ON THE REGULARITY OF p-HARMONIC FUNCTIONS IN THE HEISENBERG GROUP

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In this thesis we first implement iteration methods for fractional difference quotients of weak solutions to the p-Laplace equation in the Heisenberg group. We obtain that $Tu \in L^p_{\text{loc}}(\Omega)$ for $1 < p < 4$, where u is a p-harmonic function. Then we give detailed proofs for $HW^{2,2}$-regularity for p in the range $2 \leq p < 4$ and $HW^{2,p}$-regularity in the case $\frac{\sqrt{17}-1}{2} \leq p \leq 2$ for ε-approximate p-harmonic functions in the Heisenberg group. These last estimates however are not uniform in ε. The method to prove uniform estimates is based on Cordes type estimates for subelliptic linear partial differential operators in non-divergence form with measurable coefficients in the Heisenberg group. In this way we establish interior $HW^{2,2}$-regularity for p-harmonic functions in the Heisenberg group $\mathbb{H}^n$ for p in an interval containing 2. We will also show that the $C^{1,\alpha}$ regularity is true for p in a neighborhood of 2.

In the last chapter we extend our results to the more general case of Carnot groups.

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1.0 INTRODUCTION

The Heisenberg group plays an important role in several branches of mathematics such as representation theory, harmonic analysis, complex variables, partial differential equations and quantum mechanics. It can be constructed in many different ways, for example, as a group of unitary operators acting on $L^2(\mathbb{R}^n)$, or it can be identified with the group translations of the Siegel upper half space in $\mathbb{C}^n$, or it can be realized as a group of unitary operators generated by the exponentials of the position and momentum operators in quantum mechanics.

In the Heisenberg groups we find an abstract form of the commutation relations for the quantum-mechanical position and momentum operators. The commutation relations will be present in the form of the noncommutatitvity of first order differential operators, more exactly of the horizontal left invariant vector fields.

The number of the horizontal vector fields we use is $2n$ in a $2n+1$ dimensional space. The horizontal vector fields and their commutators span the tangent space at any point, so they form a completely nonholonomic or bracket-generating family. According to the Rashevsky-Chow theorem, we can connect any two points in the Heisenberg group using curves that have tangent vectors at each point in the subspace generated by the horizontal vector fields. This is a very important fact in control theory and has important consequences in the regularity of weak solutions of partial differential equations. The study of regularity of weak solutions is needed because it is difficult to find classical solutions that match real world situations. Therefore, we have to extend the search and first get solutions in a very general class of functions. After that one has to show that it has the required properties.

Let us consider the Heisenberg group $\mathbb{H}^n$ as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the group multiplication

$$(x_1, ..., x_{2n}, t) \cdot (y_1, ..., y_{2n}, u) = \left(x_1 + y_1, ..., x_{2n} + y_{2n}, t + u - \frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_i y_{n+i})\right).$$

With respect to this operation the neutral element is $\mathbf{1} = (0, \ldots, 0)$ and the inverse is given by

$$(x_1, \ldots, x_{2n}, t)^{-1} = (-x_1, \ldots, -x_{2n}, -t).$$

The conjugation by $x = (x_1, \ldots, x_{2n}, t)$ is defined as $\mathbf{Ad}((x_1, \ldots, x_{2n}, t)) : \mathbb{H}^n \to \mathbb{H}^n$,

$$\mathbf{Ad}((x_1, \ldots, x_{2n}, t))(y_1, ..., y_{2n}, s) = \left(y_1, ..., y_{2n}, s - \sum_{i=1}^{n}(x_{n+i}y_i - x_i y_{n+i})\right).$$

The tangent space at $\mathbf{1}$ and at the same time the Lie algebra of the Heisenberg group is $\mathbb{R}^{2n+1}$, hence the differential of $\mathbf{Ad}(x_1, ..., x_{2n}, t)$ at $\mathbf{1}$ is

$$\mathrm{Ad}(x_1, ..., x_{2n}, t) = D_{\mathbf{1}}\mathbf{Ad}(x_1, ..., x_{2n}, t) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$$

given in matrix form by

$$\mathrm{Ad}((x_1, ..., x_{2n}, t) \;=\; \begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \cdots & 0 \\ 0 & \ldots & 1 & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \cdots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 1 & 0 \\ -x_{n+1} & \ldots & -x_{2n} & x_1 & \ldots & x_n & 1 \end{pmatrix}$$

Therefore we can consider the mapping $\mathrm{Ad} : \mathbb{R}^{2n+1} \to \mathbf{GL}(\mathbb{R}^{2n+1})$ and its differential at 0, $\mathrm{ad} = D_{(0,...,0,0)}\mathrm{Ad} : \mathbb{R}^{2n+1} \to \mathbf{L}(\mathbb{R}^{2n+1}, \mathbb{R}^{2n+1})$ given by

$$\mathrm{ad}(X_1, ..., X_{2n}, T) \;=\; \begin{pmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \cdots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \cdots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ -X_{n+1} & \ldots & -X_{2n} & X_1 & \ldots & X_n & 0 \end{pmatrix}.$$

The Lie bracket or commutator of $X, Y \in \mathbb{R}^{2n+1}$ is given by

$$[X,Y] = \mathrm{ad}(X)(Y) = \left(0,...,0,-\sum_{i=1}^{n}(X_{n+i}Y_i - X_iY_{n+i})\right).$$

The left multiplication by $x = (x_1, \ldots, x_{2n}, t)$ is defined by $L_x : \mathbb{H}^n \to \mathbb{H}^n$,

$$L_x(y) = x \cdot y = \left(x_1 + y_1, ..., x_{2n} + y_{2n}, t + u - \frac{1}{2}\sum_{i=1}^{n}(x_{n+i}y_i - x_iy_{n+i})\right),$$

and its differential at $\mathbf{1}$ is

$$D_{\mathbf{1}}L_x = \begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \cdots & 0 \\ 0 & \ldots & 1 & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \cdots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 1 & 0 \\ -\frac{1}{2}x_{n+1} & \ldots & -\frac{1}{2}x_{2n} & \frac{1}{2}x_1 & \ldots & \frac{1}{2}x_n & 1 \end{pmatrix}$$

For each $v = (v_1, \ldots, v_{2n}, s) \in \mathbb{R}^{2n+1}$ corresponds a left invariant vector field X_v given by

$$X_v(x) = D_{\mathbf{1}}L_x(v) =$$

$$= v_1\frac{\partial}{\partial x_1} + \ldots + v_{2n}\frac{\partial}{\partial x_{2n}} + \left(s - \frac{1}{2}\sum_{i=1}^{n}(x_{n+i}v_i - x_iv_{n+i})\right)\frac{\partial}{\partial t}.$$

Therefore, if $i \in \{1,...,n\}$ and $e_i \in \mathbb{R}^{2n+1}$ is the vector with the i^{th} component 1 and the others 0, we have the corresponding left invariant vector field

$$X_i(x) = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t}.$$

For e_{n+i} we have

$$X_{n+i}(x) = \frac{\partial}{\partial x_{n+i}} + \frac{x_i}{2}\frac{\partial}{\partial t},$$

while for $e_{2n+1} = (0, \ldots, 0, 1)$ we have

$$T(x) = \frac{\partial}{\partial t}.$$

The commutators of the horizontal vector fields X_i satisfy $[X_i, X_{n+i}] = T$, otherwise $[X_i, X_j] = 0$. Therefore the horizontal vector fields X_i and their commutators span the tangent space of $\mathbb{H}^n$ at each point and hence satisfy the Hörmander's condition of hypoellipticity.

Let Ω be a domain in $\mathbb{H}^n$ and let $p > 1$. Recall that the Haar measure in $\mathbb{H}^n$ is the Lebesque measure of $\mathbb{R}^{2n+1}$, therefore the space $L^p(\Omega)$ is defined in the usual way. Consider the following Sobolev space with respect to the horizontal vector fields X_i

$$HW^{1,p}(\Omega) = \Big\{ u \in L^p(\Omega) \ : \ X_i u \in L^p(\Omega), \text{ for all } \ i \in \{1, ..., 2n\} \Big\} .$$

$HW^{1,p}(\Omega)$ is a Banach space with respect to the norm

$$||u||_{HW^{1,p}} = ||u||_{L^p} + \sum_{i=1}^{2n} ||X_i u||_{L^p} .$$

We denote by $HW_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$. We will also use the local Sobolev space

$$HW^{1,p}_{\text{loc}}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \ : \ \eta u \in HW^{1,p}(\Omega), \text{ for all } \ \eta \in C_0^\infty(\Omega) \Big\} .$$

Consider the p-Laplace equation:

$$-\sum_{i=1}^{2n} X_i \left(|Xu|^{p-2} X_i u \right) = 0 \, , \ \text{ in } \ \Omega \tag{1.0.1}$$

where $Xu = (X_1 u, ..., X_{2n} u)$ is the horizontal gradient of u.

A function u from the horizontal Sobolev space $HW^{1,p}_{\text{loc}}(\Omega)$ is called a p-harmonic function if it is a weak solution of equation (1.0.1), that is

$$\int_\Omega |Xu(x)|^{p-2} \langle Xu(x), X\varphi(x) \rangle \, dx = 0 \, , \ \text{ for all } \varphi \in HW_0^{1,p}(\Omega) \, . \tag{1.0.2}$$

Together with equation (1.0.1) we will consider for $\varepsilon > 0$ small the approximating equations

$$-\sum_{i=1}^{2n} X_i \left(\left(\varepsilon + |Xu|^2 \right)^{\frac{p-2}{2}} X_i u \right) = 0 \, , \ \text{ in } \ \Omega \tag{1.0.3}$$

and their weak solutions $u_\varepsilon \in HW^{1,p}_{\text{loc}}(\Omega)$ which we will call ε-approximate p-harmonic functions.

In the case $p = 2$ the left hand side of equation (1.0.1) is the Kohn-Hörmander Laplacian and the C^∞-regularity of the weak solutions u and u_ε follows from Hörmander's celebrated theorem [12].

In the case $p \neq 2$ the equation degenerates. In the classical Euclidean case we know that $u_\varepsilon \in C^\infty(\Omega)$ and $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for $1 < p < \infty$ and $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ for p close to 2. In the case of the Heisenberg group or in general in the subelliptic case there are no definite answers yet. We can mention the results from the papers of Capogna [2, 3], Capogna and Garofalo [4] and Marchi [17, 18, 19]. In the papers [2, 3, 4] the a priori assumption on the boundedness of the horizontal gradient allows the use of some aspects of linear theory like L^2 spaces or fractional derivatives defined via Fourier transform to gain control on difference quotients and prove interior C^∞ regularity for the weak solutions of (1.0.1). Due to the noncommutativity of the horizontal vector fields in the Heisenberg group, the first thing to be proved is the differentiability in the non-horizontal direction T. Under the boundedness condition of the horizontal gradient it is possible to prove for any $p \geq 2$ not just that $Tu_\varepsilon \in L^2_{\mathrm{loc}}(\Omega)$ but $Tu_\varepsilon \in HW^{1,2}_{\mathrm{loc}}(\Omega)$. This opens the way to the proof of $u_\varepsilon \in HW^{2,2}_{\mathrm{loc}}(\Omega)$ and then differentiating equation (1.0.1) we can prove C^∞-regularity.

In the general case proving $Tu \in L^p_{\mathrm{loc}}(\Omega)$ is more difficult. Marchi [17, 18, 19] proved this for $1 + \frac{1}{\sqrt{5}} < p < 1 + \sqrt{5}$. She used the fractional difference quotients to show that a weak solution is in some truncated versions of fractional Besov and Bessel-potential spaces. Marchi used the embedding among these spaces (see [21, 23, 24, 25]) to obtain more information on the differentiability of weak solutions.

It is clear that the way we manage the fractional difference quotients constitutes a key point in the further development of this theory. We propose a direct method to bound the first order difference quotients. Using the semi-group properties hidden in the second order difference quotients we will be able to control the first order fractional difference quotients and hence to get a complete nonlinear treatment of the regularity problems. Among our main contributions are Lemma 2.2.1 and the implementation of several iteration schemes on fractional difference quotients. The point here is that using an appropriate test function, and exploiting the geometry of vector fields in the Heisenberg group described by the Baker-Campbell-Hausdorff formula, we get information on the second order difference quotients.

Using Lemma 2.2.1 we transfer this information to the first order difference quotients and do our iterations. In this way first we will extend Marchi's results by proving that $Tu \in L^p_{\text{loc}}(\Omega)$ for $1 < p < 4$. Our method can be used also to give a new proof of $Tu \in HW^{1,2}_{\text{loc}}(\Omega)$ for $1 < p < \infty$ under the boundedness assumption of the papers [2, 3, 4].

Once we have the differentiability in the T direction we can prove second order differentiability in the horizontal directions. We do modified, and at the same time relatively simple versions of Marchi's proofs, that are independent of the embedding properties of Besov and Bessel-potential spaces.

We remark that our $HW^{2,2}$ estimates for $2 \leq p < 4$ and the $HW^{2,p}$ estimates for $\frac{\sqrt{17}-1}{2} \leq p \leq 2$ are essential to be able to differentiate equation (1.1) and use the Cordes conditions in order prove uniform $HW^{2,2}$ bounds, which leads to interior $HW^{2,2}$ and $C^{1,\alpha}$ regularity of p-harmonic functions in intervals that contain $p = 2$ and depend on n.

Here is the plan of this thesis. In the next chapter we prove that $Tu \in L^p_{\text{loc}}(\Omega)$ for $1 < p < 4$. Our main contributions are Lemma 2.2.1 and the implementation of several iteration schemes in the T-direction. Lemma 2.2.1 presents a direct proof based on a classical argument of A. Zygmund used for Hölder-Zygmund spaces of one variable functions [30].

In the third chapter we prove $HW^{2,2}$ estimates for $2 \leq p < 4$ and the $HW^{2,p}$ estimates for $\frac{\sqrt{17}-1}{2} < p \leq 2$ of the ϵ-approximate p-harmonic functions.

In the fourth chapter we use the Cordes condition [5, 28] and Strichartz's spectral analysis [27] to establish $HW^{2,2}$ estimates for linear subelliptic partial differential operators with measurable coefficients. As an application we obtain uniform $HW^{2,2}$ bounds for the ε-approximate p-harmonic functions for p in a range that depends on the dimension of the Heisenberg group $\mathbb{H}^n$. Using a stronger version of Cordes condition we prove $C^{1,\alpha}$ regularity of the p-harmonic functions for p close to 2.

In the last chapter we extend the results from the previous chapters to the case of Carnot groups of an arbitrary step.

2.0 DIFFERENTIABILITY ALONG THE T-DIRECTION

2.1 PRELIMINARIES

In this section we introduce the first and second order difference quotients and state the first results involving them. In the next section we prove the lemma about the connection between second order and first order fractional difference quotients. The third section is devoted to the iteration scheme in the T-direction for $2 \leq p < 4$, while in the fourth section we discuss the case $1 < p < 2$.

Let us rewrite equation (1.0.1) in the following way

$$-\sum_{i=1}^{2n} X_i \left(a_i(Xu)\right) = 0\,, \text{ in } \Omega \tag{2.1.1}$$

where

$$a_i(\xi) = |\xi|^{p-2}\xi_i\,, \text{ for all } \xi \in \mathbb{R}^{2n}\,.$$

A p-harmonic function $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution of equation (2.1.1), i.e.

$$\sum_{i=1}^{2n} \int_\Omega a_i(Xu(x))\; X_i\varphi(x)dx = 0\,, \text{ for all } \varphi \in HW_0^{1,p}(\Omega)\,. \tag{2.1.2}$$

For $\varepsilon > 0$ small the ε-approximating equation to (2.1.1) is

$$-\sum_{i=1}^{2n} X_i \left(a_i^\varepsilon(Xu)\right) = 0\,, \text{ in } \Omega \tag{2.1.3}$$

where

$$a_i^\varepsilon(\xi) = \left(\varepsilon + |\xi|^2\right)^{\frac{p-2}{2}}\;\xi_i\,, \text{ for all } \xi \in \mathbb{R}^{2n}\,.$$

We will use the following properties of the functions a_i and a_i^ε:

(i) There exists a constant $c > 0$ such that

$$c|\xi|^{p-2}|q|^2 \leq \sum_{i,j=1}^{2n} \frac{\partial a_i(\xi)}{\partial \xi_j} q_i q_j \,, \text{ for all } \xi, q \in \mathbb{R}^{2n} \tag{2.1.4}$$

and

$$c\left(\varepsilon + |\xi|^2\right)^{\frac{p-2}{2}} |q|^2 \leq \sum_{i,j=1}^{2n} \frac{\partial a_i^\varepsilon(\xi)}{\partial \xi_j} q_i q_j \,, \text{ for all } \xi, q \in \mathbb{R}^{2n} \,. \tag{2.1.5}$$

(ii) there exists a constant $c > 0$ such that

$$\left|\frac{\partial a_i(\xi)}{\partial \xi_j}\right| \leq c|\xi|^{p-2} \,, \text{ for all } \xi \in \mathbb{R}^{2n} \tag{2.1.6}$$

and

$$\left|\frac{\partial a_i^\varepsilon(\xi)}{\partial \xi_j}\right| \leq c\left(\varepsilon + |\xi|^2\right)^{\frac{p-2}{2}} \,, \text{ for all } \xi \in \mathbb{R}^{2n} \,. \tag{2.1.7}$$

If Z is a left invariant vector field then for some

$$z = (z_H, z_T) = (z_1, ..., z_{2n}, z_T)$$

we can write

$$Z = \sum_{i=1}^{2n} z_i X_i + z_T T \,.$$

The exponential mapping in canonical coordinates is defined by

$$e^Z = z \,.$$

In particular,

$$e^{X_1} = (1, 0, ..., 0, 0) \,, ..., \; e^{X_{2n}} = (0, 0, ..., 1, 0) \,, \text{ and } \; e^T = (0, 0, ..., 0, 1) \,.$$

Recall that in the Heisenberg group the Baker-Campbell-Hausdorff formula for two left invariant vector fields $Z = \sum_{i=1}^{2n} z_i X_i + z_T T$ and $V = \sum_{i=1}^{2n} v_i X_i + v_T T$ is

$$e^Z e^V = e^{Z+V+\frac{1}{2}[Z,V]} = z \cdot v \,.$$

Let $\Omega \subset \mathbb{H}^n$ be a bounded domain. For $x \in \Omega$, a left invariant vector field Z, $s \in \mathbb{R}$ sufficiently small, $0 < \alpha, \theta \leq 1$, and $u : \Omega \to \mathbb{R}$ let us define:

$$\begin{aligned}
\triangle_{Z,s} u(x) &= u(x \cdot e^{sZ}) - u(x)\,,\\
\triangle^2_{Z,s} u(x) &= u(x \cdot e^{sZ}) + u(x \cdot e^{-sZ}) - 2u(x)\,,\\
D_{Z,s,\theta} u(x) &= \frac{u(x \cdot e^{sZ}) - u(x)}{|s|^\theta}\,,\\
D_{Z,-s,\theta} u(x) &= \frac{u(x \cdot e^{-sZ}) - u(x)}{-|s|^\theta}\,.
\end{aligned}$$

Then

$$D_{Z,-s,\alpha} D_{Z,s,\theta} u(x) = D_{Z,s,\theta} D_{Z,-s,\alpha} u(x) = \frac{u(x \cdot e^{sZ}) + u(x \cdot e^{-sZ}) - 2u(x)}{|s|^{\alpha+\theta}} = \frac{\triangle^2_{Z,s} u(x)}{|s|^{\alpha+\theta}}\,.$$

We will use the following result [2, 12]:

Proposition 2.1.1. *Let $\Omega \subset \mathbb{H}^n$ be an open set, K a compact set included in Ω, Z a left invariant vector field and $u \in L^p_{\text{loc}}(\Omega)$. If there exist σ and C two positive constants such that*

$$\sup_{0<|s|<\sigma} \int_K |D_{Z,s,1} u(x)|^p \, dx \leq C^p$$

then $Zu \in L^p(K)$ and $||Zu||_{L^p(K)} \leq C$.

Conversely, if $Zu \in L^p(K)$ then for some $\sigma > 0$

$$\sup_{0<|s|<\sigma} \int_K |D_{Z,s,1} u(x)|^p \, dx \leq \Big(2||Zu||_{L^p(K)}\Big)^p\,.$$

The following result is a direct consequence of the Baker-Campbell-Hausdorff formula (see [2, 12]). We will use the notation $\bar{s} = (0, ..., 0, s)$ and

$$D_{\bar{s},\alpha} u(x) = D_{T,s,\alpha} u(x)\,.$$

Proposition 2.1.2. *Let $\Omega \in \mathbb{H}^n$ be an open set, $1 \leq p < \infty$, $u \in HW^{1,p}_{\text{loc}}(\Omega)$, $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, 3r) \subset \Omega$. Then there exists a positive constant c independent of u such that*

$$\int_{B(x_0,r)} \left| D_{\bar{s},\frac{1}{2}} u(x) \right|^p dx \leq c \int_{B(x_0,2r)} (|u|^p + |Xu|^p)\, dx\,. \tag{2.1.8}$$

Remark 2.1.1. Let us observe that if g is a cut-off function between $B(x_0, r)$ and $B(x_0, 2r)$ then

$$\int_{B(x_0,r)} \left|D_{\bar{s},\frac{1}{2}}\, u(x)\right|^p dx \le \int_{B(x_0,2r)} \left|D_{\bar{s},\frac{1}{2}}\,(g^2 u)(x)\right|^p dx$$
$$\le c \int_{B(x_0,2r)} (|u|^p + |Xu|^p)\, dx\,. \quad (2.1.9)$$

2.2 FRACTIONAL DIFFERENCE QUOTIENTS

In this section we will prove a lemma that will help us handle the second order fractional difference quotients. The classical method is to use the interpolation properties or equivalent norms of Besov (or Lipschitz) spaces, and Bessel potential (or Triebel-Lizorkin)spaces. However, our approach requires a truncated version of these spaces. Rather than referring the reader to a modified version of Theorem 2.5.1 on page 189 [24], we present a direct proof based on a classical argument of A. Zygmund (Theorem 3.4 [30]).

