducing both volume (i.e., long-distance) and ultraviolet (i.e., short-distance) cutoffs reduces the theory to a finite number of degrees of freedom, in which case Haag's theorem is inapplicable (see Section 2.2 of Chapter 1). In the interaction picture with both cutoffs renormalization is required, but all renormalization counterterms–including the vacuum self-energy–are finite. However, introduction of both types of cutoffs is overkill. The volume cutoff alone is a necessary and sufficient response to the *reductio* posed by Haag's theorem.[8] A number of authors concur with this assessment (Wightman 1967a, pp. 255, 257-8, 259; Ginibre

[8]But note that this claim does not entail that ultraviolet divergences are not responsible for vacuum polarization. Vacuum polarization occurs in the $(\phi^4)_3$ theory with spatial cutoff (to which Haag's theorem is inapplicable) as a result of ultraviolet divergences (Glimm and Jaffe 1973, p. 408).

and Velo 1970, p. 65).

The fact that a volume cutoff is necessary for this purpose and that an ultraviolet cutoff is insufficient follows from the Streit-Emch version of the theorem (see Section 1.5 of Chapter 1). Recall that the Streit-Emch theorem does not require Lorentz covariance and imposes a condition that is more general than Euclidean covariance (i.e., the algebraic state is required to be G-invariant and η-clustering). Consequently, as Streit points out, "the notion of space (Euclidean vs. e.g. lattice) need not be specified" (Streit 1969, p. 674). That is, the Streit-Emch theorem applies to the interaction picture *with* an ultraviolet cutoff (i.e., on a lattice of infinite extent). Thus, Haag's theorem cannot be circumvented by introducing an ultraviolet cutoff into the interaction picture. The fact that Haag's theorem holds for interactions (e.g., the scalar $(\phi^4)_2$ self-interaction) in which the interaction picture representation exhibits an infinite volume divergence but no ultraviolet divergence is another indication that a volume cutoff is relevant but that an ultraviolet cutoff is not.

The introduction of a volume cutoff is also a sufficient response to Haag's theorem. A volume cutoff can be introduced in one of two ways, each of which involves the rejection of a different premise of the Hall-Wightman theorem. One method is to multiply the interaction term of the Hamiltonian by a spatial cutoff function $g(\mathbf{x})$ ($g(\mathbf{x}) = 1$ in some finite bounded region of space and $g(\mathbf{x}) = 0$ outside this region). For concreteness, consider the Hamiltonian derived from a Lagrangian with a scalar ϕ^4 self-interaction term:

$$H(g) = H_F + \lambda_0 \int :\phi^4(\mathbf{x}): g(\mathbf{x}) d\mathbf{x} \tag{2.15}$$

$H(g)$ is associated with the "cutoff" field equation

$$\left(\Box + m_0^2\right) \phi(x) + 4g\lambda_0 \phi^3(x) = 0 \tag{2.16}$$

The "cutoff field" $\phi(x)$ that satisfies this field equation is clearly not invariant under spatial translations:

$$e^{-i\mathbf{P}\cdot\mathbf{x}}\phi(\mathbf{x},t)e^{i\mathbf{P}\cdot\mathbf{x}} \neq \phi(\mathbf{x}+\mathbf{x}',t) \tag{2.17}$$

$$\text{for } \mathbf{x} \text{ such that } g(\mathbf{x}) = 1,\ \mathbf{x}' \text{ such that } g(\mathbf{x}') = 0$$

Thus, introduction of a spatial cutoff function entails rejection of premise I(iii) of the Hall-Wightman theorem.

The second method of introducing a volume cutoff is to compactify space: for example, if space has a single dimension, take R to be a circle (Glimm and Jaffe 1971, p. 5). (This is also known as "quantization in a box with periodic boundary conditions" (ibid)). In this case, it is more difficult to identify the premise(s) of Haag's theorem that are violated. Consider the interaction picture representation of an interaction with an ultraviolet cutoff. As noted above, the Streit-Emch theorem holds in this case. Compactifying space reduces the degrees of freedom to a finite number (because the fields are defined on a lattice of finite extent), so no version of Haag's theorem holds, including the Streit-Emch version (see Section 2.2 of Chapter 1). All versions of Haag's theorem fail because there is not a unique normalizable state invariant under Euclidean transformations; there exists more than one such state. More specifically, when there are an infinite number of degrees of freedom, there is a unique normalizable state $|0\rangle$ such that $\mathbf{P}|0\rangle$ (i.e., that is annihilated by the generator of spatial translations) (Hall and Wightman 1957, p. 38; Bogolubov et. al. 1975, p. 551). Thus, it seems likely that in compacitfying space, one responds to the Haag's theorem *reductio* by denying that there exists a unique normalizable Euclidean invariant state. The Hall-Wightman version of the theorem includes this assumption as a premise (I(iv)), but the Streit-Emch theorem deduces the existence of a unique normalizable Euclidean invariant state from more general assumptions. One of these assumptions is η-clustering, which Emch glosses as follows

> Intuitively, this condition is satisfied in relativistic and Galilean theories when G is taken to be the group of spatial translations because observables [say, A and B] commute when they are far enough apart, so "on average" B commutes with all the translations of A (Emch 1972, p. 176).

This suggests that in a compactified space η-clustering would fail since, intuitively, there is a limit to how far apart observables can be translated.

2.4 AN ALTERNATIVE APPROACH

In contrast to the indirect approaches outlined in the preceding two sections, one could respond to Haag's theorem directly. That is, faced with the *reductio* posed by Haag's theorem, one could choose to reject at least one of its premises and reformulate QFT accordingly. Since QFT amalgamates quantum mechanics and special relativity, it makes sense to choose to reject a premise that does not encapsulate an important feature of either theory. So, for example, if possible we should retain the assumption of Euclidean covariance. This is admittedly an imprecise criterion, but it turns out that in this case there is one premise that is more palatable to reject than the others: the assumption that there is a time at which the free and interacting representations are unitarily equivalent (premise I(v) of the Hall-Wightman theorem). This assumption does hold in ordinary quantum mechanics, but this seems to be an accidental feature of the theory rather than an essential one. Recall that the Stone-von Neumann applies in ordinary quantum mechanics, which guarantees that all standard representations of the ETCCR's are unitarily equivalent (see Chapter

1). However, the Stone-von Neumann theorem does not hold in QFT because in QFT, unlike ordinary quantum mechanics, there are an infinite number of degrees of freedom. Consequently, QFT admits unitarily inequivalent representations of the ETCCR's. This means that it is possible for the free and interacting representations to be unitarily inequivalent.

The decision to reject this same premise of the Hall-Wightman theorem can also be reached via a different chain of reasoning. Instead of choosing to reject the premise of Haag's theorem that seems most dispensable, approach the problem from the perspective of formal mathematics. The consequence of employing the interaction picture in spite of Haag's theorem is vacuum polarization: $H|0\rangle_F = \infty$. Formally, this means that $|0\rangle_F$ is not in the domain of H. Furthermore, perturbation theory predicts that no other vector in $\mathcal{F}$, the Fock representation for the free system is in the domain of H either (with the trivial exception of the zero vector) (Glimm 1969b, p. 106). Thus, H is not a well-defined operator on $\mathcal{F}$. Recall that the interaction picture posits the existence of a unitary operator $V(t, t_0)$ that relates the Heisenberg and interaction pictures, where t_0 is the time at which the interaction and Heisenberg pictures coincide:

$$V(t, t_0) = e^{iH_F(t-t_0)}e^{-iH(t-t_0)} \tag{2.18}$$

From a formal mathematical point of view, $V(t, t_0)$ is not a well-defined unitary transformation because H, H_F are not simultaneously well-defined operators on the same Hilbert space representation. The tacit assumption, employed in the seemingly harmless move from (2.2) to (2.3) in Section 1 above, is that H, H_F are both well-defined operators on the interaction picture. H_F is certainly well-defined on the interaction picture at $t = \pm\infty$ because, by assumption, the interaction picture coincides with $\mathcal{F}$. However, H is not a well-defined operator on $\mathcal{F}$. By definition

$H = H_F + H_I$, but, as Klauder remarks, "the '+' sign is ... strictly formal and should not be read as the sum of two well-defined expressions" (Klauder 2000, p. 131; see also Glimm and Jaffe 1970a, p. 209). He further comments that this "is an elementary illustration of what is generally called Haag's theorem" (p. 131, note). Thus, from the perspective of formal mathematics, the obvious response to Haag's theorem is to point out that $V(t, t_0)$ does not exist. In terms of the Hall-Wightman theorem, the premise that should be rejected is that there is a time at which the free and interacting representations are related by a unitary transformation.

Further support for blaming this assumption can be found by considering the simplest circumstance in which the Hall-Wightman theorem can be applied. It has already been pointed out that the Hall-Wightman theorem is not dependent on the specific form taken by the Lagrangian or Hamiltonian for a theory; as long as the other assumptions are satisfied, it is applicable to any two systems with nonidentical dynamics. The simplest case to which the theorem applies is two free systems described by scalar fields $\phi_{m_1}(x), \phi_{m_2}(x)$ governed by Klein-Gordon field equations containing different mass parameters, $m_1 \neq m_2$:

$$(\Box + m_1^2)\phi_{m_1}(x) = 0 \qquad (2.19a)$$

$$(\Box + m_2^2)\phi_{m_2}(x) = 0 \qquad (2.19b)$$

Fock representations for $\phi_{m_1}(x), \phi_{m_2}(x)$ can be given in the usual way. A particular Fock representation satisfies all of the applicable assumptions of the Hall-Wightman theorem: it gives a representation of the ETCCR's, the fields transform appropriately under the Poincaré transformations, the no-particle state is the unique Poincaré-invariant vacuum state in the representation, and there are no states of negative energy. The assumption of the Hall-Wightman theorem that is not true of these two Fock representations is the assumption that there is a time at which the representations are unitarily equivalent. The representations are unitarily inequivalent at all times (Reed and Simon 1975, Theorem X.46). In this case, it would seem unreasonable to deny that the Fock representations are the correct representations to use for the free fields $\phi_{m_1}(x), \phi_{m_2}(x)$ in the face of the Hall-Wightman theorem. The obvious response to the theorem is to deny the assumption that there is a time at which these Fock representations are unitarily equivalent.[9]

The approach sketched in this section is adopted by proponents of the axiomatic and constructive programs for QFT. As a response to the Haag's theorem *reductio*, it raises two concerns. First, one might wonder whether it is possible to carry out the proposed reformulation of QFT without the assumption that the interacting representation is unitarily equivalent to the Fock representation for a free field. For the example of two free systems with different masses, this is straightforward, but interactions complicate matters. Second, while from a theoretical perspective it seems natural to drop the assumption of unitary equivalence, from a practical perspective it is difficult to drop this assumption. As we saw in Section 1, the unitary transformation $V(t, t_0)$ is crucial for the evaluation of S-matrix elements. According to the formal point of view, the Möller wave operators $\Omega^{\pm} = \lim_{t\to\mp\infty} e^{iHt}e^{-iH_F t}$ do not exist.[10] This position is viable only if an alternative formulation of scattering theory is possible. Both of these concerns will be addressed in the next chapter. Briefly, the first concern has been answered by the construction of a Hilbert space model

[9]Though alternative responses have also been investigated; Guenin and Velo 1968 construct a representation in which Poincaré transformations are induced by non-unitary operators.

[10]Again, the reason for this is that H, H_F are not well-defined on the same Hilbert space representation. Consequently, $\Omega^{\pm}$ are ill-defined regardless of whether the limits $t \to \pm\infty$ are taken in the strong or the weak topology. In fact, as Haag himself explains, the interaction picture is already fatally flawed at finite times, before the infinite limits are taken: "Dyson's matrix $[V(t_1, t_2)]$ for finite t_1 or t_2 cannot exist" (1955, p. 1). This is contrary to the analysis in Bain (2000, pp. 384-6). This point is discussed further in Section 2 of Chapter 3.

for a $(\phi^4)_2$ interaction which is unitarily inequivalent to Fock representations for free fields. The second worry is assuaged by Haag-Ruelle scattering theory.

2.5 CONCLUSION

In Roman's words, "[t]he most sobering consequence of Haag's theorem is that *the interaction picture of canonical field theory cannot exist unless there are no interactions*" (p. 391).[11] Haag's theorem gives a *reductio* of the interaction picture: if the interaction picture is mathematically well-defined, then it necessarily represents a free system. One response to this *reductio* is to relax the requirement that the interaction picture be mathematically well-defined and 'renormalize' it by introducing an infinite vacuum self-energy counterterm. This is, of course, the path taken in introductory textbooks, which present the canonical formulation of QFT. A second response is to introduce a volume cutoff into the interaction picture. If the volume cutoff takes the form of a spatial cutoff function inserted in the Hamiltonian and field equation, then it entails rejection of the assumption that the fields are covariant under spatial translations; the volume cutoff is implemented by compactifying space, then it entails rejection of the assumption that there is a unique Euclidean invariant state. A third response to the *reductio* is to reject the assumption that there is a time at which the interacting and free representations are related by a unitary transformation. This response, which has been pursued by axiomatic and constructive field theorists, will be considered further in the next chapter.

The content of Haag's theorem can be further clarified by identifying irrelevant

[11] Perhaps he has taken this from Streater and Wightman, who write "Haag's theorem is very inconvenient; it means that the interaction picture exists only if there is no interaction" (2000, p. 166).

factors. The fact that the interaction picture employs a perturbation series is not relevant because the premises Haag's theorem pertain to the exact expression $V(t, t_0)$. The infinite limits in the expression for the S-matrix are not relevant either; even at finite times, $V(t, t_0)$ is not well-defined on the interaction picture. Also, the ultraviolet divergences that typically crop up in the interaction picture representation are unrelated to Haag's theorem in virtue of the fact that introduction of a volume cutoff into the interaction picture is necessary and sufficient to render Haag's theorem inapplicable (i.e., a necessary and sufficient response to the reductio).

3.0 CHAPTER 3: ALTERNATIVE FORMULATIONS OF QFT

This chapter is intended to provide reassurance about the foundations of QFT in the aftermath of the *reductio* of the canonical framework without renormalization. One question raised by Haag's theorem is whether there exists an alternative[1] formalism for QFT on infinite, continuous space that is mathematically well-defined and consistent. Teller expresses the fear that the answer is "no": "[a]ccording to something called Haag's theorem there appears to be no known consistent formalism within which interacting quantum field theory can be expressed" (1995, p. 115). Fortunately, this fear is not warranted by Haag's theorem. For at least one interacting theory, a consistent alternative formalism has been found; furthermore, this formalism is a Hilbert space representation with all of the same attributes as the interaction picture with the sole exception that it is unitarily inequivalent to the Fock representation for a free field at all times. In this case, responding to Haag's theorem does not entail substantial revisions to the usual formalism. It is also possible to construct a consistent framework for scattering theory starting from a set

[1] A reminder about my terminology: "canonical QFT" refers to the formulation of QFT developed by Feynman, Dyson and their colleagues that is presented in most introductory QFT textbooks. At points, it will be important to distinguish between canonical QFT prior to the introduction of renormalization counterterms ("unrenormalized canonical QFT") and canonical QFT post-renormalization ("renormalized canonical QFT"). "Canonical QFT with cutoffs" is canonical QFT with volume (i.e., long-distance) and/or ultraviolet (i.e., short-distance) cutoffs. (Which sorts of cutoffs are imposed will be specified on a case-by-case basis.)

of consistent, mathematically well-defined axioms for QFT. The first part of this chapter is devoted to describing, first, the Hilbert space representation that has been constructed for the $(\phi^4)_2$ theory and, second, Haag-Ruelle scattering theory.

In the second part of the chapter, the formulations are compared on empirical grounds. It is reassuring that all three formulations of QFT turn out to be empirically equivalent.

3.1 PART I: MATHEMATICALLY WELL-DEFINED AND CONSISTENT ALTERNATIVES TO THE INTERACTION PICTURE

3.1.1 The $(\phi^4)_2$ model

In principle, the Haag's theorem *reductio* can be resolved by abandoning the assumption that there is a time at which the interacting field is unitarily equivalent to a free field, which is an assumption of all rigorous versions of Haag's theorem. It would be reassuring to know that, in practice, a Hilbert space representation for an interacting field satisfying all of the other assumptions is available. This issue must be addressed on a case-by-case basis. The existence of a Hilbert space model for at least one non-trivial interaction demonstrates that Haag's theorem does not rule out the existence of well-defined Hilbert space representations for all interactions.

Historically, the first Hilbert space model to be constructed was for bosons with a Lagrangian containing a ϕ^4 scalar self-interaction term in two spacetime dimensions. This is the simplest non-trivial and physically meaningful interaction. A ϕ^2 self-interaction is, of course, trivial because the wave equation is linear; it is equivalent to a finite mass renormalization of the free field:

$$\left(\Box + m^2\right)\phi = \lambda\phi \tag{3.20}$$

$$\left[\Box + (m^2 - \lambda)\right]\phi = 0 \tag{3.21}$$

A ϕ^3 self-interaction term is not physically meaningful because even the classical wave equation has energy that is unbounded from below and has singular solutions (Jaffe 1999, p. 139; Keller 1957). A ϕ^4 self-interaction term yields the following field equation, which is non-trivial and physically meaningful:

$$\left(\Box + m^2\right)\phi + 4\lambda\phi^3 = 0 \tag{3.22}$$

This Lagrangian approximately describes pion-pion scattering (Bogolubov and Shirkov 1983, p. 104). The restriction to two spacetime dimensions results in the simplification that the ultraviolet divergences can be completely removed by normal-ordering (Glimm and Jaffe 1968, p. 175).

Glimm and Jaffe's strategy for constructing the $(\phi^4)_2$ model is to prove that the theory with a spatial cutoff is well-defined on the Fock space for the corresponding field, and then to use algebraic methods to remove the spatial cutoff. The field equation with fixed spatial volume cutoff $g(\mathbf{x})$ ($g(\mathbf{x}) = 1$ in some finite bounded region of space and $g(\mathbf{x}) = 0$ outside this region) is[2]

$$\left(\Box + m_0^2\right)\phi + 4g\lambda\phi^3 = 0 \tag{3.23}$$

The spatial cutoff Hamiltonian is

$$H(g) = H_F + \lambda \int :\phi^4(\mathbf{x}): g(\mathbf{x}) d\mathbf{x} - E_g \tag{3.24}$$

[2] m_0 is the bare mass. Mass and field strength renormalization constants are left out because perturbation theory indicates that they are finite for the constructed $\left(\phi^4\right)_2$ model without spatial cutoff (Glimm and Jaffe 1970a, pp. 249-50).

where E_g is chosen so that the lowest energy eigenvalue is zero. The spatial cutoff theory is well-defined on the Fock space for the corresponding free field: the field

$$\phi_g(\mathbf{x}, t) = e^{iH(g)t}\phi(\mathbf{x}, 0)e^{-iH(g)t} \tag{3.25}$$

satisfying (3.23) is a self-adjoint operator when its spatial variable is smeared with a test function; $\phi_g(f, t)$, $\frac{\partial \phi_g(f,t)}{\partial t}$satisfy the ETCCR's;[3] $H(g)$ is a self-adjoint operator and bounded from below; and there exists a unique (up to constant) normalizable vacuum state Ω_g such that

$$H(g)\Omega_g = 0 \tag{3.26}$$

(Glimm and Jaffe 1968, 1970a; Glimm and Jaffe 1970b, p. 208). Of course, as dictated by Haag's theorem, the Hamiltonian H without the spatial cutoff is not a well-defined self-adjoint operator on the Fock space and it does not possess a vacuum vector Ω in the Fock space (see Chapter 2).

To obtain a Hilbert space representation $\mathcal{F}_{ren}$ for which H is well-defined and Ω exists, Glimm and Jaffe define

$$\omega_g(A) = \langle \Omega_g | A | \Omega_g \rangle \tag{3.27}$$

where A is a bounded function of the smeared field operators $\phi(f, t)$. ω_g is a state on a C^*-algebra (see the Appendix): the set of A forms a C^*-algebra $\mathcal{U}$; ω_g is in $\mathcal{U}^*$, the dual space of $\mathcal{U}$; and ω_g is positive and has a norm of one (Glimm and Jaffe 1970b, p. 206). The Hilbert space representation for the theory without a spatial cutoff is obtained by taking the limit $g(\cdot) \to 1$ of $\omega_g(A)$ and then applying the GNS construction to the limiting state $\omega(A)$.

[3] In fact, these properties also hold for the fields without spatial cutoffs (Glimm and Jaffe 1970b).

It has been established that $\mathcal{F}_{ren}$ satisfies all of the Wightman and Haag-Kastler axioms (Glimm, Jaffe, and Spencer 1974; Glimm and Jaffe 1970a; Cannon and Jaffe 1970; Glimm and Jaffe 1970b, p. 208). This entails that $\mathcal{F}_{ren}$ is a well-defined Hilbert space representation for the interaction in the sense that all of the assumptions of the Hall-Wightman theorem hold with the sole exception of the unitary equivalence of $\mathcal{F}_{ren}$ to the Fock representation for a free field. Thus, in the case of the non-trivial $(\phi^4)_2$ interacting theory, it can be demonstrated that there exists a well-defined Hilbert space representation even though Haag's theorem is in force.

The $(\phi^4)_2$ Hilbert space representation is the simplest to construct because the attempt to represent the Hamiltonian on the Fock space for the corresponding free field is only subject to infinite volume divergences. These divergences are associated with Haag's theorem (see Section 4 of Chapter 2); therefore, this construction needs to overcome only the problem identified by Haag's theorem. Interactions which take different forms are afflicted with other divergences in addition to the infinite volume ones (e.g., ultraviolet divergences). It has proven to be more difficult to obtain well-defined Hilbert space representations for these. It would be nice to know that the infinite volume divergences associated with Haag's theorem can always be remedied in principle, even if in practice a well-defined Hilbert space representation cannot be found because the other divergences are intractable. This would be the case if Glimm and Jaffe's method for removing the infinite volume divergences in the $(\phi^4)_2$ model could be applied to any other interacting theory once the other divergences have been removed. Unfortunately, this is not the case, as the construction of the model for the $(\phi^4)_3$ theory illustrates. The method for obtaining the infinite volume limit in the $(\phi^4)_2$ case relies on the fact that the interaction Hamiltonian with spatial cutoff is well-defined on the Fock representation for the corresponding free field (the "locally Fock" property). In the $(\phi^4)_3$ theory (and, more generally, in any case in which infinite wave function renormalization is required), the locally Fock property does not hold (Jaffe 1999, p. 140 n. 2); the Hamiltonian with spatial cutoff can only be defined on a representation that is unitarily equivalent to the Fock representation for the corresponding free field. Consequently, another method for removing the volume cutoff had to be found. This seems to have been nontrivial;[4] a model without ultraviolet divergences is described in Glimm (1968), but the first infinite volume models were not in print until 1976 (Glimm and Jaffe 1987, p. 461). Consequently, success in obtaining an infinite volume model in the $(\phi^4)_2$ case does not give us reason to believe that infinite volume models should exist in more complicated cases.

3.1.2 Haag-Ruelle scattering theory

The challenge that Haag's theorem poses for scattering theory is to construct a mathematically consistent framework for treating scattering problems without adopting as postulates all of the assumptions of the theorem. One of the important assumptions underpinning scattering theory is that in the limit of infinitely early or late time an interacting system tends to a free system. This assumption is particularly important in QFT because it allows a scattering process to be interpreted as the scattering of quanta of particular types. The Fock representation for the asymptotic free system can be given an interpretation in terms of noninteracting quanta; Chapter 4 will establish that interacting systems do not possess Fock representations. The trick is to introduce the assumption that at infinitely early or late times the states of the system are in some way identifiable as free states for noninteracting quanta without running afoul of Haag's theorem.

Haag-Ruelle scattering theory furnishes a framework for scattering theory that

[4] Though Glimm and Jaffe characterize the proof that the method for removing the ultraviolet cutoff is successful as "the fundamental step in the existence proof" (1987, p. 461).

avoids the pitfalls pointed out by Haag's theorem (Reed and Simon 1979, p. 317; Streater 1975, p. 797). The strategy is to adopt the Wightman axioms for QFT and then to prove that there exist operators and states that asymptotically tend to operators and states for a free system in an appropriate sense. Haag-Ruelle scattering theory was first proposed in Haag (1958) and a rigorous proof for the key theorem was subsequently published in Ruelle (1962).

Haag himself does not describe Haag-Ruelle scattering theory as a reaction to Haag's theorem, which might seem curious. In fact, this is natural because, by 1958, this problem had already been solved: the LSZ scattering formalism, which also circumvents Haag's theorem, had already been published by Lehmann, Symanzik, and Zimmerman in 1955.[5] As Haag explains in the Introduction to his 1958 paper, his aim was to synthesize and unify existing techniques for treating the scattering of composite particles in order to obtain a framework for scattering theory that could accommodate such particles (p. 669). ("Composite particles" are particles that "are not related to one of the basic fields via an asymptotic condition" (Haag 1996, p. 84).) He had previously criticized Wightman's axiomatization for being inadequate in this respect (Haag 1959, pp. 152-3). The innovation that allows composite particles to be treated is the abandonment of the assumption that there is a one-to-one correspondence between particles and fields. The focus of the exposition of Haag-Ruelle scattering theory in this section will be another virtue, that of being rigorously well-defined. Haag may be alluding to this virtue when he remarks that "the discussion incidentally also gives a deeper understanding of the customary asymptotic condition in a simple field theory in which there is a one-to-one correspondence between particles and fields" (1958, p. 669).

