A SHEAF THEORETIC APPROACH TO MEASURE THEORY

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Submitted to the Graduate Faculty of

the Department of Mathematics in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2006

UNIVERSITY OF PITTSBURGH

DEPARTMENT OF MATHEMATICS

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ABSTRACT

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University of Pittsburgh, 2006

The topos $\mathrm{Sh}(\mathcal{F})$ of sheaves on a σ-algebra $\mathcal{F}$ is a natural home for measure theory. The collection of measures is a sheaf, the collection of measurable real valued functions is a sheaf, the operation of integration is a natural transformation, and the concept of almost-everywhere equivalence is a Lawvere-Tierney topology.

The sheaf of measurable real valued functions is the Dedekind real numbers object in $\mathrm{Sh}(\mathcal{F})$ (Scott [24]), and the topology of "almost everywhere equivalence" is the closed topology induced by the sieve of negligible sets (Wendt [28]) The other elements of measure theory have not previously been described using the internal language of $\mathrm{Sh}(\mathcal{F})$. The sheaf of measures, and the natural transformation of integration, are here described using the internal languages of $\mathrm{Sh}(\mathcal{F})$ and $\widehat{\mathcal{F}}$, the topos of presheaves on $\mathcal{F}$.

These internal constructions describe corresponding components in any topos $\boldsymbol{\mathcal{E}}$ with a designated topology j. In the case where $\boldsymbol{\mathcal{E}} = \widehat{\mathcal{L}}$ is the topos of presheaves on a locale, and j is the canonical topology, then the presheaf of measures is a sheaf on $\mathcal{L}$. A definition of the measure theory on $\mathcal{L}$ is given, and it is shown that when $\mathrm{Sh}(\mathcal{F}) \simeq \mathrm{Sh}(\mathcal{L})$, or equivalently, when $\mathcal{L}$ is the locale of closed sieves in $\mathcal{F}$ this measure theory coincides with the traditional measure theory of a σ-algebra $\mathcal{F}$. In doing this, the interpretation of the topology of "almost everywhere" equivalence is modified so as to better reflect non-Boolean settings.

Given a measure μ on $\mathcal{L}$, the Lawvere-Tierney topology that expresses the notion

of "μ-almost everywhere equivalence" induces a subtopos $\mathrm{Sh}_{\mu}(\mathcal{L})$. If this subtopos is Boolean, and if μ is locally finite, then the Radon-Nikodym theorem holds, so that for any locally finite $\nu \ll \mu$, the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ exists.

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PREFACE

Although I am listed as the author of this dissertation, I owe a great deal to many people who have helped me get through the process of writing it.

Firstly, I must thank my partner, Fiona Callaghan. Fiona's love and support was essential to the act of writing this thesis, and I am looking forward to loving and supporting her, through her own thesis, and beyond.

Thanks are also due to my advisor, Professor Steve Awodey. It is often said that a student's relationship with his or her advisor is paramount in determining the success of a dissertation. It is certainly true in my case. Steve has managed the delicate balancing act of providing support, encouragement and advice, whilst still leaving me the freedom to explore ideas in my own way, and at my own pace. I am honoured to have had Steve as an advisor, and I value his friendship.

Thank you also to the other members of my committee: Professors Paul Gartside, Bob Heath, Chris Lennard, and Dana Scott. All of these committe members have provided invaluable assistance, support and suggestions. In particular, Paul Gartside and Bob Heath have shared in the process of mentoring me. I am deeply appreciative of their help. To this group I should add Professor George Sparling, who sat on my comprehensive exam committee and has helped me with a lot of geometric insight.

More generally, I wish to thank the faculty, staff, and graduate students of the Department of Mathematics at the University of Pittsburgh. I have always felt welcome, respected and valued here. I have greatly enjoyed my time at Pitt, and am proud to be a University of Pittsburgh alumnus.

Matthew Jackson

April, 2006

1.0 INTRODUCTION

1.1 OVERVIEW

A reoccurring technique in pure mathematics is to take a well known mathematical structure and find an abstraction of this structure that captures its key properties. As new structures are developed, further abstractions become possible, leading to deeper insights.

In this dissertation we develop an abstraction of measure theory (which is itself an abstraction of integration theory). The framework that we use to do this is category theory. More precisely, we use the apparatus of categorical logic to establish connections between the analytic ideas of measure theory and the geometric ideas of sheaf theory.

We start with some of the key definitions from these three areas of mathematics. Results in these sections will be presented without proof, as they are part of the standard literature of the respective fields. After establishing these definitions, we present the structure of this dissertation.

1.2 SOME CATEGORY THEORY

We can study a class of algebraic objects by investigating the functions between members of this class that that preserve the algebraic structure. For example, we can study groups by investigating group homomorphisms, we can study sets by investigating functions, and we can study topological spaces by investigating continuous functions. Categories are algebraic structures that capture the relationships between similar types of objects, and so allow us to formalize this notion.

The study of category theory allows for the development of techniques that can apply simultaneously in all of these settings. Categories have been studied extensively, for example in Mac Lane [18], Barr and Wells [1, 2], and McLarty [20].

Definition 1. A category $\mathbb{C}$ consists of a collection $\mathcal{O}_{\mathbb{C}}$ of objects and a collection of $\mathcal{M}_{\mathbb{C}}$ of arrows, or morphisms, such that

1. Each arrow f is assigned a pair of objects; the domain of f, written $\mathrm{dom}(f)$, and the codomain of f, written $\mathrm{cod}(f)$. If $A = \mathrm{dom}(f)$, and $B = \mathrm{cod}(f)$, then we write $f : A \to B$.
2. If $f : A \to B$ and $g : B \to C$ are two arrows in $\mathbb{C}$, then there is an arrow $g \circ f : A \to C$, called the composition of g and f.
3. Every object A is associated with an identity arrow $\mathrm{id}_A : A \to A$. This arrow is the identity with respect to composition, so that if $f : A \to B$ and $g : Z \to A$, then $f \circ \mathrm{id}_A = f$, and $\mathrm{id}_A \circ g = g$.

There are many examples of categories. The prototypical example is the category Set. The objects of Set are sets, and the arrows are functions, with domain, codomain, composition, and identity defined in the obvious ways. More generally, *any* model of ZFC constitutes a category in this way.

Two other important examples are the category Grp, whose objects are groups and whose arrows are group homomorphisms, and the category Top, whose objects are topological spaces, and whose arrows are continuous functions.

These are all examples of categories where the objects can be considered as "sets with structure" (although in the case of Set, the structure is trivial). Not all categories have this property. Categories are classified according to the following taxonomy:

Definition 2. Let $\mathbb{C}$ be a category.

1. $\mathbb{C}$ is called *small* if the collection of arrows $\mathcal{M}_{\mathbb{C}}$ is a set (and not a proper class).
2. $\mathbb{C}$ is called *large* if $\mathbb{C}$ is not small.
3. $\mathbb{C}$ is called *locally small* if for any pair of objects C and D, the collection of arrows with domain C and codomain D is a set (and not a proper class).

For a locally small category $\mathbb{C}$, and objects C_1 and C_2 of $\mathbb{C}$, we refer to the set of arrows of $\mathbb{C}$ with domain C_1 and codomain C_2 as the "homomorphism set", or "hom set", denoted $\mathrm{Hom}_{\mathbb{C}}(C_1, C_2)$.

The categories Set, Grp, and Top share the same taxonomic classification from Definition 2; they are all large, locally small, categories. They are also all examples of concrete categories (categories whose objects are "sets with structure" and whose arrows are functions from these underlying sets). However, there are categories that are small, and there are categories that are not concrete.

For example, let $\mathcal{G} = \langle G, \oplus, -, e\}$ be a group. Then we can represent $\mathcal{G}$ as a category with one object $*$, and whose arrows are elements of G. Composition of arrows corresponds to the group operation, so that the composition $g \circ h$ is just $c \oplus h$. Note that the identity arrow is just e. This idea can obviously also be applied to represent monoids as categories.

As another example, let $(\mathbb{P}, \leq)$ be any poset. Then we can view $\mathbb{P}$ as a category. The objects of $\mathbb{P}$ are just the elements of $\mathbb{P}$, and the arrows are witnesses to the "$\leq$" relation. Between any two elements of $\mathbb{P}$, there is at most one arrow.

For example, $\mathbb{N}$, the natural numbers, constitute a category:

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

Note that there are other implicit arrows here, for example from 0 to 2. This arrow is the composition of the arrows from 0 to 1 and from 1 to 2.

Given a category $\mathbb{C}$, there is category $\mathbb{C}^{\mathrm{op}}$, called the dual, or opposite category of $\mathbb{C}$. The dual category has the same objects and arrows as $\mathbb{C}$, but the domain and codomain operations are reversed.

Definition 3. Let $f : C \to D$ and $g : D \to C$ be two arrows in $\mathbb{C}$ such that $g \circ f = \mathrm{id}_C$ and $f \circ g = \mathrm{id}_D$. Then we say that f and g are *isomorphisms*, and that $C \cong D$.

Since every element of a group has an inverse, it follows that if we represent the group $\mathcal{G}$ as a category, every arrow is an isomorphism. This observation leads to the following definitions: A category $\mathbb{C}$ is called a groupoid if every arrow of $\mathbb{C}$ is an isomorphism. $\mathbb{C}$ is called a group if $\mathbb{C}$ is a groupoid with only one object.

The concept of an isomorphism is essential in category theory. The cancellation properties of isomorphisms, together with the fact that the idiosyncratic properties of objects are inaccessible to a categorical analysis except insofar as they are reflected in the arrows of the category, mean that in category theory critical objects are only defined up to isomorphism.

In the category Set, for example, isomorphisms are just bijections. Since in Set every set is isomorphic to exactly one cardinal, we can think of every set represented by its cardinality. As an illustration of this, every singleton is a terminal object (see Definition 8 below). There is therefore a proper class of terminal objects in Set. However, the terminal object is unique, up to isomorphism. From a categorical point of view it doesn't matter which singleton we are considering, only that the set is indeed a singleton.

Definition 4. An arrow $f : C \to D$ in $\mathbb{C}$ is called a *monomorphism* if for any $g, h : B \to C$ such that $f \circ g = f \circ h$, we must have $g = h$. In this case we call C (or more properly the diagram $f : C \to D$) a subobject of D. Monomorphisms are indicated by the special arrow "$\rightarrowtail$", so that we write $f : C \rightarrowtail D$.

In the category Set, monomorphisms correspond to injections. Thus we say that, $f : A \rightarrowtail B$ is a subobject of B, even though A need not be an actual subset of B. However, it does follow that A is isomorphic to a subobject of B. In fact, in the category Set, A is a subobject of B (for some monomorphism) if and only if $|A| \leq |B|$.

Group homomorphisms are functions that preserve the group structure. A corresponding role in category theory is taken by functors.

Definition 5. Let $\mathbb{C} = \langle \mathcal{O}_{\mathbb{C}}, \mathcal{M}_{\mathbb{C}} \rangle$ and $\mathbb{D} = \langle \mathcal{O}_{\mathbb{D}}, \mathcal{M}_{\mathbb{D}} \rangle$ be two categories. A *functor* $F : \mathbb{C} \to \mathbb{D}$ consists of two functions, $F_{\mathcal{O}} : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{D}}$, and $F_{\mathcal{M}} : \mathcal{M}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{D}}$, such that all the categorical structure (domain, codomain, composition, identity) is preserved.

There are many examples of functors. For any concrete category $\mathbb{C}$, there is a "forgetful" functor $U : \mathbb{C} \to$ Set, which takes every "set with structure" to the underlying set. If $\mathbb{P}_1$ and $\mathbb{P}_2$ are two posets, viewed as categories, then a functor from $\mathbb{P}_1$ to $\mathbb{P}_2$ is just an order preserving map.

One way to think of a functor $F : \mathbb{J} \to \mathbb{C}$ is as a diagram. F describes a copy of the category $\mathbb{J}$ inside $\mathbb{C}$. For example, suppose that $\mathbb{J}$ is the category given by the following diagram:

$$\begin{tikzcd} J \arrow[r, "j"] & L & K \arrow[l, "k"'] \end{tikzcd}$$

then we have a diagram inside $\mathbb{C}$:

$$\begin{tikzcd} F(J) \arrow[r, "F(j)"] & F(L) & F(K) \arrow[l, "F(k)"'] \end{tikzcd}$$

Using this terminology, we can define limits in $\mathbb{C}$.

Definition 6. Let $F : \mathbb{J} \to \mathbb{C}$ be a functor.

1. A *cone* for F consists of an object C of $\mathbb{C}$, together with a family of arrows

$$\mathbf{f} = \langle f_J : C \to F(J) | J \in \mathcal{O}_{\mathbb{J}} \rangle$$

   such that for any arrow $j : J \to K$ in $\mathbb{J}$, the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "f_J"'] \arrow[dr, "f_K"] & \\ F(J) \arrow[rr, "F(j)"'] & & F(K) \end{tikzcd}$$

2. A *limiting cone* for F is a cone $(C, \mathbf{f})$ is a cone such that for any other cone $(D, \mathbf{g})$ there is a unique arrow $h : D \to C$ such that for any $J \in \mathcal{O}_{\mathbb{J}}$ we have $f_J \circ h = g_J$.

   Such an arrangement looks something like this:

$$\begin{tikzcd} & D \arrow[d, dotted, "h"] \arrow[ddl, "g_J"'] \arrow[ddr, "g_K"] & \\ & C \arrow[dl, "f_J"'] \arrow[dr, "f_K"] & \\ F(J) \arrow[rr, "F(j)"'] & & F(K) \end{tikzcd}$$

Definition 7. Let $F : \mathcal{J} \to \mathbb{C}$ be a functor. Then, viewing F as a diagram in $\mathbb{C}$, the *limit* of the diagram, denoted

$$\lim_{\leftarrow \mathbb{J}} F$$

is the limiting cone.

Limits are also sometimes called "inverse limits". A limit in $\mathbb{C}^{\mathrm{op}}$ is called a colimit in $\mathbb{C}$, or a "direct limit". A colimit can be thought of as a limiting cocone:

$$\begin{tikzcd}
F(J) \arrow[rr, "F(j)"'] \arrow[dr, "f_J"] \arrow[ddr, "g_J"'] & & F(K) \arrow[dl, "f_K"'] \arrow[ddl, "g_K"] \\
& C \arrow[d, dotted, "h"] & \\
& D &
\end{tikzcd}$$

Sometimes we will refer to a category having all limits of a certain class. This usually refers to the index category $\mathbb{J}$. For example, a category has all finite limits if every functor F from a finite category $\mathbb{J}$ into $\mathbb{C}$ has a limit.

Definition 8. 1. A *product* is the limit of a discrete category $\mathbb{J}$. It consists of a single object P and an arrow $\pi_J : P \to F(J)$ for each object $J \in \mathbb{J}$. such that for any object Z, and arrows $\langle f_J : Z \to F(J) | J \in \mathcal{O}_{\mathbb{J}} \rangle$, there is a unique $u : Z \to P$ such that for each J, we have $\pi_J \circ u = f_J$. It is easy to see that this definition of a product coincides with the usual definition of the product in Set, in Grp, and in Top. In a poset category $\mathbb{P}$, the product of elements A and B is their greatest lower bound.

Arrows into products are generally written by pairing the arrows into the factors. For example:

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[d, dotted, "\langle f, g \rangle"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B
\end{tikzcd}$$

Thus we write $\langle f, g\rangle : Z \to A \times B$. Occasionally we will have arrows from one product to another. In this case, we will sometimes drop the projection maps. For example, in the following diagram, we will write *timesg* $: A \times B \to C \times D$, rather than $\langle f \circ \pi_A, g \circ \pi_2\rangle : A \times B \to C \times D$.

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"] \arrow[d, dotted, "f \times g"] & B \arrow[d, "g"] \\
C & C \times D \arrow[l, "\pi_C"'] \arrow[r, "\pi_D"'] & D
\end{tikzcd}$$

2. A *terminal* object is a special product. It is the limit of the empty diagram. Since every object of $\mathbb{C}$ is a cone for the empty diagram, the terminal object is just an object $\mathbf{1}$ such that for any object C of $\mathbb{C}$ for which there is a unique arrow $! : C \to 1$. In a poset, the terminal object, if it exists, is the top element. In Set any singleton is a terminal object. In Grp, the terminal objects are the trivial groups; that is, groups with only one element. Note that although there may be more than one terminal object, all of the terminal objects in $\mathbb{C}$ are isomorphic to one another.
3. An *equalizer* is a limit of a diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

A cone for such a diagram consists of an object Z together with an arrow $z : Z \to A$, such that $f \circ z = g \circ z$. Hence an equalizer consists of an object E and an arrow $e : E \to A$ such that for any such Z, there is a $u : Z \to E$ such that $z = e \circ u$. The arrow e is always a monomorphism. In the category of sets, E is the set $\{x \in A | f(x) = g(x)\}$.
4. A *pullback* is a limit of a diagram of the form

$$\begin{tikzcd}
B \arrow[r, "f"] & A & C \arrow[l, "g"']
\end{tikzcd}$$

The pullback is usually expressed as a commutative square:

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd}$$

In the category of sets, the pullback is the subset of $B \times C$ given by

$$P = \{\langle b, c \rangle \in B \times C | f(b) = g(c)\}$$

A functor F from $\mathbb{C}^{op} \to \mathbb{D}$ is sometimes called a "contravariant functor" from $\mathbb{C}$ to $\mathbb{D}$. This terminology is something of a misnomer, as F is not a functor from $\mathbb{C}$ to $\mathbb{D}$.

Given two categories, $\mathbb{C}$ and $\mathbb{D}$, there is a category $\mathbb{D}^{\mathbb{C}}$, whose objects are the functors from $\mathbb{C}$ to $\mathbb{D}$. In order to understand this category, we need a notion of an arrow from one functor to another. Such an arrow is called a "natural transformation".

Definition 9. Given two functors $F, G : \mathbb{C} \to \mathbb{D}$, a *natural transformation* $\eta : F \to G$ consists of a family of arrows $\left\langle \eta^C | C \in \mathcal{O}_{\mathbb{C}} \right\rangle$ such that for any $f : C_1 \to C_2$ in $\mathbb{C}$, the following square commutes in $\mathbb{D}$:

$$\begin{tikzcd}
F(C_1) \arrow[r, "F(f)"] \arrow[d, "\eta^{C_1}"'] & F(C_2) \arrow[d, "\eta^{C_2}"] \\
G(C_1) \arrow[r, "G(f)"'] & G(C_2)
\end{tikzcd}$$

The arrow η^C is called the "component of η at C".

Suppose that $\mathbb{C}$ and $\mathbb{D}$ are two categories, and $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$ are two functors. Then we can compute the composites, to get to functors $GF : \mathbb{C} \to \mathbb{C}$ and $FG : \mathbb{D} \to \mathbb{D}$. These compositions are objects in the categories $\mathbb{C}^{\mathbb{C}}$ and $\mathbb{D}^{\mathbb{D}}$ respectively. Each of these categories also has an identity functor, $\mathbf{id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$, in $\mathbb{C}^{\mathbb{C}}$ and $\mathbf{id}_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ in $\mathbb{D}^{\mathbb{D}}$.

Definition 10. If $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$ are functors such that GF is isomorphic to $\mathbf{id}_{\mathbb{C}}$ (in $\mathbb{C}^{\mathbb{C}}$), and FG is isomorphic to $\mathbf{id}_{\mathbb{D}}$ (in $\mathbb{D}^{\mathbb{D}}$), then we say that $\mathbb{C}$ and $\mathbb{D}$ are *equivalent* and write $\mathbb{C} \simeq \mathbb{D}$.

From above, it is clear that $\textsc{Set} \simeq \textsc{Card}$, where $\textsc{Card}$ is the subcategory of $\textsc{Set}$ whose objects are cardinals. It is often said that an equivalence is "isomorphic to an isomorphism".

Equivalence is a special case of a more general relation between functors. Let $\mathbb{C}$ and $\mathbb{D}$ be two categories, and let $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$ be two functors. We say that F is the left adjoint of G, or the G is the right adjoint of F (written $F \dashv G$), if for any objects $C \in \mathcal{O}_{\mathbb{C}}$ and $D \in \mathcal{O}_{\mathbb{D}}$ there is an isomorphism between $\mathrm{Hom}_{\mathbb{C}}(C, GD)$ and $\mathrm{Hom}_{\mathbb{D}}(FC, D)$ (natural in both C and D). Given an adjunction $F \dashv G$ there are two natural transformations $\eta : \mathrm{id}_C \to GF$ and $\epsilon : FG \to \mathrm{id}_D$, called the unit and counit of the adjunction respectively.

The unit and counit are universal, in the sense that for any objects C in $\mathbb{C}$ and D in $\mathbb{D}$, and every arrow $f : C \to G(D)$, there is a unique arrow $h : F(C) \to D$ in $\mathbb{D}$ such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "\eta^C"] \arrow[dr, "f"'] & GF(C) \arrow[d, "G(h)"] \\
 & G(D)
\end{tikzcd}$$

Adjunctions occur in many contexts. For example, the "forgetful" functor $U : \textsc{Grp} \to \textsc{Set}$ has a left adjoint F, which takes a set X to the free group on X. The unit of this adjunction embeds a set X into the underlying set of the free group on X. The counit takes an element of the free group on the letters taken from the underlying set of a group G (which is a string of elements of G) to the product of that string in G. Many more examples of adjunctions are given in Mac Lane [18].

One specific example of an adjunction that is important here is in the construction of exponentials. Let $\mathbb{C}$ be a category, and fix an object C in $\mathbb{C}$. Then there is a functor $P_C : \mathbb{C} \to \mathbb{C}$ with the action $B \mapsto B \times C$. If this functor has a right adjoint, that adjoint is an exponentiation functor, E_C, given by the action $B \mapsto C^B$. All of the key properties of an exponential are deduced from the properties of the adjunction.

The counit of this adjunction is particularly interesting. For a given B, ϵ^B is an arrow from $B^C \times C$ to B. In the $\textsc{Set}$, this arrow represents function application. An element of B^C

is a function f from C to B, so an element of $B^C \times C$ can be thought of as an ordered pair $\langle f, c\rangle$. Then η^B applied to this pair is just $f(c) \in B$.

This counit also has an important role in a Lindenbaum algebra of logical formulas. In this case, the product is conjunction, and the exponential is the conditional. Hence we write $B \wedge C$, rather than $B \times C$, and $C \Rightarrow B$, in place of B^C. In this context, arrows correspond to the provability predicate, so we get the inferential law *modus ponens*.

$$C \wedge (C \Rightarrow B) \vdash B$$

The unit also has familiar interpretations. The component of the unit at C takes B to $(B \times C)^C$. Interpreting this in Set gives us the following

$$\begin{array}{ccc} B & \rightarrow & (B \times C)^C \\ b & \mapsto & \lambda x.\langle b, x\rangle \end{array}$$

Applying the unit in the Lindenbaum algebra gives us the following inference (a form of implication introduction):

$$B \vdash C \Rightarrow (B \wedge C)$$

1.3 SOME SHEAF AND TOPOS THEORY

Certain functor categories arise frequently. Presheaves are an example:

Definition 11. Let $\mathcal{X} = (X, \tau)$ be a topological space. A *presheaf* on $\mathcal{X}$ is a contravariant functor from τ (viewed as a poset category) to Set. The category of presheaves on τ is $\textsc{Set}^{\tau^{op}}$. This category is often denoted $\widehat{\tau}$.

Since functors act on arrows as well as objects, a presheaf P can be thought of as a τ-indexed family of sets, together with functions between them. Since the arrows in τ correspond to subsets, if follows that if $V \subseteq U$, then P includes a function $\rho^U_V : P(U) \rightarrow P(V)$. This function is called a “restriction map”.

In fact, presheaves can be studied more generally. If $\mathbb{C}$ is any small category, then the category $\mathrm{Set}^{\mathbb{C}^{\mathrm{op}}}$ is called the category of presheaves on $\mathbb{C}$, and is usually denoted $\widehat{\mathbb{C}}$.

As the name suggests, one reason for the significance of presheaves is that they are related to sheaves. Unfortunately, it is difficult to give a single definition of a sheaf, as different settings require different languages. Here we give three presentations of the definition of a sheaf, in order of increasing generalization.

The most specific of these examples is a sheaf on a topological space. To understand this concept, we start with the idea of a matching family.

Definition 12. Suppose that P is a presheaf on τ, and $U \in \tau$.

1. A *sieve* $\mathcal{I}$ on τ is any family of open subsets of U which is "downward closed", meaning that if $W \subseteq V \subseteq U$ and $V \in \mathcal{I}$, then $W \in \mathcal{I}$.
2. A sieve $\mathcal{I}$ *covers* U if $\bigcup_{V \in \mathcal{I}} V = U$.
3. A *family* for P and $\mathcal{I}$ is a element $\mathbf{x} \in \prod_{V \in \mathcal{I}} P(V)$
4. A family $\mathbf{x} = \langle x_V | V \in \mathcal{I} \rangle$ is a *matching family* if for any $V, W \in \mathcal{I}$ we have

$$\rho^V_{V \cap W}(x_V) = \rho^W_{V \cap W}(x_W)$$

5. $x \in P(U)$ is an *amalgamation* for a matching family $\mathbf{x}$ if for every $V \in \mathcal{I}$ we have

$$\rho^U_V(x) = x_V$$

6. P is a *sheaf* if for every $U \in \tau$, and for every covering sieve $\mathcal{I}$ of U, and for every matching family $\mathbf{x}$ for $\mathcal{I}$, there is a unique amalgamation $x \in P(U)$.

The arrows of $\mathrm{Sh}(\tau)$ are just natural transformations between sheaves, so that $\mathrm{Sh}(\tau)$ is a full subcategory of $\widehat{\tau}$. The inclusion of $\mathrm{Sh}(\tau)$ into $\widehat{\tau}$ is a functor $\mathbf{i}$:

$$\mathbf{i} : \mathrm{Sh}(\tau) \rightarrowtail \widehat{\tau}$$

This functor has a left adjoint, $\mathbf{a}$, called the associated sheaf, or sheafification functor. The component at P of the unit of this adjunction is a natural transformation $\eta^P : P \to \mathbf{a}P$. It is immediate from the definition of the unit of an adjunction that for any sheaf F and any natural transformation $\phi : P \to F$, there is a unique natural transformation $\overline{\phi} : \mathbf{a}P \to F$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathbf{a}P \arrow[d, dotted, "\overline{\phi}"] \\ P \arrow[ur, "\eta P"] \arrow[r, "\phi"'] & F \end{tikzcd}$$

The concept of a sheaf on a topological space can be generalized. Let $\mathbb{C}$ be any small category. A sieve on an object C is a set $\mathcal{I}$ of arrows, all with codomain C, such that if $f \in \mathcal{I}$, and g and h are any arrows such that $g = f \circ h$, then g is also in $\mathcal{I}$. In the case that $\mathbb{C}$ is a poset category, a sieve is just a lower or downward closed set.

Definition 13. Ω is the presheaf of sieves on $\mathbb{C}$. Hence $\Omega(C)$ is the set of sieves on C. The restriction operation is given by

$$\rho_f(\mathcal{I}) = \{g \in \mathcal{M}_{\mathbb{C}} | f \circ g \in \mathcal{I}\}$$

$\rho_f(\mathcal{I})$ can be thought of as the arrows of $\mathcal{I}$ that factor through f. Note that if $f \in \mathcal{I}$, then $\rho_f(\mathcal{I})$ is the maximal sieve on $\mathrm{dom}(f)$.

Definition 14. A *Grothendieck topology* is a subfunctor $J \rightarrowtail \Omega$ that assigns to each object C of $\mathbb{C}$ a set of sieves on C that cover C. In order to be a Grothendieck topology, J must satisfy the following axioms:

1. Maximality: The maximal sieve $\mathcal{I} = \{f \in \mathcal{M}_{\mathbb{C}} | \mathrm{cod}(f) = C\}$ is a cover. (Note that the maximal sieve on C is the principal sieve generated by id_C.)
2. Transitivity: If $\mathcal{I} \in J(C)$ and for each $f \in \mathcal{I}$, $\mathcal{J}_f \in J(\mathrm{dom}(f))$, then

$$\bigcup_{f \in \mathcal{I}} \{f \circ g | g \in \mathcal{J}_f\}$$

is a cover for C.

It is usually required that J also satisfy the stability condition:

If $\mathcal{I} \in J(C)$ and $f : D \to C$, then $\{g \in \mathcal{M}_{\mathbb{C}} | f \circ g \in \mathcal{I}\} \in J(D)$

However, this follows directly from the fact that J is a subfunctor of Ω.

A small category $\mathbb{C}$, together with a Grothendieck topology J is called a *site* (Mac Lane and Moerdijk [19]) or a *coverage* (Johnstone [13]). Given a site $(\mathbb{C}, J)$ a sheaf on the site is defined in a way that is analogous to the way that a sheaf on a topological space; a sheaf is a presheaf that has unique amalgamations for every matching family for every cover.

It is easy to verify that the usual notion of a cover of an open set is a Grothendieck topology. Hence the definition of a sheaf on a site extends the definition of a sheaf on a topological space. In fact, the usual Grothendieck topology on a topological space has a special name; it is called the *canonical topology*.

It is also worth noting that any presheaf category $\widehat{\mathbb{C}}$ is also a sheaf category. Let J be the smallest Grothendieck topology on $\mathbb{C}$, so that the only sieve that covers C is the maximal sieve. Then for every covering sieve $\mathcal{I} \in J(C)$, and every matching family $\mathbf{x}$ for P there must be an amalgamation, namely x_{id_C}. Thus all presheaves are sheaves.

The sheaf categories that we have constructed have more structure than categories have in general. They are "toposes" (or "topoi" – there is no consensus on the plural of "topos", Johnstone [12, 14, 15], Lambeck and Scott [17], and McLarty [20] use "toposes", Mac Lane and Moerdijk [19], and Goldblatt [10] use "topoi"). A topos is a category $\mathcal{E}$ where one can "do mathematics".

Definition 15. Let $\mathbb{C}$ be a category, A *subobject classifier* in $\mathbb{C}$ is an object Ω, together with an arrow $\top : \mathbf{1} \to \Omega$, called "true", such that for any monomorphism $S \rightarrowtail A$ in $\mathbb{C}$, there is a unique arrow $\chi_S : A \to \Omega$ such that for any arrow $f : Z \to A$, there is a unique arrow u making the following diagram commute:

$$\begin{tikzcd}
Z \arrow[dr, dotted, "u"] \arrow[drr, "!"] \arrow[ddr, "f"'] & & \\
& S \arrow[r, "!"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, "\top"] \\
& A \arrow[r, "\chi_S"'] & \Omega
\end{tikzcd}$$

In other words, S is the pullback of "true" along χ_S.

In Set, the subobject classifier is just the two point set $\{\bot, \top\}$, and the characteristic maps are just the usual characteristic functions. More generally, we think of Ω as the object of truth values in $\mathcal{E}$. The subobject classifier is the key to building an internal logic inside a topos.

We can now give a formal definition of a topos:

Definition 16. A *topos* is a category $\mathcal{E}$ such that $\mathcal{E}$ has all finite limits and colimits, exponential objects, and a subobject classifier. A topos that is equivalent to the topos of sheaves on some site $(\mathbb{C}, J)$ is called a "Grothendieck topos".

Since we can take exponentials in a topos, we can compute Ω^A, the "power object" of A. In Set, this is just the set of all characteristic functions of subsets of A. Note that the counit in this case is just the "element of" relation, thus justifying the use of the letter "ϵ" to denote the counit. Rather than writing it as an exponential, we denote the power object of A by $\mathcal{P}A$.

The internal logic of a topos is higher order intuitionistic logic (Lambeck and Scott [17]). The objects of $\mathbb{C}$ represent the types of the logic. There are arrows of the form $\Omega \times \Omega \to \Omega$ corresponding to $\vee$, $\wedge$, and $\Rightarrow$, together with a negation arrow $\neg : \Omega \to \Omega$. An arrow from $C \to \Omega$ represents a logical formula with free variable of type C. If $\phi : C \to \Omega$ is some formula, then the arrow corresponding to the negation of ϕ is just the following composition:

$$C \xrightarrow{\phi} \Omega \xrightarrow{\neg} \Omega$$

Likewise, if $\phi : A \to \Omega$ and $\psi : B \to \Omega$ are two formulas, then we find the formula $\phi(a) \wedge \psi(b)$ by means of the following diagram:

$$A \times B \xrightarrow{\phi \times \psi} \Omega \times \Omega \xrightarrow{\wedge} \Omega$$

Since toposes have a lot of features in common with Set, it is not surprising that toposes have a similar feel to Set. In particular, viewed internally, we can think of a topos as a model for an intuitionistic set theory. A topos will not, in general, satisfy all of

ZFC. However, a Grothendieck topos will satisfy most of the set theory IZF (intuitionistic Zermelo Frankel set theory, see Fourman [8]), except for the axiom of foundation. Hence we can reason about a topos by considering the objects to be sets, and using intuitionistic logic.

It is possible to study the internal logic of more general categories that do not have subobject classifiers. See Crole [6] for more information on how to do this.

