TOPOLOGY OF FUNCTION SPACES

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This dissertation is a study of the relationship between a topological space X and various higher–order objects that we can associate with X. In particular the focus is on $C(X)$, the set of all continuous real–valued functions on X endowed with the topology of pointwise convergence, the compact–open topology and an admissible topology. The topological properties of continuous function universals and zero set universals are also examined. The topological properties studied can be divided into three types (i) compactness type properties, (ii) chain conditions and (iii) sequential type properties.

The dissertation begins with some general results on universals describing methods of constructing universals. The compactness type properties of universals are investigated and it is shown that the class of metric spaces can be characterised as those with a zero set universal parametrised by a σ–compact space. It is shown that for a space to have a Lindelof–Σ zero set universal the space must have a σ–disjoint basis.

A study of chain conditions in $C_k(X)$ and $C_p(X)$ is undertaken, giving necessary and sufficient conditions on a space X such that $C_p(X)$ has calibre (κ, λ, μ), with a similar result obtained for the $C_k(X)$ case. Extending known results on compact spaces it is shown that if a space X is ω–bounded and $C_k(X)$ has the countable chain condition then X must be metric. The classic problem of the productivity of the countable chain condition is investigated in the C_k setting and it is demonstrated that this property is productive if the underlying space is zero–dimensional. Sufficient conditions are given for a space to have a continuous function universal parametrised by a separable space, ccc space or space with calibre ω_1.

An investigation of the sequential separability of function spaces and products is undertaken. The main results include a complete characterisation of those spaces such that $C_p(X)$ is sequentially separable and a characterisation of those spaces such that $C_p(X)$ is strongly sequentially separable.

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PREFACE

This dissertation is dedicated to the memory of my mother, Úna Marsh.

I would like to express my sincerest gratitude to my adviser Professor Paul Gartside for his patience and generosity. His support and advise throughout the last few years have been invaluable.

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1.0 INTRODUCTION

In this thesis we will examine the topological properties of various "higher–order" objects that we can associate with a given topological space. For example if we take the set of all continuous real–valued functions on a space X then we can create a number of new spaces by topologising this set in different ways. In the case of universals we attempt to find a space that in some appropriate sense parametrises all the objects of certain class, for example all the zero sets of a space X. We are mainly interested in how the topological properties of the higher–order object relate to the topological properties of the underlying space.

The general questions we ask and approach we take are similar to those of C_p–Theory.

Question 1. If the higher order object (i.e. function space or universal) satisfies some given property what can we imply about the underlying space?

Question 2. If the underlying space satisfies some given property what can we imply about the higher order object?

The collection of properties we consider can be broadly divided into two categories: (i) properties related to separability, such as sequential separability or chain conditions, (ii) properties related to compactness such as σ–compactness or the Lindelof property. Ultimately we seek to obtain a complete characterisation of when a given higher–order object associated with a space has a given topological property. An examination of the results of C_p–Theory makes one thing clear: certain properties will be easy to deal with while others will be extremely complex. For example we can characterise those spaces X such that $C_p(X)$ is separable as those with a coarser separable metric topology, while there is no known characterisation of when $C_p(X)$ is Lindelof.

1.1 NOTATION

Most of our notation is standard and follows Engelking in [7] and Kunen in [20]. Unless otherwise stated all spaces will be Tychonoff. This certainly seems the natural class of spaces to deal with when one considers function spaces as it asserts the existence of many continuous real–valued functions. There is an example of a T_3 space X such that the only continuous real–valued functions on X are the constant functions demonstrating that the class of T_3 spaces is somewhat deficient when studying function spaces.

Definitions of any topological properties that are not defined in this thesis can be found in [7]. Many of the results in this thesis can be found in [10, 11, 8]. As these are joint papers we have tried to include only those results that are due to the author. However in certain circumstances other results from these papers are needed and we make it clear throughout this thesis if a result is not the work of the author.

We think of each ordinal α as the set of all ordinals preceeding α, and for this reason we may write $\beta \in \alpha$ instead of $\beta < \alpha$. This applies to finite ordinals and so for example $2 = \{0, 1\}$ and $n+1 = \{0, 1, \ldots, n\}$. Every cardinal is the least ordinal of a given cardinality and so for example we may write ω_1 instead of $\aleph_1$.

If X is a topological space we will use $C(X)$ to denote the set of all continuous real–valued functions on X. Whenever we need to specify the topology τ we will write $C(X, \tau)$ instead of $C(X)$. A *zero–set* of X is some $Z \subset X$ such that there exists $f \in C(X)$ with $Z = f^{-1}(\{0\})$ and a *co–zero* set is the complement of a zero–set.. An F_σ subset of X is a set that can be written as the countable union of closed subsets of X. We say Z is a regular F_σ set if there exist open sets U_n for each $n \in \omega$ such that $Z = \bigcup_{n\in\omega} U_n = \bigcup_{n\in\omega} \overline{U_n}$.

Given two functions f and g if we write $f = g$ we mean that the domain of f is the same as the domain of g, and for every x in this domain $f(x) = g(x)$. We will use $\mathcal{P}(X)$ to denote the power set of X.

1.2 OUTLINE AND STRUCTURE OF THESIS

Chapter 2 of this thesis deals with some general results on function spaces and universals that will be needed when studying specific topological properties of these objects. With the aim of making this work as self–contained as possible we have included some well–known results about $C_p(X)$ and $C_k(X)$. The rest of the results from this section are new and address the problem of constructing universals.

The main body of the work is divided up according to topological property. Chapter 3 deals with compactness, σ–compactness and the Lindelof–Σ property in universals. In Chapter 4 we deal with chain conditions in function spaces and universals, giving a complete characterisation of when $C_p(X)$ or $C_k(X)$ has a given chain condition. Finally Chapter 5 contains our results on the (strong) sequential separability of $C_p(X)$. At the very end, in Chapter 6, we collect the unsolved problems.

2.0 FUNCTION SPACES AND UNIVERSALS

The study of function spaces began in the 19th century with attempts to understand the convergence of sequences of functions. The idea is to view the collection of all functions in a certain class, such as all continuous real–valued functions on a space X, as a space in its own right. By varying the topology on this space of functions we can change what it means for a sequence of functions to converge. We focus in particular on two topologies, one giving pointwise convergence of functions the other convergence on compact sets.

With universals we take the same higher–order approach. A universal is a space that in some appropriate sense parametrises a collection of objects associated with a given topological space. For example we might be interested in parametrising all the open subsets of $\mathbb{R}$ with some space. In general we will try to find the nicest possible parametrising space. The study of universals began with the examination of the length of the Borel hierarchy. Universals for the Borel sets were used to calculate this length. In more recent years Gartside and others have looked at universals for many other types of objects, such as zero–sets, F_σ sets and continuous real–valued functions (see [10, 12, 13, 14]).

Here we collect many of the fundamental results regarding function spaces and universals that we will need in the rest of the thesis. The results on the space $C_p(X)$ are well known and can be found in [2]. The two results, Lemma 6 and Lemma 8 have probably appeared before. However we have been unable to find either a statement or proof of either in the literature. As results later in this thesis rely on these theorems and as the proof of Lemma 6 in particular is non–trivial we give full proofs of Lemma 6 and Lemma 8.

2.1 FUNCTION SPACES

If we fix a space X we can view the set $C(X)$ as a topological space in a number of natural ways. In this section we will define two topologies (i) the topology of pointwise convergence and (ii) the topology of compact convergence. In addition we will discuss the idea of an admissible topology on $C(X)$. First some notation that will allow us to define these topologies.

If $A \subset X$ and $B \subset \mathbb{R}$ then we define $[A, B] \subset C(X)$ and $[A, B]' \subset \mathbb{R}^X$ as

$$\begin{aligned} [A, B] &= \{f : f \in C(X), f[A] \subset B\}, \\ [A, B]' &= \{f : f \in \mathbb{R}^X, f[A] \subset B\}. \end{aligned}$$

If $\mathcal{A} = \langle A_0, \ldots, A_n\rangle$ consists of subsets of X and $\mathcal{B} = \langle B_0, \ldots, B_n\rangle$ consists of subsets of $\mathbb{R}$ then we define $W(\mathcal{A}, \mathcal{B}) \subset C(X)$ and $W'(\mathcal{A}, \mathcal{B}) \subset \mathbb{R}^X$ as

$$\begin{aligned} W(\mathcal{A}, \mathcal{B}) &= \bigcap\{[A_i, B_i] : i \leq n\}, \\ W'(\mathcal{A}, \mathcal{B}) &= \bigcap\{[A_i, B_i]' : i \leq n\}. \end{aligned}$$

2.1.1 The space $C_p(X)$

We use $C_p(X)$ to denote the space with underlying set $C(X)$ and the topology of pointwise convergence. If there is a need to specify the topology τ on X we will write $C_p(X, \tau)$. This topology has as a subbasis $\{[\{x\}, U] : x \in X, U \subset \mathbb{R}, U \text{ is open}\}$. For convenience we will write $[x, U]$ or $W(\langle x_0, \ldots, x_n\rangle, \langle U_0, \ldots, U_n\rangle)$ when in fact we should write $[\{x\}, U]$ or $W(\langle\{x_0\}, \ldots, \{x_n\}\rangle, \langle U_0, \ldots, U_n\rangle)$. The space $C_p(X)$ can be viewed as a subspace of the Tychonoff product $\mathbb{R}^X$. The study of the topological properties of $C_p(X)$ has been ongoing for many years. See [2] for an introduction to this area and proofs of the results that we will mention in this section. In addition the study of $C_p(X)$ is closely related to the study of Banach spaces in their weak topology, every Banach space in its weak topology is homeomorphic to a closed subspace of some $C_p(X)$.

Another way to specify the topology on $C_p(X)$ is as follows. If $f \in C(X)$, $\epsilon > 0$ and $x_0, \ldots, x_n \in X$ then we define

$$B(f, x_0, \ldots, x_n, \epsilon) = \{g \in C(X) : \forall i \leq n |f(x_i) - g(x_i)| < \epsilon\}.$$

We can generate the topology on $C_p(X)$ by taking the collection of all such $B(f, x_0, \ldots, x_n, \epsilon)$ to be a basis.

The following theorem gives a further justification for restricting ourselves to the class of Tychonoff spaces.

Theorem 1 *If X is a Tychonoff space then so is $C_p(X)$.*

Of course the space $C_p(X)$ has an algebraic structure in addition to its topology. We can define $f + g$ for $f, g \in C(X)$ by $(f + g)(x) = f(x) + g(x)$ for all $x \in X$. Similarly we can multiply two functions f, g to get fg where $(fg)(x) = f(x)g(x)$. Defining addition and multiplication of functions in this way makes $C_p(X)$ a topological ring. One of the key results of C_p–Theory is the following theorem that shows that the combined topological and algebraic structure of $C_p(X)$ completely determines the topology of X.

Theorem 2 *[J. Nagata] If the topological rings $C_p(X)$ and $C_p(Y)$ are topologically isomorphic then the spaces X and Y are homeomorphic.*

However the topology of $C_p(X)$ does not suffice to determine X. In fact even if we view $C_p(X)$ as a topological group with addition as the group operation this is not enough. It was shown by Okunev that $C_p(\mathbb{R})$ and $C_p(X)$ where X is the disjoint sum of ω–many copies of $\mathbb{R}$ are linearly homeomorphic. In studying how the properties of X are related to the properties of $C_p(X)$ we are particularly interested in seeing which properties of X are decided by the topology of $C_p(X)$ and which are decided by the topology plus some algebraic structure.

There is a canonical way in which to embed X into $C_pC_p(X)$ that can be very useful. Note that if we fix X then each $x \in X$ can be viewed as a continuous function g_x on $C_p(X)$ by setting $g_x(f) = f(x)$ for each $f \in C_p(X)$. Moreover we can embed X in $C_pC_p(X)$ in this way.

Theorem 3 *The function $\psi : X \to C_pC_p(X)$ defined by $\psi(x) = g_x$ for each $x \in X$ is an embedding.*

There is an obvious relationship between $C_p(X)^2$ and $C_p(X \oplus X)$ as shown in the following theorem.

Theorem 4 *For every space X there is a canonical homeomorphism between $C_p(X)^2$ and $C_p(X \oplus X)$.*

We finish with the following theorem.

Theorem 5 *Let τ and σ be two topologies on a set X and assume that $\sigma \subset \tau$. Then $D = \{f \in C_p(X, \tau) : f$ is σ–continuous$\}$ is dense in $C_p(X, \tau)$.*

2.1.2 The space $C_k(X)$

A topology finer than that of $C_p(X)$ is the *compact–open* topology on $C(X)$. It arises as a natural generalisation of the metric that we can define on $C(X)$ when X is compact. While this topology will give us fewer compact sets it does improve the metric properties of the function space.

Fixing a space X we define the space $C_k(X)$ to have as its underlying set $C(X)$ and a subbasis for the topology $\{[K, U] : K \subset X, U \subset \mathbb{R}, K \text{ is compact}, U \text{ is open}\}$. A typical basic open set will be $W(\mathcal{K},\mathcal{U})$ where $\mathcal{K} = \langle K_0, \ldots, K_n \rangle$ consists of compact subsets of X and $\mathcal{U} = \langle U_0, \ldots, U_n \rangle$ consists of basic open subsets of $\mathbb{R}$.

As with $C_p(X)$ we can define a basis where each basic open set depends on a function f and $\epsilon > 0$. Fixing compact $K \subset X$ we define

$$B(f, K, \epsilon) = \{g : \forall x \in K |f(x) - g(x)| < \epsilon\}.$$

Just as for $C_p(X)$ the collection of all such basic open sets generates the topology on $C_k(X)$.

We will now define a notion, the *type* of a basic open set, that will be useful when dealing with this topology. First we define, for every collection $\mathcal{K} = \langle K_0, \ldots, K_n \rangle$ of compact subsets of X, a function $t_\mathcal{K}$ called the type of $\mathcal{K}$. For each $A \subset n + 1$ we define $t_\mathcal{K}(A) = 1$ if $\bigcap_{i \in A} K_i \neq \emptyset$ and $t_\mathcal{K}(A) = 0$ if $\bigcap_{i \in A} K_i = \emptyset$. In this way we define $t_\mathcal{K} : \mathcal{P}(n + 1) \to 2$. If $B = W(\mathcal{K},\mathcal{U})$ is a basic open subset of $C_k(X)$ then we define the type of B to be the type of $\mathcal{K}$ i.e. $t_\mathcal{K}$.

We say that $\mathcal{K}$ is of *linear type* if and only if for all $i, j \leq n$ we have $K_j \cap K_i \neq \emptyset$ implies that $|i - j| = 1$. We say that $\mathcal{K}$ is of *discrete type* if and only if for all $i, j \leq n$ with $i \neq j$ we have $K_j \cap K_i = \emptyset$, and we will use d_n to denote this discrete type.

Our first theorem shows that we get basic open subsets of all types.

Theorem 6 *Let X be a Tychonoff space. Let $\mathcal{K} = \langle K_i : i \leq n \rangle$ consist of subsets of X and $\mathcal{U} = \langle U_i : i \leq n \rangle$ be a collection of basic open subsets of $\mathbb{R}$. If there exists $f' \in W'(\mathcal{K}, \mathcal{U})$ and either (i) each K_i is compact or (ii) each K_i is a zero–set, then there exists $f \in W(\mathcal{K}, \mathcal{U})$.*

Before we prove this theorem a few observations are in order. At first glance it would appear that we could prove this by repeated applications of Urysohn's Lemma, recursively defining functions f_i for each $i \leq n$ such that $f_i[K_j] \subset U_j$ when $j \leq i$. However this approach seems to create some difficulties, so we need to take a similar but more complicated approach. The following lemma will simplify this proof.

Lemma 7 *Let $\mathcal{K} = \langle K_i : i \leq n \rangle$ consist of subsets of X. Define $E_A = \bigcap_{j \in A} K_j \setminus \bigcup_{j \notin A} K_j$ and for each $A \subset n+1$ define $o(A) = |(n+1) \setminus A|$. Let $I \subset \mathcal{P}(n+1)$ satisfy: there exists $k \leq n$ such that $o(A) \leq k+1$ for all $A \in I$ and if $o(A) \leq k$ then $A \in I$ (we say that such an I is* downward closed*). Then $\bigcup\{E_A : A \in I\} = \bigcup\{\bigcap_{j \in A} K_j : A \in I\}$.*

Proof. ■

Now we are ready to prove Theorem 6

Proof. We will only give the proof for the case (i) where each K_i is compact. Case (ii) can be proved in an almost identical fashion.

Let $\mathcal{K}$ and $\mathcal{U}$ be as in the statement of the lemma case (i). Assume that there exists some $f' \in W'(\mathcal{K}, \mathcal{U})$. We will construct $f \in W(\mathcal{K}, \mathcal{U})$. For each $i \leq n$ we will recursively define a continuous function f_i satisfying: for all $A \subset n+1$ such that $o(A) \leq i$ and for all $x \in E_A$ we have $h_i(x) \in \bigcap\{U_j : j \in A\}$. This will suffice as defining $f = f_n$ we must have that $f \in W(\mathcal{K}, \mathcal{U})$.

To construct f_0: there is only one $A \subset n+1$ such that $o(A) = 0$, that is $A = n+1$. If $E_A \neq \emptyset$ then we can choose $r_0 \in \bigcap \mathcal{U}$. We define a function f_0 by setting for each $x \in X$ that $f_0(x) = r_0$. If $E_A = \emptyset$ then any choice of f_0 will do.

Assume that for some $k < n$ and for all $i \leq k$ we have the required function f_i.

To construct f_{k+1} : Let $\langle A_0, \ldots, A_l \rangle$ be an ordering of the set $\{A : o(a) = |k+1|\}$. We claim that for each $s \leq l$ we can recursively define a continuous function f^s_{k+1} satisfying: (i) for all $A \subset n+1$ such that $o(A) \leq k$ and for all $x \in E_A$ we have $f^s_{k+1}(x) \in \bigcap\{U_j : j \in A\}$ and (ii) for all $i \leq s$ and for all $x \in E_{A_i}$ we have $f^s_{k+1}(x) \in \bigcap\{U_j : j \in A_i\}$. Then defining $f_{k+1} = f^l_{k+1}$ we will have constructed the required f_{k+1}.

All that remains to be shown is that the claim is true. Let $f^{-1}_{k+1} = f_k$. Assume that for some $s < l$ and all $i \leq s$ we have defined the required f^i_{k+1}. Let $Z^{s+1}_{k+1} = \{x : \exists j \leq n(x \in K_j \wedge f^s_{k+1}(x) \notin U_j)\}$ and note that Z^{s+1}_{k+1} is a compact set. To see this we can rewrite Z^{s+1}_{k+1} as

$$Z^{s+1}_{k+1} = \bigcup_{j \leq n} (K_j \cap (f^s_{k+1})^{-1}(\mathbb{R} \setminus U_j)).$$

To define h^{s+1}_{k+1}: if $E_{A_{s+1}} \cap Z^{s+1}_{k+1} = \emptyset$ then let $f^{s+1}_{k+1} = f^s_{k+1}$ and note that this function satisfies (i) and (ii) as described in the previous paragraph. If not then find $r^{s+1}_{k+1} \in \bigcap\{U_j : j \in A_{s+1}\}$. By Lemma 7 the set $\bigcup\{E_A : o(A) \leq k\} \cup \bigcup\{E_{A_i} : i \leq s\}$ is compact and from the definitions is disjoint from Z^{s+1}_{k+1}.

We can now find a continuous function p^{s+1}_{k+1} such that $p^{s+1}_{k+1} \restriction Z^{s+1}_{k+1} = 1$ and $p^{s+1}_{k+1} \restriction (\bigcup\{E_A : o(A) \leq k\} \cup \bigcup\{E_{A_i} : i \leq s\}) = 0$ and $p^{s+1}_{k+1}[X] \subset [0,1]$.

We define the function f^{s+1}_{k+1} by setting for each $x \in X$ that $f^{s+1}_{k+1}(x) = f^s_{k+1}(x) - f^s_{k+1}(x)p^{s+1}_{k+1}(x) + r^{s+1}_{k+1}p^{s+1}_{k+1}(x)$. This function is certainly continuous. We must check that it satisfies (i) and (ii) as described earlier.

Fix A such that $o(A) \leq k$ and $x \in E_A$. Then $f^{s+1}_{k+1}(x) = h^s_{k+1}(x) \in \bigcap\{U_j : v \in A\}$ and so (i) is satisfied. Fix $i \leq s$ and $x \in E_i$. Again $f^{s+1}_{k+1}(x) = f^s_{k+1}(x) \in \bigcap\{U_j : j \in A\}$. Finally look at $x \in E_{A_{s+1}}$. If $x \in Z^{s+1}_{k+1}$ then $f^{s+1}_{k+1}(x) = r^{s+1}_{k+1} \in \bigcap\{U_j : j \in A_{s+1}\}$. If $x \notin Z^{s+1}_{k+1}$ then $f^s_{k+1}(x) \in \bigcap\{U_j : j \in A_{s+1}\}$. Defining $a = \min\{f^s_{k+1}(x), r^{s+1}_{k+1}\}$ and $b = \max\{f^s_{k+1}(x), r^{s+1}_{k+1}\}$ we get that $f^{s+1}_{k+1} \in [a,b] \subset \bigcap\{U_j : j \in A_{s+1}\}$, completing our proof that (ii) is satisfied. ■

Our next theorem shows that we don't in fact need all types of basic open sets to have a basis.

Theorem 8 *Let X be Tychonoff. Then $\{W(\mathcal{K},\mathcal{U}) : \mathcal{K}$ is of linear type$\}$ forms a basis for $C_k(X)$.*

Proof. Fix a function f, compact set K and $n > 0$. Assume without loss of generality that $f[K] \subset [0,1]$. We will find a collection $\mathcal{C}$ of compact subsets of X of linear type and a collection $\mathcal{U}$ of basic open subsets of $\mathbb{R}$ such that

$$f \in W(\mathcal{C},\mathcal{U}) \subset \{g : \forall x \in K |g(x) - f(x)| < \frac{1}{n}\}.$$

Let $C_i = f^{-1}[\frac{i-1}{2n}, \frac{i}{2n}]$ for each $i = 1, \ldots, 2n$ and let $U_i = (\frac{i-1}{2n} + \epsilon, \frac{i}{2n} + \epsilon)$ where $\epsilon < \frac{1}{4n}$. Let $\mathcal{C} = \langle C_1, \ldots, C_{2n} \rangle$. It is easily verified that $\mathcal{C}$ is of linear type. That $f \in W(\mathcal{C},\mathcal{U})$ follows directly from the definition of $\mathcal{C}$ and $\mathcal{U}$. We will check that for all $g \in W(\mathcal{C},\mathcal{U})$ and $x \in K$ we have $|g(x) - f(x)| < \frac{1}{n}$.