Let us continue to denote by $\bar{s} = (0, ..., 0, s) \in \mathbb{R}^{2n+1}$. Although our lemma will be stated in $\mathbb{R}^{2n+1}$ we will be able to use it in the Heisenberg group, because the group multiplication by $\bar{s}$ is just the addition in the last variable. Let us observe that a similar proof can be carried out if we replace the Euclidean space by a nilpotent stratified Lie group and the translations by the flow of a left invariant vector field. Let us recall our notations for the following lemma:

$$\triangle_{\bar{s}} u(x) = u(x + \bar{s}) - u(x)$$

$$\triangle_{\bar{s}}^2 u(x) = u(x + \bar{s}) + u(x - \bar{s}) - 2u(x)\,.$$

Lemma 2.2.1. *Let $u \in L^p(\mathbb{R}^{2n+1})$, $0 < \alpha$, $0 < \sigma$ and $M \ge 0$. Suppose that*

$$\sup_{0<|s|\le\sigma} \frac{||\triangle_{\bar{s}}^2 u||_{L^p}}{|s|^\alpha} \le M\,. \quad (2.2.1)$$

Then for all $0 < \beta \le \min\{1, \alpha\}$ if $\alpha \ne 1$ and for all $0 < \beta < 1$ if $\alpha = 1$ there exists $c > 0$ independent of u and $0 < \sigma' \le \sigma$ such that

$$\sup_{0<|s|\le\sigma'} \frac{||\triangle_{\bar{s}} u||_{L^p}}{|s|^\beta} \le c(||u||_{L^p} + \frac{M}{2^\alpha})\,. \quad (2.2.2)$$

Proof. Using $u \in L^p(\mathbb{R}^{2n+1})$ we have that $\triangle_{\bar{s}} u \in L^p(\mathbb{R}^{2n+1})$ and $||\triangle_{\bar{s}} u||_{L^p} \leq 2||u||_{L^p}$ for all $0 < |s| \leq \sigma$. Let us denote $g(s)(x) = u(x+\bar{s}) - u(x)$. Condition (2.2.1) implies that

$$||u(\cdot + \bar{s}) + u(\cdot - \bar{s}) - 2u(\cdot)||_{L^p} \leq M\,|s|^\alpha\,.$$

Without loss of generality we can work just with $s > 0$. Replacing s by $\frac{s}{2}$, denoting $M' = \frac{m}{2^\alpha}$ and then changing the variables $x \to x + \frac{s}{2}$ in the integral gives

$$\left\| u\left(\cdot + \bar{s}\right) + u(\cdot) - 2u\left(\cdot + \frac{\bar{s}}{2}\right) \right\|_{L^p} \leq M'\,s^\alpha\,,$$

and hence

$$\left\| g(s) - 2g\left(\frac{s}{2}\right) \right\|_{L^p} \leq M'\,s^\alpha\,. \tag{2.2.3}$$

Replacing s by $\frac{s}{2}$ in formula (2.2.3) we get

$$\left\| g\left(\frac{s}{2}\right) - 2g\left(\frac{s}{2^2}\right) \right\|_{L^p} \leq M'\,\frac{s^\alpha}{2^\alpha}\,,$$

and hence

$$\left\| 2g\left(\frac{s}{2}\right) - 2^2 g\left(\frac{s}{2^2}\right) \right\|_{L^p} \leq M'\,s^\alpha 2^{1-\alpha}\,. \tag{2.2.4}$$

Repeating this procedure we get

$$\left\| 2^{n-1} g\left(\frac{s}{2^{n-1}}\right) - 2^n g\left(\frac{s}{2^n}\right) \right\|_{L^p} \leq M'\,s^\alpha 2^{(1-\alpha)(n-1)}\,. \tag{2.2.5}$$

Adding the above inequalities we get

$$\left\| g\left(s\right) - 2^n g\left(\frac{s}{2^n}\right) \right\|_{L^p} \leq M'\,s^\alpha \sum_{k=0}^{n-1} 2^{(1-\alpha)k}\,. \tag{2.2.6}$$

If $0 < \alpha < 1$ then

$$\left\| g\left(s\right) - 2^n g\left(\frac{s}{2^n}\right) \right\|_{L^p} \leq M'\,s^\alpha\,\frac{2^{(1-\alpha)n} - 1}{2^{1-\alpha} - 1} \leq M'\,s^\alpha\,\frac{2^{(1-\alpha)n}}{2^{1-\alpha} - 1}$$

and hence

$$\left\| g\left(\frac{s}{2^n}\right) \right\|_{L^p} \leq \frac{1}{2^n} 2||u||_{L^p} + cM' s^\alpha\,2^{-\alpha n}\,.$$

Consider now $0 < a < \frac{\sigma}{2}$ fixed and $s \in \left[\frac{a}{2}, a\right]$. For all $h > 0$ sufficiently small there exist $n \in \mathbb{N}$ and $s \in \left[\frac{a}{2}, a\right]$ such that $h = \frac{s}{2^n}$. Then

$$||g(h)||_{L^p} \leq \frac{4h}{a}||u||_{L^p} + cM'h^\alpha \, .$$

Dividing this last inequality by h^α we get (2.2.2).

If $\alpha = 1$, then inequality (2.2.6) implies that

$$\left\|g\left(s\right) - 2^n g\left(\frac{s}{2^n}\right)\right\|_{L^p} \leq M' \, s \, n \, . \tag{2.2.7}$$

Consider now $h = \frac{s}{2^n}$ in a similar way as for the previous case and observe that $n = O(\log h)$ to get

$$||g(h)||_{L^p} \leq 2h||u||_{L^p} + hO(\log h) \, , \tag{2.2.8}$$

and hence we can use any $\beta < 1$ to get (2.2.2).

If $\alpha > 1$ then inequality (2.2.6) implies that

$$\left\|g\left(s\right) - 2^n g\left(\frac{s}{2^n}\right)\right\|_{L^p} \leq M' \, s^\alpha \, \frac{1}{1 - 2^{(1-\alpha)}} \, . \tag{2.2.9}$$

Therefore, we have

$$||g\left(\frac{s}{2^n}\right)|| \leq \frac{1}{2^n}2||u||_{L^p} + \frac{1}{2^n}M's^\alpha \frac{1}{1 - 2^{(1-\alpha)}} \, ,$$

and hence for $h = \frac{s}{2^n}$ and $s \in [\frac{a}{2}, a]$ we obtain

$$||g(h)||_{L^p} \leq \frac{4h}{a}||u||_{L^p} + \frac{2h}{a}M'\frac{1}{1 - 2^{(1-\alpha)}}a^\alpha \, . \tag{2.2.10}$$

Now we can use $\beta = 1$ to get (2.2.2). □

Remark 2.2.1. Proposition 2.1.1 together with Lemma 2.2.1 implies that if u has compact support K and (2.2.1) is satisfied with $\alpha > 1$, then $Tu \in L^p(K)$.

2.3 ITERATIONS IN THE T-DIRECTION FOR $P \geq 2$.

We prove a general lemma, that constitutes the key step in our iteration. In an informal way, we can say that if u_ε has locally $\frac{1}{2}+\alpha$ derivatives in the T direction, then it also has $\frac{1}{2}+\frac{1}{p}+\frac{2}{p}\alpha$ derivatives in the same direction.

Lemma 2.3.1. *Let $u_\varepsilon \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution of (2.1.3), $x_0 \in \Omega$, $r > 0$ such that $B(x_0, 3r) \Subset \Omega$. Let us suppose that there exists constants $c > 0$, $\sigma > 0$ and $\alpha \in [0, \frac{1}{2})$ such that*

$$\sup_{0 \neq |s| \leq \sigma} \int_{B(x_0,r)} \left| D_{\bar{s},\frac{1}{2}+\alpha}(u_\varepsilon) \right|^p dx \quad \leq \quad c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2 \right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx \,. \quad (2.3.1)$$

If we have

$$\frac{1+2\alpha}{p} < \frac{1}{2}$$

then for possibly different $c > 0$, $\sigma > 0$ holds

$$\sup_{0 \neq |s| \leq \sigma} \int_{B(x_0,\frac{r}{2})} \left| D_{\bar{s},\frac{1}{2}+\frac{1}{p}+\frac{2}{p}\alpha}(u_\varepsilon) \right|^p dx$$
$$\leq c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2 \right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx \,. \quad (2.3.2)$$

In the case

$$\frac{1+2\alpha}{p} > \frac{1}{2}$$

we have that

$$\int_{B(x_0,\frac{r}{2})} |Tu_\varepsilon(x)|^p \, dx \leq c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2 \right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx \,. \quad (2.3.3)$$

Otherwise,

$$\frac{1+2\alpha}{p} = \frac{1}{2}$$

and we have that

$$\int_{B(x_0,\frac{r}{4})} |Tu_\varepsilon(x)|^p \, dx \leq c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2 \right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx \,. \quad (2.3.4)$$

Proof. Consider

$$\gamma = \frac{1}{2} + \alpha\,,$$

and let g be a cut-off function between $B(x_0, \frac{r}{2})$ and $B(x_0, r)$. We use now the test function

$$\varphi = D_{-\bar{s},\gamma}\ \left(g^2 D_{\bar{s},\gamma}\, u_\varepsilon\right) \tag{2.3.5}$$

to get

$$\sum_{i=1}^{2n} \int_\Omega a_i^\varepsilon(Xu_\varepsilon(x))\ X_i\Big(D_{-\bar{s},\gamma}\ \left(g^2 D_{\bar{s},\gamma}\, u_\varepsilon(x)\right)\Big)dx = 0$$

and from here, by the fact that X_i commutes with $D_{\bar{s},\gamma}$ and $D_{-\bar{s},\gamma}$, we obtain

$$\sum_{i=1}^{2n} \int_\Omega D_{\bar{s},\gamma}\ a_i^\varepsilon(Xu_\varepsilon(x))\ g^2(x)\ D_{\bar{s},\gamma}\ (X_i u_\varepsilon(x))\,dx$$
$$+ \sum_{i=1}^{2n} \int_\Omega D_{\bar{s},\gamma}\ a_i^\varepsilon(Xu_\varepsilon(x))\ D_{\bar{s},\gamma}\, u_\varepsilon(x)\ 2g(x)\ X_i g(x)\,dx = 0\,. \tag{2.3.6}$$

We can use now similar arguments as in Marchi's proof [17, 19], involving the properties of the functions a_i^ε and Lemma 8.3 [11] to get

$$\int_{B(x_0,r)} g^2(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |D_{\bar{s},\gamma}\, Xu_\varepsilon(x)|^2\ dx$$

$$\le c \int_{B(x_0,r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |D_{\bar{s},\gamma}\, Xu_\varepsilon(x)|$$
$$\cdot\, |D_{\bar{s},\gamma}\, u_\varepsilon(x)|\ \ |g(x)|\ |Xg(x)|dx\,.$$

Using the fact that $p \geq 2$ we get

$$\int_{B(x_0,r)} g^2(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |D_{\bar{s},\gamma}\, Xu_\varepsilon(x)|^2\ dx$$
$$\le c \int_{B(x_0,r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |D_{\bar{s},\gamma}\, u_\varepsilon(x)|^2\ \ |Xg(x)|^2\,dx\,. \tag{2.3.7}$$

Denoting by RHS the right hand side of (2.3.7) we have that

$$RHS \le c \int_{B(x_0,r)} \Big(\left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p}{2}} + |D_{\bar{s},\gamma}\, u_\varepsilon(x)|^p\Big)\ dx\,.$$

Using (2.3.1) we get that

$$RHS \leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx$$

and therefore

$$\int_{B(x_0,r)} g^2(x) \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p-2}{2}} \ |D_{\bar{s},\gamma}\, Xu_\varepsilon(x)|^2 \ dx$$
$$\leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, . \quad (2.3.8)$$

From the inequality

$$|s^\gamma D_{s,\gamma} Xu_\varepsilon(x)| \leq \sqrt{2}\sqrt{\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2}$$

we get

$$\int_{B(x_0,r)} g^2(x) \quad s^{(p-2)\gamma} \quad |D_{\bar{s},\gamma}\, Xu_\varepsilon(x)|^p \ dx \ \ \leq \ \ c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} \ + \ |u_\varepsilon(x)|^p \, dx \, .$$

Since

$$D_{\bar{s},\gamma}\, X(g^2 u_\varepsilon)(x) = D_{\bar{s},\gamma}\, X(g^2)(x) \ u_\varepsilon(x \cdot \bar{s}) + X(g^2)(x) \ D_{\bar{s},\gamma}\, u_\varepsilon(x)$$
$$+ D_{\bar{s},\gamma}\, g^2(x) \ Xu_\varepsilon(x \cdot \bar{s}) + g^2(x) \ D_{\bar{s},\gamma}\, Xu_\varepsilon(x)$$

it follows that

$$\int_{B(x_0,r)} \left| D_{\bar{s},\frac{2\gamma}{p}} X(g^2 u_\varepsilon)(x)\right|^p \, dx \quad \leq \quad c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) \, dx \, . \quad (2.3.9)$$

Let us denote the right hand side of (2.3.9) by M^p. Using Proposition 2.1.2 we get

$$\int_{B(x_0,r)} \left| D_{-\bar{s},\frac{1}{2}} \, D_{\bar{s},\frac{2\gamma}{p}} (g^2 u_\varepsilon)(x)\right|^p \, dx \leq M^p \, . \quad (2.3.10)$$

Therefore, for all s sufficiently small we have

$$\frac{\|\triangle^2_{\bar{s}}(g^2 u_\varepsilon)\|_{L^p(\mathbb{H}^n)}}{s^{\frac{1}{2}+\frac{1+2\alpha}{p}}} \leq M \, ,$$

so there exists $\sigma > 0$ such that

$$\sup_{0<|s|\leq\sigma} \frac{\|\triangle^2_{\bar{s}}(g^2 u_\varepsilon)\|_{L^p(\mathbb{H}^n)}}{s^{\frac{1}{2}+\frac{1+2\alpha}{p}}} \leq M \, . \quad (2.3.11)$$

If it happens that

$$\frac{1+2\alpha}{p} < \frac{1}{2}$$

then by Lemma 2.2.1 we get (2.3.2).

If we have

$$\frac{1+2\alpha}{p} > \frac{1}{2}$$

then by Lemma 2.2.1 we have $Tu \in L^p_{\text{loc}}(\Omega)$ and estimate (2.3.3) is valid.

In the remaining case

$$\frac{1+2\alpha}{p} = \frac{1}{2}$$

and then using that $\alpha \in [0, \frac{1}{2})$ we get

$$0 \leq \frac{p-2}{4} < \frac{1}{2}$$

which gives $2 \leq p < 4$. Lemma 2.2.1 implies that we can use (2.3.1) with α' arbitrarily close to $\frac{1}{2}$, in particular $\alpha' > \frac{p-2}{4}$, to get back (2.3.11) with

$$\frac{1+2\alpha'}{p} > \frac{1}{2}$$

and then use the previous case. $\square$

Proposition 2.1.2 implies that we can start with $\alpha_0 = 0$ in the assumption (2.3.1) to get $\alpha_1 = \frac{1}{p}$ in (2.3.2). Now we can use α_1 in (2.3.1) to get

$$\alpha_2 = \frac{1}{p} + \frac{2}{p}\alpha_1$$

such that estimate (2.3.2) is true. In general, if we already found $\alpha_1, ..., \alpha_k$, then we get

$$\alpha_{k+1} = \frac{1}{p} + \frac{2}{p}\alpha_k = \frac{1}{p} + ... + \frac{2^{k-2}}{p^{k-1}} + \frac{2^{k-1}}{p^{k-1}}\alpha_1 = \frac{1}{p}\sum_{i=0}^{k-1}\left(\frac{2}{p}\right)^i = \frac{1}{p}\,\frac{1-\left(\frac{2}{p}\right)^k}{1-\frac{2}{p}}.$$

Therefore, for a given $p > 2$ the supremum for the numbers α_k, $k \in \mathbb{N}$ is given by

$$\frac{1}{p-2}.$$

Hence, for $p \in [2, 4)$, after a number sufficiently large of k iterations, we get that $\alpha_k \geq \frac{1}{2}$ and this means that $Tu_\varepsilon \in L^p_{\text{loc}}(\Omega)$.

Remark 2.3.1. If we ask for $\alpha_2 \geq \frac{1}{2}$ then we get the inequality

$$p^2 - 2p - 4 \leq 0$$

that leads to Marchi's result $p \in [2, 1+\sqrt{5})$.

We can summarize our results from this section by the following theorem that extends the results of Marchi [17].

Theorem 2.3.1. *If $2 \leq p < 4$, then for any weak solution u_ε of (2.1.3) we have that $Tu_\varepsilon \in L^p_{\mathrm{loc}}(\Omega)$ with bounds locally independent of ε.*

In the case $p \geq 4$ our proof gives the following result.

Theorem 2.3.2. *For $p \geq 4$ and weak solutions u_ε of (2.1.3) we have*

$$\sup_{0 \neq |s| \leq \sigma} \int_{B(x_0, \frac{r}{2^k})} \left| D_{\bar{s}, \frac{1}{2}+\alpha'}(u_\varepsilon) \right|^p dx$$
$$\leq c \int_{B(x_0, 2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx\,. \quad (2.3.12)$$

for $c > 0$ independent of ε, α' less then, but arbitrarily close to $\frac{1}{p-2}$, and a corresponding number k of iterations.

2.4 ITERATIONS IN THE T-DIRECTION FOR $1 < p < 2$.

Theorem 2.4.1. *Let $1 < p < 2$ and $u_\varepsilon \in HW^{1,p}_{loc}(\Omega)$ be a weak solution of (2.1.3). Then $Tu_\varepsilon \in L^p_{loc}(\Omega)$ with bounds locally independent of ε.*

Proof. Let us consider arbitrary $x_0 \in \Omega$, $r > 0$ such that $B(x_0, 3r) \subset \Omega$ and let g be a cut off function between $B(x_0, \frac{r}{2})$ and $B(x_0, r)$. We can follow then the proof of Lemma 2.3.1 for $\alpha = 0$ and $\gamma = \frac{1}{2}$ until we get

$$\int_{B(x_0,r)} g^2(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p-2}{2}}\ |D_{\bar{s},\gamma}\, Xu_\varepsilon(x)|^2\, dx$$
$$\leq c \int_{B(x_0,r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p-2}{2}}\ |D_{\bar{s},\gamma}\, Xu_\varepsilon(x)|$$
$$\cdot\, |D_{\bar{s},\gamma}\, u_\varepsilon(x)|\ |g(x)|\ |Xg(x)| dx\,. \quad (2.4.1)$$

Let us denote by RHS the right hand side of (2.4.1). We will keep using γ instead of $\frac{1}{2}$ to get a general iteration formula. Then

$$RHS \leq \frac{c}{s^\gamma} \int_{B(x_0,r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p-2}{2}} \cdot |Xu_\varepsilon(x \cdot \bar{s}) - Xu_\varepsilon(x)| \, |D_{\bar{s},\gamma}\, u_\varepsilon(x)| \, dx$$

$$\leq \frac{c}{s^\gamma} \int_{B(x_0,r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p-2}{2}} \cdot \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{1}{2}} \; |D_{\bar{s},\gamma}\, u_\varepsilon(x)| \, dx$$

$$= \frac{c}{s^\gamma} \int_{B(x_0,r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p-1}{2}} \; |D_{\bar{s},\gamma}\, u_\varepsilon(x)| \, dx$$

$$\leq \; \frac{c}{s^\gamma} \left(\int_{B(x_0,r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p}{2}} \; dx \right)^{\frac{p-1}{p}} \cdot \left(\int_{B(x_0,r)} |D_{\bar{s},\gamma}\, u_\varepsilon(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$\leq \frac{c}{s^\gamma} \left(\int_{B(x_0,r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p}{2}} \; dx \right)^{\frac{p-1}{p}} \cdot \left(\int_{B(x_0,2r)} \left(|u_\varepsilon|^p + |Xu_\varepsilon|^p\right) dx \right)^{\frac{1}{p}}$$

$$\leq \frac{c}{s^\gamma} \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, .$$

Therefore,

$$\int_{B(x_0,r)} g^2(x) \; \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p-2}{2}} \; |Xu_\varepsilon(x \cdot \bar{s}) - Xu_\varepsilon(x)|^2 \; dx$$
$$\leq c\, s^\gamma \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, . \quad (2.4.2)$$

We need the following inequalities used initially in the Euclidean case (see [16]).

$$
\begin{aligned}
&\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p}{2}}\\
&\leq \left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p}{2}-1}\ \left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})|^2\right)\\
&\leq \left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p}{2}-1}\cdot\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})-Xu_\varepsilon(x)|^2\right)\\
&\leq 3\left(\varepsilon+|Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}}+3\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p}{2}-1}\cdot|Xu_\varepsilon(x\cdot\bar{s})-Xu_\varepsilon(x)|^2
\end{aligned}
$$

We can suppose $s\leq 1$ and then

$$
\begin{aligned}
&\int_{B(x_0,r)} g^2(x)\ \left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p}{2}}\ dx\\
&\leq 3\int_{B(x_0,r)} g^2(x)\ \left(\varepsilon+|Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}}\ dx+\ c\int_{B(x_0,2r)}\left(\varepsilon+|Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}}+|u_\varepsilon(x)|^p\,dx\\
&\leq c\int_{B(x_0,2r)}\left(\varepsilon+|Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}}+|u_\varepsilon(x)|^p\,dx\,.
\end{aligned}
$$

