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Sequential bargaining with pure common values

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Sequential bargaining with pure common values*

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Abstract

We study the alternating-offers bargaining problem of assigning an indivisible and commonly valued object to one of two players in return for some payment among players. The players are asymmetrically informed about the object's value and have veto power over any settlement. There is no depreciation during the bargaining process which involves signalling of private information. We characterise the perfect Bayesian equilibrium of this game which is essentially unique if offers are required to be strictly increasing. Equilibrium agreement is reached gradually and non-deterministically. The better informed player obtains a rent. (JEL *C73, C78, D44, D82, J12.* Keywords: *Sequential bargaining, Common values, Incomplete information, Repeated games.*)

Introduction

We study an alternating-offers bargaining situation where a privately informed player signals his information to the uninformed opponent through his bargaining behaviour. There is an indivisible object that is of either high or low value. Both players know these possible values. Only one player (P1) knows the true state of Nature, ie. the true value of

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the object, while all the other player (P2) knows is a probability distribution over the possible values. The players are infinitely patient and possess similar bargaining power as offers are made alternatingly. In the essentially unique equilibrium, the informed player obtains an information rent even when the object is worthless and the uninformed player finds it interim individually rational to participate.

Many economic applications lend themselves to our interpretation of bargaining. Our study of non-depreciating common values complements the analysis of depreciating private values by Ståhl (1972) or Rubinstein (1982). Their assumption of depreciation typically leads to immediate or temporally finely tuned agreement in subgame perfect equilibrium. This phenomenon, however, is often not observed in bargaining situations in which signalling of private information matters. Some examples captured within our framework are: (1) A partnership dissolution problem where two asymmetrically informed players jointly own a firm. (2) A fight for control between two owners of a single firm. (3) Agreeing on a profit sharing rule between two firms involved in a joint venture. (4) The 'buying out' of parties holding dispersed property rights (or patents) needed for the production of some good or service. (5) Deciding whether to spin-off some yet-to-be-proven innovation ('selling the project to the manager') or developing it inside the firm. (6) Splitting an inheritance (eg. an Amish farm or company) under the provision of maintaining it as a unit. (7) Agreeing on payments for hosting some airport or waste disposal site in one community although two or more communities profit from it.[1]

Bargaining models under incomplete information are typically plagued by a plethora of equilibria. One might expect that the signalling aspect introduced by the common value nature of the object further accentuates this problem. This is, however, incorrect. Indeed, we argue that the finite version of our game where a player's offer must be strictly higher than his previous offer[2] has an *essentially unique* perfect Bayesian equilibrium:[3]

[1] This involves an economic 'bad' taking a negative value in its 'high' state. Consequently players can offer payments for the opponent to accept the object. Here it suffices to analyse the game with absolute high value and then reverse the sign on the outcome.

[2] We are grateful to an anonymous referee for pointing out that the requirement for strictly increasing offers corresponds to the "bargaining in good faith" stipulation present in many countries' labour codes. These are often interpreted as not allowing a party to withdraw offers or to at least restrict a party's ability to make offers which get worse as time passes.

[3] We call an equilibrium 'essentially' unique if, roughly, all stage actions but P1's final action are unique and beliefs are uniquely determined whenever they can lead to the uninformed player's continuation.

When the value of the object is high, the informed P1 keeps increasing his offers by the minimal unit at a time up to approximately half the true value of the object and then quits. When the value is low, P1 keeps increasing his offers with probability 1 up to approximately a quarter of the high value of the object (this depends on P2's prior). After that, he mixes between minimally increasing his offer and quitting until he reaches the stage where also the high type P1 quits. The uninformed P2 mixes between minimally increasing her offer and quitting over the same range of stages and increases minimally with probability 1 before. With minor modifications, this equilibrium remains an equilibrium of the extended game where we allow also for non-increasing bids. Uniqueness, though, is lost in this potentially infinite game.

The reason for the essential equilibrium uniqueness in the finite game is that P1 cannot use an equilibrium action which induces beliefs which cause an action by the uninformed P2 which is beneficial only to a certain type of P1 (eg. immediate quitting after a jump-bid). For at each stage where P2 moves, it is beneficial for the high-type P1 to induce P2 to quit immediately and it is beneficial for the low-type P1 to induce P2 to continue with probability one. Hence the only belief about the value of the object which does not separate states through the induced belief is when P2 is *exactly indifferent between quitting and continuing*. Thus observing any on- or off-equilibrium-path action by the informed P1, the uninformed P2's *equilibrium* response must contain a belief which makes her indifferent between accepting the current offer by quitting and continuing to make a (minimally) higher own offer. There are only exceptions to this rule when such beliefs do not exist. This suffices to force essentially unique on- and off-equilibrium-path beliefs. The eventual agreement is reached gradually and stochastically over a stretch of multiple rounds of offers and counteroffers. The same logic applies to the infinite game but there, inserting any finite or infinite sequence of non-increasing offers at any stage, may still constitute an equilibrium.

Our departures from the standard model are threefold: (*i*) we analyse incomplete information over a *commonly* valued pie, (*ii*) there is *no depreciation* of the object's value during the bargaining process, and (*iii*) the finite game assumes strictly increasing offers. The first assumption focusses attention on the signalling of private information during the bargaining process. To that end, introducing a private value element changes nothing. The second assumption—shared with Calcagno and Lovo (2006)—is made because we are mainly interested in situations where the object's value does not change significantly during the negotiation period. The final departure from the literature—our 'activity rule'—is made for technical reasons. As pointed out above, however, there is a class of "bargaining in good faith" applications where the requirement for strictly increasing offers is natural in its own right. At any rate, we relax the third assumption in the analysis of the infinite game where we allow for general offers.

Related literature

Our dynamic game can be interpreted as a repeated game of incomplete information as defined by Aumann and Maschler (1966) and subsequently developed by Mertens, Sorin, and Zamir (1994). Indeed, our model poses questions similar to those addressed there. That literature, however, typically derives *average* payoffs from long interactions which do not arise naturally in our context. Nevertheless we use repeated game terminology throughout.

There is a rich literature on bargaining with incomplete information. Most contributions such as Grossman and Perry (1986) or Watson (1994), however, are concerned with the players' incomplete information over the opponents discount rate and not over the pie itself for which valuations are usually taken to be independent. Hence incomplete information bargaining is typically very sensitive to discounting considerations. If a (sequential) bargaining model considers private information on the pie—such as Sobel and Takahashi (1983), Cramton (1984), Fudenberg, Levine, and Tirole (1987), Evans (1989), or Deneckere and Liang (2006)—the problem of strategic communication is usually avoided by allowing offers only by the uninformed players. Ausubel and Deneckere (1989) are an exception in characterising the full set of sequential equilibria of seller-offer, buyer-offer and alternating-offer bargaining games with one-sided incomplete information. They consider private values while imposing ex-post individual rationality on bargaining mechanisms. Our setting features incomplete information over a common value and interim individual rationality. A full survey of this literature is Ausubel, Cramton, and Deneckere (2002) but, to date, there is no full analysis of bargaining under incomplete information over an object's *pure common value* with alternating offers by *both* players.

Compte and Jehiel (2004) study complete information *gradual* bargaining and contribution games where players can opt out at each stage of bargaining. They obtain history dependent quitting payoffs by assuming that these outside options depend on the players' bargaining concessions. Our game also has a certain war of attrition flavour with the difference that the loser is not typically paid the winning bid in a war of attrition. The common element, however, is that parties mix over quitting or remaining with probabilities balanced such as to keep the opponent indifferent.

A bargaining game can be viewed as an auction with balanced budget among participants. Thus our game can be interpreted as an ascending auction where the highest bidder wins the object and pays his bid to the loser. From this point of view, our analysis addresses questions similar to those explored by Milgrom and Weber (1982) and Engelbrecht-Wiggans, Milgrom, and Weber (1983) who characterise the incentives for additional information acquisition (or communication) in a standard two-bidder, sealed-bid auction.[4] In contrast to their setting, players have mutual veto power here as results, for instance, from joint ownership. Introducing such considerations into the auction context leads to effects reminiscent of the study of auctions with toeholds as discussed by Bulow, Huang, and Klemperer (1999) or Ettinger (2003). There, a bidder with a toehold bids more aggressively than without because his bid is at the same time an offer for the remaining part of the object and a demand for his own holdings.

We share our interest in strategic information transmission in repeated bidding games with a number of recent papers: Deneckere and Liang (2006) analyse an infinite horizon bargaining game with an information structure similar to ours but offers by the uninformed player only. They allow for interdependent players' valuations and analyse the

[4] Similarly, our mechanism relates to Japanese (ascending-clock) auctions. The main problem in formulating an ascending-clock auction in our framework is the case where *both players quit simultaneously* because this is a fundamentally non-bargaining event. Sákovics (1993) illustrates that even in the one-shot, complete-information case, simultaneous offers permit any outcome. If this problem is resolved by the auctioneer alternatingly giving preference to one of the players, we are back in the current setup (without jump-bids).

efficiency of their generically unique stationary equilibrium outcome in comparison to the literature on the Coase conjecture. They find that the limiting bargaining outcome may be worse than even the outcome of the one-period bargaining game. Calcagno and Lovo (2006) are concerned with strategic information transmission between informed and uninformed market makers through a finite sequence of bid-ask quotes and the resulting inventory optimisation problem. In our setup, this corresponds to sharing the undiscounted common value of a set of stocks between two asymmetrically informed players. In strategic terms their result is similar to ours: the informed player profits from equilibria in semi-separating strategies. Hörner and Jamison (2003) analyse an infinitely repeated sequence of first-price auctions where a common value object is sold at every stage and players maximise their discounted average payoffs. For the case of incomplete information on one side, they find that all information is gradually revealed in finite time and, surprisingly, that the uninformed player is doing better than the informed bidder. This result, however, cannot be readily compared to ours because their bidders' payoffs stem only from the value of the objects and not also from payments between bidders as in our case.

1 The model

We consider two identical, risk-neutral players {P1,P2} and two possible common values for an indivisible object $\theta \in \{\underline{\theta}, \bar{\theta}\}$, $\bar{\theta} \in \mathbb{R}$. We normalise $\underline{\theta} = 0$ and assume $\bar{\theta} \geq 3$ to avoid trivialities. P1 is assumed to know the realisation of θ. We denote the high-type informed player by $\overline{\text{P1}}$ and the low-type by $\underline{\text{P1}}$. P2 only has a public prior of $\varphi_2^0 = \text{pr}(\theta = \bar{\theta}) = {}^1/_2$.[5] This prior is refined into P2's beliefs φ_2^t on the basis of P1's observed bids.