In a presheaf topos, the subobject classifier is just Ω, the presheaf of sieves. Let $(\mathbb{C}, J)$ be a site. A sieve $\mathcal{I} \in \Omega(C)$ is called "closed" if every for every $f : D \to C$ such that $\{g \in \mathcal{O}_{\mathbb{C}} | f \circ g \in \mathcal{I}\} \in J(D)$, we have $f \in \mathcal{I}$. In other words, a sieve is closed if it contains all that arrows that it covers. The object of closed sieves is denoted Ω_j, and is a subobject of Ω. In fact, Ω_j is a sheaf, and is the subobject classifier in $\mathrm{Sh}(\mathbb{C}, J)$.

Since Ω_j is a subobject of Ω, it follows that there is a characteristic map $j = \chi_{\Omega_j} : \Omega \to \Omega$. This map is called the "closure map", and is the key to the most general notion of a sheaf.

Definition 17. Let $\mathcal{E}$ be a topos, and let Ω be the subobject classifier of $\mathcal{E}$. Then $j : \Omega \to \Omega$ is called a *Lawvere-Tierney topology* (or *local operator*, in Johnstone [14]) if the following diagrams commute:

$$\begin{tikzcd} \mathbf{1} \arrow[r, "\top"] \arrow[dr, "\top"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

$$\begin{tikzcd} \Omega \arrow[r, "j"] \arrow[dr, "j"'] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "j \times j"] \arrow[d, "\wedge"'] & \Omega \times \Omega \arrow[d, "\wedge"] \\ \Omega \arrow[r, "j"'] & \Omega \end{tikzcd}$$

Definition 18. An object E of $\mathcal{E}$ is a j-sheaf if for any $S \rightarrowtail E$ such that $j \circ \chi_S = \top$ and for any arrow $f : S \to F$, there is a unique arrow $\overline{f} : E \to F$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[d, tail] \arrow[dr, "f"] & \\ E \arrow[r, "\overline{f}"'] & F \end{tikzcd}$$

The topos of j-sheaves in $\mathcal{E}$ is denoted $\mathrm{Sh}_j(\mathcal{E})$.

As before, the inclusion functor $\mathbf{i} : \mathrm{Sh}_j(\mathcal{E}) \rightarrowtail \mathcal{E}$ has a left adjoint $\mathbf{a}$, called the "associated sheaf" or "sheafification" functor. The subobject classifier in $\mathrm{Sh}_j(\mathcal{E})$ is Ω_j, which is given by the following equalizer:

$$\begin{tikzcd} \Omega_j \arrow[r, tail] & \Omega \arrow[r, shift left, "\mathrm{id}_\Omega"] \arrow[r, shift right, "j"'] & \Omega \end{tikzcd}$$

If $\mathcal{E}$ is a topos, and j is a Lawvere-Tierney topology in $\mathcal{E}$, then we say that $\mathrm{Sh}_j(\mathcal{E})$, the topos of j-sheaves, is a "subtopos of $\mathcal{E}$".

It turns out that if $\mathcal{E}$ is a presheaf topos, then the Grothendieck topologies on $\mathcal{E}$ correspond to the Lawvere-Tierney topologies, and the two notions of sheaf coincede. Therefore, this new notion of a sheaf does indeed generalize the notion of a sheaf on a site.

There is one very special class of toposes that arise frequently in this dissertation. A locale is a type of lattice (specifically, a complete Heyting algebra). Locales arise often in topology, as the algebra of open sets in a topological space is a locale. Point-free topology is generally construed as the study of locales (see Johnstone [13]). However, locales need not be spatial. To recognize this, we use the symbols "$\vee$","$\wedge$", and "$\leq$" to refer to the lattice operations of a locale $\mathcal{L}$, and "$\top$" and "$\bot$" to refer to the top and bottom elements of $\mathcal{L}$.

In any Grothendieck topos, the object Ω forms an internal locale object. However, the significance of locales does not stop there. Any topos that is equivalent to the topos of sheaves on a locale (with the canonical topology) is called a "localic topos". Localic toposes have many useful properties (see Mac Lane and Moerdijk [19]), but the most useful here is that if $\mathbb{P}$ is any poset, and J is any Grothendieck topology on $\mathbb{P}$, then $\mathrm{Sh}(\mathbb{P}, J)$ is a localic topos, and moreover, is equivalent to the topos of sheaves on the locale of closed sieves in $\mathbb{P}$.

Another important class of toposes with which we need to be familiar are the Boolean toposes. A topos $\mathcal{E}$ is Boolean if the internal logic satisfies the law of the excluded middle:

$$\mathcal{E} \models \phi \vee (\neg\phi)$$

This is equivalent to the subobject classifier of $\mathcal{E}$ being an internal *Boolean* algebra object. For any topos, the "double negation arrow" $\neg\neg : \Omega \to \Omega$ is a Lawvere-Tierney topology, and the resulting subtopos is Boolean. This construction is related to the double negation translation between intuitionistic and classical logic (see Van Dalen [26]).

1.4 SOME MEASURE THEORY

Classical measure theory (see, for example, Billingsley [3], Royden [22], or Rudin [23]) begins with the following definitions.

Definition 19. Let X be a set. Then $\mathcal{F} \subseteq \mathcal{P}X$ is called a σ-field on X if

1. $\mathcal{F}$ is closed under complements.
2. $\mathcal{F}$ is closed under countable unions.

Note that $\emptyset \in \mathcal{F}$, since

$$\emptyset = \bigcup_{A \in \emptyset} A$$

and $X \in \mathcal{F}$, since $X = \neg\emptyset$.

Definition 20. Let $\mathcal{F}$ be a σ-field. Then a function $\mu : \mathcal{F} \to [0, \infty]$ is called a measure if for any countable antichain $\mathcal{A} = \langle A_i | i < \alpha \leq \omega \rangle$ in $\mathcal{F}$,

$$\sum_{i<\alpha} \mu(A_i) = \mu\left(\bigcup_{i<\alpha} A_i\right)$$

Note that it is a consequence of this that $\mu(\emptyset) = 0$.

Definition 21. A measure space consists of a triple $(X, \mathcal{F}, \mu)$, where X is a set, $\mathcal{F}$ is a σ-field on X, and μ is a measure on $\mathcal{F}$.

There are a number of special subclasses of the set of measures on $\mathcal{F}$. The most important for our needs is the class of σ-finite measures.

Definition 22. Let $(X, \mathcal{F}, \mu)$ be a measure space. Then μ is called σ-finite if there is a countable partition of $\langle X_i \in \mathcal{F} | i < \omega \rangle$ of X such that for each i, $\mu(X_i) < \infty$.

Definition 23. Let $\mathcal{X} = (X, \mathcal{F}, \mu)$ and $\mathcal{Y} = (Y, \mathcal{G}, \nu)$ be two measure spaces. A measurable function from $\mathcal{X}$ to $\mathcal{Y}$ is a function $f : X \to Y$ such that for any $G \in \mathcal{G}$, $f^{-1}[G] \in \mathcal{F}$.

Measure theory can also be studied in a point-free way (see, for example, Fremlin [9]). The point-free approach to measure theory focuses on the algebraic properties of the σ-field. Correspondingly, the underlying sets X and Y are de-emphasized. The distinction is made explicit in the following Definition:

Definition 24. A σ-algebra is a countably complete Boolean algebra.

Many authors use the terms "σ-algebra" and "σ-field" interchangeably, usually to mean what we have referred to as a σ-field. Our terminology here echoes the distinction between a Boolean algebra, and a field of sets (that is, a collection of subsets of some universe X that contains $\emptyset$ and is closed under the operations of union, intersection, and complementation. Every field of sets is a Boolean algebra, but the converse is not true. Likewise, a σ-field is necessarily a σ-algebra, but σ-algebras are not necessarily σ-fields.

The well known Stone representation theorem (see Johnstone [13], or Koppelberg [16]) shows that every Boolean algebra $\mathcal{B}$ is isomorphic to a field of sets (the underlying set being the set of ultrafilters of $\mathcal{B}$). There is no direct analogue for the relationship between σ-algebras and σ-fields. The closest that we can get is the Loomis-Sikorski theorem (see Sikorski [25] or Koppelberg [16]). This theorem says that every σ-algebra is isomorphic to the quotient of some σ-field $\mathcal{F}$ by some countably complete ideal $\mathcal{I} \subseteq \mathcal{F}$.

In order to emphasize that the σ-algebras that we refer to are not necessarily spatial, we use the symbols "$\sqcap$", "$\sqcup$", and "$\sqsubseteq$" to denote the meet and join operations, and the partial ordering in a σ-algebra $\mathcal{F}$, and "$\perp$" and "$\top$" to denote the smallest and largest elements of $\mathcal{F}$. In the special case where $\mathcal{F}$ is a σ-field, we revert to the usual set theoretic symbols: "$\cup$", "$\cap$", etc.

If $(X, \mathcal{F}, \mu)$ is a measure space, and $f \to [0, \infty)$ is a measurable function, (when $\mathbb{R}$ is equipped with the σ-field of Lebesgue measurable sets, and the Lebesgue measure), then we can find the integral $\int f\,d\mu$. This integral is itself a measure ν, given by

$$\nu(A) = \int_A f\,d\mu$$

The process of calculating the integral, Lebesgue integration, takes several steps. The integral of a constant function is found through multiplication:

$$\int_A c\,d\mu = c \cdot \mu(A)$$

The integral of a measurable function with a finite range (ie, a simple function) is computed by exploiting the additive property of measures: Suppose that $\langle X_i | i = 1 \ldots n \rangle$ is a partition of X, and that for all $x \in X_i$, $s(x) = s_i$. Then

$$\int_A s\,d\mu = \sum_{i=1}^{n} s_i \cdot \mu(X_i \cap A)$$

Finally, the integral of a measurable function f is calculated by taking the limit of the integrals of an increasing sequence of simple functions converging to f.

In addition to the usual (pointwise) partial ordering on the measures, there is also an important preordering, the "absolute continuity" ordering:

$$\nu \ll \mu \iff ((\mu(A) = 0) \Rightarrow \nu(A) = 0)$$

This ordering allows us to state the Radon-Nikodym Theorem, one of the central results in Measure Theory:

Theorem 1 (Radon-Nikodym Theorem). *If $\nu \ll \mu$ are two σ-finite measures, then there is a measurable function f such that*

$$\nu(-) = \int_{-} f\,d\mu$$

The function f is called "the Radon-Nikodym derivative of ν with respect to μ", and is often denoted $\frac{d\nu}{d\mu}$. It is important to note that the derivative is not necessarily unique. Two functions f_1 and f_2 can both be derivatives of ν with respect to ν if

$$\mu\left(\{x \in X | f_1(x) \neq f_2(x)\}\right) = 0$$

Consequently, we say that the Radon-Nikodym derivative is defined only up to "almost everywhere" equivalence.

1.5 MORE DETAILED OVERVIEW

A number of connections have been observed between the measure theory of a σ-algebra $\mathcal{F}$, and the geometry of the topos $\mathrm{Sh}(\mathcal{F})$ (where the Grothendieck topology is the *countable join topology*). Breitsprecher [5, 4] observed that the functor $\mathbb{M} : \mathcal{F}^{\mathrm{op}} \to \textsc{Set}$ of measures is in fact an object of $\mathrm{Sh}(\mathcal{F})$. Scott [24] (referred to in Johnstone [12]) showed that the Dedekind real numbers object in $\mathrm{Sh}(\mathcal{F})$ is the sheaf of measurable real valued functions. Combining these two observations, it is obvious that integration can be represented as a natural transformation $\int : \mathbb{D} \times \mathbb{M} \to \mathbb{M}$, where $\mathbb{D}$ is the sheaf of non-negative measurable real numbers. More recently, Wendt [27, 28] showed that the notion of almost everywhere equivalence corresponds to a certain Grothendieck topology.

Between them, these results suggest that there are some strong connections between measure theory and the topos of sheaves on a σ-algebra. In this dissertation, we ground these connections in the internal logic of the sheaf topos, and then extend them to create a measure theory for an arbitrary localic topos.

In Chapter 2, we present a measure theory for a locale $\mathcal{L}$. This measure theory is based around the object of measures, the sheaf of measurable real numbers, and an integration arrow. The object of measures is constructed in the presheaf topos $\widehat{\mathcal{L}}$, but is a sheaf. Thus the measure theory of $\mathcal{L}$ exists in $\mathrm{Sh}(\mathcal{L})$,

Simultaneously, we show that when $\mathcal{L}$ is the locale of closed sieves in the σ-algebra $\mathcal{F}$ (in other words, when $\mathrm{Sh}(\mathcal{F}) \simeq \mathrm{Sh}(\mathcal{L})$), this localic measure theory restricts to the usual measure theory on $\mathcal{F}$. We also show that when the constructions of the sheaf of measures and the integration arrow are carried out in $\widehat{\mathcal{F}}$ and Sh($\mathcal{F}$), we arrive at the same objects of Sh($\mathcal{L}$) as we did when building a localic measure theory.

The construction of $\mathbb{M}$ starts in the presheaf topos with the construction of a "presheaf of semireals". These objects act as functionals from the underlying locale $\mathcal{L}$ to $[0, \infty]$. We construct the sheaf of measures $\mathbb{M}$ by taking only those semireals that are both additive and semicontinuous. The construction of $\int$ mimics the usual construction of the Lebesgue integral, starting with constant functions, proceeding to locally constant functions, and then, by limits, to measurable functions.

One immediate generalization of classical measure theory that follows from this framework is that it is possible to consider integration theory for non-spatial σ-algebras. Since Dedekind real numbers take the role of measurable functions, there is no need to have an underlying set in order to integrate.

In Chapter 3, we investigate subtoposes of Sh($\mathcal{L}$), and Sh($\mathcal{F}$). We generalize Wendt's construction of the "almost everywhere" topology so that it has a more natural interpretation in localic toposes. Equipped with this topology, we prove a generalization of the Radon-Nikodym Theorem: A locally finite measure μ that induces a Boolean subtopos has all Radon-Nikodym derivatives.

Finally, in Chapter 4, we discuss some unanswered questions, and opportunities for further research.

2.0 MEASURE AND INTEGRATION

2.1 MEASURES ON A LOCALE

The definition of a measure on a σ-algebra (Definition 20) can be extended to a locale:

Definition 25. Let $(\mathcal{L}, \leq, \perp, \top)$ be a locale. Then a function $\mu : \mathcal{L} \to [0, \infty]$ is a called a *measure* if it satisfies the following conditions:

1. μ is order preserving
2. $\mu(A) + \mu(B) = \mu(A \curlywedge B) + \mu(A \curlyvee B)$
3. For any directed family $\mathcal{D} \subseteq \mathcal{L}$ we have

$$\mu\left(\bigcurlyvee_{D \in \mathcal{D}} D\right) = \bigvee_{D \in \mathcal{D}} \mu(D)$$

Note that the last condition implies that $\mu(\perp) = 0$, since $\perp = \bigcurlyvee \emptyset$.

In order to justify calling such things measures, there needs to be some sort of connection between these localic measures and traditional σ-algebra measures.

Let $(\mathcal{F}, \sqsubseteq, \perp, \top)$ be a σ-algebra. A countably complete sieve in $\mathcal{F}$ is a set $\mathcal{I} \subseteq \mathcal{F}$ which is downward closed and closed under countable joins. The collection of all countably closed sieves forms a locale $\mathcal{L}$. Clearly all subsets of $\mathcal{F}$ of the form $\downarrow A = \{B \in \mathcal{F} | B \sqsubseteq A\}$ are countably closed, so we have an embedding $\mathcal{F} \rightarrowtail \mathcal{L}$.

Lemma 1. *Let μ be a measure on $\mathcal{L}$, and let μ' be the restriction of μ to $\mathcal{F}$ (so that $\mu'(A) = \mu(\downarrow A)$. Then μ' is a measure on $\mathcal{F}$.*

Proof. We need to show that μ' satisfies the following conditions:

1. $\mu'(\perp) = 0$

2. If $A \sqcap B = \perp$ then $\mu'(A) + \mu'(B) = \mu'(A \sqcup B)$
3. If $\mathcal{A} = \langle A_i | i < \omega \rangle$ is a countable increasing sequence, then

$$\mu'\left(\bigsqcup_{i<\omega} A_i\right) = \bigvee_{i<\omega} \mu'(A_i)$$

For the first condition, note that $\downarrow\perp = \perp$. Therefore $\mu'(\perp) = \mu(\perp) = 0$, as required.

For the second condition, take $A, B \in \mathcal{F}$ with $A \sqcap B = \perp$.

$$\begin{aligned}
\mu'(A) + \mu'(B) &= \mu(\downarrow A) + \mu(\downarrow B) \\
&= \mu((\downarrow A) \wedge (\downarrow B)) + \mu((\downarrow A) \vee (\downarrow B))
\end{aligned}$$

But $(\downarrow A) \wedge (\downarrow B) = \perp$ and $(\downarrow A) \vee (\downarrow B) = \downarrow(A \sqcup B)$, so we get

$$\begin{aligned}
\mu'(A) + \mu'(B) &= \mu(\perp) + \mu\left(\downarrow(A \sqcup B)\right) \\
&= \mu'(A \sqcup B)
\end{aligned}$$

For the final condition, let $\mathcal{A} = \langle A_i | i < \omega \rangle$ be an increasing sequence in $\mathcal{F}$. Observe that

$$\bigcurlyvee_{i<\omega} A_i = \downarrow\left(\bigsqcup_{i<\omega} A_i\right)$$

Using this observation, we can write:

$$\begin{aligned}
\mu'\left(\bigsqcup_{i<\omega} A_i\right) &= \mu\left(\downarrow\left(\bigsqcup_{i<\omega} A_i\right)\right) \\
&= \mu\left(\bigcurlyvee_{i<\omega} (\downarrow A_i)\right) \\
&= \bigvee_{i<\omega} \mu(\downarrow A_i) \\
&= \bigvee_{i<\omega} \mu'(A_i)
\end{aligned}$$

and we are done.

□

Lemma 2. *Let μ be a measure on $\mathcal{F}$. Define $\overline{\mu} : \mathcal{L} \to [0, \infty]$ by*

$$\overline{\mu}(\mathcal{I}) = \bigvee_{A \in \mathcal{I}} \mu(A)$$

Then $\overline{\mu}$ is a measure on the locale $\mathcal{L}$.

Proof. It is obvious that $\overline{\mu}$ is order preserving.

To see that $\overline{\mu}$ satisfies the additivity condition, start by taking two countably complete sieves $\mathcal{I}, \mathcal{J} \in \mathcal{L}$, and set $\epsilon > 0$. Then there exist $B_{\mathcal{I}} \in \mathcal{I}$ and $B_{\mathcal{J}} \in \mathcal{J}$ such that

$$\begin{aligned} \overline{\mu}(\mathcal{I}) &< \mu(B_{\mathcal{I}}) + \frac{\epsilon}{2} \\ \overline{\mu}(\mathcal{J}) &< \mu(B_{\mathcal{J}}) + \frac{\epsilon}{2} \end{aligned}$$

Furthermore, there exist $B_{\mathcal{I}\vee\mathcal{J}} \in \mathcal{I} \vee \mathcal{J}$ and $B_{\mathcal{I}\wedge\mathcal{J}} \in \mathcal{I} \wedge \mathcal{J}$ such that

$$\begin{aligned} \overline{\mu}(\mathcal{I} \vee \mathcal{J}) &< \mu(B_{\mathcal{I}\vee\mathcal{J}}) + \frac{\epsilon}{2} \\ \overline{\mu}(\mathcal{I} \wedge \mathcal{J}) &< \mu(B_{\mathcal{I}\wedge\mathcal{J}}) + \frac{\epsilon}{2} \end{aligned}$$

Since $\mathcal{I} \vee \mathcal{J}$ is the set of all elements of $\mathcal{F}$ that can be expressed as the join of an element of $\mathcal{I}$ and an element of $\mathcal{J}$, we know that there exist $B^1_{\mathcal{I}\vee\mathcal{J}} \in \mathcal{I}$ and $B^2_{\mathcal{I}\vee\mathcal{J}} \in \mathcal{J}$ such that

$$B^1_{\mathcal{I}\vee\mathcal{J}} \sqcup B^2_{\mathcal{I}\vee\mathcal{J}} = B_{\mathcal{I}\vee\mathcal{J}}$$

Furthermore, since $\mathcal{I} \wedge \mathcal{J} = \mathcal{I} \cap \mathcal{J}$, we know that $B_{\mathcal{I}\wedge\mathcal{I}} \in \mathcal{I} \cap \mathcal{J}$. Now, let $B_1 \in \mathcal{I}$ and $B_2 \in \mathcal{J}$ be defined by

$$B_1 = B_{\mathcal{I}} \sqcup B^1_{\mathcal{I}\vee\mathcal{J}} \sqcup B_{\mathcal{I}\wedge\mathcal{J}} \quad B_2 = B_{\mathcal{J}} \sqcup B^2_{\mathcal{I}\vee\mathcal{J}} \sqcup B_{\mathcal{I}\wedge\mathcal{J}}$$

Now

$$\begin{aligned} \mu(B_1) + \mu(B_2) &\leq \overline{\mu}(\mathcal{I}) + \overline{\mu}(\mathcal{J}) \\ &\leq \mu(B_1) + \mu(B_2) + \epsilon \\ \mu(B_1) + \mu(B_2) &= \mu(B_1 \sqcup B_2) + \mu(B_1 \sqcap B_2) \\ &\leq \overline{\mu}(\mathcal{I} \vee \mathcal{J}) + \overline{\mu}(\mathcal{I} \wedge \mathcal{J}) \\ &\leq \mu(B_1 \sqcup B_2) + \mu(B_1 \sqcap B_2) + \epsilon \\ &= \mu(B_1) + \mu(B_2) + \epsilon \end{aligned}$$

Hence

$$\left|(\overline{\mu}(\mathcal{I} \vee \mathcal{J}) + \overline{\mu}(\mathcal{I} \wedge \mathcal{J})) - (\overline{\mu}(\mathcal{I}) + \overline{\mu}(\mathcal{J}))\right| \leq \epsilon$$

and so $\overline{\mu}$ satisfies the additivity condition.

Now, to see that $\overline{\mu}$ satisfies the semicontinuity condition, take a directed family $\mathcal{S} = \langle \mathcal{I}_i | i \in I \rangle$ of countably complete sieves in $\mathcal{L}$, and let $\mathcal{I} = \bigvee \mathcal{S}$ be the join of the $\mathcal{I}_i$'s.

We know that $A \in \mathcal{I}$ if and only if there is a countable sequence $\langle A_\alpha | \alpha < \omega \rangle$ contained in $\bigcup_{i \in I} \mathcal{I}_i$ such that $\bigsqcup_{\alpha < \omega} A_\alpha = A$.

Take $\epsilon > 0$. Then there is an $A \in \mathcal{I}$ such that $\overline{\mu}(\mathcal{I}) \leq \mu(A) + \epsilon$. Let $C = \langle A_\alpha | \alpha < \omega \rangle$ be the sequence described in the above paragraph. We may assume without loss of generality that C is an directed sequence. Then since C is countable, we can write

$$\begin{aligned} \bigvee_{\alpha < \infty} \mu(A_\alpha) &\leq \bigvee_{i \in I} \overline{\mu}(\mathcal{I}_i) \\ &\leq \overline{\mu}(\mathcal{I}) \\ &\leq \mu(A) + \epsilon \\ &= \bigvee_{\alpha < \infty} \mu(A_\alpha) + \epsilon \end{aligned}$$

Hence

$$\bigvee_{i \in I} \overline{\mu}(\mathcal{I}_i) = \overline{\mu}\left(\bigvee_{i \in I} \mathcal{I}_i \right)$$

and so $\overline{\mu}$ is a (localic) measure.

□

Theorem 2. *The operations in Lemmas 1 and 2 are inverse to one another. Hence the set of measures on $\mathcal{L}$ is isomorphic to the set of measures on $\mathcal{F}$.*

Proof. Let μ be a measure on $\mathcal{L}$ and let ν be a measure on $\mathcal{F}$. We must show that $\overline{\mu'} = \mu$ and $(\overline{\nu})' = \nu$.

For the first of these, take $\mathcal{I} \in \mathcal{L}$. Then

$$\begin{aligned} \overline{\mu'}(\mathcal{I}) &= \bigvee_{A \in \mathcal{I}} \mu'(A) \\ &= \bigvee_{A \in \mathcal{I}} \mu(\downarrow A) \end{aligned}$$

However, it is immediate that

$$\mathcal{I} = \bigvee_{A \in \mathcal{I}} {\downarrow} A$$

and that $\{{\downarrow}A | A \in \mathcal{I}\}$ is a directed set, and so we have

$$\overline{\mu'}(\mathcal{I}) = \mu(\mathcal{I})$$

Now, take $A \in \mathcal{F}$. Then

$$\begin{aligned} (\overline{\nu})'(A) &= \overline{\nu}({\downarrow}A) \\ &= \bigvee_{B \sqsubseteq A} \nu(B) \\ &= \nu(A) \end{aligned}$$

□

As a consequence of this Theorem, we know that we can study measures on locales in a way that generalizes the study of measures on σ-algebras.

Theorem 2 tells us that the notion of a measure on a locale generalizes the notion of a measure on a σ-algebra. It is natural to ask a related question: If $\mathcal{L}$ is the locale of open sets in some topological space $(X, \mathcal{L})$ and μ is a measure on $\mathcal{L}$, can μ be uniquely extended to the measure space $(X, \sigma(\mathcal{L}))$, where $\sigma(\mathcal{L})$ is the smallest σ-field on X containing $\mathcal{L}$, namely the *Borel* algebra?

The following Theorem gives sufficient conditions for the measures on $\mathcal{L}$ to correspond with the measures on $\sigma(\mathcal{L})$.

Theorem 3. *Let $(X, \mathcal{L})$ be a metrizable Lindelöf space. Then every locally finite measure μ on $\mathcal{L}$ can be uniquely extended to $\sigma(\mathcal{L})$.*

Proof. Take a locally finite μ on $\mathcal{L}$.

Since $(X, \mathcal{L})$ is Lindelöf, and since μ is locally finite, it follows that there is a countable cover of X, with μ finite on each part. We work in the subspace induced by one of these μ-finite sets.

$\mathcal{L}$ is closed under finite intersections and is thus a π-system generating $\sigma(\mathcal{L})$. We can therefore apply Dynkin's π–λ theorem (see Billingsley [3]) to conclude that if μ has an extension, it must be unique.

We work to extend μ recursively, through the Borel heirachy (see Jech [11]).

Definition 26. 1. Σ_0 is the set of open sets in $(X, \mathcal{L})$

2. Π_α is the set $\{X \setminus A | A \in \Sigma_\alpha\}$
3. $\Sigma_{\alpha+1}$ is the set of countable unions of subsets of Π_α
4. When γ is a limit ordinal, then $\Sigma_\gamma = \bigcup_{\alpha<\gamma} \Sigma_\alpha$

Note that there is a duality here between the Σ_αs and the Π_αs. We could just have well taken Π_0 to be the set of closed sets, defined Σ_α as the set of complements of elements of Π_α, and Π_{α_1} as the intersections of countable subsets of Σ_α.

The following properties of the Borel heirachy are useful:

Proposition 3. *1. For any $\alpha < \beta$ we have*

$$(\Sigma_\alpha \cup \Pi_\alpha) \subseteq \left(\Sigma_\beta \cap \Pi_\beta\right)$$

2. *Σ_α and Π_α are closed under finite unions and intersections.*
3. *$\sigma(\mathcal{L}) = \Sigma_{\omega_1}$*

Proof. 1. It is immediate that $\Pi_\alpha \subseteq \Sigma_{\alpha+1}$ and that $\Sigma_\alpha \subseteq \Pi_{\alpha+1}$. We prove that $\Sigma_\alpha \subseteq \Sigma_{\alpha+1}$ and $\Pi_\alpha \subseteq \Pi_{\alpha+1}$ by induction.

Put $\alpha = 0$. Then Σ_1 is the set of F_σ sets, that is, countable unions of closed sets. It is well known that every open set in a metric space is F_σ. Hence $\Sigma_0 \subseteq \Sigma_1$. Similarly, since Π_1 is the set of G_δ sets, it follows that $\Pi_0 \subseteq \Pi_1$.

Now suppose that

$$(\Sigma_\alpha \cup \Pi_\alpha) \subseteq (\Sigma_{\alpha+1} \cap \Pi_{\alpha+1})$$

An element A of $\Sigma_{\alpha+1}$ is the union of a countable family elements of Π_α, and hence the union of a countable family of elements of $\Pi_{\alpha+1}$. Thus A is an element of $\Sigma_{\alpha+2}$. Likewise, an element B of $\Pi_{\alpha+1}$ is the intersection of a countable family in Σ_α, and hence the intersection of a countable family in $\Sigma_{\alpha+1}$. Therefore B is an element of $\Pi_{\alpha+2}$.

Thus we have shown that

$$(\Sigma_{\alpha+1} \cup \Pi_{\alpha+1}) \subseteq (\Sigma_{\alpha+2} \cap \Pi_{\alpha+2})$$

It only remains to show that

$$\Sigma_\alpha \cup \Pi_\alpha \subseteq \Sigma_\beta \cap \Pi_\beta$$

for $\alpha < \beta$, where β is a limit.

It is immediate that $\Sigma_\alpha \subseteq \Sigma_\beta$. If we can show that $\Pi_\alpha \subseteq \Pi_\beta$, we will be finished. But this is also immediate, since an element of Π_α is the complement of an element of Σ_α, and thus the complement of an element of Σ_β, as required.

2. We start by showing that Σ_α is closed under finite intersections, and Π_α is closed under finite unions. We proceed by induction. The result is immediate for $\alpha = 0$, since Σ_0 is the set of open sets, and Π_0 is the set of closed sets. Assume that Π_α is closed under finite unions. Then it follows from DeMorgan's laws that Σ_α is closed under finite intersections. Likewise, if we assume that Σ_α is closed under finite intersections, it follows that Π_α is closed under finite unions.

   To check the results at limits, suppose that γ is a limit ordinal. Since Σ_γ is the union of an expanding sequence of sets, each closed under finite intersections, it follows that Σ_γ is also closed under finite intersections. The fact that Π_γ is closed under finite unions follows directly.

   Now to verify that Σ_α is closed under finite unions, and that Π_α is closed under finite intersections. We again proceed by induction. The base case is immediate. For the successor case, observe that each $\Sigma_{\alpha+1}$ is the union of countably many elements of Π_α, it is trivial that $\Sigma_{\alpha+1}$ is closed under finite unions. Likewise, it is immediate that $\Pi_{\alpha+1}$ is closed under finite intersections. The limit case is similar.

   In fact, we have shown that Σ_α is closed under *countable* unions, and that Π_α is closed under *countable* intersections, except possibly at limit stages.

3. Since the cofinality of ω_1 is uncountable, it follows that Σ_{ω_1} is closed under countable unions and complements. Therefore Σ_{ω_1} is a σ-field containing $\mathcal{L} = \Sigma_0$. Hence

$$\sigma(\mathcal{L}) \subseteq \Sigma_{\omega_1}$$

It is easy to prove, by induction, that each Σ_α is a subset of $\sigma(\mathcal{L})$, and so we have

$$\Sigma_{\omega_1} \subseteq \sigma(\mathcal{L})$$

$\square$

Now, let μ be a finite measure with $\mu(X) = M$. We extend μ through the heirachy.

- $\mu_0 : \Sigma_0 \to [0, M]$ is just μ
- $\mu^*_\alpha : \Pi_\alpha \to [0, M]$ is given by

$$\mu^*_\alpha(F) = M - \mu_\alpha(X \setminus F)$$

- $\mu_{\alpha+1} : \Sigma_{\alpha+1} \to [0, M]$ is given by

$$\mu_{\alpha+1}\left(\bigcup_{i=1}^{\infty} F_i\right) = \bigvee \left\{ \mu^*_\alpha(F) \middle| F \in \Pi_\alpha \wedge \left(F \subseteq \bigcup_{i=1}^{\infty} F_i\right)\right\}$$

- For a limit β, $\mu_\beta(A) = \mu_\alpha(A)$ for some $\alpha < \beta$ satisfying $A \in \Sigma_\alpha$

We must verify that this construction of μ_{ω_1} is well defined, and is indeed a measure (in the σ-algebra sense). Note that $\mu_{\alpha+1}(A)$ does not depend on the choice of countable family $\langle F_i | i < \omega \rangle$ in Π_α.

We start by proving that all the μ_αs are additive, in the sense that

$$\mu_\alpha(A) + \mu_\alpha(B) = \mu_\alpha(A \cup B) + \mu_\alpha(A \cap B)$$

It is immmediate that μ_0 is additive, as it is a measure (in the localic sense) on $\mathcal{L} = \Sigma_0$. Assume that μ_α is a additive. Then it is immediate from DeMorgan's laws that μ^*_α is also additive.

