GOVERNANCE AND THE EFFICIENCY OF ECONOMIC SYSTEMS
GESY

Discussion Paper No. 139

Contests for Status

Benny Moldovanu*
Aner Sela**
Xianwen Shi***

July 2005

*Benny Moldovanu, Department of Economics, University of Bonn, Lennéstr. 37, 53113 Bonn, Germany mold@uni-bonn.de
**Aner Sela, Department of Economics, Ben Gurion University, Beer Sheva, Israel, anersela@bgu.ac.il
***Xianwen Shi, Department of Economics, Yale University, New Haven, CT 06511, USA; xianwen.shi@yale.edu

Financial support from the Deutsche Forschungsgemeinschaft through SFB/TR 15 is gratefully acknowledged.

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4.7.2005

Abstract

We study the optimal design of organizations under the assumption that agents in a contest care about their relative position. A judicious definition of status categories can be used by a principal in order to influence the agents' performance. We first consider a pure status case where there are no tangible prizes. Our main results connect the optimal partition in status categories to various properties of the distribution of ability among contestants. The top status category always contains an unique element. For distributions of abilities that have an increasing failure rate, a proliferation of status classes is optimal, while in other cases the optimal partition involves some coarseness. Finally, we modify the model to allow for status categories that are endogenously determined by monetary prizes of different sizes. If status is solely derived from monetary rewards, we show that the optimal partition in status classes contains only two categories.

*We wish to thank Pradeep Dubey, John Geanakoplos, Rongzhu Ke, Hans Haller, Heidrun Hoppe, Dana Moldovanu, Stephen Morris and Moshe Shaked for helpful remarks. Moldovanu is grateful to the Max Planck Research Prize and to the German Science Foundation for financial support, and to Yale University for its hospitality. Shi thanks the Cowles Foundation for financial support. Moldovanu: Department of Economics, University of Bonn, Lennestr. 37, 53113 Bonn, Germany; e-mail: mold@uni-bonn.de. Sela: Department of Economics, Ben Gurion University, Beer Sheva, Israel; e-mail: anersela@bgu.ac.il. Shi: Department of Economics, Yale University, New Haven, CT 06511, USA; e-mail: xianwen.shi@yale.edu.

1 Introduction

One of the earliest designed society structures was that of Solon's (ca. 638 BC - 558 BC) *timokratia,* an oligarchy with a sliding scale of status determined by precisely defined ranges of measured economic output (fruit, grain, oil, etc.). Solon divided the entire population of Attica into four status classes,[1] and attached various, more or less tangible rights, to each class. Higher classes had more rights but were also expected to contribute more to the state.

For hundreds of years, the kings and queens of feudal states awarded titles of nobility such as *duke (or duchess), marquis, earl, count, viscount, baron, baronet* in return for special services to the crown[2]. Initially there was a strong link between such titles and tangible assets, such as land and serfs. But, this link weakened over time. For example in 18th century France there were about 4000 public offices conferring some title of nobility, and these offices were often re-sold by incumbents after nobility was achieved. In time of financial distress the king sold even blank letters of nobility to be further sold by his provincial administrators.

If the above example seems old-fashioned, consider today's large corporations (such as large banks) that have on offer, besides a single *president* , several *executive vice presidents,* tens of *senior vice-presidents* , and several hundred "mere" *vice-presidents.* Or the New York Metropolitan Museum of Art that offers eight different donor categories[3] for corporate members (such as "*Chairman's Circle*" for donations above $100000, "*Director's Circle*" for donations between $60,000 and $100,000, and so on) and 10 similar categories for private members[4].

The common denominator to the above (and many other[5]) examples is that agents' care about social status, and that a self-interested principal is usually able to divert (or "manipulate") this concern to an avenue that is beneficial to himself/herself.

In this paper we study the optimal design of organizations under the assumption that agents care about their relative position. We show how a judicious definition of the number and size of

[1] These were the Pentakosiomedimnoi, the Hippeis, the Zeugitai and the Thetes.

[2] Even today's citizens of the United Kingdom are eligible for more than 50 orders and decorations, awarded for special services to the "queen". These are structured in a strict precedence system, and seem to play an important role in some parts of the public.