The game starts with P1 offering a payment o_1^1 (subscripts are players, superscripts time periods) to P2 for sole ownership of the object.[6] Pure offers o_i^t, $i \in \{1, 2\}$, $t > 0$ are restricted to the set of admissible bids $B = \{0, 1, \ldots, \bar{B}\} \subset \mathbb{N}$ where $\bar{B} > \bar{\theta}$ ('all the money in the world'). This defines the minimal offer increase to be 1 (currency unit). The

[5] This prior is chosen for simplicity; increasing P2's prior reduces P1's information rent through (A.18).

[6] Since only P1 holds private information, having P2 start the game just inserts a trivial stage at the beginning.

terms offers and bids are used synonymously. If P$-i$ accepts Pi's offer, Pi pays the offered amount to P$-i$, Pi gets the object and the game is over. If P$-i$ does not accept Pi's offer, nothing is paid, and P$-i$ makes an own offer. Players go on making alternating offers until one player accepts an offer by quitting.

We set $o_2^0 = o_1^{-1} = 0$ equal to the low value of the object and, for the finite game $\mathcal{Q}$, we require offers to be strictly increasing over time, ie. all continuation *increments* over the last own offer $o_i^t - o_i^{t-2} > 0$. Conversely, $o_i^t - o_i^{t-2} \leq 0$ is interpreted as quitting and denoted 'q'. We define a *jump bid* $j_i^t = o_i^t - o_i^{t-2} - 1$ as an offer which increases the last own offer by more than the minimal amount. We keep a running sum of player i's jump bids as $J_i^t = \sum_{\hat{t}=i}^{(t+i)/2} j_i^{2\hat{t}-i}$. Mixed offers attach probability α_i^t to the pure continuation bid o_i^t and the complementary probability to quitting. We denote such mixed actions by $\beta_i^t = [\alpha_i^t : o_i^t, q]$.[7] The requirement for offers to be strictly increasing is lifted for the infinite game $\mathcal{Q}_\infty$. Non-increasing offers are then denoted by '0' and a succession of '0'-bids by both players is called a *cycle*.

Pi's (repeated game) strategy β_i consists of the sequence of potentially mixed stage actions for each possible plan of the opponent. Players observe the opponents' realised offers and enjoy perfect recall. The players' final expected payoffs are written $u_i(\beta|\theta)$ and consist of the object's value minus payment made for the winner of the object and the payment received for the loser. Player i's quitting payoff when accepting an offer at t is written as $u_i^t(q)$.

To sum up, our 'queto' model is a standard alternating-offer bargaining game with incomplete information over common values and no discounting.[8] We now state the definitions required to formulate our results in section 3. There, we first analyse the game with strictly increasing bids and then extend our results to the infinite game allowing for non-increasing offers. All proofs and details are presented in appendix A, an example is shown in appendix B.

[7] We refrain from a more general definition of a mixed stage action (over a larger support of pure actions) because lemma 6 shows that we need nothing more complicated than the above.

[8] The name derives from the player's stage actions of either *qu*itting or v*eto*ing the current proposal. The idea of our game is similar to the *quitting games* introduced by Vieille and Solan (2001) in the context of complete information stochastic games. They define quitting games as sequential games in which, at any stage, each player has the choice between a single continuation bid and quitting.

Definition 1. *A perfect Bayesian equilibrium is called essentially unique if it comprises of (i) arbitrary beliefs at information sets where every equilibrium prescribes quitting for any belief, (ii) uniquely determined beliefs at every other information set of P2, and (iii) an arbitrary final stage equilibrium action for* $\overline{P1}$ *if* $\lfloor \bar{\theta} \rfloor = \bar{\theta}$.[9]

Definition 2. *The informed agent's strategy is called non-separating if, at every decision node of P1, there is no pure continuation bid which is in the support of the strategy of one type and not of the other. Conversely, the informed agent's strategy is called separating if, at some stage, it reveals the informed player's type with probability one.*

Definition 3. *A strategy is called minimal-increment strategy if all its constituent continuation actions increase the previous own bid by the minimal admissible amount.*

2 Equilibrium definition and discussion

In this section we define the perfect Bayesian equilibrium candidate β^* which we will analyse in the following section. Since the formal definition (1) is rather involved, we first state the most important equilibrium (path) properties of β^* verbally: (1) Starting with $o_1^1 = 1$, both players keep increasing their last offer o_i^t by the minimum admissible increment of 1 with probability 1 until P2's quitting payoff exceeds her prior-based payoff expectation from β^*. The period before, $\underline{\text{P1}}$ starts mixing with probability $\alpha_1^{t_s}$ between his minimal-increment continuation and quitting. (We call this period t_s and the periods $t \in [1, t_s)$ the 'preplay'-phase.) This and all following mixture probabilities are such as to induce next-period beliefs for P2 which make her precisely indifferent between quitting and bidding her minimal-increment continuation bid. (2) $\overline{\text{P1}}$ bids his minimal-increment continuation bid with probability 1 as long as his quitting payoff o_2^{t-1} is below what he can get if P2 quits the following period (ie. $\bar{\theta} - o_1^t$). Thus as soon as $o_2^{t-1} > \bar{\theta} - o_1^t$, both players quit with probability 1. As soon as $\underline{\text{P1}}$ starts mixing, the uninformed P2 is made indifferent between quitting and minimally increasing her offer. Hence she is willing to mix with precisely the probability α_2 which $\underline{\text{P1}}$ requires to mix himself.

[9] The notation $\lfloor x \rfloor$ denotes the next integer below x. Similarly, $\lceil x \rceil$ is the integer directly above x.

The above is an informal definition of the equilibrium path. It is made precise by lemma 7 and, in particular, (A.13) on P2's mixtures, (A.14) on P1's mixtures and (A.15) on P2's beliefs as part of the equilibrium β^*. Notice that these probabilities apply both on and off-equilibrium-path. The continuation payoff $u_2^{t+1}(\beta^*|\bar{\theta})$ is defined, on and off-equilibrium-path, in (A.19) and the quitting payoff is just last period's offer $u_i^t(q) = o_{-i}^{t-1}$. The definition is not circular because P2's beliefs and payoffs—and thus P1's mixture probabilities—are defined from the last stage of the game forward while P1's payoffs—and thus P2's mixture probabilities—are defined from the first stage backwards. The point in time where both meet determines when P1 starts to mix. Thus we define the equilibrium profile $\beta^* = (\beta_1^*, \beta_2^*)$, for all $t > 0$ and $o_i^t \in B$, as

$$
\begin{aligned}
\beta_1^*(\bar{\theta}) &= \begin{cases} o_1^t = \text{`q'} & \ldots \text{if } o_1^{t-2} + o_2^{t-1} > \bar{\theta} - 1, \\ o_1^t = o_1^{t-2} + 1 & \ldots \text{otherwise}, \end{cases} \\
\beta_1^*(\underline{\theta}) &= \begin{cases} o_1^t = \text{`q'} & \ldots \text{if } o_1^{t-2} + o_2^{t-1} \geq \bar{\theta} - 2, \\ \beta_1^t = \left[\alpha_1^t : o_1^t = o_1^{t-2} + 1, q\right] & \ldots \text{if } u_2^{t+1}(q) > {}^1/_2 u_2^{t+1}(\beta^*|\bar{\theta}) + {}^1/_2 u_2^t(q), \\ o_1^t = o_1^{t-2} + 1 & \ldots \text{otherwise}, \end{cases} \\
\beta_2^*(\theta) &= \begin{cases} o_2^t = \text{`q'} & \ldots \text{if } o_2^{t-2} + o_1^{t-1} \geq \bar{\theta} - 1 \text{ for any } \varphi_2^t, \\ \beta_2^t = \left[\alpha_2^t : o_2^t = o_2^{t-2} + 1, q\right] & \ldots \text{if } u_2^t(\beta^*) = u_2^t(q) \text{ and } \varphi_2^t \text{ is given by (A.15)}, \\ o_2^t = o_2^{t-2} + 1 & \ldots \text{if } u_2^t(\beta^*) > u_2^t(q) \text{ and } \varphi_2^t = \varphi_2^0. \end{cases}
\end{aligned}
\tag{1}
$$

Notice that, in equilibrium, P2's beliefs are uniquely defined at each information set. Her beliefs are not fully defined, however, when she quits with probability 1 for *any* belief. This situation cannot occur in equilibrium: P2 only quits with probability 1 following a fully separating action by P1 which implies that $\varphi_2 = 1$ if a continuation is observed. Off-equilibrium-path, this situation cannot be ruled out and we set $\varphi_2 = 1$ at all information sets where any belief leads to quitting, ie. where P2's beliefs do not matter.

An equilibrium argument removes a related problem in the preplay-phase. If P2 observes an off-equilibrium-path bid, her beliefs cannot be deduced from Bayes' rule. But as long as P2's continuation payoff is larger than her current quitting payoff for *any* belief φ_2 (which is the definition of the preplay-phase), then mixing by P1 (over any set of pure actions) cannot be an equilibrium action since it is not followed by mixing of P2. Hence the only *equilibrium* response consistent with P2 not mixing is that neither of P1's types mixes. But then P2 does not update her belief and $\varphi_2 = \varphi_2^0$ throughout the preplay-phase. Thus *equilibrium* preplay-beliefs are pinned down uniquely.

The equilibrium candidate of the infinite game β^*_∞ extends β^* with reactions to a non-increasing bid (denoted by '0') by the opponent—all other actions are identical to β^*. (β^*_∞ is defined in proposition 3.) Thus non-increasing bids are never made along the equilibrium path of the infinite game. The main idea of these reactions is that, following $o_i^t =$ '0', P$-i$ plays '0' as well which takes the game exactly to the point where the first deviation occurred. Since there is no depreciation, any number of such cycles gives just the continuation payoff which follows the cycles, ie. the equilibrium payoff. Thus players' simply ignore any non-increasing offer. To fully define payoffs we need an assumption about the payoff of playing '0' forever. We define this payoff as the limit of what is obtained from any number of finite cycles.

3 Results

In the course of solving our bargaining game, the first group of lemmata determines properties of any equilibrium of the game. Lemma 6 employs these for constructing a path through the game based on a terminal quitting condition for $\overline{\text{P1}}$. Proposition 1 then shows that this path is the equilibrium β^* defined in (1) and, moreover, shows that β^* is the essentially unique equilibrium of $\mathcal{Q}$. Proposition 2 determines when the informed player starts signalling his type and calculates payoffs. Up to the final proposition 3 we only deal with the finite game and strictly increasing offers.

The first lemma shows that the sum of the two bidders' most recent offers determines when each (type of) player quits with probability one.[10]

Lemma 1. *In any equilibrium of $\mathcal{Q}$, (i) $\overline{\text{P1}}$ quits wp1 at $\hat{t}$ if $o_2^{\hat{t}-1} + o_1^{\hat{t}-2} > \bar{\theta} - 1$, (ii) P2 quits wp1 at $\hat{t}$ if $o_2^{\hat{t}-2} + o_1^{\hat{t}-1} \geq \bar{\theta} - 1$, and (iii) $\underline{\text{P1}}$ quits wp1 at $\hat{t}$ if $o_2^{\hat{t}-1} + o_1^{\hat{t}-2} \geq \bar{\theta} - 2$.*

[10] The abbreviations wp1 and wpp are used for 'with probability one' and 'with positive probability,' respectively.