Fix such a g and x. Now $x \in C_i$ for at least one i. We know that $g(x) \in (\frac{i-1}{2n} - \epsilon, \frac{i}{2n} + \epsilon)$ and that $f(x) \in [\frac{i-1}{2n}, \frac{i}{2n}]$. So

$$\begin{aligned} |f(x) - g(x)| &\leq \max\{|\frac{i-1}{2n} - (\frac{i}{2n} + \epsilon)|, |\frac{i}{2n} - (\frac{i-1}{2n} - \epsilon)|\} \\ &\leq \frac{1}{2n} + \epsilon \leq \frac{3}{4n}. \end{aligned}$$

■

We have theorems analogous to Theorem 1, Theorem 4 and Theorem 5 for $C_k(X)$.

Theorem 9 *If X is a Tychonoff space then so is $C_k(X)$.*

Theorem 10 *For every space X there is a canonical homeomorphism between $C_k(X)^2$ and $C_k(X \oplus X)$.*

Theorem 11 *Let τ and σ be two topologies on a set X and assume that $\sigma \subset \tau$. Then $D = \{f \in C_k(X,\tau) : f$ is σ–continuous$\}$ is dense in $C_k(X,\tau)$.*

2.1.3 Admissible topologies for $C(X)$

The study of admissible topologies on $C(X)$ began with Arens and Dugundji in [1]. A topology τ on $C(X)$ is *admissible* if and only if the evaluation map $e : X \times (C(X), \tau) \to \mathbb{R}$ defined as $e(x, f) = f(x)$ for each $x \in X$, $f \in C(X)$ is continuous. In [1] it is shown that the topology on $C_k(X)$ is an admissible topology if and only if X is locally compact. In the case where X is not locally compact we may still seek an admissible topology on $C(X)$. Although of course simply giving the set $C(X)$ the discrete topology will give us an admissible topology we seek coarser topologies as this will give the space nicer global properties and yield more compact sets. We are somewhat limited as in [1] it is shown that any admissible topology must be finer that the topology on $C_k(X)$.

One of the most appealing aspects of an admissible topology τ is that the space $(C(X), \tau)$ and the continuous function e parametrise every continuous real–valued function on X. In this thesis we take this idea further by examining continuous function universals and demonstrating that admissible topologies are just special cases of continuous function universals.

2.2 SET AND FUNCTION UNIVERSALS

Let us assume that we have a space X and associated with that set we have a collection of objects $\mathcal{T}(X)$. For example $\mathcal{T}(X)$ might be all open subsets of X. A universal for this collection of objects will consist of a space Y and an object in $\mathcal{T}(X \times Y)$ that in some appropriate sense parametrises $\mathcal{T}(X)$. More specifically we can define a *continuous function universal* as follows.

Definition 12 *Given a space X we say that a space Y parametrises a continuous function universal for X via the function F if $F : X \times Y \to \mathbb{R}$ is continuous and for any continuous $f : X \to \mathbb{R}$ there exists some $y \in Y$ such that $F(x, y) = f(x)$ for all $x \in X$. We will use F^y to denote the corresponding function on X.*

It is clear that if τ is an admissible topology on $C(X)$ then the space $(C(X), \tau)$ parametrise a continuous function universal for X via the evaluation mapping e. However the idea of a continuous function universal is more general as it allows each function to appear more than once in the parametrisation.

We are also interested in the following three types of set universal.

Definition 13 *Given a space X we say that a space Y parametrises a zero–set (respectively, open F_σ, open regular F_σ) universal for X if there exists $\mathcal{U}$, a zero–set (respectively, open F_σ, open regular F_σ) in $X \times Y$ such that for all $A \subset X$ with A a zero–set (open F_σ, open regular F_σ) there exists $y \in Y$ such that $\mathcal{U}^y = \{x \in X : (x, y) \in \mathcal{U}\} = A$.*

Of course we can similarly define universals for any type of subset. Note that the complement of a zero–set universal is a cozero–set universal. For more on open set universals and Borel set universals see [10, 12, 13].

The type of questions that we are interested in are similar to those in C_p and C_k–Theory. It Y parametrises a universal for X and Y has some property P then what properties will X have? On the other hand if X has some property P then what sort of spaces can we find to parametrise a universal for X? Since we can always give the collection of objects to be parametrised the discrete topology it is clear that the local properties of the universal can tell us nothing about the properties of X.

2.2.1 Construction of universals

Universals present us with a difficulty that function spaces such as $C_k(X)$ and $C_p(X)$ do not. For example in the case of $C_p(X)$ we know exactly what the space is. If we are dealing with a continuous function universal then we really must construct the continuous function universal. We develop and utilise a number of techniques for creating universals. In this section we describe these techniques.

The first result shows how we can create a continuous function universal from a zero set universal. As this result is due to Gartside we do not include the proof.

Theorem 14 *Suppose Y parametrises a zero–set universal for a space X. Then some subspace of Y^ω parametrises a continuous function universal for X.*

The following lemma is useful for constructing universals as it allows us to partition the class of subsets to be parametrised and parametrise each piece separately. We will use $\mathcal{T}$ to denote a type of subset of a space. For example $\mathcal{T}$ could be open F_σ. Let $\mathcal{T}_X$ denote all subsets of a space X of type $\mathcal{T}$.

Lemma 15 *Fix a Tychonoff space X and $\mathcal{T}$, a type of subset of X. For each $n \in \omega$ let A_n be a subset of $\mathcal{T}_X$ such that all the following holds. Assume that for all $V \in \mathcal{T}_X$ there exist $V_n \in A_n$ for each $n \in \omega$ such that $V = \bigcup\{V_n : n \in \omega\}$. Furthermore assume that we have spaces $\{Y_n : n \in \omega\}$ and $\{\mathcal{U}_n : n \in \omega\}$ where $\mathcal{U}_n \subset X \times Y_n$ and $\mathcal{U}_n \in \mathcal{T}_{X \times Y_n}$ such that for each $V \in A_n$ there exists $y \in Y_n$ with $(\mathcal{U}_n)^y = V$. Then $Y = \prod_{n\in\omega} Y_n$ parametrises a $\mathcal{T}$ universal for X when $\mathcal{T}$ is any of the types: (i) cozero–set, (ii) open regular F_σ, (iii) open F_σ.*

Proof. We begin by defining $\mathcal{U} \subset X \times Y$ by $\mathcal{U} = \bigcup\{\mathcal{U}_n \times \prod_{j\neq n} Y_j : n \in \omega\}$. Note that for any $y = (y_n)_{n\in\omega} \in Y$ we have $\mathcal{U}^y = \bigcup\{(\mathcal{U}_n)^y : n \in \omega\}$. Now since each of the three types of set in question are closed under countable unions we see that $\{\mathcal{U}^y : y \in Y\} = \mathcal{T}_X$. It remains to be shown that in fact $\mathcal{U}$ is a set of type $\mathcal{T}$ in $X \times Y$.

(i) $\mathcal{T}$ = cozero–set : For each $n \in \omega$ we have $f_n : X \times Y_n \to \mathbb{R}$ such that $f_n(x, y) = 0$ if and only if $(x, y) \notin \mathcal{U}_n$. Define $F : X \times Y \to \mathbb{R}$ by $F(x, y) = \sum_{n\in\omega} 2^{-n} f_n(x, y_n)$ for $x \in X$ and $y \in Y$ (here y_n is the n^{th} component of $y \in \prod Y_n$). Note that F is continuous and $F^{-1}(\{0\}) = (X \times Y) \setminus \mathcal{U}$.

(ii) & (iii) $\mathcal{T}$ = open F_σ or open regular F_σ: We know that for each $n \in \omega$ we have $\mathcal{U}_n = \bigcup_{m\in\omega} F_n^m$ where $F_n^m \subset F_n^{m+1}$ for all $m \in \omega$, each F_n^m is closed and in the case of open regular F_σ has non–empty interior. Now define $F_n = \bigcup\{F_i^n \times \prod_{j\neq i} Y_j : i \leq n\}$. Then $\mathcal{U} = \bigcup_{n\in\omega} F_n$ and each F_n is closed with non–empty interior if each F_n^m does. ■

There is a canonical way of creating a continuous function universal for a space in the case where the space X has a K–coarser topology.

Definition 16 *Let τ, σ be two topologies on a set X with $\tau \subset \sigma$. We say that τ is a K–coarser topology if (X, σ) has a neighbourhood basis consisting of τ–compact neighbourhoods.*

The existence of a K–coarser topology τ on a space (X, σ) will allow us to construct a continuous function universal for (X, σ) by refining the topology on $C_k(X)$ without adding "too many" open sets.

Fix a space (X,σ). Let $\mathfrak{U} = \{(r,q) : r,q \in \mathbb{Q}, r < q\}$ and $\mathfrak{U}_{\mathbb{Q}} = \mathcal{U} \cup \{\{q\} : q \in \mathbb{Q}\}$. Fix $\mathcal{C} = \langle C_0, \ldots, C_n \rangle$ where each $C_i \subset X$ and $\mathcal{U} = \langle U_0, \ldots, U_n \rangle$ where each $U_i \subset \mathbb{R}$.

If $\mathfrak{B} \subset \mathcal{P}(\mathbb{R})$ and τ, σ are topologies on X we define the space $C_{k_\tau}((X,\sigma),\mathfrak{B})$ to have as its underlying set $C(X,\sigma)$ and a subbasis

$$\mathcal{S} = \{W(\mathcal{C},\mathcal{U}) : \mathcal{C} \subset \mathcal{P}(X)^{<\omega}, \mathcal{U} \subset \mathfrak{B}^{<\omega}, |\mathcal{C}| = |\mathcal{U}|, \forall C \in \mathcal{C}(C \text{ is } \tau\text{–compact})\}.$$

For any set A the set $A^{<\omega}$ is the collection of all finite partial functions from ω into X whose domain consists of some initial segment of ω. Note that $C_{k_\sigma}((X,\sigma),\mathfrak{U})$ is simply the space $C_k(X,\sigma)$.

Let τ be a K–coarser topology on (X,σ). The space $C_{k_\tau}((X,\sigma),\mathfrak{U}_{\mathbb{Q}})$ parametrises a continuous function universal for (X,σ) via the evaluation map. In addition this space is T_2 and 0–dimensional, and so the space is normal. Although it may be easier to work with the space $C_{k_\tau}((X,\sigma),\mathfrak{U})$ it is difficult to see how one would show that this space is Tychonoff. We summarise with the following theorem.

Theorem 17 *Let τ be a K–coarser topology on (X,σ).*

(i) The space $C_{k_\tau}((X,\sigma),\mathfrak{U}_{\mathbb{Q}})$ parametrises a continuous function universal for (X,σ) via the evaluation map.

(ii) $C_{k_\tau}((X,\sigma),\mathfrak{U}_{\mathbb{Q}})$ is T_2 and 0–dimensional, and hence is normal.

Proof. (i) We must check the continuity of the evaluation map. Fix open $U \subset \mathbb{R}$, $x \in X$ and $f \in C(X,\sigma)$ such that $f(x) \in U$. There exists K, a τ–compact neighbourhood of x such that $f[K] \subset U$. For all $(x', f') \in K \times [K, U]$ we know that $e(x', f') = f'(x') \subset U$ verifying the continuity of e.

(ii) First we check that $C_{k_\tau}((X,\sigma),\mathfrak{U}_{\mathbb{Q}})$ is T_2. Fix $f, g \in C(X,\sigma)$ such that $f \neq g$. So we can find $x \in X$ such that $f(x) \neq g(x)$. The sets $[\{x\}, (f(x) - \frac{|f(x)-g(x)|}{4}, f(x) + \frac{|f(x)-g(x)|}{4})]$ and $[\{x\}, (g(x) - \frac{|f(x)-g(x)|}{4}, g(x) + \frac{|f(x)-g(x)|}{4})]$ are disjoint open sets separating f and g.

To show that $C_{k_\tau}((X,\sigma),\mathfrak{U}_{\mathbb{Q}})$ is 0–dimensional it will suffice to check that each of the subbasic open sets are in fact closed. Let S be a subbasic open set. Assume first of all that $S = [K, (r,q)]$ where K is τ–compact and $r, q \in \mathbb{Q}$. If $f \notin S$ then there exists $x \in K$ such that $f(x) \notin (r,q)$. If $f(x) = r$ then $[\{x\}, \{r\}]$ witnesses that f is not in the closure of S.

The case where $f(x) = q$ can be dealt with in the same way. If $f(x) \notin [r, q]$ then the open set $[\{x\}, (f(x) - \frac{\min\{|f(x)-r|,|f(x)-q|\}}{2}, f(x) + \frac{\min\{|f(x)-r|,|f(x)-q|\}}{2})]$ witnesses that f is not in the closure of S.

Now assume that $S = [K, \{q\}]$ where K is τ–compact and $q \in \mathbb{Q}$. If $f \notin S$ then there is some $x \in K$ with $f(x) \neq q$. The open set $[\{x\}, (f(x) - \frac{|f(x)-q|}{2}, f(x) - \frac{|f(x)-q|}{2})]$ witnesses that f is not in the closure of S.

■

We will now show that $C_{k_\tau}((X, \tau), \mathfrak{U}_{\mathbb{Q}})$ is a dense subspace of $C_{k_\sigma}((X, \sigma), \mathfrak{U}_{\mathbb{Q}})$. Towards this end we have the following theorem that is closely related to Theorem 6.

Theorem 18 *Let X be a Tychonoff space. Let $\mathcal{C} = \langle C_0, \ldots, C_n\rangle$ consist of subsets of X and let $\mathcal{U} = \langle U_0, \ldots, U_n\rangle$ where each $U_i \in \mathfrak{U}$. Assume $\mathcal{D} = \langle D_0, \ldots, D_m\rangle$ consists of subsets of X and that $\mathcal{V} = \langle\{q_0\}, \ldots, \{q_m\}\rangle$ where each $q_j \in \mathbb{Q}$.*

If either (i) each C_i and D_i is compact or (ii) each C_i and D_i are zero–sets and if there exists $f \in W'(\mathcal{C}, \mathcal{U}) \cap W'(\mathcal{D}, \mathcal{V})$ then there exists $g \in W(\mathcal{C}, \mathcal{U}) \cap W(\mathcal{D}, \mathcal{V})$.

Proof. As for Theorem 6 we only give the proof for case (i). Let $\mathcal{C}, \mathcal{U}, \mathcal{D}$ and $\mathcal{V}$ be as in the statement of the lemma, case (i).

Assume that there exists some $f \in W'(\mathcal{C}, \mathcal{U}) \cap W'(\mathcal{D}, \mathcal{V})$. Theorem 6 tells us that there is some $h \in W(\mathcal{C}, \mathcal{U})$.

We will now recursively define for each $k \leq m$ a continuous function g_k such that $g_k \in W(\mathcal{C}, \mathcal{U}) \cap W(\langle D_0, \ldots, D_k\rangle, \langle\{q_0\}, \ldots, \{q_k\}\rangle)$. Let $g_{-1} = h$. Assume that there is $k < m$ such that for each $i \leq k$ we have defined the required g_i. Find a continuous function p_{k+1} that satisfies: for all $x \in D_{k+1}$ we have $p_{k+1}(x) = 1$, for all $i \leq k$ and $x \in D_i$ we have $p_{k+1}(x) = 0$ and for all $j \leq n$ such that $D_{k+1} \cap C_j = \emptyset$ and $x \in C_j$ we have $p_{k+1}(x) = 0$. Now we define the function g_{k+1} by setting for each $x \in X$ that $g_{k+1}(x) = g_k(x) - p_{k+1}(x)g_k(x) + p_{k+1}(x)q_{k+1}$. It is easily verified that $g_{k+1} \in W(\mathcal{C}, \mathcal{U}) \cap W(\langle D_0, \ldots, D_{k+1}\rangle, \langle\{q_0\}, \ldots, \{q_{k+1}\}\rangle)$. Now defining $g = g_m$ we have constructed the required function.

■

The next result follows almost immediately from Theorem 18.

Corollary 19 *Fix a space (X,σ) and let τ be a K–coarser topology. Then $C_{k_\tau}((X,\tau),\mathfrak{U}_{\mathbb{Q}})$ is a dense subspace of $C_{k_\tau}((X,\sigma),\mathfrak{U}_{\mathbb{Q}})$.*

Proof. Let $W(\mathcal{C},\mathcal{U})$ be a non–emptyset basic open subset of $C_{k_\tau}((X,\sigma),\mathfrak{U}_{\mathbb{Q}})$. Theorem 18 tells us that since there is a function in $W(\mathcal{C},\mathcal{U})$ then there is a τ continuous function in $W(\mathcal{C},\mathcal{U})$. ■

2.3 RELATIONSHIPS BETWEEN THE SPACES

Most of the function spaces and universals that we have defined are closely related. For example it is clear that the identity function from $C_k(X)$ to $C_p(X)$ is continuous.

Fix a space X. Let Y parametrise a continuous function universal for X, Z parametrise a zero set universal for X and let τ be an admissible topology for $C(X)$. In this section we discuss some relationships between these universals and $C(X)$ with any of the topologies discussed in the previous section.

Theorem 20 *Let Y parametrise a continuous function universal for X via the function F. Then the mapping $q : Y \to C_k(X)$ defined as $q(y) = F^y$ for each $y \in Y$ is continuous.*

Proof. Fix compact $K \subset X$ and U a basic open subset of $\mathbb{R}$. We need to show that $q^{-1}([K,U])$ is open in Y. Fix $y \in q^{-1}([K,U])$. Note that $F[K \times \{y\}] \subset U$. For each $x \in K$ we can find open $A_x \subset X$ and open $B_x \subset Y$ such that $(x,y) \in A_x \times B_x$ and $F[A_x \times B_x] \subset U$. The compact set K can be covered by finitely many of the A_x's, say $\{A_{x_i} : i \leq n\}$. Let $V = \bigcap\{B_{x_i} : i \leq n\}$. Then V is open, $y \in V$ and $q[V] \subset [K,U]$ completing the proof. ■

For the zero–set universal of course we don't get a continuous map onto $C_k(X)$, but we get the following. A set $A \subset C(X)$ *separates points from zero–sets* if given any zero–set $Z \subset X$ and a point $x \notin Z$ there is some $f \in A$ such that $f(x) \notin f[Z]$.

Theorem 21 *Let Z parametrise a zero set universal for X. Then there exists a continuous function from Z onto a subspace of $C_k(X)$ that separates points from zero–sets.*

Proof. Let the function F witness that Z parametrises a zero set universal for X. We define $e : Z \to C_k(X)$ as $e(z) = F^z$ for each $z \in Z$. Continuity of e can be demonstrated in the same way as for continuity of q in Theorem 20. It is easily seen that the image of Z under e must separate points from zero–sets. ■

If τ is an admissible topology on $C(X)$ then as already noted the space $(C(X), \tau)$ parametrises a continuous function universal for X via the evaluation mapping. In this scenario the mapping q defined in the statement of Theorem 20 is of course the identity mapping. That the identity mapping from $C_k(X)$ onto $C_p(X)$ is continuous is clear from the definitions of both topologies. We summarise all these relationships in Figure 1.

$$
\begin{array}{ccccc}
 & & Y & & \\
 & & \big\downarrow{\scriptstyle q} & & \\
(C(X), \tau) & \xrightarrow{\ i\ } & C_k(X) & \xrightarrow{\ i\ } & C_p(X) \\
 & & {\scriptstyle e}\big\uparrow & & \\
 & & Z & &
\end{array}
$$

Figure 1: Maps between spaces

3.0 COMPACTNESS TYPE PROPERTIES

Compactness is one of the most useful and powerful topological properties. Given any space X it is natural to ask what the compact subsets of X are. If X is compact these will of course be the closed subsets, but in general this will not be the case. It is clear from the off that function spaces and continuous function universals will never be compact as they cannot even be pseudocompact. However much is known about the compact subsets of $C_p(X)$ particularly Eberlein compacta i.e. the case where X is compact.

There are many weakenings of compactness that we can look at in the context of function spaces. A space is σ–compact if and only if it is the countable union of compact subsets. This again seems too strong a property for function spaces to satisfy as demonstrated by the following theorem (see [2] p. 28 for a proof).

Theorem 22 (N. V. Velicho) *The space $C_p(X)$ is σ–compact if and only if X is finite.*

A weakening of σ–compactness that has been very important in the context of $C_p(X)$ is the Lindelof–Σ property. The class of Lindelof Σ–spaces is the smallest class containing all compact spaces, all separable metric spaces, and which is closed under countable products, closed subspaces and continuous images. There are many results in C_p–Theory regarding this property. In this thesis we will focus on these properties in universals. We will show that for set universals compactness and σ–compactness, far from being overly restrictive properties actually yield interesting characterisations of metrisability. In fact we obtain a complete characterisation of those spaces with set universals parametrised by compact or σ–compact universals. We make significant progress in the study of Lindelof–Σ universals but do not obtain a complete characterisation. It is worth noting that no characterisation of when $C_p(X)$ is Lindelof–Σ is known.

3.1 COMPACTNESS AND σ–COMPACTNESS

We begin by investigating what happens when X has a zero set universal, open regular F_σ universal or open F_σ universal parametrised by a compact or σ–compact space. First we define the following universal type object. Let X, Y be spaces and $\mathcal{U} \subset X \times Y$. We will refer to the topology τ with subbasis $\{\mathcal{U}^y : y \in Y\}$ (where $\mathcal{U}^y = \{x \in X : (x, y) \in \mathcal{U}\}$) as the *topology generated by* $(Y,\mathcal{U})$.

The following general lemma will be very useful.

Lemma 23 *Let X be a T_1 space. If there exists a σ–compact space Y and open regular F_σ set $\mathcal{U} \subset X \times Y$ such that the topology generated by $(Y,\mathcal{U})$ coincides with the topology on X, then X is metrisable.*

Proof. Without loss of generality we can assume that $Y = \bigcup_{n\in\omega} K_n$, where each K_n is compact and open and for all $n \in \omega$ we have $K_n \subset K_{n+1}$. Let $\mathcal{U} \subset X \times Y$ be as in the statement of the lemma. For each $m \in \omega$ let $\mathcal{U}_m \subset X \times Y$ be open and assume that $\mathcal{U} = \bigcup\{\overline{\mathcal{U}_m} : m \in \omega\}$. For each $x \in X$ and $n, m \in \omega$ define

$$V(n, m, x) = \bigcup\{A \subset X : x \in A, A \text{ is open}, A \times (\overline{\mathcal{U}_m} \cap (X \times K_n))_x \subset \mathcal{U} \cap (X \times K_n)\}.$$

Note that each $V(n, m, x)$ is an open neighbourhood of x. By the Collins–Roscoe metrisation theorem (see [5]) it suffices to show:

1. For all $x \in X$ the collection $\{V(n, m, x) : n, m \in \omega\}$ is a local base.
2. For all $n, m \in \omega$ and $x \in X$ there exists an open $S \subset X$ with $x \in S$ such that $x \in V(n, m, x')$ for all $x' \in S$.
3. For all $x \in X$ and all open S with $x \in S$ there exist $n, m\omega$ and open T with $x \in T$ such that $x' \in T$ implies $V(n, m, x') \subset S$.