[3] See Glazer and Konrad (1996) for some empirical evidence and a theoretical model that focuses on conspicuous giving.

[4] Besides various tangible benefits such as free entry to various events, the status symbols are ever present: For example, memebership in the top categories (either corporate or private) comes with "reservation coupons for the Trustees Dining Room, the Museum's exclusive restaurant overlooking Central Park"

[5] Learned societies, like the Econometric society, have different status classes as well (e.g., members and fellows).

status classes can be used by a principal in order to maximize the agents' performance.

The tournament literature has shown how prizes based on rank-orders of performance can be effectively used to provide incentives (see Lazear and Rosen, 1981, Green and Stokey, 1983, and Nalebuff and Stiglitz, 1983). O'Reilly et al. (1988) have emphasized the important role of status in executive compensation, and Hambrick and Cannela (1993) use relative standing as the main factor for explaining departures rates of executives of acquired firms. Bognanno (2001) studies the empirical relation between the number of executive board members and the CEO's compensation in "corporate tournaments". Moldovanu and Sela (2001, 2005) developed a convenient contest model that can easily accommodate several prizes of different size. Using their methodology, it is a natural step to analyze the incentive effect of "status prizes," and the interplay between such prizes and tangible ones.

The general importance of status concerns for explaining behavior has been long recognized by sociologists and economists (see Weber, 1978, Coleman, 1990, Veblen, 1934, Duesenberry, 1949 , Friedman and Savage, 1948, and Friedman 1953 for some early contributions). Frank (1985) offers an entertaining account of some of the issues. Recent happiness research shows how wage rank affects workers' well-being (Brown et al. 2004), and experimental studies pointed out that social status may play a role also in market exchanges (Ball et al., 2001).

Biologists view status as an almost synonym for the ability of winning contests. The possibility that concerns about relative standing are biologically "hard-wired" is discussed in Postlewaite (1998). Indeed, there are multi-faceted and intriguing interactions between social status and several metabolic processes (in particular the production of various hormones such as serotonin) - this is the topic of a growing literature in the bio-sciences (see, for example, the findings about bees, crustaceans, lizards, fish, rats, birds and rats reported in Larson and Summers, 2001, and the elegant experiments performed with primates and humans by Raleigh et al., 1991 and by Mazur and Lamb, 1980). In the "biological" vein, Cole et al. (1992) present a model where agents care about relative wealth because parents' relative wealth affects the mating prospects of children, and Samuelson (2004) applies an evolutionary argument to justify relative consumption effects.

In our model, several agents are privately informed about their ability engage in a contest, and are then partitioned into status categories (or classes) according to their performance. A status category consists of all contestants who have performances in a specified quantile, e.g., the top status class may consist of the individual with the highest output, the second class of individuals with the next three highest outputs, and so on...Each individual cares about the number of contestants in classes above and below him. We choose a convenient functional formulation that captures well the "zero-sum game" nature of concerns for relative position: if an individual gets higher (lower) status, one or more individuals must get lower (higher) status.

A designer (or principal) determines the number of status classes and their size in order to maximize total output. Since the contest equilibrium only depends on the structure of status classes, and not directly on the designer's goal, our type of analysis can, in principle, be performed for a variety of other goals.

We first analyze the "pure status" case where there are no other tangible prizes to motivate the contestants. We then extend our model to investigate a setting where the designer awards monetary prizes, and where status is purely derived from the differences in monetary compensation, i.e., having a higher monetary prize *per se* implies higher status (see Robson, 1992 for another model where status is defined by wealth). These two models represent opposite extremes, and reality is often somewhere in the middle. In most cases, we think that individuals in organizations are, at least partly, motivated by status concerns, but that status is not solely derived from the monetary payoffs attached to various activities. For example, Fershtmann and Weiss (1993) relate status to the length of the education necessary for a specific occupation (their motto is Adam Smith's nicely circular: "Honour makes a great part of the reward of all honourable professions").