Also, by Hölder's inequality we get

$$
\int_{B(x_0,r)} g^2(x)\ |Xu_\varepsilon(x\cdot\bar{s})-Xu_\varepsilon(x)|^p\ dx
$$

$$
\begin{aligned}
=\int_{B(x_0,r)}&\left(g^2(x)\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p}{2}-1}\ |Xu_\varepsilon(x\cdot\bar{s})-Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}}\\
&\cdot\left(g^{\frac{4}{p}}(x)\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})|^2\right)\right)^{\left(1-\frac{p}{2}\right)\frac{p}{2}}\ dx
\end{aligned}
$$

$$
\begin{aligned}
\leq&\left(\int_{B(x_0,r)} g^2(x)\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p}{2}-1}\ |Xu_\varepsilon(x\cdot\bar{s})-Xu_\varepsilon(x)|^2\ dx\right)^{\frac{p}{2}}\\
&\cdot\left(\int_{B(x_0,r)}\left(g^{\frac{4}{p}}(x)\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot\bar{s})|^2\right)\right)^{\frac{p}{2}}\ dx\right)^{1-\frac{p}{2}}
\end{aligned}
$$

$$\leq \left(c\, s^{\gamma} \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \right)^{\frac{p}{2}}$$

$$\cdot \left(\int_{B(x_0,r)} g^2(x) \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p}{2}} \, dx \right)^{1-\frac{p}{2}}$$

$$\leq c\, s^{p\frac{\gamma}{2}} \left(\int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \right)^{\frac{p}{2}}$$

$$\cdot \left(\int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \right)^{1-\frac{p}{2}}$$

$$\leq \; c\, s^{p\frac{\gamma}{2}} \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, .$$

Therefore,

$$\int_{B(x_0,r)} g^2(x) \left| D_{\bar{s},\frac{\gamma}{2}} Xu_\varepsilon(x) \right|^p \, dx \leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, .$$

In the same way as we obtained inequality (2.3.9), we get

$$\int_{B(x_0,r)} \left| D_{\bar{s},\frac{\gamma}{2}} X(g^2 u_\varepsilon)(x) \right|^p \, dx \leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, . \tag{2.4.3}$$

Proposition 2.1.2 implies that

$$\int_{B(x_0,r)} \left| D_{-\bar{s},\frac{1}{2}} \, D_{\bar{s},\frac{\gamma}{2}} (g^2 u_\varepsilon)(x) \right|^p \, dx \;\; \leq \;\; c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} \; + \; |u_\varepsilon(x)|^p \, dx \, , \tag{2.4.4}$$

and this means for a sufficiently small σ

$$\sup_{0<|s|\leq\sigma} \frac{||\triangle_{\bar{s}}^2 (g^2 u_\varepsilon)||_{L^p(\mathbb{H}^n)}}{|s|^{\frac{1}{2}+\frac{\gamma}{2}}} \leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, . \tag{2.4.5}$$

We started with $\gamma = \frac{1}{2}$ therefore in (2.4.3) we have a power of $\frac{1}{4}$ for s while in (2.4.5) we have a power of $\frac{3}{4}$. Using Lemma 2.2.1 we can do iterations to obtain after k steps and corresponding cut off functions between $B(x_0, \frac{r}{2^k})$ and that $B(x_0, \frac{r}{2^{k-1}})$ that

$$\int_{B(x_0,\frac{r}{2^{k-1}})} \left| D_{\bar{s},\frac{2^k-1}{2^{k+1}}} X(g^2 u_\varepsilon)(x) \right|^p \, dx$$

$$\leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, , \tag{2.4.6}$$

and

$$\sup_{0<|s|\leq\sigma} \frac{||\triangle_{\bar{s}}(g^2 u_\varepsilon)||_{L^p(\mathbb{H}^n)}}{|s|^{\frac{2^{k+1}-1}{2^{k+1}}}} \leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx\,. \tag{2.4.7}$$

Let us consider now $k \in \mathbb{N}$ such that

$$\frac{1}{2^k - 1} < p - 1\,.$$

Then for

$$a = \frac{2^k - 1}{2^{k+1}} \quad \text{and} \quad b = \frac{2^{k+1} - 1}{2^{k+1}}$$

we have

$$a(p-1) + b > 1\,.$$

Let us consider now

$$\gamma = \frac{a(p-1)+b}{2} > \frac{1}{2}$$

and return to (2.4.1) with a cut off function g between $B(x_0, \frac{r}{2^{k+1}})$ and $B(x_0, \frac{r}{2^k})$. Then

$$\begin{aligned} RHS \leq c \int_{B(x_0,\frac{r}{2^k})} & \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}} \; |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^{2-p} \\ & \cdot \frac{|Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^{p-1}}{s^{a(p-1)}} \; |D_{\bar{s},b}u_\varepsilon(x)| dx \\ \leq \quad & \int_{B(x_0,\frac{r}{2^k})} \frac{|Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^{p-1}}{s^{a(p-1)}} \qquad |D_{\bar{s},b}u_\varepsilon(x)| dx \\ \leq \quad & \left(\int_{B(x_0,\frac{r}{2^k})} \frac{|Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^{p}}{s^{ap}}\, dx\right)^{\frac{p-1}{p}} \qquad \left(\int_{B(x_0,\frac{r}{2^k})} |D_{\bar{s},b}u_\varepsilon(x)|^p \, dx\right)^{\frac{1}{p}} \\ \leq \quad & c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} \quad + \quad |u_\varepsilon(x)|^p \, dx\,. \end{aligned}$$

Therefore,

$$\int_{B(x_0,\frac{r}{2^k})} g^2(x) \; \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}} \; |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^2 \; dx \leq$$

$$\leq c\, s^{2\gamma} \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, . \tag{2.4.8}$$

Doing a similar proof as we did starting from formula (2.4.2) we get that

$$\sup_{0<|s|\leq\sigma} \frac{||\triangle^2_{\bar{s}}(g^2 u_\varepsilon)||_{L^p(\mathbb{H}^n)}}{|s|^{\frac{1}{2}+\gamma}} \leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, . \tag{2.4.9}$$

Using the fact that $\frac{1}{2} + \gamma > 1$, Lemma 2.2.1 implies now that

$$\int_{B(x_0,\frac{r}{2^{k+1}})} |Tu_\varepsilon(x)|^p \, dx \leq c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) \, dx \tag{2.4.10}$$

and therefore $Tu_\varepsilon \in L^p_{loc}(\Omega)$. $\square$

3.0 SECOND ORDER HORIZONTAL DIFFERENTIABILITY OF THE APPROXIMATING p-HARMONIC FUNCTIONS

3.1 CASE $p \geq 2$

In this section we prove the $HW^{2,2}$ regularity of the approximate p-harmonic functions u_ε. As immediate consequences of the results from the previous section we can prove that:

Proposition 3.1.1. *With estimates depending on ε we have the following two regularity properties.*

(1) For all $p \geq 2$ we have $Tu_\varepsilon \in L^2_{\text{loc}}(\Omega)$.

(2) For $2 \leq p < 4$ we have that also $XTu_\varepsilon \in L^2_{\text{loc}}(\Omega)$.

Proof. For $2 \leq p < 4$ we know that $Tu_\varepsilon \in L^p_{\text{loc}}(\Omega) \subset L^2_{\text{loc}}(\Omega)$. Theorem 2.3.2 implies that for all $p \geq 4$, $x_0 \in \Omega$ and $r > 0$ sufficiently small we can choose an $\alpha > 0$, a cut off function g between $B(x_0, \frac{r}{2^{k+1}})$ and $B(x_0, \frac{r}{2^k})$ and repeat the proof of Lemma 2.3.1 until we obtain for $\gamma = \frac{1}{2} + \alpha$ and

$$M^p = \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx$$

we have

$$\varepsilon^{\frac{p-2}{2}} \int_{B(x_0,\frac{r}{2^k})} g^2(x) \quad \left|D_{\bar{s},\gamma} \, Xu_\varepsilon(x)\right|^2 \, dx \leq cM^p \, .$$

From here we obtain

$$\int_{B(x_0,\frac{r}{2^k})} \left|D_{\bar{s},\gamma} \, X\left(g^2 u_\varepsilon\right)(x)\right|^2 \, dx \leq c\varepsilon^{\frac{2-p}{2}} (M^p + M^2) \, . \tag{3.1.1}$$

Proposition 2.1.2 implies that

$$\int_{B(x_0,\frac{r}{2^k})} \left|D_{-\bar{s},\frac{1}{2}} \, D_{\bar{s},\gamma} \left(g^2 u_\varepsilon\right)(x)\right|^2 \, dx \leq c\varepsilon^{\frac{2-p}{2}} (M^p + M^2) \, dx$$

and hence for some $\sigma > 0$ holds

$$\sup_{0<|s|\leq\sigma} \frac{\|\triangle_{\bar{s}}^2(g^2u_\varepsilon)\|_{L^2(\mathbb{H}^n)}}{s^{1+\alpha}} \leq \left(c\varepsilon^{\frac{2-p}{2}}(M^p + M^2)\right)^{\frac{1}{2}}.$$

Lemma 2.2.1 gives now that $Tu_\varepsilon \in L^2_{\text{loc}}(\Omega)$.

To prove the second part let us observe that in the case $2 \leq p < 4$ we can start the proof with $\gamma = 1$ and get

$$\int_{B(x_0,\frac{r}{2^k})} \left|D_{\bar{s},1}X\big(g^2u_\varepsilon\big)(x)\right|^2 \, dx \leq c\varepsilon^{\frac{2-p}{2}}(M^p + M^2), \tag{3.1.2}$$

and hence by Proposition 2.1.1 we have $TXu_\varepsilon = XTu_\varepsilon \in L^2_{\text{loc}}(\Omega)$. □

Theorem 3.1.1. *Let $2 \leq p < 4$ and $u_\varepsilon \in HW^{1,p}_{\text{loc}}(\Omega)$ be a weak solution of (2.1.3). Also consider $x_0 \in \Omega$, $r > 0$ such that $B(x_0, 3r) \in \Omega$ and let k be the number of iterations depending only on p. Then we have*

$$\int_{B(x_0,\frac{r}{2^{k+2}})} \big(\varepsilon + |Xu_\varepsilon(x)|^2\big)^{\frac{p-2}{2}} \left|X^2u_\varepsilon(x)\right|^2 \, dx \leq \; c\int_{B(x_0,2r)} \Big(\varepsilon + |Xu(x)|^2\Big)^{\frac{p}{2}} + |u(x)|^p \, dx. \tag{3.1.3}$$

Proof. For the proof we use a simplified version of Marchi's method [17] and use the extended range of $2 \leq p < 4$ obtained in the previous chapter.
For $i_0 \in \{1, ..., n\}$, $h > 0$, let us denote $h_{i_0} = (0, ..., h, ...0, 0) \in \mathbb{H}^n$ with the h in the i_0th place. We will use the notation

$$D_{h_{i_0}} = D_{X_{i_0},h,1} \text{ and } D_{-h_{i_0}} = D_{X_{i_0},-h,1}$$

and the test function

$$\varphi = D_{-h_{i_0}}D_{h_{i_0}}(g^4u_\varepsilon)$$

where g is a cut-off function between $B(x_0, \frac{r}{2^{k+2}})$ and $B(x_0, \frac{r}{2^{k+1}})$.

For $i \neq i_0 + n$ we have

$$X_i\left(D_{-h_{i_0}}D_{h_{i_0}}\big(g^4u_\varepsilon\big)\right) = D_{-h_{i_0}}D_{h_{i_0}}\Big(X_i\big(g^4u_\varepsilon\big)\Big),$$

while for $i = i_0 + n$ we have

$$X_{i_0+n}\left(D_{-h_{i_0}} D_{h_{i_0}}\left(g^4 u_\varepsilon\right)\right)(x) = D_{-h_{i_0}} D_{h_{i_0}}\Big(X_{i_0+n}\left(g^4 u_\varepsilon\right)\Big)(x) - \frac{1}{h}\Big(T\left(g^4 u_\varepsilon\right)\left(x \cdot h_{i_0}\right) - T\left(g^4 u_\varepsilon\right)\left(x \cdot h_{i_0}^{-1}\right)\Big)\,. \tag{3.1.4}$$

To see that formula (3.1.4) is true it is enough to observe that

$$X_{i_0+n}\Big(g^4 u_\varepsilon\left(x \cdot h_{i_0}\right)\Big) = X_{i_0+n}\left(g^4 u_\varepsilon\right)\left(x \cdot h_{i_0}\right) - hT\left(g^4 u_\varepsilon\right)\left(x \cdot h_{i_0}\right)$$

and

$$X_{i_0+n}\Big(g^4 u_\varepsilon\left(x \cdot h_{i_0}^{-1}\right)\Big) = X_{i_0+n}\left(g^4 u_\varepsilon\right)\left(x \cdot h_{i_0}^{-1}\right) + hT\left(g^4 u_\varepsilon\right)\left(x \cdot h_{i_0}^{-1}\right)\,.$$

Using the test function φ in the weak form of the equation (2.1.3) we get

$$\sum_{i=1}^{2n} \int_\Omega a_i^\varepsilon\left(Xu_\varepsilon(x)\right)\, D_{-h_{i_0}} D_{h_{i_0}} X_i\left(g^4 u_\varepsilon\right)(x)\, dx = \\ = \int_\Omega a_{i_0+n}^\varepsilon\left(Xu_\varepsilon(x)\right)\, \frac{1}{h}\, \Big(T\left(g^4 u_\varepsilon\right)\left(x \cdot h_{i_0}\right) - T\left(g^4 u_\varepsilon\right)\left(x \cdot h_{i_0}^{-1}\right)\Big) dx\,. \tag{3.1.5}$$

Therefore,

$$\sum_{i=1}^{2n} \int_\Omega D_{h_{i_0}} a_i^\varepsilon\left(Xu_\varepsilon(x)\right)\, D_{h_{i_0}} X_i\left(g^4 u_\varepsilon\right)(x)\, dx = \\ = -\int_\Omega a_{i_0+n}^\varepsilon\left(Xu_\varepsilon(x)\right)\, \Big(D_{h_{i_0}} T\left(g^4 u_\varepsilon\right)(x) + D_{-h_{i_0}} T\left(g^4 u_\varepsilon\right)(x)\Big) dx\,.$$

We use that

$$
\begin{aligned}
D_{h_{i_0}} X_i\big(g^4 u_\varepsilon\big)(x) &= D_{h_{i_0}}\ \Big(4g^3(x)\ X_i g(x)\ u_\varepsilon(x) + g^4(x)\ X_i u_\varepsilon(x)\Big) = \\
&= 4 D_{h_{i_0}} g(x)\ g^2(x\cdot h_{i_0})\ X_i g(x\cdot h_{i_0})\ u(x\cdot h_{i_0}) \\
&+ 4g(x)\ D_{h_{i_0}} g(x)\ g(x\cdot h_{i_0})\ X_i g(x\cdot h_{i_0})\ u_\varepsilon(x\cdot h_{i_0}) \\
&+ 4g^2(x)\ D_{h_{i_0}} g(x)\ X_i g(x\cdot h_{i_0})\ u_\varepsilon(x\cdot h_{i_0}) \\
&+ 4g^3(x)\ D_{h_{i_0}} X_i g(x)\ u(x\cdot h_{i_0}) \\
&+ 4g^3(x)\ X_i g(x)\ D_{h_{i_0}} u_\varepsilon(x) \\
&+ D_{h_{i_0}} g(x)\ g^3(x\cdot h_{i_0})\ X_i u_\varepsilon(x\cdot h_{i_0}) \\
&+ g(x)\ D_{h_{i_0}} g(x)\ g^2(x\cdot h_{i_0})\ X_i u_\varepsilon(x\cdot h_{i_0}) \\
&+ g^2(x)\ D_{h_{i_0}} g(x)\ g(x\cdot h_{i_0})\ X_i u_\varepsilon(x\cdot h_{i_0}) \\
&+ g^3(x)\ D_{h_{i_0}} g(x)\ X_i u_\varepsilon(x\cdot h_{i_0}) \\
&+ g^4(x)\ D_{h_{i_0}} X_i u_\varepsilon(x)
\end{aligned}
$$

Therefore, equation (3.1.5) has the form

$$
\sum_{i=1}^{2n} \int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ D_{h_{i_0}} X_i u_\varepsilon(x)\ g^4(x)\,dx = \tag{L1}
$$

$$
= \int_\Omega D_{-h_{i_0}} a^\varepsilon_{i_0+n}\big(Xu_\varepsilon(x)\big)\ T\big(g^4 u_\varepsilon(x)\big)\,dx + \int_\Omega D_{h_{i_0}} a^\varepsilon_{i_0+n}\big(Xu_\varepsilon(x)\big)\ T\big(g^4 u_\varepsilon(x)\big)\,dx \tag{R1}
$$

$$
-\sum_{i=1}^{2n} \int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ 4D_{h_{i_0}} g(x)\ g^2(x\cdot h_{i_0})\ X_i g(x\cdot h_{i_0})\ u(x\cdot h_{i_0})\,dx \tag{R2}
$$

$$-\sum_{i=1}^{2n}\int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ 4g(x)\ D_{h_{i_0}}g(x)\ g(x\cdot h_{i_0})X_i g(x\cdot h_{i_0})\ u_\varepsilon(x\cdot h_{i_0})\,dx \tag{R3}$$

$$-\sum_{i=1}^{2n}\int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ 4g^2(x)\ D_{h_{i_0}}g(x)\ X_i g(x\cdot h_{i_0})\ u_\varepsilon(x\cdot h_{i_0})\,dx \tag{R4}$$

$$-\sum_{i=1}^{2n}\int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ 4g^3(x)\ D_{h_{i_0}}X_i g(x)\ u(x\cdot h_{i_0})\,dx \tag{R5}$$

$$-\sum_{i=1}^{2n}\int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ 4g^3(x)\ X_i g(x)\ D_{h_{i_0}}u_\varepsilon(x)\,dx \tag{R6}$$

$$-\sum_{i=1}^{2n}\int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ D_{h_{i_0}}g(x)\ g^3(x\cdot h_{i_0})\ X_i u_\varepsilon(x\cdot h_{i_0})\,dx \tag{R7}$$

$$-\sum_{i=1}^{2n}\int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ g(x)\ D_{h_{i_0}}g(x)\ g^2(x\cdot h_{i_0})\ X_i u_\varepsilon(x\cdot h_{i_0})\,dx \tag{R8}$$

$$-\sum_{i=1}^{2n}\int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ g^2(x)\ D_{h_{i_0}}g(x)\ g(x\cdot h_{i_0})\ X_i u_\varepsilon(x\cdot h_{i_0})\,dx \tag{R9}$$

$$
-\sum_{i=1}^{2n}\int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ g^3(x)\ D_{h_{i_0}} g(x)\ X_i u_\varepsilon(x\cdot h_{i_0})\,dx
\tag{R10}
$$

We estimate now each of the above lines. We will use $\delta > 0$ as a sufficiently small number.

$$
(L1) \geq c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{2}}\ \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|^2\ g^4(x)\ dx\,.
$$

$$
\begin{aligned}
(R1) \leq{}& c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0}^{-1})|^2\right)^{\frac{p-2}{2}}\ \left|D_{-h_{i_0}} Xu_\varepsilon(x)\right|\ g^4(x)|Tu_\varepsilon(x)|\ dx\\
&+ c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0}^{-1})|^2\right)^{\frac{p-2}{2}}\ \left|D_{-h_{i_0}} Xu_\varepsilon(x)\right|\ 4|g^3(x)||Tg(x)|\ |u_\varepsilon(x)|\ dx\\
&+ c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{2}}\ \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|\ g^4(x)\ |Tu_\varepsilon(x)|\ dx\\
&+ c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{2}}\ \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|\ 4|g^3(x)|\ |Tg(x)|\ |u_\varepsilon(x)|\ dx
\end{aligned}
$$

$$
\begin{aligned}
\leq{}& \delta\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0}^{-1})|^2\right)^{\frac{p-2}{2}}\ \left|D_{-h_{i_0}} Xu_\varepsilon(x)\right|^2\ g^4(x)\,dx\\
&+ c(\delta)\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0}^{-1})|^2\right)^{\frac{p-2}{2}}\ g^4(x)\ |Tu_\varepsilon(x)|^2\,dx\\
&+ c(\delta)\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0}^{-1})|^2\right)^{\frac{p-2}{2}}\ g^2(x)\ |Tg(x)|^2\ |u_\varepsilon(x)|^2\,dx\\
&+ \delta\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{2}}\ \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|^2\ g^4(x)\,dx\\
&+ c(\delta)\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{2}}\ g^4(x)\ |Tu_\varepsilon(x)|^2\,dx\\
&+ c(\delta)\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{2}}\ g^2(x)\ |Tg(x)|^2\ |u_\varepsilon(x)|^2\,dx
\end{aligned}
$$

$$(R2) \le c \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right| \left|D_{h_{i_0}} g(x)\right|$$
$$g^2(x)\, |Xg(x \cdot h_{i_0})|\, |u_\varepsilon(x \cdot h_{i_0})|\, dx$$
$$+ c \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right| \left|D_{h_{i_0}} g(x)\right|$$
$$h \left| \frac{g^2(x \cdot h_{i_0}) - g^2(x)}{h} \right| |Xg(x \cdot h_{i_0})|\, |u_\varepsilon(x \cdot h_{i_0})|\, dx$$

$$\le \delta \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|^2 g^4(x)\, dx$$
$$+ c(\delta) \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \left|D_{h_{i_0}} g(x)\right|^2 |Xg(x \cdot h_{i_0})|^2 |u_\varepsilon(x \cdot h_{i_0})|^2\, dx$$
$$+ c \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-1}{2}} \left|D_{h_{i_0}} g(x)\right|^2 |Xg(x \cdot h_{i_0})|\, |u_\varepsilon(x \cdot h_{i_0})|\, dx$$

The estimates for (R3) - (R5) are similar to that of (R2).

$$(R6) \le \delta \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|^2 g^4(x)\, dx$$
$$+ c(\delta) \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} g^2(x)\, |Xg(x)|^2 \left|D_{h_{i_0}} u_\varepsilon\right|^2 dx$$

$$(R7) \le c \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right| \left|D_{h_{i_0}} g(x)\right|$$
$$|g^3(x)|\, |Xu_\varepsilon(x \cdot h_{i_0})|\, dx$$
$$+ c \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right| \left|D_{h_{i_0}} g(x)\right|$$
$$h \left| \frac{g^3(x \cdot h_{i_0}) - g^3(x)}{h} \right| |Xu_\varepsilon(x \cdot h_{i_0})|\, dx$$

$$
\begin{aligned}
&\leq \delta \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|^2 \; g^4(x)\, dx \\
&+ c(\delta) \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \; |Xu_\varepsilon(x\cdot h_{i_0})|^2 \; \left|D_{h_{i_0}} g(x)\right|^2 \; g^2(x)\, dx \\
&+ c \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-1}{2}} \left|D_{h_{i_0}} g(x)\right|^2 \; |Xu_\varepsilon(x\cdot h_{i_0})|\, dx
\end{aligned}
$$

The estimates for (R8)-(R10) are similar to that of (R7). We can go back now to the beginning of the proof and use a test function

$$
\varphi = D_{h_{i_0}} D_{-h_{i_0}} (g^4 u_\varepsilon)
$$

to get similar results with $x \cdot h_{i_0}$ changed to $x \cdot h_{i_0}^{-1}$. Adding the two inequalities, embedding the terms with δ coefficient into the left hand side and using Theorem 2.3.1 we get that for all $h > 0$ sufficiently small we have

$$
\begin{aligned}
&\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \; \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|^2 \; g^4(x)\, dx \\
&+ \int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0}^{-1})|^2\right)^{\frac{p-2}{2}} \; \left|D_{-h_{i_0}} Xu_\varepsilon(x)\right|^2 \; g^4(x)\, dx \\
&\leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu(x)|^2\right)^{\frac{p}{2}} + |u(x)|^p \; dx
\end{aligned}
$$

We can repeat the proof for $n < i_0 \leq 2n$ and then we get that $X^2 u_\varepsilon \in L^2_{\mathrm{loc}}(\Omega)$ and this leads to (3.1.3). $\square$

Remark 3.1.1. Theorem 3.3 shows that $u_\varepsilon \in HW^{2,2}_{\mathrm{loc}}(\Omega)$, even if for this case the bounds for the second order horizontal derivatives have bounds dependent on ε.