Suppose that μ_α is additive, and consider $A, B \in \Sigma_{\alpha+1}$. Then

$$
\begin{aligned}
& \mu_{\alpha+1}(A) + \mu_{\alpha+1}(B) \\
= \ & \bigvee \{M - \mu_\alpha(F) | F \in \Sigma_\alpha \wedge F \cap A = \emptyset\} + \bigvee \{M - \mu_\alpha(G) | G \in \Sigma_\alpha \wedge G \cap B = \emptyset\} \\
= \ & 2M - \bigwedge \left\{\mu_\alpha(F) + \mu_{\alpha(G)} | F, G \in \Sigma_\alpha \wedge (F \cap A) = (G \cap B) = \emptyset\right\} \\
= \ & 2M - \bigwedge \left\{\mu_\alpha(F \cup G) + \mu_{\alpha(F \cap G)} | F, G \in \Sigma_\alpha \wedge (F \cap A) = (G \cap B) = \emptyset\right\} \\
= \ & 2M - \bigwedge \left\{\mu_\alpha(D) + \mu_{\alpha(E)} | D, E \in \Sigma_\alpha \wedge (D \cap (A \cup B)) = (E \cap (A \cap B)) = \emptyset\right\} \\
= \ & \bigvee \{M - \mu_\alpha(D) | D \in \Sigma_\alpha \wedge D \cap (A \cup B) = \emptyset\} + \bigvee \{M - \mu_\alpha(E) | E \in \Sigma_\alpha \wedge E \cap (A \cap B) = \emptyset\} \\
= \ & \mu_{\alpha+1}(A \cap B) + \mu_{\alpha+1}(A \cup B)
\end{aligned}
$$

The fact that μ_α is additive at limit stages is immediate.

Now that we have shown that the μ_αs are additive, it is immediate that for $\alpha < \beta$, μ_β extends μ_α. In turn, this result shows that μ_β is well defined for limit ordinals β.

Finally, the fact that the μ_αs have the required continuity condition is also immediate from the definition, and the fact that the Π_αs are closed under finite unions.

□

2.2 THE PRESHEAF $\mathbb{S}$

In this section, we make the following notational conventions. $\mathcal{E}$ is a topos (with natural numbers object), $\mathbb{Q}$ is the object of positive rational numbers in $\mathcal{E}$, Ω is the subobject classifier in $\mathcal{E}$, $(\mathcal{L}, \preceq, \top, \bot)$ is a locale, (possibly, although not necessarily, the locale of countably complete sieves on some σ-algebra), and $\widehat{\mathcal{L}}$ is the topos of presheaves on $\mathcal{L}$.

We construct an object $\mathbb{S}$ of $\mathcal{E}$. $\mathbb{S}$ is called the semireal numbers object.

Definition 27. The object $\mathbb{S}$ of *semireals* in $\mathcal{E}$ is the subobject of $\mathcal{P}\mathbb{Q}$ characterized by the formula

$$\phi(S) \equiv \forall q \in \mathbb{Q} \ (q \in S \iff \forall r \in \mathbb{Q} \, q + r \in S)$$

These objects are called semireal numbers because they contain half the data of a Dedekind real; they have an upper cut, but no lower cut. Johnstone [15] and Reichman [21] call them semicontinuous numbers, but that terminology is confusing here as we are using a different notion of semicontinuity to discuss measures.

The justification for calling these numbers "semicontinuous" stems from the fact that if they are interpreted in the topos of sheaves on a topological space, then these numbers do indeed correspond to semicontimuous real valued functions, just as Dedekind real numbers correspond to continuous real valued functions (see Mac Lane and Moerdijk [19]).

Although the semireals can be interpreted in any topos (with natural numbers object), they have a special interpretation in the topos of presheaves over some poset $\mathbb{P}$.

Definition 28. Let $(\mathbb{P}, \leq)$ be a poset, and let $\widehat{\mathbb{P}}$ be the topos of presheaves on $\mathbb{P}$. Say that $\mathbb{S}'$ is the *presheaf of order preserving functionals* on $\mathbb{P}$ if

$$\mathbb{S}'(P) = \{s : \downarrow P \to [0, \infty] | A \leq B \Rightarrow s(A) \leq s(B)\}$$

Theorem 4. *Let $\mathbb{P}$ be a poset, and let $\widehat{\mathbb{P}}$ be the topos of presheaves on $\mathbb{P}$. Then inside $\widehat{\mathbb{P}}$ we have*

$$\mathbb{S} \cong \mathbb{S}'$$

In order to study the elements of $\mathbb{S}(P)$, we use the following Lemma:

Lemma 4. *Assume that $\mathcal{E} = \widehat{\mathbb{P}}$ for some poset $\mathbb{P}$. A subfunctor $S \rightarrowtail \mathbb{Q}$ is a semireal if and only if for every $A \in \mathbb{P}$, $S(A)$ is a topologically closed upper segment of the positive rationals.*

Note that a "topologically closed upper segment of the positive rationals" is the same thing as "the set of all positive rationals greater than or equal to some extended real $x \in [0, \infty]$".

Proof. $\mathbb{S}$ is a subobject of $\mathcal{P}\mathbb{Q}$. Therefore if $S \in \mathbb{S}(A)$, then S is a subfunctor of $\mathbb{Q}$ satisfying $S(B) \subseteq S(C)$, whenever $C \leq B \leq A$ (and $S(D) = \emptyset$ for and $D \nleq A$). We can interpret the formula $\phi(S)$ that characterizes $\mathbb{S}$ by using Kripke-Joyal sheaf semantics. In this framework, we can write $P \Vdash \phi(S)$ for $S \in \mathbb{S}(P)$.

$$
\begin{aligned}
& P \Vdash \forall q \in \mathbb{Q}\ (q \in S \iff \forall r \in \mathbb{Q}\, q + r \in S) \\
\longleftrightarrow\quad & \text{for every } Q \leq P \text{ and every } q \in \mathbb{Q} \\
& Q \Vdash q \in S' \iff \forall r \in \mathbb{Q}\, q + r \in S' \\
& \text{where } S' \text{ is the restriction of } S \text{ to } Q \\
\longleftrightarrow\quad & \text{for every } q \in \mathbb{Q} \text{ and every } Q \leq P \text{ we have} \\
& Q \Vdash (q \in S) \Rightarrow (\forall r \in \mathbb{Q}\, q + r \in S) \text{ and} \\
& Q \Vdash \forall r \in \mathbb{Q}\ (q + r \in S \Rightarrow q \in S)
\end{aligned}
$$

If $Q \Vdash (q \in S) \Rightarrow (\forall r \in \mathbb{Q}\, q + r \in S)$ then for every $R \leq Q$ such that $q \in S(R)$, we must have

$$R \Vdash \forall r \in \mathbb{Q}\, (q + r) \in S'$$

But this is just equivalent to saying that if q is an element of $S(R)$ then all rationals greater than q are also elements of $S(R)$. Hence $S(R)$ is an upper segment of rationals.

Now, if $Q \Vdash \forall r \in \mathbb{Q}\, (q + r) \in S \Rightarrow q \in S$, then for every $R \leq Q$ such that

$$\forall r \in \mathbb{Q}\, (q + r) \in S(R)$$

we must have $q \in S(R)$. This means that if all the rationals greater than q are elements of $S(R)$, then q must also be an element of $S(R)$. Hence $S(R)$ is (topologically) closed.

Since R is an arbitrary element of $\downarrow P$, it follows thta $S(P)$ is a topologically closed upper segment of rationals.

□

We can now prove Theorem 4.

Proof. Fix $P \in \mathbb{P}$. We construct a bijection between $\mathbb{S}(P)$ and $\mathbb{S}'(P)$. Take $S \in \mathbb{S}(P)$. Then let the order preserving functional $s : \downarrow P \to [0, \infty]$ be given by

$$s(Q) = \bigwedge S(Q)$$

Now, given an order preserving functional $t \in \mathbb{S}'(P)$, we define $T \in \mathbb{S}(P)$ by

$$T(Q) = \{q \in \mathbb{Q} | t(Q) \leq q\}$$

It is immediate that the two operations are inverse to one another. The fact that s is an order preserving map is a consequence of the fact that S is a subobject of $\mathbf{y}A\times\mathbb{Q}$: $R \leq Q \leq P$ implies that $S(R) \supseteq S(Q)$, and so that $s(R) \leq s(Q)$. $\square$

As with other number systems, the semireals have a number of important properties.

Proposition 5. *There is an embedding* $\mathbb{Q} \to \mathbb{S}$ *given by*

$$q \mapsto \{r \in \mathbb{Q} | q \leq r\}$$

Proof. First note that if we are working in the topos of presheaves on a poset, then the result is an immediate consequence of Lemma 4.

We work internally in $\mathcal{E}$. Fix $q \in \mathbb{Q}$. We show that $\{r \in \mathbb{Q}|q \leq r\}$ is a semireal.

$$\mathbf{q} = \{r \in \mathbb{Q}|q \leq r\} = \{r \in \mathbb{Q}|q < r \vee q = r\}$$

We need to show that $(r \in \mathbf{q}) \iff \forall s \in \mathbb{Q}\,(r + s \in \mathbf{q})$. Suppose that $r \in \mathbf{q}$. Then $q < r$ or $q = r$. In either case, $q < r + s$, and so $r + s \in \mathbf{q}$.

Conversely, suppose that $\forall s \in \mathbb{Q}\,(r + s) \in \mathbf{q}$. To show that $r \leq q$, we exploit the fact that the rationals are totally ordered, and so satisfy the following formula:

$$\forall r \in \mathbb{Q}(r < q) \vee (q \leq r)$$

Suppose that $r < q$. Then let $s = \frac{q-r}{2}$. Then $r + s < q$. Hence $(r + s) \notin \mathbf{q}$. This is a contradiction, and so we must have $q \leq r$, as required. $\square$

In view of Theorem 4, it would obviously be convenient to have some form of evaluation operation for the functionals. Unfortunately, there is no natural way to do this directly. Suppose we were to try for a morphism of the form $\Omega \times \mathbb{S} \to \mathbb{R}$, where Ω is the subobject classifier, and $\mathbb{R}$ the object of Dedekind real numbers. In order for such a morphism to be a natural transformation, we would need the following diagram to commute (for $R \leq Q$):

$$\begin{tikzcd} \Omega(Q) \times \mathbb{S}(Q) \arrow[r] \arrow[d, "\rho"'] & \mathbb{R} \arrow[d, "="] \\ \Omega(R) \times \mathbb{S}(R) \arrow[r] & \mathbb{R} \end{tikzcd}$$

However, the object of Dedekind real numbers in a presheaf topos is just the constant functor (see Lemma 6 below), and so the right hand restriction here is just equality. But since $s(Q) \neq s(R)$, in general, our evaluation map wold not be compatible with this restriction.

Lemma 6. *Let $\widehat{\mathbb{C}}$ be a presheaf topos. Then the object $\mathbb{R}$ of Dedekind reals in $\widehat{\mathbb{C}}$ is a constant functor whose value at every object C of $\mathbb{C}$ is just the set of real numbers.*

Proof. It is well known that in a presheaf topos $\widehat{\mathbb{C}}$, the rational numbers object $\mathbb{Q}$ is just the presheaf $\Delta\mathbf{Q}$, whose action at every object C of $\mathbb{C}$ is just the set $\mathbf{Q}$ of rationals.

Let $\mathbb{D}$ be the Dedekind real numbers object of $\widehat{\mathbb{C}}$. Then an element of $\mathbb{D}(C)$ is a pair $\langle L, U\rangle$ of subfunctors of $\mathbf{y}C \times \mathbb{Q}$. For any object D, $L(D)$ is a family $\langle S_f | f \in \mathrm{Hom}(D, C)\rangle$ of open lower sets of rationals. Likewise $U(D)$ is a family $\langle T_f | f \in \mathrm{Hom}(D, C)\rangle$ of open upper sets of rationals. Following the arguments in Theorem 4 we can construct functionals l and u from t_C, the maximal sieve on C, to $\mathbf{R}$, the set of reals. These functional are given by

$$l(f) = \bigvee S_f \qquad s(f) = \bigwedge T_f$$

(Note that S_f and T_f are members of $L(\mathrm{dom}(f))$ and $U(\mathrm{dom}(f))$ respectively.)

There is a preordering on t_C. Write $f \leq g$ if there is an $h : \mathrm{dom}(f) \to \mathrm{dom}(g)$ such that $f \circ h = g$:

$$\begin{tikzcd}
& C & \\
\mathrm{dom}(g) \arrow[ur, "g"] & & \\
& & \mathrm{dom}(f) \arrow[ull, dotted, "h"] \arrow[uul, "f"']
\end{tikzcd}$$

Note that id_C is the top element of this preorder. If $\mathbb{C}$ has an initial object $\mathbf{0}$, then the initial map $! : \mathbf{0} \to C$ is the minimal element. It is easy to see that l is an order reversing functional, and that u is order preserving.

We have used most of the axioms of a Dedekind real in order to build l and u. However, we have not used the disjointness and apartness axioms:

$$\begin{aligned}
\widehat{\mathbb{C}} &\models \forall q \in \mathbb{Q}\, \neg\, (q \in L \wedge q \in U) \\
\widehat{\mathbb{C}} &\models \forall q, r \in \mathbb{Q}\, (q < r) \Rightarrow (q \in L \vee r \in U)
\end{aligned}$$

Since elements of $\mathbb{Q}$ are just constant rational numbers, we can represent a rational as a constant functional $\tilde{q} : t_C \to \mathbf{R}$. The above conditions can now be rewritten in terms of the functionals l and u:

$$\begin{aligned}
\widehat{\mathbb{C}} &\models \forall q \in \mathbb{Q}\, \neg\, (l < \tilde{q}) \wedge (\tilde{q} < u)) \\
\widehat{\mathbb{C}} &\models \forall q, r \in \mathbb{Q}\, (q < r) \Rightarrow ((\tilde{q} < l) \vee (\tilde{r} < u))
\end{aligned}$$

Since $\widehat{C}$ is a presheaf topos, it follows that for every arrow $f \in t_C$, we must have

$$\forall q \in \mathbf{Q}\, \neg\, (l(f) < q) \wedge (q < u(f)))$$

$$\forall q, r \in \mathbf{Q}\, (q < r) \Rightarrow ((q < l(f)) \vee (r < u(f)))$$

But this implies that for any f, $\langle S_f, T_f \rangle$ is a Dedekind real in $\textsc{Set}$ (that is, a real number). Furthermore, S_f and T_f must be independent of f, as $\langle S_{\mathrm{id}_C}, T_{\mathrm{id}_C} \rangle$ is also a Dedekind real, and we must have

$$S_f \supseteq S_{\mathrm{id}_C} \qquad T_f \supseteq T_{\mathrm{id}_C}$$

□

So, we cannot have a direct evaluation map for the semireals. However, we do have an indirect evaluation map. We can use the following composition:

$$\mathbb{Q} \times \mathbb{S} \rightarrowtail \mathbb{Q} \times \mathcal{P}\mathbb{Q} \xrightarrow{\in} \Omega$$

In the special case where $\mathcal{E} = \widehat{\mathbb{P}}$, the "element of" map takes a rational q and a semireal S to the sieve $\mathcal{I} = \{P \in \mathbb{P} | q \in S(P)\}$. But applying Theorem 4, we see that $\mathcal{I} = \{P \in \mathbb{P} | s(P) \leq q\}$, where s is the functional associated with S by Theorem 4. This map, the "element of" map will serve as our evaluation map, taking a rational and a semireal to the sieve where the the semireal is smaller that q.

There is a natural partial ordering on $\mathbb{S}$, extending the usual ordering on $\mathbb{Q}$.

Definition 29. Let S and T be two semireals. Then

$$S \leq T \equiv S \supseteq T$$

Note that in the event that $\mathcal{E} = \widehat{\mathbb{P}}$, then this coincides with the usual ordering of functionals on $\mathbb{P}$.

This ordering is just the reverse of the inclusion inherited from $\mathcal{P}\mathbb{Q}$. It turns out that with this ordering, $\mathbb{S}$ is internally a complete lattice:

Proposition 7. *Take $\mathcal{S} \subseteq \mathbb{S}$ and define $\bigvee \mathcal{S}$ and $\bigwedge \mathcal{S}$ by*

$$\begin{aligned} \bigvee \mathcal{S} &= \{q \in \mathbb{Q} | \forall S \in \mathcal{S}\, q \in S\} \\ \bigwedge \mathcal{S} &= \{q \in \mathbb{Q} | \forall r \in \mathbb{Q} \exists S \in \mathcal{S}\, q + r \in S\} \end{aligned}$$

Then $\bigvee$ and $\bigwedge$ are the supremum and infimum operators on $\mathbb{S}$ respectively.

Proof. It is immediate from Definition 29 that if $\bigvee \mathcal{S}$ and $\bigwedge \mathcal{S}$ are indeed semireals then they must be the supremum and infimum of $\mathcal{S}$ respectively.

Hence it suffices to show that they are semireals. But this is immediate from their definitions. □

There are also a number of algebraic operations defined on $\mathbb{S}$: Semireals can be added together, multiplied by a rational, and restricted to a truth value (or sieve, when working externally). All of these operations are defined using the internal logic of $\mathcal{E}$, treating semireals as certain sets of rationals.

We define addition first:

Definition 30.

$$S + T = \{q \in \mathbb{Q} | \forall r \in \mathbb{Q}\, \exists s \in S\, \exists t \in T\ (s + t = q + r)\}$$

Proposition 8. *The addition of two semireals, as defined above, does indeed yield a semireal.*

Proof. Let S and T be two semireals.

Firstly, we show that if $q \in S + T$ and $u \in \mathbb{Q}$, then $(q + u) \in (S + T)$. Take $r \in \mathbb{Q}$. Then there exist $s \in S$ and $t \in T$ such that $q + (u + r) = s + t$. But this is all that is needed to show that $(q + u) \in (S + T)$. This shows that $S + T$ is an upper segment.

Now, assume that $(q + u) \in (S + T)$ for every $u \in \mathbb{Q}$. We need to show that

Take $r \in \mathbb{Q}$. We know that $\left(q + \frac{r}{2}\right) \in (S + T)$, so there must be $s \in S$ and $t \in T$ such that

$$\left(q + \frac{r}{2} + \frac{r}{2}\right) = s + t$$

Consequently,

$$\forall r \in \mathbb{Q}\, \exists s \in S\, \exists t \in T\, (q + r) = (s + t)$$

But this implies that $q \in S + T$, as required.

□

Multiplication of a semireal by a rational is also defined internally:

Definition 31. Take $a \in \mathbb{Q}$ and $S \in \mathbb{S}$, Then the product $a \times S$ is given by:

$$a \times S = \left\{q \in \mathbb{Q} \middle| \frac{q}{a} \in S\right\}$$

Note that the right hand side here is just $\{a \cdot q | q \in S\}$.

Proposition 9. *Multiplication of a semireal S by a rational a as described above does indeed yield a semireal.*

Proof. Take $q \in a \times S$, and $r \in \mathbb{Q}$. Since $\frac{q}{a} \in S$, it follows that $\frac{q+r}{a} \in S$. But this is just what is needed to prove that $(q + r) \in a \times S$.

Now assume that for every $r \in \mathbb{Q}$ we have $(q + r) \in (a \times S)$. Then for every $r \in \mathbb{Q}$ we have $\frac{q}{a} + \frac{r}{a} \in S$. Putting $s = \frac{r}{a}$, this is equivalent to saying that for every $s \in \mathbb{Q}$ we have $\frac{q}{a} + s \in S$. Since S is a semireal, this in turn implies that $\frac{q}{a} \in S$, whence $q \in a \times S$, as required. $\square$

It is clear that $\mathbb{Q}$ is a commutative division semiring (a field, except without additive inverses, and without zero).

Proposition 10. *The object $\mathbb{S}$ of semireals is a semimodule over $\mathbb{Q}$, with the operations of addition and scalar multiplication as defined above.*

Proof. $\langle \mathbb{S}, + \rangle$ is clearly an abelian monoid (associativity of addition is easy to check). Thus we need only show that for any $a, b \in \mathbb{Q}$ and $S, T \in \mathbb{S}$, we have

1. $a \times (S + T) = (a \times S) + (a \times T)$
2. $(a + b) \times S = a \times S + b \times S$
3. $a \times (b \times S) = (a \cdot b) \times S$
4. $1 \times S = S$

1. Suppose that $q \in a \times (S + T)$. Then $\frac{q}{a} \in S + T$. This means that for any $r \in \mathbb{Q}$, there exist $s \in S$ and $t \in T$ such that $s + t = \frac{q}{a} + r$. Consequently, $a \cdot s \in a \times S$ and $a \cdot t \in a \times T$. Hence

$$a \cdot s + a \cdot t = q + a \cdot r$$

   which, since a is fixed, and r is arbitrary, shows that $q \in a \times S + a \times T$.
   For the converse direction, suppose that $q \in a \times S + a \times T$. Then for any $r \in \mathbb{Q}$, there exist $s \in q \times S$ and $t \in a \times T$ such that $s + t = q + r$. Since $\frac{s}{a} \in S$, and $\frac{t}{a} \in T$, it follows that $\frac{q}{a} \in S + T$, whence $q \in a \times (S + T)$, as required.

2. Suppose that $q \in a \times S + b \times S$. Then for any $r \in \mathbb{Q}$ we know that there exist $s_1, s_2 \in S$ such that $as_1 + bs_2 = q + r$. Since the rationals are totally ordered, we may assume without loss of generality that $s_1 \leq s_2$, so that $s_2 = s_1 + d$. Hence $(a+b)s_1 \leq q + r$ and so, for every $r \in \mathbb{Q}$, we have
$$s_1 \leq \frac{q+r}{a+b}$$
Therefore $q + r \in (a+b) \times S$, whence $q \in (a+b) \times S$, as required.
For the converse direction, suppose that $q \in (a+b) \times S$. The $s = \frac{q}{a+b} \in S$. It will suffice to find s_1 and s_2 in S such that $a \cdot s_1 + b \cdot s_2 = q$. Put $s_1 = s_2 = s$. Then
$$\begin{aligned} a \cdot s_1 + b \cdot s_2 &= a \cdot s + b \cdot s \\ &= (a+b) \cdot s \\ &= (a+b)\frac{q}{a+b} \\ &= q \end{aligned}$$
as required.
3. This is immediate.
4. This is also immediate.

□

With this semimodule structure established, we can now study the restriction operation.

Definition 32. The restriction operator $\rho : \mathbb{S} \times \Omega \to \mathbb{S}$ is defined internally:

$$\rho(\mathcal{I}, S) = \{q \in \mathbb{Q} | \mathcal{I} \Rightarrow q \in S\}$$

Lemma 11. *Take $\mathcal{I} \in \Omega$ and $S \in \mathbb{S}$. Then $\rho(S, \mathcal{I})$, as described above, is indeed a semireal.*

Proof.

$$\begin{aligned} \mathcal{I} \Rightarrow q \in S &\leftrightarrow \mathcal{I} \Rightarrow (\forall r \in \mathbb{Q} q + r \in S) \\ &\leftrightarrow \forall r \in \mathbb{Q}\ [\mathcal{I} \Rightarrow (q + r \in S)] \\ &\leftrightarrow \forall r \in \mathbb{Q}\ [q + r \in \rho(S, \mathcal{I})] \end{aligned}$$

□

The restriction operation, $\rho : \mathbb{S} \times \Omega \to \mathbb{S}$ can be thought of as an Ω indexed family of linear maps from the semimodule $\mathbb{S}$ to itself.

Proposition 12. *1. For any $\mathcal{I} \in \Omega$, the operation $\rho(-, \mathcal{I}) : \mathbb{S} \to \mathbb{S}$ is a linear map.*

2. *For a fixed $S \in \mathbb{S}$, the operation $\rho(S, -) : \Omega \to \mathbb{S}$ is an order preserving map.*
3. *For a fixed $S \in \mathbb{S}$, we have $\rho(S, \top) = S$ and $\rho(S, \bot) = \mathbf{0}$, where $\mathbf{0}$ is the bottom element of $\mathbb{S}$.*

Proof. 1. To see that $\rho(-, \mathcal{I})$ preserves sums, note that the following argument is intuitionistically valid:

$$\begin{array}{rl} & q \in \rho(S + T, \mathcal{I}) \\ \equiv & \mathcal{I} \Rightarrow (q \in S + T) \\ \longleftrightarrow & \mathcal{I} \Rightarrow \forall r \in \mathbb{Q}\, \exists s, t \in \mathbb{Q}\ (s \in S \wedge t \in T) \wedge (s + t = q + r) \\ \longleftrightarrow & \forall r \in \mathbb{Q}\, \exists s, t \in \mathbb{Q}\ [\mathcal{I} \Rightarrow (s \in S \wedge t \in T)] \wedge (s + t = r) \\ \equiv & q \in \rho(S, \mathcal{I}) + \rho(T, \mathcal{I}) \end{array}$$

The fact that $\rho(-, \mathcal{I})$ preserves scalar multiplication is immediate.

2. Suppose that $\mathcal{I} \leq \mathcal{J}$. Then $\mathcal{J} \Rightarrow q \in S$, implies that $\mathcal{I} \Rightarrow q \in S$, so that $\rho(S, \mathcal{I}) \supseteq \rho(S, \mathcal{J})$, as required.
3. This is immediate.

□

Our goal is to provide a logical construction (in $\mathcal{E}$) of $\mathbb{M}$, the object of measures. We take as our data not just $\mathcal{E}$, but also a topology j on $\mathcal{E}$. This means that we say that $\mathbb{M}$ is the measure object of $\mathcal{E}$, *relative to the topology* j. In the special case where $\mathcal{E} = \widehat{\mathcal{L}}$, and j is the canonical topology on $\mathcal{L}$, then $\mathbb{M}$ is a j-sheaf.

It turns out that we will only ever need to take the restriction to closed truth values (or closed sieves, in the external view). Hence we take ρ to have $\Omega_j \times \mathbb{S} \rightarrowtail \Omega \times \mathbb{S}$ as its domain.

In the case that $\mathcal{E} = \widehat{\mathbb{P}}$ (of course, this case subsumes the case where $\mathcal{E} = \widehat{\mathcal{L}}$) all of the operations that we have defined on $\mathbb{S}$ have the natural interpretations when applied to the associated functionals:

Proposition 13. *Suppose that $\mathcal{E} = \widehat{\mathbb{P}}$ is the topos of presheaves on some poset $\mathbb{P}$. Take $S, T \in \mathbb{S}(P)$, $\{S_i | i \in I\} \subseteq \mathbb{S}(P)$, $\mathcal{I} \in \Omega(P)$ and $a \in \mathbb{Q}(P)$, and let $s, t, \{s_i | i \in I\}$ be the associated functionals. Then:*

1. *The associated functional of $S + T$ is $s + t$*
2. *The associated functional of $a \times S$ is $a \cdot s$*
3. *The associated functional of $\rho(S, \mathcal{I})$ is given by*

$$\rho(s, \mathcal{I})(Q) = \bigvee_{R \in \downarrow Q \cap \mathcal{I}} s(R)$$

4. *$S \leq T$ if and only if $s \leq t$.*
5. *The associated functional of $\bigvee_{i \in I} S_i$ is $\bigvee_{i \in I} s_i$*

Proof. Except for part 3, this is immediate from Theorem 4.

For part 3, we can use sheaf semantics. Recall that $s(P) \leq q \leftrightarrow P \Vdash q \in S$. Then

$$\begin{aligned} \rho(s, \mathcal{I})(P) \leq q &\leftrightarrow P \Vdash q \in \rho(S, \mathcal{I}) \\ &\leftrightarrow P \Vdash \mathcal{I} \Rightarrow q \in S \\ &\leftrightarrow \text{for all } R \in \mathcal{I} \text{ such that } R \leq P, R \Vdash q \in S \\ &\leftrightarrow \text{for all } R \in \mathcal{I} \text{ such that } R \leq P, S(R) \leq q \\ &\leftrightarrow \bigvee_{R \in \mathcal{I} \cap \downarrow P} S(R) \leq q \end{aligned}$$

□

Note that in the case where $\mathbb{P}$ is a meet semilattice, part 3 can be rewritten

$$\rho(s, \mathcal{I})(Q) = \bigvee_{R \in \mathcal{I}} s(R \wedge Q)$$

Furthermore, if $\mathbb{P}$ is a locale, and $\mathcal{I}$ is a closed sieve, then there is an $I \in \mathcal{L}$ such that $\mathcal{I} = \downarrow I$ and we can write

$$\rho(s, \mathcal{I})(Q) = s(I \wedge Q)$$

This observation provides the motivation for calling ρ the "restriction" operation.

2.3 THE CONSTRUCTION OF $\mathbb{M}$

In this section, we work in a topos $\mathcal{E}$ (with natural numbers object). Ω is the subobject classifier in $\mathcal{E}$, $\mathbb{Q}$ is the object of positive rationals in $\mathcal{E}$, and $\mathbb{S}$ is the object of semireal numbers in $\mathcal{E}$. We assume that there is a designated topology $j : \Omega \to \Omega$, which induces the sheaf topos $\mathrm{Sh}_j(\mathcal{E})$. The subobject classifier in $\mathrm{Sh}_j(\mathcal{E})$ is denoted Ω_j. We refer to $\mathcal{E}$ as the "presheaf topos", and $\mathrm{Sh}_j(\mathcal{E})$ as the "sheaf topos". Sometimes, we make the additional assumption that $\mathcal{E}$ is the topos of presheaves on some locale $\mathcal{L}$. In this case, j will be the canonical topology on $\mathcal{L}$.

In this section we construct a subobject $\mathbb{M}$ of $\mathbb{S}$. In the special case where $\mathcal{E} = \widehat{\mathcal{L}}$, $\mathbb{M}$ is the presheaf of measures (Definition 25), and is in fact a sheaf (Theorem 6).

To construct $\mathbb{M}$, we find logical formulas that pick out those semireals satisfying the additivity and semicontinuity conditions of Definition 25.

We start with additivity.

Definition 33. The *additive semireals* are semireals satisfying the following formula:

$$\phi(S) \equiv \forall \mathcal{I}, \mathcal{J} \in \Omega_j \ [\rho(S, \mathcal{I}) + \rho(S, \mathcal{J}) = \rho(S, \mathcal{I} \wedge \mathcal{J}) + \rho(S, \mathcal{I} \vee \mathcal{J})]$$

(where "$\wedge$" and "$\vee$" are the meet and join in Ω_j.)

The presheaf of additive semireals is denoted $\mathbb{S}_A \rightarrowtail \mathbb{S}$

Proposition 14. *Suppose that $\mathcal{E} = \widehat{\mathcal{L}}$ is the topos of presheaves on some locale $\mathcal{L}$. Let j be the cannonical topology. Then a semireal $S \in \mathbb{S}(A)$ is additive if and only if the associated functional $s : \downarrow A \to [0, \infty]$ satisfies*

$$s(B) + s(C) = s(B \wedge C) + s(B \vee C)$$

Proof. $\Rightarrow$ This direction follows immediately from Proposition 13, by considering the closed sieves $\downarrow B$ and $\downarrow C$.

$\Leftarrow$ For the reverse direction, we can use the fact that any closed sieves $\mathcal{I}$ and $\mathcal{J}$ are in fact principal sieves $\downarrow I$ and $\downarrow J$ respectively. Then taking an arbitrary $A \in \mathcal{L}$, we have

$$\begin{aligned}
\rho(s,\mathcal{I})(A) + \rho(s,\mathcal{J})(A) &= \rho(s,\downarrow I)(A) + \rho(s,\downarrow J)(A) \\
&= s(A \wedge I) + s(A \wedge J) \\
&= s\left((A \wedge I) \wedge (A \wedge J)\right) + s\left((A \wedge I) \vee (A \wedge J)\right) \\
&= s\left(A \wedge (I \wedge J)\right) + s\left(A \wedge (I \vee J)\right) \\
&= \rho\left(s, \downarrow(I \wedge J)\right)(A) + \rho\left(s, \downarrow(I \vee J)\right)(A) \\
&= \rho\left(s, \mathcal{I} \wedge \mathcal{J}\right)(A) + \rho\left(s, \mathcal{I} \vee \mathcal{J}\right)(A)
\end{aligned}$$

$\square$

Characterizing the semicontinuity condition requires some preparatory steps.

Definition 34. $\mathcal{I} \in \Omega$ is called "directed" if it satisfies the condition:

$$\forall \mathcal{J},\mathcal{K} \in \Omega_j \, \mathcal{J} \vee \mathcal{K} \leq \mathcal{I} \Rightarrow \overline{\mathcal{J} \vee \mathcal{K}} \leq \mathcal{I}$$

Using this formula, we find an object $\Omega_{\mathcal{D}} \rightarrowtail \Omega$ of directed truth values. It is easy to see that in $\widehat{\mathcal{L}}$, $\Omega_{\mathcal{D}}(A)$ consists of the ideals in $\downarrow A$.

Definition 35. A sieve $\mathcal{I} \in \Omega$ is called "directed closed" if it satisfies the condition:

$$\forall \mathcal{J} \in \Omega_{\mathcal{D}} \; \mathcal{J} \leq \mathcal{I} \Rightarrow \overline{\mathcal{J}} \leq \mathcal{I}$$

If $\mathcal{E} = \widehat{\mathcal{L}}$, then the directed closed sieves are those that are closed under directed joins. As an example of a directed closed sieve that is not closed, fix $A, B \in \mathcal{L}$, and let

$$\mathcal{I} = (\downarrow A) \cup (\downarrow B)$$

The following is immediate:

Proposition 15.

$$\Omega_j \rightarrowtail \Omega_{\mathcal{DC}} \rightarrowtail \Omega_{\mathcal{D}} \rightarrowtail \Omega$$

Before we describe the semicontinuity condition, we need to introduce a notational convention. One feature of topos logic is that logical formulas are themselves arrows in the topos (arrows into the subobject classifier Ω). Since the Ω is object of truth values, the arrow into Ω can be thought of as representing the truth value of a formula. To avoid ambiguity, when we refer to the truth value of a formula ϕ, we shall use Scott brackets: $[\![\phi]\!]$. For example, if q is a rational number, and S is a semireal, then $[\![q \in S]\!]$ is the truth value of the formula $q \in S$. If we are working in a presheaf topos $\widehat{\mathcal{L}}$, then $[\![q \in S]\!]$ is the sieve of those $A \in \mathcal{L}$ such that $q \in S(A)$.