[5]Note, however, that as Streater points out, this asymptotic condition was not formulated in a rigorously defined way and, more importantly, it was postulated, not proven (1975, p. 829).

For simplicity, consider a theory in which there are spinless particles with strictly positive masses m_1 and m_2 and their interaction is described by a single scalar field $A(x)$. Assume that all of the Wightman axioms are satisfied (Streater and Wightman 2000, Sec. 3.1; Reed and Simon 1979, Sec. IX.8). At $t = \pm\infty$ there are states represented by Heisenberg state vectors of the form $|\alpha_1 ... \alpha_k, \rho_1 ... \rho_l\rangle^{\pm}$ where α_i indicates that the state of the i^{th} particle is described by the positive-energy solution f_α to the Klein-Gordon equation with mass m_1 and, analogously, ρ_j indicates that the state of the j^{th} particle is described by the positive-energy solution f_ρ to the Klein-Gordon equation with mass m_2. Furthermore, Haag-Ruelle scattering theory relies on the assumption there is a "single-particle space" $\mathcal{H}^{(1)}$ in $\mathcal{H}$, the Hilbert space representation for the interacting field $A(x)$, that contains "all state vectors which describe one particle alone in the world" (Haag 1996, p. 76). More formally, $\mathcal{H}^{(1)}$ is specified using Wigner's classification of representations of the Poincaré group. Assume that the spectrum of the energy-momentum operator has the same structure as that for a theory with free fields of masses m_1 and m_2: i.e., the only discrete eigenvalues are $p = 0$ (corresponding to the vacuum state) and $p^2 = -m_1^2, p^2 = -m_2^2$ (corresponding to the one-particle states) (Haag 1958, p. 669; Jost 1965, p. 120). $\mathcal{H}^{(1)}$ contains the discrete eigenstates of P with eigenvalues m_1, m_2.

The problem is then to specify the relationship between interacting states and the free states in $\mathcal{H}^{(1)}$. From a practical point of view, the goal is to "to express all the incoming and outgoing fields in terms of the basic quantities $A(x)$" since "[i]f this is done, it is clear how to calculate the S-matrix elements for all possible processes" (Haag 1958, p. 670). The trick is to specify the sense in which states in $\mathcal{H}$ approach free states in the limits $t \to \pm\infty$ without adopting all of the assumptions of Haag's theorem.

The strategy is to construct operators out of the basic field $A(x)$ that act like creation operators at $t = \pm\infty$. The following is a key formula:

$$C_i(f_i, t) = i \int_{x^0=t} \{C_i(\mathbf{x}, x^0)\frac{\partial f_i}{\partial x^0} - \frac{\partial C_i(\mathbf{x}, x^0)}{\partial x^0} f_i(\mathbf{x}, x^0)\} d^3\mathbf{x} \tag{3.28}$$

where f_i is a positive-energy solution to the Klein-Gordon equation with mass m_i $(i = 1, 2)$. In the free field case, if $C_i(\mathbf{x}, x^0) = \phi^{in}(\mathbf{x}, x^0)$, then $C_i(f_i, t) = a^{in}(f_i)^\dagger$:

$$a^{in}(f_i)^\dagger = i \int_{x^0=t} \{\phi^{in}(\mathbf{x}, x^0)\frac{\partial f_i}{\partial x^0} - \frac{\partial \phi^{in}(\mathbf{x}, x^0)}{\partial x^0} f_i(\mathbf{x}, x^0)\} d^3\mathbf{x} \tag{3.29}$$

Haag and Ruelle demonstrated that the intuitive idea that free particles are recovered in the limit of $t \to \pm\infty$ can be made precise in two different ways: (1) in terms of a strong limit, and (2) in terms of a weak limit. Both results will be explicated here in order to elucidate the importance of choosing a strong or weak limit for the purpose of evading Haag's theorem.

1. Strong limit version

Choose *any* operators q_i $(i = 1, 2)$ that satisfy the following two conditions: (q_i will be the analogues of the creation operators)

i. q_i is an *almost local* operator (Haag 1958, p. 670; Haag 1996, p. 84):

$$q_i = \sum_n \int f^n(x_1, ...x_n) A(x_1)...A(x_n) \Pi d^4 x_j \tag{3.30}$$

where f^n decrease faster than any power if any point x_j moves to infinity. The intuitive motivation for this condition is that at $t = \pm\infty$ free behaviour will be recoverable because the particle states are almost-localized very far apart.

ii. Denote the one-particle subspace of the interacting field Hilbert space $\mathcal{H}$ by $\mathcal{H}^{(1)}$. Choose a reference state vector $|\Psi_i^R\rangle$ in $\mathcal{H}^{(1)}$ for each of the two species of particle

($i = 1, 2$), where $|\Psi_i^R\rangle$ has a momentum space wave function $\tilde{f}_i^R(p)$ that is smooth[6] and nowhere vanishing on the mass hyperboloid (Haag 1996, pp. 86-7). Choose any q_i such that

$$q_i|\Psi_0\rangle = (2\pi)^{-3/2}|\Psi_i^R\rangle \tag{3.32}$$

where $|\Psi_0\rangle$ is the vacuum state in $\mathcal{H}$.

Variants of the theorems stated below can be proven for any q_i satisfying these conditions.[7] However, for the sake of simplicity, consider a case in which the following choice for q_i can be made (Jost 1965, p. 120; Reed and Simon 1979, pp. 319-20):

$$\tilde{q}_i(p) \equiv h(p^2)\tilde{A}(p) \tag{3.33}$$

where[8]

$$h(y) = 1 \text{ in a neighbourhood of } y = m^2 \tag{3.34a}$$

$$h(y) = 0 \text{ for } |y - m^2| > \frac{m^2}{2} \tag{3.34b}$$

$$0 \leq h(y) \leq 1 \tag{3.34c}$$

(Roughly, this guarantees that the vectors $Q_i(f_j, t)|\Psi_0\rangle$ specified in the next two lines are not all orthogonal to the one-particle states (Reed and Simon 1979, p. 319)).

Define

$$Q_i(f_j, t) := i\int_{x^0=t} \{q_i(\mathbf{x}, x^0)\frac{\partial f_j}{\partial x^0} - \frac{\partial q_i(\mathbf{x}, x^0)}{\partial x^0} f_j(\mathbf{x}, x^0)\} d^3\mathbf{x} \tag{3.35}$$

[6]A smooth solution is a solution of the form

$$f(x) = (2\pi)^{-\frac{3}{2}} \int \delta(p^2 - m^2)\left[e^{-ip\cdot x}\theta(p)g_+(\mathbf{p}) + e^{ip\cdot x}\theta(p)g_-(\mathbf{p})\right] d^4p \tag{3.31}$$

where $g_\pm \in \mathcal{D}$ (Jost 1965, p. 121).

[7]See Haag (1996, pp. 84-94) for a treatment of the general case.

[8]Presumably the last line in Jost is a typo?

where f_j is a smooth solution of the Klein-Gordon equation.

Consider states of the form

$$|\Psi(t)\rangle := Q_i(f_1,t)Q_i(f_2,t)\cdots Q_i(f_n,t)|\Psi_0\rangle \tag{3.36}$$

$Q_i(f_j,t)|\Psi_0\rangle$ satisfies the Klein-Gordon equation for mass m_i (Reed and Simon 1979, p. 320). This state is time-independent; however, a state 'created' by applying a product of Q_i's to $|\Psi_0\rangle$ is time-dependent. Intuitively, this reflects the fact that the particles interact.

The main result of Haag-Ruelle scattering theory can be stated in two parts:

Haag-Ruelle Theorem, Part I: (Jost 1965, p. 121; Reed and Simon 1979, pp. 320-1) The asymptotic states $|\Psi\rangle^{\pm}$ of $|\Psi(t)\rangle$ (which take the form specified in (3.36)) exist as strong limits, independent of the coordinate system and the choice of $h(p^2)$:

$$s - \lim_{t\to\pm\infty} || \; |\Psi(t)\rangle - |\Psi\rangle^{\pm}|| = 0 \tag{3.37}$$

Let $\mathcal{H}^{in}$ and $\mathcal{H}^{out}$ denote the closed spans of $|\Psi\rangle^{-}$ and $|\Psi\rangle^{+}$, respectively.

Haag-Ruelle Theorem, Part II: (Jost 1965, p. 122; Reed and Simon 1979, pp. 321) There exist operator-valued distributions ϕ_i^{ex} ($ex = in, out$) such that (where the limit is understood in the strong limit sense)

$$s - \lim_{t\to\pm\infty} Q_i(f_1,t)Q_i(f_2,t)...Q_i(f_n,t)|\Psi_0\rangle = a_i^{ex}(f_1)^{\dagger}a_i^{ex}(f_2)^{\dagger}\cdots a_i^{ex}(f_n)^{\dagger}|\Psi_0\rangle \tag{3.38}$$

where

$$a_i^{ex}(f_j)^{\dagger} = \int_{x^0=t}\{\phi_i^{ex}(\mathbf{x},x^0)\frac{\partial f_j}{\partial x^0} - \frac{\partial\phi_i^{ex}(\mathbf{x},x^0)}{\partial x^0}f_j(\mathbf{x},x^0)\}d^3\mathbf{x} \tag{3.39}$$

Furthermore, ϕ_i^{ex} can be considered free fields in the following sense: the representations $\langle\mathcal{H}_i^{ex},U,\phi_i^{ex}\rangle$ are respectively unitarily equivalent to the representations $\langle\mathcal{H}^F,U^F,\phi_{m_i}\rangle$ for a free field of mass m_i, where U and U^F are the strongly continuous unitary representations of the restricted Poincaré group on the respective Hilbert spaces[9] (Reed and Simon 1979, p. 321; Reed and Simon 1975, p. 62).

[9]That is, "U restricted to $[\mathcal{H}_i^{ex}]$ is unitarily equivalent to the $[U^F]$ of the free field" (Reed and Simon 1979, p. 321).

The background assumptions originally used to prove these theorems are that all the Wightman axioms are satisfied, that the single particle subspace $\mathcal{H}^{(1)}$ is not empty,[10] that there is a mass gap in the theory,[11] and that the $Q_i(f_j, t)$ have the properties specified above (Haag 1996, pp. 84-5; Reed and Simon 1979, p. 319). However, it has since been established that not all of these assumptions are necessary (Haag 1996, p. 85).

Part II of the theorem makes precise the sense in which $Q_i(f_j, t)$ can be regarded as a free creation operator at $t = \pm\infty$. The states $a_i^{ex}(f_1)^\dagger a_i^{ex}(f_2)^\dagger \cdots a_i^{ex}(f_n)^\dagger |\Psi_0\rangle$ are the states representing non-interacting particles defined above (e.g., $|\alpha_1 ... \alpha_k, \rho_1 ... \rho_l\rangle^\pm$). Consequently, $a_i^{ex}(f_j)^\dagger$ can also be defined in the usual way in terms of its action on states of this form: for example,

$$a_i^{ex}(f_{\alpha_m})^\dagger |\alpha_1 ... \alpha_k, \rho_1 ... \rho_l\rangle = (n(\alpha_m) + 1)|\alpha_m \alpha_1 ... \alpha_k, \rho_1 ... \rho_l\rangle \tag{3.40}$$

where $n(\alpha_m)$ is the number of particles fitting the description α_m in $|\alpha_1 ... \alpha_k, \rho_1 ... \rho_l\rangle$ (Haag 1958, p. 670).

Haag-Ruelle scattering theory is not subject to Haag's theorem, even though Part II of the theorem establishes that $\langle \mathcal{H}_i^{ex}, U, \phi_i^{ex}\rangle$ (where $\mathcal{H}_i^{ex}$ is a subspace of $\mathcal{H}$, the Hilbert space for the interaction) is unitarily equivalent to the free representation $\langle \mathcal{H}^F, U^F, \phi_{m_i}\rangle$. Haag's theorem is not applicable because it is neither proved nor assumed that $A(x)$ is unitarily equivalent to $\phi_{m_i}(x)$ at $t = \pm\infty$. The existence at some time of a unitary transformation relating the interacting field to a free field is a premise of all formal versions of the theorem. $\phi_i^{ex}(x)$ is at least two degrees of separation away from $A(x)$: $Q_i(f_j, \pm\infty)$ only coincides with $a_i^{ex}(f_j)^\dagger$ when applied

[10] That is, the mass operator $M = (P_\mu P^\mu)^{\frac{1}{2}}$ has some discrete eigenvalues ((Haag 1996, p. 85); (Reed and Simon 1979, p. 319)).

[11] That is: "[i]n the subspace orthogonal to the vacuum the spectrum of M has a lower bound $m_0 > 0$" (Haag 1996, p. 85). See the previous footnote for the definition of M.

to $|\Psi_0\rangle$ (the vacuum) and it is not possible to define $Q_i(f_j, \pm\infty)$ (or q_i) so that it is identical to $A(x)$.

2. Weak limit version (Haag 1958, pp. 671-672; Haag 1996, pp. 94-95)

Choose any operator $B(x)$[12] that satisfies the following conditions:

i. $B(x)$ is almost local.

ii. $\langle\Psi_0|B(x)|\Psi_0\rangle = 0$

iii. $\langle\Psi(t)|B(x)|\Psi_0\rangle \neq 0$ where $\Psi(t) = Q_i(f_j, t)|\Psi_0\rangle$ for all f_j

Usually it is easy to find a $B(x)$ that is a polynomial in the basic field $A(x)$ (Haag 1996, p. 95). Define

$$B_i(f_i, t) := i\int_{x^0=t}\{B(\mathbf{x}, x^0)\frac{\partial f_i}{\partial x^0} - \frac{\partial B(\mathbf{x}, x^0)}{\partial x^0} f_i(\mathbf{x}, x^0)\}d^3\mathbf{x} \tag{3.41}$$

where f_i is a smooth solution to the Klein-Gordon equation. Note that, in contrast to the strong limit case, $B_i(f_i, t)|\Psi_0\rangle$ is not a single particle state (Haag 1996, p. 94). The weak limit counterpart of the theorems stated above is that the following weak limit exists:

$$w - \lim_{t\to\pm\infty}\langle\Psi_1(t)|B_i(f_i, t)|\Psi_2(t)\rangle =_{ex} \langle\Psi_1|a^{ex}(f_i)^\dagger|\Psi_2\rangle_{ex} \tag{3.42}$$

where $|\Psi_1(t)\rangle$ and $|\Psi_2(t)\rangle$ are states of the form $|\Psi(t)\rangle := Q_i(f_1, t)Q_i(f_2, t)\cdots Q_i(f_n, t)|\Psi_0\rangle$ and $a^{ex}(f_i)^\dagger$ is defined as above.[13]

[12] A note on the notation: Haag (1958) uses $B(x)$ in the context of the weak limit version of the theorem, but Jost (1965) and Reed and Simon (1979) use $B(x)$ to represent the operator figuring in the strong limit.

[13] In the Haag (1996) presentation it is also assumed that the $f_1, \ldots f_n$ that are arguments of Q do not overlap.

In special cases it is possible to choose $B(x) = A(x)$.[14] If it is also assumed that there is only one type of particle in the theory,[15] the weak limit (3.42) is identical to the LSZ asymptotic condition (where $A_i(f_i, t) = B_i(f_i, t)$ as defined above with $B(\mathbf{x}, x^0) = A(\mathbf{x}, x^0)$)

$$w - \lim_{t\to\pm\infty} [_{ex}\langle\Psi_1|A_i(f_i, t)|\Psi_2\rangle_{ex}] =_{ex} \langle\Psi_1|a^{ex}(f_i)^\dagger|\Psi_2\rangle_{ex} \tag{3.43}$$

(Haag 1996, pp. 94-5). Thus, LSZ scattering theory is a special case of the weak limit version of Haag-Ruelle scattering.

In contrast to the strong limit formulation, in certain cases it is possible for the weak limit to relate the interacting field (or, more precisely $A_i(f_i, t)$, which is the creation operator derived from the interacting field $A(x)$) $A_i(f_i, t)$ directly to $a^{ex}(f_i)^\dagger$. However, the weak limit version of the theory also successfully evades Haag's theorem. Since $|\Psi(t)\rangle$ are the same states that figure in the strong limit, it is still the case that the representation $\langle\mathcal{H}^{ex}, U, \phi^{ex}\rangle$ is unitarily equivalent to the representation $\langle\mathcal{H}^F, U^F, \phi_m\rangle$. The unitary transformation $V : \mathcal{H}^{ex} \to \mathcal{H}^F$ relates $a^{ex}(f_i)^\dagger$, $a^F(f_i)^\dagger$ (Reed and Simon 1979, p. 326):

$$V a^{ex}(f_i)^\dagger V^{-1} = a^F(f_i)^\dagger \tag{3.44}$$

Inserting $VV^{-1} = V^{-1}V = I$ into (3.42) and applying (3.44) yields (for arbitrary states $|\Psi\rangle_{ex}$)

$$_{ex}\langle\Psi|V^{-1}VA_i(f_i, \pm\infty)V^{-1}V|\Psi\rangle_{ex} = {}_{ex}\langle\Psi|V^{-1}Va^{ex}(f_i)^\dagger V^{-1}V|\Psi\rangle_{ex} \tag{3.45}$$

$$_{F}\langle\Psi|VA_i(f_i, \pm\infty)V^{-1}|\Psi\rangle_{F} = {}_{F}\langle\Psi|a^F(f_i)^\dagger|\Psi\rangle_{F} \tag{3.46}$$

[14] According to Haag, this is always possible when there is no selection rule in the theory (Haag 1958, p. 671); however, the argument could be made that we should not expect $A(x)$ to be almost local, even when it is suitably smeared. See a footnote below for an alternative derivation of the LSZ asymptotic condition.

[15] Otherwise, additional normalization constants must be taken into account. See Haag (1958, p. 671).

But this does not imply that

$$VA_i(f_i,\pm\infty)V^{-1} = a^F(f_i)^\dagger \tag{3.47}$$

Consider the result of applying each side of (3.47) to the vacuum $|\Psi_0\rangle_F \in \mathcal{H}^F$:

$$V\left[A_i(f_i,\pm\infty)|\Psi_0\rangle_{ex}\right] = a^F(f_i)^\dagger|\Psi_0\rangle_F \tag{3.48a}$$

$$A_i(f_i,\pm\infty)|\Psi_0\rangle_{ex} = V^{-1}\left[a^F(f_i)^\dagger|\Psi_0\rangle_F\right] \tag{3.48b}$$

(3.48) does not hold because

$$V^{-1}\left[a^F(f_i)^\dagger|\Psi_0\rangle_F\right] = a^{ex}(f_i)^\dagger|\Psi_0\rangle_{ex} \tag{3.49}$$

and

$$A_i(f_i,\pm\infty)|\Psi_0\rangle_{ex} \neq a^{ex}(f_i)^\dagger|\Psi_0\rangle_{ex} \tag{3.50}$$

That is, $A_i(f_i,t)|\Psi_0\rangle_{ex}$ does not converge to $a^{ex}(f_i)^\dagger|\Psi_0\rangle_{ex}$ in the strong limit sense. As Haag puts it, more generally "[t]he vectors $\left[A_i(f_i,t) - a^{ex}(f_i)^\dagger\right]|\Psi_0\rangle_{ex}$ do not decrease in length as $t \to -\infty$ but only keep 'moving away' so that they become orthogonal to every fixed vector in the limit" (1958, p. 671; translated into my notation). Therefore, $VA_i(f_i,\pm\infty)V^{-1} \neq a^F(f_i)^\dagger$ and Haag's theorem is not applicable because there is no time at which $A(x)$ is related to a free field by a unitary transformation.[16]

[16]Hepp (1964) takes a different approach to establishing a connection between the Haag-Ruelle and LSZ scattering conditions. He introduces the following local (but not almost local) 'creation' operator:

$$\breve{B}(f_j,t) = \int dp\tilde{f}_j(p)A(-p)\exp\left(i(p^0-\omega_{\mathbf{p}})t\right)$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2}$ and the exponential factor ensures that the correct energy $\omega_{\mathbf{p}}$ holds in the limit $t \to \pm\infty$ (Streater 1975, p. 829). $\breve{B}(f_j,t)$ is well-defined in the Wightman framework. Under the assumption that $\mathcal{H}^{ex}$ is dense in $\mathcal{H}$, Hepp proves that the weak limit

$$w - \lim_{t\to\pm\infty}\langle\Phi_1|\breve{B}(f_1,t)\cdots\breve{B}(f_n,t)|\Phi_2\rangle = \langle\Phi_1|a^{ex}(F_1)^\dagger\cdots a^{ex}(F_n)^\dagger|\Phi\rangle$$

Bain argues that LSZ scattering theory evades Haag's theorem in virtue of its employment of a weak rather than a strong limit (2000, pp. 384-6). While it is true that the strong limit version of (3.43) would imply that $A(x)$ is a free field, the preceding analysis of the LSZ scattering theory in the more general context of Haag-Ruelle scattering theory shows that this account is incomplete. The Haag-Ruelle asymptotic condition can be formulated as either a weak limit or a strong limit; the important thing is that the operators that figure in the limit are carefully chosen so that the condition does not imply that the interacting field is equal to a free field at any time.

Recall that Haag's theorem entails that the Möller wave operators

$$\Omega^{\pm} = \lim_{t \to \pm\infty} e^{iHt} e^{-iH_F t} \tag{3.51}$$

are not well-defined (see Chapter 2). The strong limit version of Haag-Ruelle scattering theory permits the definition of well-defined analogues of the Möller wave operators. For simplicity, consider the case in which the interacting field $A(x)$ describes an interaction between particles of the same mass. Define the map $J : \mathcal{H}^F \to \mathcal{H}$ as follows (Reed and Simon 1979, p. 321):

$$J\left(\prod_{i=1}^{n} a^F(f_j)^{\dagger} |\Psi_0\rangle_F\right) = \prod_{i=1}^{n} Q(f_j, 0)|\Psi_0\rangle \tag{3.52}$$

---

holds, where $\tilde{F}_j(p) = \theta(p^0)\delta(p^2 - m^2)\tilde{f}_j(p, \omega_{\mathbf{p}})$ (ibid). He also proves (without the assumption that $\mathcal{H}^{ex}$ is dense in $\mathcal{H}$) that the strong limit holds when the f_i do not overlap in velocity space (i.e., for $(\mathbf{p}, \omega_{\mathbf{p}})$ such that $f(\mathbf{p}, \omega_p) \neq 0$ and for $(\mathbf{q}, \omega_{\mathbf{q}})$ such that $f(\mathbf{q}, \omega_{\mathbf{q}}) \neq 0$,

$$\frac{\mathbf{p}}{\omega_{\mathbf{p}}} \neq \frac{\mathbf{q}}{\omega_{\mathbf{q}}}$$

) (Streater 1975, pp. 829, 800; Hepp 1964, p. 642). However, I take it that Haag's theorem is not violated because the restriction to non-overlapping f_i means that the strong limit cannot be used to define a unitary transformation on all of $\mathcal{H}$ (or even on a dense subspace of $\mathcal{H}$?). See Haag (1958, p. 673).

It can then be proven that the following strong limits exist and define partial isometries $\tilde{\Omega}^{\pm} : \mathcal{H}^F \rightarrow \mathcal{H}^{ex}$ (Reed and Simon 1979, p. 322):

$$\tilde{\Omega}^{\pm} = s - \lim_{t \to \mp\infty} e^{iHt} J e^{-iH_F t} \tag{3.53}$$

If asymptotic completeness holds (i.e., $\mathcal{H}_{in} = \mathcal{H}_{out} = \mathcal{H}$), an S-matrix can be defined in terms of $\tilde{\Omega}^{\pm}$ and the asymptotic fields ϕ^{ex} defined above:

$$\phi^{out} = S^{-1} \phi^{in} S \tag{3.54}$$

$$\text{where } S = \tilde{\Omega}^{+} (\tilde{\Omega}^{-})^{\dagger} \tag{3.55}$$

The difference between these wave operators and the Möller wave operators is that the definition of the former does not presume that H, H_F are defined on the same Hilbert space. Instead, H, H_F are defined on unitarily inequivalent Hilbert space representations $\mathcal{H}$ and $\mathcal{H}^F$ and the definition of $\tilde{\Omega}^{\pm}$ employs the map $J : \mathcal{H}^F \rightarrow \mathcal{H}$. This is why Reed and Simon characterize Haag-Ruelle scattering as a "two Hilbert space formalism" (1979, p. 318; see also Koshmanenko 1978, 1979). It is this feature that allows $\tilde{\Omega}^{\pm}$ to be defined in a manner compatible with Haag's theorem.

Haag-Ruelle scattering theory demonstrates that it is possible to formulate scattering theory for QFT in a rigorously well-defined way without violating Haag's theorem. The assumption of Haag's theorem which does not hold in this approach is the assumption that there is a time at which the canonical interacting fields are unitarily equivalent to canonical free fields. Haag and Ruelle proved that, given the Wightman axioms and a few additional assumptions, the asymptotic states and fields exist. It is reassuring that Haag's theorem does not undermine all formulations of scattering theory, but only the canonical formulation based on the interaction picture. Haag-Ruelle scattering theory retains all of the features that are expected of such a framework.