3.2 CASE $1 < p < 2$.

Let us use in equation (2.1.3) a test function

$$\varphi(x) = \triangle_{-\bar{s}}\Big(g^2(x)\triangle_{\bar{s}}u_\varepsilon(x)\Big)$$

where g is a cut-off function between $B(x_0, \frac{r}{2^{k+2}})$ and $B(x_0, \frac{r}{2^{k+1}})$ to get

$$\int_\Omega g^2(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^2\ dx \leq$$
$$\leq c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|\ 2|g(x)|$$
$$|Xg(x)|\ |u_\varepsilon(x\cdot\bar{s}) - u_\varepsilon(x)|\, dx \tag{3.2.1}$$

Following a method from [11, 18] and using Young's inequality we estimate the right hand side as follows.

$$RHS = c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}+\frac{2-p}{2p}}$$
$$\left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2p}}\ |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|$$
$$2|g(x)|\ |Xg(x)|\ |u_\varepsilon(x\cdot\bar{s}) - u_\varepsilon(x)|\, dx \leq$$

$$\leq c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{(p-2)(p-1)}{2p}}\ |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^{\frac{2(p-1)}{p}}$$
$$|g(x)|\ |Xg(x)|\ |u_\varepsilon(x\cdot\bar{s}) - u_\varepsilon(x)|\, dx$$

$$\leq \delta\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^2\ g^2(x)\, dx$$
$$+ c(\delta)\int_\Omega |g(x)|^{2-p}\ |Xg(x)|^p\ |u_\varepsilon(x\cdot\bar{s}) - u_\varepsilon(x)|^p\, dx$$

Therefore,

$$\int_\Omega g^2(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^2\ dx$$
$$\leq c\int_{B(x_0,\frac{r}{2^{k+1}})} |u_\varepsilon(x\cdot\bar{s}) - u_\varepsilon(x)|^p\, dx\,.$$

The method used in the previous section for handling the left hand side gives

$$\int_\Omega g^2(x)\ |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^p\ dx \leq c\left(\int_{B(x_0,\frac{r}{2^{k+1}})} |u_\varepsilon(x\cdot\bar{s}) - u_\varepsilon(x)|^p dx\right)^{\frac{p}{2}}$$

Using Theorem 2.4.1 and Proposition 2.1.1 we get that

$$\int_\Omega \left|D_{\bar{s},\frac{p}{2}} Xu_\varepsilon(x)\right|^p \leq M^p \tag{3.2.2}$$

where we denote

$$M^p = c\int_{B(x_0,2r)} \left(\Lambda + |Xu(x)|^2\right)^{\frac{p}{2}} + |u(x)|^p\, dx\,.$$

This shows that Xu_ε has locally $\frac{p}{2}$ derivatives in the T direction. Now we use Proposition 2.1.2 to get that for a sufficiently small $\sigma > 0$ we have

$$\sup_{0<s<\sigma} \frac{||\triangle^2_{\bar{s}}(g^2 u_\varepsilon)||_{L^p}}{s^{\frac{1+p}{2}}} \leq M\,. \tag{3.2.3}$$

We will use the fact that for a for small $\delta > 0$ we have u_ε is locally C^δ (see [1]) and that for $\frac{\sqrt{17}-1}{2} \leq p \leq 2$ we have

$$2 - \frac{p}{2} - \frac{p^2}{2} \leq 0\,.$$

Therefore, for all $0 < s < \sigma$ and for $\delta' = \delta(2-p)$ we have

$$\int_\Omega \frac{\left|\triangle^2_{\bar{s}}(g^2 u_\varepsilon(x))\right|^2}{|s|^{2+\delta'}}\, dx$$
$$= \int_\Omega \frac{\left|\triangle^2_{\bar{s}}(g^2 u_\varepsilon(x))\right|^p}{|s|^{\frac{p}{2}+\frac{p^2}{2}}}\ \frac{\left|\triangle^2_{\bar{s}}(g^2 u_\varepsilon(x))\right|^{2-p}}{|s|^{2+\delta'-\frac{p}{2}-\frac{p^2}{2}}}\, dx \leq c\, M^p\, ||g^2 u_\varepsilon||^{2-p}_{C^\delta(\Omega)}\,.$$

Theorem 2.2.1 shows now that $Tu_\varepsilon \in L^2_{\text{loc}}(\Omega)$.

Therefore we have proved the following lemma:

Lemma 3.2.1. *Let* $\frac{\sqrt{17}-1}{2} \leq p < 2$ *and* $u_\varepsilon \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ *be a weak solution of (1.1). Let* $x_0 \in \Omega$, $r > 0$ *such that* $B(x_0, 3r) \in \Omega$, *and let* k *be the number of iterations from the proof of Theorem 2.4.1. that depends only on* p. *Then we have* $Tu_\varepsilon \in L^2_{\mathrm{loc}}(\Omega)$ *and*

$$\int_{B(x_0,\frac{r}{2^{k+2}})} |Tu_\varepsilon(x)|^2\, dx \\ \leq c\Bigg(||u||^{2-p}_{C^\delta(B(x_0,\frac{r}{2^{k+1}}))} \int_{B(x_0,2r)} \Big((\varepsilon + |Xu_\varepsilon(x)|^2)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \Big)\, dx \\ + ||u_\varepsilon||^2_{L^2(B(x_0,\frac{r}{2^{k+1}}))} \Bigg). \quad (3.2.4)$$

As an immediate corollary of the above lemma we have:

Corollary 3.2.1. *For* $\frac{\sqrt{17}-1}{2} \leq p < 2$ *we have* $XTu_\varepsilon \in L^p_{\mathrm{loc}}(\Omega)$ *with bounds depending on* ε.

Proof. Lemma 3.2.1 allows us to estimate the right hand side of (3.2.1) in the following way.

$$RHS \leq \delta \int_\Omega g^2(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^2\, dx \\ + c(\delta) \int_\Omega |Xg(x)|^2\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |u_\varepsilon(x\cdot\bar{s}) - u_\varepsilon(x)|^2\, dx\,.$$

Therefore,

$$\int_\Omega g^2(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot\bar{s})|^2\right)^{\frac{p-2}{2}}\ |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^2\, dx \\ \leq c(\varepsilon) \int_\Omega |Xg(x)|^2\ |u_\varepsilon(x\cdot\bar{s}) - u_\varepsilon(x)|^2\, dx$$

and hence

$$\int_\Omega g^2(x) \quad |Xu_\varepsilon(x\cdot\bar{s}) - Xu_\varepsilon(x)|^p\ dx \quad \leq \quad c(\varepsilon)\left(\int_{B(x_0,2r)} |u_\varepsilon(x\cdot\bar{s}) - u_\varepsilon(x)|^2\, dx\right)^{\frac{p}{2}} \quad (3.2.5)$$

which gives $XTu_\varepsilon \in L^p_{\mathrm{loc}}(\Omega)$. □

We will prove now a theorem on estimates of the second order horizontal derivatives.

Theorem 3.2.1. *Let* $\frac{\sqrt{17}-1}{2} \leq p \leq 2$ *and* $u_\varepsilon \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ *be a weak solution of (1.1). Consider* $x_0 \in \Omega$, $r > 0$ *such that* $B(x_0, 3r) \Subset \Omega$, *and let* k *be the number of iterations from the proof of Theorem 2.4.1 that depends only on* p. *Then for each* $i_0 \in \{1, ..., 2n\}$ *and* $s > 0$ *sufficiently small we have*

$$\int_{B(x_0, \frac{r}{2^{k+3}})} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|^2 \, dx$$
$$\leq c\Bigg(\varepsilon^{\frac{p-2}{2}} \, ||u||^{2-p}_{C^\delta(B(x_0, \frac{r}{2^{k+1}}))} \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx$$
$$+ \varepsilon^{\frac{p-2}{2}} \, ||u||^2_{L^2(B(x_0, \frac{r}{2^{k+1}}))} + c\int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx \Bigg), \tag{3.2.6}$$

and hence $u_\varepsilon \in HW^{2,p}_{\mathrm{loc}}(\Omega)$.

Proof. Let g be a cut-off function between $B(x_0, \frac{r}{2^{k+3}})$ and that $B(x_0, \frac{r}{2^{k+2}})$. The proof begins in the same way as the proof of Theorem 3.1.1, until we get the extended form of our inequality with the lines (L1) and (R1)-(R10). We can remark that although we could use a test function $\varphi = D_{-h_{i_0}}\left(g^2 D_{h_{i_0}} u_\varepsilon\right)$, we cannot avoid estimates similar to that of line (R6). For the line (L1) the estimate is the same as in the proof of Theorem 3.1.1. For the lines (R1)-(R5) we keep again the same estimates and use Lemma 3.2.1 with the facts that for $p < 2$ we have

$$\left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot \bar{s})|^2\right)^{\frac{p-2}{2}} \leq \varepsilon^{\frac{p-2}{2}}.$$

For (R7) we have

$$(R7) \leq c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-1}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right| \left|D_{h_{i_0}} g(x)\right| |g^3(x \cdot h_{i_0})| \, dx$$

$$= c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-1}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right| \left|D_{h_{i_0}} g(x)\right| \, |g^3(x)| \, dx$$
$$+ c\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-1}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right| \left|D_{h_{i_0}} g(x)\right|$$
$$h \left| \frac{g^3(x \cdot h_{i_0}) - g^3(x)}{h} \right| dx$$

$$
\begin{aligned}
&= c\int_{\Omega} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{4}} \ \left|D_{h_{i_0}} Xu_\varepsilon(x)\right| \ g^2(x) \\
&\qquad\qquad \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p}{4}} \ \left|D_{h_{i_0}} g(x)\right| \ |g(x)|\, dx \\
&+ c\int_{\Omega} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-1}{2}} \ \left|D_{h_{i_0}} Xu_\varepsilon(x)\right| \ \left|D_{h_{i_0}} g(x)\right| \\
&\qquad\qquad h\ \left|\frac{g^3(x\cdot h_{i_0}) - g^3(x)}{h}\right| \, dx \\
&\le \delta \int_{\Omega} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \ \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|^2 \ g^4(x)\, dx \\
&+ c(\delta)\int_{\Omega} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p}{2}} \ \left|D_{h_{i_0}} g(x)\right|^2 \ g^2(x)\, dx \\
&+ c\int_{\Omega} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot h_{i_0})|^2\right)^{\frac{p}{2}} \left|D_{h_{i_0}} g(x)\right|^3 \, dx
\end{aligned}
$$

The estimates for (R8)-(R10) are similar. It is left to estimate the line (R6). Following the methods in [11, 18] we consider for small $h > 0$ and a.e. $x \in B(x_0, 4r)$

$$
\alpha_i(x) = \int_0^1 a_i^\varepsilon\big(Xu_\varepsilon(x\cdot (th_{i_0})\big)\, dt
$$

and

$$
Y(x) = \int_0^1 \left(\varepsilon + |Xu_\varepsilon(x\cdot (th_{i_0})|^2\right)^{\frac{p-1}{2}} \ dt\,.
$$

In the distributional sense we have

$$
D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big) = X_{i_0}\alpha_i(x)\,.
$$

Also,

$$
|\alpha_i(x)| \le Y(x)\,, \quad \text{a.e } \ x \in B(x_0, 4r)\,.
$$

Therefore, we can estimate (R6) in the following way.

$$
\begin{aligned}
(R6) &= \sum_{i=1}^{2n} \int_\Omega D_{h_{i_0}} a_i^\varepsilon\big(Xu_\varepsilon(x)\big)\ 4g^3(x)\ X_i g(x)\ D_{h_{i_0}} u_\varepsilon(x)\, dx \\
&= 4 \sum_{i=1}^{2n} \int_\Omega \alpha_i(x)\ X_{i_0}\Big(g^3(x)\ X_i g(x)\ D_{h_{i_0}} u_\varepsilon(x)\Big)\, dx \\
&= 4 \sum_{i=1}^{2n} \int_\Omega \alpha_i(x)\ 3g^2(x)\ X_{i_0} g(x)\ X_i g(x)\ D_{h_{i_0}} u_\varepsilon(x)\, dx \\
&\quad + 4 \sum_{i=1}^{2n} \int_\Omega \alpha_i(x)\ g^3(x)\ X_{i_0} X_i g(x)\ D_{h_{i_0}} u_\varepsilon(x)\, dx \\
&\quad + 4 \sum_{i=1}^{2n} \int_\Omega \alpha_i(x)\ g^3(x)\ X_i g(x)\ D_{h_{i_0}} X_{i_0} u_\varepsilon(x)\, dx \\
&\le c \int_\Omega g^2(x)\ Y_i(x)\ \big|D_{h_{i_0}} u_\varepsilon(x)\big|\ dx \qquad (R6_1) \\
&\quad + c \int_\Omega \big|g^3(x)\big|\ Y_i(x)\ \big|D_{h_{i_0}} X u_\varepsilon(x)\big|\ dx \qquad (R6_2)
\end{aligned}
$$

Because of

$$
Y_i \in L^{\frac{p}{p-1}}_{\text{loc}}(\Omega)
$$

and $Xu_\varepsilon \in L^p_{\text{loc}}(\Omega)$ we get that $(R6_1)$ is finite. To estimate $(R6_2)$ we follow the method from [11]. Therefore,

$$
\begin{aligned}
(R6_2) &= \int_\Omega g^2(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{4}}\ \big|D_{h_{i_0}} X u_\varepsilon(x)\big| \\
&\qquad\qquad |g(x)|\ Y_i(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{2-p}{4}}\ dx \\
&\le \delta \int_\Omega g^4(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}}\ \big|D_{h_{i_0}} X u_\varepsilon(x)\big|^2\ dx \\
&\qquad\qquad + c(\delta) \int_\Omega g^2(x)\ Y_i^2(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{2-p}{2}}\ dx \\
&\le \delta \int_\Omega g^4(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}}\ \big|D_{h_{i_0}} X u_\varepsilon(x)\big|\ dx \\
&\qquad\qquad + c(\delta) \int_\Omega g^2(x)\ \Big(Y_i^{\frac{p}{p-1}}(x) + \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p}{2}}\Big)\, dx
\end{aligned}
$$

We can now continue the proof in the same way as we did in the case $p \geq 2$, going back to the beginning of the proof and using a test function

$$\varphi = D_{h_{i_0}} D_{-h_{i_0}}(g^4 u_\varepsilon)\,,$$

then adding the two inequalities and embedding the terms with δ coefficients into the left hand side. Therefore, we get

$$\int_\Omega g^4(x)\ \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}}\ \left|D_{h_{i_0}} X u_\varepsilon(x)\right|^2\ dx \leq M(\varepsilon)$$

where by $M(\varepsilon)$ we denote the right hand side of the inequality (3.2.6). Quoting again the method in section 2.4 we get that

$$\int_\Omega g^4(x)\ \left|D_{h_{i_0}} X u_\varepsilon(x)\right|^p\ dx \leq M(\varepsilon)\,, \tag{3.2.7}$$

and this proves that $X^2 u_\varepsilon \in L^p_{\text{loc}}(\Omega)$.

$\square$

4.0 CORDES CONDITIONS AND UNIFORM ESTIMATES IN THE HEISENBERG GROUP

4.1 BOUNDING THE SECOND ORDER HORIZONTAL DERIVATIVES BY THE SUBELLIPTIC LAPLACIAN

We denote by X^2u the matrix of the second order horizontal derivatives and by $\Delta_H u = \sum_{i=1}^{2n} X_i X_i u$ the subelliptic Laplacian associated to the horizontal vector fields X_i.

Lemma 4.1.1. *For all $u \in HW_0^{2,2}(\Omega)$ we have*

$$||X^2u||_{L^2(\Omega)} \leq c_n ||\Delta_H u||_{L^2(\Omega)} ,$$

where

$$c_n = \sqrt{1 + \frac{2}{n}} ,$$

and it is a sharp constant when $\Omega = \mathbb{H}^n$.

Proof. We follow the spectral analysis of Δ_H developed by Strichartz [27]. Let us recall the fact that $-\Delta_H$ and iT commute, and share the same system of eigenvectors

$$\begin{aligned} \Phi_{\lambda,k,l}(z,t) &= \frac{\lambda^n}{(2\pi)^{n+1}(n+2k)^{n+1}} \cdot \exp\left(-\frac{il\lambda t}{n+2k}\right) \\ &\quad \cdot \exp\left(-\frac{\lambda|z|^2}{4(n+2k)}\right) \cdot L_k^{n-1}\left(\frac{\lambda|z|^2}{2(n+2k)}\right) , \end{aligned}$$

where $l = \pm 1$, $k \in \{0, 1, 2, ...\}$ and L_k^{n-1} is the Laguerre polynomial

$$L_k^{n-1}(t) = \frac{e^t}{t^{n-1}} \cdot \frac{1}{k!} \cdot \frac{d^k}{dt^k} \left(e^{-t} t^{k+n-1}\right) .$$

For the eigenvalues, we have the following relations

$$\begin{aligned} iTu * \Phi_{\lambda,k,l} &= \frac{l\lambda}{n+2k} u * \Phi_{\lambda,k,l} && (4.1.1) \\ -\Delta_H u * \Phi_{\lambda,k,l} &= \lambda u * \Phi_{\lambda,k,l} \,, && (4.1.2) \end{aligned}$$

where $*$ denotes the group convolution. Therefore, the spectral decomposition of $\Delta_H u$ for $u \in C_0^\infty(\Omega)$, the Plancherel formula, and relations (4.1.1)-(4.1.2) give

$$\begin{aligned} ||\Delta_H u||^2_{L^2(\Omega)} &= 2\pi \sum_{k=0}^{\infty} \sum_{l=\pm 1} (n+2k) \int_0^\infty \int_{\mathbb{C}^n} |\Delta_H u * \Phi_{\lambda,k,l}(z,0)|^2 \, dz d\lambda \\ &= 2\pi \sum_{k=0}^{\infty} \sum_{l=\pm 1} (n+2k) \int_0^\infty \int_{\mathbb{C}^n} \left| \frac{n+2k}{l} iTu * \Phi_{\lambda,k,l}(z,0) \right|^2 dz d\lambda \\ &\geq n^2 ||Tu||^2_{L^2}(\Omega) \end{aligned}$$

Therefore, for all $u \in C_0^\infty(\Omega)$ we have

$$||Tu||_{L^2(\Omega)} \leq \frac{1}{n} ||\Delta_H u||_{L^2(\Omega)} \,. \qquad (4.1.3)$$

In the following we will use the fact that the formal adjoint of X_k is $-X_k$. Let $u \in C_0^\infty(\Omega)$. For $k \in \{1, ..., n\}$ and $j \neq k+n$, X_k and X_j commute, therefore

$$\int_\Omega (X_k X_j u(x))^2 \, dx = \int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) dx \,.$$

For $j = k+n$ we have

$$\begin{aligned} & \int_\Omega (X_k X_j u(x))^2 \, dx = \int_\Omega X_k X_j u(x) \cdot (X_j X_k u(x) + Tu(x)) \, dx \\ = & \int_\Omega X_k X_j u(x) \cdot X_j X_k u(x) \, dx + \int_\Omega X_k X_j u(x) \cdot Tu(x) \, dx \\ = & -\int_\Omega X_j u(x) \cdot X_k X_j X_k u(x) \, dx + \int_\Omega X_k X_j u(x) \cdot Tu(x) \, dx \\ = & -\int_\Omega X_j u(x) \cdot (X_j X_k + T) X_k u(x) \, dx + \int_\Omega X_k X_j u(x) \cdot Tu(x) \, dx \\ = & -\int_\Omega X_j u(x) \cdot X_j X_k X_k u(x) \, dx + 2 \int_\Omega X_k X_j u(x) \cdot Tu(x) \, dx \\ = & \int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) \, dx + 2 \int_\Omega X_k X_j u(x) \cdot Tu(x) \, dx \,. \end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_\Omega (X_j X_k u(x))^2 \, dx \\
= & \int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) \, dx - 2 \int_\Omega X_j X_k u(x) \cdot T u(x) \, dx \,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& ||X^2 u||^2_{L^2(\Omega)} = \sum_{k,j=1}^{2n} ||X_k X_j u||^2_{L^2(\Omega)} = \\
= & \sum_{k,j=1}^{2n} \int_\Omega X_k X_k u(x) \cdot X_j X_j u(x) \, dx + 2 \sum_{k=1}^{n} \int_\Omega [X_k, X_{k+n}] u(x) \cdot T u(x) \, dx \\
= & \int_\Omega \left(\sum_{k=1}^{2n} X_k X_k u(x) \right)^2 dx + 2n \int_\Omega (T u(x))^2 \, dx \\
\leq & \left(1 + 2n \frac{1}{n^2}\right) ||\Delta_H u||^2_{L^2(\Omega)} = \left(1 + \frac{2}{n}\right) ||\Delta_H u||^2_{L^2(\Omega)} \,.
\end{aligned}$$

The constant $\sqrt{1+\frac{2}{n}}$ is sharp when $\Omega = \mathbb{H}^n$, because for $v = \Phi_{\lambda,0,1}$ we have $Tv = \frac{i}{n}\Delta_H v$. $\square$

4.2 CORDES CONDITIONS FOR SECOND ORDER SUBELLIPTIC PDE OPERATORS IN NON-DIVERGENCE FORMS WITH MEASURABLE COEFFICIENTS

Let us consider now

$$\mathcal{A}u = \sum_{i,j=1}^{2n} a_{ij}(x) X_i X_j u$$

where the functions $a_{ij} \in L^\infty(\Omega)$. Let us denote by $A = (a_{ij})$ the $2n \times 2n$ matrix of coefficients.