We can now describe the semicontinuity condition for semireals.

Definition 36. The object $\mathbb{S}_C$ of semicontinuous semireals is defined by the following formula

$$\mathbb{S}_C = \left\{ S \in \mathbb{S} \middle| \forall q \in \mathbb{Q} \;\; [\![q \in S]\!] \in \Omega_{\mathcal{D}C} \right\}$$

In the case where $\mathcal{E} = \widehat{\mathbb{P}}$, an order preserving functional $s : \downarrow A \to [0, \infty]$ corresponds to an element of $\mathbb{S}_C(A) \subseteq \mathbb{S}(A)$ if for every rational q, the set

$$\{B \leq A | s(B) \leq q\}$$

is closed under directed joins.

It follows immediately that:

Theorem 5. *Let $\mathcal{E} = \widehat{\mathcal{L}}$ be the topos of presheaves on a locale $\mathcal{L}$. Then $\mathbb{M}$, the presheaf of measures is defined by the following pullback:*

$$\begin{tikzcd}
\mathbb{M} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}_C \arrow[d, tail] \\
\mathbb{S}_A \arrow[r, tail] & \mathbb{S}
\end{tikzcd}$$

Equivalently, a measure is a semireal that is both additive and semicontinuous. Internally, this can be written:

$$\mathbb{M} = \mathbb{S}_A \cap \mathbb{S}_C \subseteq \mathbb{S}$$

Theorem 6. *Let $\mathbb{M}$ be the presheaf of measures in the topos $\mathcal{E} = \widehat{\mathcal{L}}$. Then $\mathbb{M}$ is a sheaf.*

The are two approaches to proving this Theorem. The first option is to use the external interpretation of $\mathbb{M}$ as the presheaf of measures, and show that measures can be amalgamated in the appropriate ways. The second approach is to use the logical characterization of $\mathbb{M}$.

We use a combination of the two approaches. Explicit reference is not made to the fact that the elements of $\mathbb{S}(A)$ are measures. However, we do make use of the fact that $\mathrm{Sh}_j(\mathcal{E})$ is a localic topos.

Proof. We must show that given $A \in \mathcal{L}$, a cover C for A, and a matching family $\mathcal{M} = \langle \mu_C \in \mathbb{M}(C) | C \in C \rangle$ for $\mathbb{M}$ and C, there is a unique $\mu \in \mathbb{M}(A)$ which is an amalgamation for $\mathcal{M}$.

We do this by looking at the nature of C. Recall that in a locale $\mathcal{L}$, a sieve $C \subseteq \mathcal{L}$ is a cover for A if and only if $\curlyvee C = A$.

First, consider the case where C is directed. In this case, since measures must satisfy the semicontinuity condition, it follows that if μ is an amalgamation, we must have

$$\mu(B) = \bigvee \{\mu(B \wedge C) | C \in C\}$$

Hence, if an amalgamation exists, it must be unique. We now show that the μ defined above is a measure, and that it is an amalgamation of $\mathcal{M}$. The latter follows immediately from the fact that $\mathcal{M}$ is a matching family. To see that μ is a measure, take $B_1, B_2 \leq A$. Then

$$\mu(B_1) + \mu(B_2) = \bigvee \{\mu(B_1 \wedge C) | C \in C\} + \bigvee \{\mu(B_2 \wedge C) | C \in C\}$$

Fix an $\epsilon > 0$. Then there exists C_1, C_2, C_3, C_4 in C such that

$$\begin{array}{rcccl} \mu(B_1 \wedge C_1) & \leq & \mu(B_1) & \leq & \mu(B_1 \wedge C_1) + \dfrac{\epsilon}{2} \\ \mu(B_2 \wedge C_2) & \leq & \mu(B_2) & \leq & \mu(B_2 \wedge C_2) + \dfrac{\epsilon}{2} \\ \mu((B_1 \wedge B_2) \wedge C_3) & \leq & \mu(B_1 \wedge B_2) & \leq & \mu((B1 \wedge B_2) \wedge C_3) + \dfrac{\epsilon}{2} \\ \mu((B_1 \vee B_2) \wedge C_4) & \leq & \mu(B_1 \vee B_2) & \leq & \mu((B1 \vee B_2) \wedge C_4) + \dfrac{\epsilon}{2} \end{array}$$

Since C is directed, there is a $C_0 \in C$ such that for all $i \in \{1, \ldots, 4\}$, $C_i \leq C_0$. Then

$$\begin{array}{rcl} \mu(B_1) + \mu(B_2) & \leq & \mu(B_1 \wedge C_0) + \mu(B_1 \wedge C_0) + \epsilon \\ \mu(B_1 \vee B_2) + \mu(B_1 \wedge B_2) & \leq & \mu((B_1 \wedge B_2) \wedge C_0) + \mu((B_1 \vee B_2) \wedge C_0) + \epsilon \end{array}$$

Since $\mu(B_1 \wedge C_0) + \mu(B_1 \wedge C_0) = \mu((B_1 \wedge B_2) \wedge C_0) + \mu((B_1 \vee B_2) \wedge C_0)$, it follows that the difference between $\mu(B_1) + \mu(B_2)$ and $\mu(B_1 \wedge B_2) + \mu(B_1 \vee B_2)$ must be less than ϵ. Hence the amalgamation is additive.

We must show that μ satisfies the semicontinuity condition. Take an increasing chain $\mathcal{A}$ in $\downarrow A$. Then

$$
\begin{aligned}
\mu\left(\Upsilon \mathcal{A}\right) &= \bigvee \left\{\mu\left(\left(\Upsilon \mathcal{A}\right) \wedge C\right) | C \in \mathcal{C}\right\} \\
&= \bigvee \left\{\mu\left(\Upsilon \mathcal{A} \wedge C\right) | C \in \mathcal{C}\right\} \\
&= \bigvee \left\{\bigvee \{\mu\,(A \wedge C)\,|C \in \mathcal{C}\}\,|A \in \mathcal{A}\right\} \\
&= \bigvee \left\{\bigvee \{\mu\,(A \wedge C)\,|A \in \mathcal{A}\}\,|C \in \mathcal{C}\right\} \\
&= \bigvee \{\mu(A)|A \in \mathcal{A}\}
\end{aligned}
$$

This shows that if $\mathcal{I}$ is a *directed* sieve, then the sieve of elements of $\mathcal{L}$ for which a matching family for $\mathbb{M}$ on $\mathcal{I}$ can be amalgamated contains $\overline{\mathcal{I}}$.

We now show that given *any* sieve $\mathcal{I}$, and matching family, that family can be uniquely extended to some directed sieve that contains $\mathcal{I}$, namely the closure of $\mathcal{I}$ under the *finite* join topology. To do this it suffices to show that if B and C are elements of $\mathcal{L}$ satisfying $B \vee C = A$, and μ_B and μ_C are measures on $\downarrow B$ and $\downarrow C$ respectively, which match on $(\downarrow B) \wedge (\downarrow C)$, then there is a unique $\mu \in \mathbb{M}(A)$ such that the restriction of μ to B is μ_B and the restriction of μ to C is μ_C. Define μ by

$$
\mu(D) = \begin{cases} \mu_B(B \wedge D) + \mu_C(C \wedge D) - \mu_B(B \wedge C \wedge D) & \text{if} \quad \mu_B(B \wedge C \wedge D) < \infty \\ \infty & \text{otherwise} \end{cases}
$$

It is clear that this is the only possible amalgamation. We merely need to verify that this is indeed a measure. However, both the additivity and semicontinuity conditions follow immediately from the fact that μ_B and μ_C satisfy these conditions.

Hence, given a matching family for $\mathbb{M}$ on a sieve $\mathcal{I}$, that matching family may be uniquely extended to a certain directed sieve containing $\mathcal{I}$, and then amalgamated to $\Upsilon \mathcal{I}$. Hence $\mathbb{M}$ is a sheaf.

□

Corollary 16 (Breitsprecher [5]). *Let $\mathcal{F}$ be a σ-algebra, and let $\mathbb{M}$ be the presheaf of measures on $\mathcal{F}$. Then $\mathcal{F}$ is a sheaf with respect to the countable join topology.*

Proof. Breitsprecher's original proof of this result involved an argument explicitly using measures on a σ-algebra. However, the result can now be proved using the fact that the presheaf of measures on a locale is a sheaf.

Let $\mathcal{F}$ be a σ-algebra, and let $\mathcal{L}$ be the locale of countably complete sieves on $\mathcal{F}$. Then every presheaf on $\mathcal{L}$ can be restricted to a presheaf on $\mathcal{F}$. When this restriction is carried out on a sheaf on $\mathcal{L}$, the result is a sheaf on $\mathcal{F}$. In fact, this operation is one direction of the equivalence $\mathrm{Sh}(\mathcal{F}) \simeq \mathrm{Sh}(\mathcal{L})$.

But, according to Theorem 2, applying this restriction to $\mathbb{M}$, the presheaf of measures on $\mathcal{L}$ yields the presheaf of measures on $\mathcal{F}$. Since $\mathbb{M}$ is in fact a sheaf, it follows that the corresponding presheaf of measures on $\mathcal{F}$ is a sheaf on $\mathcal{F}$. $\square$

We can go further.

Theorem 7. *Let $\mathcal{F}$ be a σ-algebra, and let j be the countable join topology. Then carrying out the logical construction of $\mathbb{M}$ in the topos $\widehat{\mathcal{F}}$ (relative to the topology j) yields the sheaf of (σ-algebra) measures.*

Proof. The argument is the same as for showing that in $\widehat{\mathcal{L}}$, $\mathbb{M}$ is the (pre)sheaf of localic measures: $\mathbb{M}_A$ is easily seen to be the presheaf of finitely additive measures, and $\mathbb{M}_C$ is the sheaf of semicontinuous measures. $\square$

Corollary 17. *Let $\mathcal{B}$ be a Boolean algebra, and let j be the* finite *join topology. Then carrying out the logical construction on $\mathbb{M}$ in the topos $\widehat{\mathcal{B}}$ (relative to the topology j) yields the presheaf of finitely additive measures. Furthermore, $\mathbb{M}$ is a sheaf.*

2.4 PROPERTIES OF $\mathbb{M}$

Throughout this section we work in a fixed elementary topos $\mathcal{E}$ (with natural numbers object). We will designate a Lawvere-Tierney topology $j : \Omega \to \Omega$, which induces a subtopos $\mathrm{Sh}_j(\mathcal{E})$.

$\mathcal{E}$ shall be referred to as the presheaf topos, and $\mathrm{Sh}_j(\mathcal{E})$ as the sheaf topos. Ω shall denote the subobject classifier in $\mathcal{E}$, and Ω_j shall denote the subobject classifier in $\mathrm{Sh}_j(\mathcal{E})$. $\mathbb{S}$ will denote the object of semireals in $\mathcal{E}$, and $\mathbb{M}$ the object of measures as defined in the previous Sections.

We will assume that $\mathbb{M}$ is a j-sheaf. As a consequence of Theorem 6, we know that the case where $\mathcal{E}$ is the topos of sheaves on a locale $\mathcal{L}$, and j is the canonical topology on $\mathcal{E}$, then $\mathbb{M}$ is automatically a sheaf, and so working in $\mathcal{E}$ and $\mathrm{Sh}_j(\mathcal{E})$ can be thought of as generalizing this case.

Beyond the assumption that $\mathbb{M}$ is a sheaf, we will not assume anything about the structure of $\mathcal{E}$ or $\mathrm{Sh}_j(\mathcal{E})$.

$\mathbb{M}$ is a subobject of $\mathbb{S}$ (in $\mathcal{E}$), and inherits many of its properties:

Lemma 18. *The following arrows factor through* $\mathbb{M} \rightarrowtail \mathbb{S}$*:*

1.

$$\mathbb{M} \times \Omega_j \rightarrowtail \mathbb{S} \times \Omega_j \xrightarrow{\rho} \mathbb{S}$$

2.

$$\mathbb{M} \times \mathbb{M} \rightarrowtail \mathbb{S} \times \mathbb{S} \xrightarrow{+} \mathbb{S}$$

3.

$$\mathbb{Q} \times \mathbb{M} \rightarrowtail \mathbb{Q} \times \mathbb{S} \xrightarrow{\times} \mathbb{S}$$

Proof. 1. Fix $\mathcal{I} \in \Omega_j$, and $\mu \in \mathbb{M}$. We want to show that $\rho(\mu, \mathcal{I}) \in \mathbb{M}$. To do this, we must show that $\rho(\mu, \mathcal{I})$ is both additive and semicontinuous.

For additivity, first note that for arbitrary $\mathcal{J} \in \Omega_j$, we have

$$\begin{aligned}
\rho(\rho(\mu, \mathcal{I}), \mathcal{J}) &= \{q \in \mathbb{Q} | \mathcal{J} \Rightarrow (\mathcal{I} \Rightarrow q \in \mu)\} \\
&= \{q \in \mathbb{Q} | (\mathcal{I} \wedge \mathcal{J}) \Rightarrow q \in \mu\} \\
&= \rho(\mu, \mathcal{I} \wedge \mathcal{J})
\end{aligned}$$

Now, fix $\mathcal{J}, \mathcal{K} \in \Omega_j$.

$$\begin{aligned}
& \rho(\rho(\mu, \mathcal{I}), \mathcal{J}) + \rho(\rho(\mu, \mathcal{I}), \mathcal{K}) \\
= \ & \rho(\mu, \mathcal{I} \wedge \mathcal{J}) + \rho(\mu, \mathcal{I} \wedge \mathcal{K}) \\
= \ & \rho(\mu, (\mathcal{I} \wedge \mathcal{J}) \wedge (\mathcal{I} \wedge \mathcal{K})) + \rho(\mu, (\mathcal{I} \wedge \mathcal{J}) \vee (\mathcal{I} \wedge \mathcal{K})) \\
= \ & \rho(\mu, \mathcal{I} \wedge (\mathcal{J} \wedge \mathcal{K})) + \rho(\mu, \mathcal{I} \wedge (\mathcal{J} \vee \mathcal{K})) \\
= \ & \rho(\rho(\mu, \mathcal{I}), \mathcal{J} \wedge \mathcal{K}) + \rho(\rho(\mu, \mathcal{I}), \mathcal{J} \vee \mathcal{K})
\end{aligned}$$

To verify that $\rho(\mu, \mathcal{I})$ is semicontinuous, we need to show that for an arbitrary $q \in \mathbb{Q}$, the truth value

$$[\![q \in \rho(\mu, \mathcal{I})]\!]$$

is directed closed.

We start by noting that

$$\begin{aligned}
[\![q \in \rho(\mu, \mathcal{I})]\!] &= [\![\mathcal{I} \Rightarrow q \in \mu]\!] \\
&= [\![\mathcal{I}]\!] \Rightarrow [\![q \in \mu]\!]
\end{aligned}$$

Since $[\![\mathcal{I}]\!]$ is closed, and $[\![q \in \mu]\!]$ is directed closed, the result will follow from the following Lemma:

Lemma 19. *Let $\mathcal{I}$ be a closed truth value, and let $\mathcal{J}$ be a directed closed truth value. Then $\mathcal{I} \Rightarrow \mathcal{J}$ is also directed closed.*

Proof. Since we are working explicitly with truth values, we do not need the Scott brackets to distinguish between formulas and their truth values.

In this proof, the objects of discourse are truth values. The argument is similar to an argument in propositional logic. It is possible to notate the following argument in terms of the "$\leq$", rather than the "$\Rightarrow$" symbol. Likewise, we could write "$=$" for "$\iff$". Such substitutions would be natural when thinking of the truth values as elements of a Heting algebra. However, for the sake of consistency, we work here with logical connectives.

The topology j is a unary logical connective satisfying the axioms:

$$\begin{aligned} \mathcal{I} &\Rightarrow j\mathcal{I} \\ jj\mathcal{I} &\Rightarrow j\mathcal{I} \\ j(\mathcal{I} \wedge \mathcal{J}) &\iff \jmath\mathcal{I} \wedge j\mathcal{J} \end{aligned}$$

A truth value $\mathcal{I}$ is closed if and only if

$$j\mathcal{I} \Rightarrow \mathcal{I}$$

Recall that a truth value $\mathcal{K}$ is directed if for any closed truth values $\mathcal{A}$ and $\mathcal{B}$ we have

$$[(\mathcal{A} \Rightarrow \mathcal{K}) \wedge (\mathcal{B} \Rightarrow \mathcal{K})] \Rightarrow [j\,(\mathcal{A} \vee \mathcal{B}) \Rightarrow \mathcal{K}]$$

A truth value $\mathcal{Z}$ is directed closed, if for all directed $\mathcal{K}$ such that $\mathcal{K} \Rightarrow \mathcal{Z}$, we have $j(\mathcal{K}) \Rightarrow \mathcal{Z}$.

Start by taking truth values $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{I}$ is closed and $\mathcal{J}$ is directed closed. In order to show that $\mathcal{I} \Rightarrow \mathcal{I}$ is directed closed, we take a directed truth value $\mathcal{K}$, assume that $\mathcal{K} \Rightarrow (\mathcal{I} \Rightarrow \mathcal{J})$, and prove that $j\mathcal{K} \Rightarrow (\mathcal{I} \Rightarrow \mathcal{J})$. But

$$\mathcal{K} \Rightarrow (\mathcal{I} \Rightarrow \mathcal{J}) \longleftrightarrow (\mathcal{I} \wedge \mathcal{K}) \Rightarrow \mathcal{J}$$

and

$$j\mathcal{K} \Rightarrow (\mathcal{I} \Rightarrow \mathcal{J}) \longleftrightarrow j(\mathcal{I} \wedge \mathcal{K}) \Rightarrow \mathcal{J}$$

since $\mathcal{I}$ is closed.

Hence, in order to show that $\mathcal{I} \Rightarrow \mathcal{J}$ is directed closed, we just need to show that

$$[(\mathcal{I} \wedge \mathcal{K}) \Rightarrow \mathcal{J}] \Rightarrow [j(\mathcal{I} \wedge \mathcal{K}) \Rightarrow \mathcal{J}]$$

Since $\mathcal{J}$ is assumed to be directed closed, it suffices to prove that $\mathcal{I} \wedge \mathcal{K}$ is directed. But this is trivial since $\mathcal{I}$ and $\mathcal{J}$ are both directed (since closed sieves are directed closed, and directed closed sieves are closed).

$\square$

2. Take $\mu, \nu \in \mathbb{M}$. To see that $\mu + \nu$ is a measure, we must verify that $\mu + \nu$ is both additive and semicontinuous.

For additivity, take $\mathcal{I}, \mathcal{J} \in \Omega_j$. Then

$$\begin{aligned}
\rho(\mu+\nu,\mathcal{I}) + \rho(\mu+\nu,\mathcal{J}) &= \rho(\mu,\mathcal{I}) + \rho(\nu,\mathcal{I}) + \rho(\mu,\mathcal{J}) + \rho(\nu,\mathcal{J}) \\
&= \rho(\mu,\mathcal{I}\wedge\mathcal{J}) + \rho(\mu,\mathcal{I}\vee\mathcal{J}) + \rho(\nu,\mathcal{I}\wedge\mathcal{J}) + \rho(\nu,\mathcal{I}\vee\mathcal{J}) \\
&= \rho(\mu+\nu,\mathcal{I}\wedge\mathcal{J}) + \rho(\mu+\nu,\mathcal{I}\vee\mathcal{J})
\end{aligned}$$

To verify the semicontinuity condition, fix $q \in \mathbb{Q}$, $\mu, \nu \in \mathbb{M}$, and a directed family of truth values $\mathcal{D} \subseteq \Omega$, such that for each $D \in \mathcal{D}$, we have

$$D \Rightarrow q \in (\mu + \nu)$$

Fix an $r \in \mathbb{Q}$. Then for each $D \in \mathcal{D}$, there must exist rationals m_D and n_D such that

$$\left(m_D + n_D = q + \frac{r}{3}\right) \wedge [D \Rightarrow (m_D \in \mu) \wedge (n_D \in \nu)]$$

Let k and l be the smallest natural numbers such that

$$\forall D \in \mathcal{D} \exists E \in \mathcal{D}\, (D \Rightarrow E) \wedge m_E < k \cdot \tfrac{r}{3}$$
$$\forall D \in \mathcal{D} \exists E \in \mathcal{D}\, (D \Rightarrow E) \wedge n_E < l \cdot \tfrac{r}{3}$$

Put $m = \frac{kr}{3}$ and $n = \frac{lr}{3}$. Note that for every $Q \in \mathcal{D}$, we have

$$E \Rightarrow (m \in \mu) \wedge (n \in \nu)$$

Furthermore, we have $m < m_E + \frac{r}{3}$ and $n < n_E + \frac{r}{3}$.

Our goal is to show that $j\mathcal{D} \Rightarrow q \in \mu + \nu$. We know that

$$\begin{aligned}
m + n &\leq m_E + n_E + \frac{2r}{3} \\
m_E + n_E &= q + \frac{r}{3}
\end{aligned}$$

and so $m + n \leq q + r$.

Finally, we just need to show that $j\mathcal{D} \Rightarrow (m \in \mu) \wedge (n \in \nu)$. But for every $D \in \mathcal{D}$, we have

$$D \Rightarrow (m \in \mu) \wedge (n \in \nu)$$

Since both μ and ν are measures, and hence semicontinuous, the result holds.

3. Suppose that $\mu \in \mathbb{S}$ is a measure, and $q \in \mathbb{Q}$ is a rational. To see that $q \times \mu$ is a measure, we must show that

$$\theta = \left\{ r \in \mathbb{Q} \middle| \frac{r}{q} \in \mu \right\}$$

is both additive and semicontinuous.

Take $\mathcal{I}, \mathcal{J} \in \Omega_j$. Then

$$\begin{aligned}
& s \in \rho(\theta, \mathcal{I}) + \rho(\theta, \mathcal{J}) \\
\longleftrightarrow \quad & \exists t, u \in \mathbb{Q}\ [(\mathcal{I} \Rightarrow t \in \theta) \wedge (\mathcal{J} \Rightarrow u \in \theta) \wedge (t + u = s)] \\
\longleftrightarrow \quad & \exists t, u \in \mathbb{Q} \left[\left(\mathcal{I} \Rightarrow \frac{t}{q} \in \mu \right) \wedge \left(\mathcal{J} \Rightarrow \frac{u}{q} \in \mu \right) \wedge (t + u = s) \right] \\
\longleftrightarrow \quad & \exists t, u \in \mathbb{Q} \left[\left(\mathcal{I} \Rightarrow \frac{t}{q} \in \mu \right) \wedge \left(\mathcal{J} \Rightarrow \frac{u}{q} \in \mu \right) \wedge \left(\frac{t}{q} + \frac{t}{q} = \frac{s}{q} \right) \right] \\
\longleftrightarrow \quad & \exists v, w \in \mathbb{Q} \left[(\mathcal{I} \Rightarrow w \in \mu) \wedge (\mathcal{J} \Rightarrow w \in \mu) \wedge \left(v + w = \frac{s}{q} \right) \right] \\
\longleftrightarrow \quad & \frac{s}{q} \in \rho(\mu, \mathcal{I}) + \rho(\mu, \mathcal{J}) \\
\longleftrightarrow \quad & \frac{s}{q} \in \rho(\mu, \mathcal{I} \vee \mathcal{J}) + \rho(\mu, \mathcal{I} \wedge \mathcal{J}) \\
\longleftrightarrow \quad & s \in \rho(\theta, \mathcal{I} \vee \mathcal{J}) + \rho(\theta, \mathcal{I} \wedge \mathcal{J})
\end{aligned}$$

Hence θ is additive.

To verify that θ is semicontinuous, note that

$$[\![r \in \theta]\!] = \left[\!\!\left[\frac{r}{q} \in \mu \right]\!\!\right]$$

where the right hand side is known to be directed closed, since μ is semicontinuous.

$\square$

Proposition 20. $\mathbb{M}$ *is a semimodule over* $\mathbb{Q}$*, and for any* $\mathcal{I} \in \Omega_j$*,* $\rho(-, \mathcal{I})$ *is a linear operator on* $\mathbb{M}$*.*

$\mathbb{M}$ also has certain joins. Let $\mathfrak{O} \rightarrowtail \Omega_j^{\mathbb{M}}$ be the presheaf of *totally ordered* subsheaves of $\mathbb{M}$. To define a "supremum" arrow $\bigvee : \mathfrak{O} \to \mathbb{M}$, we just restrict the usual infimum arrow from $\Omega^{\mathbb{S}} \to \mathbb{S}$:

Proposition 21. *The supremum of an ordered family of measures, computed in $\widehat{\mathcal{F}}$, is again a measure. In other words, there is an arrow making the following diagram commute:*

$$\begin{tikzcd}
\mathfrak{O} \arrow[r, dotted] \arrow[d, tail] & \mathbb{M} \arrow[dd, tail] \\
\Omega_j^{\mathbb{M}} \arrow[d, tail] & \\
\Omega^{\mathbb{S}} \arrow[r, "\bigvee"'] & \mathbb{S}
\end{tikzcd}$$

Proof. Let $\mathfrak{O}$ be the set of totally ordered subsets of $\mathbb{M}$. The supremum of some $\mathcal{O} \in \mathfrak{O}$ is just the intersection:

$$\bigvee \mathcal{O} = \{q \in \mathbb{Q} | \forall \mu \in \mathcal{O}\, q \in \mu\} = \bigcap \mathcal{O}$$

We just need to show that this set of rationals is a measure.

For additivity, take closed truth values $\mathcal{I}$ and $\mathcal{J}$. We need to show that

$$q \in \rho\left(\bigvee \mathcal{O}, \mathcal{I}\right) + \rho\left(\bigvee \mathcal{O}, \mathcal{J}\right) \iff q \in \rho\left(\bigvee \mathcal{O}, \mathcal{I} \vee \mathcal{J}\right) + \rho\left(\bigvee \mathcal{O}, \mathcal{I} \wedge \mathcal{J}\right)$$

But

$$\begin{aligned}
& q \in \rho\left(\bigvee \mathcal{O}, \mathcal{I}\right) + \rho\left(\bigvee \mathcal{O}, \mathcal{J}\right) \\
\longleftrightarrow \quad & \exists a, b \in \mathbb{Q}\, (a + b = q) \wedge \left(\mathcal{I} \Rightarrow a \in \bigvee \mathcal{O}\right) \wedge \left(\mathcal{J} \Rightarrow b \in \bigvee \mathcal{O}\right) \\
\longleftrightarrow \quad & \exists a, b \in \mathbb{Q}\, (a + b = q) \wedge (\forall \mu \in \mathcal{O}\, \mathcal{I} \Rightarrow a \in \mu) \wedge (\forall \mu \in \mathcal{O} \mathcal{J} \Rightarrow b \in \mu) \\
\longleftrightarrow \quad & \exists a, b \in \mathbb{Q}\, (a + b = q) \wedge \forall \mu \in \mathcal{O}\, (\mathcal{I} \Rightarrow a \in \mu) \wedge (\mathcal{J} \Rightarrow b \in \mu) \\
\longleftrightarrow \quad & \exists c, d \in \mathbb{Q}\, (c + d = q) \wedge \forall \mu \in \mathcal{O}\, ((\mathcal{I} \vee \mathcal{J}) \Rightarrow c \in \mu) \wedge ((\mathcal{I} \wedge \mathcal{J}) \Rightarrow d \in \mu) \\
\longleftrightarrow \quad & \exists c, d \in \mathbb{Q}\, (c + d = q) \wedge (\forall \mu \in \mathcal{O}\, (\mathcal{I} \vee \mathcal{J}) \Rightarrow c \in \mu) \wedge (\forall \mu \in \mathcal{O} (\mathcal{I} \wedge \mathcal{J}) \Rightarrow d \in \mu) \\
\longleftrightarrow \quad & q \in \rho\left(\bigvee \mathcal{O}, (\mathcal{I} \vee \mathcal{J})\right) + \rho\left(\bigvee \mathcal{O}, (\mathcal{I} \wedge \mathcal{J})\right)
\end{aligned}$$

Now, to verify the semicontinuity condition, we must show that $[\![q \in \bigvee \mathcal{O}]\!]$ is directed closed. But

$$\begin{aligned} & \left[\!\!\left[q \in \bigvee \mathcal{O} \right]\!\!\right] \\ = \quad & [\![\forall \mu \in \mathcal{O}\, q \in \mathcal{O}]\!] \\ = \quad & \bigwedge_{\mu \in \mathcal{O}} [\![q \in \mu]\!] \end{aligned}$$

Since this is itself the meet of a decreasing family of truth values, each of which is directed closed, it follows that $[\![q \in \bigvee \mathcal{O}]\!]$ is a directed closed truth value, as required.

$\square$

In fact, we can do slightly better. We do not require that $\mathcal{O}$ be totally ordered in $\mathcal{E}$, but only that it be *locally* totally ordered (totally ordered in $\mathrm{Sh}_j(\mathcal{E})$), for sheafifying the arrow $\bigvee : \mathfrak{O} \to \mathbb{M}$ yields an arrow whose domain is the sheaf of those subsheaves of $\mathbb{M}$ which are totally ordered in $\mathrm{Sh}_j(\mathcal{E})$.

2.5 INTEGRATION

In order to generalize measure theory, it is necessary to find a way to discuss integration. With the framework that has been set up here, we can mimic the standard approach to defining the Lebesgue integral (see for example Billingeley [3] or Royden [22]). As in Section 2.4, we work in $\mathcal{E}$, with a designated topology $j : \Omega \to \Omega$, and we assume that $\mathbb{M}$ is a j-sheaf.

In this Section, we build an arrow in $\mathrm{Sh}_j(\mathcal{E})$ that captures the operation of integration. We know that when $\mathcal{E} = \widehat{\mathcal{F}}$ is the topos of presheaves on a σ-algebra $\mathcal{F}$, $\mathbb{M}$ is the sheaf of measures on $\mathcal{F}$. Our goal in this Section is find a logical characterization of the integration arrow in $\widehat{\mathcal{F}}$. In the next Section we use this logical characterization to find elementary properties of this arrow.

The classical approach to integrating measurable functions is to first integrate constant functions, then locally constant ("simple") functions, and finally to integrate measurable functions.

Scott [24] (referred to in [12]) showed that in the topos of sheaves on a σ-algebra $\mathcal{F}$, the sheaf of measurable real valued functions on the measurable space $(X, \mathcal{F})$ is just the Dedekind real numbers object in $\mathrm{Sh}(\mathcal{F})$. We will therefore consider integration as acting on the sheaf $\mathbb{D}$ of nonnegative Dedekind real numbers:

$$\begin{array}{rcl} \int : \mathbb{D} \times \mathbb{M} & \longrightarrow & \mathbb{M} \\ \langle f, \mu \rangle & \longmapsto & \int_{-} f \, d\mu \end{array}$$

Since we are working with positive Dedekind reals, we modifiy the definition slightly. A Dedekind real consists of a pair $\langle L, U \rangle$ of subsheaves of the sheaf of *positive* rationals. We do not assume that L is nonempty, as the pair $\langle \emptyset, \mathbb{Q} \rangle$ corresponds to the zero function. We do retain the assumption that U is non-empty, so that the corresponding measurable function is locally finite.

First, we verify that $\int$, as described above, is indeed a natural transformation. This is an immediate consequence of the following well known result:

Lemma 22. *Let $(X, \mathcal{F})$ be a measurable space, take $B \subseteq A$ in $\mathcal{F}$, let μ be a measure on $(X, \mathcal{F})$, and let f be a positive real valued measurable function defined on the subspace $(A, \downarrow A)$. Then for any $C \subseteq B$ in $\mathcal{F}$, we have*

$$\int_C f \, d\mu = \int_C (f \upharpoonright_B) \, d\,(\mu \upharpoonright_B)$$

where $(f \upharpoonright_B)$ is the restriction of f to B and $(\mu \upharpoonright_B)$ is the restriction of μ to B.

Proof. Both the left and right hand sides of the above equation can be rewritten as

$$\int_X f \cdot \chi_C \, d\mu$$

where χ_C is the characteristic function of C. □

Corollary 23. *If $\mathcal{F}$ is a σ-algebra, then working in the topos* $\mathrm{Sh}(\mathcal{F})$*, the operation of Lebesgue integration is a natural transformation* $\int : \mathbb{D} \times \mathbb{M} \to \mathbb{M}$.

In our framework, constant functions can be considered to be elements of the presheaf $\mathbb{Q}$, and locally constant functions as elements of $\mathbf{a}\mathbb{Q}$.

As we extend the definition of the integral, we need to ensure that at each stage, the interpretation of the integral coincides with the usual definition of the Lebesgue integral in the toposes $\widehat{\mathcal{F}}$ and $\mathrm{Sh}(\mathcal{F})$.