From a theoretical point of view, Haag-Ruelle scattering theory is impeccable; however, from a practical point of view, it is not of much use for generating predictions for scattering experiments. The Wightman axioms were postulated and then (given additional assumptions) certain asymptotic limits were proven to exist in general. Since it is not clear how to represent a particular interacting system in the Wightman framework, the asymptotic limits cannot be used to calculate S-matrix elements for a given system.

3.2 PART II: EMPIRICAL EQUIVALENCE OF ALTERNATIVE FORMULATIONS OF QFT

There is more than one way of responding to the *reductio* posed by Haag's theorem, and different responses spawn different formulations of QFT. Three alternatives were surveyed in Chapter 2: renormalized canonical QFT, canonical QFT with cutoffs, and axiomatic QFT. This raises an obvious question: is there a principled grounds for privileging one of these formulations over the others? The simplest possibility is that one of the formulations will turn out to be empirically more successful than the others. The remainder of the chapter will be devoted to ruling out this possibility.

On the theme of reassurance, the empirical equivalence of renormalized canonical QFT and the other formulations offers some reassurance about the renormalization procedure. *Prima facie*, the technique of infinite renormalization seems like an unsatisfactory means of extracting predictions from canonical QFT. Philosophers and physicists have felt the need to offer a justification for the procedure. However, it is not clear what sort of explanation our sense of vague uneasiness calls for. A virtue of Haag's theorem is that it clarifies this situation by pinpointing a source of one of the infinities: namely, that the unrenormalized canonical formulation embraces all of the assumptions of the Hall-Wightman theorem. It is disconcerting that renormalization addresses this problem not by refining the assumptions, but by rendering the canonical framework mathematically ill-defined so that the Hall-Wightman theorem cannot be proven. Furthermore, the question of whether renormalized canonical QFT is consistent is up in the air (see Chapter 5). The fact that the renormalized canonical QFT yields the same predictions as more principled and manifestly consistent responses to Haag's theorem provides some reassurance about the predictions that result from renormalization.

Empirical equivalence is reassuring, but unfortunately not explanatory. One would like to know *why* it is that the three formulations generate the same predictions. This larger explanatory project is important, but I will not tackle it here. This project is distinct from another sort of explanatory project, which arises from worries about the mechanics of the renormalization procedures. The goal is to explain how renormalization procedures can be used to deduce consistent results. (This would be an especially pressing problem if the underlying theoretical principles of renormalized canonical QFT are indeed inconsistent; again, see Chapter 5.) This project is undertaken in Huggett and Weingard (1995), which addresses the apparent "mystery" that "renormalization must be ad hoc in some sense and, since it apparently involves manipulating infinite quantities, must show that the predictions of QFT are not logical consequences of the core theory" (Huggett and Weingard 1996, pp. S162-3). In Huggett (2002), which continues this project, the worry is explained as follows: that "the need to renormalize in perturbation theory raises [grave doubts] that the phenomenological cross-sections are deductive consequences of exact path-integral QFT" (p. 269).

The next subsection compares the predictions of renormalized canonical QFT to those of canonical QFT with cutoffs; the following section compares the predictions of renormalized canonical QFT to those of axiomatic QFT. Since in the context of QFT predictions take the form of S-matrix elements, the strategy in both cases will be to establish that the sets of S-matrix elements yielded by the theories are in very close agreement.

3.2.1 Renormalized canonical QFT vs. QFT with a volume cutoff

As explained in Section 3 of Chapter 2, Haag's theorem can be circumvented by placing the system under consideration in a box of finite spatial volume. For the purposes of this subsection, place the system in a box of finite spacetime volume. That is, stipulate that the fields are free outside some spacetime region by introducing a spacetime cutoff function: multiply the interaction term in the Lagrangian by $g(x)$, a function of spacetime such that $g(x) = 1$ in some finite bounded region of spacetime and $g(x) = 0$ outside this region. This renders Haag's theorem inapplicable because the fields are not covariant under spacetime translations. This is sufficient to take care of Haag's theorem, which is the focus here. However, Haag's theorem concerns only the infinite volume divergences; in general, QFT's with interactions also exhibit ultraviolet divergences. I will return to this point in a moment. The goal of this section is to show that, as the volume cutoff is removed, the predictions of the interaction picture with the volume cutoff rapidly approach those of the renormalized interaction picture. The conclusion will be that renormalized canonical QFT and canonical QFT with a volume cutoff are empirically equivalent in the sense that the sets of S-matrix elements that they generate can be brought into arbitrarily close agreement by choosing a large enough volume cutoff.

As Haag's theorem establishes, an infinite vacuum self-energy counterterm is required in the renormalized interaction picture. However, the infinite vacuum self-energy term does not render the expression for the S-matrix mathematically ill-defined. The infinite vacuum self-energy counterterm is taken care of by dividing the expression for the S-matrix (see Section 1 of Chapter 2) by ${}_F\langle 0|0\rangle_I$, where $|0\rangle_F$ is the free vacuum and $|0\rangle_I$ is the vacuum for the full interaction Hamiltonian H (Peskin and Schroeder 1995, p. 87, 4.30 to 4.31). In terms of Feynman diagrams, this is equivalent to disregarding disconnected diagrams (Peskin and Schroeder 1995, pp. 96-8). Of course, since vacuum polarization occurs, this is an illicit division by an infinite quantity. However, for present purposes, the provenance of the expression is not important; the important point is that it does not contain infinite vacuum self-energy counterterms.

Thus, it is not necessary to worry about vacuum self-energy renormalization rendering the expression for an S-matrix in the renormalized interaction picture ill-defined. The only source of discrepancy between the S-matrix elements for the renormalized and cutoff interaction picture representations is that in the region in which $g(x) = 0$ the fields in the renormalized representation interact, but in the cutoff representation the fields are free. In Wightman's framework for QFT, it is possible to prove that this discrepancy approaches zero–and therefore that the S-matrix elements of the two formulations approach one another–as the vacuum cutoff is removed.

But before entering into the details, a concern about employing the Wightman framework must be addressed. The concern is that it is inappropriate to appeal to the Wightman formulation of QFT in this context because the expressions of interest in canonical QFT are not rigorously well-defined and are therefore not elements of these formulations. For example, the Wightman axioms stipulate that each field $\phi_\alpha(f)$ is smeared with a function of spacetime (Streater and Wightman 2000, pp.

98, 33):

$$\phi_\alpha(f) = \int f(x)\phi_\alpha(x)dx \text{ where } f \in \ell, \tag{3.56}$$

ℓ is the set of all complex-valued, infinitely differentiable functions that approach infinity faster than any power of the euclidean distance

Furthermore, expectation values of products of fields are tempered distributions (ibid). In canonical QFT, S-matrix elements are expressed as a perturbative expansion of Green's functions, which are vacuum expectation values of time-ordered products of fields which are smeared with functions of space only. Green's functions are not tempered distributions. Consequently, neither the Green's functions nor the fields that they contain are admissible expressions in the Wightman formulation. This is a problem; however, for present purposes it can be set aside. The source of the difficulty is the ultraviolet divergences and the focus of this section is on Haag's theorem and the infinite volume divergences. Thus, it seems tolerable to employ the Wightman framework in the present context. (For certain types of interactions, renormalization group methods provide a means of establishing that the S-matrix elements of the interaction picture with an ultraviolet cutoff approach the S-matrix elements of the infinitely renormalized interaction picture as the ultraviolet cutoff is removed (see Huggett (2002) and Peskin and Schroeder (1995, Section 12.1) for a discussion of renormalization group methods)).

In the Wightman framework, the Haag-Ruelle scattering theorem establishes that at $t = -\infty$ and $t = +\infty$ there is a set of states in and operators on the Hilbert space for the interaction that is unitarily equivalent to the Fock representation for a free field (see Section 2 above). For the rest of this paragraph, consider the intuitive explanation for this result and the result that we seek. At $t = -\infty$ these states represent clusters of quanta that do not interact because they are infinitely spacelike separated. Similarly for $t = +\infty$. For large but finite $t = -T$ or $t = +T$, the clusters will not be infinitely spacelike separated; in fact, they can be contained in a sufficiently large but finite box. Set the cutoff $g(x)$ to be non-zero in the region containing the clusters of quanta at times $t = -T$ and $t = +T$ and zero for $|t| > T$. The clusters of quanta may be interacting, but we expect that the interaction will be weak if the interaction is local and the clusters of particles are far enough apart.

More rigorously, introduction of the cutoff $g(x)$ has the effect of switching off the interaction at $t = -T$ and $t = +T$. What is needed is a means of measuring the error that results from ignoring the contribution of the interaction term for $|t| > T$. In the Wightman framework, the cluster decomposition property of the Wightman functions $\mathcal{W}(x_1, \ldots x_j)$ suits this purpose (Bogolubov et. al. 1975, p. 272; Streater and Wightman 2000, p. 111):

$$\lim_{\lambda \to \pm\infty} \mathcal{W}(x_1, \ldots x_j, x_{j+1} + \lambda a, \ldots x_n + \lambda a) = \mathcal{W}(x_1, \ldots x_j)\mathcal{W}(x_{j+1}, \ldots x_n) \quad (3.57)$$

$$\text{where } a \text{ is a spacelike vector} \quad (3.58)$$

In terms of vacuum expectation values (VEV's) for a theory with one scalar field (neglecting the complication of smearing with a function of spacetime):[17]

$$\lim_{\lambda \to \pm\infty} \langle 0|\phi(x_1) \cdots \phi(x_j)\phi(x_{j+1} + \lambda\alpha) \cdots \phi(x_n + \lambda\alpha)|0\rangle = \langle 0|\phi(x_1) \cdots \phi(x_j)|0\rangle\langle 0|\phi(x_{j+1}) \cdots \phi(x_n)|0\rangle \quad (3.59)$$

[17] With smearing (under the assumption that there is a mass gap) (Bogolubov et. al. (1975, p. 275):

$$\lim_{\rho \to \pm\infty} \rho^l \int [\langle 0|\phi(x_1) \cdots \phi(x_j)\phi(x_{j+1} + \rho\alpha) \cdots \phi(x_n + \rho\alpha)|0\rangle - \langle 0|\phi(x_1) \cdots \phi(x_j)|0\rangle\langle 0|\phi(x_{j+1}) \cdots \phi(x_n)|0\rangle] u(x_1, \ldots x_n) dx_1 \cdots dx_n = 0$$

for all $l \geq 0$, spacelike a.

Intuitively, "if two clusters of particles separate to a great [spacelike] distance from each other, then they will not interact" (Bogolubov et. al. 1975, p. 272). For finite λ, the VEV's on the RHS may be correlated. The introduction of a volume cutoff $g(x)$ amounts to ignoring these correlations for fields in the region in which $g(x) = 0$. The Wightman reconstruction theorem establishes (given its assumptions) that a complete set of Wightman functions determines a (Heisenberg picture) Hilbert space representation for an interaction that is unique up to unitary equivalence, and thereby a complete set of S-matrix elements (in the Heisenberg picture). Since the correlations between the VEV's on the RHS of (3.59) tend to zero as $\lambda \to \pm\infty$ (i.e., as the volume cutoff is taken to infinity), the S-matrix elements in the theory with volume cutoff tend to those in the theory without volume cutoff in the limit of infinite volume.

The proof of the cluster decomposition property is an important component of the proof of the Haag-Ruelle theorem. The cluster decomposition principle is a consequence of the assumption of local commutativity (i.e., that fields at spacelike separation[18] commute) and the assumption that the vacuum is unique (i.e., the state of zero energy is nondegenerate) (Bogolubov et. al. 1975, p. 272). Thus, these two assumptions guarantee that, as the volume cutoff is removed, the S-matrix elements in the theory with volume cutoff tend to those in the theory without a volume cutoff. If it is also assumed that the theory has a mass gap and energy is positive, the correlations drop off exponentially for $m > 0$[19] (Bogolubov et al. 1975, p. 281; Haag 1996, p. 103). Thus, the S-matrix elements in the theory with volume cutoff rapidly approach those in the theory without a volume cutoff as the box is

[18] When two fields are smeared with functions of spacetime $f(x)$, $g(x)$, this means that $f(x)g(y) = 0$ for all spacelike separated (x, y) (Streater and Wightman 2000, p. 100).

[19] For $m = 0$ correlations drop off as $\frac{1}{r^2}$ (Bogolubov et. al. 1975, p. 281).

enlarged. Renormalized canonical QFT and canonical QFT with a volume cutoff are empirically equivalent in the sense that the sets of S-matrix elements that they generate can be brought into arbitrarily close agreement by choosing a large enough volume cutoff.

3.2.2 Renormalized canonical QFT vs. extant rigorous models for particular interactions

For present purposes, a drawback of the axiomatic approach is precisely that it is axiomatic: the axioms are postulated to hold for all interacting systems, but this can only be verified by constructing models of the axioms for particular interactions. The Wightman axiomatization cannot be employed to calculate S-matrix elements for an arbitrary interacting system, unlike renormalized canonical QFT. Thus, it is only possible to compare the predictions of renormalized canonical QFT to those of, say, the Wightman axiomatization for those special cases in which models of the Wightman axioms have been constructed.

Consider Glimm and Jaffe's $(\phi^4)_2$ model (see Section 1 above). Jaffe writes that the S-matrix elements for this model "are exactly the ones computed by the perturbation expansion based on Feynman diagrams and the textbook methods" (Jaffe 1999, p. 140). However, this statement must be carefully interpreted because the perturbation expansion in the coupling constant λ gives an approximation to the exact solution (for $\lambda \neq 0$) and because it seems likely that most renormalized perturbation expansions diverge (Haag 1996, p. 213). (Jaffe (1965) contains a proof that the perturbation expansions for theories with polynomial interactions in two space-time dimensions (e.g., the $(\phi^4)_2$ theory) do diverge). The more precise statement of Jaffe's claim is that the renormalized perturbation expansion is asymptotic to the exact solution given by the model: for any N, as $\lambda \to 0$

$$\langle\psi|S|\chi\rangle = \sum_{n=0}^{N} \lambda^n S_{f,i}(n) - O\left(\lambda^{N+1}\right) \tag{3.60}$$

where $\langle\psi|S|\chi\rangle$ is an S-matrix element evaluated in the model and $S_{f,i}(n)$ is the perturbation coefficient given by the Feynman diagrams (Jaffe 1999, p. 140; see Wightman (1986, p. 213)). According to Rivasseau (2000), for most models which have been constructed "the relationship of the nonperturbative construction to the perturbative one has been elucidated, the nonperturbative Green's functions being the Borel sum of the corresponding perturbative expansion" (p. 3765; see also Glimm and Jaffe 1987, pp. 461-2).[20]

Of course, this could be turned around. We must confirm that a proposed model for an interaction is the correct one. Glimm describes the goal of constructive field theory as solving "the problem of finding rigorously defined mathematical objects which correspond to the operators discussed in [canonical] Quantum Field Theory" (1969b, p. 103). He notes that one way of recognizing a correct solution is to test whether it satisfies some set of axioms for QFT (ibid). He also says that he would expect agreement between the values for S-matrix elements given by the model and the values given by canonical QFT (ibid). Thus, it seems likely that, by design, any model constructed would replicate the predictions of renormalized canonical QFT.

As an aside, there is another approach to comparing the S-matrix elements yielded by renormalized canonical QFT to those generated from some set of axioms.

[20] An alternative approach would be to establish that the Hilbert space for a given model is locally (but not globally) unitarily equivalent to the Fock representation for the corresponding free field and then to argue that measurements are represented by local operators. But: (1) this 'locally Fock' property does not even hold for the $(\phi^4)_3$ model because the ultraviolet divergences also necessitate the use of a representation that is unitarily inequivalent to the Fock representation (even more roughly, the problem is that infinite wave function renormalization is needed) (see Jaffe 1999, p. 140, n. 2) and (2) the claim that measurements are represented by local operators is controversial.

Instead of constructing a model for the set of axioms and calculating exact S-matrix elements, the method is to consider perturbative expansions for S-matrix element and to use the axioms as constraints on the form that can be taken by the terms of the expansion. This strategy of deriving the terms of the renormalized perturbation expansion using axioms was pursued by (among others) Bogolubov and Parasiuk, Zimmerman, Symanzik, Hepp, Epstein and Glaser, and Steinmann from the late 1950's into the 1970's (Haag 1996, p. 71, n. 7). More recently, this project has been taken up by algebraic field theorists (see, for example, Brunetti and Fredenhagen (2000)).

3.3 CONCLUSION

Haag's theorem undermines the unrenormalized interaction picture representation, but this is not cause for despair about the mathematical foundations of QFT. The existence of the $(\phi^4)_2$ model demonstrates that there is at least one interacting QFT that can be rigorously defined in the customary Hilbert space framework. Haag-Ruelle scattering theory is a mathematically well-defined alternative formalism for scattering theory. Furthermore, Haag's theorem does not undermine the predictions of renormalized canonical QFT. The renormalized interaction picture is empirically equivalent to the interaction picture with a volume cutoff in the sense that, in the limit as the volume cutoff is removed, the S-matrix elements of the interaction picture with volume cutoff approach the S-matrix elements of the renormalized interaction picture. Thus, the sets of S-matrix elements generated by these two formulations can be brought into arbitrarily close agreement by choosing a large enough volume cutoff. The sets of S-matrix elements yielded by extant rigorous models are also in agreement with those yielded by renormalized canonical QFT (modulo the complications arising from the perturbative expansion).

One might wonder whether the empirical equivalence of renormalized canonical QFT and canonical QFT with cutoffs undermines the efforts of constructive field theorists. The interaction picture with cutoffs is a mathematically well-defined formulation of QFT which replicates the predictions of the renormalized interaction picture, to good approximation. What, then, is the point of devoting so much energy to tackling the very difficult problems of constructing well-defined models without cutoffs and formulating consistent sets of axioms that are physically relevant? Is this a puzzle for pure mathematicians that has no relevance for physics? Wightman considers this reaction to Haag's theorem, and offers the following response: "I disagree with this to this extent: the existence of the Hamiltonian and its properties seem to me to be an important part of the physics" (1967a, p. 251). Chapter 5 will take up the issue of whether the programs of axiomatic and constructive QFT are relevant to the foundations and interpretation of QFT.

4.0 CHAPTER 4: THE FATE OF 'PARTICLES' IN QUANTUM FIELD THEORIES WITH INTERACTIONS

Most philosophical discussion of the particle concept that is afforded by quantum field theory has focused on free systems. This chapter is devoted to a systematic investigation of whether the particle concept for free systems can be extended to interacting systems. The possible methods of accomplishing this are considered and all are found unsatisfactory. Therefore, an interacting system cannot be interpreted in terms of particles. As a consequence, quantum field theory does not support assigning particles a fundamental place in our ontology. In contrast to much of the recent discussion on the particle concept derived from quantum field theory, this argument does not rely on the assumption that a particulate entity be localizable.

4.1 INTRODUCTION

Quantum field theory (QFT) is the basis of the branch of physics known as "particle physics." However, the philosophical question of whether quantum field theories genuinely describe particles is not straightforward to answer. Since QFT's are formulated in terms of fields (i.e., mathematical expressions that associate quantities with points of spacetime), the issue is whether the formalism can be interpreted in terms of a particle notion. What is at stake is whether QFT, one of our current best physical theories, supports the inclusion of particles in our ontology as fundamental entities. This chapter advances an argument that, because systems which interact cannot be given a particle interpretation, QFT does not describe particles.

Even proponents of a particle interpretation of QFT acknowledge that the particle concept inherent in a QFT would differ from the classical particle concept in many ways. To distinguish the QFT concept from the classical one, the former has been dubbed the 'quanta' concept (e.g., Teller 1995, p. 29). Redhead and Teller (1992) argue that one way in which quanta differ from classical particles is that quanta are not capable of bearing labels.[1] That is, they lack a property that is variously termed "haecceity," "primitive thisness," or "transcendental individuality." However, Teller argues that the quanta notion should still be considered a particlelike notion because quanta are aggregable (Teller 1995, p. 30). There are, for example, states in which we definitely have two quanta and states in which we definitely have three quanta, and these can be combined to yield a state in which we definitely have five quanta. Quanta are also particlelike insofar as they possess the same energies as classical, relativistic, non-interacting particles.

This minimal notion of quanta will be employed in the following investigation. These quanta may significantly differ from classical particles in other respects. For example, the question of whether there is an appropriate sense in which quanta are localized has been the subject of recent debate (see Malament (1996), Halvorson and Clifton (2002), and Fleming (2001)). However, for our purposes, this debate can be set aside. The particlelikeness of the quanta notion will not challenged; instead, the arguments presented below aim to show that the domain of application of the quanta

[1] For a dissenting view on the metaphysics of classical particles, see Huggett (1999).

concept is so strictly limited that quanta cannot be admitted into our ontology as fundamental entities.

As Teller acknowledges, his argument that quanta are aggregable is based on special properties of the mathematical representation for free systems in QFT. Free fields describe the world in the absence of interactions. But in the real world there are always interactions. This raises a crucial question: can the quanta interpretation be extended to interacting systems? The analysis presented here aims to supply a comprehensive answer to this question. Ultimately, the answer is "no;" an interacting system cannot be described in terms of quanta. This inquiry is in the same spirit as recent discussions (which are restricted to free systems) of whether a unique quanta notion is available for accelerating observers or in more general spacetime settings (i.e., nonstationary spacetimes) (Clifton and Halvorson 2001, Arageorgis et. al. 2003; Arageorgis et. al. 2002). The present investigation adopts the opposite approach: the restriction to free systems will be dropped, and the restriction to inertial observers on flat Minkowski spacetime will be retained. The commonality is that the interpretive conclusions rest on the employment of unitarily inequivalent representations of the canonical commutation relations. In this case, the fact that the representations for free and interacting systems are necessarily unitarily inequivalent is invoked in the first stage of the argument. However, the structure of the argument diverges from the above-mentioned discussions after this first stage; further work is required to establish that the unitarily inequivalent representation for the interacting field cannot possess the relevant formal properties.

After a brief review of the Fock representation for a free system and the standard argument that it supports a quanta interpretation, three methods for obtaining a quanta interpretation for an interacting system will be evaluated. The first method is simply to use the Fock representation for a free system to represent an interacting system. Since this method proves unsuccessful, it is necessary to generalize the definition of Fock representation so that it is applicable to interacting systems. In order to distinguish these definitions, I will reserve the term "Fock representation" for free systems and refer to the results of attempts to formulate analogous representations for interacting systems as "ΦOK representations."[2] In principle, there are two methods for extending the definition of a Fock representation which are allied with the two approaches to defining a Hilbert space representation in quantum field theory: the 'constructive' method of applying to an interacting field the same quantization procedure that generates a Fock representation from a classical free field and the 'axiomatic' method of specifying a Hilbert space representation by stipulating formal conditions. The former method will be investigated in Section 4 and the latter in Section 5. Following the failure of both methods, a final, last-ditch attempt to retain a quanta interpretation for interacting systems will be critiqued in Section 6. The implications of the conclusion that interacting systems cannot be described in terms of quanta for metaphysics and for the foundations of QFT will be assessed in Section 7.

4.2 THE FOCK REPRESENTATION FOR A FREE FIELD

Every introductory QFT textbook contains a discussion of how to construct a Fock space representation of the equal-time canonical commutation relations (ETCCR's) for a free field. The construction proceeds by effecting a positive-negative frequency Fourier decomposition of a classical free field and then promoting the coefficients to

[2]This expression is borrowed from the early history of QFT, when what is here referred to as a "Fock representation" was sometimes labeled a "ΦOK representation." (See, for example, Wightman (1967b)).

operators. The details of this construction will be discussed in Section 4. In this section, the properties of the final product of the construction–a Fock representation for a free field–will be reviewed. Strictly speaking, the textbook characterization of a Fock representation is not well-defined. This can be remedied by making a few modifications. However, for the purposes of this section, the textbook treatment will be used because the argument that a Fock representation supports a quanta interpretation is most naturally formulated using the unrigorous presentation. Details of the rigorous version can be found in the footnotes.

A Fock representation for a free bosonic real scalar field with $m > 0$ on Minkowski spacetime[3] possesses the following formal properties:[4]

1. There exist well-defined annihilation and creation operators $a(\mathbf{k}, t)$, $a^\dagger(\mathbf{k}, t)$ where $\mathbf{k}^2 = k_0^2 - m^2$. $a(\mathbf{k}, t)$, $a^\dagger(\mathbf{k}, t)$ obey the ETCCR's

$$[a(\mathbf{k}, t), a(\mathbf{k}', t)] = 0,\ [a^\dagger(\mathbf{k}, t), a^\dagger(\mathbf{k}', t)] = 0,\ [a(\mathbf{k}, t), a^\dagger(\mathbf{k}', t)] = \delta^3(\mathbf{k} - \mathbf{k}') \quad (4.1)$$

At any given time t (where $\omega_{\mathbf{k}}^2 = k_0^2 = \mathbf{k}^2 + m^2$)

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\mathbf{k}}}}[a^\dagger(\mathbf{k}, t)e^{ik\cdot x} + a(\mathbf{k}, t)e^{-ik\cdot x}] \quad (4.2)$$

[3]The free bosonic real scalar field is treated for simplicity; other types of free fields also possess Fock representations. The restriction to $m > 0$ and (flat) Minkowski spacetime ensures that there exists a Fock representation that is unique up to unitary equivalence and supports a quanta interpretation in the manner described below. For accounts of what happens in other spacetime settings see Wald (1994, §§4.3, 4.4) and Arageorgis, Earman & Ruetsche (2002).