Definition 4.2.1. [5, 28] *We say that A satisfies the Cordes condition $K_{\varepsilon,\sigma}$ if there exists $\varepsilon \in (0,1]$ and $\sigma > 0$ such that*

$$0 < \frac{1}{\sigma} \leq \sum_{i,j=1}^{2n} a_{ij}^2(x) \leq \frac{1}{2n-1+\varepsilon} \left(\sum_{i=1}^{2n} a_{ii}(x) \right)^2 \,, \ a.e. \ x \in \Omega \,. \tag{4.2.1}$$

Theorem 4.2.1. *Let $0 < \varepsilon \leq 1$, $\sigma > 0$ such that $\gamma = \sqrt{1-\varepsilon}\, c_n < 1$ and A satisfies the Cordes condition $K_{\varepsilon,\sigma}$. Then for all $u \in HW_0^{2,2}(\Omega)$ we have*

$$||X^2 u||_{L^2} \leq \sqrt{1+\frac{2}{n}}\,\frac{1}{1-\gamma}\,||\alpha||_{L^\infty}||\mathcal{A}u||_{L^2}\,, \tag{4.2.2}$$

where

$$\alpha(x) = \frac{\langle A(x), I\rangle}{||A(x)||^2}\,.$$

Proof. We denote by I the identity $2n \times 2n$ matrix, by $\langle A, B\rangle = \sum_{i,j=1}^{2n} a_{ij}b_{ij}$ the inner product and by $||A|| = \sqrt{\sum_{i,j=1}^{2n} a_{ij}^2}$ the Euclidean norm in $\mathbb{R}^{2n\times 2n}$ for matrices A and B. The Cordes condition $K_{\varepsilon,\sigma}$ implies that

$$\frac{\langle A(x), I\rangle^2}{||A(x)||^2} \geq 2n - (1-\varepsilon) \tag{4.2.3}$$

for all $x \in \Omega' \subset \Omega$, where the Lebesgue measure of $\Omega \setminus \Omega'$ is 0.
Let be now $x \in \Omega'$ arbitrary, but fixed. Consider the quadratic polynomial

$$P(\alpha) = ||A(x)||^2\alpha^2 - 2\langle A(x), I\rangle\alpha + 2n - (1-\varepsilon)\,.$$

Inequality (4.2.3) shows that

$$\min_{\alpha\in\mathbb{R}} P(\alpha) = P\left(\frac{\langle A(x), I\rangle}{||A(x)||^2}\right) \leq 0\,. \tag{4.2.4}$$

Therefore there exists

$$\alpha(x) = \frac{\langle A(x), I\rangle}{||A(x)||^2} \tag{4.2.5}$$

such that $P(\alpha(x)) \leq 0$. Observing that

$$||I - \alpha(x)A(x)||^2 = ||A(x)||^2\alpha^2(x) - 2\langle A(x), I\rangle\alpha(x) + 2n$$

we get that (4.2.4) implies that

$$||I - \alpha(x)A(x)||^2 \leq 1-\varepsilon\,,$$

which is equivalent to

$$|\langle I - \alpha(x)A(x), M\rangle| \leq \sqrt{1-\varepsilon}||M||\,,\ \text{for all } M \in \mathcal{M}_{2n}(\mathbb{R})\,. \tag{4.2.6}$$

Condition (4.2.6) can be written also as

$$\left|\sum_{i=1}^{n} m_{ii} - \alpha(x)\sum_{i,j=1}^{n} a_{ij}(x)m_{ij}\right| \leq \sqrt{1-\varepsilon}\left(\sum_{i,j=1}^{n} m_{ij}^2\right)^{1/2} \tag{4.2.7}$$

for all $M \in \mathcal{M}_{2n}(\mathbb{R})$.

Formula (4.2.7) and Lemma 4.1.1 imply that for all $u \in HW_0^{2,2}(\Omega)$ we have

$$\int_\Omega |\Delta_H u(x) - \alpha(x)\mathcal{A}u(x)|^2\, dx \leq (1-\varepsilon)\int_\Omega \sum_{i,j=1}^{2n} (X_iX_ju(x))^2\, dx \leq$$

$$\leq (1-\varepsilon)c_n^2 \int_\Omega |\Delta_H u(x)|^2\, dx\,.$$

Therefore, for $\gamma = \sqrt{1-\varepsilon}\, c_n < 1$ we get

$$||\Delta_H u - \alpha\mathcal{A}u||_{L^2(\Omega)} \leq \gamma||\Delta_H u||_{L^2(\Omega)}$$

which shows that

$$||X^2u||_{L^2(\Omega)} \leq c_n||\Delta_H u||_{L^2(\Omega)} \leq$$

$$\leq \frac{c_n}{1-\gamma}||\alpha\mathcal{A}u||_{L^2(\Omega)} \leq \frac{c_n}{1-\gamma}||\alpha||_{L^\infty(\Omega)}||\mathcal{A}u||_{L^2(\Omega)}\,.$$

$\square$

4.3 $HW^{2,2}$-INTERIOR REGULARITY FOR p-HARMONIC FUNCTIONS IN $\mathbb{H}^n$

Let $\Omega \in \mathbb{H}^n$ be a domain, $h \in HW^{1,p}(\Omega)$ and $p > 1$. Consider the problem of minimizing the functional

$$\Phi(u) = \int_\Omega |Xu(x)|^p \, dx$$

over all $u \in HW^{1,p}(\Omega)$ such that $u - h \in HW_0^{1,p}(\Omega)$. The Euler equation for this problem is the p-Laplace equation

$$-\sum_{i=1}^{2n} X_i \left(|Xu|^{p-2} X_i u\right) = 0 \,, \text{ in } \Omega \,. \tag{4.3.1}$$

A function $u \in HW^{1,p}(\Omega)$ is called a weak solution for (4.3.1) if

$$\sum_{i=1}^{2n} \int_\Omega |Xu(x)|^{p-2} X_i u(x) \cdot X_i \varphi(x) dx = 0 \,, \ \forall \, \varphi \in HW_0^{1,p}(\Omega) \,. \tag{4.3.2}$$

Φ is a convex functional on $HW^{1,p}$, therefore weak solutions are minimizers for Φ and vice-versa.

For $m \in \mathbb{N}$ let us define now the approximating problems of minimizing functionals

$$\Phi_m(u) = \int_\Omega \left(\frac{1}{m} + |Xu(x)|^2\right)^{\frac{p}{2}} dx$$

and the corresponding Euler equations

$$-\sum_{i=1}^{2n} X_i \left(\left(\frac{1}{m} + |Xu|^2\right)^{\frac{p-2}{2}} X_i u\right) = 0 \,, \text{ in } \Omega \,. \tag{4.3.3}$$

The weak form of this equation is

$$\sum_{i=1}^{2n} \int_\Omega \left(\frac{1}{m} + |Xu(x)|^2\right)^{\frac{p-2}{2}} X_i u(x) \cdot X_i \varphi(x) dx = 0 \,, \quad \text{for all } \ \varphi \in HW_0^{1,p}(\Omega) \,. \tag{4.3.4}$$

The differentiated version of equation (4.3.3) has the form

$$\sum_{i,j=1}^{2n} a_{ij}^m \, X_i X_j u = 0 \,, \text{ in } \Omega \tag{4.3.5}$$

where

$$a_{ij}^m(x) = \delta_{ij} + (p-2)\frac{X_iu(x)\,X_ju(x)}{\frac{1}{m}+|Xu(x)|^2}\,.$$

Let us consider a weak solution $u_m \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ of equation (4.3.3). Then $a_{ij}^m \in L^\infty(\Omega)$. Define the mapping $L_m : HW_0^{2,2}(\Omega) \to L^2(\Omega)$ by

$$L_m(v)(x) \;=\; \sum_{i,j=1}^{2n} a_{ij}^m(x)X_iX_jv(x)\,. \tag{4.3.6}$$

We will check the validity of Theorem 4.2.1 for L_m. We have

$$\sum_{i=1}^{2n} a_{ii}^m(x) = 2n + (p-2)\frac{|Xu_m|^2}{\frac{1}{m}+|Xu_m|^2}\,,$$

and

$$\sum_{i,j=1}^{2n}\left(a_{ij}^m(x)\right)^2 = 2n + 2(p-2)\frac{|Xu_m|^2}{\frac{1}{m}+|Xu_m|^2} + (p-2)^2\frac{|Xu_m|^4}{\left(\frac{1}{m}+|Xu_m|^2\right)^2}\,.$$

Denote

$$(p-2)\frac{|Xu_m|^2}{\frac{1}{m}+|Xu_m|^2} = \Lambda\,.$$

Therefore, for an $\varepsilon \in (1-\frac{1}{c_n^2}, 1)$ we need

$$2n + 2\Lambda + \Lambda^2 \le \frac{1}{2n-1+\varepsilon}\left(2n+\Lambda\right)^2\,.$$

This leads to

$$(2n-1)\Lambda^2 \le (1-\varepsilon)\left(2n+2\Lambda+\Lambda^2\right) < \frac{1}{c_n^2}\left(2n+2\Lambda+\Lambda^2\right)\,.$$

Hence,

$$\left((2n-1)c_n^2 - 1\right)\Lambda^2 - 2\Lambda - 2n < 0\,.$$

Solving this inequality we get

$$\Lambda \in \left(\frac{1-\sqrt{2n\left((2n-1)\,c_n^2-1\right)+1}}{(2n-1)c_n^2-1}\,,\;\frac{1+\sqrt{2n\left((2n-1)\,c_n^2-1\right)+1}}{(2n-1)c_n^2-1}\right)\,. \tag{4.3.7}$$

Using $c_n^2 = \frac{n+2}{n}$ and the fact that $\frac{|Xu_m|^2}{\frac{1}{m}+|Xu_m|^2} < 1$ we have that for all $m \in \mathbb{N}$ we have

$$p-2 \in \left(\frac{n-n\sqrt{4n^2+4n-3}}{2n^2+2n-2}\,,\;\frac{n+n\sqrt{4n^2+4n-3}}{2n^2+2n-2}\right)\,, \tag{4.3.8}$$

and that the operators L_m satisfies the assumptions of Theorem 4.2.1 uniformly in m.

Let us remark that in the case $n = 1$ our methods gives

$$p \in \left(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2} \right).$$

Theorem 4.3.1. *Let*

$$2 \leq p < 2 + \frac{n + n\sqrt{4n^2+4n-3}}{2n^2+2n-2}.$$

If $u \in HW^{1,p}(\Omega)$ is a p-harmonic function then $u \in HW^{2,2}_{loc}(\Omega)$.

Proof. The case $p = 2$ it is well known, so let us suppose $p \neq 2$. Consider $x_0 \in \Omega$ and $r > 0$ such that $B_{4r} = B(x_0, 4r) \subset\subset \Omega$. We need a cut-off function $\eta \in C_0^\infty(B_{2r})$ such that $\eta = 1$ on B_r. Also consider minimizers u_m for Φ_m on $HW^{1,p}(B_{2r})$ subject to the condition $u_m - u \in HW_0^{1,p}(B_{2r})$. Then $u_m \to u$ in $HW^{1,p}(B_{2r})$ as $m \to \infty$.

By Theorem 3.1.1 we get that for $2 \leq p < 4$ we have $u_m \in HW^{2,2}_{loc}(\Omega)$, but with bounds depending on m, and also that u_m satisfies equation $L_m(u_m) = 0$ a.e. in B_{2r}. So, in B_{2r} we have a.e.

$$X_iX_j(\eta^2 u_m) = X_iX_j(\eta^2)u_m + X_j(\eta^2)X_iu_m + X_i(\eta^2)X_ju_m + \eta^2 X_iX_ju_m$$

and hence

$$L_m(\eta^2 u_m) = u_m\, L_{m,u_m}(\eta^2) + \sum_{i,j=1}^{2n} a_{ij}^m(x)\Big(X_j(\eta^2)X_iu_m + X_i(\eta^2)X_ju_m\Big).$$

By Theorem 4.2.1 it follows that

$$\begin{aligned} ||X^2u_m||_{L^2(B_r)} &\leq ||X^2(\eta^2 u_m)||_{L^2(B_{2r})} \leq c||L_m(\eta^2 u_m)||_{L^2(B_{2r})} \\ &\leq c||u_m||_{HW^{1,p}(B_{2r})} \leq c||u||_{HW^{1,p}(B_{2r})} \end{aligned}$$

where c is independent of m. Therefore, $u \in HW^{2,2}(B_r)$. □

Remark 4.3.1. Observe that the range for p given by Theorem 4.3.1 is shrinking from $[2, \frac{5+\sqrt{5}}{2})$ to $[2, 3]$ as n increases from 1 to ∞.

For the case $p < 2$ we need the following lemmas. The first lemma is an interpolation result and its proof is based on integration by parts.

Lemma 4.3.1. *For all $u \in C_0^\infty(\Omega)$ and for all $\delta > 0$ there exists $c(\delta) > 0$ such that*

$$||Xu||^2_{L^2(\Omega)} \leq \delta ||X^2 u||^2_{L^2(\Omega)} + c(\delta)||u||^2_{L^2(\Omega)}\,.$$

Proof.

$$\begin{aligned}
&||Xu||^2_{L^2(\Omega)} = \sum_{i=1}^{2n} \int_\Omega X_i u(x)\ X_i u(x)\,dx = -\sum_{i=1}^{2n} \int_\Omega u(x)\ X_i X_i u(x) dx = \\
&= -\int_\Omega u(x)\ \Delta_H u(x)\,dx \leq \frac{\delta}{2n} \int_\Omega |\Delta_H u(x)|^2\,dx + c(\delta) \int_\Omega u^2(x)\,dx \\
&\leq \delta \int_\Omega |X^2 u(x)|^2\,dx + c(\delta) \int_\Omega u^2(x)\,dx
\end{aligned}$$

□

From Lemma 4.3.1 and the higher order extension results available for the Sobolev spaces on the Heisenberg group [15, 20] we get the following result.

Lemma 4.3.2. *For all $u \in HW^{2,2}(B_r)$ and all $\delta > 0$ there exists $c(\delta) > 0$ such that*

$$||Xu||^2_{L^2(B_r)} \leq \delta ||X^2 u||^2_{L^2(B_r)} + c(\delta)||u||^2_{L^2(B_r)}\,.$$

By Lemmas 4.3.1 and 4.3.2 we can use a method similar to the proof of Theorem 9.11 [10] to get the following result.

Lemma 4.3.3. *Let us suppose that the operator $\mathcal{A}$ satisfies the assumptions of Theorem 4.2.1 and that $B_{3r} \subset \Omega$. Then*

$$||X^2 u||_{L^2(B_r)} \leq c\Big(||\mathcal{A}u||_{L^2(B_{2r})} + ||u||_{L^2(B_{2r})}\Big)\,,$$

for all $u \in HW^{2,2}_{\mathrm{loc}}(B_{3r})$.

Proof. Let $\eta \in C_0^\infty(B_{2r})$, $0 < \sigma < 1$ and $\sigma' = \frac{1+\sigma}{2}$ such that η is a cut-off function between $B_{\sigma 2r}$ and $B_{\sigma' 2r}$ satisfying

$$|X\eta| \leq \frac{2}{(1-\sigma)r} \text{ and } |X^2\eta| \leq \frac{4}{(1-\sigma)^2 r^2}\,.$$

Then we can use Theorem 4.2.1 for ηu to get

$$\begin{aligned}
&||X^2u||_{L^2(B_{\sigma 2r})} \leq ||X^2(\eta u)||_{L^2(B_{2r})} \leq c||\mathcal{A}(\eta u)||_{L^2(B_{2r})} \\
&= c\left\|\eta \mathcal{A}u + u\,\mathcal{A}(\eta) + \sum_{i,j=1}^{2n} a_{ij}\Big(X_j(\eta)X_i u + X_i(\eta)X_j u\Big)\right\|_{L^2(B_{2r})} \\
&\leq c\Big(||\mathcal{A}u||_{L^2(B_{2r})} + \frac{1}{(1-\sigma)r}||Xu||_{L^2(B_{\sigma' 2r})} + \frac{1}{(1-\sigma)^2 r^2}||u||_{L^2(B_{\sigma' 2r})}\Big)
\end{aligned}$$

For $k \in \{0, 1, 2\}$ let us use the seminorms

$$|||u|||_k = \sup_{0<\sigma<1} (1-\sigma)^k r^k ||X^k u||_{L^2(B_{\sigma 2r})}\,.$$

Then

$$|||u|||_2 \leq c\Big(r^2||\mathcal{A}u||_{L^2(B_{2r})} + |||u|||_1 + |||u|||_0\Big)\,.$$

Lemma 4.3.2 implies that for $\delta > 0$ small we have

$$|||u|||_1 \leq \delta|||u|||_2 + c(\delta)|||u|||_0\,.$$

Therefore,

$$|||u|||_2 \leq c\Big(r^2||\mathcal{A}u||_{L^2(B_{2r})} + |||u|||_0\Big)$$

and hence

$$||X^2u||_{L^2(B_{\sigma 2r})} \leq \frac{c}{(1-\sigma)^2 r^2}\Big(r^2||\mathcal{A}u||_{L^2(B_{2r})} + ||u||_{L^2(B_{2r})}\Big)\,.$$

For $\sigma = \frac{1}{2}$ we get the desired inequality. □

Theorem 4.3.2. *Let us consider the Heisenberg group $\mathbb{H}^1$ and*

$$\frac{\sqrt{17}-1}{2} \leq p \leq 2\,.$$

If $u \in HW^{1,p}(\Omega)$ is a p-harmonic function then $u \in HW^{2,2}_{loc}(\Omega)$.