With this goal, the following Lemmas provide a logical characterization of the integration arrow in $\widehat{\mathcal{F}}$ and $\mathrm{Sh}(\mathcal{F})$.

Lemma 24. *The arrow describing integration of constant functions is just multiplication. This is expressed by stating that the following diagram commutes:*

$$\begin{tikzcd} \mathbb{Q} \times \mathbb{M} \arrow[d, tail] \arrow[dr, "\times"] & \\ \mathbb{D} \times \mathbb{M} \arrow[r, "\int"'] & \mathbb{M} \end{tikzcd}$$

The embedding $\mathbb{Q} \rightarrowtail \mathbb{D}$ is given by

$$q \mapsto \langle \{r \in \mathbb{Q} | r < q\}, \{s \in \mathbb{Q} | q < s\} \rangle$$

Note that $\mathbb{D}$ is a sheaf, and that $\mathbb{Q}$ is a presheaf. The logical description $\langle \{r \in \mathbb{Q} | r < q\}, \{s \in \mathbb{Q} | q < s\} \rangle$ is interpreted in $\mathrm{Sh}(\mathcal{F})$.

Proof. Just as $\mathbb{D}$ is the sheaf of measurable real valued functions, $\mathbb{Q}$ is the presheaf of constant, rational valued functions, and the embedding $\mathbb{Q} \rightarrowtail \mathbb{D}$ is the natural embedding.

By definition,

$$\int_B q \, d\mu = q \cdot \mu(B)$$

But since the multiplication arrow $\mathbb{Q} \times \mathbb{S} \to \mathbb{S}$ is just pointwise multiplication of the corresponding functional, the right hand side of the above equation is the product of q with μ, and we are done. □

Lemma 25. *The arrow describing integration of locally constant rational valued functions is the image of the multiplication arrow under the sheafification functor. This is expressed by stating that the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{a}\mathbb{Q} \times \mathbb{M} \arrow[d, tail] \arrow[dr, "\mathbf{a}\times"] & \\
\mathbb{D} \times \mathbb{M} \arrow[r, "\int"'] & \mathbb{M}
\end{tikzcd}$$

The following Corollary exploits the fact that integration of rationals is just sheafified multiplication, and makes this relationship explicit:

Corollary 26. *Working in the presheaf topos $\widehat{\mathcal{F}}$, take $q, r \in \mathbb{Q}$ and $\mu \in \mathbb{M}$. Then*

$$q \in \int(r, \mu) \iff \frac{q}{r} \in \mu$$

Before we can prove Lemma 25, we first prove the following useful fact about sheafification in σ-algebras.

Lemma 27. *Let $(\mathcal{F} \ \sqsubseteq, \perp, \top, \neg)$ be a σ-algebra, and let P be a presheaf on $\mathcal{F}$. Then $\mathbf{a}P$ is given by*

$$\mathbf{a}(P)(A) = \coprod_{\mathcal{P} \in \mathfrak{P}(A)} \left(\prod_{B \in \mathcal{P}} P(B) \right) \Big/ \sim$$

where $\mathfrak{P}(A)$ is the set of all countable partitions of A, and

$$\langle x_B | B \in \mathcal{P}_1 \rangle \sim \langle x_C | C \in \mathcal{P}_2 \rangle$$

if for any $B \in \mathcal{P}_1$ and $C \in \mathcal{P}_2$ we have $\rho^B_{B \sqcap C}(x_B) = \rho^C_{B \sqcap C}(x_C)$. This is just the usual notion of equivalence of two matching families.

The substance of this Lemma is in two parts. Firstly, it says that to find the associated sheaf of a presheaf, you need only apply the Grothendieck "+" construction one time. Secondly, it says that we need only consider countable *partitions* of A, rather than *all* countably generated covers of A.

Proof. The fact that we need only investigate partitions follows directly from the observation that any countable cover in a σ-algebra can be refined to form a partition, since then the countable partitions will form a basis (see [19]) for the countable join topology.

Let $\mathcal{C} = \langle C_i | i < \omega \rangle$ be a countable cover for A. Let $\mathcal{P} = \langle P_i | i < \omega \rangle$ be defined recursively:

$$\begin{aligned} P_0 &= C_0 \\ P_{k+1} &= C_{k+1} \sqcap \left(\prod_{i=0}^{k} \neg C_k \right) \end{aligned}$$

It is immediate that $\mathcal{P}$ is a partition, that $\mathcal{P}$ is a refinement of $\mathcal{C}$, and that for any k,

$$\bigsqcup_{i \leq k} P_i = \bigsqcup_{i \leq k} C_i$$

The advantage of using a partition is that *every* family on the partition is a matching family,

So now, in order to show that $\mathbf{a}P$ is the associated sheaf of P, all we must show is that $\mathbf{a}P$ is a sheaf. Since the countable partitions form a basis for the topology, and since a single application of the Grothendieck "+" construction provides a separated presheaf, it suffices to show that given any countable partition $\mathcal{P}$ of A, and any family $\mathbf{x} = \langle x_B \in \mathbf{a}P(B) | B \in \mathcal{P} \rangle$, there is an amalgamation for $\mathbf{x}$ in $\mathbf{a}P(A)$.

But each x_B is itself (or at least, can be represented by) a family $\langle x_C \in P(C) | C \in \mathcal{P}_B \rangle$ for some countable partition $\mathcal{P}_C$ of C. The union of these families over all $B \in \mathcal{P}$ provides a matching family for the partition

$$\bigcup_{B \in \mathcal{P}} \mathcal{P}_B$$

But this union is itself a family over a countable partition, and so corresponds to an element of $\mathbf{a}P(A)$, as required.

□

Corollary 28. *Let $(X, \mathcal{F})$ be a measurable space, let $\mathbb{Q}$ be the presheaf of positive rational numbers (in $\widehat{\mathcal{F}}$), and let $\mathbf{a}$ be the sheafification functor for the countable join topology. Then $\mathbf{a}\mathbb{Q}$ is the sheaf of measurable rational valued functions.*

Proof. We know from Lemma 27 that $\mathbf{a}\mathbb{Q}(B)$ is the set of matching families of rational numbers for partitions (modulo a trivial equivalence). Let $\mathcal{P}$ be such a partition. Then a matching family for $\mathcal{P}$ consists of a family $\langle q_P \in \mathbb{Q} | P \in \mathcal{P} \rangle$ of rational numbers. But this is equivalent to the locally constant rational valued function

$$q(x) = q_P$$

whenever $x \in P$ □

We can now prove Lemma 25

Proof. Since $\mathbf{a}\mathbb{Q} \times \mathbb{M}$ is the associated sheaf of $\mathbb{Q} \times \mathbb{M}$, we know that there is a limiting map $\mathbf{a}\times : \mathbf{a}\mathbb{Q} \times \mathbb{M} \to \mathbb{M}$ making the following diagram commute:

$$\begin{tikzcd}
 & \mathbf{a}\mathbb{Q} \times \mathbb{M} \arrow[d, "\mathbf{a}\times"] \\
\mathbb{Q} \times \mathbb{M} \arrow[ur, dotted, "\eta^{\mathbb{Q}\times\mathbb{M}}"] \arrow[r, "\times"'] & \mathbb{M}
\end{tikzcd}$$

Given $\mathbf{q} = \langle q_P | P \in \mathcal{P} \rangle \in \mathbf{a}\mathbb{Q}(A)$, the natural transformation $\mathbf{a}\times$ agrees with $\times$ on each element $P \in \mathcal{P}$. Hence whenever there is a $P \in \mathcal{P}$ such that $B \subseteq P$, we have

$$\int(\mathbf{q}, \mu)(B) = \int(q_P, \mu)(B)$$

We now extend $\int(\mathbf{q}, \mu)$ to all measurable $B \subseteq A$, by using the countable additivity property of $\mathbb{M}$:

$$\sum_{P \in \mathcal{P}} \int(q_P, \mu)(B \cap P) = \int(\mathbf{q}, \mu)(B)$$

But this is the usual definition of the integral of a simple function, and so $\mathbf{a}\times$ does indeed coincide with the classical notion of the integral. □

We can now define the integral of a nonnegative Dedekind real. A Dedekind real consists of a pair $f = \langle L, U \rangle$ of subsheaves of $\mathbb{Q}$ (in $\mathrm{Sh}(\mathcal{F})$). Since $\mathbf{a}\mathbb{Q}$ is totally ordered in $\mathrm{Sh}(\mathcal{F})$, and since integration preserves order, it follows that for fixed $f = \langle L, U \rangle$ and μ, the set of measures

$$\left\{ \int q, \mu) \middle| q \in L \right\}$$

is totally ordered in $\mathrm{Sh}(\mathcal{F})$. Hence this family of measures has a supremum, by Proposition 21.

Definition 37. Let $f = \langle L, U \rangle$ be a Dedekind real, and let μ be a measure. Then set:

$$\int (f, \mu) = \bigvee_{q \in L} \int (q, \mu) = \bigvee\nolimits_{q \in L} \mathbf{a} \times (q\mu)$$

.

Note that the right hand side here is makes sense in an arbitray topos $\boldsymbol{\mathcal{E}}$ with topology j.

2.6 INTEGRABILITY AND PROPERTIES OF INTEGRATION

One might ask why we define the integral of a Dedekind real $f = \langle L, U \rangle$ as the supremum of the integrals of the rationals in L, rather than as the infimum of the rationals in U. The difficulty is that in general, it is hard to construct the infimum of an ordered family of measures. The approach that we used to construct the supremum of such a family was to construct the supremum in $\mathbb{S}$, and then show that this supremum is indeed in $\mathbb{M}$. The corresponding argument is not valid for infima.

However, we can work around this problem for the special case of a Dedekind real:

Theorem 8. *Let μ be a measure, and let $f = \langle L, U \rangle$ be a Dedekind real satisfying $\mathrm{Sh}_j(\boldsymbol{\mathcal{E}}) \models \exists q \in L$. Then working in $\mathbb{S}$, we have*

$$\bigwedge_{r \in U} \int (r, \mu) = \bigvee_{q \in L} \int (q, \mu)$$

Corollary 29. *Let μ be a measure, and let $f = \langle L, U \rangle$ be a Dedekind real satisfying*

$$\mathrm{Sh}_j(\mathcal{E}) \models (\exists q \in L)$$

Then the semireal

$$\bigwedge_{r \in U} \int (r, \mu)$$

is a measure.

Proof. Proposition 21 asserts that

$$\bigvee_{q \in L} \int (q, \mu)$$

is a measure, and Theorem 8 asserts that

$$\bigwedge_{r \in U} \int (r, \mu) = \bigvee_{q \in L} \int (q, \mu)$$

□

So now, to prove Theorem 8:

Proof. We work using the internal logic of the "presheaf topos" $\mathcal{E}$. Let $\langle L, U \rangle$ be a Dedekind real in the sheaf topos $\mathrm{Sh}_j(\mathcal{E})$, and let μ be a measure. Note that L and U are subpresheaves of $\mathbb{Q}$, as is μ.

For simplicity of notation, let ν_L and ν_U be the semireals given by

$$\begin{aligned} \nu_L &= \bigvee_{q \in L} \int (q, \mu) \\ \nu_U &= \bigwedge_{r \in U} \int (r, \mu) \end{aligned}$$

Then it follows from Proposition 7 that

$$\nu_L = \left\{ s \in \mathbb{Q} \middle| \forall q \in L\, s \in \int (q, \mu) \right\} = \left\{ s \in \mathbb{Q} \middle| \forall q \in L\, \frac{s}{q} \in \mu \right\}$$

Likewise,

$$\nu_U = \left\{ s \in \mathbb{Q} \middle| \forall t \in \mathbb{Q} \exists r \in U\, (s + t) \in \int (r, \mu) \right\} = \left\{ s \in \mathbb{Q} \middle| \forall t \in \mathbb{Q} \exists r \in U\, \frac{s + t}{r} \in \mu \right\}$$

It is immediate that

$$\nu_L \leq \nu_U$$

Hence we need only show the reverse inequality, or equivalently:

$$\nu_L \subseteq \nu_U$$

where both the terms in the above expression are viewed as subobjects of $\mathbb{Q}$.

Since $\mathrm{Sh}_j(\mathcal{E}) \models \exists r \in U$, we (still working in $\mathcal{E}$) know that there is an $\mathcal{I} \in \Omega$ such that $j\mathcal{I} = \top$, and

$$\mathcal{I} \Rightarrow (\exists q \in L \wedge \exists r \in U)$$

Let q_0 and r_0 be any witnesses to this statement, so

$$\mathcal{I} \Rightarrow (q_0 \in L \wedge r_0 \in U)$$

It follows that $q_0 < r_0$ since if $q_0 \geq r_0$, we would have $q_0 \in L \cap U$, which cannot happen.

We need to show that any $s \in \nu_L$, satisfies $s \in \nu_U$. Start by taking $t \in \mathbb{Q}$. We must find $r \in U$ such that

$$\frac{s+t}{r} \in \mu$$

We know that $\langle L, U \rangle$ is a Dedekind real in $\mathrm{Sh}(\mathcal{L})$. Therefore we know that

$$j\,(\forall q, r \in \mathbb{Q}\, q < r \Rightarrow q \in L \vee r \in U)$$

In other words, for any pair of rationals $q_1 < r_1$, there is a dense $\mathcal{I}_1 \in \Omega$ for which the following apartness property holds:

$$\mathcal{I}_1 \Rightarrow q_1 \in L \vee r_1 \in U$$

Using q_0 as our starting point, we will construct two pair of recursively defined sequences: The first pair, is given by

$$\begin{aligned} m_n &= \frac{s}{q_n} \\ q_{n+1} &= \frac{s+t}{m_n} \end{aligned}$$

We can rewrite q_{n+1} as

$$q_{n+1} = \frac{s}{m_n} + \frac{t}{m_n} = q_n + \frac{t}{m_n}$$

so it is immediate that $\langle q_n \rangle$ is an increasing sequence, and $\langle m_n \rangle$ is a decreasing sequence. Furthermore,

$$\begin{aligned} q_{n+1} &\geq q_n + \frac{t}{m_0} \\ &\geq q_0 + \frac{(n+1)t}{m_0} \end{aligned}$$

The second pair of sequences $\langle q'_n \rangle$ and $\langle m'_n \rangle$ are given by the same recurrence relation. The only difference is that q'_0 is chosen such that

$$q_0 < q'_0 < q_1$$

and $q'_0 \in L$. (We know that there is a $q' \in L$ such that $q_0 < q'$. Let q'_0 be the minimum of q' and $\frac{q_0+q_1}{2}$.)

An easy induction argument shows that for any n, we have

$$q_n < q'_n < q_{n+1} \qquad m_{n+1} < m'_n < m_n$$

Now, suppose that n satisfies $n \geq \frac{m_0(r_0-q_0)}{t}$. It follows from above that $q_n \geq r_0$, whence $q_n \in U$. Likewise, for such an n, we would have $q'_n \in U$.

For each n, since $q_n < q_{n+1}$, we know that we can find an $\mathcal{I}_{n+1}$ such that $j\mathcal{I}_{n+1} = \top$, and

$$\mathcal{I}_{n+1} \Rightarrow (q_n \in L) \vee (q_{n+1} \in U) \wedge \mathcal{I}_n$$

Likewise, we can always find $\mathcal{J}_{n+1}$ such that $j\mathcal{J}_{n+1} = \top$ and

$$\mathcal{J}_{n+1} \Rightarrow (q'_n \in L) \vee \left(q'_{n+1} \in U\right) \wedge \mathcal{J}_n$$

(let $\mathcal{J}_0 = \top$).

Let N be the smallest natural number such that $\mathcal{J}_{N+1} \Rightarrow q'_{N+1} \in U$.

We first show that

$$\mathcal{I}_{N+1} \wedge \mathcal{J}_{N+1} \Rightarrow q_N \in L$$

For convenience, let $\mathcal{K} = \mathcal{I}_{N+1} \wedge \mathcal{J}_{N+1}$. Note that we have $j\mathcal{K} = \top$.

We know that either $\mathcal{K} \Rightarrow q_N \in L$ or $\mathcal{K} \Rightarrow q'_N \in U$. But from our choice of N, it follows that q'_{N+1} is the first term in the sequence $\langle q'_n \rangle$ satisfying $\mathcal{K} \Rightarrow q'_n \in U$. Hence we cannot have $\mathcal{K} \Rightarrow q'_N \in U$, and so $\mathcal{K} \Rightarrow q_N \in L$.

Now, since $\mathcal{K} \Rightarrow q_N \in L$, we must have $\mathcal{K} \Rightarrow \frac{s}{q'_N} \in \mu$, since $\mathcal{K} \Rightarrow s \in \nu_L$. But this means that $\mathcal{K} \Rightarrow \frac{s+t}{q'_{N+1}} \in \mu$, since

$$\begin{aligned} \frac{s}{q'_N} &= m'_N \\ &= \frac{s+t}{q'_{N+1}} \end{aligned}$$

Hence, we have shown that for arbitrary $s \in \nu_L$, and $t \in \mathbb{Q}$, there is a dense $\mathcal{K} \in \Omega$ such that $\mathcal{K} \Rightarrow s + t \in \nu_U$. Hence $s \in \nu_U$, as required.

$\square$

It is possible to extend the definition of the integral arrow a little further. The object $\mathbb{R}_M$ of *McNeille real numbers* is somewhat more general than the object of $\mathbb{R}_D$ of Dedekind real numbers (see Johnstone [15]).

Like a Dedekind real number, a McNeille real number consists of a pair $\langle L, U \rangle$ of subsheaves of $\mathbb{Q}$. Most of the axioms for a McNeille real number are the same as for a Dedekind real number. The exception is the "apartness condition". For a Dedekind real number, this is stated as

$$\forall q, r \in \mathbb{Q}\, (q < r) \Rightarrow (q \in L \vee r \in U)$$

The equivalent condition for McNeille real numbers is the conjunction of the following two formulas

$$\begin{aligned} \forall q, r \in Q\, (q < r \wedge q \notin L) &\Rightarrow r \in U \\ \forall q, r \in Q\, (q < r \wedge r \notin U) &\Rightarrow q \in L \end{aligned}$$

It is obvious from this definition that every Dedekind real number is a McNeille real number. Hence $\mathbb{R}_D \rightarrowtail \mathbb{R}_M$.

The converse is not intuitionistically valid. In fact, $\mathbb{R}_D \cong \mathbb{R}_M$ if and only if DeMorgan's law holds:

$$\neg (A \wedge B) \vdash \neg A \vee \neg B$$

(The other three of DeMorgan's laws are intuitionistically valid.) DeMorgan's law is strictly weaker than the law of the excluded middle, so all classical logical systems satisfy DeMorgan's law, but conversely, there are non-classical toposes where DeMorgan's law is satisfied.

The principal reason that McNeille real numbers are studied is that they are the order completion of the Dedekind real numbers. Hence DeMorgan's law holds if and only if the Dedekind reals numbers satisfy Bolzano–Weierstrass completeness.

This also allows us to construct McNeille real numbers that are not Dedekind real numbers. Let $(\mathbb{R}, \mathcal{L})$ be the measurable space consisting of the real numbers and the Lebesgue measurable functions. Let Z be a non-Lebesgue measurable set. Then for each $z \in Z$, the characteristic function $\chi_{\{z\}}$ is measurable, and so is a Dedekind real number in $\mathrm{Sh}(\mathcal{L})$. However, the supremum of these characteristic functions is the characteristic function χ_Z, which is evidently not a measurable function, and so not a Dedekind real number.

As a result, we could define the integral of a McNeille real f (relative to a measure μ) as the supremum of the integrals of the rationals in the lower cut of f. However Theorem 8 does not apply in such a case, and so a McNeille real is not *integrable*, in the usual sense.

Finally, we present some important properties of the integration arrow.

Proposition 30. *Integration is an order preserving map.*

Proof. We need to show two things here. Firstly, if $f \leq g$, then

$$\int (f, \mu) \leq \int (g, \mu)$$

and secondly, if $\mu \leq \nu$, then

$$\int (f, \mu) \leq \int (g, \mu)$$

For the first, note that if $f = \langle L_f, U_f \rangle$ and $g = \langle L_g, U_g \rangle$, then

$$f \leq g \iff L_f \subseteq U_f$$

Since

$$\int(f,\mu) = \bigvee_{q \in L_f} \int(q,\mu)$$

and

$$\int(g,\mu) = \bigvee_{r \in L_g} \int(r,\mu)$$

it follows that

$$\int(f,\mu) \leq \int(g,\mu)$$

as required.

Now suppose that $\mu \leq \nu$. Then for each $q \in L_f$, it follows immediately from Corollary 26 that

$$\int(q,\mu) \leq \int(q,\nu)$$

Hence

$$\bigvee_{q \in L_f} \int(q,\mu) \leq \bigvee_{q \in L_f} \int(q,\nu)$$

as required. □

Theorem 9 (Monotone Convergence Theorem). *Suppose that $f_\alpha \uparrow f$ is an increasing family of Dedekind reals, converging to another Dedekind real f. Then*

$$\int(f_\alpha,\mu) \to \int(f,\mu)$$

Proof. We first show that the result holds for increasing families of rationals, and then for increasing families of Dedekind reals.

Let $Q = \langle q_i | i \in I \rangle$ be a directed family of rational numbers, and let $q = \bigvee Q$. Every rational number a can be represented by the semireal $\{s \in \mathbb{Q} | a \leq s\}$. The arrow $\int$ is just multiplication, so

$$
\begin{aligned}
a \in \bigvee_{i \in I} \int(q_i, \mu) &\longleftrightarrow \forall i \in I \, a \in \int(q_i, \mu) \\
&\longleftrightarrow \forall i \in I \, \exists m \in \mu \, a = m q_i \\
&\longleftrightarrow \forall i \in I \, \frac{a}{q_i} \in \mu \\
&\longleftrightarrow \forall d \in \mathbb{Q} \frac{a}{q} + d \in \mu \\
&\longleftrightarrow \frac{a}{q} \in \mu \\
&\longleftrightarrow a \in \int(q, \mu)
\end{aligned}
$$

Now, let $\mathcal{D} = \langle f_i | i \in I \rangle$ be a directed family of Dedekind real numbers (defined in $\mathrm{Sh}(\mathcal{L})$), and let $f = \langle L, U \rangle = \bigvee \mathcal{D}$. Then

$$
\begin{aligned}
a \in \bigvee_{i \in I} \int(f_i, \mu) &\longleftrightarrow \forall i \in I \, a \in \int(f_i, \mu) \\
&\longleftrightarrow \forall i \in I \left[\forall q \in L_i \left(a \in \int(q, \mu) \right) \right] \\
&\longleftrightarrow \forall i \in I \left[\forall q \in L_i \left(\frac{a}{q} \in \mu \right) \right] \\
&\longleftrightarrow \left[\forall q \in \bigcup_{i \in I} L_i \left(\frac{a}{q} \in \mu \right) \right] \\
&\longleftrightarrow \forall q \in \bigcup_{i \in I} L_i \left(\frac{a}{q} \in \mu \right) \\
&\longleftrightarrow \forall q \in L \left(\frac{a}{q} \in \mu \right) \\
&\longleftrightarrow \forall q \in L \left(a \in \int(q, \mu) \right) \\
&\longleftrightarrow a \in \int(f, \mu)
\end{aligned}
$$

□

Theorem 10. $\mathbb{M}$ *is a semimodule over the semiring* $\mathbb{D}$*, with the action of scalar multiplication given by integration.*

Proof. We know from Proposition 10 that $\mathbb{S}$ is a semimodule over the semiring of positive rationals. Since $(\mathbb{M}, +)$ is a subalgebra of $(\mathbb{S}, +)$, it follows that $\mathbb{M}$ is an abelian monoid. Hence we just need to show the following, for arbitrary $\mu, \nu \in \mathbb{M}$ and $f = \langle L_1, U_1 \rangle, g = \langle L_2, U_2 \rangle \in \mathbb{D}$:

1. $\int(f, (\mu + \nu)) = \int(f, \mu) + \int(f, \nu)$
2. $\int((f + g), \mu) = \int(f, \mu) + \int(g, \mu)$
3. $\int\left(f, \int(g, \mu)\right) = \int(f \cdot g, \mu)$
4. $\int(1, \mu) = \mu$

To prove these:

1.

$$\begin{aligned} \int(f, \mu + \nu) &= \bigvee_{q \in L_1} \int(q, \mu + \nu) \\ &= \bigvee_{q \in L_1} \int(q, \mu) + \int(q, \nu) \end{aligned}$$

Since L_1 is totally ordered, we can apply distributivity here, to get

$$\begin{aligned} \bigvee_{q \in L_1} \int(q, \mu) + \int(q, \nu) &= \bigvee_{q \in L_1} \int(q, \mu) + \bigvee_{q \in L_1} \int(q, \nu) \\ &= \int(q, \mu) + \int(q, \nu) \end{aligned}$$

2. First, note that $f + g = \langle L_1 \oplus L_2, U_1 \oplus U_2 \rangle$, where

$$A \oplus B = \{a + b | \langle a, b \rangle \in A \times B\}$$

$$
\begin{aligned}
\int(f+g,\mu) &= \bigvee_{s\in L_1\oplus L_2} \int(s,\mu) \\
&= \bigvee_{q\in L_1}\bigvee_{r\in L_2} \int(q+r,\mu) \\
&= \bigvee_{q\in L_1}\bigvee_{r\in L_2} \int(q,\mu)+\int(r,\mu) \\
&= \left(\bigvee_{q\in L_1}\bigvee_{r\in L_2} \int(q,\mu)\right)+\left(\bigvee_{q\in L_1}\bigvee_{r\in L_2} \int(r,\mu)\right) \\
&= \left(\bigvee_{q\in L_1} \int(q,\mu)\right)+\left(\bigvee_{r\in L_2} \int(r,\mu)\right) \\
&= \int(f,\mu)+\int(g,\mu)
\end{aligned}
$$

3. This time, note that $f \cdot g = \langle L_1 \otimes L_2, U_1 \otimes U_2\rangle$, where

$$A \otimes B = \{a \cdot b | \langle a, b\rangle \in A \times B\}$$

$$
\begin{aligned}
\int\left(f,\int(g,\mu)\right) &= \bigvee_{q\in L_1} \int\left(q,\int(g,\mu)\right) \\
&= \bigvee_{q\in L_1} \int(q\cdot g,\mu) \\
&= \bigvee_{q\in L_1}\bigvee_{r\in L_2} \int(q\cdot r,\mu) \\
&= \bigvee_{s\in L_1\otimes L_2} \int(s,\mu) \\
&= \int(f\cdot g,\mu)
\end{aligned}
$$

4. Observe that the Dedekind real number 1 is a rational number, and so we can apply Proposition 10 here.

□

3.0 DIFFERENTIATION

3.1 SUBTOPOSES OF LOCALIC TOPOSES

Let $\mathbb{C}$ be an arbitrary category, let $\widehat{\mathbb{C}}$ be the topos of presheaves on $\mathbb{C}$, and let Ω denote the subobject classifier in $\widehat{\mathbb{C}}$. Then a Grothendieck topology on $\widehat{\mathbb{C}}$ is a presheaf $J \rightarrowtail \Omega$ of "covering sieves"; a sieve $\mathcal{I} \in \Omega(C)$ is a cover for C if and only if $\mathcal{I} \in J(C)$. Grothendieck topologies correspond to the Lawvere-Tierney topologies $j : \Omega \to \Omega$ which characterize them. A (Lawvere-Tierney or Grothendieck) topology induces a subtopos of $\widehat{\mathbb{C}}$, the topos of sheaves on the site $(\mathbb{C}, J)$, denoted $\mathrm{Sh}(\mathbb{C}, J)$. Such sheaf toposes have been extensively studied (see for example, Mac Lane and Moerdijk [19]).

This result connects Grothendieck topologies on a presheaf topos with Lawvere-Tierney topologies on the same presheaf topos. A more general situation is given by the case where $\mathcal{E} = \mathrm{Sh}(\mathbb{C}, J)$ is a Grothendieck topos, and j is a Lawvere-Tierney topology in $\mathcal{E}$. In this case, there is a relationship between Lawvere-Tierney topologies in $\mathcal{E}$ and certain Grothendieck topologies on $\mathbb{C}$.

Definition 38. Let j and k be two Lawvere-Tierney topologies in an elementary topos $\mathcal{E}$. Then k is said to be finer than j if $k \circ j = k$.

Lemma 31. *Suppose that k is finer than j. Then $j \circ k = k$.*

Proof. It is obvious that the composition of Lawvere-Tierney topologies is also a Lawvere-Tierney topology (see Definition 17).

Start by assuming that $k \circ j = k$, and let l be the topology $j \circ k$.

$$\begin{aligned} l \circ k &= j \circ k \circ k \\ &= j \circ k \\ &= l \end{aligned}$$

Hence l is a finer topology than k.

Conversely, consider $k \circ l$:

$$\begin{aligned} k \circ l &= k \circ j \circ k \\ &= k \circ k \\ &= k \end{aligned}$$

Hence k is finer than l. Since the "finer" relationship on topologies is inherited from the usual ordering on Ω, it follows that the "finer" relation is in fact a partial ordering. Hence $k = l$, whence

$$k = j \circ k$$

□

The condition $j \circ k = k$ is exactly the condition needed to infer that k factors through $i : \Omega_j \to \Omega$. The reason for this is given by the following diagram:

$$\begin{tikzcd} \Omega \arrow[d, "k_1"'] \arrow[dr, "k"] & & \\ \Omega_j \arrow[r, tail, "i"'] & \Omega \arrow[r, shift left, "\mathrm{id}_\Omega"] \arrow[r, shift right, "j"'] & \Omega \end{tikzcd}$$

Since i is the equalizer of the arrows $j, \mathrm{id} : \Omega \rightrightarrows \Omega$, it follows that $j \circ k = k$ if and only if k factors through i.

We start with the following two results (which are given as exercises in [19]).

Proposition 32. *Let $\mathcal{E}$ be an elementary topos, and let $j, k : \Omega \rightrightarrows \Omega$ be two Lawvere-Tierney topologies on $\mathcal{E}$, such that k is finer than j. Let Ω_j be the subobject classifier in* $\mathrm{Sh}_j(\mathcal{E})$*, and define k' as the composition $k_1 i$, where k_1 and i are given by the following diagram:*

$$\begin{tikzcd}
\Omega_j \arrow[r, tail, "i"] & \Omega \arrow[r, "k_1"] \arrow[dr, "k"'] & \Omega_j \arrow[d, tail, "i"] \\
& & \Omega
\end{tikzcd}$$

Then k' is a Lawvere-Tierney topology in the topos $\mathrm{Sh}_j(\mathcal{E})$.

Proof. To see that k' is a Lawvere-Tierney topology, we need to show that k' satisfies the usual three commutative properties (see Definition 17) in $\mathrm{Sh}_j(\mathcal{E})$.

To do this, we introduce an arrow $j_1 : \Omega \to \Omega_j$. We know that $i : \Omega_j \rightarrowtail \Omega$ is the equalizer of the arrows $j : \Omega \to \Omega$ and $\mathrm{id}_\Omega : \Omega \to \Omega$. Since $j \circ j = j$, there must be an arrow j_1 making the following diagram commute:

$$\begin{tikzcd}
\Omega \arrow[d, dotted, "j_1"'] \arrow[dr, "j"] & & \\
\Omega_j \arrow[r, tail, "i"'] & \Omega \arrow[r, shift left, "\mathrm{id}_\Omega"] \arrow[r, shift right, "j"'] & \Omega
\end{tikzcd}$$

First we observe that j_1 has the following properties:

Lemma 33.

$$i \circ j_1 = j \qquad j_1 \circ i = id_{\Omega_j}$$

Proof. The first property is immediate from the above diagram.

For the second property, we show that $i \circ j_1 \circ i : \Omega_j \to \Omega$ is an equalizer for j and id_Ω. Since equalizers are unique up to isomorphism, we have $i \circ j_1 \circ i = i$. Finally, since i is a monic, we get $j_1 \circ i = \mathrm{id}_{\Omega_j}$.