1. Fix U open and $x \in U$. There exists some $\underline{y} = \{y_1, \ldots, y_r\} \in Y^r$ such that $x \in \cap_{i=1}^r \mathcal{U}^{y_i} \subset U$. We will use $\mathcal{U}^{\underline{y}}$ to denote $\cap_{i=1}^r \mathcal{U}^{y_i}$. We know that there is some $n \in \omega$ such that $y_i \in K_n$ for all $i = 1, \ldots, r$. There is also some $m \in \omega$ such that $(x, y_i) \in (\overline{\mathcal{U}_m} \cap (X \times K_n))$ for all $i = 1, \ldots, r$.. Note that since $(\overline{U_m} \cap (X \times K_n))_x$ is compact we can find open A with $x \in A$ such

that $A\times(\overline{\mathcal{U}_m}\cap(X\times K_n))_x \subset \mathcal{U}\cap(X\times K_n)$. But if $x' \in V(n,m,x)$ then $(x', y_i) \in \mathcal{U}\cap(X\times K_n)$ for all $i = 1,\ldots,r$ and so $x' \in \mathcal{U}^{\underline{y}} \subset U$. This shows that $x \in V(n,m,x) \subset U$.

2. Fix $n, m \in \omega$ and $x \in X$. If $(\overline{\mathcal{U}_m}\cap(X\times K_n))_x = \emptyset$ then $(\{x\}\times K_n)\cap(\overline{\mathcal{U}_m}\cap(X\times K_n)) = \emptyset$. Since $\{x\}\times K_n$ is compact we get an open S with $x \in S$ such that $(S\times K_n)\cap(\overline{\mathcal{U}_m}\cap(X\times K_n)) = \emptyset$. So for all $x' \in S$ we have $V(n,m,x') = X$. Now assume that $(\overline{\mathcal{U}_m}\cap(X\times K_n))_x \neq \emptyset$. We can find open S with $x \in S$ and open $W_1, W_2 \subset K_n$ such that

- $W_1 \cup W_2 = K_n$,
- $K_n \setminus (\mathcal{U}\times K_n)_x \subset W_1$,
- $(\overline{\mathcal{U}_m}\cap(X\times K_n))_x \subset W_2$,
- $(S\times W_1)\cap(\overline{\mathcal{U}_m}\cap(X\times K_n)) = \emptyset$,
- $S\times W_2 \subset \mathcal{U}\cap(X\times K_n)$.

If $x' \in S$ we know from the fourth condition that $((\overline{\mathcal{U}_m}\cap(X\times K_n))_{x'} \subset W_2$ and so $S\times((\overline{\mathcal{U}_m}\cap(X\times K_n))_{x'} \subset \mathcal{U}\cap(X\times K_n)$ implying that $x \in S \subset V(n,m,x')$. 3. Fix $x \in X$ and open T with $x \in T$. As in the proof of part 1 we can fin $\underline{y} \in Y^r$ such that $x \in \mathcal{U}^{\underline{y}} \subset T$. Then there exist $n, m \in \omega$ such that $(x, y_i) \in \overline{\mathcal{U}_m}\cap(X\times K_n)$ for all $i = 1,\ldots,r$. As before find open S with $x \in S$ and open W_1, W_2 satisfying the same five properties as in part 2. In addition assume that $S \subset (\overline{\mathcal{U}_m}\cap(X\times K_n))^{\underline{y}}$. Now take $x' \in S$. Since $x' \in S \subset (\overline{\mathcal{U}_m}\cap(X\times K_n))^{\underline{y}}$ we have $y_i \in (\overline{\mathcal{U}_m}\cap(X\times K_n))_{x'}$ for all $i = 1,\ldots,r$ and so $V(n,m,x') \subset \mathcal{U}^{\underline{y}} \subset T$. ∎

If the parametrising space is almost σ–compact (i.e. has a dense σ–compact subspace) then we have the following.

Theorem 24 *Let X be a Tychonoff space. If X has a zero set universal or a open regular F_σ universal parametrised by Y, an almost σ–compact space, then X is submetrisable.*

Proof. Let $\mathcal{U}$ witness the universal. Let $D \subset Y$ be a dense σ–compact subspace. Now D generates a topology on X by letting each $\mathcal{U}^y$ be open for each $y \in D$. It is routine to check that this must be a T_1 topology and so by Lemma 23 we are done. ∎

Lemma 23 also simplifies the proof of the following.

Theorem 25 *Let X be a Tychonoff space. Then the following are equivalent:*

1. *X is metrisable.*
2. *X has a zero set universal parametrised by a compact Hausdorff space.*

3. *X has a open regular F_σ universal parametrised by a compact Hausdorff space.*

4. *X has a zero set universal parametrised by a σ–compact Hausdorff space.*

5. *X has a open regular F_σ universal parametrised by a σ–compact Hausdorff space.*

Proof. The fact that any of (2),(3),(4) or (5) imply (1) follows from Lemma 23. It is also clear that (2) implies (4) and that (3) implies (5). It remains to show that (1) implies (2) and (4). Let X be a metric space. Since all open sets are cozero–sets and also open regular F_σ sets it will suffice to find a continuous real–valued function $F : X \times 2^{\mathcal{B}} \to \mathbb{R}$ such that for all open U there is a $y \in 2^{\mathcal{B}}$ with $U = \{x \in X : F(x,y) \neq 0\}$, where $\mathcal{B}$ is a basis for X of minimal cardinality. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n \in \omega\}$ be a σ–discrete basis for X. For each $n \in \omega$ we define the function $f_n : X \times 2^{\mathcal{B}_n} \to \mathbb{R}$ by letting

$$f_n(x,y) = d(x, X \setminus \bigcup\{U \in \mathcal{B}_n : y(U) = 1\}.$$

We now define $F(x,y) = \sum_{n<\omega} 2^{-n} f_n(x, y \restriction_{\mathcal{B}_n})$. Then $F(x,y) = 0$ if and only if $f_n(x, y \restriction_{\mathcal{B}_n}) = 0$ for all $n \in \omega$. This holds precisely when $x \notin \bigcup\{U : y(U) = 1\}$. So this does in fact give us all open subsets of X. It remains to show that F is continuous. It will suffice to check that each f_n is continuous. Fix $n \in \omega, x \in X$ and $y \in 2^{\mathcal{B}_n}$. If $f_n(x,y) \neq 0$ then $x \in B$ for some $B \in \mathcal{B}_n$ with $y(B) = 1$. Then for all $y' \in 2^{\mathcal{B}_n}$ such that $y'(B) = y(B) = 1$ and all $x' \in B$ we have $f_n(x',y') = f_n(x,y)$. Now assume that $f_n(x,y) = 0$. There exists open U with $x \in U$ such that U intersects at most one $B \in \mathcal{B}_n$. If U intersects none then for all $x' \in U$ and all $y' \in 2^{\mathcal{B}_n}$ we have $f_n(x',y') = 0$. If U intersects some $B \in \mathcal{B}_n$ and $y(B) = 0$ then $f_n(x',y') = 0$ for all $x' \in U$ and for all y' such that $y'(B) = 0$. If $y(B) = 1$ then $f_n(x',y') \in (-\epsilon, \epsilon)$ for all $x' \in B(x,\epsilon)$ and for all y' such that $y'(B) = 1$.

■

In complete contrast to Theorem 25 it is true that every space has an open universal parametrised by a compact space. In fact for any space X the space $2^{w(X)}$ will suffice (see [14]).

A space X is *functionally perfect* if and only if there exists compact $K \subset C_p(X)$ such that for every pair $x, y \in X$ with $x \neq y$ there exists $f \in K$ such that $f(x) \neq f(y)$. It is known that every metric space is functionally perfect. But as a corollary to Theorem 25 and Theorem 21 we get the following much stronger result.

Corollary 26 *If X is a metric space then there is, Y a compact subset of $C_k(X)$ such that for every $x \in X$ and zero–set $Z \subset X$ with $x \notin Z$ there exists $y \in Y$ with $y[Z] = 0$ and $y(x) \neq 0$.*

We finish this section by examining the case for continuous function universals. Clearly X can never have a continuous function universal parametrised by a compact space. However the situation for σ–compactness is not much better.

Theorem 27 *Assume that Y parametrises a continuous function universal for a Tychonoff space X. Then Y is σ–compact if and only if X is finite.*

Proof. We know that $C_p(X)$ is the continuous image of Y and so $C_p(X)$ is σ–compact if Y is. But then by Theorem 22 X must be finite. The converse is obvious as if X is finite $\mathbb{R}^n$ parametrises a continuous function universal for X. ■

3.2 LINDELOF AND LINDELOF–Σ SPACES

We will examine what properties X must have if it has universals parametrised by a Lindelof–Σ space or a Lindelof space. But first we have the following result.

Theorem 28 *X is separable metric if and only if X has a continuous function universal parametrised by a separable metric space.*

Proof. If X is separable metric then it has a zero–set universal parametrised by 2^ω (see the proof of Theorem 25). Now by Lemma 14 some subspace of $(2^\omega)^\omega$ parametrises a continuous function universal for X. If Y parametrises a continuous function universal for X then it is straightforward to check that $w(X) \leq nw(Y)$. In fact if $\{B_\alpha : \alpha \in \kappa\}$ is a network for Y then defining

$$C_\alpha = \{U \subset X : U \text{ is open}, F(x, y) \neq 0 \forall x \in U \forall y \in B_\alpha\}$$

we have that $\{C_\alpha : \alpha \in \gamma\}$ is a basis for X. ■

Theorem 29 *If a Tychonoff space X has a zero set universal, open regular F_σ universal or an open F_σ universal parametrised by a Lindelof space Y then X is first countable.*

Proof. Let $\mathcal{U} \subset X \times Y$ be the relevant universal. We know that $\mathcal{U} = \bigcup\{F_n : n \in \omega\}$ where each F_n is closed. Fix $x \in X$ and $n \in \omega$. The set $(F_n)_x$ is a closed subset of Y and so $\{x\} \times (F_n)_x$ is Lindelof. Cover $\{x\} \times (F_n)_x$ with countably many $U(x,n,m) \times V(x,n,m)$ such that $U(x,n,m) \times V(x,n,m) \subset \mathcal{U}$. So we have that $(F_n)_x \subset \bigcup\{V(x,n,m) : m \in \omega\}$ and $x \in U(x,n,m)$ for all $m \in \omega$. We claim that $\{U(x,n,m) : n,m \in \omega\}$ is a local basis at x. Fix open U with $x \in U$. There is some $y \in Y$ such that $x \in \mathcal{U}^y \subset U$. There are $n,m \in \omega$ such that $(x,y) \in U(x,n,m) \times V(x,n,m)$ and hence $x \in U(x,n,m) \subset \mathcal{U}^y \subset U$. ■

However we know that having a continuous function universal parametrised by a Lindelof space will not necessarily give metrisability. For example $C_k(\omega_1)$ is Lindelof (see [17]) and since ω_1 is locally compact we know that the evaluation mapping $e : \omega_1 \times C_k(\omega_1) \to \mathbb{R}$ defined by $e(\alpha, f) = f(\alpha)$ is continuous. So $C_k(\omega_1)$ parametrises a continuous function universal for ω_1. We now look at a subclass of the class of Lindelof spaces, that of Lindelof–Σ spaces. We have the following characterisation of Lindelof–Σ spaces.

Definition 30 *A space Y is a Lindelof–Σ space if and only if there exists a cover of Y by compact sets $\{K_\alpha : \alpha \in \kappa\}$ and a countable collection of sets $\{S_n : n \in \omega\}$ such that for all $\alpha \in \kappa$ and open $U \supset K_\alpha$ there is $n \in \omega$ such that $K_\alpha \subset S_n \subset U$.*

Since a space with a zero–set universal (or open regular F_σ universal) parametrised *either* by a compact space *or* by a separable metric space is metric (separable metric in the second case) it would be plausible to suppose that weakening 'compact' and 'separable metric' to 'Lindelof Σ' would also give metric. This is not the case. We will however get some generalised metric properties.

The following property was defined by Bennett in [4].

Definition 31 *A quasi–development for a space X is a collection $\{\mathcal{G}_n : n \in \omega\}$ where each $\mathcal{G}_n$ is a collection of open subsets of X such that for all $x \in X$ and any open neighbourhood U of x there is $n = n(x,U) \in \omega$ such that $x \in St(x, \mathcal{G}_n) \subset U$. Any space with a quasi–development is called quasi–developable.*

In [6] it is shown that the next class of spaces, the *strongly quasi–developable* spaces, is the same as the class of spaces with a σ–disjoint basis.

Definition 32 *A strong quasi–development for a space X is a collection $\{\mathcal{G}_n : n \in \omega\}$ where* *each $\mathcal{G}_n$ is a collection of open subsets of X such that for all open U and $x \in U$ there exists open $V = V(x, U)$ with $x \in V \subset U$ and $n = n(x, V) \in \omega$ such that $x \in \bigcup \mathcal{G}_n$ and $x \in St(V, \mathcal{G}_n) \subset U$. Any space with a strong quasi–development is called strongly quasi–developable.*

The class of quasi–developable spaces is strictly larger than the class of strongly quasi–developable spaces. As is noted in [6] the space $\psi(\omega)$ is a developable (and hence quasi–developable) space that has no σ–disjoint basis. To define the space $\psi(\omega)$ one takes a maximal almost disjoint family of infinite subsets of ω, say $\mathcal{A}$. Then $\psi(\omega) = \omega \cup \mathcal{A}$. Each point in ω is isolated and a typical basic open neighbourhood of $A \in \mathcal{A}$ is $\{A\} \cup (A \setminus F)$ where F is finite.

We now see that a space having a zero–set universal or a open regular F_σ universal parametrised by a Lindelof Σ–space must have a strong quasi–development, but need not be metrisable (or even developable). Further there is a strongly quasi–developable space with no zero–set universal or open regular F_σ universal parametrised by a Lindelof Σ–space.

Lemma 33 *Let X be a Tychonoff space.*

(i) If X has a zero–set universal or a open regular F_σ universal parametrised by Y, a Lindelof–Σ space, then X has a strong quasi–development.

(ii) If X has an open F_σ universal parametrised by a Lindelof–Σ space, then X has a quasi–development.

Proof. Assume that we have collections of sets $\{K_\alpha : \alpha \in \kappa\}$ and $\{S_n : n \in \omega\}$ as in the definition of Lindelof–Σ, where each is a collection of subsets of Y. Note that if Y is any of the three relevant types of universal then there is an open $\mathcal{U} \subset X \times Y$ and a collection of closed subsets $\{\mathcal{F}_m : m \in \omega\}$ such that

$$\mathcal{U} = \bigcup \{\mathcal{F}_m : m \in \omega\}$$

and $\{\mathcal{U}^y : y \in Y\}$ is a basis for X. Now for each $n, m \in \omega$ and $x \in X$ find open $A(x, n, m), B(x, n, m) \subset Y$ and open $V(x, n, m) \subset X$ with $x \in V(x, n, m)$ satisfying: (a) $A(x, n, m) \cup B(x, n, m) \supset S_n$, (b)$(V(x, n, m) \times A(x, n, m)) \cap \mathcal{F}_m = \emptyset$, and (c) $V(x, n, m) \times B(x, n, m) \subset \mathcal{U}$. Of course we may not be able to find such sets and in this case we define

$V(x,n,m) = \emptyset$. Let $\mathcal{G}_{n,m} = \{V(x,n,m) : x \in X\}$. We claim that $\mathcal{G} = \{\mathcal{G}_{n,m} : n,m \in \omega\}$ is a strong quasi–development for X if $\mathcal{U}$ is a zero–set universal or a open regular F_σ universal. If $\mathcal{U}$ is an open F_σ universal then $\mathcal{G}$ is a quasi–development.

First we check that it is a basis. Fix $x \in X$ and open U with $x \in U$. Without loss of generality we can assume that $U = \mathcal{U}^y$ for some $y \in Y$. There is some $m \in \omega$ with $(x,y) \in \mathcal{F}_m$ and there is some $\alpha \in \kappa$ such that $y \in K_\alpha$. We can find open $V_1 \subset X$ with $x \in V_1$ and open $A \subset Y$ such that

$$\{x\} \times ((Y \setminus \mathcal{U}_x) \cap K_\alpha) \subset V_1 \times A$$

and $(V_1 \times A) \cap \mathcal{F}_m = \emptyset$. Now find open $V_2 \subset X$ with $x \in V_2$ and open $B \subset Y$ such that

$$\{x\} \times ((Y \setminus A) \cap K_\alpha) \subset V_2 \times B$$

and $V_2 \times B \subset \mathcal{U}$. Find S_n such that $K_\alpha \subset S_n \subset A \cup B$. This shows that $V(x,n,m)$ is indeed an open neighbourhood of x (as $V(x,n,m) = V_1 \cap V_2$ is one possibility). Also if $x' \in V(x,n,m)$ then since $y \in S_n$ we have $y \in A(x,n,m)$ or $y \in B(x,n,m)$. We know that $y \in B(x,n,m)$ since $(x,y) \in \mathcal{F}_m$. Then $(x',y) \in \mathcal{U}$ and so $x' \in \mathcal{U}^y = U$.

Now we fix open $U \subset X$ and $x \in U$ and will show that there exist $n,m \in \omega$ such that $x \in st(x, \mathcal{G}_{n,m}) \subset U$. As before, assume $U = \mathcal{U}^y$ for some y and find $n,m \in \omega$ such that $(x,y) \in \mathcal{F}_m$, $y \in S_n$ and $x \in V(x,n,m) \subset U$. If $x \in V(x',n,m)$ then we must have $y \in B(x',n,m)$ since $(x,y) \in \mathcal{F}_m$. But then condition (c) implies that $V(x',n,m) \subset \mathcal{U}^y = U$. This completes the proof of part (ii).

In the cases where $\mathcal{U}$ is a zero–set universal or a open regular F_σ universal we can assume that there is a collection of open sets $\{\mathcal{U}_m : m \in \omega\}$ such that $\mathcal{F}_m = \overline{\mathcal{U}}_m$ for all $m \in \omega$ and also

$$\mathcal{U} = \bigcup \{\mathcal{U}_m : m \in \omega\}.$$

Again fix open U, $x \in U$ and $y \in Y$ such that $U = \mathcal{U}^y$. Find $n,m \in \omega$ such that $(x,y) \in \mathcal{U}_m$, $y \in S_n$ and $x \in V(x,n,m) \subset U$. Furthermore find open V with $x \in V \subset V(x,n,m)$ such that there is an open $W \subset Y$ with $y \in W$ and $V \times W \subset \mathcal{U}_m$. Let $x_2 \in X$ be such that $V \cap V(x_2,n,m) \neq \emptyset$. Let $x_1 \in V(x_2,n,m) \cap V$. If we can show that $y \in B(x_2,n,m)$ then since $V(x_2,n,m) \times B(x_2,n,m) \subset \mathcal{U}$ we must have $V(x_2,n,m) \subset U$. If $y \in A(x_2,n,m)$ then

$(x_1, y) \notin \mathcal{U}_m$. But we know that $(x_1, y) \in \mathcal{U}_m$ so $y \notin A(x_2, n, m)$. We know that $y \in S_n$. But $y \notin A(x_2, n, m)$ means that $y \in B(x_2, n, m)$. This shows that $x \in st(V, \mathcal{G}_{n,m}) \subset U$. ■

We have fallen somewhat short of our original aim of proving that X must be metric. The following example shows that this in fact will not be the case.

Example 34 *There is a non–developable Tychonoff space X and a Lindelof–Σ space Y, such that Y parametrises a zero–set universal, a regular F_σ universal, and an open F_σ universal for X.*

Proof. Our aim here is to create a non–metrisable space by refining the topology on the real–line. However we don't want to add too many zero–sets as we don't want to make the parametrisation too difficult. Let $B \subset \mathbb{R}$ be a Bernstein set. Let $A = \mathbb{R} \setminus B$. We define the topology on $X = A \cup B$ by isolating all points in B. Note that X has a σ–disjoint base and is not developable. We can express an arbitrary open subset of X as $U \cup V$ where $U \cap V = \emptyset$, $V \subset B$ and U is open with respect to the Euclidean topology. If $|V| \leq \omega$ then $U \cup V$ is a co–zero subset of X (and hence an open F_σ, regular F_σ). We will show that if $|V| > \omega$ then $U \cup V$ is not an open F_σ subset. Assume that $|V| > \omega$. We know that $U \cup V = \bigcup\{C_n : n \in \omega\}$ where each C_n is closed. Then for some $n \in \omega$ we know that $|V \cap C_n| > \omega$. This set must have a limit point in A, say x. But then $x \in C_n \subset U \cup V$. We must have $x \in U$ and so we get $U \cap V \neq \emptyset$ contradicting our assumption.

We know that we can parametrise all the Euclidean open sets by a compact space Y_1. We will parametrise all the one point subsets of B by a Lindelof–Σ space. So by Lemma 15 we will have that $Y_1 \times Y_2^\omega$ parametrises a zero–set universal for X, (and by our previous arguments, regular F_σ and open F_σ universals). We will use B_d to denote the set B with the discrete topology and B_u to denote B with the Euclidean topology. For any space Z let αZ denote the Alexandroff one–point compactification of X. We define $Y_2 \subset \alpha B_d \times B_u$ by

$$Y_2 = \{(b, b) : b \in B\} \cup \{(\infty, b) : b \in B\}.$$

Note that Y_2 is closed and hence a Lindelof–Σ space. We define $\mathcal{U} \subset X \times Y_2$ to be

$$\mathcal{U} = \{(b, b, b) : b \in B\}.$$

First note that $\mathcal{U}$ does in fact parametrise all the one point subsets of B. If we can show that $\mathcal{U}$ is closed and open then it must be a co–zero subset of $X \times Y_2$. To show that $\mathcal{U}$ is open fix $(b, b, b) \in \mathcal{U}$. We know that $\{b\}$ is open in X and $\{b\}$ is open in αB_d. Then $(\{b\} \times \{b\} \times B_u) \cap X \times Y_2 = \{(b, b, b)\}$ is open.