Proof. We start the proof in the same way as we did in the proof of Theorem 4.3.1. Consider $x_0 \in \Omega$ and $r > 0$ such that $B_{4r} = B(x_0, 4r) \subset\subset \Omega$. We need a test function $\eta \in C_0^\infty(B_{3r})$. Also consider minimizers u_m for Φ_m on $HW^{1,p}(B_{3r})$ subject to the condition $u_m - u \in HW_0^{1,p}(B_{3r})$. Then $u_m \to u$ in $HW^{1,p}(B_{3r})$ as $m \to \infty$. We use the facts that

$$\frac{4}{3} < \frac{5-\sqrt{5}}{2} < \frac{\sqrt{17}-1}{2} < 2,$$

the homogeneous dimension of $\mathbb{H}^1$ is $Q = 4$, and

$$2 \leq \frac{4p}{4-p} \quad \text{for all } \frac{4}{3} \leq p < 2\,.$$

The Sobolev embeddings result in the subelliptic setting [1] says that

$$HW_0^{1,p}\big(B_{3r}\big) \hookrightarrow L^q\big(B_{3r}\big)\,, \ \text{ for } \ 1 \leq q \leq \frac{4p}{4-p}\,.$$

Therefore, $u_m \to u$ in $L^2\big(B_{3r}\big)$. Also, using Theorem 3.2.1 we have for $\frac{\sqrt{17}-1}{2} \leq p < 2$ that $u_m \in HW^{2,p}_{\text{loc}}\big(B_{3r}\big)$ we get that $Xu_m \in L^2_{\text{loc}}\big(B_{3r}\big)$. Let us remark that these bounds of $X^2 u_m$ in L^p may depend on m and that $L_m(u_m) = 0$ a.e. in B_{3r}. Moreover,

$$||L_m\big(\eta^2 u_m\big)||_{L^2(B_{3r})} = c\left\|u_m\, L_m(\eta^2) + \sum_{i,j=1}^{2n} a_{ij}^{m,u}(x)\Big(X_j(\eta^2)X_i u_m + X_i(\eta^2)X_j u_m\Big)\right\|_{L^2(B_{3r})}$$

$$\leq c\Big(||u_m||_{L^2(\text{supp}\eta)} + ||Xu_m||_{L^2(\text{supp}\eta)}\Big) < +\infty\,.$$

and hence $u_m \in HW^{2,2}_{\text{loc}}\big(B_{3r}\big)$. By Lemma 4.3.3 for all m sufficiently large we have

$$||X^2\big(u_m\big)||_{L^2(B_r)} \leq c||u_m||_{L^2(B_{2r})} \leq 2c||u||_{L^2(B_{2r})}$$

which shows that $X^2 u_m$ is uniformly bounded in $HW^{2,2}(B_r)$, hence $u \in HW^{2,2}(B_r)$. □

4.4 $C^{1,\alpha}$-REGULARITY FOR p-HARMONIC FUNCTIONS IN THE HEISENBERG GROUP FOR p NEAR 2

In this section we use previous results regarding the Calderón-Zygmund theory in Heisenberg group (see [9, 13, 14], the $HW^{2,2}$ regularity of p-harmonic functions from Chapter 3 and the properties of second order PDE operators that are near to the subelliptic Laplacian, to prove $C^{1,\alpha}$ regularity for p-harmonic functions in the Heisenberg group for p in a neighborhood of 2. In the Euclidean case this result is known for $1 < p < \infty$, while in the Heisenberg group there is no definite answer yet. Our result constitutes the first indication that the $C^{1,\alpha}$ regularity for p-harmonic functions in the Heisenberg group is possible.

We keep the general setting from the previous section given by formulas (4.3.1) - (4.3.6) and update the working methods from those corresponding to L^2 to those corresponding to L^s with $s > 1$.

The Calderón-Zygmund theory gives the following lemma (see the theorem on page 917 in [9]).

Lemma 4.4.1. *For all $1 < s < \infty$ there exists $C_{n,s} \geq 1$ such that for all $u \in HW_0^{2,s}(\Omega)$ we have*

$$||X^2u||_{L^s(\Omega)} \leq C_{n,s}||\Delta_X u||_{L^s(\Omega)}\,.$$

Recall that in the case $s = 2$ we have

$$C_{n,2} = \sqrt{1 + \frac{2}{n}}$$

and this is a sharp constant as shown in the previous section.

Let us consider now

$$\mathcal{A}u = \sum_{i,j=1}^{2n} a_{ij}(x)X_iX_ju$$

where the functions $a_{ij} \in L^\infty(\Omega)$ and denote by $A = (a_{ij})$ the $2n \times 2n$ matrix of coefficients.

Theorem 4.4.1. *Let $0 < \varepsilon \leq 1$, such that $\varepsilon \cdot C_{n,s} < 1$ and suppose that*

$$|\Delta_X u(x) - \mathcal{A}u(x)| \leq \varepsilon \left|X^2u(x)\right| \tag{4.4.1}$$

for a.e. $x \in \Omega$ *and for all* $u \in HW_0^{2,s}(\Omega)$. *Then* $\mathcal{A} : HW_0^{2,s}(\Omega) \to L^s(\Omega)$ *is an isomorphism and there exists* $c > 0$ *such that*

$$||X^2 u||_{L^s(\Omega)} \leq c||\mathcal{A}u||_{L^s(\Omega)} \tag{4.4.2}$$

for all $u \in HW_0^{2,s}(\Omega)$.

Proof. The proof follows from the fact that Lemma 4.4.1 and formula (4.4.1) shows that $\mathcal{A} : HW_0^{2,s}(\Omega) \to L^s(\Omega)$ satisfies the relation

$$||\Delta_X u - \mathcal{A}u||_{L^s(\Omega)} \leq \varepsilon \cdot C_{n,s}\, ||\Delta_X u||_{L^s(\Omega)}$$

which proves that $\mathcal{A}$ is near to Δ_X and hence it is an isomorphism. For the properties inherited by operators that are near to each other we quote [6, 7, 29]. □

We need the following result which involves interpolation inequalities and higher order extensions of functions over the boundaries of homogeneous balls (see [15]).

Lemma 4.4.2. *Let* $u \in HW_{\text{loc}}^{2,s}(\Omega)$, $x_0 \in \Omega$ *and* $r > 0$ *such that* $B(x_0, 2r) \subset \Omega$. *Then for all* $\delta > 0$ *there exists* $c(\delta) > 0$ *such that*

$$||Xu||_{L^s(B(x_0,r))} \leq \delta||X^2 u||_{L^s(B(x_0,r))} + c(\delta)||u||_{L^s(B(x_0,r))}\,.$$

We can use now Theorem 4.4.1, Lemma 4.4.2 and a method similar to the proof of Theorem 9.11 in [10] and Lemma 4.3.3 to get the following result.

Theorem 4.4.2. *Let us suppose that the operator* $\mathcal{A}$ *satisfies the assumptions of Theorem 4.4.1 and that* $B(x_0, 3r) \subset \Omega$. *Then*

$$||X^2 u||_{L^s(B(x_0,r))} \leq c\Big(||\mathcal{A}u||_{L^s(B(x_0,2r))} + ||u||_{L^s(B(x_0,2r))}\Big)$$

for all $u \in HW_{\text{loc}}^{2,s}(\Omega)$.

For $\gamma > 0$ small but fixed, let us denote by

$$\tilde{c} = \max\left\{C_{n,s}\,,\ s \in \Big(\frac{\sqrt{17}-1}{2}\ ,\ 2n+2+\gamma\Big)\right\}.$$

Theorem 4.4.3. *For*

$$\max\Big\{\frac{\sqrt{17}-1}{2}\,,\,2-\frac{1}{\tilde{c}\,n}\Big\}\le p\le 2+\frac{1}{\tilde{c}\,n}$$

and a p-harmonic function u in $\mathbb{H}^n$ *there exists* $0<\alpha<1$ *such that we have the interior regularity* $u\in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

Proof. **The case $2\le p$.**

Theorems 4.4.2 and 3.1.1 shows that $Xu_\varepsilon\in HW^{1,2}_{\mathrm{loc}}(\Omega)$ with uniform bounds. We use the embedding

$$HW^{1,2}_{\mathrm{loc}}(\Omega)\hookrightarrow L^{q_0}_{\mathrm{loc}}(\Omega)$$

where

$$q_0=\frac{(2n+2)\cdot 2}{2n+2-2}=\frac{2n+2}{n}\,.$$

For corresponding cut-off function η between homogeneous the balls B_r and B_{2r} we have

$$\begin{aligned}
&||L_m\big(\eta^2u_m\big)||_{L^{q_0}(B_{3r})}\\
&=c\left\|u_m\,L_m(\eta^2)+\sum_{i,j=1}^{2n}a^{m,u}_{ij}(x)\Big(X_j(\eta^2)X_iu_m+X_i(\eta^2)X_ju_m\Big)\right\|_{L^{q_0}(B_{3r})}\\
&\le c\Big(||u_m||_{L^{q_0}(\mathrm{supp}\eta)}+||Xu_m||_{L^{q_0}(\mathrm{supp}\eta)}\Big)<+\infty
\end{aligned}\tag{4.4.3}$$

Therefore, by Theorems 4.4.1 and 4.4.2 we have that $u_m\in HW^{2,q_0}_{\mathrm{loc}}(\Omega)$ with locally uniform bounds. Repeating this procedure k times we get that $u_m\in HW^{2,q_k}_{\mathrm{loc}}(\Omega)$ for

$$q_k=\frac{2n+2}{n-k}\,.$$

We stop after $n-1$ steps and get $q_{n-1}=2n+2$ which is the homogeneous dimension of $\mathbb{H}^n$ and obtain $u\in HW^{2,2n+2}_{\mathrm{loc}}(\Omega)$. Let us choose now $1<\beta<2$ close enough to 1 such that

$$(2n+2)\frac{\beta}{2-\beta}\le 2n+2+\gamma\,.$$

Then we use the embedding

$$HW^{1,\frac{2n+2}{2}\beta}_{\mathrm{loc}}(\Omega)\hookrightarrow L^{(2n+2)\frac{\beta}{2-\beta}}_{\mathrm{loc}}(\Omega)$$

and inequalities similar to (4.4.3) to conclude that

$$u_m \in HW_{\text{loc}}^{2,(2n+2)\frac{\beta}{2-\beta}}(\Omega)\,.$$

The embedding

$$HW_{\text{loc}}^{1,(2n+2)\frac{\beta}{2-\beta}}(\Omega) \hookrightarrow C^{\frac{2\beta-2}{\beta}}$$

shows that u_m has interior regularity $C^{1,\alpha}$ where

$$\alpha = \frac{2\beta - 2}{\beta}\,.$$

Because of the estimates for u_m are uniform, we can conclude that $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$.

The case $\mathbf{p \leq 2}$.

Theorems 4.4.2 and 3.2.1 implies that $Xu_m \in HW^{1,p}_{\text{loc}}(\Omega)$ with uniform bounds. Then we can start with $q_0 = p$ and follow the proof of the previous case until we get the first k with

$$q_k = \frac{(2n+2)p}{2n+2-kp} > \frac{2n+2}{2}\,.$$

Let us choose now $\beta > 1$ enough close to 1 such that

$$(2n+2)\frac{\beta}{2-\beta} \leq 2n+2+\gamma$$

and

$$(2n+2)\frac{\beta}{2} \leq q_k\,.$$

The rest is similar to the last part of the previous case. □

5.0 REGULARITY OF p-HARMONIC FUNCTIONS IN CARNOT GROUPS

In this chapter we generalize our results from the previous chapters to the more general case of a Carnot group of arbitrary step. Note that the Heisenberg group is a Carnot group of step 2 and the methods elaborated for it will be used heavily at each step of our iterations.

5.1 BASIC FACTS ABOUT LIE GROUPS

Definition 5.1.1. *A Lie group is a group $\mathcal{G}$ that is a finite dimensional smooth manifold such that the group operations*

$$\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}\,,\ \mu(x,y) = x \cdot y$$

and

$$\mathrm{inv} : \mathcal{G} \to \mathcal{G}\,,\ \mathrm{inv}(x) = x^{-1}$$

are smooth mappings.

We denote the identity element of $\mathcal{G}$ by $\mathbf{1}$ and the tangent space of $\mathcal{G}$ at $\mathbf{1}$ by $\mathfrak{g}$, which is a vector space, having the same dimension as $\mathcal{G}$. Looking to the differential of μ at $(\mathbf{1},\mathbf{1})$ we find

$$D_{(\mathbf{1},\mathbf{1})}\mu(X,Y) = X + Y\,.$$

At the same time

$$D_{\mathbf{1}}\mathrm{inv}(X) = -X$$

which shows that first order derivatives do not reflect the noncommutativity of $\mathcal{G}$. Therefore we have to turn to the level of second order derivatives.

For each $x \in \mathcal{G}$ let us consider the conjugation by x, that is

$$Ad_x : \mathcal{G} \to \mathcal{G}\,,\ Ad_x(y) = x \cdot y \cdot x^{-1}\,.$$

In the case of a commutative Lie group Ad_x is the identity mapping of $\mathcal{G}$ for all $x \in \mathcal{G}$. The infinitesimal conjugation by x on $\mathfrak{g}$ is defined as the differential of Ad_x at $\mathbf{1}$, that is

$$\mathrm{A}d_x = D_{\mathbf{1}}(Ad_x) : \mathfrak{g} \to \mathfrak{g}\,.$$

The chain rule for differentiation shows that

$$\mathrm{A}d_{x \cdot y} = \mathrm{A}d_x \circ \mathrm{A}d_y$$

and therefore

$$\mathrm{A}d : \mathcal{G} \to GL(\mathfrak{g})$$

is a homomorphism of groups, called the adjoint representation of $\mathcal{G}$. In the case of a commutative group $\mathrm{A}d$ is the trivial homomorphism.

Taking the differential of $\mathrm{A}d$ at $\mathbf{1}$ we get the mapping

$$\mathrm{a}d : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})\,.$$

For each $X, Y \in \mathfrak{g}$ we define the Lie bracket of X and Y by

$$[X, Y] = \mathrm{a}dX(Y)\,.$$

The Lie bracket satisfies the anti-symmetry

$$[X, Y] = -[Y, X]$$

and the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

and therefore $\mathfrak{g}$ has the structure of a Lie algebra.

For each $x \in \mathcal{G}$ we define the left and right multiplications by

$$L_x : \mathcal{G} \to \mathcal{G}\,,\ L_x(y) = x \cdot y$$

$$R_x : \mathcal{G} \to \mathcal{G}\,,\ R_x(y) = y \cdot x\,.$$

A vector field X is called left invariant if

$$X(L_x(y)) = D_y L_x(X(y))$$

and right invariant if

$$X(R_x(y)) = D_y R_x(X(y))\,.$$

It turns out that left and right invariant vector fields are completely determined by their value at $\mathbf{1}$, namely

$$X(x) = D_{\mathbf{1}} L_x(X(\mathbf{1}))$$

for left invariant, and

$$X(x) = D_{\mathbf{1}} R_x(X(\mathbf{1}))$$

for right invariant vector fields. Conversely, any element X of $\mathfrak{g}$ determines a left invariant vector field by the formula

$$X(x) = D_{\mathbf{1}} L_x X$$

and also a right invariant vector field by

$$X(x) = D_{\mathbf{1}} R_x X\,.$$

From this moment we will concentrate our attention on left invariant vector fields. We can introduce the Lie bracket of left invariant vector fields by

$$[X, Y](x) = D_{\mathbf{1}} L_x [X(\mathbf{1}), Y(\mathbf{1})]\,.$$

So, we can identify the space of left invariant vector fields by $\mathfrak{g}$ and talk about the Lie algebra of left invariant vector fields, that is isomorphic to $\mathfrak{g}$. Therefore, we will identify a left invariant vector field X by its value at $\mathbf{1}$.

For every $X \in \mathfrak{g}$ there exists a unique differentiable homomorphism

$$\Phi_X : (\mathbb{R}, +) \to (\mathcal{G}, \cdot)$$

that satisfies

$$\frac{d\Phi_X}{dt}(t) = X(\Phi_X(t)), \ \forall\, t \in \mathbb{R}\,.$$

The mapping $t \rightarrowtail \Phi_X(t)$ is an integral curve of the left invariant vector field X satisfying $\Phi_X(0) = \mathbf{1}$.

Definition 5.1.2. *We define the exponential mapping* $\exp : \mathfrak{g} \to \mathcal{G}$ *by*

$$\exp X = \Phi_X(1)\,.$$

By the uniqueness of solutions for initial value problems for ordinary differential equations we get that

$$\Phi_X(st) = \Phi_{tX}(s), \ \forall\, s, t \in \mathbb{R}\,.$$

Therefore we have

$$\exp(tX) = \Phi_X(t)$$

and hence $t \rightarrowtail \exp(tX)$ is a homomorphism between $(\mathbb{R}, +)$ and $(\mathcal{G}, \cdot)$. Differentiating and using the definition and the differential equation of Φ_X we get

$$\frac{d}{dt}\exp(tX) = X(\exp(tX))$$

and hence the differential of exp at 0 is the identity mapping of $\mathfrak{g}$. Using the inverse function theorem we get the following result.

Theorem 5.1.1. *There exist open neighborhoods of U of 0 in $\mathfrak{g}$ and V of $\mathbf{1}$ in $\mathcal{G}$ such that*

$$\exp|_U : U \to V$$

is a diffeomorphism and the inverse mapping

$$\log : V \to U$$

is called a logarithmic chart for $\mathcal{G}$.

We recall now the Baker-Hausdorff-Campbell-Dynkin formula for the exponential mapping. For all X and Y from a small neighborhood U of 0 in $\mathfrak{g}$ we have

$$\exp(X) \cdot \exp(Y) = \exp(\mu(X, Y))$$

where

$$\begin{aligned} \mu(X, Y) &= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y] - [Y, [X, Y]) + \\ &+ \text{commutators of order four and higher} \end{aligned}$$

If we look for the expansion up to order 2 we get

$$\mu(X, Y) = X + Y + \frac{1}{2}[X, Y] + O(|(X, Y)|^3)\,, \ \textit{as}\ (X, Y) \to (0, 0)\,.$$

5.2 NILPOTENT LIE GROUPS

Let $\mathcal{G}$ be a simply connected Lie group and $\mathfrak{g}$ its Lie algebra. For $U, V \subset \mathfrak{g}$ we denote by $[U, V]$ the subspace of $\mathfrak{g}$ generated by the elements of the form $[X, Y]$, where $X \in U$, $Y \in V$. For $A, B \subset \mathcal{G}$ we denote by $[A, B]$ the subgroup of $\mathcal{G}$ generated by elements of the form $a \cdot b \cdot a^{-1} \cdot b^{-1}$, where $a \in A$ and $b \in B$. The lower central series of $\mathfrak{g}$ are defined by

$$\mathfrak{g}_{(1)} = \mathfrak{g}\,, \quad \mathfrak{g}_{(j)} = [\mathfrak{g}, \mathfrak{g}_{(j-1)}]\,.$$

The lower central series of $\mathcal{G}$ are defined by

$$\mathcal{G}_{(1)} = \mathcal{G}\,, \quad \mathcal{G}_{(j)} = [\mathcal{G}, \mathcal{G}_{(j-1)}]\,.$$

Then each $\mathcal{G}_{(j)}$ is a connected Lie normal subgroup of $\mathcal{G}$ and has its Lie algebra $\mathfrak{g}_{(j)}$. The lower central series form a descending chain of subspaces, respectively of normal subgroups.

Definition 5.2.1. *If there exists $\nu \in \mathbb{N}$ such that $g_{(\nu+1)} = \{0\}$, or equivalently $\mathcal{G}_{(\nu+1)} = \{\mathbf{1}\}$, then $\mathcal{G}$ is called a nilpotent Lie group and $\mathfrak{g}$ is called a nilpotent Lie algebra.*

We recall two basic properties of nilpotent Lie groups.

Proposition 5.2.1. *Let $\mathcal{G}$ be a nilpotent Lie group. Then*

(1) *The exponential map is a diffeomorphism from* $\mathfrak{g}$ *to* $\mathcal{G}$.

(2) *If* λ *denotes the Lebesgue measure on* $\mathfrak{g}$, *then* $\lambda \circ exp^{-1}$ *is a bi-invariant Haar measure on* $\mathcal{G}$.

Definition 5.2.2.

(1) *We say that* $\mathfrak{g}$ *is a graded Lie algebra of step* $\nu \in \mathbb{N}$, *if there exist subspaces* V_i, $i \in \{1, ..., \nu\}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \bigoplus_{i=1}^{\nu} V_j$, $[V_i, V_j] \subset V_{i+j}$ *if* $i + j \leq \nu$ *and* $[V_i, V_j] = 0$ *if* $i + j > \nu$.

(2) *A simply connected Lie group with a graded Lie algebra is called stratified if* V_1 *generates* $\mathfrak{g}$ *as an algebra. A simply connected nilpotent Lie group with stratified Lie algebra of step* ν *is called a Carnot group of step* ν.

Definition 5.2.3. *The homogeneous dimension of a Carnot group of step* ν *is defined as*

$$Q = \sum_{i=1}^{\nu} i\, d_i \,,$$

where $d_i = \dim V_i$.

Let us choose an orthonormal basis $X_{i,j}$, $j \in \{1, ..., d_i\}$ for each V_i. For each $x \in \mathcal{G}$ we can give a unique set of scalars $\{c_{1,1}, ..., c_{\nu,d_\nu}\}$, called the exponential coordinates of x, such that

$$\exp\left(\sum_{i=1}^{\nu} \sum_{j=1}^{d_i} c_{i,j} X_{i,j}\right) = x \,.$$

For $r > 0$ we define the dilations $\delta_r : \mathcal{G} \to \mathcal{G}$ by

$$\delta_r(x) = \exp\left(\sum_{i=1}^{\nu} \sum_{j=1}^{d_i} r^i c_{i,j} X_{i,j}\right)$$

.

The natural metric is determined by homogeneous norms.

Definition 5.2.4. *A homogeneous norm on* $\mathcal{G}$ *is continuous function*

$$|\cdot| : \mathcal{G} \to [0, +\infty)$$

such that

(1) $|x| = 0$ *if an only if* $x = \mathbf{1}$.

(2) $|x^{-1}| = |x|$ *and* $|\delta_r x| = r|x|$, $\forall\, x \in \mathcal{G}$, $r > 0$.