So, it only remains to show that $i \circ j_1 \circ i : \Omega_j \to \Omega$ is the required equalizer. Take $f : Z \to \Omega$ such that $j \circ f = f$. Since f factors through i, we get $f = i \circ f_1$, so

$$\begin{aligned} f &= j \circ i \circ f_1 \\ &= i \circ j_1 \circ i \circ f_1 \end{aligned}$$

But this tells us that f factors through $i \circ j_1 \circ i$, as required. ☐

Note also that i and k_1 satisfy the following properties:

$$k' = k_1 \circ i \qquad k = i \circ k_1$$

The first condition that we need to show is inflationarity:

$$\begin{tikzcd} \mathbf{1} \arrow[r, "\top"] \arrow[dr, "\top"'] & \Omega_j \arrow[d, "k'"] \\ & \Omega_j \end{tikzcd}$$

Note that $\mathbf{1}$ is also the terminal object in $\boldsymbol{\mathcal{E}}$, and $i \circ \top : \mathbf{1} \to \Omega$ is the "top" map in $\boldsymbol{\mathcal{E}}$. The result follows from the following diagram chase:

$$\begin{aligned} k' \circ \top &= k_1 \circ i \circ \top \\ &= j_1 \circ i \circ k_1 \circ i \circ \top \\ &= j_1 \circ k \circ i \circ \top \\ &= j_1 \circ i \circ \top \\ &= \top \end{aligned}$$

The second condition is idempotence:

$$\begin{tikzcd} \Omega_j \arrow[r, "k'"] \arrow[dr, "k'"'] & \Omega_j \arrow[d, "k'"] \\ & \Omega_j \end{tikzcd}$$

Again, we engage in a diagram chase:

$$\begin{aligned} k' \circ k' &= k_1 \circ i \circ k_1 \circ i \\ &= k_1 \circ k \circ i \\ &= j_1 \circ i \circ k_1 \circ k \circ i \\ &= j_1 \circ k \circ k \circ i \\ &= j_1 \circ k \circ i \\ &= j_1 \circ i \circ k_1 \circ i \\ &= k_1 \circ i \\ &= k' \end{aligned}$$

Finally, we must verify that k' commutes with the meet operator $\wedge_j$ on Ω_j. This will follow if we can show that the outer rectangle in the following diagram commutes:

$$\begin{tikzcd} \Omega_j \times \Omega_j \arrow[r, "\langle i,i \rangle"] \arrow[d, "\wedge_j"'] & \Omega \times \Omega \arrow[r, "\langle k_1,k_1 \rangle"] \arrow[d, "\wedge"'] & \Omega_j \times \Omega_j \arrow[d, "\wedge_j"] \\ \Omega_j \arrow[r, tail, "i"'] & \Omega \arrow[r, "k_1"'] & \Omega_j \end{tikzcd}$$

The fact that the left hand square commutes follows directly from the fact that j is a topology.

To see that the right hand square commutes, consider the following diagram:

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "\langle k_1,k_1 \rangle"] \arrow[d, "\wedge"'] & \Omega_j \times \Omega_j \arrow[r, tail, "\langle i,i \rangle"] \arrow[d, "\wedge_j"'] & \Omega_j \times \Omega_j \arrow[d, "\wedge"] \\ \Omega \arrow[r, "k_1"'] & \Omega_j \arrow[r, tail, "i"'] & \Omega \end{tikzcd}$$

The right hand square commutes, as it is the same as the left hand square of the previous diagram. Since the top and bottom sides of the large rectangle are just $\langle k,k \rangle$

and k, respectively, it follows (from the fact the k is a topology) that the outer rectangle commutes. Hence:

$$\begin{aligned} i \circ k_1 \circ \wedge &= \wedge \circ \langle i, i \rangle \circ \langle k_1, k_1 \rangle \\ &= i \circ \wedge_j \circ \langle k_1, k_1 \rangle \end{aligned}$$

Since i is a monomorphism, it follows that

$$k_1 \circ \wedge = \wedge_j \circ \langle k_1, k_1 \rangle$$

But this is just what we needed to make the right hand square in the first diagram commute.

This completes the proof that k' is a Lawvere-Tierney topology in $\mathrm{Sh}_j(\mathcal{E})$. □

Proposition 34. *Suppose that $\mathcal{E}$ is an elementary topos, j is a Lawvere-Tierney topology in $\mathcal{E}$, and k is a Lawvere-Tierney topology in the sheaf topos $\mathrm{Sh}_j(\mathcal{E})$. Let Ω be the subobject classifier in $\mathcal{E}$, let Ω_j be the subobject classifier in $\mathrm{Sh}_j(\mathcal{E})$, let $i : \Omega_j \rightarrowtail \Omega$ be the natural inclusion, and let $j_1 : \Omega \to \Omega_j$ be the closure map of j. Let $\tilde{k} = i \circ k \circ j_1$ denote the following composition:*

$$\Omega \xrightarrow{j_1} \Omega_j \xrightarrow{k} \Omega_j \overset{i}{\rightarrowtail} \Omega$$

Then

1. *$\tilde{k}$ is a Lawvere-Tierney topology in $\mathcal{E}$.*
2. *$\tilde{k}$ is a finer topology than j.*
3. $\mathrm{Sh}_k\big(\mathrm{Sh}_j(\mathcal{E})\big) \simeq \mathrm{Sh}_{\tilde{k}}(\mathcal{E})$

Proof. 1. As in Proposition 32, we need to show that j satisfies the required commutative diagrams.

Firstly, to check inflationarity, it is enough to realize that each of the triangles in the following diagram commutes:

$$\begin{tikzcd} \mathbf{1} \arrow[d, "\top"'] \arrow[dr, "\top"] \arrow[drr, "\top"] \arrow[drrr, "\top"] & & & \\ \Omega \arrow[r, "j_1"'] & \Omega_j \arrow[r, "k"'] & \Omega_j \arrow[r, tail, "i"'] & \Omega \end{tikzcd}$$

Next, to check that $\tilde{k}$ is idempotent, we use the fact that $j_1 \circ i = \mathrm{id}_{\Omega_j}$:

$$\begin{aligned} \tilde{k} \circ \tilde{k} &= i \circ k \circ j_1 \circ i \circ k \circ j_1 \\ &= i \circ k \circ k \circ j_1 \\ &= i \circ k \circ j_1 \\ &= \tilde{k} \end{aligned}$$

Finally, to check that $\tilde{k}$ commutes with products, we need to show that the outer rectangle in the following diagram commutes:

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "\langle j_1, j_1 \rangle"] \arrow[d, "\wedge"'] & \Omega_j \times \Omega_j \arrow[r, "\langle k, k \rangle"] \arrow[d, "\wedge_j"'] & \Omega_j \times \Omega_j \arrow[r, tail, "\langle i, i \rangle"] \arrow[d, "\wedge_j"] & \Omega \times \Omega \arrow[d, "\wedge"] \\ \Omega \arrow[r, "j_1"'] & \Omega_j \arrow[r, "k"'] & \Omega_j \arrow[r, tail, "i"'] & \Omega \end{tikzcd}$$

That the right hand square commutes is an immediate consequence of the fact that j is a topology, and that topologies preserve meets.

That the middle square commutes follows from the fact that k is itself a topology in $\mathrm{Sh}_j(\boldsymbol{\mathcal{E}})$.

To see that the left hand square commutes, consider the following diagram:

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "\langle j_1, j_1 \rangle"] \arrow[d, "\wedge"'] & \Omega_j \times \Omega_j \arrow[r, tail, "\langle i, i \rangle"] \arrow[d, "\wedge_j"] & \Omega \times \Omega \arrow[d, "\wedge"] \\ \Omega \arrow[r, "j_1"'] & \Omega_j \arrow[r, tail, "i"'] & \Omega \end{tikzcd}$$

The outer square commutes, since the compositions along the top and the bottom are just $\langle j, j \rangle$, and j, respectively. The right hand square is the same right hand square as in the previous diagram. Hence we can write

$$i \circ \wedge_j \circ \langle j_1, j_1 \rangle = i \circ j_1 \circ \wedge$$

However, since i is a monomorphism, we can factor it out of the above equation, yielding

$$\wedge_j \circ \langle j_1, j_1 \rangle = j_1 \circ \wedge$$

which is exactly what we need to show that the left hand square also commutes.

2. To see that $\tilde{k} \circ j = \tilde{k}$, note that

$$\tilde{k} \circ j = i \circ k \circ j_1 \circ j$$

It will suffice to show that $j_1 \circ j = j_1$. But

$$\begin{aligned} j_1 \circ j &= j_1 \circ i \circ j_1 \\ &= \mathrm{id}_{\Omega_j} \circ j_1 \\ &= j_1 \end{aligned}$$

as required.

3. We show that an object F of $\boldsymbol{\mathcal{E}}$ is both a j-sheaf, and a k-sheaf in $\mathrm{Sh}_j(\boldsymbol{\mathcal{E}})$ if and only if F is a $\tilde{k}$-sheaf in $\boldsymbol{\mathcal{E}}$.

Suppose that F is a j-sheaf, and a k-sheaf in $\mathrm{Sh}_j(\boldsymbol{\mathcal{E}})$. Take a pair of objects $A \rightarrowtail E$ in $\boldsymbol{\mathcal{E}}$, such that A is a $\tilde{k}$-dense subobject of E, with $\chi_E : A \to \Omega$ denoting the characteristic map of A, and let $f : A \to F$ be an arbitrary arrow. Let $A_j \rightarrowtail E$ be the closure of A under the topology j. Then, since F is a j-sheaf, f has a unique extension $\overline{f} : A_j \to F$. Now consider the closure of $A_k \rightarrowtail E$, the closure of A_j under the topology k. We know that $\overline{f}$ must have a unique extension $\tilde{f} : A_k \to F$, since F is a k sheaf.

The characteristic map of A_k is $i \circ k \circ j_1 \circ \chi_A$. But this is just $\tilde{k} \circ \chi_A$. Since we are assuming that A is a $\tilde{k}$ dense subobject of E, it follows that $A_k = E$, and so $\tilde{E} : E \to F$ is the extension of f required to show that F is a $\tilde{k}$-sheaf.

Now suppose that F is a $\tilde{k}$-sheaf. We first show that F is a j-sheaf. Suppose that $A \rightarrowtail E$ is a j-dense subobject in $\boldsymbol{\mathcal{E}}$, and let $f : A \to F$ be an arbitrary arrow. Let $\chi_A : E \to \Omega$ denote the characteristic arrow for A. Then $j \circ \chi_A = \top$. But then $\tilde{k} \circ \chi_A = \top$, since

$$\begin{aligned} \tilde{k} \circ \chi_A &= \tilde{k} \circ j \circ \chi_A \\ &= \tilde{k} \circ \top \\ &= \top \end{aligned}$$

Thus A is a $\tilde{k}$-dense subobject of E. Since F is a $\tilde{k}$-sheaf, it follows that f has a unique extension $\hat{f} : E \to F$. Hence F is a j-sheaf.

Now take a pair of objects $A \rightarrowtail E$ in $\mathrm{Sh}_j(\mathcal{E})$ such that A is k-dense, together with an arrow $f : A \to F$. Since A and E are j-sheaves, it follows that when we interpret $A \to E$ in $\mathcal{E}$, the characteristic map $\chi_A : E \to \Omega$ will factor through i. Hence there is a map $\chi_A^j : E \to \Omega_j$ such that $i \circ \chi_A^j : E \to \Omega$ is just χ_A. (χ_A^j is just the characteristic map of $A \rightarrowtail E$ in $\mathrm{Sh}_j(\mathcal{E})$.)

Since A is a k-dense subobject of E, it follows that in $\mathrm{Sh}_j(\mathcal{E})$, $k \circ \chi_A^j = \top$ (or $i \circ k \circ \chi_A^j = \top$ in $\mathcal{E}$). Since $j_1 \circ i = \mathrm{id}_{\Omega_j}$, we get

$$\begin{aligned} \top &= i \circ k \circ \chi_A^j \\ &= i \circ k \circ j_1 \circ i \circ \chi_A^j \\ &= \tilde{k} \circ \chi_A \end{aligned}$$

Hence A is a $\tilde{k}$-dense subobject of E, and so there is a unique $\overline{f} : E \to F$ extending f. This F is a k-sheaf in $\mathrm{Sh}_j(\mathcal{E})$.

$\square$

The following result connects these two Propositions.

Theorem 11. *The processes in Propositions 32 and 34 are inverse to one another. Hence topologies in* $\mathrm{Sh}_j(\mathcal{E})$ *correspond to those topologies in* $\mathcal{E}$ *which are finer than* j.

Proof. We start with a Lawvere-Tierney topology $k : \Omega \to \Omega$ in $\mathcal{E}$, with k satisfying $j \circ k = k$. The Lawvere-Tierney topology in $\mathrm{Sh}_j(\mathcal{E})$ corresponding to k is $k' : \Omega_j \to \Omega_j$, given by

$$k' = k_1 \circ i$$

The Lawvere-Tierney topology in $\mathcal{E}$ corresponding to k' in $\mathcal{E}$ is the arrow $\widetilde{k'} : \Omega \to \Omega$ given by

$$\widetilde{k'} = i \circ k' \circ j_1$$

We must show that $\widetilde{k'} = k$.

$$\begin{aligned}
(k')' &= i \circ k' \circ j_1 \\
&= i \circ k_1 \circ i \circ j_1 \\
&= k \circ i \circ j_1 \\
&= k \circ j \\
&= k
\end{aligned}$$

Now, suppose that $k : \Omega_j \to \Omega_j$ is a Lawvere-Tierney topology in $\mathrm{Sh}_j(\mathcal{E})$. Then the Lawvere-Tierney topology in $\mathcal{E}$ associated with k is given by

$$\tilde{k} = i \circ k \circ j_1 : \Omega_j \to \Omega_j$$

To find the Lawvere-Tierney topology in $\mathrm{Sh}_j(\mathcal{E})$ associated with $\tilde{k}$, first note that $\left(\tilde{k}\right)'$ is defined by the following diagram:

$$\begin{tikzcd}
\Omega \arrow[rrr, dotted, "\left(\tilde{k}\right)'"] \arrow[d, "j_1"'] & & & \Omega \\
\Omega_j \arrow[r, tail, "i"'] & \Omega \arrow[r, "k"'] & \Omega_{j_1} \arrow[r, tail, "i"'] & \Omega_j \arrow[u, tail, "i"']
\end{tikzcd}$$

But since $j = i \circ j_1$, it follows that $\left(\tilde{k}\right)' = j \circ k \circ k$. Applying the fact that k is assumed to be finer than j, this just reduces to k, as required. □

For most of this Chapter, we will be considering the case where $\mathcal{E} = \mathrm{Sh}(\mathcal{L})$ is the topos of sheaves on a locale (ie, $\mathcal{E}$ is a localic topos), and j is some topology in $\mathcal{E}$. In this case, $\mathrm{Sh}_j(\mathcal{L})$ has some useful properties.

Proposition 35. *Let $\mathcal{E} = \mathrm{Sh}(\mathcal{L})$ be a localic topos, and let k be a Lawvere-Tierney topology in $\mathcal{E}$. Then*

1. *There is a Grothendieck topology K on $\mathcal{L}$ such that $\mathrm{Sh}(\mathcal{L}, K) \simeq \mathrm{Sh}_k(\mathcal{E})$*
2. *$\mathrm{Sh}_k(\mathcal{E})$ is a localic topos.*

Proof. First note that the second clause follows immediately from the first.

To prove the first clause, let j be the canonical topology on $\mathcal{L}$. Then $\mathcal{E} \simeq \mathrm{Sh}_j(\mathcal{L})$. Hence, we can apply Proposition 34 to find a topology $\tilde{k}$ on $\mathcal{L}$ such that $\mathrm{Sh}_{\tilde{k}}(\mathcal{L}) \simeq \mathrm{Sh}_k(\mathcal{E})$. □

For any elementary topos $\mathcal{E}$ and Lawvere-Tierney topology j, there is a functor $\mathbf{i} : \mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$, embedding the sheaf topos in $\mathcal{E}$. This functor has a left adjoint $\mathbf{a} : \mathcal{E} \to \mathrm{Sh}_j(\mathcal{E})$, called the sheafification, or "associated sheaf" functor. If $\mathcal{E} = \widehat{\mathbb{C}}$ is a presheaf topos, then the usual method for constructing $\mathbf{a}P$, the associated sheaf of some presheaf P, is to apply the "Grothendieck +" construction twice (once to arrive at a separated presheaf, and then again to arrive at a sheaf). In light of Proposition 34, this method can clearly be extended to the case where $\mathcal{E}$ is any Grothendieck topos.

However, in the case that $\mathcal{E}$ is a localic topos, the process can be simplified considerably. Note that in this case, a topology j is a closure opeator on $\mathcal{L}$.

Lemma 36. *Let $\mathcal{E} = \mathrm{Sh}(\mathcal{L})$ be a localic topos, and let $j : \Omega \to \Omega$ be a Lawvere-Tierney topology in $\mathcal{E}$. Then a sheaf F is a j-sheaf if and only if we have*

$$F(A) = F(jA)$$

for any $A \in \mathcal{L}$.

Proof. We know already that a sheaf F is a contravariant functor from $\mathcal{L}$ to $\textsc{Set}$, and hence an object of the presheaf topos $\widehat{\mathcal{L}}$. We also know that F is a j-sheaf if and only if it is a $\tilde{j} = i \circ j \circ c_1$ sheaf in $\widehat{\mathcal{L}}$ (where c is the cannonical topology on $\widehat{\mathcal{L}}$).

But this just means that for every matching family for some $\tilde{j}$-covering family, there is a unique amalgamation. Take $A \in \mathcal{L}$, and suppose that C is a $\tilde{j}$-cover for A. Since F is a c-sheaf, we know that matching families for C correspond to elements of $F(\bigvee C)$. Hence F is a $\tilde{j}$-sheaf if and only if

$$F\left(\bigvee C\right) = F(A)$$

for any $\tilde{j}$-cover C of A.

But this is equivalent to requiring that $F(B) = F(jB)$ for any B.

□

Equipped with this convenient characterization of j-sheaves, we can easily find the associated sheaf $\mathbf{a}_j F$ of F:

Theorem 12. *Let $\mathcal{E}$ be a localic topos, and let j be a Lawvere-Tierney topology on $\mathcal{E}$. Then for any object F of $\mathcal{E}$, we have*

$$\mathbf{a}_j F(A) = \coprod_{j(B)=j(A)} F(B) \Big/ \sim$$

where $x_1 \in F(B_1)$ and $x_2 \in F(B_2)$ are related by $\sim$ if

$$\rho^{B_1}_{B_1 \wedge B_2}(x_1) = \rho^{B_2}_{B_1 \wedge B_2}(x_2)$$

Proof. We prove this result by showing firstly that that $\mathbf{a}_j F$, as defined above, is indeed a sheaf, and secondly that every natural transformation $\psi : F \to G$ for some j-sheaf G factors through the evident natural transformation $\eta : F \to \mathbf{a}_j F$.

$\eta^A : F(A) \to \mathbf{a}_j F(A)$ is given as the following composition:

$$F(A) \rightarrowtail \coprod_{jB=jA} F(B) \twoheadrightarrow \coprod_{jB=j(A)} F(B) \Big/ \sim$$

To see that $\mathbf{a}_j F$ is a sheaf, it suffices to observe that for any $A \in \mathcal{L}$, we have

$$\mathbf{a}_j F(jA) = \mathbf{a}_j F(A)$$

But this is immediate from the definition of $\mathbf{a}_j F$, as $j(B) = j(A) \iff j(A) = j(j(A))$.

Now take an arbitrary j-sheaf G, and an arbitrary natural transformation $\psi : F \to G$. Then since G is a j-sheaf, it follows that $G(A) = G(jA)$ for any A. Thus the following square must commute:

$$\begin{tikzcd} F(j(A)) \arrow[r, "\psi^{jA}"] \arrow[d] & G(jA) \arrow[d, equal] \\ F(A) \arrow[r, "\psi^A"'] & G(A) \end{tikzcd}$$

For each A, we can extend ψ^A to $\overline{\psi^A} : \mathbf{a}_j F(A) \to G(A)$ as follows: Take $x \in \mathbf{a}_j(F)(A)$. Then there is a B such that $jB = jA$ and for which $x \in F(B)$. Then let

$$\begin{aligned} \overline{\psi^A}(x) &= \psi^B(x) \\ &\in G(B) \\ &= G(A) \end{aligned}$$

To verify that this operation is independent of the choice of B, it suffices to observe that if $x_1 \sim x_2$, for some $x_1 \in F(B_1)$ and $x_2 \in F(B_2)$ (with $jB_1 = jB_2 = A$), then the following diagram commutes:

$$\begin{tikzcd} F(B_1) \arrow[r] \arrow[d, "\psi^{B_1}"'] & F(B_1 \wedge B_2) \arrow[d, "\psi^{B_1 \wedge B_2}"'] & F(B_2) \arrow[l] \arrow[d, "\psi^{B_2}"] \\ G(B_1) \arrow[r, equal] & G(B_1 \wedge B_2) \arrow[r, equal] & G(B_2) \end{tikzcd}$$

It is easy to see that this is the unique extension of ψ to $\mathbf{a}_j F$. Thus $\mathbf{a}_j F$ is indeed the associated j-sheaf of F, and $\mathbf{a}_j : \mathcal{E} \to \mathrm{Sh}_j(\mathcal{E})$ is the sheafification functor.

□

3.2 ALMOST EVERYWHERE COVERS

Suppose that $(X, \mathcal{F}, \mu)$ is a measure space (where $\mathcal{F}$ denotes the σ-algebra $(\mathcal{F}, \sqsubseteq, \top, \bot, \neg)$) and that $\mathcal{E} = \mathrm{Sh}(\mathcal{F})$ is the topos of sheaves on $\mathcal{F}$. Then the notion of formula being true "almost everywhere" induces a topology on $\mathrm{Sh}(\mathcal{F})$. Let $\mathcal{N} \subseteq \mathcal{F}$ be the ideal of μ-negligible sets

$$\mathcal{N} = \{A \in \mathcal{F} | \mu(A) = 0\}$$

Then $\mathcal{N}$ is a countably closed sieve, and so corresponds to an arrow

$$\mathcal{N} : \mathbf{1} \to \Omega$$

in $\mathcal{E}$.

Define a map $j : \Omega \to \Omega$ by

$$\Omega \xrightarrow{\langle \mathrm{id}_\Omega, \mathcal{N} \rangle} \Omega \times \Omega \xrightarrow{\vee} \Omega$$

j is just the "closed topology" induced by $\mathcal{N}$ (see[15]). The idea of j is that it takes a sieve $\mathcal{I}$ to the set

$$\{A \in \mathcal{F} | \exists I \in \mathcal{I}\ \exists N \in \mathcal{N}\ (A = I \sqcup N)\}$$

A sieve $\mathcal{I} \in \Omega(A)$ covers A if there is an $I \in \mathcal{I}$ such that $\mu(A \sqcap \neg I) = 0$.

The subtopos of $\mathcal{E}$ induced by this topology is easily seen to be the topos of sheaves on the quotient algebra $\mathcal{F}/\mathcal{N}$.

Unfortunately, this conception of an "almost everywhere topology" will not translate to the setting of measures on a locale. The problem is that when we use the expression "almost everywhere", we mean that everything of significance is included. The "closed topology" interpretation above captures the notion that everything that is not included is insignificant. These notions coincide only in certain Boolean settings, in this case because $\mathcal{F}$ is a Boolean algebra.

Since a locale $\mathcal{L}$ is not in general a Boolean algebra, we must find a more direct way to capture the idea of "everything of significance" than "everything that isn't negligible". Such a formulation is the first step towards extending "almost everywhere" equivalence to localic measure theory.

A number of the proofs in this Section make explicit use of sheaf semantics. Hence we assume that we are working in a localic topos $\mathcal{E} = \mathrm{Sh}(\mathcal{L})$, for some locale $(\mathcal{L}, \preceq, \top, \bot)$. Occasionally, we shall also be referring back to the case where $\mathcal{E} = \mathrm{Sh}(\mathcal{F})$ is the topos of sheaves on a σ-algebra, as we will want to ensure that we are generalizing the classical notions outlined above.

The first step is to build a notion of "everything significant". To do this, we use the restriction operator ρ.

Definition 39. Take $\mu \in \mathbb{M}$ and $\mathcal{I} \in \Omega$. Say that $\mathcal{I}$ is dense for μ if

$$\rho(\mu, \mathcal{I}) = \mu$$

This logical formula has two free variables (μ and $\mathcal{I}$), and so can be thought of as an arrow

$$[\![\rho(\mathcal{I}, \mu) = \mu]\!] : \mathbb{M} \times \Omega \to \mathbb{M}$$

We can define the collection of μ-dense sieves, by taking the transpose of this arrow:

$$\frac{\mathbb{M} \times \Omega \longrightarrow \Omega}{\mathbb{M} \longrightarrow \mathcal{P}\Omega}$$

In fact, this arrow factors through the object $\mathcal{T} \rightarrowtail \mathcal{P}\Omega$ of topologies.

Theorem 13. *Fix $\mu \in \mathbb{M}$. Then the (internal) set $\left\{\mathcal{I} \in \Omega \middle| \rho(\mu, \mathcal{I}) = \mu\right\}$ is a topology in $\boldsymbol{\mathcal{E}}$.*

Proof. We need to show that this arrow satisfies the usual algebraic conditions for a topology. We write these conditions as follows:

1. Inflationarity:

$$\rho(\mu, \top) = \mu$$

2. Idempotence:

$$(\rho\,(\mu, [\![\rho(\mu, \mathcal{I}) = \mu]\!]) = \mu) \iff (\rho(\mu, \mathcal{I}) = \mu)$$

3. Commutativity with $\wedge$:

$$(\rho\,(\mu, \mathcal{I} \wedge \mathcal{J}) = \mu) \iff ((\rho(\mu, \mathcal{I}) = \mu) \wedge (\rho(\mu, \mathcal{J}) = \mu))$$

The first of these conditions is immediate.

For the two remaining conditions, we will use the fact that if we interpret a measure as a functional on the underlying locale,

$$\rho(\mu, \mathcal{I})\,(B) = \mu(B \wedge C)$$

whenever $\mathcal{I} = \downarrow C$. Note that in $\mathrm{Sh}(\mathcal{L})$, *all* closed sieves are principal.

We simultaneously use the semantics of the topos $\widehat{\mathcal{L}}$. In particular, we use the fact that the following sequent is a reversible inference in the sheaf semantics:

$$\frac{B \Vdash (\downarrow C) \Rightarrow A}{\overline{B \wedge C \Vdash A}}$$

Here we are taking advantage of the fact that the formula $\downarrow C$ is a closed sieve, and so a principal sieve. This allows us to view C dually as an element of the base category $\mathcal{L}$ (since we can compute $B \wedge C$) and a truth value in $\mathcal{E}$ (since we can consider the formula $(\downarrow C) \Rightarrow A$). We use this duality a number of times in the remainder of this proof. To make this duality more explicit, we adopt the convention of writing $C \Rightarrow A$, rather than $(\downarrow C) \Rightarrow A$. Likewise, there is no reason to maintain the distinction between the internal conjunction operation "$\wedge$", and the meet operator "$\wedge$" in $\mathcal{L}$. We shall use "$\wedge$" to denote both operations, and reserve "$\wedge$" for *external* conjunction, and occasionally the join operator in $[0, \infty]$.

Let U be the formula (and hence the element of $\mathcal{L}$) given by

$$U \equiv \rho(\mu, \mathcal{I}) = \mu$$

Note that $\mathcal{I} \leq U$, and that

$$U \Vdash (\mathcal{I} \Rightarrow q \in \mu) \iff (q \in \mu)$$

$$\begin{aligned}
q \in \rho(\mu, U)(B) &\longleftrightarrow B \Vdash q \in \rho(\mu, U) \\
&\longleftrightarrow B \Vdash U \Rightarrow q \in \mu \\
&\longleftrightarrow B \wedge U \Vdash q \in \mu \\
&\longleftrightarrow B \wedge U \Vdash \mathcal{I} \Rightarrow q \in \mu \\
&\longleftrightarrow B \wedge U \wedge \mathcal{I} \Vdash q \in \mu \\
&\longleftrightarrow B \wedge \mathcal{I} \Vdash q \in \mu \\
&\longleftrightarrow B \Vdash \mathcal{I} \Rightarrow q \in \mu \\
&\longleftrightarrow B \Vdash q \in \rho(\mu, \mathcal{I}) \\
&\longleftrightarrow q \in \rho(\mu, \mathcal{I})(B)
\end{aligned}$$

For the third condition, we again use the semantic interpretation of $\rho(\mu, B)$. Suppose that $\rho(\mu, \mathcal{I} \wedge \mathcal{J}) = \mu$. Then for any $B \in \mathcal{L}$ we have

$$\begin{aligned} \mu(B) &= \rho(\mu, \mathcal{I} \wedge \mathcal{J})(B) \\ &= \mu\left((\mathcal{I} \wedge \mathcal{J}) \wedge B\right) \\ &\leq \mu(\mathcal{I} \wedge B) \\ &\leq \mu(B) \end{aligned}$$

Hence $\rho(\mu, \mathcal{I}) = \mu$, and likewise for $\mathcal{J}$. Hence

$$\rho(\mu, \mathcal{I} \wedge \mathcal{J}) = \mu \vdash (\rho(\mu, \mathcal{I}) = \mu) \wedge (\rho(\mu, \mathcal{J}) = \mu)$$

For the reverse inequality, suppose that $\rho(\mu, \mathcal{I}) = \mu = \rho(\mu, \mathcal{J})$. Consequently, we have

$$\begin{aligned} \mu(B) &= \rho(\mu, \mathcal{I} \wedge \mathcal{J}) \\ &\leq \rho(\mu, \mathcal{I} \vee \mathcal{J})(B) \\ &\leq \mu(B) \end{aligned}$$

Then:

$$\begin{aligned} \rho(\mu, \mathcal{I} \wedge \mathcal{J})(B) &= \mu((\mathcal{I} \wedge \mathcal{J}) \wedge B) \\ &= \mu((\mathcal{I} \wedge B) \wedge (\mathcal{J} \wedge B)) \\ &= \mu(\mathcal{I} \wedge B) + \mu(\mathcal{J} \wedge B) - \mu((\mathcal{I} \wedge B) \vee (\mathcal{J} \wedge B)) \\ &= \rho(\mu, \mathcal{I})(B) + \rho(\mu, \mathcal{J})(B) - \mu((\mathcal{I} \vee \mathcal{J}) \wedge B) \\ &= \mu(B) + \mu(B) - \mu(B) \\ &= \mu(B) \end{aligned}$$

Hence $\rho(\mu, \mathcal{I} \wedge \mathcal{J}) = \mu$, as required. □

Now, we must verify that this topology coincides with the closed topology induced by $\mathcal{N}$, when $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{F})$ is the topos of sheaves on the σ-algebra $(\mathcal{F}, \sqsubseteq, \top, \bot, \neg)$.

Proposition 37. *Let μ be a measure on $\mathcal{F}$. Then the closed topology on* Sh($\mathcal{F}$) *induced by the topology*

$$\mathcal{I} \mapsto \mathcal{N} \vee \mathcal{I}$$

coincides with the topology in Definition 39.

Proof. Viewing Sh($\mathcal{F}$) as a localic topos Sh($\mathcal{L}$), the underlying locale is the collection of countably closed sieves in $\mathcal{F}$. There is potential for notational confusion here, as a sieve in $\mathcal{L}$ is a sieve of sieves on $\mathcal{F}$. For this reason, for the remainder of this Section, we adpot the following conventions: We use roman script $A, B, C, \ldots$ to denote elements of $\mathcal{F}$. We use the usual calligraphic script, $\mathcal{I}, \mathcal{J}, \mathcal{K}, \ldots$ to denote sieves in $\mathcal{F}$ (that is, *elements* of $\mathcal{L}$), and bold face $\mathbf{A}, \mathbf{B}, \mathbf{C}$ to denote sieves in $\mathcal{L}$. We use $\sqcap$ and $\sqcup$ to denote the meet and join in $\mathcal{F}$, $\wedge$ and $\vee$ to denote the meet and join in $\mathcal{L}$ (that is, the meet and join in the locale of countably closed sieves in $\mathcal{F}$), and, when needed, $\Cap$ and $\Cup$ to denote the meet and join in the locale of sieves in $\mathcal{L}$

In this setting, μ is a measure on $\mathcal{F}$, and $\overline{\mu}$ is the corresponding measure on $\mathcal{L}$, given by

$$\overline{\mu}(\mathcal{I}) = \bigvee_{A \in \mathcal{I}} \mu(A)$$

Fix $\mathcal{I} \in \mathcal{L}$. In $\widehat{\mathcal{L}}$, a sieve on $\mathcal{I}$ is some set $\mathbf{A}$ of countably complete sieves. However, since we are working in Sh($\mathcal{L}$), $\mathbf{A}$ must be a principal sieve $\mathbf{A} = \downarrow\mathcal{J}$, for some $\mathcal{J} \leq \mathcal{I}$.

One element of $\mathcal{L}$ is the sieve $\mathcal{N}$ of elements of $\mathcal{F}$ with measure 0. The traditional "almost everywhere" topology is given by

$$\mathbf{A} \mapsto (\downarrow\mathcal{N}) \Cup \mathbf{A}$$

The topology described in Definition 39, is given by

$$\begin{aligned}
\mathbf{A} &\mapsto [\![\rho(\overline{\mu}, \mathbf{A}) = \overline{\mu}]\!] \\
\downarrow\mathcal{J} &\mapsto [\![\rho(\overline{\mu}, \mathcal{J}) = \overline{\mu}]\!] \\
&= \left\{\mathcal{K} \in \mathcal{L} \middle| \overline{\mu}\left(\mathcal{K} \wedge \mathcal{J}\right) = \overline{\mu}(\mathcal{K})\right\} \\
&= \left\{\mathcal{K} \in \mathcal{L} \middle| \left(\bigvee_{\langle J,K\rangle \in \mathcal{J}\times\mathcal{K}} \mu(K \sqcap J)\right) = \left(\bigvee_{K \in \mathcal{K}} \mu(K)\right)\right\}
\end{aligned}$$

Hence a sieve $\mathcal{K}$ is in the closure of $\downarrow\mathcal{J}$ if and only if for every $K \in \mathcal{K}$, and for every natural number n, there is a $J_n \in \mathcal{J}$ such that

$$\mu(K \sqcap J_n) \geq \mu(K) - \frac{1}{n}$$

Since $\mathcal{J}$ is countably complete, it follows that there is a $J = \bigsqcup_n J_n \in \mathcal{J}$ such that $\mu(K \sqcup J) = \mu(K)$. Hence $\mu(K \sqcap \neg J) = 0$. Thus $K = J \sqcup N$ for some $N \in \mathcal{N}$. Thus the closure of $\downarrow(\mathcal{J})$ is contained in $(\downarrow\mathcal{J}) \Cup \downarrow(\mathcal{N})$.