[4]A note on how to make these properties rigorously well-defined. The consequence of allowing the integrals to range over all of space is that the n-particle states are not normalizable. In addition, the integral in equation (4.2) does not converge (Wald 1994, p. 35). To fix the first problem, it is often stipulated that $\phi(\mathbf{x}, t)$ is confined to a box or satisfies periodic boundary conditions (see, e.g., Roman (1969, p. 50)), but this is not necessary. Both problems can be resolved by 'smearing' the fields in their spatial variables. See the definition of a ΦOK_2 representation in Section 5.

and, using $\pi(\mathbf{x},t) = \frac{\partial\phi(\mathbf{x},t)}{\partial t}$ and $\frac{\partial a(\mathbf{k},t)}{\partial t} = \frac{\partial a^\dagger(\mathbf{k},t)}{\partial t} = 0$,

$$\pi(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_\mathbf{k}}}[i\omega_\mathbf{k} a^\dagger(\mathbf{k},t)e^{ik\cdot x} - i\omega_\mathbf{k} a(\mathbf{k},t)e^{-ik\cdot x}] \tag{4.3}$$

Inverting and solving for $a(\mathbf{k},t)$, $a^\dagger(\mathbf{k},t)$ gives

$$a(\mathbf{k},t) = \int \frac{d^3x}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_\mathbf{k}}}e^{ik\cdot x}[\omega_\mathbf{k}\phi(\mathbf{x},t) + i\pi(\mathbf{x},t)] \tag{4.4a}$$

$$a^\dagger(\mathbf{k},t) = \int \frac{d^3x}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_\mathbf{k}}}e^{ik\cdot x}[\omega_\mathbf{k}\phi(\mathbf{x},t) - i\pi(\mathbf{x},t)] \tag{4.4b}$$

2. There exists a unique (up to phase factor) 'no-particle state' $|0\rangle$ such that

$$a(\mathbf{k},t)|0\rangle = 0 \text{ for all } \mathbf{k}$$

3. Number operators $N(\mathbf{k})$ can be defined for any t:

$$N(\mathbf{k}) = a^\dagger(\mathbf{k},t)a(\mathbf{k},t) \tag{4.5}$$

$$N(\mathbf{k})[a^\dagger(\mathbf{k},t)^n|0\rangle] = n[a^\dagger(\mathbf{k},t)^n|0\rangle]$$

where $n = \{0,1,2,\ldots\}$.[5]

In addition, for any t, the total number operator N given by $N = \int d^3kN(\mathbf{k}) = \int d^3ka^\dagger(\mathbf{k},t)a(\mathbf{k},t)$ is well-defined.[6]

4. The one-particle Hilbert space $\mathcal{H}$ has as a basis the set of vectors generated from $|0\rangle$ by single applications of $a^\dagger(\mathbf{k},t)$ (for any $\mathbf{k}$ satisfying $\mathbf{k}^2 = k_0^2 - m^2$). The Fock space $\mathcal{F}$ for $\phi(\mathbf{x},t)$ is obtained by taking the direct sum of the n-fold symmetric tensor product of $\mathcal{H}$: $\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty}(\otimes^n\mathcal{H})$ (Wald 1994, p. 192). $|0\rangle$ is cyclic with respect to the $a^\dagger(\mathbf{k},t)$'s.

[5]When normalized, the n-particle state becomes $\frac{a^\dagger(\mathbf{k},t)^n|0\rangle}{\sqrt{n!}}$.

[6]That is, when the number operators are properly defined using a test function space $\mathcal{T}$, as in Section 5, $N = \sum_{j=1}^{\infty} a^\dagger(f_j)a(f_j)$ converges in the sense of strong convergence on the domain of N where $\{f_j\}$ is an orthonormal basis of $\mathcal{T}$ and N exists only if N exists and is the same for every choice of orthonormal basis $\{f_j\}$ (Dell'Antonio, Doplicher & Ruelle (1966), pp. 225-226).

Before turning to the quanta interpretation, note that this set of formal properties prescribes a physically appropriate representation for a free system. $a(\mathbf{k}, t)$, $a^\dagger(\mathbf{k}, t)$, $N(\mathbf{k})$, and N are relativistically covariant, as is required of physically significant quantities in a relativistic theory like QFT. Also, this set of formal properties uniquely characterizes a Fock representation. As a consequence of the failure of the Stone-von Neumann theorem for an infinite number of degrees of freedom,[7] there are

[7] See Ruetsche (2002) for an exposition.

uncountably many unitarily inequivalent[8] representations of the ETCCR's.[9] It has been proven that (when suitably rigorized), these properties pick out an irreducible representation of the ETCCR's that is unique up to unitary equivalence (Gårding and Wightman 1954, pp. 624-5; Wightman and Schweber 1955, pp. 819-22). More precisely, if ϕ_m is the free field on Minkowski spacetime satisfying the Klein-Gordon equation with mass m ($m > 0$), then the representation of the ETCCR's for $\phi_m(\mathbf{x}, t)$, $\pi_m(\mathbf{x}, t)$ that possesses properties (1) to (4) is unique (up to unitary equivalence).

[8]Two representations of the ETCCR's $(\mathcal{H}_1, \{O_1^i\})$ and $(\mathcal{H}_2, \{O_2^i\})$ (where $\{O_n^i\}$ is the collection of operators appearing in the ETCCR's) are unitarily equivalent if and only if there exists some unitary mapping U from Hilbert space $\mathcal{H}_1$ to Hilbert space $\mathcal{H}_2$ that such that for each operator $O_1^j \in \{O_1^i\}$ there exists an operator $O_2^j = UO_1^jU^{-1} \in \{O_2^i\}$ (Wald 1994, p. 19). The practical importance of unitary equivalence is that–assuming $\{O_1^i\}$ and $\{O_2^i\}$ include all physically significant operators–if two representations are unitarily equivalent then they will be physically equivalent in the sense that both produce the same expectation values for all physically significant operators. See Ruetsche (2002) for further discussion.

[9]A representation of the Weyl form of the ETCCR's is given by the pair of families of unitary operators $U(f,t)$, $V(g,t)$ (f,$g \in \mathcal{T}$) in the usual way, where $U(f,t) = e^{i\phi(f,t)}$, $V(g,t) = e^{i\pi(g,t)}$ and $U(f,t)$ and $U(g,t)$ satisfy the usual conditions (Wightman 1967b, p. 189)

$$\begin{aligned} U(f,t)U(g,t) &= U(f+g,t) \\ V(f,t)V(g,t) &= V(f+g,t) \\ U(f,t)V(g,t) &= e^{i(f,g)}V(g,t)U(f,t) \end{aligned}$$

and $U(\alpha f, t)$, $V(\alpha g, t)$ are continuous in the real number α. The Weyl form of the ETCCR's is more convenient than the standard form

$$[\phi(f,t), \phi(g,t)] = [\pi(f,t), \pi(g,t)] = 0, [\phi(f,t), \pi(g,t)] = i(f,g)$$

because, unlike $\phi(f,t)$ and $\pi(f,t)$, the exponentiated operators $U(f,t)$ and $V(f,t)$ are bounded, so they are automatically defined on the same domain. Of course, 'unsmearing' this form of the ETCCR's yields the familiar relations

$$[\phi(\mathbf{x},t), \phi(\mathbf{x}',t)] = 0,\ [\pi(\mathbf{x},t), \pi(\mathbf{x}',t)] = 0,\ [\phi(\mathbf{x},t), \pi(\mathbf{x}',t)] = i\delta^3(\mathbf{x}-\mathbf{x}')$$

It may turn out that the ETCCR's do not hold for physically realistic interactions because the arguments of the field operators must be smeared in time as well as space (see Streater and Wightman 2000, p. 168). However, this potential complication will be overlooked here. Note that this is not a universal feature of interacting theories; there are rigorous Hilbert space models for non-trivial interactions in which the ETCCR's do hold (e.g., for a Lagrangian with a scalar ϕ^4 interaction term in two spacetime dimensions) (Glimm and Jaffe 1970a).

However, it should also be noted that the Fock representation of the ETCCR's for the field $\phi_{m'}(\mathbf{x}, t)$, $m' \neq m$, is unitarily inequivalent to the Fock representation of the ETCCR's for $\phi_m(\mathbf{x}, t)$ (Reed & Simon 1975, pp. 233-5). Therefore, there is such a thing as *the* Fock representation for a free field that satisfies a specified Klein-Gordon equation; there are a multitude of unitarily inequivalent Fock representations when the field equation is left unspecified.

The existence of a well-defined operator N with a spectrum consisting of the non-negative integers is not in itself sufficient for a quanta interpretation. Quantum theories contain discrete properties. How, then, do we know that N is counting entities rather than, say, energy levels? An argument needs to be made that N counts quanta, i.e., $|0\rangle$ is a state in which there are no quanta, $a^\dagger(\mathbf{k},t)|0\rangle$ is a state in which there is one quantum, etc. A Fock representation supports a quanta interpretation because the eigenvectors of N, the total number operator, possess properties that are appropriate for states containing definite numbers of particles. These properties arise from all of the continuous and discrete symmetries of the field equation, but, for the purposes of the ensuing discussion, we will restrict our attention to time translations and energy. In the Fock representation, the Hamiltonian operator H is (for any t)[10]

$$H = \int d^3k \omega_{\mathbf{k}} a^\dagger(\mathbf{k},t) a(\mathbf{k},t) = \int d^3k \omega_{\mathbf{k}} N(\mathbf{k}) \tag{4.6}$$

This expression for H can be used to establish that the eigenvectors of N have the correct energies for states in which the indicated number of quanta is present.

1. $|0\rangle$, *the no-particle state* It can easily be verified that $H|0\rangle = 0$. Since the spectrum of H does not contain any negative eigenvalues,[11] this means that $|0\rangle$

[10]After normal ordering; i.e., moving all the $a^\dagger(\mathbf{k}, t)$'s to the left of the $a(\mathbf{k}, t)$'s without using the ETCCR's in (1).

[11]Because $a^\dagger(k) = 0$ when $k_0 < 0$. For details, see the explication of the positive-negative frequency decomposition of $\psi(x)$ in Section 4.

is the ground state of H. Moreover, we can infer from property 2 that $|0\rangle$ is the unique ground state of H. Since the state in which there are no particles would presumably have the lowest energy, if $|0\rangle$ cannot be interpreted as the no-particle state, then there is no other candidate in a Fock representation.

Furthermore, $|0\rangle$ is invariant under time translations: $e^{iHt}|0\rangle = |0\rangle$. More generally, $|0\rangle$ is the unique state which carries the trivial, one-dimensional representation of all symmetry groups[12] (Roman 1969, p. 79). Consequently, $|0\rangle$ is the physical vacuum state. As Streater and Wightman remark, since $|0\rangle$ is invariant under the unitary operators that give a representation of the Poincaré group and we are operating in Minkowski spacetime, $|0\rangle$ "looks the same to all observers" (i.e., observers in inertial motion) (2000, p. 21). Nothing–i.e., the absence of particles–is something that we would expect to look the same to all inertial observers. Thus, it is reasonable to interpret $|0\rangle$ as a state in which there are no particles because it is the unique state with the desired properties: it is the lowest energy state and looks the same to all inertial observers.

2. $a^{\dagger}(\mathbf{k}, t)|0\rangle$ *(where* $\mathbf{k}^2 = k_0^2 - m^2$*), the one-particle states* The one-particle states furnish the definitive argument for the quanta interpretation. States of the form $a^{\dagger}(\mathbf{k}, t)|0\rangle$ are also eigenstates of H: $H[a^{\dagger}(\mathbf{k}, t)|0\rangle] = \sqrt{\mathbf{k}^2 + m^2}[a^{\dagger}(\mathbf{k},t)|0\rangle]$. Special relativity dictates that $\sqrt{\mathbf{k}^2 + m^2}$ is the correct energy for a single non-interacting particle with momentum $\mathbf{k}$ and mass m. This correspondence between the energy eigenvalues associated with these quantum states and the relativistic energies of single classical particles provides a justification for interpreting $a^{\dagger}(\mathbf{k}, t)|0\rangle$ as a state containing a single quantum of momentum $\mathbf{k}$ and mass m.[13]

[12] The Poincaré group and, when applicable, a symmetry group associated with charge and any discrete symmetry groups.

[13] This correspondence to relativistic energies holds for sharp values of $\mathbf{k}$. However, to make the characterization of the Fock representation for a free field rigorous, the momentum argument of

3. $a^\dagger(\mathbf{k},t)^n|0\rangle$ *($n \geq 2$), the n-particle states* All of the other eigenvectors of $N(\mathbf{k})$ also have the correct relativistic energies for states in which there are particular numbers of non-interacting particles. In general, $H\left[a^\dagger(\mathbf{k},t)^n|0\rangle\right] = n\omega_\mathbf{k}\left[a^\dagger(\mathbf{k},t)^n|0\rangle\right]$ (where $\omega_\mathbf{k} = \sqrt{\mathbf{k}^2 + m^2}$), so $a^\dagger(\mathbf{k},t)^n|0\rangle$ is an eigenstate of H with the correct relativistic energy for a state in which there are n non-interacting particles, each with momentum $\mathbf{k}$. This analysis generalizes to eigenvectors of the total number operator N in a straightforward way:

$$H\left[a^\dagger(\mathbf{k}_1,t)^{n_1}a^\dagger(\mathbf{k}_2,t)^{n_2}\cdots a^\dagger(\mathbf{k}_l,t)^{n_l}|0\rangle\right] = \\ (n_1\omega_{\mathbf{k}_1} + n_2\omega_{\mathbf{k}_2} + \ldots + n_l\omega_{\mathbf{k}_l})\left[a^\dagger(\mathbf{k}_1,t)^{n_1}a^\dagger(\mathbf{k}_2,t)^{n_2}\cdots a^\dagger(\mathbf{k}_l,t)^{n_l}|0\rangle\right]$$

therefore, $a^\dagger(\mathbf{k}_1,t)^{n_1}a^\dagger(\mathbf{k}_2,t)^{n_2}\cdots a^\dagger(\mathbf{k}_l,t)^{n_l}|0\rangle$ has the correct relativistic energy for a state in which there are n_1 non-interacting particles with momentum $\mathbf{k}_1$, n_2 non-interacting particles with momentum $\mathbf{k}_2$, etc.

Thus the energy eigenvalues provide a physical justification for interpreting any n-particle eigenvector of N as a state in which there are n non-interacting quanta.

There is a formal analogy between the Fock representation for a free field and an infinite array of quantum mechanical simple harmonic oscillators (SHO's) (e.g., a quantum mass on a spring). This analogy does not do any philosophical work in this chapter, but it is useful background because philosophers often couch their discussions of QFT in terms of this analogy. Associated with a single SHO are the

$a^\dagger(\mathbf{k},t)$ must be smeared: $a^\dagger(\tilde{f},t) = \int a^\dagger(\mathbf{k},t)\tilde{f}(\mathbf{k})d\mathbf{k}$ where $\tilde{f}$ is the Fourier transform of $f \in \mathcal{T}$. For this formulation, an argument could made that $\int \tilde{f}(\mathbf{k})\sqrt{\mathbf{k}^2 + m^2}d\mathbf{k}$ is the correct relativistic energy for a single particle with unsharp momentum distributed according to $\tilde{f}(\mathbf{k})$.

"ladder operators" (or "lowering" and "raising" operators):

$$a = x\sqrt{\frac{m\omega}{2}} + ip\sqrt{\frac{1}{2m\omega}} \text{ where } \omega^2 \text{ is the frequency of the SHO} \quad (4.7a)$$

$$a^\dagger = x\sqrt{\frac{m\omega}{2}} - ip\sqrt{\frac{1}{2m\omega}} \quad (4.7b)$$

Note the formal similarity between (4.7a, 4.7b) and $a(\mathbf{x}, t)$, $a^\dagger(\mathbf{x}, t)$ (the Fourier transforms of (4.4a,4.4b)). Written in terms of a, $a^\dagger$, the Hamiltonian for a SHO is

$$H = \frac{1}{2}\varpi(a^\dagger a + aa^\dagger) \quad (4.8)$$

Prior to normal ordering and when the system is placed in a box, the Hamiltonian for a free quantum field is an infinite sum of SHO Hamiltonians of different frequencies:

$$H = \sum_k \frac{\omega_{\mathbf{k}}}{2}[a^\dagger(\mathbf{k},t)a(\mathbf{k},t) + a(\mathbf{k},t)a^\dagger(\mathbf{k},t)] \quad (4.9)$$

In this manner, the Fock representation for a free system can sustain a quanta interpretation. But is it also the case that an interacting system possesses a Hilbert space representation that admits a quanta interpretation? In the next section, the possibility of using the Fock representation for a free system to represent an interacting system will be ruled out. This leaves the alternative strategy of attempting to generalize the definition of a Fock representation for a free system so that it can also be applied to an interacting system. Two approaches fall into this category: focusing on the motivation behind the relationship between $\phi(\mathbf{x}, t)$, $\pi(\mathbf{x}, t)$ and $a(\mathbf{k}, t)$, $a^\dagger(\mathbf{k}, t)$ expressed in identities (4.2), (4.3) and focusing on the four formal properties of the free Fock representation listed above. The motivation for identity (4.2) is that it is the output of the quantization procedure that generates operator expressions for the free quantum fields from the free classical fields. The relevant step in the quantization procedure is the positive-negative frequency decomposition of the classical field. One approach to extending the definition of Fock representation to interacting fields is to apply the same quantization procedure to an interacting field (i.e., to decompose the classical interacting field into positive and negative frequency parts). To avoid confusing the representation that results with the Fock representation for a free field, the former will be labelled a "ΦOK_1 representation." This approach will be investigated in Section 4; as we shall see, it is not feasible.

The second approach to extending the definition of Fock representation to interacting systems is to focus attention on the formal properties of a Fock representation listed above, without regard for their origins. Instead of defining a representation by quantizing a classical field, a unique (up to unitary equivalence) representation of the ETCCR's is picked out by stipulating formal properties. As noted, the four formal properties listed above do pick out a unique (up to unitary equivalence) representation of the ETCCR's for a given set of canonical free fields (ϕ_F, π_F). If the same set of properties could be used to pick out a unique representation of the ETCCR's for a given set of canonical interacting fields (ϕ_I, π_I), then the representation singled out in this way could be considered the 'Fock' representation for this interacting field. Work by Wightman and his collaborators provides a starting point for extending the definition of Fock representation to interacting fields in this way. Such a representation will be dubbed a "ΦOK_2 representation." The question of whether a ΦOK_2 representation can sustain a quanta interpretation for an interacting system will be the subject of Section 5. Once again, the conclusion will be that a ΦOK_2 representation fails to yield a quanta interpretation for an interacting field.

To motivate the search for an extension of the definition of Fock representation to interacting fields, the next section advances an argument based on Haag's theorem that a Fock representation for a free field cannot sustain a quanta interpretation for an interacting field.

4.3 METHOD #1: USING THE FOCK REPRESENTATION FOR A FREE FIELD

The simplest strategy for obtaining a quanta interpretation for an interacting field would be to use the Fock representation for a free system to represent a given interacting system, and then to try to extract a quanta interpretation. That is, the hope is that the Hilbert space spanned by n-particle states for the free field contains states that can be interpreted as containing n quanta in the presence of the interaction. As we shall see, this simple strategy fails because there is no state in the Fock representation for a free field that can be interpreted as containing zero quanta.

The background to this argument is the existence of different approaches to QFT with interactions. "Canonical QFT" is the version of the theory that is found in most introductory QFT textbooks. It was developed by Feynman, Dyson, and their colleagues. Mathematical physicists concerned with the lack of mathematical rigor in canonical QFT adopted an axiomatic approach to the theory. Prominent contributors to this tradition include Wightman and Haag.

Haag's theorem pinpoints the source of the problem with the strategy of obtaining a quanta interpretation for an interacting system from the Fock representation for some free system. The consensus among axiomatic quantum field theorists is that Haag's theorem entails that a Fock representation for a free field cannot be used to represent an interacting field (e.g., Wightman (1967a, p.250; 1967b, p. 193; 1989, p. 610), Haag (1992, p. 55), Lopuszanski (1961, p. 747), Bratteli and Robinson (1996, p. 218), Bogolubov et al. (1975, p. 560); also Heathcote (1989, p. 91)).[14]

[14]For example, Wightman writes that "[a] *necessary* condition that a[n interacting] theory make physical sense is therefore that one use a strange representation of the commutation relations" (1967a, p. 250). A "strange representation" is any representation of the ETCCR's for a free field that is unitarily inequivalent to the Fock representation (Bogolubov et. al. 1975, p. 560).

Translated into mathematical terms, the assumption that a given interacting field *can* be represented using the Fock representation for a free field amounts to the assumption that the Fock representation $\mathcal{F}$ of the ETCCR's for the canonical free fields $(\phi_F(x), \pi_F(x))$ is unitarily equivalent to the Hilbert space representation $\mathcal{H}_I$ of the ETCCR's for the interacting fields $(\phi_I(x), \pi_I(x))$ at all times:

For all t there exists a unitary transformation $U(t) : \mathcal{F} \to \mathcal{H}_I$ such that

$$\phi_I(\mathbf{x}, t) = U(t)\phi_F(\mathbf{x}, t)U^{-1}(t); \quad \pi_I(\mathbf{x}, t) = U(t)\pi_I(\mathbf{x}, t)U^{-1}(t) \tag{4.10}$$

This assumption underlies the interaction picture representation employed in canonical QFT. The interaction picture is undermined by Haag's theorem. In short, the theorem states that if all the assumptions of the interaction picture are accepted, then $(\phi_I(\mathbf{x}, t), \pi_I(\mathbf{x}, t))$ are necessarily free fields. As Streater and Wightman put it, "the interaction picture exists only if there is no interaction" (2000, p. 166).

The version of Haag's theorem proven in Hall and Wightman (1957) establishes that, given certain physically reasonable assumptions, $U(t)$ does not exist at any time. That is, any irreducible Hilbert space representation of the ETCCR's for genuinely interacting fields $(\phi_I(x), \pi_I(x))$ is unitarily *in*equivalent to a Fock representation for any free fields at all times. The physically reasonable premises of the theorem are as follows:

(i) No states of negative energy exist.
(ii) Poincaré transformations on $\mathcal{F}$ and $\mathcal{H}_I$ are represented by unitary operators P_i $(i = F, I)$, respectively, which follows from Wigner's theorem concerning the representation of symmetries (Streater and Wightman 2000, pp. 7-8).
(iii) $(\phi_i(x), \pi_i(x))$ are covariant under Poincaré transformations P_i, which is a prerequisite from relativity theory for $(\phi_i(x), \pi_i(x))$ to be independent of inertial reference frame.
(iv) There exist unique normalizable vacuum states $|\Phi_i\rangle$ such that $P_i|\Phi_i\rangle = 0$, which follows from Wigner's classification of representations of the Poincaré group (Streater and Wightman 2000, pp. 21-2).

The final assumption is also underwritten by the quanta interpretation for a free field: the no-particle state $|0_F\rangle$ in a Fock representation coincides with the vacuum state $|\Phi_F\rangle$; the invariance of $|0_F\rangle$ under Poincaré transformations means that this state looks the same to all inertial observers.

What would happen if one paid no heed to the lesson of Haag's theorem and insisted on retaining the assumption that the Hilbert space representation for (ϕ_I, π_I) is unitarily equivalent to the Fock representation for (ϕ_F, π_F)? This is the course taken by canonical quantum field theorists, who continue to employ the interaction picture. The answer suggested by Haag's theorem is that $H_I|\Phi_I\rangle = \infty$ where H_I is the total interaction Hamiltonian. The implied expectation value $\langle_I\Phi|H_I|\Phi_I\rangle = \infty$ has been confirmed by non-perturbative calculations for bosonic $(\phi^{2n})_{s+1}$ $(n \geq 1)$ self-interaction terms in spacetime of dimension $(s+1) \geq 2$ (Glimm and Jaffe 1969). As a consequence, this representation cannot sustain a quanta interpretation for the interacting field. Recall that the justification for regarding n-particle states in a Fock representation for a free field as states in which a definite number of quanta are present is that these states possess the appropriate energy eigenvalues (see Section 2). The formal[15] meaning of $H_I|\Phi_I\rangle = \infty$ is that $|\Phi_I\rangle$ is not in the domain of H_I. This means that $|\Phi_I\rangle$ is not associated with an energy expectation value, which undercuts the justification for regarding $|\Phi_I\rangle$ as a state in which there are no quanta. However, $|\Phi_I\rangle$ is the only state that is invariant under Poincaré transformations; since $|\Phi_I\rangle$ is the only state that looks the same to all inertial observers, it is the only candidate no quanta state. Therefore, there is no state in the interaction picture that, in the presence of the interaction, can reasonably be interpreted as a state in

[15] An informal reading of $H_I|\Phi_I\rangle = \infty$ yields the same conclusion. Informally, the expression $H_I|\Phi_I\rangle = \infty$ indicates that $|\Phi_I\rangle$ is a state of infinite energy. Therefore, $|\Phi_I\rangle$ is not a physically possible state.

which no quanta are present, and the interaction picture does not support a quanta interpretation for the interacting field. Put in different terms, the Fock representation for a free field does not furnish a quanta interpretation for an interacting field.