To show that $\mathcal{U}$ is closed we fix $(x, y, b) \notin \mathcal{U}$ and look at two cases. (i) $(y \neq \infty)$. If $y \neq b$ then find Euclidean open V with $b \in V$ and $y \notin V$. Then $(X \times \{y\} \times V) \cap \mathcal{U} = \emptyset$. If $y = b$ then $x \neq b$. Find disjoint Euclidean open V_1, V_2 with $x \in V_1$, $b \in V_2$. This gives $(V_1 \times \{b\} \times V_2) \cap \mathcal{U} = \emptyset$. (ii) $(y = \infty)$. If $x \neq b$ then again find disjoint Euclidean open V_1, V_2 with $x \in V_1$, $b \in V_2$. This gives $(V_1 \times \alpha B_d \times V_2) \cap \mathcal{U} = \emptyset$. If $x = b$ then choose some open neighbourhood V of ∞ such that $b \notin V$. So $(\{b\} \times V \times B_u) \cap \mathcal{U} = \emptyset$. ∎

We must now eliminate the possibility that we could reverse the implications in Lemma 33. In other words find a strongly developable space that cannot have a zero set universal parametrised by a Lindelof–Σ space.

Example 35 *There is a Tychonoff space X that is strongly quasi–developable that cannot have a zero–set universal, a open regular F_σ universal or an open F_σ universal parametrised by a Lindelof–Σ space.*

Proof. We begin by defining a preliminary space Z. Let $Z = \prod\{\beth_n : n \in \omega\}$ where each cardinal $\beth_n$ has the discrete topology. Note that $|Z| = (\beth_\omega)^\omega > \beth_\omega$ and that $w(Z) = \beth_\omega$. Also there are $(\beth_\omega)^\omega$ closed subsets of size $|Z|$. In fact any closed subset of cardinality greater than $\beth_\omega$ has cardinality $(\beth_\omega)^\omega$ (see [19]). We can construct a Bernstein set in Z, i.e. $B \subset Z$ such that $|B| = (\beth_\omega)^\omega$ satisfying the condition that both B and $Z \setminus B$ intersect every closed subset of cardinality $(\beth_\omega)^\omega$. To do this we enumerate all closed subsets of cardinality $(\beth_\omega)^\omega$ as $\mathcal{C} = \{C_\alpha : \alpha < (\beth_\omega)^\omega\}$. For each $\alpha < (\beth_\omega)^\omega$ we choose $b_\alpha, c_\alpha \subset C_\alpha$ such that $b_\alpha \neq c_\lambda$ for all $\lambda \leq \alpha$. Let $A = Z \setminus B$. We define the topology on $X = A \cup B$ by isolating all the points in B. It remains to show that X can have no open F_σ universal parametrised by Y, a Lindelof–Σ space. It will suffice to assume that Y is a closed subspace of $K \times M$ where K is compact and M is second–countable.

If X does have such a parametrising space then in particular it must parametrise all the one–point sets from B (i.e. there is some open F_σ set $\mathcal{U} \subset X \times Y$ such that for all

$b \in B$ there is a $y_b \in Y$ with $\mathcal{U}^{y_b} = \{b\}$). Note that the collection $\{y_b : b \in B\}$ is a discrete subspace of Y. Let $C = \{(b, y_b) : b \in B\}$. If we can show for all $m \in M$ that $|\pi_Y(C) \cap (K \times \{m\}) \cap Y| < 2^{\beth_\omega}$ then we are done as $|M| \leq 2^\omega$. Now assume that there exists an $m \in M$ such that $|\pi_Y(C) \cap (K \times \{m\}) \cap Y| = 2^{\beth_\omega}$. Let $D = \pi_Y(C) \cap (K \times \{m\}) \cap Y$. We know that $\{(b, y_b) : y_b \in D\} \subset \mathcal{U}$. Since $\mathcal{U}$ is the countable union of closed subsets $\{F_n : n \in \omega\}$ we know for some $n \in \omega$ that $|\{(b, y_b) : y_b \in D\} \cap F_n| = 2^{\beth_\omega}$. Let $E = \overline{\{(b, y_b) : y_b \in D\}} \cap F_n$. Note that E is a closed subspace of $X \times Y$ and that $\overline{\pi_Y(E)} \subset K \times \{m\}$. Hence E is compact. Now the collection $\pi_X(E)$ must have a limit point outside B, say x. We will show that for some $y \in \overline{\pi_Y(E)}$ we must have $(x, y) \in E$. (Such a y can clearly not be one of the y_b's). If for every $y \in \overline{\pi_Y(E)}$ there exist open V_y, W_y such that $(x, y) \in V_y \times W_y$ and $(V_y \times W_y) \cap E = \emptyset$ then find some countable subcover of $\overline{\pi_Y(E)}$ say $\{W_{y_i} : i = 1, \ldots, j\}$. Since $(V_{y_i} \times W_{y_i}) \cap E = \emptyset$ then defining $V = \bigcap\{V_{y_i} : i = 1, \ldots, j\}$ we get $V \cap \pi_X(E) = \emptyset$ which is a contradiction. To finish we note that $(x, y) \in \mathcal{U}$. So there is some open $S \times T$ with $(x, y) \in S \times T \subset \mathcal{U}$. But then we have $(b, y_b) \in S \times T$ and so $(x, y_b) \in \mathcal{U}$ contradicting the fact that $\mathcal{U}^{y_b} = \{b\}$. ∎

Problem 36 *Characterise the spaces with a zero–set universal parametrised by a Lindelof–Σ space.*

We have been unable to answer the following question.

Problem 37 *If a Tychonoff space X has a continuous function universal parametrised by a Lindelof–Σ space then is X metrisable?*

The following observations may be of use in answering this question. Note that if Y parametrises a continuous function universal for X then so does any space which can be continuously mapped onto Y. In addition every Lindelof–Σ space is the continuous image of some space which is a closed subspace of $K \times M$ for some compact K and second–countable M (see [16]) and so we can restrict our attention to parametrisation by such spaces. Also, if there is a non–metrisable space with a continuous function universal parametrised by such a space then the following result does place some restrictions on the parametrisation.

Lemma 38 *Fix a Tychonoff space X, compact K, separable metrisable M and closed $Y \subset K \times M$. Assume that Y parametrises a continuous function universal for X via $F : X \times Y \to$*

$\mathbb{R}$. *If there is some continuous* $f : X \to \mathbb{R}$ *that only appears finitely many times in the parametrisation then* $s(X) = \omega$. *Hence (by Lemma 33)* X *is second–countable.*

Proof. Since Y is a closed subspace of $K \times M$ there is a cover of Y by pairwise disjoint compact sets $\{K_\alpha : \alpha \in \kappa\}$ and a countable collection of open sets $\mathcal{U} = \{U_n : n \in \omega\}$ such that for all $\alpha \in \kappa$ and open $U \supset K_\alpha$ there exists $n \in \omega$ such that $K_\alpha \subset U_n \subset U$. Also assume that $\mathcal{U}$ is closed under finite unions. Assume that $f : X \to \mathbb{R}$ appears only finitely many times in the parametrisation and that $\{z_i : i = 1, \dots, j\}$ list all elements of Y representing f (i.e. $F^{z_i} = f$ for $i = 1, \dots, j$). Each z_i is in K_{α_i} for some $\alpha < \kappa$. Now assume that $s(X) > \omega$ and let $\{x_\beta : \beta < \omega_1\}$ be an uncountable discrete subspace. For any $r \in \mathbb{R}$ and $\gamma < \omega_1$ there is a continuous $f_{\gamma,r}$ such that $f_{\gamma,r}(x_\gamma) = r$ and $f_{\gamma,r}(x_\beta) = 0$ when $\gamma \neq \beta$. Now since each K_{α_i} is compact we can choose $r_\beta \in \mathbb{R}$ such that $F(x_\beta, y) \neq (r + f(x_\beta))$ for any $y \in \bigcup\{K_{\alpha_i} : i = 1, \dots, j\} = K$. For each of the functions $f + f_{\beta,r_\beta}$ choose some $y_\beta \in Y$ that represents the function. For each $\beta < \omega_1$ we can find $U_{n_\beta} \in \mathcal{U}$ such that $K \subset U_{n_\beta}$ and $y_\beta \notin U_{n_\beta}$. Since $\mathcal{U}$ is countable there is some U_n such that $K \subset U_n$ and $y_\beta \notin U_n$ for uncountably many β. But this gives a contradiction as this uncountable set must have a limit point but the only possible limit points are $\{z_i : i \leq j\}$. ■

Corollary 39 *Let* X *be a Tychonoff space and* τ *an admissible topology on* $C(X)$. *If* $(C(X), \tau)$ *is homeomorphic to a closed subspace of* $K \times M$ *where* K *is compact and* M *is separable metrisable then* X *is a separable metrisable space.*

Problem 40 *If a space* X *has a zero–set universal parametrised by a product of a compact and a second countable space, then is* X *metrisable? If* X *has an open regular* F_σ *universal parametrised by a product of a compact and a second countable space, then is* X *metrisable?*

4.0 CHAIN CONDITIONS

The study of chain conditions began with Souslin's attempt to characterise the real line. Souslin had shown that any separable, connected, linearly ordered space with no endpoints is homeomorphic to $\mathbb{R}$. It is easily shown that if we take an uncountable collection of open subsets of a separable space then two of them must have non–empty intersection. This property is called the countable chain condition, as an equivalent definition is that every collection of pairwise disjoint open subsets must be countable. Souslin wondered if one could weaken the property of separability in the characterisation to that of the countable chain condition. Any linearly ordered non–separable space with the countable chain condition is known as a Souslin line. Souslin's hypothesis states that there are no Souslin lines.

It has been shown that Souslin's hypothesis is consistent and independent of ZFC. In one direction it was shown by Kurepa that if X is a Souslin line then $X \times X$ fails to have the countable chain condition. However under $MA(\omega_1)$ the product of any number of ccc spaces is ccc, ruling out the existence of a Souslin line. Under $\diamondsuit$ one can construct a Souslin line. This line of enquiry demonstrated that the question of productivity of the countable chain condition is also undecidable in ZFC. The study of products has been a central theme of the study of chain conditions.

The countable chain condition can be seen to be the weakest of a whole collection of chain conditions. Other chain conditions such as Knaster's property K and Shanin's property have been extensively studied. A space X has property K if and only if given any uncountable collection of open subsets of X there is some uncountable subcollection such that any two open subsets in this subcollection have non–empty intersection. This is obviously a strengthening of the countable chain condition and this property is preserved in products. Under $MA(\omega_1)$ the countable chain condition implies property K and this is why the countable chain condition is preserved in products under this axiom. A space X has Shanin's condition if and only if given any uncountable collection of open subsets of X there is some uncountable subcollection such that any finite subset of this sucollection has non–empty intersection. This strengthens property K and is also preserved in products. The property calibre ω_1 is stronger than Shanin's condition and says that given any uncountable collection of open subsets of X there is some uncountable subcollection with non–empty intersection. The article by Todorcevic [27] is an excellent introduction to these properties.

We can consider these chain conditions to be a specific instance of the following definition.

Definition 41 *Let κ, λ, μ be cardinals with $\kappa \geq \lambda \geq \mu$.*

(i) X has calibre (κ, λ, μ) if and only if for every collection of open subsets $\{U_\alpha : \alpha \in \kappa\}$ there is some $A \subset \kappa$ with $|A| = \lambda$ such that for every $B \subset A$ with $|B| = \mu$ we have $\bigcap\{U_\alpha : \alpha \in B\} \neq \emptyset$.

(ii) X has calibre $(\kappa, \lambda, < \mu)$ if and only if for every collection of open subsets $\{U_\alpha : \alpha \in \kappa\}$ there is some $A \subset \kappa$ with $|A| = \lambda$ such that for every $B \subset A$ with $|B| < \mu$ we have $\bigcap\{U_\alpha : \alpha \in B\} \neq \emptyset$.

The countable chain condition is calibre $(\omega_1, 2, 2)$, Shanin's condition is calibre $(\omega_1, \omega_1, < \omega)$, property K is calibre $(\omega_1, \omega_1, 2)$ and calibre ω_1 is shorthand for calibre $(\omega_1, \omega_1, \omega_1)$. In this section we examine when the spaces $C_k(X)$ or $C_p(X)$ satisfy these chain conditions. We also deal with the countable chain condition in zero–set universals. At the end we give some examples.

4.1 CALIBRES IN $C_K(X)$

In [23] Nakhmanson provides necessary and sufficient on a space X for $C_k(X)$ to have a given cardinal κ as a calibre (under the assumption that the cofinality of κ is uncountable). He also deals with precalibres of $C_k(X)$. In this section we generalise these results to deal with the case where $C_k(X)$ has calibre (κ, λ, μ) or calibre $(\kappa, \lambda, < \mu)$. Towards this end we introduce the following definition. Remember from Chapter 2 that a type of basic open subset $W(\mathcal{K}, \mathcal{U})$ of $C_k(X)$ is a function $t : \mathcal{P}(n+1) \to 2$ for some $n \in \omega$ describing which $K_i, K_j \in \mathcal{K}$ have non–empty intersection. Types will be crucial to our study of chain conditions in $C_k(X)$.

Definition 42 *Let κ, λ, μ be cardinals and t a type of basic open subset of $C_k(X)$.*

(i) X has property $\mathcal{K}(\kappa, \lambda, \mu, t)$ if and only if for every collection $\{F_\alpha : \alpha \in \kappa\}$ where each F_α is a collection of compact subsets of X of type t, there is some $A \subset \kappa$ with $|A| = \lambda$ such that for every $B \subset A$ with $|B| = \mu$ there exists a collection of zero sets $\langle C_i : i \leq n\rangle$ of type t satisfying $F^i_\alpha \subset C_i$ for all $\alpha \in B$ (where the ith compact set in each F_α is F^i_α).

(ii) X has property $\mathcal{K}(\kappa, \lambda, < \mu, t)$ if and only if for every collection $\{F_\alpha : \alpha \in \kappa\}$ where each F_α is a collection of compact subsets of X of type t, there is some $A \subset \kappa$ with $|A| = \lambda$ such that for every $B \subset A$ with $|B| = \mu$ there exists a collection of zero sets $\langle C_i : i \leq n\rangle$ of type t satisfying $F^i_\alpha \subset C_i$ for all $\alpha \in B$.

(iii) If X has property $\mathcal{K}(\kappa, \lambda, \mu, t)$ for all linear types t then we write that X has $\mathcal{K}(\kappa, \lambda, \mu)$.

(iv) If X has property $\mathcal{K}(\kappa, \lambda, < \mu, t)$ for all linear types t then we write that X has $\mathcal{K}(\kappa, \lambda, \mu)$.

Theorem 43 *Let X be Tychonoff and κ, λ, μ cardinals with $\kappa \geq \lambda \geq \mu$. Assume that κ has uncountable cofinality.*

(i) $C_k(X)$ has calibre (κ, λ, μ) if and only if X has property $\mathcal{K}(\kappa, \lambda, \mu)$.

(i) $C_k(X)$ has calibre $(\kappa, \lambda, < \mu)$ if and only if X has property $\mathcal{K}(\kappa, \lambda, < \mu)$.

Proof. We will only present the proof of (i) as (ii) can be proved with minor modifications of the same argument. Assume that $C_k(X)$ has calibre (κ, λ, μ) and fix a collection $\{F_\alpha : \alpha \in \kappa\}$ where each F_α is a finite sequence of compact subsets of X of linear type t. Find for each $i \leq n$ a pair of reals (l_i, r_i) satisfying: for all $i < n$ we have $l_i < r_i$, $l_i + \frac{1}{4} < l_{i+1} < r_i$ if $F^i_\alpha \cap F^{i+1}_\alpha \neq \emptyset$ and $l_{i+1} - r_i > \frac{1}{4}$ if $F^i_\alpha \cap F^{i+1}_\alpha = \emptyset$. For each F_α find continuous f_α satisfying that $f_\alpha(x) \in [l_i, r_i]$ for all $i \leq n$ and $x \in F^i_\alpha$. Note that the definition of (l_i, r_i) is independent of α.

We get a collection of open sets $\mathcal{U} = \{B(f_\alpha, F_\alpha, \frac{1}{8}) : \alpha \in \kappa\}$. There exists $A \subset \kappa$ with $|A| = \lambda$ such that for all $B \subset A$ with $|B| = \mu$ we have $\bigcap\{U_\alpha : \alpha \in B\} \neq \emptyset$ (where $U_\alpha = B(f_\alpha, F_\alpha, \frac{1}{8})$). Fixing $B \subset A$ with $|B| = \mu$ let $f \in \bigcap\{U_\alpha : \alpha \in B\} \neq \emptyset$. Fix $\alpha \in B$. Let $i < n$. Define $C_i = f^{-1}[l_i - \frac{1}{8}, r_i + \frac{1}{8}]$ and note that the type of the collection $\langle C_0, \ldots, C_n\rangle$ is in fact t. Also $F^i_\alpha \subset C_i$ for all $\alpha \in B$ and so we are done.

Now assume that X has property $\mathcal{K}(\kappa, \lambda, \mu)$. Fix a collection of open subsets of $C_k(X)$, say $\{U_\alpha : \alpha \in \kappa\}$. We can assume that there exist $n > 0$,$\epsilon > 0$ and rationals $r_1, \ldots, r_n$ such that each U_α is of the form $U_\alpha = W(F_\alpha, V_\alpha)$ where F_α is a collection of compact subsets of X and G_α is an n–tuple of open intervals with $G^i_\alpha = (r_i - \epsilon, r_i + \epsilon)$. In addition we can assume that $t_{F_\alpha} = t_{F_\beta}$ for each $\alpha, \beta \in \kappa$ and let t denote this common type. (Since κ has uncountable cofinality we can do this). Apply $\mathcal{K}(\kappa, \lambda, \mu, t)$ to the collection $\{F_\alpha : \alpha \in \kappa\}$ we get $A \subset \kappa$ with $|A| = \lambda$ such that for all $B \subset A$ with $|B| = \mu$ we have zero sets C_i for $i \leq n$ such that $F^i_\alpha \subset C_i$ for all $\alpha \in B$ and the collection of zero–sets has type t. Fixing $B \subset A$ with $|B| = \mu$ and finding the zero sets $\langle C_i : i \leq n\rangle$ we note that there is $f' \in W'(\langle C_i : i \leq n\rangle, \{(r_i - \epsilon, r_i + \epsilon) : i \leq n\})$. So there exists, by Theorem 6, an $f \in W(\langle C_i : i \leq n\rangle, \{(r_i - \epsilon, r_i + \epsilon) : i \leq n\})$ and if we note that

$$W(\langle C_i : i \leq n\rangle, \{(r_i - \epsilon, r_i + \epsilon) : i \leq n\}) \subset \bigcap_{\alpha \in B} U_\alpha$$

then we are done. ■

Note that for uncountable κ the property $\mathcal{K}(\kappa, \kappa, \kappa, d_2)$ is the same as the property k–κ–separable as defined in [23]. It is also shown in [23] that $\mathcal{K}(\kappa, \kappa, \kappa, d_2)$ implies $\mathcal{K}(\kappa, \kappa, \kappa)$.

We can see that any property $\mathcal{K}(\kappa, \lambda, \mu)$ is preserved in subspaces.

Theorem 44 *Fix a space X and cardinals κ, λ, μ. Assume that κ has uncountable cofinality. Let $Y \subset X$.*

(i) If X has calibre (κ, λ, μ) then Y has calibre (κ, λ, μ).

(ii) If X has calibre $(\kappa, \lambda, < \mu)$ then Y has calibre $(\kappa, \lambda, < \mu)$.

The situation regarding products is not so simple. We will deal with this later in Section 4.1.2.

4.1.1 The countable chain condition and metrisability of X

The case where $C_k(X)$ has the countable chain condition (equivalently has calibre $(\omega_1, 2, 2)$) perhaps raises the most interesting questions. It is a "folklore" result that if X is compact and $C_k(X)$ is ccc then X must be metric. To see that this is true we note that if X is compact then $C_k(X)$ is metric. Every metric space with the countable chain condition must in fact have a countable network and so X must have a countable network implying that X is metrisable.

We can now ask the question:

Just how much of the property $\mathcal{K}(\omega_1, 2, 2)$ is needed to prove this result? A more precise formulation of this question is the following.

Question 45 *For which types t can we find a compact non–metrisable space X such that X has the property $\mathcal{K}(\omega_1, 2, 2, t)$?*

The following lemma provides a partial answer to this question.

Lemma 46 *Let X be a compact space. Let t be the type of a collection $\langle K_0, K_1, K_2, K_3\rangle$ where $K_i \cap K_j \neq \emptyset$ if and only if $|i - j| \leq 1$. If X has property $\mathcal{K}(\omega_1, 2, 2, t)$ then X is metrisable.*

Proof. It suffices to find a countable T_1–separating collection of closed subsets of X (see [16]). Assume that X is compact but that there exists no countable T_1–separating collection of closed subsets of X. We will show that X cannot have property $\mathcal{K}(\omega_1, 2, 2, t)$ by recursively defining $\mathcal{C} = \{K_\alpha : \alpha \in \omega_1\}$, where each $K_\alpha = \langle K^0_\alpha, K^1_\alpha, K^2_\alpha, K^3_\alpha\rangle$ has type t that satisfies $(*)$: for all $\alpha_1, \alpha_2 \in \omega_1$ either

$$\bigcup_{i=1,2} K^1_{\alpha_i} \cap \bigcup_{i=1,2} K^3_{\alpha_i} \neq \emptyset$$

or

$$\bigcup_{i=1,2} K^0_{\alpha_i} \cap \bigcup_{i=1,2} K^2_{\alpha_i} \neq \emptyset.$$

Assume that we have defined the collection $\mathcal{C}_\lambda = \{\langle K^0_\alpha, K^1_\alpha, K^2_\alpha, K^3_\alpha\rangle : \alpha \in \lambda\}$ for some $\lambda < \omega_1$ where $\mathcal{C}_\lambda$ satisfies $(*)$. In addition assume that $\bigcup\{K^i_\alpha : i = 0, 1, 2, 3\} = X$. We will show how to define $\langle K^0_\lambda, K^1_\lambda, K^2_\lambda, K^3_\lambda\rangle$ so that $\mathcal{C}_\lambda \cup \langle K^0_\lambda, K^1_\lambda, K^2_\lambda, K^3_\lambda\rangle$ satisfies $(*)$.

By assumption we know that $\mathcal{S} = \bigcup_{i=0,1,2,3}\{K^i_\alpha : \alpha \in \lambda\}$ is not a T_1 separating collection. So there exists $x_1, x_2 \in X$ such that for all $C \in \mathcal{S}$ we have $x_1 \in C$ implies $x_2 \in C$. Let $K^0_\lambda = \{x_1\}$ and let $K^3_\lambda = \{x_2\}$. Find U, an open neighbourhood of x_1 that is not closed such that $x_2 \notin \overline{U}$. Let $K^1_\lambda = \overline{U}$ and let $K^2_\lambda = X \setminus U$. Note that $\langle K^0_\lambda, K^1_\lambda, K^2_\lambda, K^3_\lambda\rangle$ has type t. Fix $\alpha \in \lambda$. Now $x_1 \in K^i_\alpha$ for some $i \in \{0, 1, 2, 3\}$. By our choice of x_1, x_2 we know that x_2 is in the same K^i_α. A straightforward check through the four possible cases shows that either

$$(K^1_\alpha \cup K^1_\lambda) \cap (K^3_\alpha \cup K^3_\lambda) \neq \emptyset$$

or

$$(K^0_\alpha \cup K^0_\lambda) \cap (K^2_\alpha \cup K^2_\lambda) \neq \emptyset.$$

This demonstrates that $\mathcal{C}_\lambda \cup \langle K^0_\lambda, K^1_\lambda, K^2_\lambda, K^3_\lambda \rangle$ satisfies $(*)$.