The reverse inclusion is immediate.

□

The following Corollary follows immediately, since the closed sieves of $\mathcal{F}$ are just elements of $\uparrow \mathcal{N} \subseteq \mathcal{P}\mathcal{F}$.

Corollary 38. *Let $\mathcal{F}$ be a σ algebra, and let μ be a measure on $\mathcal{F}$. Then*

$$\mathrm{Sh}_\mu(\mathcal{F}) \simeq \mathrm{Sh}(\mathcal{F}_\mu)$$

where $\mathcal{F}_\mu = \mathcal{F}/\mathcal{N}$ is the σ-algebra found by taking the quotient of $\mathcal{F}$ by the ideal $\mathcal{N}$ of μ-negligible sets.

Finally in this Section, we introduce a preorder $\ll$ on $\mathbb{M}$. This preorder is related to the idea of the "almost everywhere" cover.

Definition 40. Take $\mu, \nu \in \mathbb{M}$. Then say that μ *dominates* ν, or that $\nu \ll \mu$, if

$$\forall \mathcal{I} \in \Omega \; (\rho(\mu, \mathcal{I}) = \mu) \Rightarrow (\rho(\nu, \mathcal{I}) = \nu)$$

This notion of dominance captures the idea of distribution of mass. μ dominates ν if every sieve that captures all of μ's mass also captures ν's mass. In the case where $\mathcal{E} = \mathrm{Sh}(\mathcal{F})$ this can be thought of as saying that ν has no mass wherever μ has no mass. Thus $\nu \ll \mu$ if and only if $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

We can formulate this idea internally.

Definition 41. The map $\mathfrak{Null} : \mathbb{M} \to \Omega$ is defined in the presheaf topos $\widehat{\mathcal{L}}$ by

$$\mathfrak{Null}(\mu) = [\![\forall q \in \mathbb{Q}\, q \in \mu]\!]$$

Hence $\nu \ll \mu$ if and only if

$$\mathfrak{Null}(\mu) \subseteq \mathfrak{Null}(\nu)$$

This means that $\ll$ is the pullback of the usual ordering $\leq$ along $\mathfrak{Null}$:

$$\begin{tikzcd}
\ll \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \leq \arrow[d, tail] \\
\mathbb{M} \times \mathbb{M} \arrow[r, "\langle \mathfrak{Null}, \mathfrak{Null} \rangle"'] & \Omega \times \Omega
\end{tikzcd}$$

It follows from the semicontinuity condition that $\mathfrak{Null}$ factors through $\Omega_j \rightarrowtail \Omega$. However, in the case that $\mathrm{Sh}(\mathcal{L})$ is a Boolean topos, we can go further.

Proposition 39. *Take $\mu, \nu \in \mathbb{M}$. Then $\nu \ll \mu$ if and only if $\mathfrak{Null}(\mu) \leq \mathfrak{Null}(\nu)$.*

Definition 40 therefore restricts to the classical notion of one measure dominating another.

Unsurprisingly, there is a relationship between the dominance relation on measures and the "finer" relation on topologies:

Proposition 40. *Let μ and ν be two measures, and let j_μ and j_ν be the induced topologies. Then $\mu \ll \nu$ implies that j_μ is finer than j_ν.*

Proof. Note that:

$$\begin{aligned}
\mu \ll \nu &\equiv \forall \mathcal{I}\ (\rho(\nu, \mathcal{I}) = \nu) \Rightarrow (\rho(\mu, \mathcal{I}) = \mu) \\
&\equiv \forall \mathcal{I}\ (\forall q \in \mathbb{Q}\ (\mathcal{I} \Rightarrow q \in \nu) \iff q \in \nu) \Rightarrow (\forall q \in \mathbb{Q}\ (\mathcal{I} \Rightarrow q \in \mu) \iff q \in \mu) \\
&\equiv \forall \mathcal{I}\ j_\nu(\mathcal{I}) \Rightarrow j_\mu(\mathcal{I})
\end{aligned}$$

Hence we can indicate $\mu \ll \nu$ by writing $j_\nu \leq j_\mu$. Recall that j_μ is finer than j_ν if anod only if

$$j_\mu = j_\mu \circ j_\nu$$

Since j_μ is a topology, it preserves order. Hence

$$\begin{aligned} j_\nu &\leq j_\mu \\ j_\mu \circ j_\nu &\leq j_\mu \circ j_\mu \\ j_\mu \circ j_\nu &\leq j_\mu \end{aligned}$$

Likewise,

$$\begin{aligned} \mathrm{id}_\Omega &\leq j_\nu \\ j_\mu \circ \mathrm{id}_\Omega &\leq j_\mu \circ j_\nu \\ j_\mu &\leq j_\mu \circ j_\nu \end{aligned}$$

Hence $j_\mu = j_\mu \circ j_\nu$, as required.

$\square$

3.3 ALMOST EVERYWHERE SHEAFIFICATION

In this Section, we investigate some of the important properties of "almost everywhere" topologies. We adopt the following notational conventions in this Section.

$(\mathcal{L}, \leq, \top, \bot)$ is a locale, $\widehat{\mathcal{L}}$ is the topos of presheaves on $\mathcal{L}$, and $\mathrm{Sh}(\mathcal{L})$ is the topos of sheaves on $\mathcal{L}$ (relative to the canonical topology j). $\mathbb{M}$ is the sheaf of measures, $\mathbb{D}$ is the sheaf of non-negative Dedekind real numbers, Ω is the subobject classifier in $\widehat{\mathcal{L}}$, and Ω_j is the subobject classifier in $\mathrm{Sh}(\mathcal{L})$. $\int : \mathbb{D} \times \mathbb{M} \to \mathbb{M}$ is the integration arrow, and $\rho : \mathbb{M} \times \Omega_j \to \mathbb{M}$ is the restriction arrow. Both $\int$ and ρ are arrows in the sheaf topos $\mathrm{Sh}(\mathcal{L})$.

Given a measure μ, j_μ is the "almost everywhere" topology associated with μ, and $\mathrm{Sh}_\mu(\mathcal{L})$ is the subtopos of $\mathrm{Sh}(\mathcal{L})$ induced by this topology. The sheafification functor is denoted $\mathbf{a}_\mu$.

$\mathrm{Sh}_\mu(\mathcal{L})$ is itself a localic topos. Let $\mathcal{L}_\mu$ be the locale of μ closed sieves, so we have

$$\mathcal{L} \to \mathcal{L}_\mu \rightarrowtail \mathcal{L}$$

Since $\mathrm{Sh}_\mu(\mathcal{L})$ is localic, it has its own measure theory. Let $\mathbb{M}_\mu$, $\mathbb{D}_\mu$, and $\int_\mu$ denote respectively the sheaf of measures, the sheaf of non-negative Dedekind reals, and the integration arrow in $\mathrm{Sh}_\mu(\mathcal{L})$.

Finally, we consider two subobjects of $\mathbb{M}$. $\twoheaddownarrow\mu$ is interpreted in $\mathrm{Sh}(\mathcal{L})$ as the set

$$\twoheaddownarrow\mu \equiv \{\nu \in \mathbb{M} | \nu \ll \mu\}$$

and $\mathbb{M}^F$ is interpreted as

$$\mathbb{M}^F \equiv \{\nu \in \mathbb{M} | \exists q \in \mathbb{Q}\, \mu \leq q\}$$

($\mathbb{Q}$ is the object of positive rationals in $\mathrm{Sh}(\mathcal{L})$ and "$\leq$" relation is the associated sheaf of the "element of" relation on $\mathbb{Q} \times \mathbb{M}$).

$\twoheaddownarrow\mu$ is the sheaf of measures that are dominated by μ. $\mathbb{M}^F$ is the sheaf of locally finite measures. It is easy to see that if $\mathcal{L}$ is the locale of countably closed sieves in some σ-algebra $\mathcal{F}$, then $\mathbb{M}^F$ corresponds to the sheaf of σ-finite measures on $\mathcal{F}$.

The results from this Section can all be displayed in one commutative diagram in $\mathrm{Sh}(\mathcal{L})$:

$$\begin{tikzcd}
\mathbb{D} \arrow[r, equal] \arrow[d, "\langle \mathrm{id}_D, \mu \rangle"'] & \mathbb{D} \arrow[r, "\eta^{\mathbb{D}}"] \arrow[dd, dotted, "\int(-,\mu)"'] & \mathbf{a}_\mu \mathbb{D} \arrow[dd, "\mathbf{a}_\mu\left(\int(-,\mu)\right)"'] & \mathbb{D}_\mu \arrow[l, tail] \arrow[d, "\langle \mathrm{id}_{\mathbb{D}_\mu}, \mu^* \rangle"] \\
\mathbb{D} \times \mathbb{M} \arrow[d, "\int"'] & & & \mathbb{D}_\mu \times \mathbb{M}_\mu \arrow[d, "\int_\mu"] \\
\mathbb{M} & \twoheaddownarrow\mu \arrow[l, tail] \arrow[r, equal, "\eta^{\twoheaddownarrow\mu}", "\cong"'] & \mathbf{a}\twoheaddownarrow\mu \arrow[r, equal, "k", "\cong"'] & \mathbb{M}_\mu
\end{tikzcd}$$

In this diagram, η is the unit of the adjunction $\mathbf{a}_\mu \dashv \mathbf{i}$, and so $\eta^A : A \to \mathbf{a}_\mu A$. The arrow $\mu^* : \mathbf{1} \to \mathbb{M}_\mu$ picks out the measure on the j_μ-closed sieves of $\mathcal{L}$ obtained by restricting μ.

Technically $\mathbf{a}_\mu A$ is an object of the topos $\mathrm{Sh}_j(\mathcal{L})$, rather than $\mathrm{Sh}(\mathcal{L})$, and $\mathbf{ia}_{\mu(\mu)}$ is the corresponding onbect in $\mathrm{Sh}(\mathcal{L})$. However, as this whole diagram is assumed to exist in $\mathrm{Sh}(\mathcal{L})$, we can drop the $\mathbf{i}$s without fear of ambiguity.

We have five results to show:

Theorem 14. *1. The map $\int \circ \langle id_{\mathbb{D}}, \mu \rangle$ does indeed factor through $\downarrow\!\!\downarrow\mu \rightarrowtail \mathbb{M}$.*

2. $\mathbf{a}_\mu \downarrow\!\!\downarrow\mu \cong \mathbb{M}_\mu$

3. *There is a natural embedding $\mathbb{D}_\mu \rightarrowtail \mathbf{a}_\mu \mathbb{D}$.*

4. *The right hand rectangle commutes.*

5. *The sheaf $\downarrow\!\!\downarrow\mu$ is a μ-sheaf.*

Proof. 1. To prove Part 1, we need to show that for any measures μ, and any density f, we have $\int(f, \mu) \ll \mu$.

This can be rewritten as

$$(\rho(\mu, \mathcal{I}) = \mu) \Rightarrow \left(\rho\left(\int(f, \mu), \mathcal{I}\right) = \int(f, \mu)\right)$$

Start by taking $\mathcal{I} \in \Omega_j$ such that $\rho(\mu, \mathcal{I}) = \mu$. Then, since we are working in a localic topos, there must be a $B \in \mathcal{L}$ such that $\mathcal{I} = \downarrow B$. In this case, it follows that for any $A \in \mathcal{L}$,

$$\mu(A) = \mu(A \wedge B)$$

We must show that given such a B, we have

$$\int_A f\, d\mu = \int_{A \wedge B} f\, d\mu$$

The result is immediate for locally constant $q \in \mathbb{Q}$:

$$\begin{aligned} \int_A q\, d\mu &= q \cdot \mu(A) \\ &= q \cdot \mu(A \wedge B) \\ &= \int_{A \wedge B} q\, d\mu \end{aligned}$$

Now, assume that $f = \langle L, U \rangle$ is a Dedekind real. Then

$$\begin{aligned} \int_A f\, d\mu &= \bigvee_{q \in L} \int_A q\, d\mu \\ &= \bigvee_{q \in L} \int_{A \wedge B} q\, d\mu \\ &= \int_{A \wedge B} f\, d\mu \end{aligned}$$

as required.

2. We know from Theorem 12 that for any $A \in \mathcal{L}$,

$$\mathbf{a}_\mu {\downarrow\!\!\downarrow}\mu(A) = \left(\coprod_{j_\mu B = A} {\downarrow\!\!\downarrow}(B) \right) \Big/ \sim$$

where $\nu_1 \in \downarrow B_1 \sim \nu_2 \in \downarrow B_2$ if

$$\rho^{B_1}_{B_1 \sqcap B_2}(\nu_1) = \rho^{B_2}_{B_1 \sqcap B_2}(\nu_2)$$

We also know that if $A \in \mathcal{L}$, then $\mu \in \mathbb{M}_\mu(A)$ if μ is a measure on the locale $\downarrow A$.

It is helpful to consider the embedding $\mathrm{Sh}_\mu(\mathcal{L}) \rightarrowtail \mathrm{Sh}(\mathcal{L})$ a little more carefully. A functor $F \in \mathrm{Sh}(\mathcal{L})$ is a μ-sheaf if $F(A) = F(B)$ whenever $\overline{A} = \overline{B}$, or equivalently that $F(A) = F(\overline{A})$.

We can view objects in $\mathrm{Sh}_\mu(\mathcal{L})$ in two ways; as functors on $\mathcal{L}$, and as functors on $\mathcal{L}_\mu$. A functor on $\mathcal{L}$ can be restricted to give the corresponding functor on $\mathcal{L}_\mu$. Conversely, a functor on $\mathcal{L}_\mu$ can be extended to $\mathcal{L}$. If F is a μ-sheaf viewed as a functor on $\mathcal{L}_\mu$, we can define $\overline{F}$ by

$$\overline{F}(A) = F\left(\overline{A}\right)$$

We work in $\mathrm{Sh}(\mathcal{L})$. In this setting, we already understand $\mathbf{a}_\mu {\downarrow\!\!\downarrow}\mu$. However, the natural interpretation of $\mathbb{M}_\mu$ is as a functor on $\mathcal{L}_\mu$: For any $B \in \mathcal{L}_\mu$, $\mathbb{M}_\mu(B)$ is the set of measures on the sublocale $\downarrow B$. In light of the previous remarks, we will consider $\mathbb{M}_\mu$ to be a functor on $\mathcal{L}$, where for any $A \in \mathcal{L}$, $\mathbb{M}_\mu(A)$ is the set of measures on the sublocale $\downarrow\overline{A} \subseteq \mathcal{L}_\mu$.

We start by taking $\nu \in \mathbb{M}_\mu(A)$. Then ν is a measure on $\downarrow\overline{A} \subseteq \mathcal{L}_\mu$. To extend ν to a measure on $\downarrow\overline{A} \subseteq \mathcal{L}$, we define $\overline{\nu}$ by

$$\overline{\nu}(B) = \nu\left(\overline{B}\right)$$

We must verify that $\overline{\nu}$ is indeed a measure.

- To see that $\overline{\nu}$ satisfies the additivity condition, take $U, V \in \mathcal{L}$. Then

$$\begin{aligned} \overline{\nu}(U) + \overline{\nu}(V) &= \nu\left(\overline{U}\right) + \nu\left(\overline{V}\right) \\ &= \nu\left(\overline{U \wedge V}\right) + \nu\left(\overline{U \vee V}\right) \\ &= \overline{\nu}\,(U \wedge V) + \overline{\nu}(U \vee V) \end{aligned}$$

  as required.

- To see that $\overline{\nu}$ satisfies the semicontinuity condition, fix $q \in \mathbb{Q}$. We need to show that the sieve $\mathcal{I} = \{U \in \mathcal{L} | \overline{\nu} \leq q\}$ is directed closed (where closure refers to the closure operation j). Take a directed family $\mathcal{D} \subseteq \mathcal{I}$, and let $D = \bigvee \mathcal{D}$. Let $\mathcal{D}'$ be the family $\left\{\overline{B} | B \in \mathcal{D}\right\}$. Then $\mathcal{D}'$ is a directed family in $\mathcal{L}$, and in $\mathcal{L}_\mu$.

  In fact,

  $$\nu\left(\overline{D}\right) = \nu\left(\bigvee \mathcal{D}'\right) \leq q$$

  since for each $D \in \mathcal{D}$, we have

  $$\nu\left(\overline{D}\right) = \overline{\nu}(D) \leq q$$

  Finally, we get:

  $$\begin{aligned} \overline{\nu}(D) &= \nu\left(\overline{D}\right) \\ &= \nu\left(\bigvee \mathcal{D}'\right) \\ &\leq q \end{aligned}$$

  as required.

Finally, note that whenever $\overline{C} = \overline{A}$, then $\rho(\overline{\nu}, C) = \nu$, and so $\overline{\nu} \ll \mu$ (since, by definition $\overline{C} = \overline{A} \iff \rho(\mu, Z) = \mu$). Thus $\overline{\nu}$ is not just an element of $\mathbb{M}(A)$, but an element of $\left(\downarrow\!\mu\right)(A)$.

Thus we have built a monomorphism from $\mathbb{M}_\mu$ to $\downarrow\!\mu$. Composing this with the map $\eta^{\downarrow\mu} : \downarrow\!\mu \to \mathbf{a}_\mu\!\downarrow\!\mu$ gives us an arrow in $\mathrm{Sh}_\mu(\mathcal{L})$ from $\mathbb{M}_\mu$ to $\mathbf{a}_\mu\!\downarrow\!\mu$.

The reverse direction is simpler. An element of $\mathbf{a}_\mu\!\downarrow\!\mu(A)$ is a measure $\nu \ll \mu$ on $\downarrow\!\overline{A}$. In order to see that such a ν restricts to a measure on $\downarrow\!\overline{A} \subseteq \mathcal{L}_\mu$, we just need to verify that $\nu(B) = \nu\left(\overline{B}\right)$ for any $B \leq A$. But this follows immediately from the fact that $\nu \ll \mu$.

These operations are easily seen to be inverse to one another, so we have shown that

$$\mathbf{a}_\mu\!\downarrow\!\mu \cong \mathbb{M}_\mu$$

3. To see that $\mathbb{D}_\mu \rightarrowtail \mathbf{a}_\mu\mathbb{D}$, we consider $\mathbf{a}_\mu\mathbb{D}$ as a sheaf on $\mathcal{L}$. $\mathbf{a}_\mu\mathbb{D}(A)$ is the set of equivalence classes of Dedekind reals:

$$\coprod_{\overline{B}=\overline{A}} \mathbb{D}(B)/\sim$$

Now consider the sheaf $\mathbb{D}_\mu$ of Dedekind reals inside $\mathrm{Sh}_\mu(\mathcal{L}) \simeq \mathrm{Sh}(\mathcal{L}_\mu)$. Since $\mathbb{D}_\mu$ is a sheaf on $\mathcal{L}_\mu$, there is an extension, $\overline{\mathbb{D}_\mu}$, a sheaf on $\mathcal{L}$, by

$$\overline{\mathbb{D}_\mu}(A) = \mathbb{D}_\mu(\overline{A})$$

The elements of $\mathbb{D}_\mu(\overline{A})$ are themselves sheaves (in $\mathrm{Sh}(\mathcal{L}_\mu)$). These sheaves can be interpreted as sheaves on $\mathcal{L}$. It follows that an element of $\mathbb{D}_\mu(\overline{A})$ is a pair $\langle L, U\rangle$ of subsheaves of $\mathbb{Q}$.

In order for a pair of sheaves to be a Dedekind real in $\mathrm{Sh}_\mu(\mathcal{L})$, they must satisfy the satisfying the following formulas in $\mathrm{Sh}_\mu(\mathcal{L})$:

a. $\forall q \in \mathbb{Q} \neg (q \in L \wedge q \in U)$
b. $\forall q \in \mathbb{Q}\, q \in U \Rightarrow \exists r \in \mathbb{Q}\ (r < q \wedge r \in U)$
c. $\forall q \in \mathbb{Q}\, q \in L \Rightarrow \exists r \in \mathbb{Q}\ (r > q \wedge r \in L)$
d. $\forall q \in \mathbb{Q}\, \forall r \in \mathbb{Q}\ (q \in U \wedge r > q) \Rightarrow r \in U$
e. $\forall q \in \mathbb{Q}\, \forall r \in \mathbb{Q}\ (q \in L \wedge r < q) \Rightarrow r \in L$
f. $\forall q \in \mathbb{Q}\, \forall r \in \mathbb{Q}\, q < r \Rightarrow (q \in L \vee r \in U)$
g. $\exists q \in \mathbb{Q}\, q \in U$

Supposing that $\langle L, U\rangle$ satisfies these formulas in $\mathrm{Sh}_\mu(\mathcal{L})$, it is natural to ask what can we say about them in the larger topos $\mathrm{Sh}(\mathcal{L})$. In order to determine this, we first consider the nature of the object of rational numbers in $\mathrm{Sh}_\mu(\mathcal{L})$. It is known that $\mathbb{Q}_\mu$, the object of rational numbers in $\mathrm{Sh}_\mu(\mathcal{L})$ is just $\mathbf{a}_\mu\mathbb{Q}$, the associated sheaf of $\mathbb{Q}$, the object of rational numbers in $\mathrm{Sh}(\mathcal{L})$. Hence we can consider L and U to be subsheaves of $\mathbb{Q}$, rather than $\mathbb{Q}_\mu$.

The first five conditions taken together imply that for any $A \in \mathcal{L}$, $L(A)$ is an open lower set of rationals, that $U(A)$ is an open uper set of rationals, and that $L(A) \cap U(A) = \emptyset$. But this is clearly the same as if we had interpreted the formulas in $\mathrm{Sh}(\mathcal{L})$. It follows that a pair of μ-sheaves $\langle L, U\rangle$ satisfying the first five conditions will also satisfy them when interpreted as sheaves in $\mathrm{Sh}(\mathcal{L})$.

If $\langle L, U\rangle \in \mathbb{D}_\mu(A)$ (for some μ-closed sieve A) satisfy the sixth condition for some $q, r \in \mathbb{Q}$, then there must be some $B \in \mathcal{L}$ such that $\overline{B} = A$

$$B \Vdash (q \in L) \vee (r \in U)$$

hence $\langle L, U\rangle \in \mathbb{D}(B)$. Since

$$\mathbb{D}(B) \rightarrowtail \coprod_{\overline{C}=A} \mathbb{D}(C) \twoheadrightarrow \coprod_{\overline{C}=A} \mathbb{D}(C) \Big/ \sim \xrightarrow{\cong} \mathbf{a}_\mu \mathbb{D}$$

Similarly, if $\mathrm{Sh}_\mu(\mathcal{L}) \models \exists q \in \mathbb{Q}\, q \in U$, then there is some locally constant rational q_0 and a dense $B \in \downarrow A \subseteq \mathcal{L}$ such that $B \Vdash q_0 \in U$, or equivalently that $q_0 \in U(B)$. However, since B is dense, and since U is a μ-sheaf, it follows that $q_0 \in U(A)$.

We now have a map from $\mathbb{D}_\mu \to \mathbf{a}_\mu D$, and need only show that this map is monic. But this follows from the fact that if $\langle L_1, U_1\rangle$ and $\langle L_2, U_2\rangle$ are two distinct pairs of μ-sheaves, then they must differ at some μ-closed element of $\mathcal{L}$.

4. The fact that the right hand rectangle commutes follows from the construction of the integral, in $\mathrm{Sh}(\mathcal{L})$ and in $\mathrm{Sh}_\mu(\mathcal{L})$. The result is immediate for locally constant q, and the extension to Dedekind reals follows.

5. We know that $\downarrow\!\!\downarrow \mu$ is a sheaf on $\mathcal{L}$. Take $A \in \mathcal{L}$. Then a covering family for A is a certain set $C \subseteq \downarrow A \subseteq \mathcal{L}$. However, since $\downarrow\!\!\downarrow \mu$ is a sheaf, we know that matching families on C correspond to elements of $\downarrow\!\!\downarrow \mu\, (\curlyvee C)$.

   So, take a $B \in \mathcal{L}$ such that $\downarrow B$ μ-covers A, and take $\nu \in \downarrow\!\!\downarrow \mu(B)$. We have to show that ν has a unique extension $\overline{\nu} \in \downarrow\!\!\downarrow \mu(A)$.

   For each $D \leq A$, set

$$\overline{\nu}(D) = \nu(D \wedge B)$$

   It is immediate that $\overline{\nu}$ is an extension of μ, and that $\overline{\nu}$ is a measure dominated by μ. Thus we only need to show that $\overline{\nu}$ is the *unique* amalgamation of ν.

   Suppose that $\lambda \in \downarrow\!\!\downarrow \mu(A)$ is an arbitrary extension of ν. Since $\lambda \ll \mu$, it follows that for any C we have

$$(\rho(\mu, C) = \mu) \Rightarrow (\rho(\lambda, C) = \lambda)$$

Since B is μ-dense, we have $\rho(\mu, B) = \mu$, and hence $\rho(\lambda, B) = \lambda$. Therefore

$$\lambda(D) = \rho(\lambda, B)(D) = \lambda(B \wedge D) = \overline{\nu}(D)$$

Thus $\lambda = \overline{\nu}$, as $\overline{\nu}$ is unique, as required.

$\square$

These results can also be restricted to the case where we work with $\mathbb{M}^F$, rather than $\mathbb{M}$, and interpret $\downarrow\mu$ as a subobject of $\mathbb{M}^F$. There are no significant changes in the proofs.

3.4 DIFFERENTIATION IN A BOOLEAN LOCALIC TOPOS

In this section, we look at some special properties of the measure theory of a Boolean localic topos. A localic topos is a topos that is equivalent to the topos of sheaves on some locale $\mathcal{L}$. The Heyting algebra of truth values in such a topos is just $\mathcal{L}$. As a result, the topos satisfies the law of the excluded middle just in the case that $\mathcal{L}$ is a complete Boolean algebra, and not merely a Heyting algebra (note that all complete Boolean algebras are locales).

Let $\mathcal{B}$ be a complete Boolean algebra, let $\widehat{\mathcal{B}}$ be the topos of presheaves on $\mathcal{B}$, and let $\mathrm{Sh}(\mathcal{B})$ be the topos of sheaves on $\mathcal{B}$, relative to the canonical topology. Note that a measure (that is, as element of $\mathbb{M}(A)$ on $\downarrow A \subseteq \mathcal{B}$) is additive for *all* cardinalities, and not just countable cardinalities. This means that

$$\mu\left(\bigvee_{D\in\mathcal{D}}\right) = \bigvee_{D\in\mathcal{D}} \mu(D)$$

for *any* directed set $\mathcal{D} \subseteq \mathcal{B}$. Likewise:

$$\mu\left(\bigvee_{A\in\mathcal{A}}\right) = \sum_{A\in\mathcal{A}} \mu(A)$$

for *any* antichain $\mathcal{A} \subseteq \mathcal{B}$. This property can be called "complete additivity" (extending the usual measure theoretic terminology of "countable additivity").

Definition 42. Let $\mathcal{E}$ be a localic topos, and let $\mu : \mathbf{1} \to \mathbb{M}$ be a global element of $\mathbb{M}$. Then we say that μ is *differentiable*, or has Radon-Nikodym derivatives, if the following arrow in $\mathcal{E}$ has a right inverse, called $\frac{d}{d\mu}$:

$$\mathbb{D} \xrightarrow{\langle \mathrm{id}_{\mathbb{D}}, \mu \rangle} \mathbb{D} \times \mathbb{M} \xrightarrow{\int} \downarrow\mu^F$$

where $\downarrow_{\mu^F} \rightarrowtail \mathbb{M}$ is the sheaf of locally finite measures $\nu \ll \mu$.

The measure theoretic significance of Boolean localic toposes is the following Theorem:

Theorem 15 (Radon-Nikodym Theorem I). *Let $\mathcal{E}$ be a Boolean localic topos. Then every locally finite measure in $\mathcal{E}$ is differentiable.*

It is possible to view this arrangement in category theoretic terms. We prove that for locally finite μ, the arrow in Definition 42 is the surjective part of the image factorization of

$$\mathbb{D} \xrightarrow{\langle \mathrm{id}_{\mathbb{D}}, \mu \rangle} \mathbb{D} \times \mathbb{M} \xrightarrow{\int} \downarrow\mu^F$$

The derivative is therefore

$$\begin{tikzcd}
\mathbb{D} \arrow[r, two heads, shift right] \arrow[d] & \downarrow\mu^F \arrow[l, dotted, shift right, "\frac{d}{d\mu}"'] \arrow[d, tail] \\
\mathbb{D} \times \mathbb{M} \arrow[r, "\int"'] & \mathbb{M}
\end{tikzcd}$$

If we can show that the top arrow $\mathbb{D} \twoheadrightarrow \downarrow\mu^F$ is indeed an epimorphism, then the existence of the derivative is an immediate consequence of the fact that epimorphisms split in Boolean toposes.

Before proving this Theorem, some preliminary results are needed.

Definition 43. Say that two measures μ and ν are mutually singular if $\mathfrak{Null}\mu \vee \mathfrak{Null}\nu = \top$.

Proposition 41 (Hahn Decomposition Theorem). *If μ_1 and μ_2 are any two locally finite measures on a complete Boolean algebra $\mathcal{B}$, then there exists $B \in \mathcal{B}$ such that the following statements hold:*

1. $B \Vdash \nu \leq \mu$
2. $\neg B \Vdash \mu \leq \nu$

Proof. This proof is based on the usual proof of the Hahn decomposition theorem (see [3]), with some slight modifications to allow for the fact that we are not using countably additive measures on σ-fields, but rather completely additive measures on complete Boolean algebras.

Start by restricting to a cover $C = \langle C_i | i \in I \rangle$ on which both μ and ν are finite. The extension of the result to locally finite μ and ν will be immediate.

Let $\phi : \downarrow C_i \to (-\infty, \infty)$ be given by

$$\phi(B) = \mu(B) - \nu(B)$$

Since $\phi \leq \mu$, it follows that ϕ is bounded above, and so has a supremum, $\alpha < \infty$.

Suppose that there exists $B \in \downarrow C_i$ such that $\phi(C_i) = \alpha$. Then it follows that for every $D \in B_i$, we have $\phi(D) \geq 0$, or else $\phi(B \wedge \neg D) > \alpha$, which would be a contradiction. Likewise, every $D \in \downarrow(\neg B)$ must have $\phi(D) \leq 0$, for the same reason. Consequently, it will suffice to find such a B.

We know that there must be a sequence $\langle D_n | n < \omega \rangle$ such that $\phi(D_n) \uparrow \alpha$. For each n, let $F_n \subseteq \downarrow C_i$ be given by

$$\mathcal{F}_n = \left\{ \left(\bigwedge_{i \in S} D_i \right) \wedge \left(\bigwedge_{j \notin S} \neg D_j \right) \middle| S \subseteq \{0, \ldots, n\} \right\}$$

In other words, $\mathcal{F}_n$ is the set of atoms of the sub-Boolean algebra $\mathcal{B}_n$ of $\downarrow C_i$ generated by $\{D_i | i \leq n\}$.

For each n, let $\mathcal{G}_n \subseteq \mathcal{F}_n$ be given by

$$\mathcal{G}_n = \{E \in \mathcal{F}_n | \phi(E) \geq 0\}$$

and let $G_n = \bigvee \mathcal{G}_n$. Hence G_n maximizes ϕ over $\mathcal{B}_n$. Since $D_n \in \mathcal{B}_n$, it is obvious that

$$\phi(D_n) \leq \phi(G_n) \leq \alpha$$

Let $H_n = \bigcurlywedge_{i=n}^{\infty} G_i$. We can see that $\phi(H_n)$ is increasing, as

$$\left(\bigcurlyvee_{i=m}^{n} G_i\right) \wedge \neg \left(\bigcurlyvee_{i=m}^{n-1} G_i\right)$$

is the join of elements of $\mathcal{F}_n$.

Finally, let $B = \bigcurlywedge H_n$, so that

$$B = \limsup_n G_n$$

Then $H_n \downarrow B$, and so $\phi(H_n) \to \phi(B)$. Since we already know that $\phi(H_n) \to \alpha$, it follows that $\phi(B) = \alpha$, and we are done.

$\square$

Corollary 42. *If $\mathcal{E}$ is Boolean, then $\mathbb{M}^F$ is (internally) totally ordered.*

Proof. We need to show that $\mathcal{E} \models (\mu \leq \nu) \vee (\nu \leq \mu)$. But we know that

1. $\mathcal{E} \models B \Rightarrow \nu \leq \mu$
2. $\mathcal{E} \models \neg B \Rightarrow \mu \leq \nu$

Since $\mathcal{E}$ is Boolean, we also have $\mathcal{E} \models B \vee (\neg B)$, so we are done. $\square$

In order to prove the Radon-Nikodym Theorem, we will make some small modifications to the standard proof (see, for example [3]). The modified proof is based on the fact that a Boolean localic topos is the topos of sheaves on a complete Boolean lattice, since a localic topos is Boolean if and only if the underlying locale is a complete Boolean algebra, and not just a complete Heyting algebra. The main modification between the standard proof and the modified proof is that we do not assume that the Boolean algebra is a field (that is, we do not assume that the elements of $\mathcal{B}$ are sets), and so we work with Dedekind cuts rather than with measurable functions.