In fact, the true situation is probably even worse than Haag's theorem indicates. Haag's theorem only pertains to the application of H_I to the vacuum $|\Phi_I\rangle$. In general, it seems likely that interactions in four-dimensional spacetime are are such that $H_I|\Psi_I\rangle = \infty$ where $|\Psi_I\rangle = U(t)|\Psi_F\rangle$ and $|\Psi_F\rangle$ is any state in $\mathcal{F}$ (see Glimm (1969, p. 104)). This means that not only is there no zero quanta state in the interaction picture representation, but there are also no one, two, three, etc. quanta states.

Before considering proposals for generalizing the definition of Fock representation to interacting fields, I will respond to an obvious objection to this line of argument. Canonical quantum field theorists are aware that the interaction picture representation produces infinite expectation values for energy and they correct this problem by renormalizing the Hamiltonian H_I. For example, they incorporate an infinite counterterm E into the renormalized Hamiltonian H_I^{ren} ($H_I^{ren} = H_I - E$) so that (at least informally) $H_I^{ren}|\Phi_I\rangle = 0$. Then, so the objection goes, there is no longer any obstacle to a quanta interpretation; $|\Phi_I\rangle$ is an eigenstate of H_I^{ren} with the correct energy eigenvalue for a state in which there are no quanta.

A complete response to this objection would be lengthy and take me far outside the scope of this chapter.[16] The brief (but adequate) response is that H_I^{ren} is not a well-defined self-adjoint operator on the interaction picture representation in virtue of the fact that it contains an infinite term. This means that time translations are not represented in the usual way by the unitary operators $\exp(-iH_I^{ren}t)$. I urge that, unless a good reason for abandoning this *prima facie* plausible assumption about how

[16] This issue is discussed further in Chapter 5.

to formulate QFT presents itself, the assumption should be retained. Haag's theorem does not furnish a good reason for abandoning this assumption because the premise that free and interacting Hilbert space representations are unitarily equivalent is unjustified. The consequence of the failure of the Stone-von Neumann theorem in QFT is that there exist uncountably many unitarily inequivalent representations of the ETCCR's; there exist many possible Hilbert space representations for an interaction that are unitarily inequivalent to the Fock representation for any free field, and one of these may admit a well-defined self-adjoint Hamiltonian operator and a unitary operator representing time translations. Constructive quantum field theorists have taken up the project of finding such representations. Due to the formidable mathematical challenges involved, they have yet to find a representation for a physically realistic interaction in four spacetime dimensions. However, this does not mean that we should settle for the renormalized interaction picture representation.

The two methods for generalizing the definition of the Fock representation for a free field evaluated in the ensuing two sections are proposals for finding a suitable Hilbert space representation for an interaction. These methods are instances of the two different approaches to treating interactions in a mathematically well-defined framework that have been developed by mathematical physicists. The method considered in the next section follows the constructive approach of constructing a Hilbert space representation for a particular interacting system. The method discussed in the following section adopts the axiomatic approach of specifying a Hilbert space representation by stipulating formal conditions.

4.4 METHOD #2: APPLICATION OF THE CONSTRUCTION THAT GENERATES A FOCK REPRESENTATION

A Fock representation for a free system is generated from the classical free field by a quantization procedure. One approach to generalizing the definition of a Fock representation to interacting fields is to apply the same mathematical construction to a classical interacting field. The mathematical construction of a Fock representation proceeds by Fourier decomposing the free field satisfying the classical field equation into positive and negative frequency parts and then promoting the coefficients in the decomposition to operators. The proposal is that a ΦOK_1 representation is the Hilbert space representation produced when this mathematical construction is applied to an interacting field. This approach seems to be the most promising one for obtaining a Hilbert space representation that supports a quanta interpretation: intuitively, an annihilation operator annihilates a quantum because it is part of a negative frequency solution to the free field equation and a creation operator creates a quantum because it is part of a positive frequency solution to the free field equation.

For the sake of definiteness and simplicity, consider a free bosonic neutral scalar field with $m > 0$, $\psi(x)$.[17] $\psi(x)$ is a classical field satisfying the classical Klein-Gordon equation

$$(\Box + m^2)\psi(x) = 0 \tag{4.11}$$

The first step in the Fock space construction is to introduce the Fourier decomposition

[17] The following is largely based on Roman (1969, pp. 48-52, 79-97, 119).

of $\psi(x)$

$$\psi(x) = \int d^4k(a^+(k)e^{ik\cdot x} + a^-(-k)e^{-ik\cdot x}) \tag{4.12}$$

where k is a four-vector,

$a^+(k) = 0$ if $k_0 < 0$ and

$a^-(k) = 0$ if $k_0 > 0$ (so $a^-(-k) = 0$ if $k_0 < 0$)

Since we are using the convention for the metric tensor $g_{\mu\nu}$ that $g_{00} = +1$, $g_{11} = g_{22} = g_{33} = -1$, e^{ikx} and e^{-ikx} are, respectively, positive and negative frequency plane wave solutions[18] of the classical Klein-Gordon wave equation (Equation (4.11)); thus, the first term of Equation (4.12) represents the positive frequency part of $\psi(x)$ and the second term the negative frequency part. $\psi(x)$ satisfies the Klein-Gordon equation; it follows from plugging $\psi(x)$ into Equation (4.11) that $a^+(k) = 0$ and $a^-(-k) = 0$ when $k^2 \neq m^2$. That is, the coefficients are only non-zero when $k_0^2 - \mathbf{k}^2 = m^2$, the relativistic constraint for a single non-interacting particle with rest mass m. This relation also implies that, when $a^+(k)$, $a^-(k)$ are non-zero, k is timelike: $k^2 = m^2 > 0$. The fact that k is timelike guarantees that the positive-negative frequency decomposition is Lorentz covariant (see the Appendix for a proof); $a^+(k)$, $a^-(k)$ are not dependent on the inertial reference frame in which the decomposition is carried out. This means that $a^+(k)$, $a^-(k)$ are candidates for being physical scalar fields. Furthermore, when $a^+(k)$ and $a^-(k)$ are promoted to operators, the field operators $\hat{a}^+(k)$ and $\hat{a}^-(k)$ are relativistically covariant.

[18] Plane wave solutions are not normalizable on infinite four-dimensional Minkowski spacetime. This does not affect the point made here. For a version of the Fock space construction that does not make this assumption see §3.2 of Wald (1994).

Since $\hat{\psi}(x)$ is Hermitian, $\hat{a}^+(k) = (\hat{a}^-(-k))^\dagger$. Setting $\hat{a}(k) = \hat{a}^-(-k)$ then gives the four-dimensional analogue of Equation (1):

$$\hat{\psi}(x) = \int d^4k(\hat{a}^\dagger(k)e^{ik\cdot x} + \hat{a}(k)e^{-ik\cdot x}) \tag{4.13}$$

In the interacting case, the crucial difference is that, of course, the classical interacting field $\varphi(x)$ no longer obeys the homogeneous Klein-Gordon equation. It might, for example, obey the following field equation derived from a Lagrangian with a $\varphi(x)^4$ self-interaction term

$$(\Box + m^2)\varphi(x) = -4\lambda\varphi(x)^3 \tag{4.14}$$

The Fourier decomposition for $\varphi(x)$ is

$$\varphi(x) = \int d^4k(b^+(k)e^{ik\cdot x} + b^-(-k)e^{-ik\cdot x}) \tag{4.15}$$

where k is a four-vector,

$b^+(k) = 0$ if $k_0 < 0$ and

$b^-(k) = 0$ if $k_0 > 0$

It is possible to carry out this Fourier decomposition; however, plugging $\varphi(x)$ into the interacting field equation does not yield the constraint $k^2 = m^2$. The consequence is that, unlike the free field case, k will, in general,[19] not be timelike: $k^2 \neq m^2$, so there is no guarantee that $k^2 > 0$. As a result, the decomposition in terms of functions $b^+(k)$, $b^-(k)$ is typically not covariant (Roman 1969, p. 119). Furthermore, if $b^+(k)$, $b^-(k)$ were promoted to field operators, they would also fail to be covariant in general;

[19]This is a very minor qualification: for very special interacting field equations, it may happen to be the case that $k^2 > 0$.

the field operators $\hat{b}^{+}(k)$, $\hat{b}^{-}(k)$ would be inertial reference frame dependent, and therefore not candidates for physical fields.

This is a fatal flaw in the strategy of using the Fourier decomposition of an interacting field to obtain a ΦOK_1 representation for it. *A fortiori*, this procedure does not yield a quanta interpretation for an interacting system.[20]

The failure of Lorentz covariance furnishes sufficient reason to reject the ΦOK_1 representation for an interacting field; however, for the sake of completeness, it is necessary to investigate another obstacle to obtaining a quanta interpretation for this representation. In a nutshell, the other problem is that there are special circumstances which make Fourier analysis an appropriate technique to employ in the free case, but not the interacting case. This problem is unrelated to relativistic considerations. Recall that, in the free case, the definitive argument for regarding states of the form $\hat{a}^{\dagger}(k)|0\rangle$ as states in which a single quantum is present is that there is a correlation between the energies of these states and the energies of the corresponding solutions to the classical field equation. Formally, the mapping $K : S \to \mathcal{H}$ from the space of positive frequency solutions of the classical field equation S to states in the one-particle Hilbert space $\mathcal{H}$ can be set up using the Fourier decomposition. K is specified by setting $e^{ik\cdot x} \to \hat{a}^{\dagger}(k)|0\rangle$ when $k_0 > 0$ and $k^2 = m^2$ and $e^{ik\cdot x} \to 0$ when $k_0 < 0$ and $k^2 = m^2$. This stipulation fully specifies K because any solution of the classical Klein-Gordon equation can be Fourier analyzed in terms of $e^{ik\cdot x}$ with $k^2 = m^2$. It is a contingent fact that functions of the form $e^{ik\cdot x}$ $(k^2 = m^2)$ are themselves solutions of the Klein-Gordon equation; this happy accident enables us

[20]Since Fleming (2001) argues for a reference frame dependent notion of localized quanta for a free system, it is possible that he may also be willing to accept reference frame dependent annihilation and creation field operators for an interacting system. However, ultimately, this concession does not help because such a proposal would still face the additional obstacles outlined in the following paragraphs.

to describe the component S of the mapping K in terms of solutions to the Klein-Gordon equation as well as in terms of Fourier modes $e^{ik\cdot x}$ ($k^2 = m^2$).

Consider what happens when one tries to define an analogous mapping K' for an interacting field $\hat{\varphi}(x)$. Since we know that, in general, the Fourier decomposition yields operators which are not Lorentz covariant, restrict attention to a single inertial frame of reference. The Fourier decomposition gives the following mapping: $K' : X \to \mathcal{H}_I$ is fully specified by setting $e^{ik\cdot x} \to \hat{b}^+(k)|0\rangle$ where $k_0 > 0$ and k satisfies the constraint obtained by substituting the expression for $\varphi(x)$ given by the Fourier decomposition into the interacting field equation and determining the values of k for which $b^+(k)$, $b^-(k)$ are non-zero;[21] again, Fourier analysis guarantees that this fully specifies K'. In this case, functions of the form $e^{ik\cdot x}$ ($k_0 > 0$ and k satisfies the constraint) are not solutions of the interacting field equation. Consequently, X is not the set of positive frequency complex solutions to the classical interacting field equation. Fourier analysis guarantees that X contains the set of positive frequency complex solutions, but in addition X also contains other elements. One way of proceeding is to isolate the subset $\bar{S} \subset X$ that contains the positive frequency complex solutions and define the partial mapping $\bar{K} : \bar{S} \to \mathcal{H}_I$. Assuming that the vector addition operation is chosen to be arithmetic addition, $\bar{S}$ is not a vector space because it is not closed under this operation: the interacting field equation is non-linear so the sum of two solutions will in general not be a solution. Since $\bar{S}$ is not a vector space, it cannot be taken to be the vector space for the Hilbert space $\mathcal{H}_I$. This is a problem because the specification of $\mathcal{H}_I$ is an essential component of the 'Fock' quantization.

To get around this difficulty, employ the original mapping $K' : X \to \mathcal{H}_I$, which

[21] It is not necessary to use the Fourier decomposition to determine this constraint; the Fourier transform $\varphi(x) = \int d^4k b(k) e^{ikx}$ is sufficient for this purpose.

contains the vector space X, for the purpose of the quantization, but appeal to the partial mapping $\bar{K} : \bar{S} \rightarrow \mathcal{H}_I$ to justify the interpretation of $\mathcal{H}_I$ as the one-particle Hilbert space containing one-particle states. The interpretation of $\mathcal{H}_I$ is secured by establishing a correspondence between the energies of classical solutions and quantum states in $\mathcal{H}_I$. But there is at least one serious problem with this maneuver. In the free case, K maps positive frequency complex solutions of the Klein-Gordon equation of the form $e^{ik \cdot x}$ to Hilbert space states of the form $\hat{a}^{\dagger}(k)|0\rangle$ and guarantees that the energy of such a classical state coincides with the energy ascribed to the corresponding quantum state. In the interacting case, $\bar{K}$ maps an integral over $e^{ik \cdot x}$ to an element of X. In the free case, the imposition of finite volume or periodic volume boundary conditions allows the integral to be replaced by an infinite sum. This may not hold in the interacting case, but assume for the moment that it does (if not, then this could introduce the additional complication of the Hilbert space associated with X being nonseparable). Then $\bar{K}$ maps an infinite sum over $e^{ik \cdot x}$ to a superposition $b^{+}(k_1)\hat{b}^{+}(k_1)|0\rangle + b^{+}(k_2)\hat{b}^{+}(k_2)|0\rangle + ...$ There are two problems with this. First, following the same logic as in the free case, if anything should be identified as a one-particle state in $\mathcal{H}_I$, it is not a state like $\hat{b}^{+}(k_1)|0\rangle$, but a superposition of such states, $\hat{b}^{+}(k_1)|0\rangle + \hat{b}^{+}(k_2)|0\rangle + ...$ This confounds the usual quanta interpretation of a Fock representation. Second, there is no reason to expect that the set of superpositions given by $\bar{K}$ contains a set of pairwise orthogonal vectors that spans the Hilbert space because the coefficients $b^{+}(k_i)$ and the k_i's are fixed by the interacting field equation. This line of argument supplies the justification for Redhead's assertion that, for interacting fields, "the equations of motion are in general non-linear so unlike the free-field case we cannot reduce the problem to independent harmonic oscillators (the normal modes)" (1988, p. 20).

In response to the failure of the method of Fourier decomposing an interacting field to yield a quanta interpretation, one might consider generalizing the construction. Instead of Fourier decomposing the classical interacting field into functions of the form $e^{ik\cdot x}$, one might attempt to decompose it into functions of some other form. A suggestion along these lines is mooted in Huggett and Weingard (1994) and Huggett (2000). Huggett floats–but ultimately rejects[22]–the possibility of extending the oscillator analogy to the interacting case in the following way: "[f]or an interacting field the oscillators do not move independently, but as if they were interconnected: there might be further springs, one between any pair of bobs" (p. 628). Translated into the terms of the present discussion, the suggestion is that instead of decomposing the field into independent oscillators–the plane waves $e^{ik\cdot x}$–the field should be decomposed into coupled oscillators, which are represented by functions of some other form. Huggett and Weingard suspect that this is not possible (p. 376). This is a reasonable conjecture because the proposal faces significant obstacles from two sources. First, there is no guarantee that a function can be decomposed using an arbitrary set of functions; the set of functions of the form $e^{ik\cdot x}$ is special in this respect. Second, even if a workable alternative to Fourier analysis were identified, this resulting decomposition might very well fail to be Lorentz covariant. Since these challenges are both substantial, it seems safe to conclude that it is not possible to obtain an analogue of the Fock representation suited to an interacting field by applying an analogue of the mathematical construction that produces the Fock representation for a free field.

[22]He argues that, in any case, "if the bobs do not move as SHO's but in some more complex way then there are no SHO's and by analogy no quanta in the field" (p. 628). He may have in mind the point made above that the relativistic energy constraint $k^2 = m^2$ only holds for the Klein-Gordon equation, which can be interpreted as describing an infinite collection of SHO's.

4.5 METHOD #3: STIPULATION OF FORMAL CONDITIONS

The method for obtaining a ΦOK_1 representation pursued in the previous section was to quantize a given classical interacting field in the same manner in which a given classical free field is quantized to produce a Fock representation. Following the failure of this method, a different method for extending the definition of "Fock representation" to interactions will be pursued in this section. The strategy is to arrive at a Hilbert space representation of the ETCCR's by stipulating formal conditions on the field operators rather than by quantizing a classical field. This strategy is characteristic of the axiomatic approach to QFT. The definition of ΦOK_2 representation set out below was first proposed by Wightman, a leading practitioner of axiomatic QFT, and his collaborators. The focus is on the product of the quantization process for a free field instead of the quantization process itself; the formal conditions are gleaned from the properties of a Fock representation listed in Section 2. The hope is that while the quantization procedure for a free field does not produce the desired result when applied to an interacting field, the formal product of the quantization procedure for free systems can be extended to interacting systems and possesses the appropriate features. As we shall see, this wish is not fulfilled.

In order to properly define a ΦOK_2 representation, the loose characterizations of operators employed to this point must be refined. In order to be well-defined as operator-valued distributions, the fields must be smeared with test functions. Let $\mathcal{T}$ be a real vector space with scalar product (f, g) and norm $\|f\| = (f, f)^{\frac{1}{2}}$ that serves as the test function space; then, for example, $\phi(f, t) = \int \phi(\mathbf{x}, t) f(\mathbf{x}) d^3x$ for $f \in \mathcal{T}$.

Let $\phi(f, t)$ satisfy some quantum field equation[23] and let $\pi(f, t)$ be its conjugate

[23] Specification of a field equation may not be sufficient to isolate a representation that is unique up to unitary equivalence in all cases, but this potential problem is unrelated to the present discussion.

momentum field. The definition of a ΦOK_2 representation of the ETCCR's[24] for $\phi(f,t)$, $\pi(f,t)$ has two parts (Gårding & Wightman 1954; Wightman & Schweber 1955):

1. Define, for any $f \in \mathcal{T}$, the self-adjoint operator

$$c(f,t) = \frac{1}{\sqrt{2}}[\phi(f,t) + i\pi(f,t)] \tag{4.16}$$

and define $c^\dagger(f,t)$ as the adjoint of $c(f,t)$.

2. The ΦOK_2 representation is the irreducible representation of the ETCCR's for which there exists a normalizable vector (the "no-particle state") $|0\rangle$ such that[25]

$$c(f,t)|0\rangle = 0 \text{ for all } f \in \mathcal{T} \tag{4.17}$$

These conditions are familiar from the discussion of the Fock representation for a free field in Section 2. This definition is clearly not restricted to free fields; $\phi(f,t)$ may satisfy any field equation. Wightman and Schweber explicitly state that this is the case (1955, p. 816).[26] Unlike the operators $\hat{b}(k)$, $\hat{b}^\dagger(k)$, which were defined using the Fourier decomposition of the classical field, the operators $c(f,t)$, $c^\dagger(f,t)$ are Poincaré covariant since $\phi(f,t)$, $\pi(f,t)$ are Poincaré covariant.[27] These two conditions pick out a representation of the ETCCR's that is unique up to unitary

See Baez & Zhou (1992).

[24]There is an analogous definition of ΦOK representation for the ETCAR's (equal-time canonical anti-commutation relations) (Wightman & Schweber 1955, pp. 816-820).

[25]The cyclicity of $|0\rangle$ with respect to $c^\dagger(f,t)$ follows from the assumption that the Weyl representation of the ETCCR's for $\phi(f,t)$, $\pi(f,t)$ is irreducible, which implies that every non-zero vector in the representation is cyclic with respect to $e^{i\phi(f,t)}$, $e^{i\pi(f,t)}$ (Emch 1972, p. 84).

[26]They use the terminology of "Heisenberg or interaction representation quanta." In the interaction representation the field obeys the free Klein-Gordon equation and in the Heisenberg representation the field evolves with the full interaction Hamiltonian.

[27]Unlike $\hat{b}(k)$, $\hat{b}^\dagger(k)$, the operators $c(k)$, $c^\dagger(k)$ do not arise from coefficients in the Fourier decomposition of the classical interacting field.

equivalence[28] (Gårding and Wightman 1954, pp. 624-5; see also Wightman and Schweber 1955, pp. 819-22). (Of course, while the ΦOK_2 representation *for a field* $\phi(f,t)$ *satisfying a specified field equation* is unique up to unitary equivalence, the ΦOK_2 representations for fields $\phi(f,t)$ and $\phi'(f,t)$ satisfying different field equations may not be unitarily equivalent.) Since the Fock representation for a free field satisfies these two conditions (see Section 2), when $\phi(f,t)$ satisfies a Klein-Gordon equation, the ΦOK_2 representation coincides with the Fock representation.

A ΦOK_2 representation also shares some of the other properties of a Fock representation rehearsed in Section 2. It follows from the definitions in Condition 1 and the fact that $\phi(f,t)$, $\pi(f,t)$ satisfy the Weyl form of the ETCCR's that $c(f,t)$,

[28] Gårding and Wightman's proof establishes that, if the representation exists, then it is unique up to unitary equivalence, but it is not clear that it establishes that the representation necessarily exists for an interacting field (i.e., that there exists a representation containing a normalizable $|0\rangle$). I will assume that such a representation exists and argue that, even if the representation does exist, it cannot sustain a quanta interpretation.

$c^\dagger(f,t)$ satisfy the usual ETCCR's[29]

$$[c(f,t), c(g,t)] = 0 = [c^\dagger(f,t), c^\dagger(g,t)] \tag{4.18}$$
$$[c(f,t), c^\dagger(g,t)] = (f,g)$$

Most significant for our purposes is the fact that a ΦOK_2 representation has a well-defined total number operator $N(t) = \sum_{j=1}^{\infty} c^\dagger(f_j,t)c(f_j,t)$ where f_j are members of any orthonormal set in $\mathcal{T}$ (Wightman & Schweber 1955, p. 822; Gårding & Wightman 1954, p. 625; Wightman 1967b, p. 188). Moreover, any representation in which $N(t)$ is well-defined (where $c^\dagger(f_j,t)$, $c(f_j,t)$ are defined as in Condition 1) is a direct sum of representations in which both of the conditions hold (Dell'Antonio, Doplicher & Ruelle 1966).[30]. This result provides assurance that the ΦOK_2 representation is

[29]Consider the case in which $\phi(f,t)$ is a free field. Note that Wightman's definitions imply

$$\begin{aligned}\phi(\mathbf{x},t) &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2}}[c^\dagger(\mathbf{k},t)e^{ik\cdot x} + c(\mathbf{k},t)e^{-ik\cdot x}] \\ \pi(\mathbf{x},t) &= \int \frac{id^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2}}[c^\dagger(\mathbf{k},t)e^{ik\cdot x} - c(\mathbf{k},t)e^{-ik\cdot x}]\end{aligned}$$

The first expression differs from (4.2) in Section 2 by a factor of $\omega_{\mathbf{k}}^{-\frac{1}{2}}$ inside the integral and the second differs from (4.3) by a factor of $\omega_{\mathbf{k}}^{\frac{1}{2}}$ inside the integral. These factors cancel when $\phi(\mathbf{x},t)$, $\pi(\mathbf{x},t)$are multiplied, so Wightman's $\phi(f,t)$, $\pi(f,t)$obey the same ETCCR's as their counterparts in section 2. In $(\mathbf{k},t)$-space

$$[c(\mathbf{k}_1,t), c(\mathbf{k}_2,t)] = 0 = [c^\dagger(\mathbf{k}_1,t), c^\dagger(\mathbf{k}_2,t)]$$
$$[c(\mathbf{k}_1,t), c^\dagger(\mathbf{k}_2,t)] = \omega_{\mathbf{k}}\delta(\mathbf{k}_1 - \mathbf{k}_2)$$

which differs from the ETCCR's in Section 2 by a factor of $\omega_{\mathbf{k}}$.

Note also that, if $\phi(f,t)$, $\pi(f,t)$ are regarded as fields at arbitrary times rather than a fixed time, $\pi(f,t) \neq \frac{d\phi'(f,t)}{dt}$; $\pi(f,t) = \frac{d\phi'(f,t)}{dt}$ where $\phi'(\mathbf{k},t) = \frac{1}{\omega_{\mathbf{k}}}\phi(\mathbf{k},t)$.