Since, as argued above, status is a "zero-sum game", it seems, at first glance, that shifts in the allocation of status among agents should not affect total output. The missing factor in this argument is the heterogeneity in abilities. In a world of heterogenous contestants, a modification in the structure of a status class (e.g., a division in two sub-classes that elevates some individuals while lowering others) has an impact on the output of contestants most likely to fall in that particular class. Since the contestants have heterogenous abilities, and since higher ability will be, in equilibrium, associated with higher performance, modifications of classes at different levels in the hierarchy may have quite different effects. Because the expected benefit associated with a move upwards in the ranks (which is given by the expected increase in status minus the expected cost of producing an output that is sufficient for the upward move) crucially depends on the order statistics associated with the upper and lower output bounds of the quantile defining the status class, a manipulation of these bounds affects behavior, and hence total output.

Our main results in the "pure status" model relate the structural features of the optimal partition in status categories to properties of the distribution of abilities in the society.

We show that the top category in any optimal partition must contain a single agent. This agrees well with the ubiquitous structure of many human (or animal) organizations and social structures, and brings to mind familiar roles such as "queen", "alpha-male", "CEO", etc.... We then identify the main factors leading either to a proliferation of status classes (where each individual is "in a class of his/her own") or to coarse partitions where it is optimal to have a wider range of performances bunched together in the same category. A proliferation of status classes is optimal if the distribution of abilities has an increasing failure (or hazard) rate. This finding points in the same direction as the well known empirical fact that job titles do proliferate, but only in organizations with a relatively professional work-force (see Baron and Bielby, 1986). In contrast, a coarse partition of status classes (besides the top one) may be optimal if the distribution of abilities puts less and less weight on higher an higher ability ranges.

We also study the dependence of total output on the number of contestants. Given a partition in status classes, adding a new element to an arbitrary class may, in fact, reduce output. But, we show that the adoption of a policy that resembles "hiring at the lowest level" (see Baker, Gibbs, and Holmstrom, 1994) always makes an increase in the number of (ex-ante symmetric) contestants beneficial to the principal.

Finally, we introduce monetary prizes and consider status purely induced by these prizes[6]. In order to add realism, we assume that the designer is budget constrained, and that agents choose not to compete if the monetary prize is not enough to compensate them for a potential low status. In this framework, we show that the optimal structure is to have exactly two status classes: the top class consisting of the single most productive agent, while the lower class containing all other agents that get paid just enough to keep them in the contest.

Since, as illustrated above, there are many real-life examples where status classes proliferate, our results suggest that in those situations status cannot be solely and entirely induced by monetary wealth (on this topic see also Frank, 1999).

Technically, our results are obtained by combining insights derived from the general analysis of contests with multiple prizes developed by Moldovanu and Sela (2001, 2005) with powerful statistical results about stochastic dominance properties of *normalized spacings* and other functions of order statistics (Barlow and Proschan, 1966). For a large and interesting class of distribution functions it is possible to say, for example, whether normalized spacings (i.e., differences) become stochastically more (less) compressed when we climb higher in the ability range, and we show that

[6]See Bagwell and Bernheim, 1996 for a model where conspicuous consumption is used to signal status derived from wealth.

such features determine the structure of the optimal partition in status classes. The application of these statistical results to contests (or auctions) is, to the best of our knowledge, a novel enterprise.

Conceptually, the paper most closely related to ours is Dubey and Geanakoplos (2004) . These authors study optimal grading of exams in situations where students care about relative ranking. Their main finding is that status-conscious students may be better motivated to work hard by a professor who uses coarse grading (e.g., A,B,C,D rather than 100, 99,...). We have borrowed from that paper the present specification of utility functions in the pure status case. Our determination of status categories based on relative effort rank corresponds to what Dubey and Geanakoplos call in their respective context "grading on a curve". But, there are many substantial differences between their model, technique and results and ours. In particular, for their main result, Dubey and Geanakoplos assume that there is complete information, that students have discrete types, that effort choice is binary, and that the relation between effort and output is stochastic.

Many authors put "status" directly into the utility function. Fershtman and Weiss (1993) construct a simple general equilibrium model where both status and wealth are determined endogenously. Becker, Murphy and Werning (2005) consider a model where status is bought in a market (they assume that there are at least as many status class as individuals), and where status is a complement to other consumption goods. Hopkins and Kornienko (2004) study the effect of an exogenous change of income distribution on conspicuous consumption and social welfare in a model where agents care about their rank in the distribution of consumption. Harbaugh and Kornienko (2001) draw a parallel between the predictions of a decision model that assumes a concern for local status and those of prospect theory.