Notice that in the equilibrium definition (1), $\overline{\text{P1}}$'s final action is to continue if his quitting and continuation payoffs are identical. However, any mixture between quitting and minimally increasing his last bid is compatible with the above lemma and can be an equilibrium. This is the reason for point (*iii*) in definition 1 of essential uniqueness.

In the next lemma, we show that there is only one possible separating equilibrium in the game. It occurs when P2 would quit wp1 next period which is preempted by $\underline{\text{P1}}$. No other separating equilibria can exist. To see this intuitively, assume that the contrary is true and that there is a separating equilibrium involving separating strategies by P1 $\hat{\beta}_1(\underline{\theta}) = (q, \ldots)$ and $\hat{\beta}_1(\bar{\theta}) = (1, \ldots)$ with full revelation at $t = 1$. Then, whenever P1 finds himself in $\underline{\theta}$—where $\hat{\beta}$ gives a payoff of zero—he will optimally deviate from the $\hat{\beta}$-prescribed action q to mimicking his high-type by bidding 1—securing a payoff strictly higher than zero—and leading P2 astray. This contradicts $\hat{\beta}_1$ being part of an equilibrium.

Lemma 2. *There exists a separating equilibrium with first separating action by P1 at $\hat{t}$, if*

$$\bar{\theta} - 1 \geq o_2^{\hat{t}-1} + o_1^{\hat{t}-2} \geq \bar{\theta} - 2. \tag{2}$$

There is no other separating equilibrium.

The previous lemma also shows that the only revealing equilibrium action which can be played by $\underline{\text{P1}}$ is quitting. The next lemma ascertains that there is a positive probability of reaching the 'final' periods of the game determined in lemma 1 above.

Lemma 3. *In no equilibrium, Pi quits wp1 at $\hat{t}$ if*

$$o_{-i}^{\hat{t}-1} + o_i^{\hat{t}-2} < \bar{\theta} - 2. \tag{3}$$

Based on lemma 1, the next lemma shows that $\overline{\text{P1}}$ quits wpp iff the sum of P2's current offer and his own minimally increased offer exceeds $\bar{\theta} - 1$. That is, he quits as soon as his next minimally increased offer to P2 gives him a necessarily lower payoff than P2's current offer. On the equilibrium path, this is the case as soon as both players offer roughly half the object's high value.

Lemma 4. *In no equilibrium, $\overline{\text{P1}}$ quits wpp at $\hat{t}$ unless*

$$o_2^{\hat{t}-1} + o_1^{\hat{t}-2} \geq \bar{\theta} - 1. \tag{4}$$

The next lemma shows that the uninformed player's beliefs are uniquely determined on and off the equilibrium path. The reason is that, at each stage where P2 moves, it is beneficial for $\overline{\text{P1}}$ to induce P2 to quit immediately and it is beneficial for $\underline{\text{P1}}$ to induce P2 to continue with probability one. Hence the only belief about the value of the object which does not separate states (through the induced belief) is when P2 is exactly indifferent between quitting and continuing. This is true at each stage where P2 can be made indifferent, ie. as soon as her priors do not induce an expected continuation payoff which is strictly above the quitting payoff. (We call this initial phase of play the preplay-phase.) Notice that P2's equilibrium mixture after the preplay-phase precisely makes $\underline{\text{P1}}$—whose mixing determines her belief—indifferent between quitting and continuing.

Lemma 5. *In every equilibrium and as long as $o_1^{\hat{t}-1} + o_2^{\hat{t}-2} < \bar{\theta} - 1$ at $\hat{t}$, if P2's prior belief φ_2^0 does not imply her continuation wp1, she must be indifferent between quitting and continuation.*

The above lemma establishes an essentially unique belief-structure accompanying β^*. We stress that this result is rather surprising and a feature not commonly found in (repeated) signalling games. It simplifies our analysis considerably.

The following lemma makes the case against jump bidding. It consists of an induction argument which starts at a terminal node and constructs an essentially unique path through the game to the initial node. We keep—based on the terminal equilibrium quitting condition which must hold at *any* terminal node—the continuation game fixed when checking for the optimal action and assigning a compatible history. We thus, step by step, further restrict the set of possible histories by unravelling the game from the last equilibrium action.

Lemma 6. *In any equilibrium β, both $\overline{\text{P1}}$ and P2 use minimal-increment strategies.*

Notice that lemma 2 shows that the only revealing action $\underline{\text{P1}}$ can use is quitting and lemma 4 shows that $\overline{\text{P1}}$ never quits wpp before the 'last' stage defined in lemma 1. Moreover, since there are no separating equilibria before the final move by P1, the above lemma 6 allows us to conclude that the only mixing $\underline{\mathrm{P1}}$ can undertake in equilibrium is over quitting and minimally increasing. As claimed, there are no complicated mixed equilibrium actions over more than one continuation action. The next lemma completes our characterisation of the equilibrium of the game. It states the essentially unique equilibrium mixtures and beliefs which we used to define β^*.

Lemma 7. *Equilibrium mixture probabilities and beliefs are essentially uniquely determined.*

Proposition 1 summarises existence and uniqueness using the above arguments.

Proposition 1. *The profile β^* in (1) is the essentially unique perfect Bayesian equilibrium of $\mathcal{Q}$.*

Next we compute the payoffs from β^* and find an expression for the start of mixing t_s^*.

Proposition 2. *In equilibrium β^*, $\underline{\mathrm{P1}}$ starts mixing at the first odd period following $t_s = \dfrac{\lfloor\bar{\theta}\rfloor - 3}{4}$.*

The following corollary states the payoffs obtained as a consequence of proposition 2 and also provides a convenient approximation of the general result.[11] Its main virtue is to summarise P1's emerging information rent over splitting the object's value half-half and the fact that the informed player benefits from refining the bidding grid.

Corollary 1. *For t_s^* defined as the first odd period after $\dfrac{\lfloor\bar{\theta}\rfloor - 3}{4}$ and odd $\lfloor\bar{\theta}\rfloor$, payoffs are given by*[12]

$$\begin{aligned} u(\beta^*|\underline{\theta}) &= \left(\frac{t_s^*-1}{2}, -\frac{t_s^*-1}{2}\right) \approx \left(\frac{\bar{\theta}-7}{8}, -\frac{\bar{\theta}-7}{8}\right), \\ u(\beta^*|\bar{\theta}) &= \left(\bar{\theta} - u_2(\beta^*|\bar{\theta}), \frac{1-t_s}{2} + 2\frac{\Gamma(\frac{\lfloor\bar{\theta}\rfloor+2}{2})}{\Gamma(\frac{\lfloor\bar{\theta}\rfloor+1}{2})}\frac{\Gamma(\frac{t_s+1}{2})}{\Gamma(\frac{t_s}{2})}\right) \approx \left(\frac{5\bar{\theta}-11}{8}, \frac{3\bar{\theta}+11}{8}\right). \end{aligned} \tag{5}$$

Next we show that our equilibrium is preserved under much more general conditions than those assumed above. Proposition 3 shows that a variant of the equilibrium β^* remains to be an equilibrium of the infinite game allowing for non-increasing bids.

[11] The approximation is imprecise for low values of $\bar{\theta}$ and stated for convenience only.
[12] The corresponding payoffs for even $\lfloor\bar{\theta}\rfloor$ can be easily calculated from proposition 2.

Uniqueness, however, is lost. We denote a bid which does not strictly increase the previous bid by '0'. A problem arising with this extended game is to assign payoffs to an infinite sequence of '0'. There are two obvious options: one is to view this case as bargaining breakdown and assign some exogenous payoff and the other is to derive the payoff as a limit of finite cycles. We take the latter route and denote a strategy profile following β up to some period, then cycling '0' x times, and then continuing β by $\beta \cup \{\text{'0'}\}^x$. In the below proposition we reset the period counter to the period where the initial deviation was made after each complete cycle. Hence we do not count cycles.

Assumption 1. *'0'-Deviations from β^*_∞ are believed to be made with probability one.*

Assumption 2. *The payoffs from following β^* up to some period s and then cycling infinitely by each player offering '0' (with probability one) are given by the limit of finite cycles*[13]

$$\lim_{x \longrightarrow \infty} u(\beta^* \cup \{\text{'0'}\}^x) = u(\beta^*).$$

Proposition 3. *We extend the bargaining game $\mathcal{Q}$ by allowing for general bids and propose the equilibrium β^*_∞ of this game: for all $t > 0$ and any history of play, the moving Pi bids*

$$\beta_i^t = \begin{cases} {}^*\beta_i^t & \dots \text{ if } o_{-i}^{t-1} \geq {}^*o_{-i}^{t-1}, \\ {}^*\beta_i^t & \dots \text{ if } (\underbrace{o_{-i}^{t-1} = \text{'0'} \wedge o_i^{t-2} = \text{'0'}}_{\text{complete '0'-cycle}}) \wedge o_{-i}^{t-3} \geq {}^*o_{-i}^{t-3}; \text{reset } t = t-2, \\ \text{'0'} & \dots \text{ otherwise} \end{cases} \tag{6}$$

where i=1,2, ${}^*\beta_i^t$ is the (mixed) stage action prescribed by β^* and $o_2^0 = {}^*o_2^0$. The outcome of the infinite game following β^*_∞ equals that from β^* in $\mathcal{Q}$.

Conclusion

We analyse an alternating-offers, common value bargaining problem with asymmetrically informed players. Both the informed and uninformed player make offers. The special

[13] Introducing any exogenous breakdown payoff of up to and including this limit leaves our results unchanged and may fit infinite-game depreciation considerations better than the above limit.

assumptions on priors, possible bids and preferences extend easily. The implications of allowing for incomplete information on both sides change the problem fundamentally and are therefore studied separately. Generalising the type space to a larger set retains the result in the sense that some informed types always mimic the highest possible type with positive probability. Introducing depreciation is possible but numerical examples suggest that jump-bidding cannot be ruled out any more and the simple intuition of the presented equilibrium is lost.

Appendix A

Lemma 1. In any equilibrium of $\mathcal{Q}$, (*i*) $\overline{\text{P1}}$ quits wp1 at $\hat{t}$ if $o_2^{\hat{t}-1} + o_1^{\hat{t}-2} > \bar{\theta} - 1$, (*ii*) P2 quits wp1 at $\hat{t}$ if $o_2^{\hat{t}-2} + o_1^{\hat{t}-1} \geq \bar{\theta} - 1$, and (*iii*) $\underline{\text{P1}}$ quits wp1 at $\hat{t}$ if $o_2^{\hat{t}-1} + o_1^{\hat{t}-2} \geq \bar{\theta} - 2$.