[30]Chaiken (1968) proves the same result for an N specified using the exponentiated form of the ETCCR $Na^*(\varphi) = a^*(\varphi)(N+1)$ (where $a^*(\varphi)$ is the creation operator for the wavefunction φ) when the spectrum of N is bounded from below (1.3 Theorem, p. 167). (cf. Halvorson and Clifton (2002, p. 203 n. 5)).

the unique generalization of a Fock representation capable of furnishing a quanta interpretation: imposing the definition of $c^{\dagger}(f,t)$, $c(f,t)$ set out in Condition 1 and the condition that $N(t)$ be well-defined also yields a direct sum of ΦOK_2 representations.[31]

A ΦOK_2 representation for an interacting field $\phi(f,t)$ possesses a total number operator but, once again, this is an insufficient basis for an interpretation in terms of particlelike entities. In addition, it is necessary to establish that each of the eigenstates of N is a state in which the system possesses the appropriate physical properties for a state in which the indicated number of quanta is present. It turns out that this cannot be done if $\phi(f,t)$ is an interacting field. Recall from the discussion in Section 2 that, in order for the no-particle state to have the correct energy for a state in which no quanta are present and to possess the expected property of invariance under Poincaré transformations, the no-particle state must coincide with the vacuum. One would expect a state in which no quanta are present to exhibit these properties regardless of whether the system is free or interacting. If, in addition to the two conditions stated above, it is assumed that the no-particle state in a ΦOK_2 representation is the vacuum state, then it can be proven that $\phi(f,t)$ cannot be a non-trivial interacting field (Emch 1972, p. 242). Essentially,[32] this result is proven by calculating the vacuum expectation values of products of the fields. The vacuum

[31] It is worth pointing out the differences between this definition of a ΦOK_2 representation and another rigorous generalization of the definition of Fock representation that has recently been discussed by philosophers (see, e.g., Arageorgis et. al. (2002)), even though the alternative definition is inapplicable to interactions. Kay (1978) sets out a definition of Fock representation that is suitable for arbitrary globally hyperbolic spacetimes, but is only applicable to free fields. In a stationary spacetime, this definition picks out a representation that is unique up to unitary equivalence. Briefly, for free fields on Minkowski spacetime, the main differences between Kay's definition and the definition of a ΦOK_2 representation are that (1) while Kay's definition requires that the no-particle state coincide with the physical vacuum, this may not be true in a ΦOK_2 representation (see below) and (2) Condition 1 for a ΦOK_2 representation is not a part of Kay's definition.

[32] Emch states and proves this result within the algebraic framework for QFT. For a brief overview of the relevant features of the algebraic approach, see the Appendix to Earman and Fraser (2005).

expectation value of any product of n (where $n > 2$) fields satisfying (4.16) to (4.18) can be expressed solely in terms of the vacuum expectation values of products of two fields. This implies that the S-matrix is the identity (Greenberg and Licht 1963). That is, no interaction occurs; the initial state of the system is identical to the final state. This result entails that, in the presence of a non-trivial interaction, the no-particle state is not the vacuum state. Therefore, a $\Phi O K_2$ representation does not furnish a quanta interpretation for an interacting system.

To see that a $\Phi O K_2$ representation for an interacting field also fails to ascribe the correct properties to higher particle states, consider the example of a Lagrangian with a ϕ^4 interaction term. The following is the formal expression for the Hamiltonian H^{ϕ^4} in terms of $c^\dagger(\mathbf{k}, t)$, $c(\mathbf{k}, t)$, the 'unsmeared' Fourier transforms of the annihilation and creation operators defined in Condition 1 (where : : represents normal ordering):

$$\begin{aligned} H^{\phi^4} = \int dk c^\dagger(\mathbf{k}, t) c(\mathbf{k}, t) + \\ \lambda \int\int\int\int dk dk' dk'' dk''' \frac{\delta(\mathbf{k} + \mathbf{k}' + \mathbf{k}'' + \mathbf{k}''')}{4} : [c^\dagger(\mathbf{k}, t) + c(\mathbf{k}, t)] \cdots \\ [c^\dagger(\mathbf{k}''', t) + c(\mathbf{k}''', t)] : \end{aligned} \quad (4.19)$$

It can be confirmed that the no-particle state $|0\rangle$ does not coincide with the vacuum: the interaction term contains one term with four creation operators; as no other term in H^{ϕ^4} contains four creation operators, this term is not offset by any other term, which means that $H^{\phi^4}|0\rangle \neq 0$. Furthermore, $|0\rangle$ is not an eigenstate of H^{ϕ^4}, so it is not possible to make $|0\rangle$ coincide with the vacuum by adding a finite constant to H^{ϕ^4}.

The result of applying H^{ϕ^4} to a one-particle state is the following:

$$H^{\phi^4} c^\dagger(\mathbf{k}, t)|0\rangle = \sqrt{\mathbf{k}^2 + m^2} c^\dagger(\mathbf{k}, t)|0\rangle + \ldots \quad (4.20)$$

where the ellipsis stands in for the terms resulting from application of the interaction term of H^{ϕ^4}. Clearly, $c^\dagger(\mathbf{k},t)|0\rangle$ is not an eigenstate of H^{ϕ^4}. On many interpretations of quantum theory, this means that $c^\dagger(\mathbf{k},t)|0\rangle$ is a state which does not possess a definite value of energy. This seems strange for a state which is supposed to definitely contain a single quantum. However, set this issue aside, because there is a serious problem with interpreting $c^\dagger(\mathbf{k},t)|0\rangle$ as a state in which a single quantum is present that is independent of controversies surrounding the interpretation of quantum theory. This argument takes the form of a dilemma. There are two positions that could be taken on the correct relativistic energy possessed by a single quantum associated with the interacting field $\phi(x)$: either that it should be $\sqrt{\mathbf{k}^2+m^2}$, the relativistic energy for a single non-interacting particle, or that it should not be $\sqrt{\mathbf{k}^2+m^2}$ because the field equation for $\phi(x)$ contains a self-interaction term. If the former position is taken, then $c^\dagger(\mathbf{k},t)|0\rangle$ cannot be interpreted as a state in which one quantum is present because the expectation value for energy in this state is not $\sqrt{\mathbf{k}^2+m^2}$. If the latter position is adopted, we are left without any guidance from special relativity about what the correct energy for a single quantum state should be; consequently, we have no grounds for claiming that $c^\dagger(\mathbf{k},t)|0\rangle$ is a single quantum state. Therefore, $c^\dagger(\mathbf{k},t)|0\rangle$ cannot be interpreted as a state in which one quantum is present.

In sum, a ΦOK_2 representation does not sustain a quanta interpretation for any interacting field because the no-particle state is not the vacuum state. In addition, there are no grounds for interpreting one-particle states of the form $c^\dagger(\mathbf{k},t)|0\rangle$ as states in which one quantum is present.

In response to the ΦOK_2 representation's failure to support a quanta interpretation for an interacting system, one might want to further generalize the definition of Fock representation. A prerequisite for a quanta interpretation is that a total number operator $N(t)=\sum_{j=1}^{\infty} c^\dagger(f_j,t)c(f_j,t)$ exist, but there is no reason that $c^\dagger(f,t)$,

$c(f,t)$ must be defined as in Condition 1 on page 126. Further motivation for this move is that, in a Fock representation for a free field, Condition 1 is a product of the Fourier decomposition procedure for quantizing a classical free field. However, this proposal requires fleshing out: clearly, in the absence of a definition of $c^\dagger(f,t)$, $c(f,t)$ in terms of $\phi(f,t)$, $\pi(f,t)$, the requirement that a total number operator exist or that the no-particle state exists is insufficient to pick out a representation of the ETCCR that is unique up to unitary equivalence. But how should $c^\dagger(f,t)$, $c(f,t)$ be defined in terms of $\phi(f,t)$, $\pi(f,t)$ in the presence of an interaction? A well-motivated answer to this question would require revisiting the issue of how to quantize an interacting field. Once again, what is needed is an analogue of the Fourier decomposition of a classical free field that is appropriate for an classical interacting field and, as argued in the previous section, it is overwhelmingly unlikely that such a procedure could be found. The discussion in this section has raised another potential difficulty: it is not automatic that the one-particle states will (or even should!) possess the energies $\sqrt{\mathbf{k}^2+m^2}$. If not, then most compelling argument for interpreting one-particle states as states in which one quantum is present is undermined.

4.6 SCATTERING THEORY DOES NOT SUPPORT A QUANTA INTERPRETATION

To recapitulate the argument thus far, we set out to find a mathematical representation for an interacting system that sustains a quanta interpretation in the same manner that a Fock representation furnishes a quanta interpretation for a free system. All three methods for accomplishing this have proven unsuccessful. Generalizations of two of the strategies cannot be ruled out entirely, but success seems exceedingly unlikely. Since in QFT there is no known alternative for establishing that a free or interacting system exhibits particlelike properties, this would seem to be a fatal blow to the project of interpreting any realistic physical system–which is bound to interact–in terms of particlelike entities.

In a last-ditch attempt to save the quanta interpretation of QFT, one could try to find a convincing argument that the quanta interpretation for free systems is all that is needed to interpret an interacting system in terms of quanta. This position can be motivated by considering how particle physicists interpret a theory in terms of particlelike entities. For experimentalists, it is immaterial whether or not an interacting system can be given a quanta interpretation; this question is irrelevant because the only data collected from scattering experiments is for systems with negligibly small interactions. Roughly speaking, this is because the data is collected long before and long after the collision between the 'particles' occurs. The input of the experiment is the prepared system of particlelike entities of certain types with certain properties; this system is prepared in such a way that interactions are negligible[33] (e.g., the particlelike entities are very far apart).[34] Similarly, the output of the experiment is the measured system of particlelike entities of certain (possibly different) types with different properties; measurements are performed on the system at a time when interactions are negligible. These conditions are reflected in scattering theory:[35] it is assumed that the interacting system tends to a free system at asymptotically early and late times (i.e., as the time variable t tends to plus or

[33] Set aside the issue of whether the asymptotic system is genuinely free or only free to a very good approximation. If the latter, then it cannot be given a quanta interpretation for all the reasons stated in the preceding sections.

[34] Exclude quantum chromodynamics from consideration because, due to confinement, interacting systems approach free systems in the limit of small distance rather than large distance. See Redhead (1988, p. 21).

[35] For example, LSZ or Haag-Ruelle scattering theory. For a discussion of the latter see Chapter 3.

minus infinity). At these times, the free system can be given a Fock representation and a quanta interpretation in the usual way.

Bain has advanced a quanta interpretation for QFT based on scattering theory. He proposes that "a 'particle' be considered a system that minimally possesses an asymptotic state (i.e., a system that is free for all practical purposes at asymptotic times)" (Bain 2000, p. 394). Since the asymptotic state is a free state, it possesses a Fock representation and can be given an interpretation in terms of quanta. Bain urges that this is sufficient grounds for interpreting interacting states at intervening times as states in which quanta are present. Teller opposes this position, arguing that such a quanta interpretation is "severely limited" because the total number operator N_{free} defined in the Fock representation at asymptotic times does not exist at intervening finite times (1995, p. 123). Bain counters that

> a particle interpretation should not be dependent on the existence of a (free field) [total] number operator. To require otherwise seems to me to be placing undue emphasis on the free theory. ... Whether or not [a system that possesses an asymptotic state] has a corresponding [total] number operator, I would claim, is irrelevant. (Bain 2000, p. 394)

The analysis in the preceding sections provides a basis for settling this dispute about whether the existence of Fock representations for asymptotic free systems is a sufficient basis on which to regard quanta as fundamental entities in our ontology. The evidence weighs against Bain's definition of 'particle'. The underlying problem is a weakness in Bain's approach to ontology. The point at issue is whether entities with certain properties–particlelike properties–exist. More precisely, does QFT support the inclusion of particlelike entities as fundamental entities in our ontology? Bain contends that the fact that there is no free field total number operator N_{free} at intervening times is irrelevant. But, since the question is whether there are entities with particlelike properties at finite times, what is certainly relevant is whether the interacting system contains entities that are countable. In terms of the formalism, it is relevant whether a surrogate for N_{free} is available at finite times. Bain does not offer any substitute for N_{free}; he does not point to any evidence for the existence of particlelike entities in the presence of an interaction (e.g., that states of an interacting system possess the expected energies for states in which a definite number of quanta are present). Instead, Bain redefines 'quanta' to fit the evidence that QFT does provide. But lack of evidence is not the sort of thing that can be defined away in the realm of ontology. The 'quanta' in Bain's scheme may not possess any particlelike properties in the presence of interactions. Appealing to the theory of a free system with which the interacting system is associated in an idealized infinite past or future does not fill this gap in the evidence.

The investigation conducted in the preceding sections establishes that the evidence that Bain omits in fact cannot be supplied. It is not possible to produce a mathematical representation for an interacting system that is relevantly similar to the Fock representation for a free system. No substitute for N_{free} is available; it is not possible to identify n-particle states which possess the correct energies. Therefore, we cannot ascribe particlelike properties to a system in a scattering experiment at finite times, and this final attempt to extend the quanta interpretation to interacting systems is also unsuccessful.

4.7 CONCLUSION

A Fock representation sustains a quanta interpretation for a free field. The goal of this chapter was to determine whether an interacting field possesses a mathematical representation that sustains a quanta interpretation in the same manner. The simplest solution would have been to use the Fock representation for a free field to represent an interacting field. This option was ruled out by Haag's theorem. It then became necessary to find another Hilbert space representation for an interacting field. In principle, there are two ways to generalize the definition of a Fock representation to cover interacting systems, each of which is an instance of one of the two approaches to arriving at a formulation of QFT. The constructive method is to quantize a classical interacting field by carrying out the same mathematical construction that, for a free field, generates a Fock representation. The axiomatic method is to pick out a unique (up to unitary equivalence) Hilbert space representation of the ETCCR by stipulating that it share a set of formal properties with a Fock representation. Neither of these approaches is successful, and the prospects for success of generalizations of these methods are negligibly small. A Hilbert space representation cannot be constructed by Fourier decomposing a classical interacting field because the resulting expressions are not relativistically covariant, and therefore are not candidates for physical fields in relativistic QFT. The ΦOK_2 representation for an interacting field that is picked out by formal conditions does not support a quanta interpretation because, for all non-trivial interactions, the no-particle state does not coincide with the vacuum and, typically, the argument that one-particle states have the energy expectation values that special relativity assigns to single particle states is undercut.

These conclusions are significant for metaphysics as well as for the foundations of QFT. The metaphysical implication is that QFT does not support the inclusion of particlelike entities as fundamental entities in our ontology. At least on the surface, QFT is a theory of fields. The only known method of interpreting a QFT in terms of particlelike entities is the quanta interpretation that naturally arises from the Fock representation for a free system. The arguments presented here establish that it is not possible to extend this quanta interpretation to an interacting system. One response would be to find another way of interpreting interacting fields in terms of particlelike entities, one that does not require a Fock-type Hilbert space representation. But this is a program, not a solution, and even at that a program without an obvious starting point. Therefore, since in the real world there are always interactions, QFT does not furnish grounds for regarding particlelike entities as fundamental constituents of reality.

Of course, this conclusion is compatible with permitting particlelike entities some less fundamental status in our ontology. For example, one might consider the 'particle' concept to be an emergent concept (see, e.g., Wallace (2001)). Alternatively, the quanta concept may be regarded as a concept that is only approximately or ideally applicable because it is restricted to free systems. In the context of a scattering experiment, free systems occur in the idealized limits of infinitely early and late times. Teller suggests that "we regard this idealization as one of the ways in which the interpretation's similarity relation between the model and the world is only approximate" (1995, p. 124). However, the important question–which remains outstanding–is not the status of quanta, but what fundamental entities *are* allowed into our ontology by QFT.

The reasons that an interacting system cannot be given a quanta interpretation are also illuminating for the foundations of QFT. Scattered remarks in the philosophical literature on QFT often convey the impression that interacting fields do not admit Fock representations due to problems representing two-particle (and three-particle, etc.) states in such a framework (for an explicit example see Bain (2000, p. 393)). One rationale for this assessment is that, for a free system, the energy of a two-particle state is simply the sum of energies of the component one-particle states; however, this is not the case for an interacting system. This property of a free system is built into the definition of a Fock representation: a two-particle state is the direct product of component one-particle states. However, the above analysis reveals that this discrepancy is not the primary difficulty. Problems already emerge at the level of no-particle and one-particle states. Furthermore, the source of these problems is the special theory of relativity. Haag's theorem relies on relativistic premises;[36] the Fourier decomposition is not covariant under Poincaré transformations; the no-particle state in a ΦOK_2 representation is not invariant under Poincaré transformations; and special relativity may not supply the correct assignment of energies to one-particle states. For a free system, special relativity and the linear field equation conspire to produce a quanta interpretation. For an interacting system, the combination of special relativity and the nonlinear field equation is not so fortuitous; as a result, there is no quanta interpretation and there are no quanta.

4.8 APPENDIX

This Appendix contains a proof of a claim made in Section 4. To prove that if k is timelike (i.e., $k^2 = c^2k_0^2 - \mathbf{k}^2 > 0$) then the positive-negative frequency decomposition is Lorentz covariant, consider two cases: (1) there exists an inertial reference frame (IFR) in which $k_0 < 0$ so the coefficient $a^+(k) = 0$ in Equation (4.12); and, (2) there exists an IFR in which $k_0 > 0$ so the coefficient $a^-(k) = 0$. If in Case (1) $k_0 < 0$ in all IFR's and in Case (2) $k_0 > 0$ in all IFR's then the positive-negative frequency decomposition is covariant.

Case (1): $k_0 < 0$

[36] See Section 2.1 of Chapter 1 for details.

Transform to another IFR:

$$k_0 \to k_0' = \frac{k_0 - \frac{\mathbf{V}\cdot\mathbf{k}}{c^2}}{\sqrt{1 - \frac{\mathbf{V}^2}{c^2}}} \tag{4.21}$$

Then prove that $k_0' < 0$ for all possible values of $\mathbf{k}$, $\mathbf{V}$:

The denominator can be neglected because $\sqrt{1 - \frac{\mathbf{V}^2}{c^2}} > 0$
$\frac{|\mathbf{V}|}{c^2} \leq \frac{1}{c}$ because $|\mathbf{V}| \leq c$
$|\mathbf{k}| < c|k_0|$ because $k_0^2 - \mathbf{k}^2 > 0$
So $\frac{|\mathbf{V}||\mathbf{k}|}{c^2} < |k_0|$
By Schwarz's inequality, $|\frac{\mathbf{V}\cdot\mathbf{k}}{c^2}| < |k_0|$
Therefore, $k_0' < 0$ in all IFR's.

Analogously, for Case (2) $k_0 > 0$ in all IFR's. Therefore, the positive-negative frequency decomposition is Lorentz covariant.

5.0 CHAPTER 5: IMPLICATIONS OF HAAG'S THEOREM FOR THE INTERPRETATION OF QFT

5.1 INTRODUCTION

The discussion in the preceding chapters raises a *prima facie* problem for the philosophical project of interpreting QFT. (By which I mean the project of answering the question "if QFT were true, what would reality be like?") Chapter 3 explained that the canonical representation, the canonical representation with cutoffs, and the Glimm-Jaffe representation for a $(\phi^4)_2$ interaction are empirically equivalent.[1] Furthermore, one would expect the canonical and mathematically rigorous representations for any interaction to agree because this is a means of establishing that a mathematically rigorous model is a model for a particular interaction (see Glimm (1969a, p. 103)). Chapter 4 laid the groundwork for showing that different formulations support different interpretations. In short, a canonical representation, a cutoff

[1] The renormalized interaction picture is empirically equivalent to the interaction picture with a volume cutoff in the sense that, in the limit as the volume cutoff is removed, the S-matrix elements of the interaction picture with volume cutoff approach the S-matrix elements of the renormalized interaction picture. Thus, the sets of S-matrix elements generated by these two formulations can be brought into arbitrarily close agreement by choosing a large enough volume cutoff. It has been verified that the sets of S-matrix elements yielded by the Glimm-Jaffe $(\phi^4)_2$ model are in agreement with those yielded by renormalized canonical QFT (modulo the complications arising from the use of a perturbative expansion in the latter context).

representation, and a mathematically rigorous representation will disagree on the existence of quanta. (I will elaborate on this in the next section.) So it seems that our interpretation of QFT will depend on which formulation of QFT we choose to interpret. In view of the empirical equivalence of the formulations, this appears to be a classic case of underdetermination of theory by all possible evidence. How then are we to interpret QFT?

It is interesting to note that there is no precedent for this problem in ordinary, non-relativistic quantum mechanics (NRQM). There is consensus among the practitioners of NRQM about how to formulate the theory and this standard formulation is the starting point for philosophical reflection on NRQM. Of course, the by-now familiar interpretive difficulties have led some to call into question the completeness or correctness of the standard formulation, but these modifications are proposed in response to the standard formulation. For QFT, in contrast, there is no single agreed-upon standard formulation of the theory and, therefore, no natural starting point for the interpretive project. An underlying reason for this difference between NRQM and QFT is that there are only a finite number of degrees of freedom in NRQM, but an infinite number in QFT, which means that there are unitarily inequivalent Hilbert space representations of the canonical commutation relations available in QFT. Different formulations of QFT are the product of different approaches to the occurrence of unitarily inequivalent representations.

In this chapter, I will argue for a solution to the problem of how to proceed with the interpretation of QFT in light of the existence of alternative formulations. I will argue that, for the purposes of interpretation, we can pick one formulation over the others; an interpretation of QFT should be based on the mathematically rigorous model for an interaction rather than the canonical or cutoff representation for the interaction. To some philosophers of physics, this may seem to be the obvious choice. When studying the foundations of a theory, it is much simpler to work with a nice mathematically well-defined formulation of a theory than with a mathematically ill-defined one. However, for QFT, this preference raises a practical problem: to date, rigorous mathematical models have only been constructed for certain physically unrealistic interactions. For example, no model exists for any interaction on four-dimensional spacetime. For this reason, some philosophers of physics regard canonical and cutoff QFT as more informative about realistic systems than constructive or axiomatic QFT. For example, Teller explains that he does not discuss "formal and rigorous work in axiomatic field theory" in his book because "[a]lthough [axiomatic field theory] is a useful enterprise in the study of formal properties of quantum field theories, axiomatic quantum field theory as it exists today does not appear usefully to describe real physical phenomena" (Teller 1995, p. 146, fn. 22). This is a natural conclusion to draw, but I contend that it is incorrect. As counter-intuitive as it may seem, the rigorous $(\phi^4)_2$ model is a better guide to the ontology of a given real system than either the canonical or cutoff representations for that system!

The arguments for the claim that we should base our interpretation of QFT on the $(\phi^4)_2$ model rather than on canonical or cutoff QFT are laid out in sections 4 and 5, respectively. As a prelude to these arguments, Section 2 contains a discussion of the different interpretations supports by these formulations. Bear in mind that the interpretive question is "if QFT were true, what would reality be like?" The constructive and axiomatic programs for QFT are discussed in Section 3.

5.2 THE CANONICAL, CUTOFF, AND GLIMM-JAFFE REPRESENTATIONS ARE GENUINELY DISTINCT (FOR INTERPRETIVE PURPOSES)

Since the canonical, cutoff, and Glimm-Jaffe representations for the $(\phi^4)_2$ interactions yield the same predictions, one might wonder whether these representations are genuinely distinct (or whether, for example, they might merely be notational variants). It is clear that proponents of these approaches to QFT believe that they differ significantly. As Wightman described the situation in a 1962 lecture, "[t]he root-mean-square deviation from the mean of opinion on what is a sensible thing to try to do in elementary particle theory seems to be one of those unrenormalizable infinities one hears about" (Wightman 1963, p. 11). From the perspective of philosophy rather than sociology, the canonical and cutoff representations are genuinely distinct from the Glimm-Jaffe representation because they support different ontologies.

Consider the cutoff and Glimm-Jaffe representations. The upshot of Chapter 4 is that the Glimm-Jaffe representation does not admit a quanta interpretation. In contrast, the cutoff representation can be given a quanta interpretation. Recall from Chapter 4 that, essentially, the quanta interpretation rests on the fact that the Hilbert space for a free system is a Fock space representation; this representation has a basis of state vectors–the n–particle state vectors–each of which can be interpreted as representing a state in which a definite number of quanta is present. In a cutoff representation, this quanta interpretation can be extended to interacting systems. Since the number of degrees of freedom is finite, the Stone-von Neumann theorem applies and all representations of the ETCCR's are unitarily equivalent. In particular, the representation for the interacting system is unitarily equivalent to the

Fock representation for the corresponding free system. Therefore, any state vector describing the interacting system can be written in terms of the n-particle states for the corresponding free system. Thus, in the cutoff representation, the interacting system can also be given a quanta interpretation. The cutoff and the Glimm-Jaffe representations disagree about the existence of quanta.

The canonical representation can also be given a quanta interpretation. By assumption,[2] there is some time t_0 at which the Hilbert space representation for the interaction coincides with the Fock space $\mathcal{F}$ for the corresponding free system. Also by assumption, time evolution is unitary; therefore, the Hilbert space representation for the interacting system is unitarily equivalent to $\mathcal{F}$ at all times. That is, the state vector for the interacting system is always in Fock space, so it is always possible to write it in terms of the n-particle states for the free system. Thus, using the canonical representation, the interacting system can also be given a quanta interpretation. The canonical and the Glimm-Jaffe representations also disagree about the existence of quantum particles.