Postlewaite (1998) presents an excellent discussion on the advantages and disadvantage of the "direct" modeling approach versus the one where a concern for relative ranking is only implicit, or "instrumental" for other goals that are made explicit (see also Cole et al., 1992). In a nutshell, Postelwaite's argument against a direct approach is that, by adjusting utility functions at will, one can explain every phenomenon. For our purposes, the debate about the right way to model status concerns is only of secondary importance. Our main focus is on the optimal design of status classes (from an incentive point of view) given that agents care, for some direct or instrumental reason, about relative position. We view the assumed utility function as a simplification, and we ask the reader to judge the outcome by Hardy's dictum whereby good science must, at least, provide some "decent" distance between assumptions and results.

The rest of the paper is organized as follows: Section 2 presents the contest model with status concerns, and some useful facts about order statistics. In Section 3 we derive results that connect the form of the optimal partition in status categories to various properties of the distribution of ability in the population. Before analyzing the general case, we present a simple illustration where the designer can only determine two status categories (i.e., "pass" and "fail"). The question is whether the number of contestants that "fail" should be more or less than half the total number. For the general case, we first show that, by always adding new entrants to the lowest status category, the designer can ensure that his payoff is monotonically increasing in the number of contestants. Thus, potential contestants need not be excluded from competing. We next show that the top status category in any optimal partition must contain a unique element. For distribution of abilities that have an increasing hazard rate, each status category in an optimal partition will contain a unique element - thus, in this case a proliferation of status classes is optimal. Finally, we present a simple condition, stronger than having a decreasing hazard rate, ensuring that the optimal partition involves some coarseness. In Section 4 we modify the model to allow for status categories that are endogenously determined by monetary prizes of different sizes. If status is solely derived from monetary rewards, we show that the optimal partition contains only two categories, with the top category being a singleton. Section 5 concludes. Several proofs are relegated to an Appendix.

2 The Model

We consider a contest with n players where each player j makes an effort x_j. For simplicity, we postulate a deterministic relation between effort and output, and assume these to be equal. Efforts are submitted simultaneously. An effort x_j causes a cost denoted by x_j/c_j, where $c_j > 0$ is an ability parameter. The ability (or *type*) of contestant j is private information to j. Abilities are drawn independently of each other from the interval $[0, 1]$ according to a distribution function F that is common knowledge. We assume that F has a continuous density $f = dF > 0$.

Contestants are ranked according to efforts. Let $\{(0, r_1], (r_1, r_2], ...(r_{i-1}, r_i], ..., (r_{k-1}, n]\ \}$ be a partition of the integers in the interval $(0, n]$ in $k \geq 1$ status categories. Define also for convenience: $r_0 \equiv 0$ and $r_k \equiv n$. Given such a partition and the ordered list of efforts, contestants are divided into the k categories: a player is included in category i, if his effort is between the r_{i-1}-th and r_i-th highest ones.

Each player cares about the number of players in categories both below and above him, and we assume that the "pure status" prize of being in status category i is given by

$$v_i = r_{i-1} - (n - r_i).$$

Thus, a contestant is happier when he has more [less] people below [above] him. Note this formulation well captures the *zero-sum* nature of status: for any partition in status categories, the total value derived from status is given by :

$$\begin{aligned}
\sum_{i=1}^{k}(r_i - r_{i-1})v_i &= \sum_{i=1}^{k}(r_i - r_{i-1})(r_i + r_{i-1} - n) \\
&= \sum_{i=1}^{k}[(r_i)^2 - (r_{i-1})^2] - n\sum_{i=1}^{k}(r_i - r_{i-1}) \\
&= n^2 - n^2 = 0
\end{aligned}$$

We assume that each player maximizes the value of the expected status prize minus the expected effort cost, and that the designer maximizes the value of expected total effort by adjusting the partition in status classes.

2.1 Order Statistics

We use the following notation: 1) $C_{k,n}$ denotes k-th order statistic out of n independent variables independently distributed according to F (note that $C_{n,n}$ is the highest order statistic, and so on..); 2) $F_{k,n}$ denotes the distribution of $C_{k,n}$, and $f_{k,n}$ denotes its density; 3) $E(k,n)$ denotes the expected value of $C_{k,n}$.