Proof of lemma 1. As we are only interested in the highest possible bids here, we just consider the case of $\theta = \bar{\theta}$. Since B is bounded from above and given that no player has quit yet, offers in the finite game must eventually reach this upper bound $\bar{B}$. Suppose that Pi, $i = 1, 2$ makes the last admissible continuation bid $o_i^{\hat{t}} = \bar{B} > \bar{\theta}$ at some period $\hat{t}$. Then, at $\hat{t}+1$, P$-i$ must accept Pi's offer through quitting wp1. Payoffs at $\hat{t}+1$ are then $u_i = \bar{\theta} - o_i^{\hat{t}}$ and $u_{-i} = o_i^{\hat{t}}$. Since

$$u_i = \bar{\theta} - o_i^{\hat{t}} = \theta - \bar{B} < 0,$$

however, Pi can do better by quitting at $\hat{t}$ and accepting P$-i$'s offer $o_{-i}^{\hat{t}-1} > 0$. Knowing that P$-i$ will also quit if her time $t-1$ quitting payoff exceeds her time t continuation payoff, we obtain Pi's quitting condition at $\hat{t}$ wp1 as

$$\bar{\theta} - o_i^{\hat{t}} < o_{-i}^{\hat{t}-1} \text{ or } o_{-i}^{\hat{t}-1} + o_i^{\hat{t}} > \bar{\theta}. \tag{A.1}$$

Since the minimal admissible increment for continuation is 1, this corresponds to the quitting condition at $\hat{t}$ wp1 as

$$\begin{aligned} \overline{P1}: \quad & o_2^{\hat{t}-1} + o_1^{\hat{t}-2} > \bar{\theta} - 1, \\ P2: \quad & o_1^{\hat{t}-1} + o_2^{\hat{t}-2} \geq \bar{\theta} - 1 \end{aligned} \tag{A.2}$$

if $\varphi_2^{\hat{t}} < 1$. In the zero-sum branch of the game, $\underline{\text{P1}}$ will only continue wpp if P2 continues wpp at the following stage. Thus his quitting condition corresponds to looking at P2's condition above and checking whether she quits necessarily next period. Thus $\underline{\text{P1}}$ will quit at $\hat{t}$ wp1 if

$$o_2^{\hat{t}-1} + o_1^{\hat{t}} \geq \bar{\theta} - 1 \text{ or } o_2^{\hat{t}-1} + o_1^{\hat{t}-2} \geq \bar{\theta} - 2. \tag{A.3}$$

Thus no equilibrium continuation bid can be higher than $\bar{\theta} - 1$. □

Lemma 2. There exists a separating equilibrium with first separating action by P1 at $\hat{t}$, if

$$\bar{\theta} - 1 \geq o_2^{\hat{t}-1} + o_1^{\hat{t}-2} \geq \bar{\theta} - 2.$$

There is no other separating equilibrium.

Proof of lemma 2. The first part of the statement follows from the previous lemma because if P1 gets to bid at $\hat{t}$, then $\underline{\text{P1}}$ quits while $\overline{\text{P1}}$ continues. This reveals $\theta = \bar{\theta}$. For the second claim we show that in every other candidate separating equilibrium, there exists a profitable deviation for $\underline{\text{P1}}$. Let the first separating action by P1 happen at $\hat{t}$.

a) The only equilibrium of the complete information, zero-sum game for $\underline{\theta} = 0$ is for $\underline{\text{P1}}$ to quit immediately and for P2 to quit whatever P1 offers. The same is true in the continuation game after the revelation of $\underline{\theta}$. Thus any strategy which reveals P1's type as $\theta = \underline{\theta}$ must end with P1 quitting at $\hat{t}$ wp1 because, otherwise, P2 would quit wp1 at $\hat{t}+1$ which cannot be optimal for $\underline{\text{P1}}$ in a zero-sum game. Therefore, the only possible continuation separating action must reveal the value of the object as high. The players' payoff expectation from the low-value separating equilibrium branch is $u(\underline{\theta}) = (o_2^{\hat{t}-1}, -o_2^{\hat{t}-1})$.

b) So $\overline{\text{P1}}$ continues at $\hat{t}$ and we know from (A.1) that $o_1^{\hat{t}} + o_2^{\hat{t}-1} \leq \bar{\theta}$. If, at $\hat{t}+1$, P2 finds that $o_1^{\hat{t}} + o_2^{\hat{t}-1} \geq \bar{\theta} - 1$, she quits wp1 and we are in the separating equilibrium described by (2). Since we are looking for *other* separating equilibria, it must be the case that there is an admissible $\hat{o}_2^{\hat{t}+1}$ such that $o_1^{\hat{t}} + \hat{o}_2^{\hat{t}+1} < \bar{\theta}$. But then P2 will continue at $\hat{t}+1$ wp1 because by offering $\hat{o}_2^{\hat{t}+1}$, she can get at least either $u_2^{\hat{t}+2}(q) = \bar{\theta} - \hat{o}_2^{\hat{t}+1} > o_1^{\hat{t}} = u_2^{\hat{t}+1}(q)$ or $u_2^{\hat{t}+3}(q) = o_2^{\hat{t}+2} > o_1^{\hat{t}} = u_2^{\hat{t}+1}(q)$. Thus after the high-value state is revealed, P2's continuation payoff is necessarily higher than her quitting payoff at that stage. But given that P2 continues wp1, $\underline{\text{P1}}$ will mimic $\overline{\text{P1}}$'s action to obtain $o_2^{\hat{t}} > o_2^{\hat{t}-2}$ and this cannot be an equilibrium. □

Lemma 3. In no equilibrium, Pi quits wp1 at $\hat{t}$, if

$$o_{-i}^{\hat{t}-1} + o_i^{\hat{t}-2} < \bar{\theta} - 2.$$

Proof of lemma 3. a) If P2 quits wp1 at $\hat{t}$ while (3) holds, then $\underline{\text{P1}}$ quits wp1 at $\hat{t}-1$ although $o_1^{\hat{t}-1} + o_2^{\hat{t}-2} < \bar{\theta} - 2$ and thus $\overline{\text{P1}}$ continues wp1. Hence this is a separating equilibrium which violates condition (2) and can therefore not exist. The same is true if $\underline{\text{P1}}$ quits at $\hat{t}$ wp1 while (3) holds.

b) If $\overline{\text{P1}}$ quits at $\hat{t}$ wp1 while (3) holds, he gets $o_2^{\hat{t}-1}$. But since P2 continues wpp, he can get at least his strictly higher $\hat{t}+2$ quitting payoff of

$$(1 - \alpha_2^{\hat{t}+1})(\bar{\theta} - o_1^{\hat{t}}) + \alpha_2^{\hat{t}+1}(o_2^{\hat{t}-1} + j_2^{\hat{t}+1} + 1) > o_2^{\hat{t}-1}.$$ □

Lemma 4. In no equilibrium, $\overline{\text{P1}}$ quits wpp at $\hat{t}$ unless

$$o_2^{\hat{t}-1} + o_1^{\hat{t}-2} \geq \bar{\theta} - 1.$$

Proof of lemma 4. Lemma 1 shows that $\overline{\text{P1}}$ quits wp1 only under (A.2). We now prove that $\overline{\text{P1}}$ never quits wpp before the weak inequality version of the same condition holds. The reason is that each of $\overline{\text{P1}}$'s stage payoffs following $\hat{t}$ is strictly larger than $o_2^{\hat{t}-1}$ as long as $o_2^{\hat{t}-1} + o_1^{\hat{t}-2} > \bar{\theta} - 1$: This is apparent for accepting P2's offers because her offers must be strictly increasing. For $\overline{\text{P1}}$'s own choice of quitting, we have

$$u_1^{\hat{t}+1}(q) = \bar{\theta} - o_1^{\hat{t}} > o_2^{\hat{t}-1}$$

and the same is true at each period where $\overline{\text{P1}}$ contemplates quitting as long as (4) is not true. Since $\overline{\text{P1}}$'s both time $\hat{t}+1$ and $\hat{t}+2$ quitting payoffs are strictly higher than $u_1^{\hat{t}}(q) = o_2^{\hat{t}-1}$ (and so is any mixture between the two), $\overline{\text{P1}}$ will not quit wpp before $o_2^{\hat{t}-1} + o_1^{\hat{t}-2} = \bar{\theta} - 1$ is reached. If this is indeed the case at $\hat{t}$, any mixture between the minimal increase $\hat{o}_1^{\hat{t}}$ and quitting is equally good and thus an equilibrium may prescribe quitting for $\overline{\text{P1}}$ wpp. Below we disregard this equality case because we are looking only for essentially unique equilibria which leave the $\overline{\text{P1}}$'s final equilibrium action undefined. (This is condition (*iii*) in definition 1.) □

Lemma 5. In every equilibrium and as long as $o_1^{\hat{t}-1} + o_2^{\hat{t}-2} < \bar{\theta} - 1$ at $\hat{t}$, if P2's prior belief φ_2^0 does not imply her continuation wp1, she must be indifferent between quitting and continuation.

Proof of lemma 5.

a) Every equilibrium belief which would induce P2 to quit wp1 at $\hat{t}$ implies that $\underline{\text{P1}}$ quits wp1 at $\hat{t}-1$ which is a fully separating action because we now from the previous lemma that $\overline{\text{P1}}$ will not quit wpp. But we know from lemma 3 that there cannot exist a separating equilibrium as long as $o_1^{\hat{t}-1} + o_2^{\hat{t}-2} < \bar{\theta} - 1$.

b) Every belief which would induce P2 to continue wp1 at $\hat{t}$ implies that, in equilibrium, $\underline{\text{P1}}$ continues wp1 at $\hat{t}-1$ as well. Hence there is no updating of P2's beliefs because $\overline{\text{P1}}$ continues wp1 as well (lemma 4). But then P2's beliefs are such that she prefers her continuation payoff to her quitting payoff based on her unchanged prior belief $\varphi_2^{\hat{t}-2}$.

   If P2 was indifferent between quitting and continuing based on the same prior at $\hat{t}-2$, then she must prefer her necessarily increased quitting payoff at $\hat{t}$, contradicting her continuing wp1. (Since both players continue wp1, P2's continuation payoff is the same at $\hat{t}$ and $\hat{t}-2$.) Hence she must have continued wp1 at $\hat{t}-4$ as well and $\underline{\text{P1}}$ did not mix at $\hat{t}-5$ either. This argument can be repeated until we reach P2's prior φ_2^0. □

Lemma 6. In any equilibrium β, both $\overline{\text{P1}}$ and P2 use minimal-increment strategies.