Now let $\mathcal{C}$ be a collection of disjoint pairs of compact subsets of X that is maximal with respect to $(*)$ (ie $\mathcal{C}$ satisfies $(*)$, but for any collection $\mathcal{D}$, if $\mathcal{C} \subsetneq \mathcal{D}$ then $\mathcal{D}$ does not have property $(*)$). Since X has $\mathcal{K}(\omega_1, 2, 2, t)$ we must have that $\mathcal{C}$ is countable. But $\mathcal{S}$ as described above must be a T_1–separating collection, and so we are done.

■

The "folklore" result already mentioned is clearly an immediate consequence of Lemma 46 and Theorem 43. However we can weaken the assumption that X is compact to X being ω–bounded. A space is ω–bounded if and only if the closure of every countable set is compact. First we prove the following.

Lemma 47 *Let d_2 be the type of a pair of disjoint sets. If a space X is ω–bounded and X has the property $\mathcal{K}(\omega_1, 2, 2, d_2)$ then X is separable and hence compact.*

Proof. Assume that X is ω–bounded. We will show that if X is not separable then X does not have the property $\mathcal{K}(\omega_1, 2, 2, d_2)$ by recursively defining for each $\alpha \in \omega_1$ a disjoint pair of compact subsets of X, say K_α, L_α such that $\{\langle K_\alpha, L_\alpha \rangle : \alpha \in \omega_1\}$ witnesses that the property $\mathcal{K}(\omega_1, 2, 2, d_2)$ fails.

First we will recursively define two sequences of points from X say $\{x_\alpha : \alpha \in \omega_1\}$ and $\{y_\alpha : \alpha \in \omega_1\}$ satisfying: $y_\lambda \notin \overline{\{x_\alpha : \alpha \in \lambda\} \cup \{y_\alpha : \alpha \in \lambda\}}$ for every $\lambda \in \omega_1$. To begin choose two points $x_0, y_0 \in X$ such that $x_0 \neq y_0$. Assume that for some $\kappa < \omega_1$ and all $\lambda < \kappa$ we have chosen points x_λ, y_λ such that $y_\lambda \notin \overline{\{x_\alpha : \alpha \in \lambda\} \cup \{y_\alpha : \alpha \in \lambda\}}$. If κ is a successor ordinal then let $x_\kappa = y_\kappa - 1$ and choose some $y_\kappa \notin \overline{\{x_\alpha : \alpha \in \lambda\} \cup \{y_\alpha : \alpha \in \lambda\}}$. If κ is a limit then choose $x_\kappa \notin \overline{\{x_\alpha : \alpha \in \lambda\} \cup \{y_\alpha : \alpha \in \lambda\}}$ and $y_\kappa \notin \overline{\{x_\alpha : \alpha \in \lambda\} \cup \{y_\alpha : \alpha \in \lambda\}}$ such that $y_\kappa \neq x_\kappa$. Since by assumption X is not separable we can do this.

Define $K_0 = \{x_0\}$ and $L_0 = \{y_0\}$. For each $\lambda \in \omega_1$ we define $K_\alpha = \overline{\{x_\alpha : \alpha \leq \lambda\}}$ and $L_\lambda = \{y_\lambda\}$. This K_α is compact as X is ω–bounded. If we take any $\alpha, \beta < \omega_1$ then we cannot have $(K_\alpha \cup K_\beta) \cap (L_\alpha \cup L_\beta) = \emptyset$ as $y_\alpha \in K_\beta$ if $\alpha < \beta$ and $y_\beta \in K_\alpha$ if $\alpha > \beta$. This shows that the collection $\{K_\alpha, L_\alpha : \alpha \in \omega\}$ witnesses the failure of the property $\mathcal{K}(\omega_1, 2, 2, d_2)$. ■

The following is an immediate corollary to Lemma 46 and Lemma 47

Corollary 48 *If a space X is ω–bounded and $C_k(X)$ is ccc then X is metric.*

There are many properties that are known to be weaker than ω–boundedness, for example countable compactness. For which weaker properties will this metrisation theorem hold?

Question 49 *For which compactness type properties $\mathcal{P}$ can we show that every space X with property $\mathcal{P}$ such that $C_k(X)$ is ccc must be metrisable? Is it true for countable compactness? Lindelof–Σ spaces?*

4.1.2 Productivity of the countable chain condition

As we already noted the question of whether or not the product of ccc spaces is ccc cannot be decided under ZFC. Under $MA(\omega_1)$ the product of any number of ccc spaces is ccc. So this will of course be true under $MA(\omega_1)$ when we look at a subclass of spaces, those that are $C_k(X)$ for some X. However a very important question remains. Can we show in ZFC that for every space X that $C_k(X)^2$ must be ccc if $C_k(X)$ is? If not then in which models of ZFC can we construct a counterexample?

As noted in Chapter 2 the space $C_k(X)^2$ is homeomorphic to the space $C_k(X \oplus X)$. So we examine when the space $X \oplus X$ has properties $\mathcal{K}(\kappa, \lambda, \mu)$. In addition we are interested in what happens when we take products in the base space. The following lemma raises some interesting questions and demonstrates that the two questions are closely related.

Lemma 50 *Fix an infinite space X and cardinals κ, λ, μ. Assume that κ has uncountable cofinality. Then the following are equivalent:*

(i) X satisfies $\mathcal{K}(\kappa, \lambda, \mu, t)$ for every type t,

(ii) for all $n \in \omega$ the space $\bigoplus_{i \leq n} X$ satisfies $\mathcal{K}(\kappa, \lambda, \mu, t)$ for every type t,

(iii) for all $n \in \omega$ the space $\prod_{i \leq n} X$ satisfies $\mathcal{K}(\kappa, \lambda, \mu, t)$ for every type t.

Proof. (iii) implies (ii): Fix $n \in \omega$ and assume that $\prod_{i \le n} X$ satisfies $\mathcal{K}(\kappa, \lambda, \mu, t)$ for every type t. The space $\bigoplus_{i \le n} X$ can be embedded as a subspace of $\prod_{i \le n} X$. Applying Theorem 44 we are done.

(ii) implies (i): As the space X can be embedded in $\bigoplus_{i \le n} X$ for all $n \ge 1$ the result follows immediately from Theorem 44.

(i) implies (iii): It will suffice to prove this for the case $n = 2$. Assume that X, κ, λ and μ are as in the statement of the lemma and assume that X satisfies $\mathcal{K}(\kappa, \lambda, \mu, t)$ for every type t. Fix $\{K_\alpha : \alpha \in \kappa\}$ where each $K_\alpha = \langle K_\alpha^0, \ldots, K_\alpha^m \rangle$ consists of compact subsets of $X \times X$. In addition assume that each of the K_α's has the same type, which we will denote by t.

For each $\alpha \in \kappa$ and $i \le m$ find $\langle C_{\alpha,i}^0, \ldots, C_{\alpha,i}^{r(\alpha,i)} \rangle$ and $\langle D_{\alpha,i}^0, \ldots, D_{\alpha,i}^{r(\alpha,i)} \rangle$ where each $C_{\alpha,i}^j$ and $D_{\alpha,i}^j$ are compact subsets of X satisfying: (i) $K_\alpha^j \subset \bigcup \{C_{\alpha,i}^j \times D_{\alpha,i}^j : j \le r(\alpha, i)\}$ and (ii) $K_\alpha^{i_1} \cap K_\alpha^{i_2} = \emptyset$ if and only if $\bigcup \{C_{\alpha,i_1}^j \times D_{\alpha,i_1}^j : j \le r(\alpha, i_1)\} \cap \bigcup \{C_{\alpha,i_2}^j \times D_{\alpha,i_2}^j : j \le r(\alpha, i_2)\}$. By passing to a κ sized subset we can assume that $r(\alpha, i) = r_i$ for each $\alpha \in \kappa$ and $i \le m$. We can now form $T_\alpha = \langle C_{\alpha,0}^0, \ldots, C_{\alpha,0}^{r_0}, D_{\alpha,0}^0, \ldots, D_{\alpha,0}^{r_0}, \ldots, C_{\alpha,m}^0, \ldots, C_{\alpha,m}^{r_m}, D_{\alpha,m}^0, \ldots, D_{\alpha,m}^{r_m} \rangle$. Essentially T_α is just some relisting of all the $C_{\alpha,i}^j$'s and $D_{\alpha,i}^j$'s. Assume that each T_α is of the same type t'. Apply $\mathcal{K}(\kappa, \lambda, \mu, t')$ to the collection $\{T_\alpha : \alpha \in \kappa\}$ to get the required $A \subset \kappa$ with $|A| = \lambda$.

Fix $B \subset A$ with $|B| = \mu$. There exists a sequence of zero–subsets of X of type t', say $\langle Y_0^0, \ldots, Y_0^{r_0}, Z_0^0, \ldots, Z_0^{r_0}, \ldots, Y_m^0, \ldots, Y_m^{r_m}, Z_m^0, \ldots, Z_m^{r_m} \rangle$ such that for each $i \le m$, $j \le r_i$ and $\alpha \in B$ we have $C_{\alpha,i}^j \subset Y_i^j$ and $D_{\alpha,i}^j \subset Z_j^i$. Finally for each $i \le m$ define $L_i = \bigcup \{Y_i^j \times Z_i^j : j \le r_i\}$. Now the collection $\langle L_0, \ldots, L_m \rangle$ has type t. Now $K_\alpha^i \subset L_i$ for each $\alpha \in B$ and $i \le m$. We can check that if $L_i \cap L_j \ne \emptyset$ then $K_\alpha^i \cap K_\alpha^j \ne \emptyset$. This will demonstrate that the collection $\langle L_0, \ldots, L_m \rangle$ has type t, finishing the proof.

Fix such an $i, j < m$ and $\alpha \in B$. If $L_i \cap L_j \ne \emptyset$ then we can find $s \le m_i$ and $s' \le m_j$ such that $Y_i^s \times Z_i^s \cap Y_j^{s'} \times Z_j^{s'} \ne \emptyset$. This implies that $Y_i^s \cap Y_j^{s'} \ne \emptyset$ and $Z_i^s \cap Z_j^{s'} \ne \emptyset$. But this would of course imply that $C_{\alpha,i}^s \cap C_{\alpha,j}^{s'} \ne \emptyset$ and $D_{\alpha,i}^s \cap D_{\alpha,j}^{s'} \ne \emptyset$ giving us that $(C_{\alpha,i}^s \times D_{\alpha,i}^s) \cap (C_{\alpha,j}^{s'} \times D_{\alpha,j}^{s'}) \ne \emptyset$. Finally this would mean that $K_i \cap K_j \ne \emptyset$. ■

At first glance this would seem to show that all calibres are preserved in finite products in the C_k setting. However if $C_k(X)$ has calibre (κ, λ, μ) then this only guarantees that X has $\mathcal{K}(\kappa, \lambda, \mu, t)$ for every *linear* type t. We can strengthen this if we look at the class of zero–dimensional spaces.

Theorem 51 *Let X is a 0–dimensional space and fix cardinals κ, λ, μ. Assume that κ has uncountable cofinality. Then X satisfies $\mathcal{K}(\kappa, \lambda, \mu, t)$ for every discrete type t iff X satisfies $\mathcal{K}(\kappa, \lambda, \mu, t)$ for every type t.*

Proof. Let X, κ, λ and μ be as in the statement of the theorem. Assume that X satisfies $\mathcal{K}(\kappa, \lambda, \mu, t)$ for every discrete type t. Fix an arbitrary type t and let $\{K_\alpha : \alpha \in \kappa\}$ consist of $(n+1)$–tuples of compact subsets of X, where each K_α has type t. We will replace each K_α with a $D(K_\alpha)$ of discrete type.

Let $L = \langle L^0, \dots, L^n \rangle$ consist of compact subsets of X. We will demonstrate how to construct $D(L)$. To do this we will construct for each $j \leq n$ a set $L_j = \langle L^0_j, \dots, L^n \rangle$ and clopen M_j such that $M_j \cap \bigcup L_{j+1} = \emptyset$. Let $L_0 = L$ and $M_0 = \emptyset$. Assume that we have constructed L_s and M_s for all $s \leq j$. Let $A(L_j) = \{Y \subset (n+1) : \bigcap_{i \in Y} L^i_j \neq \emptyset, \forall i' \notin Y (L^{i'}_j \cap \bigcap_{i \in Y} L^i_j = \emptyset)\}$. For each $Y \in A(L_j)$ find clopen $D(L)_Y$ such that $\bigcap_{i \in Y} L^i_j \subset D(L)_Y$ but $D(L)_Y \cap D(L)_{Y'} = \emptyset$ when $Y \neq Y'$. In addition assume that $M_j \cap D(L)_Y = \emptyset$ for each Y. Define L_{j+1} by $L_{j+1} = \langle L^0_{j+1}, \dots, L^n_{j+1} \rangle$ where $L^i_{j+1} = L^i_j \cap (X \setminus (\bigcup_{Y \in A(L_j)} D(L)_Y))$. We also define $M_{j+1} = M_j \cup (\bigcup_{Y \in A(L_j)} D(L)_Y))$.

It is clear that $L^i_n = \emptyset$ for each $i \leq n$. Let I_L consist of all those $Y \subset n+1$ for which $D(L)_Y$ was defined. Let $D(L)^Y = D(L)_Y \cap \bigcup_{i \leq n} L^i$. By choosing some ordering of I_L we can now construct $D(L)$. It follows from the construction that in fact $D(L)$ is of discrete type.

For each $\alpha \in \kappa$ construct $D(K_\alpha)$. We will write D_α instead of $D(K_\alpha)$. By passing to a κ sized subcollection if necessary we can assume that $I_{K_\alpha} = I_{K_\beta}$ for each α, β and that the order chosen on these sets that is used to construct D_α and D_β are in fact the same. We use I to denote this common set. Let $m = |I|$. Apply $\mathcal{K}(\kappa, \lambda, \mu, d_m)$ to the collection $\{D_\alpha : \alpha \in \kappa\}$ to find $A \subset \kappa$ with $|A| = \lambda$ and the other properties asserted by $\mathcal{K}(\kappa, \lambda, \mu, d_m)$. Now fix $B \subset A$ with $|B| = \mu$. There is $C = \langle C^Y : Y \in I \rangle$ consisting of pairwise disjoint zero sets with $L^Y_\alpha \subset C^Y$ for all $Y \in I$ and $\alpha \in B$. Let $C_i = \bigcup \{C^Y : i \in Y\}$. Then the collection $C' = \langle C_0, \dots, C_n \rangle$ has type t and $K^i_\alpha \subset C_i$ for all $\alpha \in B$. ■

As every discrete type is clearly linear we get the following.

Corollary 52 *Let X is a 0–dimensional space and fix cardinals κ, λ, μ. Assume that κ has* *uncountable cofinality. The following are equivalent:*

(i) $C_k(X)$ *has calibre* (κ, λ, μ)*,*

(ii) $C_k(X)^n$ *has calibre* (κ, λ, μ) *for all* $n \in \omega$*,*

(iii) $C_k(X^n)$ *has calibre* (κ, λ, μ) *for all* $n \in \omega$*.*

Now we return to the general case. We know that calibre ω_1 is preserved in products in the class of all Tychonoff spaces and hence $\mathcal{K}(\omega_1, \omega_1, \omega_1)$ is preserved when we take the disjoint sum. We will show the stronger result that $\mathcal{K}(\omega_1, \omega_1, \omega_1)$ is preserved in products. First we prove the following lemma. Note that this lemma is essentially the same as the result in [23], however Nakhmanson does not deal with types.

Lemma 53 *Let* X *be a space and fix cardinals* κ *and* μ*. Assume that* κ *has uncountable cofinality.*

(i) If X *has* $\mathcal{K}(\kappa, \kappa, \mu, d_2)$ *then* X *has* $\mathcal{K}(\kappa, \kappa, \mu, t)$ *for every type* t*.*

(ii) If X *has* $\mathcal{K}(\kappa, \kappa, < \mu, d_2)$ *then* X *has* $\mathcal{K}(\kappa, \kappa, < \mu, t)$ *for every type* t*.*

Proof. Let X, κ and μ be as in the statement of the lemma. We will just prove (i) as (ii) can be proved with minor modifications of the same argument. Assume that X has $\mathcal{K}(\kappa, \kappa, \mu, d_2)$ and fix a type t.

Fix, as usual, $\{C_\alpha : \alpha \in \kappa\}$ where each C_α has type t and consists of compact subsets of X. Assume $|C_\alpha| = n + 1$. Let I consist of all unordered pairs $\{i, j\}$ where $i \neq j$, $i, j \leq n$ and $t(\{i, j\}) = 0$. In other words I lists all the pairs of disjoint sets in any collection of type t. Order this to get $\langle p_r : r \leq m \rangle$ where each $p_r = \{i_r, j_r\} \in I$.

We will recursively define the required $A \subset \kappa$ of size κ. Apply $\mathcal{K}(\kappa, \kappa, \mu, d_2)$ to the collection $\{\langle C_\alpha^{i_0}, D_\alpha^{j_0} \rangle : \alpha \in \kappa\}$ to get $A_0 \subset \kappa$ such that $|A_0| = \kappa$ and A satisfies the other properties guaranteed by $\mathcal{K}(\kappa, \kappa, \mu, d_2)$. Assume that we have defined A_l for all $l \leq r$ for some $r < m$. We can apply $\mathcal{K}(\kappa, \kappa, \mu, d_2)$ to the collection $\{\langle C_\alpha^{i_r}, D_\alpha^{j_r} \rangle : \alpha \in A_r\}$ to get $A_{r+1} \subset \omega_1$. Finally note that $A = A_m$ will suffice to witness that X satisfies $\mathcal{K}(\kappa, \kappa, \mu, t)$ in this particular instance. ■

Combining Lemma 53 and Lemma 50 we get the following corollary.

Corollary 54 *Let* X *be a space and fix cardinals* κ *and* μ*. Assume that* κ *has uncountable cofinality. The following are equivalent:*

(i) $C_k(X)$ has calibre (κ, κ, μ),

(ii) $C_k(X)^n$ has calibre (κ, κ, μ) for all $n \in \omega$,

(iii) $C_k(X^n)$ has calibre (κ, κ, μ) for all $n \in \omega$.

We will now use Martin's axiom to deal with the specific case of $\mathcal{K}(\omega_1, 2, 2)$. To begin with we will need to introduce some notation and results connecting MA with partitions. These definitions and more information on partitions can be found in [26].

Definition 55 *Let S be an uncountable set and $[S]^{<\omega} = K_0 \cup K_1$ is a partition ie $K_0 \cap K_1 = \emptyset$. We say that this is a ccc partition if and only if*

(i) $\{x\} \in K_0$ for each $x \in S$,

(ii) A subset of an element of K_0 is also in K_0,

(iii) Every uncountable subset of K_0 contains two elements whose union is in K_0.

For the application we have in mind we will fix some X and define $S = \{\langle C, D\rangle : C, D \subset X; C, D \text{ compact}, C \cap D = \emptyset\}$. Assuming that X satisfies $\mathcal{K}(\omega_1, 2, 2, d_2)$ we can form the following ccc partition. Let K_0 consist of all those finite $\{\langle C_i, D_i\rangle : i \leq n\} \subset S$ such that $\bigcup_{i \leq n} C_i \cap \bigcup_{i \leq n} D_i = \emptyset$. Then defining $K_1 = [S]^{<\omega} \setminus K_0$ we get a ccc partition.

We have the following version of Martin's axiom.

Lemma 56 *$MA(\omega_1)$ is equivalent to the following statement. Let S be a set of size ω_1 and let $[S]^{<\omega} = K_0 \cup K_1$ be a partition. Then S can be covered by countably many S_n such that $[S_n]^{<\omega} \subset K_0$ for every $n \leq \omega$.*

We will use this version of $MA(\omega_1)$ to show that $\mathcal{K}(\omega_1, 2, 2)$ is preserved in products.

Theorem 57 ($MA(\omega_1)$) *Let X be a space. If X has $\mathcal{K}(\omega_1, 2, 2)$ then X^2 has $\mathcal{K}(\omega_1, 2, 2)$.*

Proof. Let X be a spaces and assume that X satisfies $\mathcal{K}(\omega_1, 2, 2)$. We will begin by showing that X must have $\mathcal{K}(\omega_1, \omega_1, 2, d_2)$. Fix $\{(C_\alpha, D_\alpha) : \alpha \in \omega_1\}$ where each C_α and D_α are compact subsets of X and $C_\alpha \cap D_\alpha = \emptyset$. Let $S = \{C_\alpha : \alpha \in \omega_1\} \cup \{D_\alpha : \alpha \in \omega_1\}$. Let K_0 consist of all those finite $\{\langle C_i, D_i\rangle : i \leq n\} \subset S$ such that $\bigcup_{i \leq n} C_i \cap \bigcup_{i \leq n} D_i = \emptyset$. To see that this is a ccc partition it suffices to note that for all spaces the property $\mathcal{K}(\omega_1, 2, 2, d_2)$ implies $\mathcal{K}(\omega_1, n, n, d_2)$ for all $n < \omega$.

Apply Lemma 56 to this partition to get countably many S_n covering S such that $[S]^n \subset K_0$ for each $n < \omega$. There must be some S_n such that S_n is uncountable. Let $A \subset \omega_1$ satisfy $|A| = \omega_1$ and $S = \{(C_\alpha, D_\alpha) : \alpha \in A\}$. Fix $F \subset A$ with $|F| = n$. We know that $\{(C_\alpha, D_\alpha) : \alpha \in F\} \in [S]^n$ and so is in K_0, completing the proof that X must have $\mathcal{K}(\omega_1, \omega_1, 2, d_2)$.

Apply Lemma 53 to see that X satisfies $\mathcal{K}(\omega_1, \omega_1, 2, t)$ for every type t. Then Lemma 50 gives us that X^2 has $\mathcal{K}(\omega_1, \omega_1, 2)$ and so must have $\mathcal{K}(\omega_1, 2, 2)$.

■

Unfortunately we have been unable to find a consistent example of a space X such that X has $\mathcal{K}(\omega_1, 2, 2)$ but X^2, (or even $X \oplus X$) does not.