The disagreement on matters of metaphysics among the three representations can also be made out in more general terms. The representations disagree about what is possible. According to the cutoff representation, it is possible for the interacting system to be in the same state as any other system, governed by any dynamics. According to the canonical representation, it is possible for the interacting system and to be in the same state as a free system. (Indeed, scattering theory in the interaction picture works on the basis of the assumption that the free and interacting

[2] These assumptions are routinely made in textbook presentations of canonical QFT but, as will be discussed in detail below, they are invalidated by renormalization. The interpretive conclusion drawn here stands nonetheless because renormalization is accompanied by the commitment that there is an informal sense in which the state of the interacting system remains in the Fock representation for the free system (see Section 4 below).

systems are actually in the same state at $t = \pm\infty$.) This disagreement is the upshot of their respective assumptions about unitary equivalence. In contrast, according to the mathematically well-defined model, it is not possible for the interacting system to be in the same state as a system governed by any other dynamics. The metaphysical disagreement between the two representations is not limited to the existence of quanta. Thus, even if we decide that QFT does not describe quanta we can expect that the representations will still disagree on matters of metaphysics.

5.3 BUT THE GLIMM-JAFFE $(\phi^4)_2$ MODEL IS ONLY A TOY MODEL!

If we are to use the Glimm-Jaffe $(\phi^4)_2$ model as a guide to ontology, we must not lose sight of the fact that this model is a toy model. Realistic systems are not confined to two-dimensional spacetime. Similarly, it must be borne in mind that it is not known whether proposed axioms for QFT hold in models for realistic systems. The Wightman axioms have been verified in all models constructed (as of 2000), but no model for an interaction on four-dimensional spacetime has been constructed (Rivasseau 2000, p. 3765). As a preamble to the ensuing arguments that the Glimm-Jaffe model nevertheless serves as a useful guide to the ontology of QFT for realistic systems, it is helpful to consider how the constructive and axiomatic programs for QFT work.

The constructive program can be conceived as breaking down the problem of finding rigorous models for realistic interacting systems into a sequence of steps. The starting point is the unrigorous interaction picture of canonical QFT, and successive steps involve treatment of more and more complex interactions by making more and more refinements to the canonical framework. The first step was the construction of the $(\phi^4)_2$ model. This entailed dropping the assumption that there is a time at which the representation for the interacting system is unitarily equivalent to the Fock representation for a free system. The second step was the construction of the $(\phi^4)_3$ model. ... And so on.

Why expect that the assumptions of the canonical framework that are dropped or refined at early stages of the program will also be dropped or similarly refined in models for realistic interactions? In particular, why expect that the assumption that there is a time at which the representation is unitarily equivalent to the Fock representation for a free system will fail to hold not only in the $(\phi^4)_2$ model, but also in models for realistic interactions? Recall (from Chapter 3) that, for the $(\phi^4)_2$ model, Haag's theorem entails that at all times the representation for the $(\phi^4)_2$ interacting system is unitarily inequivalent to any Fock representation for a free system. However, it is unlikely that Haag's theorem will suffice to establish the same result for more complex interactions since it is likely that in these models at least one other premise of Haag's theorem will also fail (e.g., that the fields obey ETCCR's[3]). But a plausibility argument can be made. The only divergence in the $(\phi^4)_2$ interaction picture representation is a divergence in the vacuum self-energy that can be treated by introducing a long-distance cutoff. The interaction picture representation for any more complex interaction exhibits the same type of divergence. Furthermore, the interaction picture representation for any more complex interaction will also contain other types of divergences and, the more complex the interaction, the more quickly

[3]In formulating his axioms, Wightman intentionally does not assume that the ETCCR's hold because he believes that in some cases it will be necessary to smear the fields in time as well as space (Streater and Wightman 2000, pp. 100-101; Wightman 1967a, p. 178). But this is a bad example because, if the fields must be smeared in time as well as space, then it is not possible to state the claim that there is a (sharp) time at which the representations (of the ETCCR's) for the interacting and free systems are unitarily equivalent!

these tend to diverge (Glimm and Jaffe 1973, p. 408-9). Hence it seems reasonable to expect that, for more complex interactions, at least as much will have to be done to solve the problem. (e.g., it will at least be necessary to drop assumption of unitary equivalence to a Fock representation, and possibly other assumptions must be dropped or refined as well). It would be very useful for the constructive program if it were possible to prove a sequence of Haag-type no-go theorems: e.g., a $(\text{Haag})^2$ theorem which establishes that the premises of the original Haag's theorem minus the premise that there is a time at which the interacting representation is unitarily equivalent to the Fock representation for a free system implies that the system under consideration is a free system when the dimension of spacetime is greater than two. Such a sequence of theorems would reveal which assumptions of the canonical framework would need to be abandoned to treat particular interactions.

The axiomatic program adopts the opposite strategy: instead of starting from the canonical framework and making the appropriate modifications for a particular interaction, the procedure is to postulate mathematically well-defined axioms which one hopes hold in models for all interactions. The "one hopes" suggests a worry: what if no physically realistic models exist for any set of axioms for QFT?

Before this question can be answered, the question itself must be clarified. The statement "no physically realistic models exist for any set of axioms for QFT" could be interpreted in one of three ways:

> (i) The sets of axioms for QFT that have been proposed (e.g., Wightman or Haag-Kastler axioms) do not hold for physically realistic interactions.
> (ii) It is not possible to formulate a consistent, mathematically well-defined set of axioms that holds for physically realistic interactions and includes recognizable versions of all principles regarded as constitutive of a QFT. (Perhaps the principles that there are fields that satisfy commutation relations and are covariant under Poincaré transformations fall into this category. Or, more generally, that there must be some assumptions that are traceable to special relativity and NRQM and

> fields must play some role.)
> (iii) There is no consistent set of axioms of any sort that holds for physically realistic interactions.

If scenario (i) came to pass, it would not surprise the pioneers of the axiomatic program: Jost coined the term "general field theory" because he did not like the connotations of the term "axiomatic" (Jost 1965, p. xi). Haag intentionally adopts Jost's terminology because he wants to be clear that proposed axiom systems should be regarded as provisional:

> ...the word "axiom" suggests something fixed, unchangeable. This is certainly not intended here. Indeed, some of the assumptions are rather technical and should be replaced by more natural ones as deeper insight is gained. We are concerned with a developing area of physics which is far from closed and should keep an open mind for modifications of the assumptions, additional structural principles as well as information singling out a specific theory within a general frame. (Haag 1996, p. 58)

Clearly, then, refinement of the axioms in the course of constructing physically realistic models does not constitute failure of the axiomatic program.

Option (ii) captures what is entailed by the failure of the axiomatic program. As Jost explains, the "general theory of quantized fields" sets itself "the pressing task" of "analyz[ing] the general notions which underlie all relativistic quantum field theories" (p. xiii). "[R]elativistic" and "quantum" are key words. Streater and Wightman describe the "Main Problem of quantum field theory" as being "to kill it or cure it: either to show that the idealizations involved in the fundamental notions of the theory (relativistic invariance, quantum mechanics, local fields, etc.) are incompatible in some physical sense, or to recast the theory in such a form that it provides a practical language for the description of elementary particle dynamics" (p. 1). That is, to either (a) show that it is not possible to formulate mathematically well-defined versions of the relativistic and quantum tenets of QFT that are mutually consistent and admit realistic physical models or (b) find a set of mathematically well-defined, consistent axioms for QFT's in general. If the former, then the axiomatic QFT program fails; if the latter, then the axiomatic QFT program has achieved success.

Option (iii) is too strong to capture what is meant by the failure of axiomatic QFT. Failure of this sort of program does not seem to be a meaningful possibility. Presumably, since the techniques for extracting predictions from canonical QFT's do not give arbitrary results, the same set of predictions can be derived from some consistent set of assumptions. That is, surely, given complete freedom to revise or abandon any of the axioms and unrestricted access to the conceptual resources of present and future mathematics, we could find a mathematical framework which generates the predictions of canonical QFT. Of course, the resulting set of axioms might well have no physical significance (i.e., cannot plausibly be read as capturing general physical principles); the resulting axioms may not contain any recognizable quantum or relativistic principles. For this reason, the interesting sense in which the axiomatic program could fail is captured by (ii).

I will not attempt to guess whether or not the axiomatic program will be successful in the sense of (ii). For present purposes, the important point is that either a successful or an unsuccessful outcome would be interesting. Success would mean that we would know how to revise the assumptions of the canonical framework in order to render them consistent and mathematically well-defined for all types of interactions. Failure would mean that it is not possible to formulate the constitutive principles of relativistic quantum field theory in such a way as to make them well-defined and consistent. It is worthwhile to pursue the axiomatic program because either result would have significant implications for the foundations of QFT. Most obviously, which result obtains would bear on the question of whether special relativity and quantum theory are compatible. As I will argue in the remainder of this chapter, either result would also have implications for the interpretation of QFT.

5.4 GLIMM-JAFFE $(\phi^4)_2$ MODEL VS. INFINITELY RENORMALIZED CANONICAL QFT

Before launching into the argument that the Glimm-Jaffe representation is, for interpretive purposes, to be preferred to the canonical representation, it will be helpful to reflect upon the root of the differences in approach. In the canonical treatment of the $(\phi^4)_2$ interaction, the only renormalization counterterm that is infinite is the vacuum self-energy E_0 (Glimm and Jaffe 1970a, p. 205). By assumption, there is some time t_0 at which the Hilbert space representation for the interaction coincides with the Fock space $\mathcal{F}$ associated with a neutral scalar field of the same mass. The Hamiltonian for the interaction is defined as follows (Glimm and Jaffe 1968, p. 1945):

$$H = H_F + \lambda \int : \phi^4(x, t_0) : dx \qquad (5.1)$$

The vacuum self-energy counterterm E_0 is introduced to make the lowest eigenvalue of the renormalized interaction Hamiltonian zero:

$$H\Omega_0 = E_0\Omega_0 \qquad (5.2)$$

where $\Omega_0 \in \mathcal{F}$ and E_0 is the lowest eigenvalue of H

$$H_{ren} = H_F + \lambda \int : \phi^4(x, t_0) : dx - E_0 \qquad (5.3)$$

The canonical approach is to accept the infinite counterterm E_0 and to proceed with the informally defined Hamiltonian H_{ren}.

The mathematical physicists participating in the axiomatic and constructive programs, on the other hand, take a more formal approach. They interpret the infinite value of E_0 in (5.2) as an indication that the vector Ω_0 is not in the domain of H (see, for example, Glimm and Jaffe (1970a), p. 363). Furthermore, the only vector in $\mathcal{F}$ that is in the domain of H is the zero vector (Glimm 1969b). From this more formal point of view, the proper response is to find another Hilbert space on which to represent the interaction Hamiltonian (i.e., a Hilbert space representation of the canonical commutation relations that is unitarily inequivalent to $\mathcal{F}$). And this is what Glimm and Jaffe's $(\phi^4)_2$ model achieves: it supplies a Hamiltonian operator for the interaction without an infinite counterterm that is well-defined on a Hilbert space representation of the canonical commutation relations that is unitarily inequivalent to $\mathcal{F}$ at all times.

In this section, I argue that the Glimm-Jaffe representation for a $(\phi^4)_2$ interaction is a better guide to the ontology of a real system than the canonical representation for that system. As a preliminary step, I will restrict attention to a $(\phi^4)_2$ system and give two reasons for preferring the Glimm-Jaffe representation to the canonical representation for this system. Then I will extend the argument to realistic systems.

5.4.1 Reason #1: Consistency

To get a better handle on the difference between the canonical and mathematically well-defined formulations of QFT, it is necessary to take a step backwards. Consider the canonical formulation prior to renormalization. (That is, before the procedure of adding infinite terms and coefficients to the equations has been carried out.) It can be shown that, for interactions, the unrenormalized canonical formulation is inconsistent.

Recall that Haag's theorem establishes that if a theory adopts a certain set of assumptions $\{T\}$, then the theory necessarily describes a free system. For present purposes, the content of $\{T\}$ is not important; all that is relevant is that the unrenormalized canonical formulation of QFT endorses the full set $\{T\}$. (See Chapters 1 and 2 for a discussion of the content of the assumptions and documentation of their endorsement by canonical QFT). Therefore, by Haag's theorem, unrenormalized canonical QFT necessarily describes a free system. A free system is a system whose initial state is identical to its final state. That is, a free system has trivial S-matrix elements (i.e., S is the identity).

The inconsistency follows straightforwardly. Canonical QFT is routinely applied to interacting systems. In fact, all interesting applications of canonical QFT are to interacting systems, since by definition the only systems which have non-trivial S-matrix elements are the interacting (i.e., non-free) systems. Let F be the statement that the system under consideration is free. By Haag's theorem, $\{T\} \Longrightarrow F$. But, at least in the interesting case, $\neg F$. Interesting applications of unrenormalized canonical QFT thus engender a contradiction: $\{T\} + \neg F \Longrightarrow F \& \neg F$.

I contend that the fact that the mathematically well-defined model for the $(\phi^4)_2$ interaction is consistent is a reason to prefer it over the unrenormalized canonical representation. However, this is not quite the desired conclusion. The renormalized canonical formulation–not the unrenormalized formulation–is the one that is employed to make predictions in the form of S-matrix elements for interacting systems. The unrenormalized version cannot be used to compute S-matrix elements because it yields infinite values for scattering amplitudes. Renormalization solves this problem. The infinite terms and coefficients that are inserted into the equations have the effect of making the scattering amplitudes finite. Thus, the theory of interest is the renormalized canonical formulation. In the context of interactions (a qualification that will henceforth be suppressed), is renormalized canonical QFT consistent?

The answer to this question is that it is very difficult to determine whether or not renormalized canonical QFT is consistent because we cannot enlist the resources of formal mathematics to check. Renormalization procedures introduce informal mathematical reasoning. For example, strictly speaking, infinite subtractions are carried out. Therefore, renormalized canonical QFT is outside the realm of formal, rigorous mathematics. Consequently, formal mathematical methods cannot be used to test the consistency of its theoretical principles. Formal mathematical methods play an essential role in the demonstration that unrenormalized canonical QFT is inconsistent; the proof of Haag's theorem relies on advanced mathematics (e.g., the theory of analytic functions).

The question of whether renormalized canonical QFT is consistent would, of course, be settled in the negative if contradictory predictions were to be derived. To the best of my knowledge, this has not happened in the more than fifty year history of applications of the theory. I grant that it is extremely unlikely that physicists will ever unearth a contradiction.[4] However, this should not be taken as evidence in favor of the consistency of the theory. It is possible that physicists are just reasoning very carefully from an inconsistent set of theoretical principles. That is, they may be employing inferential restrictions. Many historical examples of this phenomenon have been brought to light and discussed by historians and philosophers of science,

[4]As mentioned in Chapter 3, Huggett (2002) offers an argument based on renormalization group theory that S-matrix elements follow as deductive consequences from the canonical framework. Note that renormalization group methods do not solve the problem with the canonical framework that is brought to light by Haag's theorem; however, they may fall into the category of inferential restrictions.

including the old quantum theory and Newtonian gravity (see Meheus (2002)).

So, even if renormalized canonical QFT is inconsistent, its inconsistency does not actually pose a problem for deriving predictions. Why, then, is the manifest consistency of the mathematically well-defined formulation an advantage? Consistency is an advantage, not for the sake of the predictions, but for the sake of the theoretical principles. A consistent set of theoretical principles may all be true simultaneously; an inconsistent set is certainly not.[5] This is of immediate relevance to the project of providing a metaphysical interpretation for the theory, since the set of theoretical principles is the starting point for this project. Recall the question: if QFT were true, what would the world be like? If a theory contains an inconsistent set of theoretical principles, some members of the set must be false. A metaphysical interpretation of the theory should be based on the true theoretical principles, but not the false ones. Since we do not know that renormalized canonical QFT is consistent, we must be wary of using false principles as a guide to ontology. None of these problems arise when gleaning an ontology from the consistent mathematically well-defined representation because it is manifestly consistent.

Of course, it is not unusual for an inconsistent formulation of a theory to be replaced by a consistent one in the course of its historical development. There is also historical precedent for an informally formulated theory later being given a formal reformulation. Arguably, the evolution of Newtonian mechanics from Newton's *Prin-*

[5]Unless, that is, one agrees with Priest (2002) that contradictions do occur in nature. Briefly, my response is that we would need to be driven to this by compelling considerations, and I am arguing that QFT does not supply them.

da Costa and French (2002) also argue that it is appropriate to regard inconsistent theories as true, but as partially true rather than wholly true, a notion that they explicate using model theory. This approach does not undermine my argument because we would still be left with the problem of how to determine which parts of the theory are true. In their terms, I am arguing that the mathematically well-defined representation is to be preferred because it may be wholly true, while the canonical theory may only be partially true.

cipia to its modern-day textbook formulation is an example of this. Philosophers must not overlook the fact that QFT also has a historical context; QFT is a theory that is still being developed and refined. The canonical framework was the first comprehensive formulation of the theory, but is informal and possibly inconsistent. The rigorous models that have been obtained for particular interactions constitute formal and consistent reformulations of the canonical framework. Viewed in this light, it is not surprising that the rigorous models would be the better bets as guides to ontology.

As an aside, it is worth repeating that the manifest consistency of the mathematically well-defined formulation holds another advantage for studying the foundations of the theory. An important foundational issue is whether quantum theory is consistent with special relativity. Clearly, we are in a better position to answer this question if we know whether or not relativistic quantum field theory is consistent.

5.4.2 Reason #2: Key initial assumptions of the canonical formulation turn out to be false!

Consistency aside, there is another reason to prefer the mathematically well-defined representation for the purposes of extracting an ontology. A strange (and undesirable) property of the renormalized canonical representation is that key initial assumptions about how to set up the mathematical framework turn out to be false after renormalization!

For example, a central assumption of both ordinary non-relativistic quantum mechanics and relativistic QFT is that a symmetry operation is represented by a unitary or an anti-unitary operator.[6] Indeed, Wigner proved that (assuming commutative

[6] Cases of spontaneous symmetry breaking are an important exception; in these cases, it is not

super-selection rules) any one-one mapping of states that preserves transition probabilities must be represented by a unitary or an anti-unitary operator (Streater and Wightman 2000, p. 7). In particular, the time translation operation is supposed to be represented by the unitary operator e^{-iHt} where H is the Hamiltonian operator. However, when the interaction Hamiltonian is renormalized (H_{ren}), the operator $e^{-iH_{ren}t}$ is not unitary. Recall from the preamble to this section that for the $(\phi^4)_2$ interaction $H_{ren} = H_F + \int : \phi^4(x, t_0) : dx - E_0$. In virtue of the infinity of E_0, H_{ren} is not a well-defined self-adjoint operator on $\mathcal{F}$ (Glimm and Jaffe 1970b, p. 363). (Because, strictly speaking, the domain of H_{ren} on $\mathcal{F}$ contains only the zero vector). By Stone's theorem, since H_{ren} is not self-adjoint, the operator $e^{-iH_{ren}t}$ is not unitary. Thus in the canonical representation time translations are represented by operators that are not unitary.

The fact that there are symmetries that are not represented by unitary operators is not a fatal blow to the canonical framework, but it is a drawback. It is nice to be able to identify operators that represent symmetry operations on the basis of their mathematical properties. The unitarity of the time translation operator also enables us to prove interesting theorems (e.g., the PCT theorem) (Streater and Wightman 2000, p. 143). The mathematical ill-definedness of the theory seems to be an indication that we do not have a good handle on the theoretical principles of the theory. And if we do not have a good handle on the theoretical principles, it can be difficult to interpret a theory. Perhaps we could live with these consequences if we were given no other choice; however, we do possess an alternative. The mathematically rigorous

possible to specify a unitary time translation operator (see Earman (2004)). However, spontaneous symmetry breaking differs in important respects. Spontaneous symmetry breaking can only occur in special circumstances (i.e., when the dynamics of the system admits certain symmetries, such as the $m = 0$ free scalar Klein-Gordon field). It also differs in that the unitary inequivalence of the representations is embraced (i.e., it is accepted that at distinct times vacua belong to unitarily inequivalent representations).

$(\phi^4)_2$ model comes equipped with a well-defined, self-adjoint interaction Hamiltonian and a unitary time translation operator.

From a formal point of view, another problem with using the operator $e^{-iH_{ren}t}$ to represent time translations is that time evolution takes the system from a state in one Hilbert space representation to a state in a different (i.e., unitarily inequivalent) Hilbert space representation. Even an infinitesimal time translation takes a state to a unitarily inequivalent representation: a translation through $t = \epsilon$ is effected by $I + i\epsilon H_{ren}$, $(I + i\epsilon H_{ren})|\psi(t)\rangle = \infty$. The canonical approach avoids this problem by adopting an informal point of view. The renormalization strategy is to interpret the interaction Hamiltonian as being informally defined on the Fock space for the free system (rather than finding a unitarily inequivalent Hilbert space representation on which it is formally well-defined). Put another way, subsequent to renormalization it is presumed that states of the system at different times are in the same Hilbert space (even though they actually belong to unitarily inequivalent Hilbert space representations).

5.4.3 Realistic systems

In summary, the mathematically well-defined $(\phi^4)_2$ model has several advantages over its canonical counterpart. In the first place, it is certainly consistent. Also, its theoretical principles can be clearly articulated in formal mathematical terms. An interesting feature of this case study is that these are both theoretical rather than empirical virtues; the two representations yield the same predictions. This raises the concern that the choice between the representations is made on pragmatic grounds that are divorced from the truth of the theory. This is certainly not the case for the criterion of consistency: consistency of a set of theoretical principles is a prerequisite for their simultaneously being true. The other argument is more difficult to evaluate. The fact that the renormalized canonical framework can only be stated in informal mathematical terms is an indication that we do not have a good handle on the theoretical principles of the theory. Having a better handle on the theoretical principles may eventually result in the mathematically rigorous model being put to better use and being applied in new ways, which would be an empirical advantage. Also, it seems likely that having a better handle on a theory's theoretical principles would be of heuristic value for the development of future theories.

But–if these are compelling considerations in the $(\phi^4)_2$ case–what does this tell us about how we should use QFT as a guide to the ontology of realistic systems? In virtue of Haag's theorem, we know that the complete set of formally-stated theoretical principles of unrenormalized canonical QFT cannot hold. The renormalized canonical representation for a realistic interaction will have the same defects as the renormalized canonical representation for a $(\phi^4)_2$ system: its mathematical ill-definedness prevents us from determining whether or not the theoretical principles are consistent and are an indication that we do not have a good handle on the content of the theoretical principles. The Glimm-Jaffe model for the $(\phi^4)_2$ interaction can be regarded as taking a preliminary step towards obtaining a physically realistic model; it refines the canonical framework. Since the theoretical principles of the Glimm-Jaffe $(\phi^4)_2$ model are an improvement upon the theoretical principles of the canonical framework, the Glimm-Jaffe model is a better guide to the ontology of QFT than the canonical representation.

Of course, we should not lose sight of the fact that the Glimm-Jaffe model is only a toy model. It does refine the canonical framework, but presumably further refinements will be necessary in order to obtain theoretical principles that hold in a rigorous model for a realistic interaction. Ideally, one would like to interpret a rigorously well-defined model for a realistic interaction. In practice, since we do not possess such models, we have to make do with what we do have. We can base interpretive conclusions on the theoretical principles that hold in existing rigorous models. In particular, we should take care not to accept interpretive conclusions that hinge on theoretical principles of the canonical framework that have already been abandoned in existing models. (For example, the principle that there is some time at which the Hilbert space representation for the interaction is unitarily equivalent to the Fock space for a free system, which is a casualty of the $(\phi^4)_2$ model; as a consequence, we should not accept the quanta interpretation that is predicated on this assumption.) This is not a recipe for drawing certain conclusions; conclusions drawn will be defeasible, subject to the development of further rigorous models. However, the conclusions drawn from the rigorous models are more likely to be sound than the conclusions drawn from the canonical formulation of QFT.

5.5 GLIMM-JAFFE $(\phi^4)_2$ MODEL VS. FINITELY RENORMALIZED CANONICAL QFT WITH CUTOFFS

There is a relatively simple way to solve the problem of the mathematical ill-definedness of canonical QFT: introduce short-distance and long-distance spatial cutoffs into the Lagrangian and Hamiltonian, which renders the renormalization counterterms finite. With cutoffs, canonical QFT is a mathematically well-defined theory. Clearly, this is much more straightforward than Glimm and Jaffe's construction of the $(\phi^4)_2$ model on infinite, continuous space! The introduction of cutoffs renders renormalization finite by reducing the degrees of freedom to a finite number; consequently, the Stone-von Neumann theorem applies and all standard representations of the ETCCR's are unitarily equivalent. In contrast, there are representations of the ETCCR's that are unitarily inequivalent to the Glimm-Jaffe model (e.g., the Fock representation for any free system). Introducing cutoffs also remedies the inconsistency of the unrenormalized canonical theory; Haag's theorem does not go through for only a finite number of degrees of freedom (see Chapter 1).

As a preliminary step, consider a system that undergoes a $(\phi^4)_2$ interaction. How should we proceed with the interpretation of QFT for this system? There are several obvious reasons to prefer the Glimm-Jaffe representation to the cutoff representation for the purposes of interpretation. First, in the absence of an independent motivation, the introduction of cutoffs seems to be an ad hoc solution to the problem of infinite renormalization. (An independent motivation would, for example, be independent evidence that, according to QFT, all systems are finite in spatial extent and space is discrete.) Second, once again, key theoretical principles that guided the development of QFT turn out to be false. (For example, the field operators are not Poincaré covariant (Glimm 1969b, p. 103) and do not satisfy the local commutativity condition.)