Proof of lemma 6. Our argument proceeds by induction over time t, starting with the terminal equilibrium condition (A.2) without specifying offers. The idea is that we start at *any* terminal node of the game where a player quits wp1 in equilibrium and proceed forward—*keeping this terminal action fixed*—by choosing the stage action which maximises the payoff in the continuation game. Then, again *keeping the continuation game fixed*, we go a further period ahead and determine the optimal response to this fixed continuation game. Proceeding forward, this gives a unique equilibrium path through the game whatever terminal node we start at.

a) First consider the equilibrium case where $\overline{\text{P1}}$ quits first wp1 at $\hat{t}$.

$\hat{t}$: As implied by lemmata 1 and 4, $\overline{\text{P1}}$ quits at $\hat{t}$ wp1 iff $o_2^{\hat{t}-1} + o_1^{\hat{t}-2} > \bar{\theta} - 1$. In such an equilibrium, payoffs are

$$u^{\hat{t}}(q) = (o_2^{\hat{t}-1}, \theta - o_2^{\hat{t}-1}). \tag{A.4}$$

$\hat{t}-1$: $o_2^{\hat{t}-1} = o_2^{\hat{t}-3} + 1 + j_2^{\hat{t}-1}$. Since $\overline{\text{P1}}$ quits at $\hat{t}$ (ie. the node at $\hat{t}$ is a terminal node), it must be the case that $o_1^{\hat{t}-2} + o_2^{\hat{t}-3} < \bar{\theta} - 1$ (otherwise $\hat{t}-1$ would be a terminal node). In case P2 quits at $\hat{t}-1$, payoffs are

$$u^{\hat{t}-1}(q) = (\bar{\theta} - o_1^{\hat{t}-2}, o_1^{\hat{t}-2}) \tag{A.5}$$

and for a continuation bid, payoffs are given by the $\varphi_2^{\hat{t}-1}$-weighted (A.4). Since P2's payoff is clearly decreasing in $j_2^{\hat{t}-1}$, P2 will choose to minimally increase her bid, ie. set $j_2^{\hat{t}-1} = 0$. She is indifferent between this pure bid $o_2^{\hat{t}-1}$ and quitting (and thus willing to mix with any $\alpha_2^{\hat{t}-1}$) if

$$o_1^{\hat{t}-2} = \varphi_2^{\hat{t}-1}(\bar{\theta} - o_2^{\hat{t}-1}) + (1 - \varphi_2^{\hat{t}-1})(-o_2^{\hat{t}-1}) \Rightarrow \varphi_2^{\hat{t}-1} = \frac{o_1^{\hat{t}-2} + o_2^{\hat{t}-1}}{\bar{\theta}}. \tag{A.6}$$

As is easily verified, this is the terminal equilibrium belief prescribed by (A.16).

$\hat{t}-2$: $o_1^{\hat{t}-2} = o_1^{\hat{t}-4} + 1 + j_2^{\hat{t}-2}$. Since $\overline{\text{P1}}$ does not quit wpp at $\hat{t}-2$, his continuation payoff is

$$u_1^{\hat{t}-2}(\beta) = (1 - \alpha_2^{\hat{t}-1})(\bar{\theta} - o_1^{\hat{t}-2}) + \alpha_2^{\hat{t}-1} u_1^{\hat{t}}(q) \tag{A.7}$$

where $\alpha_2^{\hat{t}-1}$ is such that $\underline{\text{P1}}$ mixes, ie. as prescribed by (A.13). Since the continuation probability (A.13) is increasing in $o_1^{\hat{t}-2}$, the probability of obtaining the next stage quitting payoff $\bar{\theta} - o_1^{\hat{t}-2} > o_2^{\hat{t}-1} = u_1^{\hat{t}}(q)$ is decreased. Thus $\overline{\text{P1}}$'s continuation payoff $u_1^{\hat{t}-2}(\beta)$ is decreasing in $o_1^{\hat{t}-2}$ and again $j_1^{\hat{t}-2} = 0$.

$\hat{t}-3$: $o_2^{\hat{t}-3} = o_2^{\hat{t}-5} + 1 + j_2^{\hat{t}-3}$. In case P2 quits at $\hat{t}-3$, payoffs are

$$u^{\hat{t}-3}(q) = (\bar{\theta} - o_1^{\hat{t}-4}, o_1^{\hat{t}-4}) \tag{A.8}$$

and for a continuation bid, payoffs are given by

$$u_2^{\hat{t}-3}(\beta) = \varphi_2^{\hat{t}-3}(o_1^{\hat{t}-2}) + (1-\varphi_2^{\hat{t}-3})\left((1-\alpha_1^{\hat{t}-2})(-o_2^{\hat{t}-3}) + \alpha_1^{\hat{t}-2} o_1^{\hat{t}-2}\right). \tag{A.9}$$

Equalisation of (A.8) and (A.9) imply indifference and give both $\underline{\text{P1}}$'s time $\hat{t}-4$ equilibrium mixture condition (A.14) and the general indifference condition on beliefs (A.15). Increasing $j_2^{\hat{t}-3} > 0$ directly reduces P2's time $\hat{t}-2$ stage payoff by the full amount of $j_2^{\hat{t}-3}$. Likewise, since now the quitting condition $o_2^{\hat{t}-1} + o_1^{\hat{t}-2} > \bar{\theta} - 1$ is reached with lower o_1^t, the continuation payoff in (A.9) is weakly decreased. Since $\underline{\text{P1}}$ mixes at each stage he moves—and thus always gets his conditional stage payoff—this cannot increase P2's payoff. (Inserting a jump-bid in (A.19) confirms this.)

t: At general $0 < t < \hat{t}-3$, both $\overline{\text{P1}}$'s and P2's problems are similar to the above and the arguments there apply unchanged. The game's initial node is reached when $o_i^{t-1} + o_{-i}^{t-2} = 0$.

b) Now consider the second possible case where P2 quits before $\overline{\text{P1}}$ wp1 in a separating equilibrium at $\hat{t}$; thus $o_2^{\hat{t}-2} + o_1^{\hat{t}-1} > \bar{\theta} - 1$. The only difference to the case of P2 moving at $\hat{t}-1$ discussed in a) is that P2's continuation payoff at $\hat{t}-2$ is now given by

$$u_2^{\hat{t}-2}(\beta) = (1-\varphi_2^{\hat{t}-2})(-o_2^{\hat{t}-2}) + \varphi_2^{\hat{t}-2}(o_1^{\hat{t}-1}) \tag{A.10}$$

and not by the $\varphi_2^{\hat{t}-1}$-weighted (A.4). This corresponds to the case where terminal beliefs are given by (A.17) for $\bar{\theta} - 1 \geq o_1^{\hat{t}-3} + o_2^{\hat{t}-2} \geq \bar{\theta} - 2$. The induction chain then proceeds as in case a) with all arguments unchanged. □

Lemma 7. Equilibrium mixture probabilities and beliefs are essentially uniquely determined.

Proof of lemma 7. We start with any equilibrium preplay-phase in which, at odd t and given P1's offer at t, P2's next period quitting payoff is lower than her expected equilibrium continuation payoff for *any* belief $\varphi_2^{t+1} \in [{}^1\!/_2, 1]$: $u_2^{t+1}(q) < {}^1\!/_2 u_2^{t+1}(\beta^*|\bar{\theta}) + {}^1\!/_2 u_2^t(q)$. This means that P2 will continue wp1 at $t+1$ and thus $\underline{\text{P1}}$ will not mix or else his expected payoffs are necessarily reduced. Since $\overline{\text{P1}}$ never mixes, P2's posterior equals her prior.

Consider any deviation from β^* by P1 which leaves P2 with a higher observed offer than expected. If there are beliefs $\varphi_2^{t+1} \in (\varphi_2^{t-1}, 1]$ for which it is possible that

$$u_2^{t+1}(q) > \varphi_2^{t+1} u_2^{t+1}(\beta^*|\bar{\theta}) + (1-\varphi_2^{t+1}) u_2^t(q) \tag{A.11}$$

then $\underline{\text{P1}}$ must mix (ie. increase φ_2^{t+1}) in order to make P2 indifferent. Because there cannot be a separating equilibrium unless (2) holds (which means that the game is over after the next move), this is only possible if P2 is indifferent between her equilibrium continuation bid and quitting. Hence her belief φ_2^{t+1} is uniquely defined (as part of her equilibrium strategy) as the belief which makes (A.11) hold with equality. There is only one mixture probability which $\underline{\text{P1}}$ can use to generate these beliefs through his observed actions and Bayes' rule—and P2 has no choice but to assume that $\underline{\text{P1}}$ uses exactly this continuation probability.

As soon as $\underline{\text{P1}}$ starts to mix, the equilibrium probabilities are obtained by inserting the equilibrium quitting payoffs into the stage indifference conditions. The players' time-t offers—and thus their opponents' following period quitting payoffs—are given through summation as

$$o_1^t = u_2^{t+1}(q) = \frac{t+1}{2} + J_1^t,\ o_2^t = u_1^{t+1}(q) = \frac{t}{2} + J_2^t. \tag{A.12}$$

Thus α_2^t is given at even t from $\underline{\text{P1}}$ mixing with any $\alpha_1^{t-1} \in (0,1)$ if $u_1^t(\beta^*|\underline{\theta}) = u_1^{t-1}(q)$ or

$$\begin{aligned} (1-\alpha_2^t)u_1^t(q) + \alpha_2^t u_1^{t+1}(\beta^*|\underline{\theta}) &= u_1^{t-1}(q) \\ (1-\alpha_2^t)\left(-o_1^{t-1}\right) + \alpha_2^t\left(o_2^t\right) &= o_2^{t-2} \\ {}^*\alpha_2^t = \frac{o_2^{t-2}+o_1^{t-1}}{o_2^t+o_1^{t-1}} = \frac{t+J_1^{t-1}+J_2^{t-2}-1}{t+J_1^{t-1}+J_2^t} &= \frac{t-1}{t} \text{ for } J_i = 0 \end{aligned} \tag{A.13}$$

while $\alpha_1^{t+1}, \varphi_2^t$ are given for even t from P2 mixing with any $\alpha_2^t \in (0,1)$ iff $u_2^{t+1}(\beta^*|\theta) = u_2^t(q)$, or

$$\begin{aligned} (1-\varphi_2^t)\left[(1-\alpha_1^{t+1})u_2^{t+1}(q|\underline{\theta}) + \alpha_1^{t+1}u_2^{t+2}(\beta^*|\theta)\right] + \varphi_2^t u_2^{t+2}(\beta^*|\theta) &= u_2^t(q) \\ (1-\varphi_2^t)\left[(1-\alpha_1^{t+1})\left(-o_2^t\right) + \alpha_1^{t+1}\left(o_1^{t+1}\right)\right] + \varphi_2^t\left(o_1^{t+1}\right) &= o_1^{t-1} \\ {}^*\alpha_1^{t+1} = \frac{(1-\varphi_2^t)(o_1^{t+1}+o_2^t) - 1 - j_1^{t-1}}{(1-\varphi_2^t)(o_1^{t+1}+o_2^t)} &= \\ \frac{(1-\varphi_2^t)(t+J_1^{t-1}+J_2^{t+1}) - \varphi_2^t(1+j_1^{t+1})}{(1-\varphi_2^t)(t+J_1^{t+1}+J_2^{t+1}+1)} = \frac{t-\varphi_2^t(t+1)}{t-\varphi_2^t(t+1)+1} &\text{ for } J_i = 0 \end{aligned} \tag{A.14}$$