Problem 58 *Is there a consistent example of a space X such that X has $\mathcal{K}(\omega_1, 2, 2)$ but X^2 does not? If not, is there a consistent example of a space X such that X has $\mathcal{K}(\omega_1, 2, 2)$ but $X \oplus X$ does not?*

See Chapter 6 for an outline of some possible approaches to this problem.

4.2 CALIBRES IN $C_P(X)$

In this section we generalise Nakhmanson's results on chain conditions in $C_p(X)$, finding a characterisation of those X where $C_p(X)$ has a given calibre or precalibre. This characterisation is very similar in nature to its equivalent in $C_k(X)$, without the added complication of dealing with types.

Definition 59 *Let κ, λ, μ be cardinals and $n < \omega$.*

(i) X has property $\mathcal{P}(\kappa, \lambda, \mu, n)$ if and only if for every collection of n–tuples of points $\{F_\alpha : \alpha \in \kappa\}$ there is some $A \subset \kappa$ with $|A| = \lambda$ such that for every $B \subset A$ with $|B| = \mu$ there exist pairwise disjoint zero sets C_i for $i \leq n$ such that $F^i_\alpha \in C_i$ for all $\alpha \in B$(where the ith coordinate in each F_α is F^i_α).

(ii) X has property $\mathcal{P}(\kappa, \lambda, < \mu, n)$ if and only if for every collection of n–tuples of points $\{F_\alpha : \alpha \in \kappa\}$ there is some $A \subset \kappa$ with $|A| = \lambda$ such that for every $B \subset A$ with $|B| < \mu$ there exist pairwise disjoint zero sets C_i for $i \leq n$ such that $F^i_\alpha \in C_i$ for all $\alpha \in B$.

(iii) If X has property $\mathcal{P}(\kappa, \lambda, \mu, n)$ for all $n < \omega$ then we write that X has $\mathcal{P}(\kappa, \lambda, \mu)$.

(iv) If X has property $\mathcal{P}(\kappa, \lambda, < \mu, n)$ for all $n < \omega$ then we write that X has $\mathcal{P}(\kappa, \lambda, < \mu)$.

Theorem 60 *Let X be Tychonoff and κ, λ, μ cardinals with $\kappa \geq \lambda \geq \mu$. Assume that κ has uncountable cofinality.*

(i) $C_p(X)$ has calibre (κ, λ, μ) if and only if X has property $\mathcal{P}(\kappa, \lambda, \mu)$.

(i) $C_p(X)$ has calibre $(\kappa, \lambda, < \mu)$ if and only if X has property $\mathcal{P}(\kappa, \lambda, < \mu)$.

Proof. We will only give the proof of (i). Assume that $C_p(X)$ has calibre (κ, λ, μ) and fix a collection of n–tuples of points $\{F_\alpha : \alpha \in \kappa\}$. For each $\alpha \in \kappa$ we define open $U_\alpha = \{g \in C(X) : |g(F_\alpha^i) - i| < \frac{1}{4} \forall i \leq n\}$. Let $\mathcal{U} = \{U_\alpha : \alpha \in \kappa\}$. There exists $A \subset \kappa$ with $|A| = \lambda$ such that for all $B \subset A$ with $|B| = \mu$ we have $\bigcap\{U_\alpha : \alpha \in B\} \neq \emptyset$. Fixing $B \subset A$ with $|B| = \mu$ let $f \in \bigcap\{U_\alpha : \alpha \in B\} \neq \emptyset$. Define $C_i = f^{-1}[i - \frac{1}{4}, i + \frac{1}{4}]$ and note that the C_i's are pairwise disjoint zero sets. Also for each $\alpha \in B$ we have $|f(F_\alpha^i) - i| < \frac{1}{4}$ and so $F_\alpha^i \in C_i$ for all $i \leq n$.

Assume that X has property $\mathcal{P}(\kappa, \lambda, \mu, n)$ for all $n < \omega$. Fix a collection of open subsets $\{U_\alpha : \alpha \in \kappa\}$. We can assume that there exist $n > 0$, an $\epsilon > 0$ and rationals $r_1, \ldots, r_n$ such that each U_α is of the form $U_\alpha = B(F_\alpha, G_\alpha)$ where F_α is an n–tuple of points of X and G_α is an n–tuple of open intervals with $G_\alpha^i = (r_i - \epsilon, r_i + \epsilon)$. (Since κ is regular and uncountable we can do this).

Applying $\mathcal{P}(\kappa, \lambda, \mu, n)$ to the collection $\{F_\alpha : \alpha \in \kappa\}$ we get $A \subset \kappa$ with $|A| = \lambda$ such that for all $B \subset A$ with $|B| = \mu$ we have pairwise disjoint zero sets C_i for $i \leq n$ such that $F_\alpha^i \in C_i$ for all $\alpha \in B$. Fixing $B \subset A$ with $|B| = \mu$ and finding the zero sets $\{C_1, \ldots, C_n\}$ we note that there is $f \in C(X)$ such that $f(x) = r_i$ for all $i \leq n$ and $x \in C_i$. Clearly $f \in \bigcap\{U_\alpha : \alpha \in B\}$. ■

We can demonstrate immediately why $C_p(X)$ has calibre $(\kappa, \kappa, < \omega)$ for all κ and every X.

Theorem 61 *For every space X the space $C_p(X)$ has calibre $(\kappa, \kappa, < \omega)$ for all κ with uncountable cofinality.*

Proof. Fix κ, λ and a space X. Choose some $\{(x_\alpha, y_\alpha) : \alpha \in \kappa\}$ where each $x_\alpha \neq y_\alpha$ and $x_\alpha, y_\alpha \in X$. We need $A \subset \kappa$ with $|A| = \kappa$ such that for all finite $F \subset A$ we have $x_\alpha \neq y_\beta$ for

$\alpha, \beta \in F$.

Fix $\alpha \in \kappa$. Note that if $x_\alpha = x_\beta$ or $y_\alpha = y_\beta$ for κ many β's then we are done. So without loss of generality we can assume that for each $\alpha \in \kappa$ we have $|\{\beta : x_\beta = x_\alpha\} \cup \{\beta : y_\beta = y_\alpha\}| \leq \omega$. Now it is straightforward to recursively define $\alpha_\beta \in \kappa$ for each $\beta \in \kappa$ such that $x_{\alpha_{\beta_1}} \neq y_{\alpha_{\beta_2}}$ for all $\beta_1, \beta_2 \in kappa$. This gives us the required $A \subset \kappa$.

It is worth noting that we have shown more than required. However this does not show that X must have $\mathcal{P}(\omega_1, \omega_1, \omega_1)$ as we have no way of knowing if we can find disjoint zero–sets C, D such that $x_{\alpha_\beta} \in C$ and $y_{\alpha_\beta} \in D$ for all $\beta \in \kappa$. ∎

Just as for property $\mathcal{K}(\kappa, \lambda, \mu)$ it is easily shown that the property $\mathcal{P}(\kappa, \lambda, \mu)$ is preserved in subspaces.

In the papers [3] and [25] Arhangelskii and Tkacuk demonstrate that $C_p(X)$ has calibre ω_1 if and only if X has a small diagonal. We say a space X has a *small diagonal* if and only if for every collection $\{x_\alpha : \alpha \in \omega_1\}$ such that each $x_\alpha \in (X \times X) \setminus \{(x, x) : x \in X\}$ there exists open $U \supset \{(x, x) : x \in X\}$ with $x_\alpha \notin U$ for uncountably many of the x_α's. We have been unable to prove directly that every space with a small diagonal also satisfies property $\mathcal{P}(\omega_1, \omega_1, \omega_1)$, however this must be true. One of the most important questions relating to this property is the following.

Question 62 *Is every compact space X with a small diagonal metrisable?*

It was shown by Hao–Zuan in [18] that it is consistent with ZFC that every compact space with a small diagonal is metrisable. However the problem remains unsolved in ZFC.

4.3 CALIBRES IN UNIVERSALS

4.3.1 Sufficient conditions

We will find some sufficient conditions for a space to have a continuous function universal parametrised by a separable space, a ccc space or a space with calibre ω_1. These rely on the idea of a K–coarser topology on a space. Recall the following definition and theorem from chapter 2.

Definition 63 *Let τ, σ be two topologies on a set X with $\tau \subset \sigma$. We say that τ is a K–coarser topology if (X, σ) has a neighbourhood basis consisting of τ–compact neighbourhoods.*

Corollary 64 *Fix a space (X, σ) and let τ be a K–coarser topology. Then $C_{k_\tau}((X,\tau), \mathfrak{U}_\mathbb{Q})$ is a dense subspace of $C_{k_\tau}((X,\sigma), \mathfrak{U}_\mathbb{Q})$.*

We also demonstrated that if τ is a K–coarser topology on (X, σ) then $C_{k_\tau}((X,\sigma), \mathfrak{U}_\mathbb{Q})$ parametrises a continuous function universal for (X, σ). If we want this continuous function universal to be ccc, separable or have calibre ω_1 then it will suffice to show that $C_{k_\tau}((X,\tau), \mathfrak{U}_\mathbb{Q})$ has these properties. From now on we will write $C_k(X, \mathfrak{U}_\mathbb{Q})$ and $C_k(X)$ as there will only one topology considered on X.

Fix an arbitrary space X. We will investigate when $C_k(X, \mathfrak{U}_\mathbb{Q})$ is ccc, separable or has calibre ω_1.

Lemma 65 *Fix a space X. Then*

(i) $C_k(X, \mathfrak{U}_\mathbb{Q})$ is separable if and only if $C_k(X)$ is separable,

(ii) $C_k(X, \mathfrak{U}_\mathbb{Q})$ is ccc if and only $C_k(X)$ is ccc,

(iii) $C_k(X, \mathfrak{U}_\mathbb{Q})$ has calibre ω_1 if and only if $C_k(X)$ has calibre ω_1.

Proof. The identity map is a continuous function from $C_k(X, \mathfrak{U}_\mathbb{Q})$ onto $C_k(X)$ and so one implication is immediate in (i), (ii) and (iii).

Part (i): Assume that $C_k(X)$ is separable and so X has a coarser second countable topology τ. Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a basis for τ. Assume that $\mathcal{A}$ is closed under finite unions. For each $n, m \in \omega$ let $f_{n,m} : X \to \mathbb{R}$ be a τ–continuous function satisfying $f_{n,m}(x) = 1$ when $x \in \overline{A_m}$ and $f_{n,m}(x) = 0$ when $x \in X \setminus A_n$. Of course this is only well–defined when $\overline{A_m} \subset A_n$ and if this does not hold then we let $f_{n,m}(x) = 0$ for all $x \in X$. The linear span of $\{f_{n,m} : n, m \in \omega\}$ over $\mathbb{Q}$ is a countable set that is dense in $C_k(X, \mathfrak{U}_\mathbb{Q})$.

Part (iii): Assume that $C_k(X)$ has calibre ω_1. Fix an uncountable collection $\mathcal{W} = \{W_\alpha : \alpha \in \omega_1\}$ of basic open non–empty subsets of $C_k(X, \mathfrak{U}_\mathbb{Q})$. So we can assume that each W_α is of the form $B(\mathcal{C}_\alpha, \mathcal{U}_\alpha) \cap B(\mathcal{D}_\alpha, \mathcal{V}_\alpha)$ where for all $\alpha \in \omega_1$: $\mathcal{C}_\alpha = \{C_i^\alpha : i \leq n_\alpha\}$ consists of zero sets of X, $\mathcal{U}_\alpha = \{U_i^\alpha : i \leq n_\alpha\}$ where each $U_i \in \mathfrak{U}$, $\mathcal{D}_\alpha = \{D_j^\alpha : j \leq m_\alpha\}$ consists of pairwise disjoint zero subsets of X and $\mathcal{V}_\alpha = \{\{q_j^\alpha\} : j \leq m_\alpha\}$ where each $q_j^\alpha \in \mathbb{Q}$.

By passing to an uncountable subcollection we can assume that for all $\alpha, \beta \in \omega_1$ we have $\mathcal{U}_\alpha = \mathcal{U}_\beta$ and $\mathcal{V}_\alpha = \mathcal{V}_\beta$. We will drop the subscripts and use $\mathcal{U}$ and $\mathcal{V}$ to denote these sets. Assume that $|\mathcal{U}| = n$ and $|\mathcal{V}| = m$. We can write $\mathcal{V}$ as $\{\{q_j\} : j \leq m$ where each $q_j \in \mathbb{Q}$. Choose $\delta > 0$ such that $4\delta < \min\{|q_i - q_j| : i, j \leq m\}$. We define a new collection $\mathcal{V}'$ by defining for each $j \leq m$ the set $V'_j = (q_j - \delta, q_j + \delta)$ and letting $\mathcal{V}' = \{V'_j : j \leq m\}$. Note that for each $\alpha \in \omega_1$ we know that $B(\mathcal{C}_\alpha, \mathcal{U}_\alpha) \cap B(\mathcal{D}_\alpha, \mathcal{V}'_\alpha)$ is a non–empty subset of $C_k(X)$. So there is some $f \in C(X)$ and $A \subset \omega_1$ such that $|A| = \omega_1$ and $f \in \bigcap\{B(\mathcal{C}_\alpha, \mathcal{U}_\alpha) \cap B(\mathcal{D}_\alpha, \mathcal{V}'_\alpha) : \alpha \in A\}$.

We define two collections of zero sets $\mathcal{C}$ and $\mathcal{D}$ by defining for each $i \leq n$, $C_i = f^{-1}(\overline{U_i})$ and for each $j \leq m$ we define $D_j = f^{-1}(\overline{V'_i})$. Note that if $j, j' \leq m$ then $D_j \cap D_{j'} = \emptyset$ when $j \neq j'$. We can define a new function f' by setting $f'(x) = f(x)$ when $x \notin \bigcup \mathcal{D}$ and $f'(x) = q_j$ when $x \in D_j$. Now we have that $f' \in B'(\mathcal{C}, \mathcal{U}) \cap B'(\mathcal{D}, \mathcal{V})$ and so applying Theorem 18 we know that there exists $g \in B(\mathcal{C}, \mathcal{U}) \cap B(\mathcal{D}, \mathcal{V})$. But

$$B(\mathcal{C}, \mathcal{U}) \cap B(\mathcal{D}, \mathcal{V}) \subset \bigcap_{\alpha \in A} W_\alpha$$

and so we are done.

Part (ii) can be proved in much the same way as part (ii). ∎

Corollary 66 *Let (X, σ) be a Tychonoff space and let τ be a K–coarser topology.*

(i) If the space (X, τ) is second–countable then (X, σ) has a continuous function universal parametrised by a separable space.

(ii) If $C_k(X, \tau)$ is ccc then (X, σ) has a continuous function universal parametrised by a ccc space.

(iii) If the space $C_k(X, \tau)$ has calibre ω_1 then (X, σ) has a continuous function universal parametrised by a space with calibre ω_1.

We can also derive sufficient conditions in the ccc or calibre ω_1 cases that do not depend on the properties of an external object (such as $C_k(X, \tau)$) by using our characterisations of when $C_k(X)$ is ccc or has calibre ω_1.

These results are useful when dealing with spaces that are not locally compact, as in the locally compact case $C_k(X)$ itself will parametrise a continuous function universal for X.

Example 67 *The Sorgenfrey line has a continuous function universal parametrised by a separable space.*

Proof. The Euclidean topology on $\mathbb{R}$ is a second–countable K–coarser on the Sorgenfrey line. So using corollary 66 we get the required result. ■

4.3.2 Necessary conditions

We will deal first with the case where a space X has a continuous function universal parametrised by a separable space. We say a space (X, σ) is co-SM if and only if there is a separable metric topology $\tau \subset \sigma$ such that (X, σ) has a neighbourhood basis of τ–closed sets.

Lemma 68 *If X has a continuous function universal parametrised by a separable space then X is co-SM.*

Proof. Let Y be a separable metric space that parametrises a continuous function universal for X via the function $F : X \times Y \to \mathbb{R}$. Let D be a countable dense subset of Y. Each $d \in D$ represents the continuous function F^d. Let τ be the coarsest topology that makes each F^d continuous and note that τ is separable metric.

Fix x in open U. Pick $y \in Y$ so that $F(x, y) = 1$ and $F[(X \setminus U) \times \{y\}] = \{0\}$. By continuity of F at (x, y) there are open V and W with $x \in V$, $y \in W$ and $F[V \times W] \subseteq (\frac{2}{3}, \frac{4}{3})$.

Claim: If $x' \notin \overline{U}$ then there is a τ–open T containing x' disjoint from V.

From the claim it follows that $\overline{V}^{\tau} \subseteq \overline{U}$, and by regularity of X, the τ–closed neighbourhoods of x form a local base — as required for co–SM.

Well suppose $x' \notin \overline{U}$, then $F(x', y) = 0$. So by continuity of F at (x', y) there are open V' and W' with $x' \in V'$ and $y \in W'$ so that $F[V' \times W'] \subseteq (\frac{-1}{3}, \frac{1}{3})$.

Pick $d \in D \cap (W \cap W')$. Then $d \in W'$ so $F(x', d) \in (\frac{-1}{3}, \frac{1}{3})$. Hence by τ–continuity of F^d at x', there is a τ–open $T \ni x'$ such that $F[T \times \{d\}] \subseteq (\frac{-1}{3}, \frac{1}{3})$. Since $d \in W$, $F[V \times \{d\}] \subseteq (\frac{2}{3}, \frac{4}{3})$. Hence V and T are disjoint — as required. ■

Note that this falls short of the sufficient condition given previously leading to the following question.

Problem 69 *Is there a Tychonoff space X such that X is co-SM but X can have no continuous function universal parametrised by a separable space?*

Turning our attention to parametrising spaces which are ccc we introduce the following two properties.

Definition 70 *A space X has the property P_1 if and only if for every pair of disjoint compact subsets (K, L) there exists a pair of open sets $U(K, L), V(K, L)$ with $K \subset U(K, L)$, $L \subset V(K, L)$ and $\overline{U(K, L)} \cap \overline{V(K, L)} = \emptyset$ satisfying the following :*

for all collections $\{(K_\alpha, L_\alpha) : \alpha \in \omega_1\}$ of pairs of disjoint compact sets there exists α_1, α_2 such that

$$\overline{\bigcup_{i=1,2} U(K_{\alpha_i}, L_{\alpha_i})} \cap \overline{\bigcup_{i=1,2} V(K_{\alpha_i}, L_{\alpha_i})} = \emptyset.$$

Definition 71 *A space X has the property P_2 if and only if for every pair of disjoint compact subsets (K, L) there exists a pair of open sets $U(K, L), V(K, L)$ with $K \subset U(K, L)$, $L \subset V(K, L)$ and $\overline{U(K, L)} \cap \overline{V(K, L)} = \emptyset$ satisfying the following :*

for all collections $\{(K_\alpha, L_\alpha) : \alpha \in \omega_1\}$ of pairs of disjoint compact sets there exists α_1, α_2 such that

$$\bigcup_{i=1,2} K_{\alpha_i} \subset \bigcap_{i=1,2} U(K_{\alpha_i}, L_{\alpha_i})$$

and

$$\bigcup_{i=1,2} L_{\alpha_i} \subset \bigcap_{i=1,2} V(K_{\alpha_i}, L_{\alpha_i}).$$

Lemma 72 *Let X be a Tychonoff space. If X has a zero set universal parametrised by a ccc space then X has property P_1 and every compact subspace has property P_2.*

Proof. Let Y be ccc and assume that Y parametrises a zero set universal for X via the continuous function $F : X \times Y \to \mathbb{R}$. Let Z be the disjoint sum of ω many copies of Y and let Y_n denote the nth copy of Y that is a subspace of Z. Define a function $F' : X \times Y \to \mathbb{R}$ by letting $F'(x, z) = nF(x, z)$ when $z \in Y_n$. Finally let $G = |F'|$. Note that Z parametrises a zero set universal for X via G, that Z is ccc and that for any pair of disjoint compact sets $K, L \subset X$ there exists $z \in Z$ such that $G^z[K] = 0$ and $G^z[L] \subset [1, \infty)$. We say that such a z separates K and L.

We will first show that X has property P_2 on its compact subspaces. Fix a compact subspace C. Let K, L be disjoint compact subsets of C. We show how to construct the required $U(K,L)$ and $V(K,L)$. Since K and L are compact we can find $z(K,L) \in Z$ that separates K and L. Let $U(K,L) = \{x \in C : G(x, z(K,L)) < \frac{1}{4}\}$ and $V(K,L) = \{x \in C : G(x, z(K,L)) > \frac{3}{4}\}$. Find open $W(K,L)$ such that $z(K,L) \in W(K,L)$ and for all $(x, z_1), (x, z_2) \in C \times W(K,L)$ we have $|G(x,z_1) - G(x,z_2)| < \frac{1}{8}$.

Now take a collection $\{(K_\alpha, L_\alpha) : \alpha \in \omega_1\}$ of pairs of disjoint compact subsets of C. Look at the corresponding collection $\{W(K_\alpha, L_\alpha) : \alpha \in \omega_1\}$. Since Z is ccc there must be $z \in Z$ and $\alpha_1, \alpha_2 \in \omega_1$ such that $z \in W(K_{\alpha_1}, L_{\alpha_1}) \cap W(K_{\alpha_2}, L_{\alpha_2})$. We claim that

$$\bigcup_{i=1,2} K_{\alpha_i} \subset \bigcap_{i=1,2} U(K_{\alpha_i}, L_{\alpha_i})$$

and

$$\bigcup_{i=1,2} L_{\alpha_i} \subset \bigcap_{i=1,2} V(K_{\alpha_i}, L_{\alpha_i})$$

as required. We will only show that $K_{\alpha_1} \subset U(K_{\alpha_2}, L_{\alpha_2})$ as the other cases can be dealt with similarly. Fix $x \in K_{\alpha_1}$. Note that $G(x,z) < \frac{1}{8}$ since $G(x, z(K_{\alpha_1}, L_{\alpha_1})) = 0$ and $z \in W(K_{\alpha_1}, L_{\alpha_1})$. But then $G(x, z(K_{\alpha_2}, L_{\alpha_2})) < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$ and so $x \in U(K_{\alpha_2}, L_{\alpha_2})$.

Now we will show that X has property P_1. The proof is similar to the P_1 case and so we will only show how to construct $U(K,L)$ and $V(K,L)$. Let K, L be disjoint compact subsets of X. Find $z \in Z$ that separates K and L. Using the compactness of K and L and the continuity of G find open $U(K,L), V(K,L), W(K,L)$ such that $K \subset U(K,L)$, $L \subset V(K,L)$ and $z \in W(K,L)$ satisfying: for all $(x, z') \in U(K,L) \times W(K,L)$, $G(x,z') < \frac{1}{4}$ and for all $(x,z') \in V(K,L) \times W(K,L)$, $G(x,z') > \frac{1}{8}$.

■

The next lemma is very similar to Lemma 46.