But what about realistic interacting systems? Does the cutoff representation, which is capable of describing a realistic system, furnish a better basis for interpretation than the Glimm-Jaffe $(\phi^4)_2$ model, which only describes a toy system? Again, a compelling argument against relying on the cutoff representation is that introducing cutoffs is an ad hoc solution to the problem of infinite renormalization. We are not justified in introducing the assumptions that space is discrete and all systems are finite in spatial extent solely because it is a simpler means of achieving the end goal of obtaining a mathematically well-defined and consistent representation. To introduce the cutoffs is to abandon the project of finding a mathematically well-defined representation on infinite, continuous space. Furthermore, it is to abandon the project of finding a mathematically well-defined representation which incorporates both relativistic and quantum principles (e.g., the field operators in the cutoff representation are not Lorentz covariant). But–if we want to interpret QFT–the end result of this project is precisely what should interest us! *If* we had good reason to believe that the axiomatic program must fail, then perhaps we would have good reason to take the cutoff representation seriously. That is, it could be argued that the impossibility of obtaining a consistent formulation of the theory with both quantum and relativistic axioms means that QFT can only be formulated on a finite lattice and that some quantum or relativistic principles must be abandoned. However, in the absence of compelling reasons to believe that the axiomatic program fails, it seems unreasonable to settle for cutoff QFT. The Glimm-Jaffe $(\phi^4)_2$ model is to be preferred for the purposes of interpreting QFT because, even though it does not describe a realistic system, it is a better guide to the ontology of realistic systems than the cutoff representation because it advances the axiomatic program. The Glimm-Jaffe model is a better bet than the cutoff representation, but since it does not apply to realistic systems, it should be treated as a fallible guide to ontology according to QFT.

The following train of thought might restore one's faith in the usefulness of the cutoff representation as a guide to ontology. Quantum gravity could well furnish reason to believe that space is not continuous and cosmology could well furnish evidence that the universe is closed. For that matter, since the effects of gravity are significant at small distance scales, we do not expect QFT–*any* variant of QFT–to be empirically adequate at small distance scales. Then (overlooking the issue of whether the universe is actually closed) why not just use the cutoff theory as a guide to ontology? Note that the metaphysical question has shifted from the one that I have been investigating–if QFT were true, what would reality be like?–to the question of what QFT, as a precursor to a final theory, gets approximately correct about nature of the real world. However, I want to argue that a cutoff QFT will not necessarily even be useful for this second purpose.

In a paper entitled "In defence of naiveté: The conceptual status of Lagrangian quantum field theory," David Wallace defends the usefulness of cutoff QFT for foundational and interpretive purposes. He is inspired by the line of reasoning sketched in the preceding paragraph. He is also motivated by the concern that "pending the discovery of a realistic interacting [model for an axiomatization of QFT]...we have only limited reason to trust that our results apply to the actual world" (Wallace 2005, p. 2). I read Wallace as arguing for a number of distinct but related theses. He frames the debate in terms of which one of two attitudes should be adopted to the foundational status of QFT:[7]

> (i) "we should reject the cutoff theories ....[8], and continue to look for nontrivial theories defined at all lengthscales," or
> (ii) "QFT's as a whole are to be regarded as only approximate descriptions of some as-yet-unkown deeper theory [theory X], which gives a mathematically self-contained description of the short-distance physics" (pp. 12-13)

I have been arguing for the first attitude, but Wallace inclines towards the second attitude. Note that the issue is not whether QFT is empirically adequate at all length scales. QFT marries special relativity and quantum mechanics, but does not incorporate general relativity. Presumably, then, QFT will not be empirically adequate at small length scales, when the effects of gravity become significant, and a theory of quantum gravity will eventually be formulated to describe this domain.[9] Note also

[7]Following Cao (1997), Wallace lists as a third option "the picture of 'an infinite tower of effective field theories'," but he does not discuss this possibility.

[8]He adds "as not mathematically well-defined," but–as Wallace himself points out–the canonical theory with cutoffs does not suffer from this problem; only the canonical theory *without* cutoffs is not mathematically well-defined.

[9]Wallace himself makes this point (pp. 14-15), but in the conclusion he characterizes the successful product of the axiomatic QFT program as "a perfectly valid choice for 'theory X'" and a

that the empirical success and fruitfulness of the program of finding applicable cutoff QFT's is not in question.

Foundational attitude (ii) can be understood in either a weak or a strong sense. The weak sense is the hypothetical sense: if the axiomatic program should fail in its quest for a continuum theory applicable to realistic interactions, then the project of interpreting QFT is not doomed because it is possible to subject canonical QFT with cutoffs to foundational analysis. Wallace explicitly endorses this thesis: he writes that canonical QFT with cutoffs is "sufficiently well-defined conceptually and mathematically that it too can be usefully subjected to foundational analysis" (p. 2). Of course, I agree that canonical QFT with cutoffs is mathematically well-defined and that its foundations *could* be analyzed; I maintain, however, that cutoff QFT is not the formulation of QFT that *should* be subject to foundational analysis. On the strong reading, attitudes (i) and (ii) are in conflict insofar as "anyone regarding [(ii)] as fully satisfactory from a foundational viewpoint would have reason to doubt whether the eventual success of the [axiomatic QFT] program is possible" (p. 13). Conversely, if we had good reason to believe that the axiomatic program will end in failure, this would weigh in favour of this strong version of attitude (ii). Wallace is noncommittal about the strong reading of (ii); in fact, it seems more accurate to characterize him as an agnostic about axiomatic QFT, given that he takes pains to point out that his views are compatible with the eventual success of algebraic QFT (see below). The fact that the axiomatic program has not achieved success is a source of motivation, but does not seem to play a role in the argument; for example, beyond noting that the algebraic program has not been successful, he does not assess

rival to string theory (p. 41). Perhaps by the latter he means only that there may be a candidate quantum theory of gravity that is a field theory and that the product of axiomatic QFT may be a subtheory of theory X?

its prospects. As I explained above, my view is that, if we had reason to believe that the axiomatic QFT program must end in failure, then we would have some reason to believe that QFT can only be defined on a lattice; however, in the absence of reason to believe that the axiomatic QFT program fails, we have little reason to endorse the strong reading of (ii).

The following thesis, which attempts to stake out middle ground, is the one that interests me: even if a continuum QFT does exist, it is still sufficient to undertake foundational analysis of QFT with cutoffs because this will reveal everything we can learn from QFT about the ontology of the real world. Wallace does not explicitly endorse this thesis, but it is compatible with his contention that cutoff QFT is "an inherently approximate, but nonetheless extraordinarily powerful tool to analyse the deep structure of the world" (p. 41) and his insistence that cutoff QFT "is not really in conflict with [axiomatic] QFT at all" (p. 41; see also pp. 2-3, 13). More importantly, his arguments supply the materials for a defence of this thesis. Roughly, the argument for this thesis is that QFT only conveys approximate information about the ontology of the world at large distance scales because we expect QFT to be empirically inadequate at short distance scales and to be superseded by some more fundamental theory X; since at large distances both QFT's with cutoffs and a continuum QFT would capture the approximate structure of theory X, either is suitable for purposes of gleaning an approximate ontology. This has obvious implications for the utility of the axiomatic program. To put the point glibbly, why should we bother investing the effort to find a continuum QFT defined at all length scales if we do not believe what QFT has to say about small distance scales anyway? I contend that it is worthwhile to continue to pursue the axiomatic program because knowledge of how to formulate the theoretical principles of QFT is useful for the purposes of developing future theories and analysis of the foundations of QFT as well as for the interpretation of QFT.

Of course, scientific revolutions often result in the abandonment or drastic overhaul of theoretical principles of the old theory, but sometimes old principles are retained. At the very least, the theoretical principles of the old theory provide a starting point for the new theory. It seems reasonable to expect that the theoretical principles of QFT will play the same role in the development of theory X. As noted above, one shortcoming of cutoff QFT is that, strictly speaking, the principles of Poincaré covariance and local commutativity are false. However, as Wallace notes, both principles do hold to a very high degree of approximation for observables averaged over a region of space that is large relative to the cutoff distance (p. 18). However, knowing the approximate principles of QFT at large scales is of limited usefulness for finding theory X. Wallace considers the implications of approximate Poincaré covariance for theory X and concludes that it is possible that either theory X is Poincaré covariant or theory X is not Poincaré covariant (p. 19). In other words, cutoff QFT does not deliver useful information about the principles of theory X. This is where the success or failure of the axiomatic program could make a contribution. If it were found that it is not possible to formulate a consistent set of quantum field theoretic axioms for realistic interactions that includes Poincaré covariance, then it would seem likely that theory X is not Poincaré covariant either. (Conversely, if there is a set of axioms including Poincaré covariance, then we do not have useful information; theory X may or may not be Poincaré covariant.) That is, getting the 'right' theoretical principles for QFT matters because the theoretical principles of QFT are a useful starting point for formulating theory X. If we settle for a cutoff QFT, then we lose out on the chance to find out this useful bit of information. The fact that a cutoff QFT in some unknown way approximates the large scale structure of theory X is not similarly useful for formulating theory X. It is worthwhile to pursue the axiomatic program for this reason. It is true that we do not expect QFT to be empirically adequate at short distance scales; however, it is too rash to infer that QFT can tell us *nothing* about theory X at short distance scales. Theory X is a theory of everything, including non-gravitational interactions. If QFT for non-gravitational interactions does not supply a starting point for formulating theory X at all distance scales, then we really are stuck!

A proponent of the 'middle ground' thesis might concede that a continuum QFT has heuristic value for finding theory X, but nevertheless insist that a cutoff QFT is just as informative as a continuum QFT about the approximate ontology of the world. After all, at the end of the day, any formulation of QFT will be empirically adequate only at large distance scales, so gives us an approximate handle only on the ontology of theory X at large distance scales. Furthermore, we expect quantum gravity to overthrow our notions of space (and time) at small scales (e.g., space may only 'emerge' at large distance scales). I do not want to contest this. What I want to point out is that it does not follow from this that the cutoff and continuum representations agree on matters of ontology at large distance scales. That is, it is not necessarily the case that the large-scale interpretation of a theory which is truncated at small scales will even approximately line up with the large-scale interpretation of the untruncated theory. This is the point made in Section 3 above. In the case of QFT, it is possible for the cutoff and continuum theories to admit different interpretations because the former theory has a finite number of degrees of freedom and all representations of the ETCCR's are unitarily equivalent and the latter theory has an infinite number of degrees of freedom and uncountably many representations of the ETCCR's. Consequently, the cutoff representation for an interacting system can be given a quanta interpretation but a continuum representation cannot.

The argument that the cutoff and continuum representations will (approximately) agree on matters of ontology at large distance scales rests on a claim about how to extract an ontology from QFT as well as the observation that QFT is only empirically adequate at large distance scales. Wallace's paper implicitly adopts a stance on the ontology of QFT that does the job:[10] structural realism plus a commitment to the Haag-Kastler algebraic axiomatization of QFT. Roughly, structural realism is the doctrine that the structure of a theory in some way (approximately) represents the structure of reality. Wallace defends a variant of structural realism elsewhere (see the references in Wallace (2005)). The assumption implicit in Wallace (2005) is that the structure of the theory that represents the structure of reality is captured by the algebraic Haag-Kastler axioms. The rest of the argument is that any cutoff representation will approximately satisfy the Haag-Kastler axioms (p. 17) and presumably any continuum representation would also satisfy the Haag-Kastler axioms, therefore the representations share the same structure and furnish the same description of the structure of reality.

Evaluating this argument would take me too far afield. My objective in this chapter is merely to argue that–in advance of figuring out what an interpretation might look like–we should hedge our bets and base our interpretation of QFT on the Glimm-Jaffe model. To attack this structural realist interpretation of QFT, I would actually have to get into the business of interpreting QFT. Let me just point out that there are many reasons to doubt that the argument in the preceding paragraph will hold up under scrutiny. First, Ruetsche (2003) and Kronz and Lupher (2005) give a number of examples which show that the algebra of observables which is specified

[10]It is not clear whether Wallace actually advances the argument I sketch below. In Section 4, he argues that the infinite number of degrees of freedom and accompanying unitarily inequivalent representations can be ignored for foundational and interpretive purposes, but he bases his conclusion on the strong thesis set out above that "real QFT's are cut off at short, though finite length scales" (p. 22).

by the Haag-Kastler axioms does not capture the full content of QFT. Second, it is also a distinct possibility that there are phenomena which can only be represented in a theory with an infinite number of degrees of freedom and unitarily inequivalent representations.[11] For example, any phenomenon which relies on the uniqueness of the vacuum state could only be represented in a theory with an infinite number of degrees of freedom because otherwise the vacuum state is not unique (see Chapter 1). All of the Haag-Kastler axioms are at least approximately true in the cutoff representation, but the same cannot be said of the Wightman axioms. The existence of a unique vacuum state is a postulate of Wightman's axiomatization and this does not even hold approximately in the cutoff representation. Third, the Haag-Kastler axioms are provisional; it is possible that, if they are to admit realistic models, they must be revised. The revised axioms could very well hold in the continuum representation, but not in even approximately in the cutoff representation. Again, any axiom which only holds when the theory has an infinite number of degrees of freedom would have this property.

Finally, getting the theoretical principles of QFT right matters for settling foundational questions. For example, the question of whether special relativity is compatible with quantum theory. Arguably, if the axiomatic program is not completable, then quantum theory and special relativity are compatible only if space is discrete and the universe is spatially finite. But, in order to establish the consequent, we need to establish the antecedent; that is, the axiomatic program must be pursued to its conclusion. Also, even if a continuum QFT does not exist, this sheds light on the question of the source of the discreteness of space. If there exists no continuum QFT

[11] Hollands and Wald (2004) argue that QFT is not merely quantum mechanics applied to an effective finite number of degrees of freedom, but it is not clear that their arguments are of any help to me here.

for non-gravitational interactions, then perhaps the discreteness of space could not be blamed entirely on quantum gravity.

If we want to err on the safe side, then we should base our interpretation of QFT on theoretical principles that hold in existing rigorously well-defined models. Our interpretive conclusions will be provisional, but at the moment this is the best that we can do. If we want secure conclusions, then we must wait until the axiomatic program has either been completed or been shown to be uncompletable. It is tempting to get impatient and declare cutoff QFT sufficient for interpretive purposes, but this is an inherently risky move. It would require an argument that the differences between cutoff and continuum QFT's are irrelevant to the interpretation of QFT.

5.6 CONCLUSION

I have argued for the somewhat counterintuitive claim that the Glimm-Jaffe $(\phi^4)_2$ toy model is a better guide to the ontology of real systems than either the canonical or cutoff representations for a real system. In both cases, the thrust of the argument is that the Glimm-Jaffe $(\phi^4)_2$ model comes closer to getting the theoretical principles right than other formulations. When canonical QFT is mathematically well-defined (i.e., prior to renormalization), its theoretical principles are demonstrably inconsistent; when canonical QFT is mathematically ill-defined (i.e., post-renormalization), it is difficult to determine whether or not its principles are consistent. This makes it difficult to interpret the theory because an interpretation should be based on principles one would have to accept in order to accept the theory and if the principles are inconsistent then all cannot be accepted simultaneously. In addition, the renormalization procedure overturns key assumptions about how the mathematical framework can be taken to represent the physical situations (e.g., symmetries are not represented by unitary operators). This leaves the impression that we do not have a good handle on the content of the renormalized canonical framework. A cutoff representation mathematically well-defined, but this comes at a cost: the representation is truncated at small and large distances and key theoretical principles come out false (e.g., Lorentz covariance). In the absence of a compelling reason to believe that it is not possible to formulate QFT on infinite, continuous space or to hold onto relativistic principles, it is difficult to take the cutoffs or the principles of cutoff QFT seriously. For example, it seems unreasonable to draw the conclusion that according to QFT space is discrete and the universe is finite. Consequently, we should not base our interpretation of QFT on canonical QFT with cutoffs.

In contrast, the Glimm-Jaffe $(\phi^4)_2$ model has a consistent set of mathematically well-defined theoretical principles which include both relativistic and quantum principles. The starting point for constructing this model was the canonical representation for a $(\phi^4)_2$ interaction. The Glimm-Jaffe model improves upon the canonical representation by refining its assumptions. Presumably, before models for more complex interactions can be constructed, further assumptions shared by the canonical and Glimm-Jaffe representations will have to be modified or abandoned. For this reason, any interpretive conclusions drawn from the Glimm-Jaffe $(\phi^4)_2$ model will be provisional. Nevertheless, we should be more confident about an interpretation of QFT based on the Glimm-Jaffe $(\phi^4)_2$ model than an interpretation based on either canonical or cutoff QFT.

6.0 CONCLUSION: HOW DOES HAAG'S THEOREM CONTRIBUTE TO OUR UNDERSTANDING OF QFT WITH INTERACTIONS?

Haag's theorem is illuminating because it is a 'no go' result: unrenormalized canonical QFT adopts a set of assumptions that entail that it is only capable of describing free systems. 'No go' results of this type have played an important role in clarifying the foundations of NRQM and have imposed substantial constraints on the interpretation of the theory. Haag's theorem makes a different sort of contribution to our understanding of QFT. It applies to an attempted formulation of QFT that–as Haag's theorem itself establishes!–is incapable of yielding predictions for interesting (i.e., interacting) systems; in contrast, the 'no go' results for NRQM apply to the standard formulation of the theory that is employed by physicists to make predictions. Consequently, 'no go' theorems for NRQM constrain the interpretation of the theory, but Haag's theorem is helpful for the preliminary task of specifying the content of QFT. Haag's theorem frames answers to the question "What is QFT?" Different formulations of QFT can be classified according to how they respond to the *reductio* posed by Haag's theorem:

1. Renormalized canonical QFT: introduction of an infinite vacuum self-energy renormalization counterterm, which renders the theory mathematically not well-

defined.

2. Canonical QFT with a volume cutoff: introduction of a volume cutoff which, depending on the form it takes, either destroys the covariance of the theory under spatial translations or entails that there is not a unique vacuum state invariant under Euclidean transformations.

3. Axiomatic QFT: rejection of an assumption common to unrenormalized canonical QFT and (all versions of) Haag's theorem. (e.g., the Glimm-Jaffe $(\phi^4)_2$ model denies the assumption that there is a time at which the representation for the interaction is unitarily equivalent to a Fock representation for a free field)

Prior to interpreting QFT, we require an answer to the question "What is QFT?" An answer should either identify one formulation of QFT as the 'real' theory or explain how the three formulations agree on the essential content of QFT. In Chapter 5, I offered an extended argument that extant mathematically rigorous models for physically unrealistic interactions capture more of the content of QFT than either renormalized canonical QFT or canonical QFT with cutoffs. The thrust of this argument derives its force from Haag's theorem. By definition, QFT is the theory that integrates the special theory of relativity and NRQM. By 1930, both special relativity and NRQM were worked out, but it remained unclear how to combine the theories. Whether it was even possible to consistently combine essential elements of the theories was an open question. Unrenormalized canonical QFT can be regarded as a failed attempt to integrate special relativity and NRQM. Haag's theorem indicates why it fails: its assumptions are incompatible with the existence of interacting systems. Historically, canonical QFT with renormalization can be regarded as the first fully worked out formulation of QFT applicable to interacting systems. But, as a response to Haag's theorem, patching up canonical QFT by renormalization seems deeply unsatisfying: instead of solving the underlying problem with canonical QFT (viz. the inconsistency of its assumptions), renormalization cheats and simply blocks the proof of Haag's theorem by rendering the theory mathematically ill-defined. The introduction of cutoffs into canonical QFT produces a mathematically well-defined theory, but in doing so it cheats by changing the rules of the game; the project was supposed to be to formulate a relativistic quantum theory on infinite, continuous space, not on a lattice of finite spatial extent. The most satisfying response to Haag's theorem is to regard unrenormalized canonical QFT as a promising start on the program of formulating a relativistic quantum theory, but as nonetheless requiring substantial modifications (i.e., at least one of the shared assumptions of Haag's theorem and unrenormalized canonical QFT must be abandoned). Axiomatic and constructive field theorists are still working on this problem. From the historical perspective, the project of finding a mathematically well-defined and consistent framework that successfully integrates both relativistic and quantum principles is ongoing.

Haag's theorem contributes to our understanding of renormalization in several ways. First, it identifies the root cause of renormalization: the inconsistency of the assumptions of canonical QFT in the context of interacting systems. It also establishes that renormalization is necessary in order to use the canonical framework (without cutoffs) to describe *any* interacting system whatsoever. Characterizing formulations of QFT in terms of their response to Haag's theorem also sheds light on the significance of renormalization. Teller distinguishes "three ways of thinking about renormalization as it is used by physicists" (Teller 1995, pp. 168-9):

1. The cutoffs approach: We regularize, absorb finite, regularized quantities, and then take the limit.
2. The real-infinities approach: We literally discard real infinities.

3. The mask-of-ignorance approach: We understand renormalization as a way of covering our ignorance of how present false theories approximate a correct, completely finite theory. [e.g., superstring theory]

These ways of thinking about renormalization can be fleshed out in various ways (see the discussion in Huggett and Weingard (1995)), but in any guise they are inspired by renormalized canonical QFT and canonical QFT with cutoffs. (This is not surprising in view of the fact that Teller explicitly excludes axiomatic QFT from consideration; see Teller (1995, p. 146, fn. 22) and the above discussion of this passage in the Introduction.) What Teller's list is missing is a fourth possibility, derived from axiomatic QFT:

4. The axiomatic approach: Renormalization and the introduction of cutoffs are responses to the inconsistency of the assumptions of canonical QFT. We understand both renormalization and the introduction of cutoffs as expedients that permit the derivation of predictions for interacting systems, but do not address the root cause of the problem (namely, the inconsistency of the initial assumptions of the canonical framework). Renormalized canonical QFT and canonical QFT with cutoffs approximate (in some unspecified way!) a correct, completely finite theory; however, this theory is not string theory or some successor to QFT, but the correct formulation of QFT itself.

The argument sketched in the preceding paragraph lends credence to this fourth way of thinking about renormalization.

There is an important interpretive consequence of accepting the view of QFT that seems natural in light of Haag's theorem–namely, the view that axiomatic and constructive QFT is a better guide to ontology than the other formulations. As I argued in Chapter 4, it is not possible to give a quanta interpretation for such a QFT with interactions. Since all real systems interact, this licenses the conclusion that, according to QFT, there is no place for quanta in our ontology of fundamental entities.

I have also responded to several skeptical challenges to the approach to interpreting QFT with interactions that I have articulated. One objection is that Haag's theorem cannot possibly have the significance that I have attributed to it because it pertains only to the infinite volume divergences and to infinite vacuum self-energy renormalization. The argument could be made that the ultraviolet divergences and the attendant renormalization counterterms are more important because they are more difficult to rectify and because they have greater physical significance. For instance, the infinite vacuum self-energy renormalization that is introduced in response to the problem identified by Haag's theorem can be interpreted as a correction to the zero point of energy which is physically insignificant since we only measure differences in energy; the fact that the infinite vacuum self-energy term drops out of the calculation of S-matrix elements reinforces this point (see Section 2.1 of Chapter 3). Haag's theorem is significant, first, because it indicates that the infinite volume divergences that afflict the unrenormalized interaction picture are generic; *any* formulation of relativistic QFT which genuinely describes *any* interacting system must come to terms with this fact. Haag's theorem is also significant because it provides guidance for revising the unrenormalized canonical framework by identifying the problematic set of assumptions. Granted, the existence of ultraviolet divergences in the unrenormalized interaction picture for most interactions indicates that further revisions to the unrenormalized canonical framework are necessary to treat these interactions. But, as I suggested in Chapter 4, Haag's theorem could provide a template for dealing with these cases: in principle it seems that it would be possible to proceed by formulating 'higher-order' Haag's theorems which identify the problematic set of assumptions and thus give some indication of how to make further revisions to the unrenormalized canonical framework.

A second skeptical response is to point out that it has proven to be very difficult to construct mathematically rigorous models for realistic interactions on infinite, continuous space and that, indeed, this task could very well prove to be impossible. What if the axiomatic program fails (in the sense defined in Section 3 of Chapter 4)? (It is worth pointing out that this would be a very difficult result to prove in light of the fact that there is such a large degree of flexibility in defining what constitutes the axioms of QFT.) Granted, if the axiomatic program fails, then the interpretive stance that I have defended is not viable. However, the failure of the axiomatic program would be an extremely interesting outcome. The view of QFT that I have sketched would be of value in understanding its significance. There are two possible ways of interpreting the result. First, it could be read as an indication that it is only possible to formulate QFT as a mathematically well-defined theory on a spatial lattice of finite extent; this would support taking the cutoffs seriously and accepting the consequence that we live in a universe which is spatially finite and in which space is discrete. The second possibility is that the failure of the axiomatic program indicates that QFT does not exist; that is, that circa 1930 it seemed that an important next step in theoretical physics would be to formulate a consistent relativistic quantum theory but that this has proven to be impossible. This is a fascinating possibility, not the least because renormalized canonical QFT has yielded such extraordinarily accurate predictions!

These arguments are variations on a single theme, which is the theme of this dissertation: axiomatic and constructive QFT carry important lessons for the foundations and interpretation of QFT with interactions, in spite of the fact that these formulations of the theory may very well fail to describe physically realistic interacting systems.

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