where P2 beliefs φ_2^t evolve according to Bayes' rule

$${}^*\varphi_2^t = \frac{(1-\alpha_1^{t+1})(o_1^{t+1}+o_2^t) - j_1^{t-1} - 1}{(1-\alpha_1^{t+1})(o_1^{t+1}+o_2^t)} \tag{A.15}$$

implying P2's terminal equilibrium beliefs for the non-separating equilibrium (even $\lfloor\bar{\theta}\rfloor$) as

$${}^*\varphi_2^{\hat{t}-1} = \frac{o_1^{\hat{t}-2}+o_2^{\hat{t}-1}}{\bar{\theta}} \text{ if } \bar{\theta} \geq o_1^{\hat{t}-2}+o_2^{\hat{t}-1} > \bar{\theta}-1 \tag{A.16}$$

and for the separating equilibrium (odd $\lfloor\bar{\theta}\rfloor$) as

$$^{*}\varphi_2^{\hat{t}-2} = \frac{o_1^{\hat{t}-3} + o_2^{\hat{t}-2}}{o_1^{\hat{t}-1} + o_2^{\hat{t}-2}} \text{ if } \bar{\theta} - 1 \geq o_1^{\hat{t}-3} + o_2^{\hat{t}-2} \geq \bar{\theta} - 2. \tag{A.17}$$

Hence on and off-equilibrium-path beliefs including P2's final move are uniquely defined. As soon as $o_1^{t-1} + o_2^{t-2} \geq \bar{\theta} - 1$ at t, P2 quits wp1 for any belief. Since then the previous *equilibrium* action by $\underline{\text{P1}}$ is to quit wp1, P2's belief is $\varphi_2 = 1$ and thus uniquely defined as well. The only problem is after an off-equilibrium jump by P1 which leads P2 to quit for *any* belief. However, our definition of essential uniqueness allows us to disregard this case. □

Proposition 1. The profile β^* in (1) is the essentially unique perfect Bayesian equilibrium of $\mathcal{Q}$.

Proof of proposition 1. Equilibrium existence is a direct consequence of the previous lemmata. Uniqueness follows from the fact that, in equilibrium, $\mathcal{Q}$ can *only end wp1* when (A.2) is reached. Although there are many possible histories leading to this condition, lemma 6 shows that only the minimal-increase profile is compatible with both (A.2) and maximisation at each stage. Lemma 7 supplies the essentially unique mixture probabilities and beliefs which turn the minimal-increase profile from lemma 6 into the equilibrium β^*. The only source of non-uniqueness of equilibria is at the terminal stage which definition 1 allows us to disregard. □

Proposition 2. In equilibrium β^*, $\underline{\text{P1}}$ starts mixing at the first odd period following $t_s = \frac{\lfloor\bar{\theta}\rfloor - 3}{4}$.

Proof of proposition 2. Denote the (odd-valued) period where $\underline{\text{P1}}$ starts mixing by t_s. In equilibrium β^*, $t_s + 1$ is the first period in which P2's *prior*-based payoff expectation from β^* is lower than her sure payoff from quitting. Therefore, $\underline{\text{P1}}$ must mix at t_s in order to manipulate P2's beliefs or else she will quit at $t_s + 1$. Therefore, on the equilibrium path P2 quits at $t_s + 1$ if

$$u_2^{t_s+1}(q) = o_1^{t_s} = \frac{t_s + 1}{2} > \varphi_2^0 u_2^{t_s+3}(\beta^*|\bar{\theta}) - (1 - \varphi_2^0) o_2^{t_s+1} = \frac{1}{2} u_2^{t_s+3}(\beta^*|\bar{\theta}) - \frac{t_s + 1}{4} = u_2^{t_s+1}(\beta^*). \tag{A.18}$$

Choosing a higher prior for P2 will increase her payoff expectation, increase t_s^* and thus decrease (increase) $\overline{\text{P1}}$'s ($\underline{\text{P1}}$'s) information rent. P2's expected continuation payoff $u_2^{t_s+3}(\beta^*|\bar{\theta})$ given a high valued object is obtained recursively

$$u_2^{t_s+3}(\beta^*|\bar{\theta}) = (1 - \alpha_2^{t_s+3}) u_2^{t_s+3}(q) + \alpha_2^{t_s+3} u_2^{t_s+5}(\beta^*|\bar{\theta}).$$

For odd $\lfloor\bar{\theta}\rfloor$, it is more convenient to rewrite this in the form

$$u_2^{t_s+1}(\beta^*|\bar{\theta}) = \sum_{\tau=1}^{\frac{\lfloor\bar{\theta}\rfloor - t_s}{2}} \left(\prod_{t=1}^{\tau-1} \alpha_2^{2t+t_s-1}\right) u_2^{2\tau+t_s-1}(q)(1-\alpha_2^{2\tau+t_s-1}) + \left(\prod_{t=1}^{\frac{\lfloor\bar{\theta}\rfloor - t_s}{2}} \alpha_2^{2t+t_s-1}\right) u_2^{\lfloor\bar{\theta}\rfloor}(q) \tag{A.19}$$

where P2's quitting payoffs and mixture probabilities are given by (A.12) and (A.13) for $J_i = 0$. Plugging these into (A.19) gives the following equivalent and exact payoff formulations[14,15]

$$\begin{aligned}
u_2^{t_s+1}(\beta^*|\bar{\theta}) &= \frac{1}{2}\sum_{\tau=1}^{\frac{\lfloor\bar{\theta}\rfloor - t_s}{2}} \left(\prod_{t=1}^{\tau-1} \frac{2t+t_s-2}{2t+t_s-1}\right) + \left(\prod_{t=1}^{\frac{\lfloor\bar{\theta}\rfloor - t_s}{2}} \frac{2t+t_s-2}{2t+t_s-1}\right) \frac{\lfloor\bar{\theta}\rfloor+1}{2} \\
&= \frac{1}{2} + \frac{(t_s-1)!!}{2(t_s-2)!!} \sum_{t=\frac{ts+1}{2}}^{\frac{\lfloor\bar{\theta}\rfloor-3}{2}} \frac{(2t-1)!!}{(2t)!!} + \frac{(\lfloor\bar{\theta}\rfloor-2)!!}{(\lfloor\bar{\theta}\rfloor-1)!!} \frac{((t_s-1)!!}{((t_s-2)!!} \frac{\lfloor\bar{\theta}\rfloor+1}{2} \\
&= \frac{1-t_s}{2} + 2^{t_s-\lfloor\bar{\theta}\rfloor-1} \frac{(\lfloor\bar{\theta}\rfloor+1)!}{\left(\left(\frac{\lfloor\bar{\theta}\rfloor+1}{2}\right)!\right)^2} \frac{\left(\left(\frac{t_s-1}{2}\right)!\right)^2}{(t_s-1)!} \frac{\lfloor\bar{\theta}\rfloor+1}{2} \\
&= \frac{1-t_s}{2} + 2^{t_s-\lfloor\bar{\theta}\rfloor-1} \binom{\lfloor\bar{\theta}\rfloor+1}{\frac{\lfloor\bar{\theta}\rfloor+1}{2}} \binom{t_s-1}{\frac{t_s-1}{2}}^{-1} \frac{\lfloor\bar{\theta}\rfloor+1}{2} \\
&= \frac{1-t_s}{2} + 2\frac{\Gamma(\frac{\lfloor\bar{\theta}\rfloor+2}{2})}{\Gamma(\frac{\lfloor\bar{\theta}\rfloor+1}{2})} \frac{\Gamma(\frac{t_s+1}{2})}{\Gamma(\frac{t_s}{2})}.
\end{aligned} \tag{A.20}$$

Notice the appearance of the central binomial coefficient (and its equivalents) in the above. Since no closed form representation of this binomial coefficient is known, we use Stirling's approximation $n! \approx \sqrt{2\pi n}n^n e^{-n}$ to approximate the above central binomial coefficient as[16]

$$\binom{2n}{n} = \frac{(2n)!}{n!^2} \approx \frac{4^n}{\sqrt{n}\sqrt{\pi}}.$$

Using this, we approximate (A.20) as

$$u_2^{t_s+1}(\beta^*|\bar{\theta}) \approx \frac{1-t_s}{2} + \frac{2\sqrt{t_s-1}}{\sqrt{\lfloor\bar{\theta}\rfloor+1}} \frac{\lfloor\bar{\theta}\rfloor+1}{2}. \tag{A.21}$$

[14] The notation !! denotes the double factorial. For even $n \geq 0$, it is defined as $n!! = n(n-2)(n-4)(n-6)\ldots(4)(2)$, and for odd $n \geq 1$ the trailing term is replaced by $\ldots(3)(1)$.

[15] The expression for even $\lfloor\bar{\theta}\rfloor$ is similar and given by replacing $\lfloor\bar{\theta}\rfloor$ by $\lfloor\bar{\theta}\rfloor+1$ in (A.20) and changing the last term from $\frac{\lfloor\bar{\theta}\rfloor+1}{2}$ to $\frac{3\lfloor\bar{\theta}\rfloor-2\bar{\theta}+2}{2}$.

[16] The asymptotic error involved in Stirling's approximation is of order $1/n$, so it vanishes as $\bar{\theta}$ gets large.