Homogeneous norms always exist, for example if we use the exponential coordinates, and we denote for $x = (c_{i,j})$, we can define

$$|x| = \left(\sum_{i=1}^{\nu} \left(\sum_{j=1}^{d_i} (c_{i,j})^2 \right)^{\frac{\nu!}{2i}} \right)^{\frac{1}{\nu!}} .$$

So, we may assume that $\mathcal{G}$ is equipped with a fixed homogeneous norm. We can define the homogeneous ball with radius r and center at x, by

$$B(x,r) = \{ y \in \mathcal{G} \ : \ |x^{-1} \cdot y| < r \} .$$

We can observe then, that $B(x,r)$ is a left translate of $B(\mathbf{1},r)$ and the Haar measure of $B(x,r)$ is a constant multiple of r^Q.
For simplicity we will denote $X_j = X_{1,j}$ and $d = d_1$. We call

$$Xu = \Big(X_1 u, ..., X_d u \Big)$$

the horizontal gradient of a corresponding function u.
Also, V_ν is the center of the Lie algebra, therefore the vector fields $X_{\nu,j}$ - which commutes with any $X_{i,j}$ - will have a special role. Let us denote $T_j = X_{\nu,j}$.
Let $1 < p < +\infty$. Consider the following Sobolev space with respect to the horizontal vector fields X_i

$$HW^{1,p}(\Omega) = \Big\{ u \in L^p(\Omega) \ : \ X_i u \in L^p(\Omega) \, , \text{ for all } i \in \{1, ..., d\} \Big\} .$$

$HW^{1,p}(\Omega)$ is a Banach space with respect to the norm

$$||u||_{HW^{1,p}} = ||u||_{L^p} + \sum_{i=1}^{d} ||X_i u||_{L^p} .$$

We denote by $HW_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$.
We consider the p-Laplace equation in a bounded domain $\Omega \subset \mathcal{G}$.

$$-\sum_{i=1}^{d} X_i \left(a_i(Xu) \right) = 0 \, , \text{ in } \Omega \tag{5.2.1}$$

where

$$a_i(\xi) = |\xi|^{p-2}\xi_i\,, \text{ for all } \xi \in \mathbb{R}^d\,.$$

Then a function $u \in HW^{1,p}_{\text{loc}}(\Omega)$ is a weak solution for (5.2.1) if

$$\sum_{i=1}^{d} \int_\Omega a_i(Xu(x))\; X_i\varphi(x)dx = 0\,, \text{ for all } \varphi \in HW^{1,p}_0(\Omega)\,. \tag{5.2.2}$$

For $\varepsilon > 0$ small, let us consider the approximating equation to (5.2.1)

$$-\sum_{i=1}^{d} X_i\left(a_i^\varepsilon(Xu)\right) = 0\,, \text{ in } \Omega \tag{5.2.3}$$

where

$$a_i^\varepsilon(\xi) = \left(\varepsilon + |\xi|^2\right)^{\frac{p-2}{2}}\, \xi_i\,, \text{ for all } \xi \in \mathbb{R}^d\,.$$

5.3 DIFFERENTIABILITY ALONG VECTOR FIELDS FROM THE CENTER OF THE LIE ALGEBRA

In this section we consider a fixed $j_0 \in \{1, ..., d_\nu\}$ and denote $T = T_{j_0}$. For $x \in \Omega$, $s \in \mathbb{R}$ sufficiently small such that $x \cdot e^{sT} \in \Omega$ and $0 < \alpha, \theta < 1$ we consider the following differences and difference quotients in a similar way as in section 2.1:

$$\begin{aligned}
\triangle_{Z,s}u(x) &= u(x \cdot e^{sZ}) - u(x)\,,\\
\triangle^2_{Z,s}u(x) &= u(x \cdot e^{sZ}) + u(x \cdot e^{-sZ}) - 2u(x)\,,\\
D_{Z,s,\theta}u(x) &= \frac{u(x \cdot e^{sZ}) - u(x)}{|s|^\theta}\,,\\
D_{Z,-s,\theta}u(x) &= \frac{u(x \cdot e^{-sZ}) - u(x)}{-|s|^\theta}\,.
\end{aligned}$$

Then

$$D_{Z,-s,\alpha}D_{Z,s,\theta}u(x) = D_{Z,s,\theta}D_{Z,-s,\alpha}u(x) = \frac{\triangle^2_{Z,s}u(x)}{|s|^{\alpha+\theta}}\,.$$

Proposition 2.1.1 remains valid also in the case of Carnot groups, while Proposition 2.1.2 has the following form.

Proposition 5.3.1. *Let $1 \le p < \infty$, $u \in HW^{1,p}_{\text{loc}}(\Omega)$, $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, 3r) \subset \Omega$. Then there exists a positive constant c independent of u such that*

$$\int_{B(x_0,r)} \left| D_{X_{ij},s,\frac{1}{i}} u(x) \right|^p dx \le c \int_{B(x_0,2r)} (|u|^p + |Xu|^p)\, dx\,. \tag{5.3.1}$$

Remark 5.3.1. If we use in Proposition 5.3.1 the vector field $X_{\nu,j_0} = T$ then we have

$$\int_{B(x_0,r)} \left| D_{T,s,\frac{1}{\nu}} u(x) \right|^p dx \le c \int_{B(x_0,2r)} (|u|^p + |Xu|^p)\, dx\,. \tag{5.3.2}$$

The following lemma is the counterpart of Lemma 2.2.1 and, as we mentioned in section 2.2, its proof needs just minor modifications.

Lemma 5.3.1. *Let $u \in L^p(\mathcal{G})$, Z a left invariant vector field, $0 < \alpha$, $0 < \sigma$ and $M \ge 0$. Suppose that*

$$\sup_{0<|s|\le\sigma} \frac{||\triangle^2_{Z,s} u||_{L^p}}{|s|^\alpha} \le M\,. \tag{5.3.3}$$

Then for all $0 < \beta \le \min\{1, \alpha\}$ if $\alpha \ne 1$ and for all $0 < \beta < 1$ if $\alpha = 1$ there exists $c > 0$ independent of u and a possibly different $\sigma > 0$ from that one in (5.3.3) such that

$$\sup_{0<|s|\le\sigma} \frac{||\triangle_{Z,s} u||_{L^p}}{|s|^\beta} \le c(||u||_{L^p} + \frac{M}{2^\alpha})\,. \tag{5.3.4}$$

5.3.1 The case: $2 \le p < \infty$

Let us fix $j_0 \in \{1, ..., d_\nu\}$ and denote $T = T_{j_0}$. The proof of the following lemma is similar to the proof Lemma 2.3.1, because T commutes with X_i and the translations $x \to x \cdot e^{sT}$ leave the integrals invariant.

Lemma 5.3.2. *Let $2 \le p < +\infty$, $u_\varepsilon \in HW^{1,p}_{\text{loc}}(\Omega)$ be a weak solution of (5.2.3), $x_0 \in \Omega$, $r > 0$ such that $B(x_0, 3r) \in \Omega$. Let us suppose that there exists a constant $c > 0$, $\sigma > 0$ and $0 \le \alpha < \frac{\nu-1}{\nu}$ such that*

$$\sup_{0\ne|s|\le\sigma} \int_{B(x_0,r)} \left| D_{T,s,\frac{1}{\nu}+\alpha}(u_\varepsilon) \right|^p dx$$
$$\le c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx\,. \tag{5.3.5}$$

If we have

$$\frac{2+2\nu\alpha}{\nu p} < \frac{\nu-1}{\nu}$$

then for possibly different $c>0$, $\sigma>0$ holds

$$\sup_{0\neq|s|\leq\sigma} \int_{B(x_0,\frac{r}{2})} \left| D_{T,s,\frac{1}{\nu}+\frac{2}{\nu p}+\frac{2}{p}\alpha}(u_\varepsilon) \right|^p dx \\ \leq c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx\,. \quad (5.3.6)$$

In the case

$$\frac{2+2\nu\alpha}{\nu p} > \frac{\nu-1}{\nu}$$

we have

$$\int_{B(x_0,\frac{r}{2})} |Tu_\varepsilon(x)|^p\, dx \leq c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx\,. \quad (5.3.7)$$

and hence $Tu \in L^p_{\mathrm{loc}}(\Omega)$.

Otherwise,

$$\frac{2+2\nu\alpha}{\nu p} = \frac{\nu-1}{\nu}$$

and we have that

$$\int_{B(x_0,\frac{r}{4})} |Tu_\varepsilon(x)|^p\, dx \leq c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx\,. \quad (5.3.8)$$

Proof. Consider

$$\gamma = \frac{1}{\nu} + \alpha\,,$$

and let g be a cut-off function between $B(x_0,\frac{r}{2})$ and $B(x_0,r)$. We use now the test function

$$\varphi = D_{T,-s,\gamma}\left(g^2 D_{T,s,\gamma} u_\varepsilon\right) \quad (5.3.9)$$

to get

$$\sum_{i=1}^{d} \int_\Omega a_i^\varepsilon(Xu_\varepsilon(x))\; X_i\Big(D_{T,-s,\gamma}\left(g^2 D_{T,s,\gamma} u_\varepsilon(x)\right)\Big) dx = 0$$

and from here, by the fact that X_i commutes with $D_{T,s,\gamma}$ and $D_{T,-s,\gamma}$, we obtain

$$\sum_{i=1}^{d}\int_\Omega D_{T,s,\gamma}\, a_i^\varepsilon(Xu_\varepsilon(x))\; g^2(x)\; D_{T,s,\gamma}\,(X_i u_\varepsilon(x))\,dx$$

$$+\sum_{i=1}^{d}\int_\Omega D_{T,s,\gamma}\, a_i^\varepsilon(Xu_\varepsilon(x))\; D_{T,s,\gamma}u_\varepsilon(x)\; 2g(x)\; X_i g(x)\,dx = 0\,. \quad (5.3.10)$$

Using the properties of the functions a_i^ε we get

$$\int_{B(x_0,r)} g^2(x)\;\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot e^{sT})|^2\right)^{\frac{p-2}{2}}\;|D_{T,s,\gamma}Xu_\varepsilon(x)|^2\;dx$$

$$\le c\int_{B(x_0,r)}\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot e^{sT})|^2\right)^{\frac{p-2}{2}}\;|D_{T,s,\gamma}Xu_\varepsilon(x)|$$

$$\cdot\,|D_{T,s,\gamma}u_\varepsilon(x)|\;\;|g(x)|\;|Xg(x)|dx\,.$$

Using the fact that $p\ge 2$ we get

$$\int_{B(x_0,r)} g^2(x)\;\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot e^{sT})|^2\right)^{\frac{p-2}{2}}\;|D_{T,s,\gamma}Xu_\varepsilon(x)|^2\;dx$$

$$\le c\int_{B(x_0,r)}\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot e^{sT})|^2\right)^{\frac{p-2}{2}}\;|D_{T,s,\gamma}u_\varepsilon(x)|^2\;\;|Xg(x)|^2\,dx\,. \quad (5.3.11)$$

Denoting by RHS the right hand side of (5.3.11) we have that

$$RHS\le c\int_{B(x_0,r)}\left(\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot e^{sT})|^2\right)^{\frac{p}{2}}+|D_{T,s,\gamma}u_\varepsilon(x)|^p\right)\;dx\,.$$

and then by assumption (5.3.5) we have

$$RHS\le c\int_{B(x_0,2r)}\left(\varepsilon+|Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}}+|u_\varepsilon(x)|^p\,dx\,.$$

Therefore

$$\int_{B(x_0,r)} g^2(x)\;\left(\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot e^{sT})|^2\right)^{\frac{p-2}{2}}\;|D_{T,s,\gamma}Xu_\varepsilon(x)|^2\;dx$$

$$\le c\int_{B(x_0,2r)}\left(\varepsilon+|Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}}+|u_\varepsilon(x)|^p\,dx\,. \quad (5.3.12)$$

From the inequality

$$|s^\gamma D_{s,\gamma}Xu_\varepsilon(x)|\le\sqrt{2}\sqrt{\varepsilon+|Xu_\varepsilon(x)|^2+|Xu_\varepsilon(x\cdot e^{sT})|^2}$$

we get

$$\int_{B(x_0,r)} g^2(x) \quad s^{(p-2)\gamma} \quad |D_{T,s,\gamma} X u_\varepsilon(x)|^p \; dx \;\; \leq \;\; c \int_{B(x_0,2r)} \left(\varepsilon + |X u_\varepsilon(x)|^2\right)^{\frac{p}{2}} \; + \; |u_\varepsilon(x)|^p \, dx \, .$$

Since

$$\begin{aligned} D_{T,s,\gamma} X(g^2 u_\varepsilon)(x) = D_{T,s,\gamma} X(g^2)(x) \; u_\varepsilon(x \cdot e^{sT}) + X(g^2)(x) \; D_{T,s,\gamma} u_\varepsilon(x) \\ + D_{T,s,\gamma} g^2(x) \; X u_\varepsilon(x \cdot e^{sT}) + g^2(x) \; D_{T,s,\gamma} X u_\varepsilon(x) \end{aligned}$$

it follows that

$$\int_{B(x_0,r)} \left| D_{T,s,\frac{2\gamma}{p}} X(g^2 u_\varepsilon)(x) \right|^p \; dx \;\; \leq \;\; c \int_{B(x_0,2r)} \left(\left(\varepsilon + |X u_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) \; dx \, . \quad (5.3.13)$$

Let us denote the right hand side of (5.3.13) by M^p. Using Lemma 5.3.1 we get

$$\int_{B(x_0,r)} \left| D_{T,-s,\frac{1}{\nu}} D_{T,s,\frac{2\gamma}{p}} (g^2 u_\varepsilon)(x) \right|^p \; dx \leq M^p \, . \qquad (5.3.14)$$

Therefore, for all s sufficiently small we have

$$\frac{\left\| \triangle^2_{T,s} (g^2 u_\varepsilon) \right\|_{L^p(\mathcal{G})}}{s^{\frac{1}{\nu} + \frac{2+2\nu\alpha}{\nu p}}} \leq M \, ,$$

so there exists $\sigma > 0$ such that

$$\sup_{0<|s|\leq\sigma} \frac{\left\| \triangle^2_{T,s} (g^2 u_\varepsilon) \right\|_{L^p(\mathcal{G})}}{s^{\frac{1}{\nu} + \frac{2+2\nu\alpha}{\nu p}}} \leq M \, . \qquad (5.3.15)$$

If it happens that

$$\frac{2 + 2\nu\alpha}{\nu p} < \frac{\nu - 1}{\nu}$$

then by Lemma 5.3.1 we get (5.3.6).

If we have

$$\frac{2 + 2\nu\alpha}{\nu p} > \frac{\nu - 1}{\nu}$$

then $T u_\varepsilon \in L^p_{\mathrm{loc}}(\Omega)$ and estimate (5.3.7) is valid.

In the remaining case

$$\frac{2 + 2\nu\alpha}{\nu p} = \frac{\nu - 1}{\nu} \, .$$

Therefore,

$$\alpha = \frac{p(\nu-1)-2}{2\nu}$$

and then using that $\alpha \in [0, \frac{\nu-1}{\nu})$ we get

$$2 \le p < \frac{2\nu}{\nu-1} .$$

We can use assumption (5.3.5) with α' arbitrarily close to $\frac{\nu-1}{\nu}$, in particular for $\alpha' > \frac{p(\nu-1)-2}{2\nu}$ to get back (5.3.15) with

$$\frac{2+2\nu\alpha'}{\nu p} > \frac{\nu-1}{\nu}$$

and then use the previous case. $\square$

Lemma 5.3.1 implies that we can start with $\alpha_0 = 0$ in the assumption (5.3.5) to get $\alpha_1 = \frac{2}{\nu p}$ in (5.3.6). Now we can use α_1 in (5.3.5) to get

$$\alpha_2 = \frac{2}{\nu p} + \frac{2}{p}\alpha_1 = \frac{2}{\nu p} + \frac{2^2}{\nu p^2}$$

such that estimate (5.3.6) is true. In general, if we already found $\alpha_1, ..., \alpha_k$, then we get

$$\alpha_{k+1} = \frac{2}{\nu p} + \frac{2}{p}\alpha_k = \frac{2}{\nu p} + ... + \frac{2^k}{\nu p^k} = \frac{2}{\nu p}\sum_{i=0}^{k-1}\left(\frac{2}{p}\right)^i = \frac{2}{\nu p}\,\frac{1-\left(\frac{2}{p}\right)^k}{1-\frac{2}{p}} .$$

Therefore, for a given $p > 2$ the upper bound for α_k is given by

$$\frac{2}{\nu}\frac{1}{p-2} .$$

Hence, for $p \in [2, \frac{2\nu}{\nu-1})$, after a sufficiently large number k of iterations, we get that $\alpha_k \ge \frac{\nu-1}{\nu}$ and this means that $Tu_\varepsilon \in L^p_{\text{loc}}(\Omega)$.

In conclusion we have the following theorem.

Theorem 5.3.1. *Let $2 \leq p < \frac{2\nu}{\nu-1}$ and $u_\varepsilon \in HW^{1,p}_{loc}(\Omega)$ be a weak solution of (2.1.3). Then for $x_0 \in \Omega$, $r > 0$ such that $B(x_0, 3r) \subset \Omega$ and for a number $k \in \mathbb{N}$ of iterations depending only on p and ν we have*

$$\int_{B(x_0,\frac{r}{2^{k+1}})} |Tu_\varepsilon(x)|^p \, dx \leq c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx \tag{5.3.16}$$

and hence $Tu_\varepsilon \in L^p_{loc}(\Omega)$.

In the case $p \geq \frac{2\nu}{\nu-1}$ our proof above gives the following result.

Proposition 5.3.2. *For $p \geq \frac{2\nu}{\nu-1}$ and weak solutions u_ε of (2.1.3) we have*

$$\sup_{0 \neq |s| \leq \sigma} \int_{B(x_0,\frac{r}{2^k})} \left| D_{T,s,\frac{1}{\nu}+\alpha'}(u_\varepsilon) \right|^p \, dx$$

$$\leq c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \right) dx \, . \tag{5.3.17}$$

for $c > 0$ independent of ε, α' less then, but arbitrarily close to $\frac{2}{\nu}\frac{1}{p-2}$, and a corresponding number k of iterations.

5.3.2 The case $1 < p < 2$.

Let us consider $u_\varepsilon \in HW^{1,p}_{\text{loc}}(\Omega)$ a weak solution of (2.1.3), $\gamma = \frac{1}{\nu}$ and the test function

$$\varphi = D_{T,-s,\gamma}\left(g^2 D_{T,s,\gamma} u_\varepsilon\right) \, . \tag{5.3.18}$$

We can follow the proof of Theorem 2.4.1 until we get

$$\int_{B(x_0,r)} \left| D_{T,s,\frac{\gamma}{2}} X(g^2 u_\varepsilon)(x) \right|^p \, dx \leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, . \tag{5.3.19}$$

By the fact that our Carnot goup has step ν we get

$$\int_{B(x_0,r)} \left| D_{T,s,\frac{1}{\nu}} D_{T,s,\frac{\gamma}{2}}(g^2 u_\varepsilon)(x) \right|^p \, dx \;\leq\; c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, , \tag{5.3.20}$$

and this leads to the fact that for a sufficiently small σ we have

$$\sup_{0<|s|\leq\sigma} \frac{||\triangle^2_{T,s}(g^2 u_\varepsilon)||_{L^p(\mathcal{G})}}{|s|^{\frac{1}{\nu}+\frac{1}{2\nu}}} \leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \, . \tag{5.3.21}$$

Therefore, knowing that we have a control on $\frac{1}{\nu}$ derivatives of u_ε in the T direction, we obtained that we can control $\frac{1}{2\nu}$ derivatives of Xu and hence $\frac{3}{2\nu}$ derivatives of u_ε. Doing iteration, and choosing corresponding cut-off functions, in general we get after k steps that

$$\int_{B(x_0,\frac{r}{2^k})} \left| D_{T,s,\frac{2^k-1}{2^k\nu}} X(g^2 u_\varepsilon)(x) \right|^p dx \leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \,, \quad (5.3.22)$$

and

$$\sup_{0<|s|\leq\sigma} \frac{||D_{T,s}(g^2 u_\varepsilon)||_{L^p(\mathcal{G})}}{|s|^{\frac{2^{k+1}-1}{2^k\nu}}} \leq c \int_{B(x_0,2r)} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p \, dx \,. \quad (5.3.23)$$

In the case of a Carnot group of step $\nu = 2$ we can continue the proof of Theorem 2.4.1 and get $Tu_\varepsilon \in L^p_{\text{loc}}(\Omega)$. In the case of $\nu \geq 3$ this method gives at most $\frac{p+3}{2\nu} < 1$ derivatives for u_ε in the T direction. In conclusion, we have

Theorem 5.3.2. *In the case of a Carnot group of step 2, for any weak solution u_ε of the approximating equation (2.1.3) we have $T_j u_\varepsilon \in L^p_{\text{loc}}(\Omega)$ for all $T_j \in V_2$.*

5.4 DIFFERENTIABILITY ALONG HORIZONTAL VECTOR FIELDS

We recall the following identities [8]:

Proposition 5.4.1. *For each $X, Y \in \mathfrak{g}$ and $x \in \mathcal{G}$ we have :*

1. $[X, Y] = \mathrm{ad}X(Y)$.
2. $\mathrm{Ad}_{e^X} = e^{\mathrm{ad}X}$.
3. $x \cdot e^X \cdot x^{-1} = e^{\mathrm{Ad}_x(X)}$.