Lemma 73 *Let X be a compact Hausdorff space. If X has property P_2 then X is metrisable.*

Proof. It suffices to find a countable T_1–separating collection of open subsets of X. Let $\mathcal{C} = \{(K_\alpha, L_\alpha) : \alpha \in I\}$ be a collection of disjoint pairs of compact subsets of X that satisfies $(*)$: for all $\alpha_1, \alpha_2 \in I$ either

$$\bigcup_{i=1,2} K_{\alpha_i} \not\subset \bigcap_{i=1,2} U(K_{\alpha_i}, L_{\alpha_i})$$

or

$$\bigcup_{i=1,2} L_{\alpha_i} \not\subset \bigcap_{i=1,2} V(K_{\alpha_i}, L_{\alpha_i}).$$

Assume that $\mathcal{S} = \{U(K_\alpha, L_\alpha) : \alpha \in I\} \cup \{V(K_\alpha, L_\alpha) : \alpha \in I\}$ is not a T_0 separating collection. We will show that we can find (K, L) such that $\mathcal{C} \cup \{(K, L)\}$ satisfies the same property $(*)$ as $\mathcal{C}$. Since $\mathcal{S}$ is not a T_1–separating collection there exists $x_1, x_2 \in X$ such that for all $C \in \mathcal{S}$ we have $x_1 \in C$ implies $x_2 \in C$. Let $K = \{x_1\}$ and let $L = \{x_2\}$. Fix $\alpha \in I$. If $x_1 \in U(K_\alpha, L_\alpha)$ and $x_2 \in V(K_\alpha, L_\alpha)$ then $x_2 \notin U(K_\alpha, L_\alpha)$, contradicting the choice of x_1, x_2. So condition $(*)$ holds for $\mathcal{C} \cup \{(K, L)\}$.

Now let $\mathcal{C}$ be a collection of disjoint pairs of compact subsets of X that is maximal with respect to $(*)$ (ie $\mathcal{C}$ satisfies $(*)$, but for any collection $\mathcal{D}$, if $\mathcal{C} \subsetneq \mathcal{D}$ then $\mathcal{D}$ does not have property $(*)$). Since X has P_2 we must have that $\mathcal{C}$ is countable. But $\mathcal{S}$ as described above must be a T_1–separating collection, and so we are done.

■

Problem 74 *Does the property P_1 imply the property P_2? If not is the property P_1 equivalent to metrisability in compact spaces?*

4.4 EXAMPLES

Our first example shows that the situation with regards to chain conditions in $C_p(X)$ and $C_k(X)$ can essentially be as bad as possible.

Example 75 *There is an example of a space X such that $C_p(X)$ does not have calibre $(\omega_1, \omega, \omega)$ and $C_k(X)$ is not ccc.*

Proof. We take $X = \alpha\aleph_1$, the one–point compactification of the discrete space of size $\aleph_1$. Let ∞ denote the extra point. As X is not even first–countable it cannot be metric and hence using Lemma 46 we know that $C_k(X)$ is not ccc. To see that $C_p(X)$ does not have calibre $(\omega_1, \omega, \omega)$, or equivalently X does not have property $\mathcal{P}(\omega_1, \omega, \omega)$ it suffices to note that ∞ is in the closure of any countable subset of $\aleph_1$. Then $\{\langle \infty, \alpha\rangle : \alpha \in \omega_1\}$ witnesses the failure of $\mathcal{P}(\omega_1, \omega, \omega)$. ■

Example 76 *There is a space X such that $C_p(X)$ does not have calibre ω_1 but does have calibre $(\omega_1, \omega_1, \omega)$. In addition $C_k(X)$ does not have calibre ω_1 but does have calibre $(\omega_1, \omega_1, \omega)$.*

Proof. The space in question is $L(\aleph_1)$, the one point Lindelofication of the discrete space of size $\aleph_1$. In other words $X = \{\infty\} \cup \aleph_1$, where each point in $\aleph_1$ is isolated and a typical open neighbourhood of ∞ is $\{\infty\} \cup (\aleph_1 \setminus C)$ where C is countable.

Note that the compact subsets of $L(\aleph_1)$ are simply the finite subsets and so $L(\aleph_1)$ has $\mathcal{P}(\kappa, \lambda, \mu)$ if and only if $L(\aleph_1)$ has $\mathcal{K}(\kappa, \lambda, \mu)$. The collection $\{\langle \infty, \alpha \rangle : \alpha \in \aleph_1\}$ shows that $L(\aleph_1)$ does not satisfy $\mathcal{P}(\omega_1, \omega_1, \omega_1)$. It takes a straightforward examination of the various cases to show that this space does satisfy $\mathcal{P}(\omega_1, \omega_1, \omega)$. ■

Example 77 *If we take $X = \omega_1$ with the order topology then $C_k(X)$ is not ccc although every compact subspace of ω_1 is metrisable.*

Proof. As every compact subset of ω_1 is countable it must be second–countable, and hence metrisable. But ω_1 is clearly ω–bounded and non–metrisable, and so using Lemma 47 we can see that ω_1 must fail to have property $\mathcal{K}(\omega_1, 2, 2)$. ■

Example 78 *There is a space X that has no continuous function universal parametrised by a separable space. However X will have a continuous function universal parametrised by a space with calibre ω_1. In fact we can ensure that $d(C_p(X))$ is a large as we want.*

Proof. Let X be the disjoint sum of $\mathfrak{c}^+$ many copies of the Sorgenfrey line. We know that X has no continuous function universal parametrised by a separable space as $C_p(X)$ is not even separable. But since X will have a K–coarser metric topology we can construct a continuous function universal parametrised by a space with calibre ω_1 (see [23]). By choosing κ sufficiently large and taking κ many copies of the Sorgenfrey line we can increase the density of $C_p(X)$ but this space will still have a continuous function universal parametrised by a space with calibre ω_1.

■

5.0 SEQUENTIAL DENSITY

The space $C_p(X)$ is rarely metric. It is easily seen that if X is uncountable then $C_p(X)$ fails to be even first countable. If X is countable then of course $C_p(X)$ embeds in $\mathbb{R}^\omega$. On the other hand $C_p(X)$ is separable if and only if X has a coarser second countable topology. This is a relatively broad class of spaces. In this section we consider weakenings of the property separable metric with the hope that $C_p(X)$ will satisfy these properties for a broader range of spaces than just the countable ones.

Definition 79 *A space X is* sequentially separable *if and only if there exists a countable $D \subset X$ such that for every $x \in X$ there exists a sequence $\langle x_n : n \in \omega \rangle$ converging to x such that $x_n \in D$ for each $n \in \omega$. We say that D is* sequentially dense *in X.*

This is a property that separable metric spaces certainly have, however the class of sequentially separable spaces is considerably broader. It includes all the cosmic spaces, those spaces that are the continuous image of a separable metric space. Although the idea of approximation of a function by functions from a fixed countable set is hardly new, as the study of Fourier series demonstrates, in this section we completely characterise those spaces X such that $C_p(X)$ is sequentially separable. We also investigate the following property in both $C_p(X)$ and $C_k(X)$.

Definition 80 *A space X is* strongly sequentially separable *if and only if X is separable and every countable dense subset of X is sequentially dense.*

Another closely related area is the study of products of (strongly) sequentially separable spaces. It a well known theorem due to Hewitt, Marczewski and Pondiczery that the product of κ many separable spaces is separable if and only if $\kappa \leq \mathfrak{c}$. For sequentially separable spaces the relevant cardinal seems to be $\mathfrak{q}$. A subset S of the real line is called a *Q–set* if each one of its subsets is a G_δ. The cardinal $\mathfrak{q}$ is the smallest cardinal so that for any $\kappa < \mathfrak{q}$ there is a Q–set of size κ. The cardinal $\mathfrak{p}$ is the smallest cardinal so that there is a collection of $\mathfrak{p}$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo–intersection.

The study of sequentially separable spaces was started by Tall [24] who showed that under MA+¬CH, the product of less than continuum many sequentially separable spaces is sequentially separable. Tall's result was extended to products of $< \mathfrak{p}$ sequentially separable spaces by Matveev [21], who also introduced the class of strongly sequentially separable spaces. Dow, Matveev and Nyikos showed that the Cantor cube 2^κ is strongly sequentially separable if and only if $\kappa < \mathfrak{p}$.

It is known that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{q} \leq 2^{\aleph_0}$, that it is consistent that $\mathfrak{p} < \mathfrak{q}$, and that the cofinality of $\mathfrak{q}$ is uncountable.. (See [28] for more on small cardinals including $\mathfrak{p}$.)

In the paper [11] Gartside shows the following.

Lemma 81 *Suppose $X = \prod_{\alpha \in \kappa} X_\alpha$ is sequentially separable. If each X_α is non–trivial, then 2^κ is sequentially separable and hence $\kappa < \mathfrak{q}$.*

One would hope to reverse Theorem 81 proving that the product of less than $\mathfrak{q}$ sequentially separable spaces is sequentially separable. We have not been able to do this however we do have a result if we assume that the spaces are in fact cosmic.

One other result from [11] due to Gartside is worth mentioning at this point.

Theorem 82 *Let X_α be non–trivial separable metrisable spaces for $\alpha \in \kappa$. Then $X = \prod_{\alpha \in \kappa} X_\alpha$ is strongly sequentially separable if and only if $\kappa < \mathfrak{p}$.*

However it seems unlikely that we could show that the product of even two arbitrary strongly sequentially separable spaces is strongly sequentially separable. Our investigations into the strongly sequentially separable property in $C_p(X)$ will provide at least a consistent example of two strongly sequentially separable spaces whose product is not strongly sequentially separable.

5.1 SEQUENTIALLY SEPARABLE PRODUCT SPACES

Here we demonstrate that the product of $< \mathfrak{q}$ cosmic spaces is sequentially separable. It may be true that this holds for products of sequentially separable spaces. When proving Theorem 83 we construct a countable sequentially dense set using the networks of the cosmic spaces. For this reason it seems unlikely that one could adapt this proof to the general, sequentially separable case.

Theorem 83 *The product of κ many non–trivial cosmic spaces is sequentially separable if and only if $\kappa < \mathfrak{q}$.*

Proof. The necessity of the condition $\kappa < \mathfrak{q}$ is immediate from Lemma 81. Let X be a set with $|X| < \mathfrak{q}$. For each $x \in X$ let (Y_x, σ_x) be cosmic. Let $\mathcal{N}_x$ be a countable network for (Y_x, σ_x). Define a new finer topology τ_x on Y_x by letting $\mathcal{N}_x$ be a base of closed and open sets. If we can construct some countable sequentially dense subset of $\prod_{x\in X}(Y_x, \tau_x)$ then we will be done since the identity $i : \prod_{x\in X}(Y_x, \tau_x) \to \prod_{x\in X}(Y_x, \sigma_x)$ is continuous.

(Y_x, τ_x) is second countable, T_2 and zero–dimensional and so it is homeomorphic to a subset of ω^ω. Then for each $x \in X$ the space (Y_x, τ_x) will certainly have a network

$$\mathcal{U}^x = \{Y^x_{i,j} : i \in \mathbb{N}, j = 1, \ldots, 2^i\}$$

such that for fixed i the collection $\{Y^x_{i,j} : j = 1, \ldots, 2^i\}$ partitions Y_x (ie $Y^x_{i,j_1} \cap Y^x_{i,j_2} = \emptyset$ when $j_1 \neq j_2$ and $\bigcup\{Y^x_{i,j} : j \leq 2^i\} = Y^x$). In case Y_x has isolated points we allow some of the $Y^x_{i,j}$'s to be empty. We also require that $Y^x_{i,j} = Y^x_{i+1,2j-1} \cup Y^x_{i+1,2j}$ so that as i increases we get finer partitions of Y^x. For each $x \in X$ choose an arbitrary point of Y_x and label this 0_x. For each non-empty $Y^x_{i,j}$ choose a $y^x_{i,j} \in Y^x_{i,j}$. $|X| < \mathfrak{q}$ and so X has a (†)basis $\mathcal{B} = \{B_n : n \in \omega\}$. Assume that $X \in \mathcal{B}$, $\mathcal{B}$ is closed under finite unions and finite intersections and $B_n \setminus B_m \in \mathcal{B}$ when $n \neq m$. We define a countable set $F \subset \prod_{x\in X}(Y_x, \tau_x)$ by insisting that $f \in F$ if and only if there exists some collection of pairwise disjoint sets, $\{B_{n_s} : s = 0, \ldots, m\}$, from $\mathcal{B}$ and some collection of indices $\{(i_s, j_s) : s = 0, \ldots, m\}$ such that $f(x) = y^x_{i_s,j_s}$ for all $x \in B_{n_s}$, and $f(x) = 0_x$ otherwise. We claim that F is sequentially dense in $\prod_{x\in X} Y_x$.

Fix $f \in \prod_{x \in X} Y_x$. For all i, j we define $X(i,j) = \{x \in X : f(x) \in Y^x_{i,j}\}$. Note that $\{X(i,j) : i \in \mathbb{N}, j \leq 2^i\}$ has the same partitioning properties as each $\{Y^x_{i,j} : i \in \mathbb{N}, j \leq 2^i\}$. For all of the sets above we have some sequence $\{B^k(i,j) : k \in \omega\}$ from $\mathcal{B}$ that (†)converges to $X(i,j)$. (If $X(i,j) = \emptyset$ then let each $B^k(i,j) = \emptyset$). Now we will inductively construct sequences $\{C^k(i,j) : k \in \omega\}$ that (†)converge to $X(i,j)$, such that for all $i \in \mathbb{N}$ we have $C^k(i,j) \cap C^k(i,j') = \emptyset$ when $j \neq j'$, and $C^k(i+1, 2j-1) \cup C^k(i+1, 2j) \subset C^k(i,j)$ when $j \leq 2^i$. Let $C^k(1,1) = B^k(1,1)$ and let $C^k(1,2) = B^k(1,2) \cap (X \setminus C^k(1,1))$ for all $k \in \omega$. $\{C^k(1,2) : k \in \omega\}$ (†)converges to $X(1,2)$ since $\{X \setminus B^k(1,1) : k \in \omega\}$ (†)converges to $X \setminus X(1,1) = X(1,2)$. Also note that $C^k(1,1) \cap C^k(1,2) = \emptyset$ for all $k \in \omega$. For arbitrary $i \in \mathbb{N}$ and $j \leq 2^i$ we define

$$\begin{aligned} C^k(i+1, 2j-1) &= B^k(i+1, 2j-1) \cap C^k(i,j), \\ C^k(i+1, 2j) &= B^k(i+1, 2j) \cap C^k(i,j) \cap (X \setminus C^k(i+1, 2j-1)). \end{aligned}$$

The fact that $\{C^k(i+1, 2j-1)\}$ is (†)convergent to $X(i+1, 2j-1)$, and $\{C^k(i+1, 2j)\}$ is (†)convergent to $X(i+1, 2j)$ follows from our inductive hypothesis and the fact that $X(i,j) = X(i+1, 2j-1) \cup X(i+1, 2j)$. It is also clear that $C^k(i+1, 2j-1) \cup C^k(i+1, 2j) \subset C^k(i,j)$. Having constructed all the C^k's at the $i+1$ level we can see that they are pairwise disjoint as $C^k(i+1, 2j-1) \cap C^k(i+1, 2j) = \emptyset$ and we have assumed that the C^k's are pairwise disjoint at the i level. Note that each $C^k(i,j)$ is in $\mathcal{B}$.

Now we construct a sequence of functions $\{f_k : k \in \omega\}$ from F, converging to f. Define f_k by setting $f_k(x) = 0_x$ if $x \notin C^k(i,j)$ for all $i \leq k$, all $j \leq 2^i$. Otherwise we let $f_k(x) = y^x_{i',j'}$ where $i' = \max\{i \leq k : x \in C^k(i,j) \text{ for some } j \leq 2^i\}$ and $x \in C^k(i',j')$.

Finally we show that this sequence does in fact converge to f. Let $x \in X$. Let U be an open subset of Y_x such that $f(x) \in U$. Then $f(x) \in Y^x_{i,j} \subset U$ for some i, j. There is some N such that $x \in C^k(i,j)$ for all $k > N$. Then for all $k > \max\{N, i\}$, we know that $f_k(x) = y_{r,s}$ for some r, s with $r \geq i$. Also $x \in C^k(r,s) \subset C^k(i,j)$ and so $Y^x_{r,s} \subset Y^x_{i,j}$. This shows that $f_k(x) \in Y^x_{i,j}$. ■

Problem 84 *Is it true that the product of less than* $\mathfrak{q}$ *many sequentially separable spaces is sequentially separable?*

5.2 SEQUENTIAL DENSITY OF $C_P(X)$

Necessary and sufficient conditions for $C_p(X)$ to be sequentially separable are given. These depend on *property* Γ which is variation on the property γ to be introduced in the context of $C_p(X)$ strongly sequentially separable.

A collection $\mathcal{C}$ of subsets of a space (X, τ) has property Γ on (X, τ) if and only if given any finite collection of disjoint cozero subsets $\{O_i\}_{i=1}^n$ there exist sequences $\{C_i^j : j \in \omega\}_{i=1}^n$, from $\mathcal{C}$ such that $C_{i'}^j \cap C_i^j = \emptyset$ when $i \neq i'$ and $O_i \subset \liminf C_i^j$.

Theorem 85 *$C_p((X, \tau))$ is sequentially separable if and only if there exists a coarser second countable topology μ for X and there exists some collection of μ–closed sets $\mathcal{C} = \{C_i : i \in \omega\}$ such that $\mathcal{C}$ has property Γ on (X, τ).*

Proof. Assume that $C_p((X, \tau))$ is sequentially separable, with F a countable sequentially dense subset. We know that X has a coarser second countable topology, μ, with basis obtained by taking all inverse images under elements of F of rational intervals in $\mathbb{R}$. Let $\{O_i\}_{i=1}^n$ be a disjoint collection of τ–cozero sets. Let f be a τ–continuous positive function satisfying: $i - 1 < f(x) < i$ for all $x \in O_i$ when $i = 1, \ldots, n$. There is a sequence $\{f_j : j \in \omega\}$ from F such that $f_j \to f$. Define $C_i^j = f_j^{-1}([i - 1 + \frac{1}{j+2}, i - \frac{1}{j+2}])$. We can see that $O_i \subset \liminf C_i^j$ and that $C_{i'}^j \cap C_i^j = \emptyset$ when $i \neq i'$. We can now clearly construct the required countable collection, $\mathcal{C}$ of μ–closed sets with property Γ.

Assume that there exists a coarser second countable topology μ for X and there exists some collection of μ–closed sets $\mathcal{C} = \{C_i : i \in \omega\}$ such that $\mathcal{C}$ has property Γ on (X, τ). For convenience we will assume that for all $f \in C_p((X, \tau))$ we have that $f(X) \subset (0, 1)$. Let F' be a countable subset of $C_p(X)$ such that given any finite collection $\{C_1, \ldots, C_n\}$ of pairwise disjoint sets from $\mathcal{C}$ and rationals $\{r_1, \ldots, r_n\}$ there is some $f \in F'$ such that $f(x) = r_i$ when $x \in C_i$ and $\min\{r_1, \ldots, r_n\} \leq f(x) \leq \max\{r_1, \ldots, r_n\}$. Also assume that given $f, g \in F'$ then the functions $\max\{f, g\}$ and $\min\{f, g\}$ are also in F', where $\max\{f, g\}(x) = \max\{f(x), g(x)\}$ for all $x \in X$ and $\min\{f, g\}(x) = \min\{f(x), g(x)\}$ for all $x \in X$. Finally assume that F' is closed under finite rational linear combinations. Now it will suffice to prove that $F = F' \cap C_p(X, (0, 1))$ is sequentially dense in $C_p(X, (0, 1))$. Fix

$f \in C_p(X,(0,1)$ and an integer $p > 1$. We claim that there exists some sequence $\{f_i^p : i \in \omega\}$ from F that converges to f except at $f^{-1}(A_p)$ where $A_p = \{\frac{i}{p^j} : j \in \mathbb{N}, i = 1, \ldots, p^j - 1\}$.

For all $j \in \mathbb{N}$ and $i = 1, \ldots, p^j - 1$ define $O_{j,i} = f^{-1}((\frac{i}{p^j}, \frac{i+1}{p^j}))$. By assumption, for fixed j we have sequences $\{B_{j,i}^k : k \in \omega\}$ from $\mathcal{C}$ such that $B_{j,i'}^k \cap B_{j,i}^k = \emptyset$ when $i \neq i'$ and $O_{j,i} \subset \liminf B_{j,i}^k$. Let $C_{1,i}^k = B_{1,i}^k$ for relevant i. For each $i \leq p^{j+1}$ define $C_{j+1,i}^k = B_{j+1,i}^k \cap C_{j,i'}^k$ when $O_{j+1,i} \subset O_{j,i'}$ (which can clearly only happen for one such i'). Define

$$f_k^p = \frac{1}{2p^k} + \sum_{j \leq k} \sum_{i < p^j} g_{j,i}^k$$

where $g_{j,i}^k(x) = \frac{i \bmod p}{p^j}$ for $x \in C_{j,i}^k$ and $g_{j,i}^k(x) \leq \frac{p-1}{p^j}$ for all $x \in X$. Note that for all $k \in \omega$ we have $f_k^p \in F$. Then $\{f_k^p : k \in \omega\}$ converges to f except at $f^{-1}(A_p)$. To prove convergence: fix $x \in X \setminus f^{-1}(A_p)$ and $\epsilon > 0$. There are i_1, j_1 such that $f(x) \in (\frac{i_1}{p^{j_1}}, \frac{i_1+1}{p^{j_1}}) \subset (f(x) - \epsilon, f(x) + \epsilon)$. There is some $N > 0$ such that $x \in C_{j_1,i_1}^k$ for all $k > N$. So it suffices to show that if $x \in C_{j_1,i_1}^k$ and $k > j_1$ then $f_k^p(x) \in (\frac{i_1}{p^{j_1}}, \frac{i_1+1}{p^{j_1}})$. To do this we split the sum $\sum_{j \leq k} \sum_{i < 2^j} g_{j,i}^k$ into two bits. First we claim that $\sum_{j \leq j_1} \sum_{i < 2^j} g_{j,i}^k = \frac{i_1}{p^{j_1}}$. We show this by induction on j_1. This is clearly true when $j_1 = 1$. If $x \in C_{j_1+1,i_1}^k$ and $x \in C_{j_1,i_2}^k$ then

$$\sum_{j \leq j_1+1} \sum_{i < 2^j} g_{j,i}^k = \frac{i_2}{p^{j_1}} + \frac{i_1 \bmod p}{p^{j_1+1}} = \frac{pi_2 + i_1 \bmod p}{p^{j_1+1}}.$$

But by the nested construction of the $C_{j,i}^k$'s we must have $pi_2 + i_1 \bmod p = i_1$. Now we also have

$$\begin{aligned} \sum_{j_1 < j \leq k} \sum_{i < 2^j} g_{j,i}^k &\leq \sum_{j_1 < j \leq k} \sum_{i < 2^j} \frac{p-1}{p^j} \\ &= \frac{1}{p^{j_1}} - \frac{1}{p^k}. \end{aligned}$$

Finally we get that

$$f_k^p(x) = \sum_{j \leq k} \sum_{i < 2^j} g_{j,i}^k + \frac{1}{2p^k} > \frac{i_1}{p^{j_1}}$$

and

$$\begin{aligned} \sum_{j \leq k} \sum_{i < 2^j} g_{j,i}^k + \frac{1}{2p^k} &\leq \frac{i_1}{p^{j_1}} + \frac{1}{p^{j_1}} - \frac{1}{p^k} + \frac{1}{2p^k} \\ &= \frac{i_1+1}{p^{j_1}} - \frac{1}{2p^k} \end{aligned}$$

and so $f_k^p(x) \in (\frac{i_1}{p^{j_1}}, \frac{i_1+1}{p^{j_1}})$.