Plugging this approximation back into (A.18)—for $u_2^{t_s+3}(\beta^*|\bar{\theta})$—gives

$$\frac{t_s+1}{2} > \frac{1}{2}\left(\frac{1-(t_s+2)}{2} + \frac{2\sqrt{(t_s+2)-1}}{\sqrt{\lfloor\bar{\theta}\rfloor+1}}\frac{\lfloor\bar{\theta}\rfloor+1}{2}\right) - \frac{1}{2}\left(\frac{t_s+1}{2}\right) \Rightarrow t_s^* > \frac{\lfloor\bar{\theta}\rfloor-3}{4}.$$

As P2 will quit at t_s+1 if her beliefs are not adjusted by a mixed action of $\underline{\text{P1}}$ at the first possible t_s, the equilibrium value of t_s is indeed given by the *first* odd period after the above t_s^*.[17] □

Corollary P2's payoff $u_2(\beta^*|\bar{\theta})$ is derived in alternative, precise formulations in (A.20). Because the high-value game is ex-post $\bar{\theta}$-sum, $\overline{\text{P1}}$'s payoff is just $u_1(\beta^*|\bar{\theta}) = \bar{\theta} - u_2(\beta^*|\bar{\theta})$. $\underline{\text{P1}}$'s payoff is given by t_s^* through $u_1(\beta^*|\underline{\theta}) = u_1^{t_s^*}(q) = o_2^{t_s^*-1} = \frac{t_s^*-1}{2}$ because all pure actions in the support of $\underline{\text{P1}}$'s first mixed action must give the same payoff. Since this branch of the game is ex-post zero-sum, $u_2(\beta^*|\underline{\theta}) = -u_1(\beta^*|\underline{\theta})$. Thus for odd $\lfloor\bar{\theta}\rfloor$ and t_s^* defined as the first odd period after $\frac{\lfloor\bar{\theta}\rfloor-3}{4}$, payoffs are indeed given by 5. The approximations are obtained by setting $\bar{\theta} \approx \lfloor\bar{\theta}\rfloor$, ignoring the required rounding up of t_s^* to the next odd integer period and plugging $t_s = \frac{\bar{\theta}-3}{4}$ into the approximation (A.21). The resulting approximation is imprecise for low $\bar{\theta}$ but conveniently summarises the obtained information rents over splitting the object's value half-half.

Proposition 3. We extend the bargaining game $\mathcal{Q}$ by allowing for general bids and propose the equilibrium β_∞^* of this game: for all $t > 0$ and any history of play, the moving Pi bids

$$\beta_i^t = \begin{cases} {}^*\beta_i^t & \ldots \text{ if } o_{-i}^{t-1} \geq {}^*o_{-i}^{t-1}, \\ {}^*\beta_i^t & \ldots \text{ if } \underbrace{(o_{-i}^{t-1} = \text{‘0’} \wedge o_i^{t-2} = \text{‘0’})}_{\text{complete ‘0’-cycle}} \wedge o_{-i}^{t-3} \geq {}^*o_{-i}^{t-3}; \text{reset } t = t-2, \\ \text{‘0’} & \ldots \text{ otherwise} \end{cases}$$

where i=1,2, ${}^*\beta_i^t$ is the (mixed) stage action prescribed by β^* and $o_2^0 = {}^*o_2^0$. The outcome of the infinite game following β_∞^* equals that from β^* in $\mathcal{Q}$.

Proof of proposition 3. We start by interpreting ‘0’ as repeating the previous own offer. In

[17] One may be concerned that the error of approximation contained in the computation of t_s^* adversely influences the result. This is not the case. In equilibrium, our approximation gives $t_s^* = \frac{\lfloor\bar{\theta}\rfloor-3}{4}$ and thus $\frac{2\sqrt{t_s^*+1}}{\sqrt{\lfloor\bar{\theta}\rfloor+1}} \equiv 1$ in (A.21) for period t_s+3. Inserting any similar *linear* candidate $t_s = \bar{\theta}/4 \pm \epsilon$, $\epsilon \geq 0$ into the decisive term in (A.20) confirms t_s^* as solution because

$$2^{t_s^*-\lfloor\bar{\theta}\rfloor+1}\binom{\lfloor\bar{\theta}\rfloor+1}{\frac{\lfloor\bar{\theta}\rfloor+1}{2}}\binom{t_s+1}{\frac{t_s+1}{2}}^{-1} \xrightarrow[\bar{\theta}\to\infty]{} 1 \equiv \frac{2\sqrt{t_s^*+1}}{\sqrt{\lfloor\bar{\theta}\rfloor+1}}.$$

The deviation of the left hand term from 1 is a measure of the error of our approximation. It is never bigger than 18% (for $\bar{\theta} = 3$) and is below 1% for $\bar{\theta} > 75$. Solving (A.18) gives the condition $t_s > \frac{2u_2^{t_s+3}(\beta^*|\bar{\theta})-3}{3}$ which is easily checked for some low $\bar{\theta} = 3, 5, 7, \ldots$ against the prediction $\frac{\lfloor\bar{\theta}\rfloor-3}{4}$. As the error of approximation diminishes, verifying the first few $\bar{\theta}$ ensures that our result is precise for all $\bar{\theta}$.

order to confirm β^*_∞ as an equilibrium of the infinite game, we first confirm optimality of the prescribed actions on the equilibrium path, then on any deviation path and then show that no deviations from the deviation path are profitable.

1. Given β^*_∞, bidding '0' wp1 cannot be a profitable deviation from the equilibrium path because this is (in equilibrium) followed by the opponent bidding '0' (wp1) which leaves the game in precisely the state before entering the '0'-cycle. Thus bidding '0' gives the equilibrium payoffs. The same is true for any finite repetition, and, through assumption 2, for any infinite repetition of cycles as well.

2. Conforming to β^*_∞ and bidding '0' after the initial '0' is optimal because any higher bid would constitute a jump bid which was shown previously not to be profitable.

3. Conforming to β^*_∞ and bidding ${}^*\beta^t_i$ after a full '0'-cycle is optimal because any higher bid would constitute a jump bid and entering another '0'-cycle cannot be profitable because any cycle gives the same payoff as the equilibrium action.

As no individual cycle has any implication on payoffs or beliefs, strategies comprising more complicated deviations than the above single-stage deviations cannot have any implication either. Since β^*_∞ prescribes the same stage actions as β^* in the finite game, the payoffs are the same. Reinterpreting '0' as any non-increasing bid does not change the above argument. Existence of β^*_∞ does not follow from our (backward) induction arguments for the finite game. It follows, however, from general arguments developed by Aumann and Maschler (1966), Mertens, Sorin, and Zamir (1994), and Simon, Spież, and Toruńczyk (1995). □

Remark: This remark attempts to elucidate the appearance of the Euler Γ-function in the above (A.20). (There are equivalent formulations based on the rising factorial and the central binomial coefficient.) From (A.17), P2's period-$\lfloor\bar{\theta}\rfloor$-1 equilibrium path beliefs for odd $\lfloor\bar{\theta}\rfloor$ are given by $\varphi_2^{\lfloor\bar{\theta}\rfloor-1} = \frac{\lfloor\bar{\theta}\rfloor-1}{\lfloor\bar{\theta}\rfloor}$. Using Bayes' rule and the equilibrium continuation probability ${}^*\alpha^t_1$ from (A.14), this can be folded back to general t. Using the Pochhammer notation $Pochhammer(a,n) = \frac{\Gamma(a+n)}{\Gamma(a)}$, P2's belief is given by

$$\begin{aligned} {}^*\underline{\varphi}^t_2 &= \left(\left(\left(\left(\frac{\lfloor\bar{\theta}\rfloor-1}{\lfloor\bar{\theta}\rfloor}\right)\frac{\lfloor\bar{\theta}\rfloor-3}{\lfloor\bar{\theta}\rfloor-2}\right)\frac{\lfloor\bar{\theta}\rfloor-5}{\lfloor\bar{\theta}\rfloor-4}\right)\underbrace{\cdots}_{\frac{\lfloor\bar{\theta}\rfloor-t}{2}\text{ times}}\right) \\ &= \prod_{\tau=1}^{\frac{\lfloor\bar{\theta}\rfloor-t+1}{2}} \frac{\lfloor\bar{\theta}\rfloor-2\tau+1}{\lfloor\bar{\theta}\rfloor-2\tau+2} = \frac{Pochhammer\left(\frac{1-\lfloor\bar{\theta}\rfloor}{2},\frac{\lfloor\bar{\theta}\rfloor-t+1}{2}\right)}{Pochhammer\left(-\frac{\lfloor\bar{\theta}\rfloor}{2},\frac{\lfloor\bar{\theta}\rfloor-t+1}{2}\right)}. \end{aligned} \tag{A.22}$$

Thus the belief process itself contains the Γ-function (or its equivalents) in an irreducible way.

Appendix B

Consider the simple case of $\Theta = \{0, 3.1\}$ in the finite game.[18] Recall that the possible bids are non-negative integers bounded by some large number $\bar{B} > 3.1$. It is easy to see (and argued in lemma 1) that in this case, no player bids in excess of 2. Thus only equilibrium offers of $\{0, 1, 2\}$ are considered below.

In principle, there could be fully revealing ('separating') equilibria corresponding to situations where, for instance, $\overline{\text{P1}}$ bids 1 and and $\underline{\text{P1}}$ always quits. This, however, is not the case. If the above were equilibrium strategies, P1 would use the *same* action in the low-value state and thus fool P2 into believing to be in the high-value state. But this cannot be equilibrium behaviour and hence all strategies which deterministically reveal the value of the object cannot be equilibria as long as P2 can still condition her response on this information (lemma 2).

Applied to the present example, the gradually revealing, semi-separating equilibrium β^* defined in the main text is

$$\begin{aligned} \beta_1^*(\underline{\theta}) &= ([\alpha_1 = {}^1\!/_2 : 1, q], q) \\ \beta_1^*(\bar{\theta}) &= \begin{cases} (1, 2, q) & \text{if } b_2^2 = 1 \\ (1, q) & \text{if } b_2^2 > 1 \end{cases} \end{aligned} \qquad \beta_2^*(\theta) = \begin{cases} ([\alpha_2 = {}^1\!/_2 : 1, q], q), \varphi = {}^2\!/_3 & \text{if } b_1^1 = 1 \\ ([\alpha_2' = {}^2\!/_3 : 1, q], q), \varphi' = {}^3\!/_{3.1} & \text{if } b_1^1 = 2 \\ (q), \varphi = 1 & \text{if } b_1^1 > 2 \end{cases} \tag{B.1}$$

where P1 chooses α_1 such that P2, given her equilibrium beliefs φ, is indifferent between continuing and quitting after his bid. P2 can only condition on the offer and chooses the equilibrium mixture probabilities α_2, α_2', which allow $\underline{\text{P1}}$ to mix. In case P2 gets to move again, she quits. Thus β^* prescribes the following sequence of play:

t=0: Nature decides on θ and sends a fully revealing signal to P1 and no signal to P2.

t=1: P1's minimum continuation offer not ending the game is $o_1^1 = 1$. Depending on the object's value, $\underline{\text{P1}}$ uses the type-dependent lottery $[\alpha_1 : 1, q]$ to mix between offering 1 and quitting in case of $\theta = 0$. He bids 1 wp1 in case of $\theta = 3.1$. Because $\underline{\text{P1}}$ plays a mixed action, he must be indifferent between the payoffs of all pure actions in the support of this mixed action, and, in particular, his quitting payoff of zero. This translates into $\underline{\text{P1}}$'s mixture requirement (A.13), or

$$0 = (1 - \alpha_2)(-1) + \alpha_2(1) \;\Leftrightarrow\; \alpha_2 = {}^1\!/_2. \tag{B.2}$$

After observing an offer of 1, P2 uses the conditional mixture probability embedded