We have denoted by $\mathrm{R}_x : \mathcal{G} \to \mathcal{G}$ the right multiplication by x. Let Z be a left invariant vector field. Then, using the identities from Proposition 5.4.1 we get

$$\begin{aligned}
&X_i u\Big(x \cdot e^{sZ}\Big) = X_i\Big(u \circ \mathrm{R}_{e^{sZ}}\Big)(x) = \frac{d}{dt}\Big|_{t=0} u \circ \mathrm{R}_{e^{sZ}}\Big(x \cdot e^{tX_i}\Big) \\
&= \frac{d}{dt}\Big|_{t=0} u\Big(x \cdot e^{tX_i} \cdot e^{sZ}\Big) = \frac{d}{dt}\Big|_{t=0} u\Big(x \cdot e^{sZ} \cdot e^{-sZ} \cdot e^{tX_i} \cdot e^{sZ}\Big) \\
&= \frac{d}{dt}\Big|_{t=0} u\Big(x \cdot e^{sZ} \cdot e^{t\mathrm{Ad}_{e^{-sZ}}(X_i)}\Big) = \mathrm{Ad}_{e^{-sZ}}(X_i)u\Big(x \cdot e^{sZ}\Big) \\
&= e^{\mathrm{ad}(-sZ)}(X_i)u\Big(x \cdot e^{sZ}\Big) \\
&= \Bigg(\Big(\mathrm{Id} - s\,\mathrm{ad}(Z) + ... + (-1)^{\nu-1}s^{\nu-1}\mathrm{ad}^{\nu-1}(Z)\Big)(X_i)\Bigg)u\Big(x \cdot e^{sZ}\Big) \\
&= X_i u\Big(x \cdot e^{sZ}\Big) - s\,\mathrm{ad}(Z)(X_i)u\Big(x \cdot e^{sZ}\Big) + ... + (-1)^{\nu-1}s^{\nu-1}\mathrm{ad}^{\nu-1}(Z)(X_i)u\Big(x \cdot e^{sZ}\Big)
\end{aligned}$$

In the same way we can prove that

$$\begin{aligned}
&X_i u\Big(x \cdot e^{-sZ}\Big) \\
&= X_i u\Big(x \cdot e^{-sZ}\Big) + s\,\mathrm{ad}(Z)(X_i)u\Big(x \cdot e^{-sZ}\Big) + ... + s^{\nu-1}\mathrm{ad}^{\nu-1}(Z)(X_i)u\Big(x \cdot e^{-sZ}\Big)
\end{aligned}$$

Therefore, we have the following result:

Lemma 5.4.1. *For any left invariant vector field Z, any $u \in C_0^\infty(\Omega)$ we have*

$$\begin{aligned}
&X_i D_{Z,-s,1} D_{Z,s,1} u(x) = \\
&= D_{Z,-s,1} D_{Z,s,1} X_i u(x) - \frac{1}{s}\Bigg([Z, X_i]u\Big(x \cdot e^{sZ}\Big) - [Z, X_i]u\Big(x \cdot e^{-sZ}\Big) \\
&\qquad + \sum_{k=2}^{\nu-1} s^{k-2}\Bigg((-1)^k ad^k(Z)(X_i)u\Big(x \cdot e^{sZ}\Big) + ad^k(Z)(X_i)u\Big(x \cdot e^{-sZ}\Big)\Bigg)\Bigg).
\end{aligned} \tag{5.4.1}$$

5.4.1 The case: $2 \le p < \frac{2\nu}{\nu-1}$

Theorem 5.4.1. *Let $u_\varepsilon \in HW^{1,p}_{\text{loc}}(\Omega)$ be a weak solution of (2.1.3), $x_0 \in \Omega$, $r > 0$ such that $B(x_0, 3r) \in \Omega$. Then there there exists a constant $0 < M < \infty$ independent of ε such that for a number $l \in \mathbb{N}$ that depends only on p and ν we have*

$$\int_{B(x_0,\frac{r}{2^l})} \left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p-2}{2}} \left|X^2 u_\varepsilon(x)\right|^2 \le M\,. \tag{5.4.2}$$

Proof. We know that we can control the derivatives in the direction of V_ν, the center of Lie algebra. We will use formula (5.4.1) to control the derivatives in the direction of $V_{\nu-1}$ and going backwards we will gain control over $V_{\nu-2}$, $V_{\nu-3}$, ... until we reach V_1.
Let $Z \in V_{\nu-1}$ and we use a test function

$$\varphi = D_{Z,-s,1} D_{Z,s,1}\Big(g^4 u_\varepsilon\Big)$$

where g is a cut-off function between $B(x_0, \frac{r}{2^{k+2}})$ and $B(x_0, \frac{r}{2^{k+1}})$ and the number k is given by Theorem 5.3.1. Formula (5.4.1) gives in this case the formula,

$$\begin{aligned} X_i D_{Z,-s,1} D_{Z,s,1}\Big(g^4 u_\varepsilon\Big)(x) &= D_{Z,-s,1} D_{Z,s,1} X_i\Big(g^4 u_\varepsilon\Big)(x) \\ &\quad - \frac{1}{s}\Bigg([Z,X_i]\Big(g^4 u_\varepsilon\Big)\Big(x\cdot e^{sz}\Big) - [Z,X_i]\Big(g^4 u_\varepsilon\Big)\Big(x\cdot e^{-sZ}\Big)\Bigg) \end{aligned} \tag{5.4.3}$$

We know that $[Z, X_i] \in V_\nu$, therefore $[Z, X_i] \in L^p_{\text{loc}}(\Omega)$, so we can use the same method of proof as for Theorem 3.1.1 to get an $M > 0$ independent of ε such that for all s sufficiently small

$$\int_\Omega \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x\cdot e^{sZ})|^2\right)^{\frac{p-2}{2}} |D_{Z,s,1}u_\varepsilon(x)|^2 \; g^4(x)\,dx \le M\,. \tag{5.4.4}$$

Hence,

$$\int_\Omega g^4(x)\; s^{p-2}\; |D_{Z,s,1}u_\varepsilon(x)|^p \le M$$

and hence

$$\int_\Omega \left|D_{Z,s,\frac{2}{p}}\;(g^4 u_\varepsilon)(x)\right|^p \le M\,.$$

Now, using the fact that $Z \in V_{\nu-1}$, we get that there exists $\sigma > 0$ such that

$$\sup_{0<|s|<\sigma} \frac{\left\| \Delta^2_{Z,s}\left(g^4 u_\varepsilon\right)\right\|_{L^p(\mathcal{G})}}{s^{\frac{1}{\nu-1}+\frac{2}{p}}} \leq M\,. \tag{5.4.5}$$

The fact that $p \leq \frac{2\nu}{\nu-1}$ implies that $\frac{1}{\nu-1} + \frac{2}{p} > 1$, so by Lemma 5.3.1 we have $Zu \in L^p_{\text{loc}}(\Omega)$.
Now let us consider $W \in V_{\nu-2}$ and a test function

$$\varphi = D_{Z,-s,1} D_{Z,s,1}\left(g^4 u_\varepsilon\right)$$

where g is a cut-off function between $B(x_0, \frac{r}{2^{k+3}})$ and $B(x_0, \frac{r}{2^{k+2}})$. Formula (5.4.1) in this case looks like

$$\begin{aligned} & X_i D_{W,-s,1} D_{W,s,1}\left(g^4 u_\varepsilon\right)(x) \\ & = D_{W,-s,1} D_{W,s,1} X_i\left(g^4 u\right)(x) - \frac{1}{s}\left([W, X_i]\left(g^4 u\right)\left(x \cdot e^{sW}\right) - [W, X_i]\left(g^4 u\right)\left(x \cdot e^{-sW}\right)\right. \\ & \quad \left. + \sum_{k=2}^{\nu-1} s^{k-2}\left((-1)^k \text{ad}^k(W)(X_i)\left(g^4 u\right)\left(x \cdot e^{sW}\right) + \text{ad}^k(W)(X_i)\left(g^4 u\right)\left(x \cdot e^{-sZ}\right)\right)\right). \end{aligned} \tag{5.4.6}$$

We observe that $[Z, X_i] \in V_{\nu-1}$ and the vector fields in the third line of formula (5.4.6) are in V_ν or are null, so we can repeat, with minor changes, the proof of the previous case to get $W u_\varepsilon \in L^p_{\text{loc}}(\Omega)$.
Continuing in this way we arrive to the case when we can use a test function

$$\varphi = D_{X_{j_0},-s,1} D_{X_{j_0},s,1}\left(g^4 u_\varepsilon\right)$$

for an arbitrary $j_0 \in \{1, ..., d\}$ and get formula 5.4.2. with $l = k + \nu$. $\square$

5.4.2 The case $1 < p < 2$ in a Carnot group of step 2

As we have seen in Theorem 5.3.2 in the case $1 < p < 2$ our methods control the derivatives in the direction of the center of the Lie Algebra if the Carnot group is of step 2. But in this case the results are essentially the same as in the Heisenberg group and require just minor modifications. For example we can prove the following theorem which is a counterpart of Theorem 3.2.1.

Theorem 5.4.2. *Let $\mathcal{G}$ be a Carnot group of step 2, $\Omega \subset \mathcal{G}$ be an open set, $\frac{\sqrt{17}-1}{2} \leq p \leq 2$ and $u_\varepsilon \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution of (5.2.3). Consider $x_0 \in \Omega$, $r > 0$ such that $B(x_0, 3r) \Subset \Omega$. Then there exists a positive number k depending only on p such that for each $i_0 \in \{1, ..., 2n\}$ and $s > 0$ sufficiently small we have*

$$\begin{aligned} &\int_{B(x_0, \frac{r}{2^{k+3}})} \left(\varepsilon + |Xu_\varepsilon(x)|^2 + |Xu_\varepsilon(x \cdot h_{i_0})|^2\right)^{\frac{p-2}{2}} \left|D_{h_{i_0}} Xu_\varepsilon(x)\right|^2 \, dx \\ &\leq c \Bigg(\varepsilon^{\frac{p-2}{2}} \, ||u_\varepsilon||^{2-p}_{C^\delta(B(x_0, \frac{r}{2^{k+1}}))} \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p\right) dx \\ &\qquad + \varepsilon^{\frac{p-2}{2}} \, ||u_\varepsilon||^2_{L^2(B(x_0, \frac{r}{2^{k+1}}))} + c \int_{B(x_0,2r)} \left(\left(\varepsilon + |Xu_\varepsilon(x)|^2\right)^{\frac{p}{2}} + |u_\varepsilon(x)|^p\right) dx \Bigg), \end{aligned} \tag{5.4.7}$$

and hence $u_\varepsilon \in HW^{2,p}_{\mathrm{loc}}(\Omega)$.

5.5 CORDES CONDITIONS IN CARNOT GROUPS

Let us consider a Carnot group $\mathcal{G}$ of step ν such that the horizontal subspace of the Lie Algebra has dimension d and for a fixed inner product on V_1 we consider an orthonormal basis $X_1, ..., X_d$. We remark that in our previous results regarding the $HW^{2,2}$ regularity of the approximating p-harmonic functions, the step of the Carnot group had the major effect on the admissible values of p, while in our next results regarding the $HW^{2,2}$ and $C^{1,\alpha}$ regularity of p-harmonic functions the dimension of V_1 will also have an important role.

We recall first the following result from the Calderón-Zygmund theory in Carnot groups (see [9, 13, 14].

Lemma 5.5.1. *For all $1 < s < \infty$ there exists $C_{\mathcal{G},s} \geq 1$ such that for all $u \in HW_0^{2,s}(\Omega)$ we have*

$$||X^2 u||_{L^s(\Omega)} \leq C_{\mathcal{G},s} ||\Delta_X u||_{L^s(\Omega)} .$$

Let us consider now the following second order linear subelliptic PDE operator in non-divergence form with measurable coefficients

$$\mathcal{A}u = \sum_{i,j=1}^{d} a_{ij}(x) X_i X_j u$$

where the functions $a_{ij} \in L^\infty(\Omega)$. Let us denote by $A = (a_{ij})$ the $d \times d$ matrix of coefficients.

Definition 5.5.1. *We say that A satisfies the Cordes condition $K_{\varepsilon,\sigma}$ if there exists $\varepsilon \in (0,1]$ and $\sigma > 0$ such that*

$$0 < \frac{1}{\sigma} \leq \sum_{i,j=1}^{d} a_{ij}^2(x) \leq \frac{1}{d-1+\varepsilon} \left(\sum_{i=1}^{d} a_{ii}(x) \right)^2 , \ a.e. \ x \in \Omega . \tag{5.5.1}$$

We will now list results similar to those from section 4.2. We will omit the proofs which differ from their counterpart just by replacing $2n$, which is the dimension of the horizontal subspace for the Heisenberg group, by d which is the dimension of the horizontal subspace in our Carnot group.

Theorem 5.5.1. *Let $0 < \varepsilon \leq 1$, $\sigma > 0$ such that $\gamma = \sqrt{1-\varepsilon}\, C_{\mathcal{G},2} < 1$ and A satisfies the Cordes condition $K_{\varepsilon,\sigma}$. Then for all $u \in HW_0^{2,2}(\Omega)$ we have*

$$||X^2 u||_{L^2} \leq C_{\mathcal{G},2} \, \frac{1}{1-\gamma} \, ||\alpha||_{L^\infty} ||\mathcal{A}u||_{L^2} , \tag{5.5.2}$$

where

$$\alpha(x) = \frac{\langle A(x), I \rangle}{||A(x)||^2} .$$

Let us recall that the p-Laplace equation is

$$-\sum_{i=1}^{d} X_i\left(|Xu|^{p-2}\, X_i u\right) = 0\,,\ \text{in}\ \Omega\,. \tag{5.5.3}$$

and for each $m \in \mathbb{N}$ we consider the approximating equations

$$-\sum_{i=1}^{d} X_i\left(\left(\frac{1}{m} + |Xu|^2\right)^{\frac{p-2}{2}} X_i u\right) = 0\,,\ \text{in}\ \Omega\,. \tag{5.5.4}$$

The differentiated version of equation (5.5.4) has the form

$$\sum_{i,j=1}^{2n} a_{ij}^m\, X_i X_j u = 0\,,\ \text{in}\ \Omega \tag{5.5.5}$$

where

$$a_{ij}^m(x) = \delta_{ij} + (p-2)\frac{X_i u(x)\, X_j u(x)}{\frac{1}{m} + |Xu(x)|^2}\,.$$

Let us consider a weak solution $u_m \in HW^{1,p}(\Omega)$ of equation (5.5.4). Then $a_{ij}^m \in L^\infty(\Omega)$. Define the mapping $L_m : W_0^{2,2}(\Omega) \to L^2(\Omega)$ by

$$L_m(v)(x) \;=\; \sum_{i,j=1}^{d} a_{ij}^m(x) X_i X_j v(x)\,. \tag{5.5.6}$$

L_m satisfies the assumptions of Theorem 5.5.1 if

$$p-2 \in \left(\frac{1-\sqrt{d\left((d-1)\,C_{\mathcal{G},2}^2 - 1\right)+1}}{(d-1)C_{\mathcal{G},2}^2 - 1}\,,\ \frac{1+\sqrt{d\left((d-1)\,C_{\mathcal{G},2}^2 - 1\right)+1}}{(d-1)C_{\mathcal{G},2}^2 - 1}\right)\,. \tag{5.5.7}$$

Taking into consideration Theorem 5.4.1 we have the following result:

Theorem 5.5.2. *Let*

$$2 \le p \le \min\left\{\frac{2\nu}{\nu-1}\,,\ \frac{1+\sqrt{d\left((d-1)\,C_{\mathcal{G},2}^2 - 1\right)+1}}{(d-1)C_{\mathcal{G},2}^2 - 1}\right\}\,.$$

Then any p-harmonic function is in $HW^{2,2}_{\mathrm{loc}}(\Omega)$.

5.6 $C^{1,\alpha}$ REGULARITY OF p-HARMONIC FUNCTIONS FOR p CLOSE TO 2

Let us consider the setting from the previous section.

Theorem 5.6.1. *Let $0 < \varepsilon \leq 1$, such that $\varepsilon \cdot C_{\mathcal{G},s} < 1$ and suppose that*

$$|\Delta_X u(x) - \mathcal{A}u(x)| \leq \varepsilon \left|X^2 u(x)\right| \tag{5.6.1}$$

for a.e. $x \in \Omega$ and for all $u \in HW_0^{2,s}(\Omega)$. Then $\mathcal{A} : HW_0^{2,s}(\Omega) \to L^s(\Omega)$ is an isomorphism and there exists $c > 0$ such that

$$||X^2 u||_{L^s(\Omega)} \leq c||\mathcal{A}u||_{L^s(\Omega)} \tag{5.6.2}$$

for all $u \in HW_0^{2,s}(\Omega)$.

Proof. The proof follows from the fact that Lemma 5.5.1 and formula (5.6.1) show that $\mathcal{A} : HW_0^{2,s}(\Omega) \to L^s(\Omega)$ satisfies the relation

$$||\Delta_X u - \mathcal{A}u||_{L^s(\Omega)} \leq \varepsilon \cdot C_{\mathcal{G},s} \, ||\Delta_X u||_{L^s(\Omega)}$$

which proves that $\mathcal{A}$ is near to Δ_X and hence is an isomorphism. For the properties inherited by operators that are near to each other we quote [7, 29]. □

We need the following interpolation result (see [15]).

Lemma 5.6.1. *Let $u \in HW_{\mathrm{loc}}^{2,s}(\Omega)$, $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, 2r) \subset \Omega$. Then for all $\delta > 0$ there exists $c(\delta) > 0$ such that*

$$||Xu||^2_{L^s(B(x_0,r))} \leq \delta||X^2 u||^2_{L^s(B(x_0,r))} + c(\delta)||u||^2_{L^s(B(x_0,r))} \, .$$

We can use now Theorem 5.6.1, Lemma 5.6.6 and a method similar to the proof of Theorem 9.11, [10] and Lemma 4.3.3 to get the following result.

Theorem 5.6.2. *Let us suppose that the operator $\mathcal{A}$ satisfies the assumptions of Theorem 4.1 and that $B(x_0, 3r) \subset \Omega$. Then*

$$||X^2 u||_{L^s(B(x_0,r))} \leq c\Big(||\mathcal{A}u||_{L^s(B(x_0,2r))} + ||u||_{L^s(B(x_0,2r))}\Big)$$

for all $u \in HW^{2,s}_{\text{loc}}(\Omega)$.

We remark that

$$|L_\lambda v(x) - \Delta_X v(x)| \leq |p-2| \, \frac{d}{2} \, |X^2 v(x)|$$

for a.e. $x \in \Omega$ and for all $v \in HW^{2,s}(\Omega)$.

For a $\gamma > 0$ arbitrary small but fixed, let us denote by

$$\tilde{c} = \sup\Big\{C_{\mathcal{G},s}\,,\ s \in (1, Q+\gamma)\Big\}\,.$$

5.6.1 The case $2 \leq p$

Theorem 5.6.3. *For*

$$2 \leq p \leq 2 + \min\Big\{\frac{2\nu}{\nu-1}\,,\,\frac{2}{\tilde{c}\,d}\Big\}$$

and a p-harmonic function u there exists $0 < \alpha < 1$ such that we have the interior regularity $u \in C^{1,\alpha}$.

Proof. Theorem 4.2 shows that $Xu_\varepsilon \in HW^{1,2}_{\text{loc}}(\Omega)$ with uniform bounds. We use the embedding

$$HW^{1,2}_{\text{loc}}(\Omega) \hookrightarrow L^{q_0}_{\text{loc}}(\Omega)$$

where

$$q_0 = \frac{2Q}{Q-2}\,,$$

and Q is the homogeneous dimension of the Carnot group. For corresponding cut-off function η between homogeneous balls B_r and B_{2r} we have

$$
\begin{aligned}
&||L_m(\eta^2 u_m)||_{L^{q_0}(B_{3r})} \\
&= c\left\|u_m\, L_m(\eta^2) + \sum_{i,j=1}^{2n} a_{ij}^{\lambda}(x)\Big(X_j(\eta^2)X_i u_m + X_i(\eta^2)X_j u_m\Big)\right\|_{L^{q_0}(B_{3r})} \\
&\le c\Big(||u_m||_{L^{q_0}(\mathrm{supp}\eta)} + ||Xu_m||_{L^{q_0}(\mathrm{supp}\eta)}\Big) < +\infty
\end{aligned}
\tag{5.6.3}
$$

Therefore, by Theorems 5.4.1 and 5.6.1 we have that $u_m \in HW^{2,q_0}_{\mathrm{loc}}(\Omega)$ with locally uniform bounds. Repeating this procedure k times we get that $u_m \in HW^{2,q_k}_{\mathrm{loc}}(\Omega)$ for

$$q_k = \frac{2Q}{Q-2k}\,.$$

We stop after l steps for the smallest l for which we get $Q - 2l < 4$. Let us choose now $1 < \beta < 2$ close enough to 1 such that

$$u \in HW^{2,\frac{Q\beta}{2}}_{\mathrm{loc}}(\Omega)$$

and

$$Q\frac{\beta}{2-\beta} \le Q + \gamma\,.$$

Then we use the embedding

$$HW^{1,\frac{Q\beta}{2}}_{\mathrm{loc}}(\Omega) \hookrightarrow L^{Q\frac{\beta}{2-\beta}}_{\mathrm{loc}}(\Omega)$$

and inequalities similar to (5.6.3) to conclude that

$$u_m \in HW^{2,Q\frac{\beta}{2-\beta}}_{\mathrm{loc}}(\Omega)\,.$$

The embedding

$$HW^{1,Q\frac{\beta}{2-\beta}}_{\mathrm{loc}}(\Omega) \hookrightarrow C^{\frac{2\beta-2}{\beta}}$$

shows that u_m has interior regularity $C^{1,\alpha}$ where

$$\alpha = \frac{2\beta-2}{\beta}\,.$$

Because the estimates for u_m are uniform in m, we can conclude that $u \in C^{1,\alpha}$. $\square$

5.6.2 The case $p \leq 2$ in a Carnot group of step 2

For a Carnot group of step 2 the results are essentially the same as in the case of Heisenberg group.

Theorem 5.6.4. *In the case of a Carnot group of step 2 and*

$$\max\left\{\frac{\sqrt{17}-1}{2}\,,\,2-\frac{2}{\tilde{c}d}\right\} \leq p \leq 2\,,$$

for any p-harmonic function u we have the interior regularity $u \in C^{1,\alpha}$ where $0 < \alpha < 1$.

Proof. Theorems 5.4.2 and 5.6.1 implies that $Xu_\varepsilon \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ uniformly in ε. Then we can start with $q_0 = p$ and follow the proof of Theorem 4.4.3 until we get the first l with

$$q_l = \frac{Qp}{Q-lp} > \frac{Q}{2}\,.$$

The rest is similar to the proof of Theorem 5.6.3. □

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