It is well-known that:

$$\begin{aligned}
F_{k,n}(s) &= \sum_{j=k}^{n}\binom{n}{j}F(s)^j[1 - F(s)]^{n-j} \\
f_{k,n}(s) &= \frac{n!}{(k-1)!(n-k)!}F(s)^{k-1}[1 - F(s)]^{n-k}f(s)
\end{aligned}$$

Let $F_i^n(s)$, $i = 1, 2, ...n$ denote the probability that a player's type s ranks exactly *i-th* highest among n random variables distributed according to F. Then

$$F_i^n(s) = \frac{(n-1)!}{(i-1)!(n-i)!}[F(s)]^{i-1}[1 - F(s)]^{n-i}$$

Defining $F_{n,n-1} \equiv 0$, and $F_{0,n-1} \equiv 1$, it is immediate that the relation between $F_{i,n}(s)$ and $F_i^n(s)$ is

$$F_i^n(s) = F_{i-1,n-1}(s) - F_{i,n-1}(s)$$

Finally, let $P_i(s)$ be the probability of a player with type s being ranked in category i, i.e., her type is between the r_i-th and $r_{i-1}-$th highest. Then :

$$P_i(s) = \sum_{j=1}^{r_i - r_{i-1}} F^n_{r_{i-1}+j}(s) = F_{r_{i-1},n-1}(s) - F_{r_i,n-1}(s)$$

3 The Optimal Partition in Status Categories

This section contains our main results about the structure of the optimal partition in status categories. We start our analysis with an illustration for the special case where the designer is constrained to partition the contestants into two categories:

3.1 How Many Students Should Fail an Exam ?

We consider here a class with a fixed number n of students where the examiner assigns either *pass* or *fail* to each student according to his performance[7]. High performers *pass* and low performers *fail*. If $(n-r)$ students *pass* while r students *fail*, then the payoff of to a student i is specified as:

$$v = \begin{cases} r & \text{if the student } \textit{passes} \\ -(n-r) & \text{if the student } \textit{fails} \end{cases}$$

A student receives a *pass* if he ranks among the $(n-r)$ best, and a *fail* otherwise. We focus on the symmetric equilibrium: assuming that all students use the same, strictly monotonic equilibrium effort function , a student i's maximization problem becomes:

$$\begin{aligned} & \max_s \{ r P_2(s) - (n-r) P_1(s) - \frac{\beta(s)}{c} \} \\ \Leftrightarrow \quad & \max_s \{ r F_{r,n-1}(s) - (n-r)[1 - F_{r,n-1}(s)] - \frac{\beta(s)}{c} \} \end{aligned} \tag{1}$$

where β denotes the equilibrium effort function.

The solution of the differential equation arising from (1) is[8]:

$$\beta(c) = n \int_0^c x f_{r,n-1}(x) dx$$

It can be then shown (see Appendix) that the total expected effort is given by

[7] Another good example is a contest where some players are eliminated, while others advance to the next stage on equal footing.

[8] See Moldovanu and Sela (2005) for details.

$$E_{total}^{(2)} = n(n-r)E(r,n)$$

The intuition for the above expression is simple: this is a contest with $(n-r)$ equal prizes (for all those who pass), and each prize is worth here n (the difference in value between *pass* and *fail*).

How many students should fail in order to best motivate status conscious students to work hard? While the exact answer depends on specific properties of the distribution of abilities (and thus may be hard to fine-tune for each application), our first result identifies a robust general property. The proof uses the following result:

Lemma 1 *(Barlow and Proschan, 1966) Assume that a distribution F with $F(0) = 0$ is convex (concave) . Then $E(i,n)/i$ is decreasing (increasing) in i for a fixed n.*

Proposition 1 *Let r^* be the division point defining the optimal partition in two status categories, i.e. the optimal number of students who should fail. If the distribution of abilities F is convex (concave) then $r^* \leq (\geq)\ n/2$.*