We first prove the following two Lemmas:

Lemma 43. *Suppose that $\nu \leq \mu \in \mathbb{M}^F(A)$. Then there exists $\lambda \in \mathbb{M}^F(A)$ such that $\nu + \lambda = \mu$.*

Proof. We know that there is a cover $\langle A_i | i \in I\rangle$ for A such that for each $i \in I$ we have $\nu(A_i) \leq \mu(A_1) < \infty$. Given such a cover, there is family of measures $\lambda_i \in \mathbb{M}^F(A_i)$. The λ_is form a matching family and so have a unique amalgamation in $\mathbb{M}^F(A)$.

First, we must find the λ_is. Let $\lambda_i(B) = \mu(B) - \nu(B)$. We need to show that λ_i is a measure on $\downarrow A_i$.

It is immediate that if the λ_i's are measures, they are finite (since they are less than or equal to the restrictions of μ). Furthermore, they obviously form a matching family, and have a locally finite amalgamation.

Take $B_1, B_2 \in \downarrow A_i$. Then

$$\begin{aligned}
\lambda_i(B_1) + \lambda_i(B_2) &= (\mu_i(B_1) + \mu_i(B_2)) - (\nu_i(B_1) + \nu_i(B_2)) \\
&= (\mu_i(B_1 \wedge B_2) + \mu_i(B_1 \vee B_2)) - (\nu_i(B_1 \wedge B_2) + \nu_i(B_1 \vee B_2)) \\
&= \lambda_i(B_1 \wedge B_2) + \lambda_i(B_1 \vee B_2)
\end{aligned}$$

Hence λ_i satisfies the additivity condition.

Since $\mathcal{L}$ is a complete Boolean topos, we can show that λ_i satisfies the semicontinuity condition by showing that for any antichain $\mathcal{A} \subseteq \downarrow A_i$ we have

$$\lambda_i\left(\bigvee \mathcal{A}\right) = \sum_{B \in \mathcal{A}} \lambda(B)$$

Since $\mu(A)$ is finite, we can assume that for all but countably many of the elements of $\mathcal{A}$ satisfy

$$\mu(B) = \nu(B) = \lambda(B) = 0$$

So, without loss of generality, we can assume that $\mathcal{A}$ is countable. Write $\mathcal{A} = \langle B_i | i < \omega\rangle$. By definition, we know that

$$\sum_{i<\omega} \lambda(B_i) = \lim_{n \to \omega} \sum_{i=0}^{n} \lambda(B_i)$$

But for each n we have

$$\begin{aligned}
\sum_{i=0}^{n} \lambda(B_i) &= \sum_{i=0}^{n} (\mu(B_i) - \nu(B_i)) \\
&= \left(\sum_{i=0}^{n} \mu(B_i)\right) - \left(\sum_{i=0}^{n} \nu(B_i)\right) \\
&\to \left(\sum_{i<\omega} \mu(B_i)\right) - \left(\sum_{i<\omega} \nu(B_i)\right) \\
&= \mu\left(\bigvee \mathcal{A}\right) - \nu\left(\bigvee \mathcal{A}\right) \\
&= \lambda\left(\bigvee \mathcal{A}\right)
\end{aligned}$$

so we are done. □

Lemma 44. *If $\mu, \nu \in \mathbb{M}^F(A)$, and μ and ν are not mutually singular, then there exists $B \in \downarrow A$ and $q \in \mathbb{Q}$ such that $\mu(B) > 0$ and*

$$B \Vdash \int(q, \mu) \leq \nu$$

Note that this Lemma can also be stated in the following contrapositive form: If there is no such B and q, then μ and ν must be mutually singular.

Proof. We shall simplify our notation by exploiting the fact that integration of a rational number is the same as multiplication, and so we can write $q \times \mu$ for

$$\int(q, \mu)$$

For each $q \in \mathbb{Q}$, apply the Hahn Decomposition Theorem (Proposition 41) to find B_q such that

$$B_q \Vdash q \times \mu \leq \nu \qquad \neg B_q \Vdash \nu \leq q \times \mu$$

Define B by

$$B = \bigvee_{q \in \mathbb{Q}} B_q$$

Then $\neg B$ is given by

$$\neg B = \bigwedge_{q \in \mathbb{Q}} \neg B_q$$

Hence $\nu(\neg B) \leq (q \times \mu)(\neg B)$ for all $q \in Q$. Since $\mu(\neg B) < \infty$, it follows that $\nu(\neg B) = 0$. This means that $\neg B \leq \mathfrak{Null}(\nu)$. Since μ and ν are not mutually singular, it follows that $\mathfrak{Null}(\mu)$ cannot be contain B. Hence $\neg(\mathfrak{Null}(\mu)) \wedge B \neq \bot$. Therefore, $\mu(B) > 0$.

But the B_q's form an increasing chain whose join is B. It cannot be the case that all of the B_q's satisfy $\mu(B_q) = 0$, since this would imply that $\mu(B) = 0$. Hence there is a $q \in \mathbb{Q}$ such that $\mu(B_q) > 0$.

This B_q also satisfies (by definition)

$$B_q \Vdash q \times \mu \leq \nu$$

as required. □

Now, to prove Theorem 15

Proof. In light of the comments at the start of this Section, it suffices to show that if μ, ν are locally finite measures on a complete Boolean algebra $\mathcal{L}$, then there is a Dedekind real f on $\mathcal{L}$ such that $\int(f, \mu) = \nu$.

Let L be the sheaf of rationals satisfying

$$q \in L \iff \int(q, \mu) \leq \nu$$

Then, since $\mathcal{E}$ is Boolean, $\mathbb{D}$ is order complete, and so there is an $f \in \mathbb{D}$ such that $f = \bigvee L$. All we need to do now is show that

$$\int(f, \mu) = \nu$$

For convenience, we shall let σ denote the measure $\int(f, \mu)$.

It follows from the Monotone Convergence Theorem that

$$\sigma = \int(f, \mu) = \bigvee \int(q, \mu) \leq \nu$$

Applying Lemma 43, we see that there must be a measure λ such that $\sigma + \lambda = \nu$. (Relative to μ, σ is called the "absolutely continuous" part of ν, and λ is called the "singular" part of ν.) We will show that $\nu \ll \mu \Rightarrow \lambda = 0$.

Suppose that λ is not constantly zero. Then since $\lambda \ll \nu \ll \mu$, and since in a Boolean topos, $(\mu_1 \ll \mu_2) \iff (\mathfrak{Null}(\mu_1) \geq \mathfrak{Null}(\mu_2))$, it follows that $\mathfrak{Null}(\lambda) \geq \mathfrak{Null}(\mu)$.

Consequently, μ and λ can only be mutually singular if λ is the zero measure. Suppose that λ is not the zero measure. Then applying Lemma 44 we find that there is a $B \in \mathcal{L}$ and a q such that $q \times \mu \leq \lambda$ on B. But then we we could add the rational q to f (only locally, at B) and get

$$\int(f+q,\mu) = \int(f,\mu) + q \times \mu \leq \sigma + \lambda \leq \nu$$

But $\int(f,\mu) \lneq \int(f+q,\mu)$, so f is not the supremum claimed in its definition. This is a contradiction, and so λ must be identically zero, whence $\sigma = \nu$ as required. □

3.5 THE RADON-NIKODYM THEOREM

In the previous Section, we saw that locally finite measures are differentiable in a Boolean localic topos. In this Section, we use these derivatives to construct more general Radon-Nikodym derivatives. Rather than requiring $\mathrm{Sh}(\mathcal{L})$ to be Boolean, we will consider the case where $\mathrm{Sh}_\mu(\mathcal{L})$ is Boolean. In this case, the derivative of μ in $\mathrm{Sh}_\mu(\mathcal{L})$ will be extended to derivatives in $\mathrm{Sh}(\mathcal{L})$.

To construct a derivative map, we use a fragment of the diagram from page 91:

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "\eta^{\mathbb{D}}"] \arrow[dd, "\int - d\mu"'] & \mathbf{a}_\mu \mathbb{D} \arrow[dd, "\mathbf{a}_\mu\left(\int(-,\mu)\right)"'] & \mathbb{D}_\mu \arrow[l, tail] \arrow[d, "\langle \mathrm{id}_{\mathbb{D}_\mu}, \mu^* \rangle"] \\
 & & \mathbb{D}_\mu \times \mathbb{M}^F_\mu \arrow[d, "\int_\mu"] \\
\downarrow\mu \arrow[r, equal, "\eta^{\downarrow\mu}", "\cong"'] & \mathbf{a}_\mu \downarrow\mu \arrow[r, equal, "k", "\cong"'] & \mathbb{M}^F_\mu
\end{tikzcd}$$

with the added assumption that $\downarrow\mu$ is computed as a subobject of $\mathbb{M}^F$.

If μ^* is differentiable (in $\mathrm{Sh}_\mu(\mathcal{L})$), then the arrow $\int_\mu - d\mu : \mathbb{D}_\mu \to \mathbb{M}^F_\mu$ has a section $\frac{d}{d\mu^*} : \mathbb{M}^F_\mu \to \mathbb{D}_\mu$. We can then write the diagram as follows:

$$
\begin{tikzcd}
\mathbb{D} \arrow[r, "\eta^{\mathbb{D}}"] \arrow[d, "\int - d\mu"'] & \mathbf{a}_\mu\mathbb{D} \arrow[d, "\mathbf{a}_\mu \int(-,\mu)"] & \mathbb{D}_\mu \arrow[l, tail, "i"'] \arrow[d, shift left, "\int_\mu - d\mu^*"] \\
\twoheaddownarrow\mu \arrow[ur, dotted, "D_\mu"] \arrow[r, equal, "\eta^{\twoheaddownarrow\mu}", "\cong"'] & \mathbf{a}_\mu\twoheaddownarrow\mu \arrow[r, equal, "k", "\cong"'] & \mathbb{M}^F_\mu \arrow[u, shift left, "\frac{d}{d\mu^*}"]
\end{tikzcd}
$$

The Radon-Nikodym differentiation map can now be defined:

Definition 44. The action of differentiation with respect to μ is given by

$$i \circ \frac{d}{d\mu^*} \circ k \circ \eta^{\twoheaddownarrow\mu} : \twoheaddownarrow\mu \to \mathbf{a}_\mu\mathbb{D}$$

This arrow is denoted $D_\mu : \twoheaddownarrow\mu \to \mathbf{a}_\mu\mathbb{D}$. It is indicated in the above diagram by the dotted arrow.

The differentiation arrow takes a measure dominated by μ to an element of $\mathbf{a}_\mu\mathbb{D}$. Since $\mathbf{a}_\mu\mathbb{D}$ consists of equivalence classes of densities, this is not surprising. In the classical case, the derivative $\frac{d\nu}{d\mu}$ is only defined up to μ-almost everywhere equivalence.

We must verify that this notion of derivative is indeed a right inverse to integration. This task reduces the Radon-Nikodym Theorem to a diagram chase.

Theorem 16 (Radon-Nikodym Theorem II). *Let μ be a measure on the locale $\mathcal{L}$, such that $\mathrm{Sh}_\mu(\mathcal{L})$ is Boolean. Let $D_\mu : \twoheaddownarrow\mu \to \mathbf{a}_\mu\mathbb{D}$ denote the Radon-Nikodym differentiation operation. Then*

$$\left(\eta^{\twoheaddownarrow\mu}\right)^{-1} \circ \left(\mathbf{a}_\mu \int\right) \circ D_\mu = id_{\twoheaddownarrow\mu}$$

Proof. Using the diagram above, we can write

$$D_\mu = i \circ \frac{d}{d\mu^*} \circ k \circ \eta^{\downarrow\mu}$$

The composition in the Theorem can therefore be written:

$$\left(\eta^{\downarrow\mu}\right)^{-1} \circ \left(\mathbf{a}_\mu \int\right) \circ i \circ \frac{d}{d\mu^*} \circ k \circ \eta^{\downarrow\mu}$$

But since $\eta^{\downarrow\mu}$ is an isomorphism, our task reduces to showing that the anticlockwise circuit of the right hand square, starting at $\mathbf{a}_\mu\downarrow\mu$ is just the identity:

$$\left(\mathbf{a}_\mu \int\right) \circ i \circ \frac{d}{d\mu^*} \circ k = \mathrm{id}_{\mathbf{a}_\mu\downarrow\mu}$$

Using the fact that

$$\left(\mathbf{a}_\mu \int\right) \circ i = k^{-1} \circ \int{}_\mu(-,\mu^*)$$

our composition can be rewritten

$$k^{-1} \circ \int{}_\mu(-,\mu^*) \circ \frac{d}{d\mu^*} \circ k$$

Now, using the fact that

$$\int_\mu(-,\mu^*) \circ \frac{d}{d\mu^*} = \mathrm{id}_{\mathbb{M}^F_\mu}$$

the composition reduces to

$$k^{-1} \circ k$$

But this is trivially $\mathrm{id}_{\mathbf{a}_\mu\downarrow\mu}$. □

The fact that the Radon-Nikodym derivative is not a density, but an equivalence class of densities, captures the idea of "almost everywhere" uniqueness of derivatives. However, there is another interesting feature of this formulation. An element of $\mathbf{a}_\mu(\mathbb{D})(A)$ is an equivalence class of densities in

$$\bigcup_{\overline{B}=A} \mathbb{D}(B)$$

It is not necessarily the case that such an equivalence class will contain an element of $\mathbb{D}(A)$. In this case, the Radon-Nikodym derivative itself is defined only "almost everywhere".

As an example of this phenomenon, let μ be the Lebesgue measure on the locale of open sets of the real line, and let ν be the restriction of this measure to the unit interval $\mathbf{I} = [0,1]$, so that $\nu(A) = \mu(A \cap \mathbf{I})$. Then $\mathbb{D}$ is the sheaf of continuous real valued functions. It is clear that there is no continuous function f on $\mathbb{R}$ such that $\int(f,\mu) = \nu$. However, if we let $A = \mathbb{R} \setminus \{0,1\}$, then there is an element $g \in \mathbb{D}(A)$ (that is, a continuous function $g : A \to \mathbb{R}$) such that $\int(g,\mu) = \nu$, namely

$$g(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x \in (-\infty,0) \cup (1,\infty) \end{cases}$$

Since A is dense in $\mathbb{R}$, it follows that there is an element of $\mathbf{a}_\mu\mathbb{D}$ corresponding to g. Thus g (together with all other continuous functions which agree with g "almost everywhere") is the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.

It may seem that the requirement that $\mathrm{Sh}_\mu(\mathcal{L})$ is Boolean is a strong condition to impose. After all, most toposes of interest are not Boolean. However, it turns out that many measures induce Boolean subtoposes.

Lemma 45. *[9] Let $(\mathcal{F}, \sqsubseteq, \top, \bot, \neg)$ be a σ-algebra, and let μ be a σ-finite measure on $\mathcal{F}$. Then the σ-algebra $\mathcal{F}_\mu = \mathcal{F}/\mathfrak{Null}(\mu)$ satisfies the countable chain condition.*

Proof. For each element $A \in \mathcal{F}_\mu$, there is an $A' \in \mathcal{F}$ such that

$$A = \left\{ (A' \sqcup N_1) \sqcap \neg N_2 \,\middle|\, N_1, N_2 \in \mathfrak{Null}(\mu) \right\}$$

We use a contrapositive argument.

Suppose that $\mathcal{F}_\mu$ does not satisfy the countable chain condition. Then there is an uncountable antichain $\mathcal{A} = \{A_\alpha | \alpha < \kappa\rangle$ for some $\kappa \geq \omega_1$. Applying the axiom of choice, there is a corresponding uncountable family $\mathcal{A}' = \langle A_\alpha | \alpha < \kappa\rangle$. The members of $\mathcal{A}'$ are pairwise almost disjoint, meaning that for $A'_\alpha \neq A'_\beta \in \mathcal{A}$ we have

$$\mu \left(A'_\alpha \sqcap A'_\beta \right) = 0$$

We now build a new antichain $\langle B_\alpha | \beta < \omega_1 \rangle$ in $\mathcal{F}$:

$$B_\alpha \quad = \quad A_\alpha \sqcap \neg \left(\bigsqcup_{\beta < \alpha} B_\beta \right)$$

Note that this recursive expression makes sense only for $\alpha < \omega_1$, as the expression $\left(\bigsqcup_{\beta<\alpha} B_\beta \right)$ is not necessarily defined if α is uncountable.

Then for each $\alpha < \omega_1$, we have $B_\alpha \sqsubseteq A_\alpha$. Furthermore, $\langle B_\alpha | \alpha < \omega_1 \rangle$ is an uncountable antichain in $\mathcal{F}$, with

$$\mu(B_\alpha) = \mu(A_\alpha) > 0$$

Hence μ is not σ-finite.

□

Proposition 46. *[25, 16] Let $(\mathcal{F}, \sqsubseteq, \top, \bot, \neg)$ be a σ-algebra, and let μ be a σ-finite measure on $\mathcal{F}$. Then $\mathrm{Sh}_\mu(\mathcal{F})$ is a Boolean topos.*

Proof. From Corollary 38, we know that $\mathcal{L}_\mu$ is the quotient of $\mathcal{F}$ by the ideal $\mathfrak{Null}(\mu)$. If μ is σ-finite, then this quotient algebra, $\mathcal{F}_\mu$ satisfies the countable chain condition, by Lemma 45. The countable chain condition here has two important consequences:

1. $\mathcal{F}_\mu$ is a complete Boolean algebra

2. If $\mathcal{A} \subseteq \mathcal{F}_\mu$, then there is a some countable $\mathcal{A}_0 \subseteq \mathcal{A}$, such that

$$\bigsqcup \mathcal{A} = \bigsqcup \mathcal{A}_0$$

As a result, it follows that $\mathrm{Sh}_\mu(\mathcal{F})$ is the topos of sheaves on a complete Boolean algebra, and is hence a Boolean topos.

To see that $\mathcal{F}_\mu$ has these properties:

1. Take $\mathcal{I} \subseteq \mathcal{F}_\mu$. Each $A \in \mathcal{I}$ consists of an equivalence class of elements of $\mathcal{F}$, where

$$B \sim C \iff \mu\left((B \sqcap \neg C) \sqcup (\neg B \sqcap C)\right) = 0$$

   Assume without loss of generality that $\mathcal{I}$ is a countably complete sieve in $\mathcal{F}$ (that is, an element of Ω_j). We will show that $\mathcal{I}$ contains a maximal element.

   Use the axiom of choice to well order the elements of $\mathcal{I}$. Define a sequence $\langle G_\alpha | \alpha < |\mathcal{I}| \rangle$ in $\mathcal{I}$ by

$$G_\alpha = \bigsqcup_{\beta < \alpha} F_\alpha$$

   In order to make sure that this is well defined, we must show that G_α exists when α has uncountable cofinality.

   Since $\langle G_\gamma | \alpha \rangle$ is a chain, it must follow that $\langle \mu(G_\gamma) | \gamma < \alpha \rangle$ is also a chain, and in fact must be increasing. However, every increasing sequence with uncountable cofinality of elements of $[0, \infty]$ must terminate, and so the sequence $\langle \mu(G_\gamma) | \gamma < \alpha \rangle$ must have countable cofinality. Since $U \nsubseteq V \Rightarrow \mu(U) \lneq \mu(V)$ in $\mathcal{F}_\mu$, it follows that $\langle G_\gamma | \gamma < \alpha \rangle$ also terminates, and so has a supremum.

2. So now, given $\mathcal{I} \subseteq \mathcal{F}_\mu$, we know that $\bigsqcup \mathcal{I}$ exists.

   First consider the case where $\mathcal{I}$ is a countable complete sieve. The argument used above shows that the sequence $\langle G_\alpha | \alpha < |\mathcal{I}| \rangle \uparrow \bigsqcap \mathcal{I}$ has countable cofinality, and so there is a countable subsequence $\langle G_{\alpha_i} | i < \omega \rangle$ in $\mathcal{I}$ whose supremum is $\bigsqcup \mathcal{I}$.

   If $\mathcal{I}$ is an not a countably complete sieve, then we have shown that there is a countable subset of $\overline{\downarrow \mathcal{I}}$ whose join is $\bigsqcup \mathcal{I}$. But each member of $\overline{\downarrow \mathcal{I}}$ is the countable join of members of $\mathcal{F}$, and so we are done.

$\square$

Proposition 47. *If* $\mathrm{Sh}(\mathcal{L})$ *is the topos of sheaves on a locale* $\mathcal{L}$*, and* μ *is a measure on* $\mathcal{L}$ *satisfying* $\mu(A) = \mu(\neg\neg A)$*, then* $\mathrm{Sh}_\mu(\mathcal{L})$ *is Boolean. (Such a measure is called "continuous".)*

Note that the Lebesgue measure, λ, is a continuous measure, as for every open set U, we have $\lambda(\partial U) = 0$.

Proof. In the case where $\mathcal{L}$ is a locale, and μ has the property that $\mu(A) = \mu(\neg\neg A)$ for all $A \in \mathcal{L}$, it follows that

$$\rho(A, \mu) = \rho(\neg\neg A, \mu)$$

Hence $j_\mu(A) = \top$ if and only if $j_\mu(\neg\neg A) = \top$.

It follows that j_μ factors through the double negation topology $\neg\neg$, and so is a topology in $\mathrm{Sh}_{\neg\neg}(\mathrm{Sh}(\mathcal{L}))$. But this sheaf topos is itself a localic topos, and is in fact the topos of sheaves on the complete Boolean algebra of $\neg\neg$-stable elements of $\mathcal{L}$. (In the event that $\mathcal{L}$ is spatial, and so is the algebra of open sets in topological space, the $\neg\neg$-stable elements of $\mathcal{L}$ are just the regular open sets — see Johnstone [13].)

In this topos, we can apply Wendt's argument (Wendt [28]) to see that the topology corresponding to j_μ is just the closed topology induced by the sieve of ($\neg\neg$-stable) elements A of $\mathcal{L}$ satisfying $\mu(A) = 0$.

But the resulting locale of closed sieves is just the quotient of a complete Boolean algebra by a complete ideal and is hence a complete Boolean algebra. Thus the topos $\mathrm{Sh}_\mu(\mathcal{L})$ is equivalent to the topos of sheaves on a complete Boolean algebra, and so is a Boolean topos, as required.

□

The following Corollary justifies calling a $\neg\neg$ stable measure "continuous".

Corollary 48. *Let* (X, τ) *be a topological space, let* $\nu \ll \mu$ *be two locally finite continuous measures on* τ*. Then there is a continuous function* f *which is defined on a* μ*-dense open set* X_0 *such that*

$$\nu = \int f \, d\mu$$

4.0 POSSIBILITIES FOR FURTHER WORK

This dissertation lays the groundwork for a locale based measure theory, and a topos based measure theory. There are several immediate possibilities for further research leading out of these ideas.

4.1 SLICING OVER $\mathbb{M}$

Rather than fixing a single measure μ, and sheafifying with respect to that measure, it would be desirable to study differentiation of all measures simultaneously. The natural way to approach this is to work in the slice topos $\mathcal{E}/\mathbb{M}$. An object of this topos is an arrow (of $\mathcal{E}$) with codomain $\mathbb{M}$. An arrow between objects $f : A \to \mathbb{M}$ and $g : B \to \mathbb{M}$ is simply an arrow $h : A \to B$ such that $g \circ h = f$.

We can build the following diagram, consisting of arrows over $\mathbb{M}$:

$$\begin{tikzcd}
\mathbb{D} \times \mathbb{M} \arrow[rr, "\left\langle \int, \mathrm{id}_{\mathbb{M}} \right\rangle"] \arrow[dr] \arrow[ddr, "\pi_2"'] & & \mathbb{M} \times \mathbb{M} \arrow[ddl, "\pi_2"] \\
& \ll \arrow[ur, hook] \arrow[d] & \\
& \mathbb{M} &
\end{tikzcd}$$

Differentiation would therefore be a section to the arrow $\mathbb{M} \times \mathbb{D} \to \ll$, taking a pair of measures $\langle \nu, \mu \rangle$ to a pair $\langle f, \mu \rangle$ such that $\int (f, \mu) = \nu$.

Further study of this topic would start with an investigation of the μ-sheafification operation in the slice topos. This should result in a single Lawvere-Tierney topology in $\mathcal{E}/\mathbb{M}$. The existence of derivatives would then be conditional on the law of the excluded middle.

4.2 EXAMPLES OF LOCALIC MEASURE THEORY

We have studied the essentials of measure theory in an arbitrary localic topos. It would be natural to investigate particular localic toposes, and determine which additional measure theoretic results hold.

A particular case that has practical implications is the case where $\mathcal{B}$ is a Boolean algebra, and J is the *finite* join topology on $\mathcal{B}$. Carrying out the construction of $\mathbb{M}$ in $\widehat{\mathcal{B}}$ yields the sheaf of finitely additive measures. The Radon-Nikodym theorem is much harder to apply to finitely additive measures (see Dunford and Schwartz [7], or Royden [22]), so it would be interesting to approach this problem using the sheaf theoretic approach.

The first steps would be to verify that the sheaf of measures in $\mathrm{Sh}(\mathcal{B})$ is equivalent to the sheaf of measures in $\mathrm{Sh}(\mathcal{L})$, where $\mathcal{L}$ is the locale of ideals (closed sieves) in $\mathcal{B}$. The next step would be to determine the nature of the μ-almost everywhere topologies, and determine which measures induce Boolean subtoposes.

4.3 STOCHASTIC PROCESSES AND MARTINGALES

We start with some definitions from probability theory (see Billingsley [3], for example):

Definition 45. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$.

1. A *stochastic process* is a sequence $\mathbf{f} = \langle f_i | i \in I \rangle$ of measurable real valued functions, for some ordered index set I
2. A *filtration* is an in increasing family $\boldsymbol{\mathcal{G}} = \langle \mathcal{G}_i | i \in I \rangle$ of sub-σ-fields of $\mathcal{F}$.

3. A stochastic process $\mathbf{f}$ is *adapted* to the filtration $\boldsymbol{\mathcal{G}}$ if for every $i \in I$ f_i is a measurable function from the measure space $(X, \mathcal{G}_i, \mu)$.

   I represents time, and is usually either $\mathbb{N}$ or $\mathbb{R}$.

The essence of this definition is that a σ-field represents partial knowledge. The finer the σ-field, the more knowledge is available. Saying that f_i is measurable with respect to $\mathcal{G}_i$ is equivalent to saying that knowledge of $\mathcal{G}_i$ provides complete knowledge of f_i. Thus a filtration represents increasing knowledge through time.

Definition 46. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$, let f be a measurable real valued function from $X \to \mathbb{R}$, and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Then the *conditional expectation* of f given $\mathcal{G}$, denoted $E[f||\mathcal{G}]$ is a measurable real valued function g such that

1. g is measurable relative to $\mathcal{G}$
2. For any $G \in \mathcal{G}$,

$$\int_G g \, d\mu = \int_G f \, d\mu$$

Note that unconditional expectation is just conditional expectation, with the conditioning done relative to the trivial σ-field $\{\emptyset, X\}$.

The existence of g is a consequence of the Radon-Nikodym theorem: Let ν be the measure on $(X, \mathcal{G}, \mu^*)$ (where μ^* is the obvious restriction of μ to $\mathcal{G}$) given by

$$\nu(G) = \int_G f \, d\mu$$

It is clear that $\nu \ll \mu^*$, and so there is a Radon-Nikodym derivative $\frac{d\nu}{d\mu^*}$ on $(X, \mathcal{G}, \mu^*)$. This derivative is g.

Definition 47. A *martingale* consists of a stochastic process $\mathbf{f} = \langle f_i | i \in I \rangle$ and a filtration $\boldsymbol{\mathcal{G}} = \langle \mathcal{G}_i | i \in I \rangle$ such that

1. $\mathbf{f}$ is adapted to $\boldsymbol{\mathcal{G}}$
2. for every $i < j$ we have

$$f_i = E[f_j || \mathcal{G}_i]$$

The machinery presented in this dissertation provides for capturing all the ingredients necessary to study martingales, except for sub-σ-fields, and filtrations. However, these objects can be embedded into the topos theoretic framework. The topos of sheaves on $\mathcal{G}$ is a subtopos of the topos of sheaves on $\mathcal{G}$. The topology J that induces this subtopos is given by saying that for any $C \in \mathcal{F}$,

$$J(C) = \{\mathcal{I} \in \Omega(C) \,|\{G \in \mathcal{G} | G \subseteq C\} \subseteq \mathcal{I}\}$$

Such a topology has a very special property. For any given C, there is a *smallest* sieve that covers C, namely the sieve

$$\{D \in \mathcal{F} | \exists G \in G\, D \subseteq G \subseteq C\} = \bigcup \{\downarrow G | G \in \mathcal{G} \cap \downarrow C\}$$

This means that the closure arrow $j : \Omega \to \Omega$ has an internal left adjoint $m : \Omega \to \Omega$. Furthermore, $m\top = \top$.

Definition 48. Say that a Lawvere-Tierney topology $j < \Omega \to \Omega$ is *nice* if it has a left adjoint $m : \Omega \to \Omega$ such that $m\top = \top$.

These "niceness" properties allow for the following important proposition:

Proposition 49. *Let $\mathcal{E}$ be a topos, and let $j : \Omega \to \Omega$ be a nice Lawvere-Tierney topology, with a left adjoint $m : \Omega \to \Omega$. Let $\mathbb{R}_j$ be the Dedekind real numbers object in $\mathrm{Sh}_j(\mathcal{E})$, and let $\mathbb{R}$ be the Dedekind real numbers object on $\mathcal{E}$. Then $\mathbb{R}_j \rightarrowtail \mathbb{R}$.*

Proof. Let $\langle L, U \rangle$ be an element of $\mathbb{R}_j$. We must show that $\langle L, U \rangle$ is a Dedekind real in $\mathcal{E}$. Let $\mathbb{R}_j$ be the Dedekind real numbers object in $\mathrm{Sh}_j(\mathcal{E})$, let $\mathbb{R}_{\mathcal{E}}$ be the Dedekind real numbers object in $\mathcal{E}$, and let $\mathbb{R}$ denote the "internal" set of Dedekind reals.

$$
\begin{aligned}
& \langle L, U\rangle \in \mathbb{R}_j \\
\leftrightarrow\ & \mathrm{Sh}_j(\boldsymbol{\mathcal{E}}) \models \langle L, U\rangle \in R \\
\leftrightarrow\ & \boldsymbol{\mathcal{E}} \models j \circ (\langle L, U\rangle \in \mathbb{R}) \\
\leftrightarrow\ & \boldsymbol{\mathcal{E}} \models \top \Rightarrow j \circ (\langle L, U\rangle \in \mathbb{R}) \\
\leftrightarrow\ & \boldsymbol{\mathcal{E}} \models m \circ \top \Rightarrow (\langle L, U\rangle \in \mathbb{R}) \\
\leftrightarrow\ & \boldsymbol{\mathcal{E}} \models \top \Rightarrow (\langle L, U\rangle \in \mathbb{R}) \\
\leftrightarrow\ & \boldsymbol{\mathcal{E}} \models \langle L, U\rangle \in \mathbb{R} \\
\leftrightarrow\ & \langle L, U\rangle \in \mathbb{R}_{\mathcal{E}}
\end{aligned}
$$

□

This proposition allows us to define a filtration as an increasing sequence of Lawvere-Tierney topologies with left adjoint's, and a stochastic process adapted to that filtration as a sequence of Dedekind reals, in the chain

$$
\mathbb{D}_0 \rightarrowtail \mathbb{D}_1 \rightarrowtail \mathbb{D}_2 \rightarrowtail \cdots
$$

The next step in the construction would be the expression of the martingale property.

4.4 MEASURE THEORY AND CHANGE OF BASIS

Measure theorists have a well known change of basis technique. If $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are measurable spaces, and $f : X \to Y$ is a measurable function, then f can "carry" a measure μ on $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ by means of the equation

$$
\nu(G) = \mu\left(f^{-1}[G]\right)
$$

This argument suggests that there is some connection between the sheaf of measures and geometric morphisms between sheaf toposes on different locales, or σ-algebras.

Wendt [29] has looked at the change of basis operation between sheaf toposes over σ-algebras in various categories of measure and measurable spaces, although he has not studied the measure theories of these sheaf toposes.

4.5 EXTENSIONS TO WIDER CLASSES OF TOPOSES

Many of the results in this dissertation apply in an arbitrary topos, with a designated topology j. The most significant exception is Theorem 6, the proof of which makes explicit reference to the fact that $\mathcal{E}$ is the topos of presheaves on a σ-algebra. Strengthening this result to apply to a wider class of Grothendieck toposes would provide an immediate extension of a great deal of measure theory to a wide class of toposes.

In a similar vein, we know that in an arbitrary topos, the double negation topology induces a Boolean subtopos. We can find the object of measures in such a topos. This object is not necessarily a sheaf (that is, an object of the Boolean subtopos), but might nonetheless have interesting properties, especially with regards to differentiation, as all measures would be Boolean.

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