Now we can construct $\{f_k : k \in \omega\}$ converging to f on all of X. We have $\{f_k^2 : k \in \omega\}$ converging except on $f^{-1}(A_2)$ and $\{f_k^3 : k \in \omega\}$ converging except on $f^{-1}(A_3)$. Let $\{f_k^{2,3} : k \in \omega\}$ be defined as $f_k^{2,3}(x) = \max\{f_k^2(x), f_k^3(x)\}$ for all $k \in \omega$. This clearly converges on $X \setminus (f^{-1}(A_2) \cup f^{-1}(A_3))$. Define $\{f_k^{5,7} : k \in \omega\}$ in the same way. Now let $\{f_k : k \in \omega\}$ be defined as $f_k(x) = \min\{f_k^{2,3}(x), f_k^{5,7}(x)\}$. Then $\{f_k : k \in \omega\}$ converges to f. Assume that $\{f_k(x) : k \in \omega\}$ does not converge for some $x \in (f^{-1}(A_2) \cup f^{-1}(A_3) \cup f^{-1}(A_5) \cup f^{-1}(A_7))$. Without loss of generality assume $x \in f^{-1}(A_2)$. So there is some $\epsilon > 0$ such that for all $N > 0$ there exists $k > N$ with $f_k(x) \notin (f(x) - \epsilon, f(x) + \epsilon)$. But $\{f_k^{5,7}(x) : k \in \omega\}$ converges to $f(x)$ so there is some $N_1 > 0$ such that for all $N > N_1$ there is some $k > N$ with $f_k(x) = f_k^{2,3}(x) \leq f(x) - \epsilon$. Using the fact that $\{f_k^3(x) : k \in \omega\}$ converges to $f(x)$ and the construction of the $f_k^{2,3}$'s we will get some $N_2 > 0$ such that for all $N > N_2$ there will be $k > N_2$ with $f_k^2(x) = f_k^{2,3}(x) \leq f(x) - \epsilon$ and $f_k^3(x) \in (f(x) - \epsilon, f(x) + \epsilon)$. This contradicts our definition of $f_k^{2,3}$. So $\{f_k : k \in \omega\}$ converges on all of X. ■

5.3 STRONG SEQUENTIAL SEPARABILITY OF $C_P(X)$

Before looking at the strong sequential separability of $C_p(X)$ we review the case for the Frechet–Urysohn property. Recall that a *Frechet–Urysohn* space is a space X such that whenever $x \in \overline{A}$, there is a sequence in A converging to x.

Definition 86 *A family α of subsets of X is called an ω–*cover *of X if for every finite $F \subset X$ there is a $U \in \alpha$ such that $F \subset U$.*

Theorem 87 (Gerlits & Nagy) *The following are equivalent:*

(i) $C_p(X)$ is Frechet–Urysohn;

(ii) X has the property γ: for any open ω–cover α of X there is a sequence $\beta \subset \alpha$ such that $\liminf \beta = X$.

For a proof of Theorem 87 see [15], and more information on the property γ see [15, 9]. This gives us immediately that if X has the property γ and has a coarser second–countable topology then $C_p(X)$ is strongly sequentially separable, as separable, Frechet Urysohn spaces are strongly sequentially separable. We also have the following.

Theorem 88 *If* $|X| < \mathfrak{p}$ *then* $C_p(X)$ *is strongly sequentially separable.*

Proof. Using Theorem 82 we know that $\mathbb{R}^X$ is strongly sequentially separable. Since $C_p(X)$ is a dense subspace of $\mathbb{R}^X$ we know that $C_p(X)$ must be strongly sequentially separable. ∎

Now we completely characterize those spaces X so that $C_p(X)$ is strongly sequentially separable, starting with the special case of X separable metric. The proof of the following result is similar to the proof of Theorem 87.

Theorem 89 *Let* X *be* T_3 *and second countable. Then* $C_p(X)$ *is strongly sequentially separable if and only if* $C_p(X)$ *is Frechet–Urysohn.*

Proof. Assume $C_p(X)$ is Frechet–Urysohn. X is second countable so $C_p(X)$ is separable. $C_p(X)$ is Frechet–Urysohn and separable and so $C_p(x)$ is strongly sequentially separable.

Now, to show that $C_p(X)$ strongly sequentially separable implies $C_p(X)$ is Frechet–Urysohn we show that $C_p(X)$ is strongly sequentially separable implies X has the property γ described in Theorem 87. Let α be an open ω–cover of X. Let $A = \{f \in C_p(X) : \overline{f^{-1}(\mathbb{R} \setminus \{0\})} \subset U \text{ for some } U \in \alpha\}$. Let $\{B_n : n \in \omega\}$ be a countable base for X. If $\overline{B_n} \subset B_m$ then let $f_n^m : X \to [0, 1]$ be a continuous function such that $f_n^m(\overline{B_n}) = 1$ and $f_n^m(X \setminus B_m) = 0$. Let B be the linear span over $\mathbb{Q}$ of all such functions. Note that B is countable. We will show that $A \cap B$ is dense in $C_p(X)$. Let $B(g, x_1, \ldots, x_n; \epsilon)$ be an arbitrary open set in $C_p(X)$. There is some $U \in \alpha$ such that $\{x_1, \ldots, x_n\} \subset U$. For each $i = 1, \ldots, n$ there are basic open V_i, W_i with $\overline{V_i} \subset W_i \subset \overline{W_i} \subset U$ and $W_i \cap W_j = \emptyset$ when $i \neq j$. We know that for each pair V_i, W_i there is some $f_i \in B$ with $f_i(\overline{V_i}) = 1$ and $f_i(X \setminus W_i) = 0$. Let $q_i \in \mathbb{Q}$ satisfy $|g(x_i) - q_i| < \epsilon$. Then

$$h = \sum_{i=1}^{n} q_i f_i \in B(g, x_1, \ldots, x_n; \epsilon).$$

Also if x is not in any W_i then $h(x) = 0$ and so $h^{-1}(\mathbb{R} \setminus \{0\}) \subset \bigcup\{W_i : i = 1, \ldots, n\}$. This shows that $\overline{h^{-1}(\mathbb{R} \setminus \{0\})} \subset U$ and so $h \in A \cap B$. Now $A \cap B$ is countable and dense in $C_p(X)$. Let f^1 denote the constant function at 1. Since $C_p(X)$ is strongly sequentially separable then there is some sequence $\{f_n : n \in \omega\}$ in $A \cap B$ (and so in A) converging to f^1. For each $n \in \omega$

we take a $U_n \in \alpha$ for which $\overline{f_n^{-1}(\mathbb{R} \setminus \{0\})} \subset U_n$. Then lim inf $\{U_n : n \in \omega\} = X$. To see this note that for any $x \in X$ we have a $n_x \in \omega$ such that $f_n \in B(f^1, x; 1)$ for all $n > n_x$. But this means that $f_n(x) > 0$ and so $x \in U_n$ for all $n > n_x$. ■

Todorcevic, in [9], has shown that, consistently, there are two subsets of the reals, X and Y say, with the γ property such that their disjoint sum $X \oplus Y$ does not have the γ property. Since $X \oplus Y$ is T_3 and second countable, and $C_p(X \oplus Y) = C_p(X) \times C_p(Y)$, we have the following example:

Example 90 (Cons(ZFC)) *There are separable metric spaces X and Y so that the topological groups $C_p(X)$ and $C_p(Y)$ are strongly sequentially separable but their product, $C_p(X) \times C_p(Y)$ is not strongly sequentially separable.*

We now remove the restriction that X be second countable.

Theorem 91 *The function space $C_p(X)$ is strongly sequentially separable if and only if X has a coarser second countable topology, and every coarser second countable topology for X has the property γ.*

Proof. Assume that $C_p(X)$ is strongly sequentially separable. We know that X has a coarser second countable topology. If τ is an arbitrary coarser second countable topology for X then we know that $C_p((X, \tau))$ embeds densely into $C_p(X)$. Also $C_p((X, \tau))$ is separable and so $C_p((X, \tau))$ is strongly sequentially separable. So Theorem 87 and Theorem 89 imply that (X, τ) has property γ.

Assume that X has a coarser second countable topology. Then $C_p(X)$ is separable. Let A be a countable dense subset of $C_p(X)$. We wish to show that for any $f \in C_p(X)$ there is some sequence $\{f_i : i \in \omega\} \subset A$ such that f is the limit of the sequence. Let $\{U_j : j \in \omega\}$ be a base for $\mathbb{R}$ and let $\{V_k : k \in \omega\}$ be the collection of all preimages of the U_j's under the functions in $A \cup \{f\}$. This collection is a base for a T_3 topology τ, on X (given $V_k = g^{-1}(U_j)$ we know that there is a $U_{j'}$ with $\overline{U_{j'}} \subset U_j$ and so $\overline{g^{-1}(U_{j'})} \subset U_j$) and clearly each function in $A \cup \{f\}$ is continuous with respect to τ. We know that $C_p(X, \tau)$ is Frechet–Urysohn and so since $f \in \overline{A}$ then we know that f is the limit of some sequence $\{f_i : i \in \omega\} \subset A$. ■

We now ask the following question.

Problem 92 *Is it possible to find a space X such that $C_p(X)$ is strongly sequentially sepa-*

rable but $C_p(X)^2$ is not strongly sequentially separable?

In other words we would need a space X such that every coarser second–countable topology on X has property γ but there are two such topologies τ_1 and τ_2 such that $(X, \tau_1) \oplus (X, \tau_2)$ fails to have the property γ.

It is consistent and independent for arbitrary X, that $C_p(X)$ is strongly sequentially separable if and only if $C_p(X)$ is Frechet–Urysohn. In fact it is consistent with ZFC that:

Corollary 93 (Cons(ZFC)) *The following are equivalent:*

(i) $C_p(X)$ is strongly sequentially separable,

(ii) X is countable,

(iii) $C_p(X)$ is separable and Frechet–Urysohn.

Proof. A space X has the property C'' if for any sequence $\{G_n : n \in \omega\}$ of open covers of X there is some $U_n \in G_n$ for all $n \in \omega$ such that $\bigcup\{U_n : n \in \omega\} = X$. If X has property γ then it has property C''. It is consistent with ZFC that the only subsets of $\mathbb{R}$ with C'' are countable. If a space X has a coarser second countable topology τ with (X, τ) having γ then $\operatorname{ind}(X, \tau) = 0$ and (X, τ) has C'', which implies (X, τ) is homeomorphic to a subset of $\mathbb{R}$ with property C''. Hence the corollary. ■

Example 94 *Assume that $\omega_1 < \mathfrak{p}$. There is a space X such that $C_p(X)$ is strongly sequentially separable but not Frechet–Urysohn.*

Proof. We simply take X to be ω_1 with the discrete topology. This has a coarser second–countable topology and so $C_p(X)$ is a separable, dense subspace of $\mathbb{R}^X$. We know from Theorem 82 that $\mathbb{R}^X$ must be strongly sequentially separable and so $C_p(X)$ is strongly sequentially separable. However $\mathbb{R}^{\omega_1}$ is not Frechet–Urysohn. ■

This example leads to the following question.

Problem 95 *Is there a consistent example of a space X, such that $C_p(X)$ is strongly sequentially separable, not Frechet–Urysohn and $\mathbb{R}^X$ is not strongly sequentially separable?*

The argument of Example 94 shows that if $\kappa < \mathfrak{p}$ then any subset of $\mathbb{R}$ of size κ has the property γ. The converse is also true (folklore) which rules out finding a solution to

Problem 95 by forcing a model of ZFC in which all $\aleph_1$ sized subsets of $\mathbb{R}$ have the γ property but $\omega_1 = \mathfrak{p}$ (so $\mathbb{R}^{\omega_1}$ is not strongly sequentially separable.)

Lemma 96 *Every subset of $\mathbb{R}$ of size κ has property γ if and only if $\kappa < \mathfrak{p}$.*

5.4 STRONG SEQUENTIAL SEPARABILITY OF $C_K(X)$

The characterisation of when $C_k(X)$ is strongly sequentially separable seems very similar to the $C_p(X)$ case. However, in contrast to the situation with regards to $C_p(X)$ we can expect $C_k(X)$ to be strongly sequentially separable for a reasonably broad class of spaces. In particular we know that $C_k(X)$ is separable metric, and hence strongly sequentially separable when X is a hemi–compact metric space.

We begin with a review of the case for the Frechet Urysohn property. See [22] for more details and proofs of the following.

Definition 97 *A family α of subsets of X is called an K–*cover *of X if for every compact $K \subset X$ there is a $U \in \alpha$ such that $K \subset U$.*

Theorem 98 *The following are equivalent: (i) $C_k(X)$ is Frechet–Urysohn; (ii) X has the property γ_K: for any open K–cover α of X there is a sequence $\{U_i : i \in \omega\}$ from α such that for all compact $K \subset X$ there exists $n_K \in \omega$ with $K \subset U_i$ for all $i > n_K$.*

As with the $C_p(X)$ case we start by assuming that X is second–countable.

Theorem 99 *Let X be T_3 and second countable. Then $C_k(X)$ is strongly sequentially separable if and only if $C_k(X)$ is Frechet–Urysohn.*

Proof. Assume $C_k(X)$ is Frechet–Urysohn. X is second countable so $C_k(X)$ is separable. $C_k(X)$ is Frechet–Urysohn and separable and so $C_k(x)$ is strongly sequentially separable.

Now we assume that $C_k(X)$ is strongly sequentially separable and show that X has the property γ_K. Let α be an open ω_K–cover of X. Let A and B be defined as in the proof of 89. We will show that $A \cap B$ is dense in $C_k(X)$. Fix $V = W(\mathcal{K}, \mathcal{U})$, a non–empty basic open subset of $C_k(X)$. So we can assume that $\mathcal{K} = \langle K_0, \ldots, K_n \rangle$ where each K_i is compact and $\mathcal{K}$ has linear type t. In addition we can assume that $\mathcal{U} = \langle U_0, \ldots, U_n \rangle$ where each U_i is an interval in $\mathbb{R}$ with rational endpoints. Let r_i denote the midpoint of the interval U_i. As $\bigcup \mathcal{K}$ is compact we know that there is $U \in \alpha$ with $K \subset U$. We will recursively define for each $i \leq n$ a continuous function $f_i \in A \cap B \cap W(\langle K_0, \ldots, K_i \rangle, \langle U_0, \ldots, U_i \rangle)$ and then f_n will be an element of $A \cap B \cap V$.

For each K_i find basic open V_i, W_i such that $K_i \subset V_i \subset \overline{V_i} \subset W_i \subset \overline{W_i} \subset U$. In addition assume that $\overline{W_i} \cap \overline{W_j} = \emptyset$ if and only if $K_i \cap K_j = \emptyset$. Let f_0 satisfy: (i) $f_0(x) = r_0$ for all $x \in \overline{V_0}$ (ii) $f_0(x) = 0$ for all $x \in X \setminus W_0$, (iii) $0 \leq f_0(x) \leq r_i$ for all $x \in X$. Then $f_0 \in W(\langle K_0 \rangle, \langle U_0 \rangle)$. As $f_0^{-1}(\mathbb{R} \setminus \{0\}) \subset W_0$ and $\overline{W_0} \subset U$ then $f_0 \in A$. We have that $f_0 \in B$ from the construction of f_0.

Now assume that there is some $i < n$ such that we have defined f_j for all $j \leq i$. We show how to construct f_{i+1}. Let g_{i+1} satisfy: (i) $g_{i+1}(x) = 1$ for all $x \in \overline{V_{i+1}}$ (ii) $g_{i+1}(x) = 0$ for all $x \in X \setminus W_{i+1}$, (iii) $0 \leq g_{i+1}(x) \leq 1$ for all $x \in X$. Now we define $f_{i+1}(x) = f_i(x) - f_i(x) g_{i+1}(x) + r_{i+1} g_{i+1}(x)$. As both f_i and g_{i+1} are in B then so is f_{i+1}. As in the proof of Lemma 6 we can see that $f_{i+1} \in W(\langle K_0, \ldots, K_{i+1} \rangle, \langle U_0, \ldots, U_{i+1} \rangle)$. Finally $x \in f_{i+1}^{-1}(\mathbb{R} \setminus \{0\}$ implies that there exist $j \leq i+1$ such that $x \in W_j$ and so $f_{i+1} \in A$.

Now $A \cap B$ is countable and dense in $C_k(X)$. Let f^1 denote the constant function at 1. Since $C_k(X)$ is strongly sequentially separable then there is some sequence $\{f_n : n \in \omega\}$ in $A \cap B$ converging to f^1. For each $n \in \omega$ we take a $U_n \in \alpha$ for which $\overline{f_n^{-1}(\mathbb{R} \setminus \{0\})} \subset U_n$. Fix compact $K \subset X$. Look at the open set $W(\langle K \rangle, \langle (0,2) \rangle)$. There is some n_K such that $f_j \in W(\langle K \rangle, \langle (0,2) \rangle)$ for all $j > n_K$. But then $K \subset U_j$ for each $j > n_K$. ■

As $C_k(X)$ will be Frechet–Urysohn for every compact X it is easy to find a space X such that $C_k(X)$ is Frechet–Urysohn but not strongly sequentially separable. Any compact space of uncountable weight will suffice.

We now remove the restriction that X be second countable. We give no proof for this theorem as it can be proven just as Theorem 91 is proven.

Theorem 100 *The function space $C_k(X)$ is strongly sequentially separable if and only if X has a coarser second countable topology ,and every coarser second countable topology for X has the property γ_K.*

Our example for the C_p section also works in the $C_k(X)$ case as for any discrete space

X we know that $C_p(X) = C_k(X) = \mathbb{R}^X$.

Example 101 *Assume that* $\omega_1 < \mathfrak{p}$. *Then if we let* X *be the discrete space of size* $\aleph_1$ *we have that* $C_k(X)$ *is strongly sequentially separable but not Frechet–Urysohn.*

We now ask the following questions.

Problem 102 *Is it possible to find spaces* X, Y *such that* $C_k(X)$*and* $C_k(Y)$ *are strongly sequentially separable but* $C_k(X) \times C_k(Y)$ *is not strongly sequentially separable?*

Problem 103 *Is it possible to find a space* X *such that* $C_k(X)$ *is strongly sequentially separable but* $C_k(X)^2$ *is not strongly sequentially separable?*

Problem 104 *Is it possible in ZFC to find a space* X *such that* $C_k(X)$ *is strongly sequentially separable but* $C_k(X)$ *is not Frechet–Urysohn?*

6.0 CONCLUSION

In addition to all the unsolved problems from the various chapters we would like to mention the following problem.

Problem 105 *For which classes of spaces can we develop general techniques for creating universals?*

In the class of spaces with a K–coarser topology we have shown that such a technique exists. If we wish to understand universals we will need more methods of creating universals.

6.1 OPEN PROBLEMS

6.1.1 Compactness properties

Problem 106 *Characterise the spaces with a zero–set universal parametrised by a Lindelof Σ space.*

Problem 107 *If a Tychonoff space X has a continuous function universal parametrised by a Lindelof–Σ space then is X metrisable?*

Problem 108 *If a space X has a zero–set universal parametrised by a product of a compact and a second countable space, then is X metrisable? If X has an open regular F_σ universal parametrised by a product of a compact and a second countable space, then is X metrisable?*

6.1.2 Chain conditions

Question 109 *For which types t can we find a compact non–metrisable space X such that X has the property $\mathcal{K}(\omega_1, 2, 2, t)$?*

Question 110 *For which compactness type properties $\mathcal{P}$ can we show that every space X with property $\mathcal{P}$ such that $C_k(X)$ is ccc must be metrisable? Is it true for countable compactness? Lindelof–Σ spaces?*

Problem 111 *Does the property P_1 imply the property P_2? If not is the property P_1 equivalent to metrisability in compact spaces?*

Surely the most interesting unsolved problem from this section is the following.

Problem 112 *Is there a consistent example of a space X such that X has $\mathcal{K}(\omega_1, 2, 2)$ but X^2 does not? If not, is there a consistent example of a space X such that X has $\mathcal{K}(\omega_1, 2, 2)$ but $X \oplus X$ does not?*

Note that the ccc in $C_k(X)$ really only depends on the partial order of compact subsets of X. To be a bit more precise we can define for every type t and space X the poset P_t with conditions $K = \langle K_0, \dots, K_n \rangle$ where each $K_i \subset X$ is compact. The order $<_t$ can be defined by $K <_t L$ if and only if $K_i \supset L_i$ for all $i \leq n$. Once we know the structure of each of these posets we can decide if $C_k(X)$ has the ccc. To create a space X where $C_k(X)$ is ccc but $C_k(X)^2$ is not ccc we will need to ensure that each P_t has some structure that allows this to happen. All this motivates the following question.

Problem 113 *Given a poset P can we find a space X such that for a given type t the poset P_t is isomorphic to P, or at least P embeds in P_t?*

6.1.3 Sequential density

Problem 114 *Is it true that the product of less than $\mathfrak{q}$ many sequentially separable spaces is sequentially separable?*

Problem 115 *Is there a consistent example of a space X, such that $C_p(X)$ is strongly sequentially separable, not Frechet–Urysohn and $\mathbb{R}^X$ is not strongly sequentially separable?*

Problem 116 *Is it possible to find spaces X, Y such that $C_k(X)$and $C_k(Y)$ are strongly sequentially separable but $C_k(X) \times C_k(Y)$ is not strongly sequentially separable?*

Problem 117 *Is it possible to find a space X such that $C_k(X)$ is strongly sequentially separable but $C_k(X)^2$ is not strongly sequentially separable?*

Problem 118 *Is it possible in ZFC to find a space X such that $C_k(X)$ is strongly sequentially separable but $C_k(X)$ is not Frechet–Urysohn?*

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