Proof. Suppose that r^* is the optimal division point. Then, total effort in the optimal partition is higher than in any other partition. In particular, it is higher than total effort in the partition where $r = n - r^*$. This yields:

$$\begin{aligned}
n\,(n-r^*)\,E\,(r^*,n) &\geq n\,[n-(n-r^*)]\,E\,(n-r^*,n) \Leftrightarrow \\
(n-r^*)\,E\,(r^*,n) &\geq r^*E\,(n-r^*,n) \Leftrightarrow \\
\frac{E\,(r^*,n)}{r^*} &\geq \frac{E\,(n-r^*,n)}{n-r^*}
\end{aligned}$$

By Barlow and Prochan's above result, we obtain that , for convex F, the last inequality above can hold only if $r^* \leq (n-r^*)$ which is equivalent to $r^* \leq n/2$. Analogously, if $F\,(x)$ is concave, it must be the case that $r^* \geq (n-r^*)$, yielding $r^* \geq n/2$. Q.E.D. ■

A simple corollary is, of course, that exactly half of the students should pass (fail) if abilities are uniformly distributed.

3.2 The General Case

We now come back to the general case with $k \geq 2$ status categories. Let the partition be defined by a family of division points $\{\ r_i\}_{i=0}^k$ where $r_0 = 0$ and $r_k = n$. Solving the contestants' maximization problem involves now considering different status prizes and different probabilities to obtain them. The calculation of equilibrium effort functions and total expected effort yields:

Theorem 1 *Assume that contestants are partitioned in k status categories according to the family $\{ r_i\}_{i=0}^{k}$. Then, total expected effort in a symmetric equilibrium is given by*

$$E_{total}^{(k)} = \sum_{i=1}^{k-1}(r_{i+1} - r_{i-1})(n - r_i)E(r_i, n)$$

Proof. See Appendix. ■

For the special case $k = 2$, note that the above formula yields

$$E_{total}^{(2)} = n(n - r_1)E\left(r_1, n\right)$$

confirming the observation in the previous section.

Given the above result, we can now formulate the designer's problem: she needs to determine the number of contestants and status categories, and the size of each category. Explicitly, we obtain the following discrete optimization problem:

$$\max_{m,k,\{r_i\}_{i=0}^{k}} [\sum_{i=1}^{k-1}(r_{i+1} - r_{i-1})(m - r_i)E(r_i, m)]$$

$$\begin{aligned} \text{subject to} \quad &: \\ \text{i) } 2 &\leq m \leq n \\ \text{ii) } 2 &\leq k \leq m \\ \text{iii) } 0 &= r_0 \leq r_1 .. \leq r_{k-1} \leq r_k = m \end{aligned}$$

3.2.1 The Optimal Number of Contestants

We first determine the optimal number of contestants by analyzing the effect of changing the number of contestants (i.e., by entry or hiring) on total expected effort. Given the zero-sum nature of status, the answer is not clear-cut, and it depends on the designer's reaction to entry (i.e., on how the size and number of status categories change.) The next example illustrates the possibility that a wrong post-entry adjustment policy may cause total effort to actually go down:

Example 1 *Let $F(x) = x^{\frac{1}{a}}$, $a > 1$, and consider partitions with two categories. Total effort is given by*

$$n(n - r)E\left(r, n\right) = n(n - r)\frac{n!\left(a + r - 1\right)!}{\left(r - 1\right)!\left(n + a\right)!}$$

where r is the division point. If we add an additional contestant to the higher category (that is, we do not change the value of r), we obtain for a high enough:

$$\begin{aligned} E_{n+1}-E_n &= (n+1)(n+1-r)\frac{(n+1)!\,(a+r-1)!}{(r-1)!\,(n+1+a)!}-n(n-r)\frac{n!\,(a+r-1)!}{(r-1)!\,(n+a)!} \\ &= \frac{(a+r-1)!n!}{(r-1)!\,(n+a)!}[\frac{(n+1)^2(n+1-r)}{(n+1+a)}-n(n-r)]<0 \end{aligned}$$

That is, for sufficiently high a, total effort decreases in the number of players.

Our main result in this section is that a designer who optimally reacts to additional entry can always ensure that total effort increases. In particular, in the proof, we identify a very simple strategy (without the need of a complex re-optimization!) ensuring that total effort does not decrease: faced with more contestants, the designer can just increase the size of the lowest status category.