[18] This simple example shows the intuition and dynamics of the equilibrium and illustrates the emergence of P1's information rent which is positive for $\bar{\theta} > 3$. The latter is confirmed by the payoffs $u(\beta^*|\bar{\theta} = 5) = (25/8, 15/8)$ and $u(\beta^*|\bar{\theta} = 7) = (77/16, 35/16)$ and for higher $\bar{\theta}$ where our payoff approximation (5) is approached. Notice, however, that this approximation is *worst* for very coarse bidding grids as in this example. Notice further that the informed player benefits from a refinement of the bidding grid. The example of $\Theta \in \{0, 2, 3\}$ with $\varphi_H = \varphi_M = \varphi_L = {}^1\!/_3$ illustrates a generalisation of our setup to more than two types.

in P1's equilibrium strategy to compute her posterior. At $t = 1$, these are $\mathrm{pr}(o_1^1 = 1|\theta = 0) = \alpha_1$, $\mathrm{pr}(o_1^1 = q|\theta = 0) = 1 - \alpha_1$ and $\mathrm{pr}(o_1^1 = 1|\theta = 3.1) = 1$, $\mathrm{pr}(o_1^1 = q|\theta = 3.1) = 0$. They thus prescribe that $\underline{\mathrm{P1}}$ should bid $o_1^1 = 1$ with probability α_1 (and quit otherwise) and for $\overline{\mathrm{P1}}$ to bid 1 wp1. Upon observing P1's bid of 1, this induces P2 to use Bayes' rule to revise her prior $\varphi_2^0 = {}^1\!/_2$ to

$$\mathrm{pr}(\theta = 3.1|o_1^1 = 1) = \frac{\mathrm{pr}(o_1^1 = 1|\theta = 3.1)\,\mathrm{pr}(\theta = 3.1)}{\mathrm{pr}(o_1^1 = 1)} = \frac{1}{1 + \alpha_1} = \varphi. \tag{B.3}$$

t=2: P2's minimum continuation offer is 1. Given her posterior of φ, P2 plays the mixed action $[\alpha_2 : 1, q]$ for any $\alpha_2 \in [0, 1]$ because—through the appropriately chosen mixture α_1—she is made indifferent between her quitting payoff of one and her continuation payoff $\varphi(2) + (1 - \varphi)(-1)$. In particular she is willing to play the mixed action $[\alpha_2 = {}^1\!/_2 : 1, q]$ which makes $\underline{\mathrm{P1}}$ indifferent between quitting and bidding 1 as required by (B.2). For the α_1-generated φ to be optimal, it has to satisfy P2's equality of payoffs for all pure actions in the support of her mixed action $1 = \varphi(2) + (1 - \varphi)(-1)$ in addition to φ resulting from the application of Bayes' rule (B.3). This implies $(\alpha_1, \varphi) = ({}^1\!/_2, {}^2\!/_3)$. Notice that φ is calculated backwards from the last stage.

t=3: Observing P2's offer of $o_2^2 = 1$, $\underline{\mathrm{P1}}$ finds it optimal to quit. $\overline{\mathrm{P1}}$ is indifferent between his minimal-increment bid of $o_1^3 = 2$, quitting, or any mixture between the two.[19] Thus P1 reveals the value of θ at this stage using a separating action.

t=4: Observing $\overline{\mathrm{P1}}$'s continuation, P2 quits.

The outcome from (B.1) is

$$u(\beta^*|\theta) = \begin{cases} (1.6, 1.5) & \text{given } \bar{\theta} = 3.1 \\ (0, 0) & \text{otherwise} \end{cases} \qquad \text{ex-ante } u(\beta^*) = (0.8, 0.75)\,.$$

For observed off-equilibrium-path bids by P1, P2's response (ie. her action complete with equilibrium mixture probability and belief) is part of her equilibrium strategy. Her beliefs are undetermined by the solution concept alone. But we know that P2 must be indifferent between quitting and continuing or else she could deduce the object's value from P1's choice of action: if he chooses a (mixed) action which makes her prefer quitting, this is beneficial *only* to $\overline{\mathrm{P1}}$ while continuation wp1 is beneficial *only* to $\underline{\mathrm{P1}}$. Since P1 cannot credibly reveal the state in this fashion, P2 must be indifferent between continuing and quitting in equilibrium. This determines her unique equilibrium belief and, in turn, the conviction that the observed jump-bid was taken by $\underline{\mathrm{P1}}$ with the probability α_1' determined from her belief through Bayes' rule. In case of observing $o_1^1 = 2$ in the example, this gives $\varphi' = {}^3\!/_{3.1}$, $\alpha_2' = {}^2\!/_3$ and $\alpha_1' = {}^1\!/_{30}$. To $\overline{\mathrm{P1}}$ this gives a payoff expectation of $3.1/3 < 3.2/2$

[19] This indifference is the reason for the 'essential' uniqueness-provision in our result: the last potential move by P1 is arbitrary for integer $\bar{\theta}$.

which renders his jump unprofitable.[20] Part of the example's extensive form is shown in

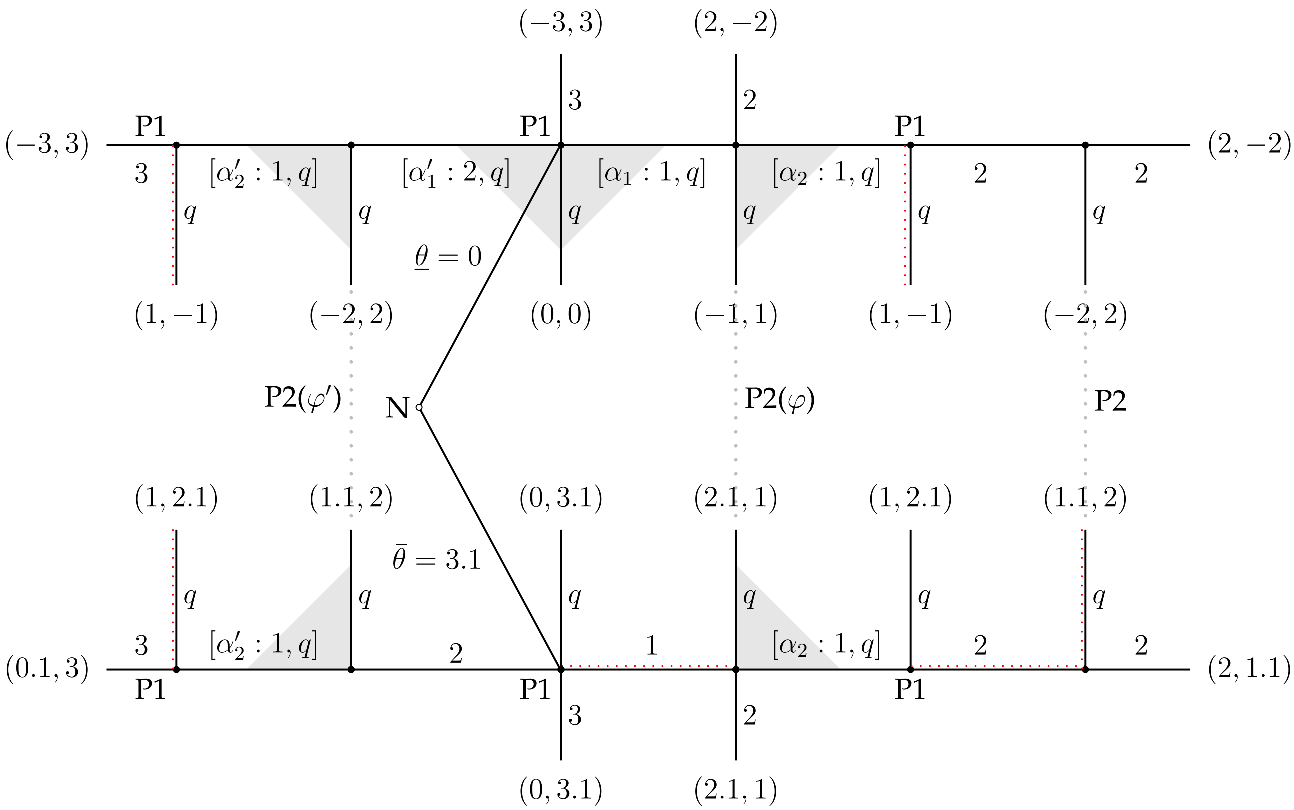

Figure 1: A partial game tree of the example of $\theta \in \{0, 3.1\}$ and equal priors.

fig. 1 where vertices on the equilibrium path are dotted and greyed triangles symbolise the range of mixed actions. The tree to the left of Nature's move shows the deviation path after P1's $b_1^1 = 2$. Checking for other deviations confirms (B.1) as an equilibrium.

Finally, we extend our example by allowing for the previous offer to be repeated indefinitely. We generically denote the repetition of the last own bid by '0'. Based on the equilibrium (B.1), for all t, any previous bidding history and for i=1,2, we examine the following equilibrium candidate β_∞^* of the infinite game:[21]

$$\beta_i^t = \begin{cases} {}^*\beta_i^t & \ldots \text{ if the previous continuation bid } o_{-i}^{t-1} \text{ was an equilibrium or jump bid,} \\ {}^*\beta_i^t & \ldots \text{ after a complete '0'-cycle started by P}i \text{ (resetting } t \text{ to before the cycle),} \\ \text{'0'} & \ldots \text{ otherwise, ie. when in a cycle.} \end{cases}$$

Repeating the last own offer (wp1, whatever the own type) is always followed by a repetition of the opponent's last offer (again wp1). Hence, after any cycle of repetitions, the game is in exactly the same state as it was before entering the '0'-cycle and especially all beliefs are unchanged. Thus, in β_∞^*, bidding '0' at any stage carries the same payoff as

[20] Because $o_1^{t-1} + o_2^t > \bar{\theta}$ after any higher jumps $o_1^{t-1} \geq 3$, P2 quits wp1 for *any* belief. Thus her off-equilibrium-path beliefs after observing such a jump-bid are undefined. Nevertheless, our equilibrium is essentially unique because definition 1 does not pin down her (terminal) beliefs when any belief leads to quitting.

[21] For a discussion of infinite deviation paths see assumption 2 in the main text.

the equilibrium action β_t^* at that stage. Therefore any mixture between the (mixed) equilibrium stage action of the finite game β_t^* and playing '0' is another equilibrium action of the game. (All leading, however, to the same outcome.) To increase one's own bid after a repetition by the opponent cannot be optimal as this would, in essence, constitute a jump bid.

A similar argument applies to allowing decreasing bids: if the above β_∞^* is extended such that '0'-offers denote not only the repetition of the last offer but *any* offer $o_i^t \leq {}^*o_i^{t-1}$, then the equilibrium β^* can again be recovered. The equilibrium-player then simply waits by playing '0' until the expected offer ${}^*\beta_i^t$ is made upon which play commences as before.

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