Theorem 2 *Total effort in an optimal partition increases in the number of contestants.*

Proof. See Appendix. ■

3.2.2 The Optimal Partition into Status Categories

Given the above result, the designer has no incentives to restrict entry in the contest, and we thus assume below that all n potential contestants are included[9].

The optimal number of status categories and the optimal size of each category generally depend on the distribution of the players' abilities (since the distribution determines the expected values of the various order statistics appearing in the maximization problem). Our first result in this section identifies a robust and general feature for any distribution:

Theorem 3 *In any optimal partition, the top status category contains an unique element.*

Proof. Suppose, by contradiction, that the k-th (top) category contains more than one element. Then, divide this category into two sub-categories, and denote by r_d the dividing point: $r_{k-1} < r_d < n$. Using the formula in Theorem 1, the difference in expected effort between the new and the

[9]See the next section were this result need not hold if the designer is budget constrained and if agents must be monetarily compensated for low status.

old partitions is given by:

$$
\begin{aligned}
E^{(k+1)}_{total} - E^{(k)}_{total} &= (n - r_{k-1})(n - r_d)E(r_d, n) - (n - r_{k-1})(n - r_d)E(r_{k-1}, n) \\
&= (n - r_{k-1})(n - r_d) \left[E(r_d, n) - E(r_{k-1}, n)\right] > 0
\end{aligned}
$$

The inequality follows since $C_{r_d,n}$ stochastically dominates $C_{r_{k-1},n}$. ■

To understand the intuition behind the result, start with a partition such that the top category contains an unique element, and consider adding one more element to the top class. The main effect of such a modification is that, while the sum of the status prizes in the top category increases from $(n-1)$ to $2(n-2)$, the competition for top status prizes diminishes since there is now more supply of them[10]. At the top, the competitive effect among the highly skilled agents is strongest, and such a change is never beneficial, even if it increases the total award to the top individuals.

Similar effects occur of course for analogous changes to other categories, but the competitive effect farther away from the top is not as strong anymore, and, as we shall see below, some additional condition is needed to ensure that all categories in the optimal partition will be singletons. In such situations, there will be a proliferation of status categories: according to effort, each individual will be in category of his/her own !

In order to get some intuition, let us start with a simple observation: If the lowest category of an optimal partition contains more than one element (i.e., $r_1 > 1$), we can divide it into two sub-categories, and denote by r_d the dividing point, $1 \leq r_d < r_1$. The comparison of total effort between the new partition (with one more category) and the initial partition yields:

$$
\begin{aligned}
E^{(k+1)}_{total} - E^{(k)}_{total} &= r_1(n - r_d)E(r_d, n) - r_d(n - r_1)E(r_1, n) > 0 \Leftrightarrow \\
\frac{E(r_d, n)}{r_d} &> \frac{(n - r_1)}{(n - r_d)} \frac{E(r_1, n)}{r_1}
\end{aligned}
$$

Recall that $E(r_i, n)/r_i$ is decreasing in r_i if the distribution of abilities F is convex. Since $\frac{(n-r_1)}{(n-r_d)} < 1$, the assumption $1 \leq r_d < r_1$ yields then for a convex F a contradiction to the assumed optimality of the initial partition. Thus, we have shown that the lowest status category must contain a unique element if F is convex.

Our next result will significantly weaken the requirement on F, and will extend the above logic to all categories. It's proof uses a non-trivial statistic result about stochastic dominance relations among *normalized spacings* of order statistics.

[10] There is also a small negative effect on the welfare of the agents in the second highest category that get two agents above them instead of one.

We first need to remind the reader some well-known concepts: The failure rate (or hazard rate) of a distribution F is defined by:

$$\lambda(c) = \frac{f(c)}{1 - F(c)}$$

A distribution function F has *increasing failure rate* (IFR) if $\lambda(c)$ is increasing or, equivalently, if $\log(1 - F(c))$ is concave. Analogously, F has *decreasing failure rate* (DFR) if $\lambda(c)$ is decreasing, or, equivalently, if $\log(1 - F(c))$ is convex. Many well known distributions belong to these important and much studied categories. The relationships between IFR, DFR, convexity and concavity of F are as follows: Convexity implies IFR , while DFR implies concavity. The only distribution that is both concave and convex is the uniform, while the only distribution that is both IFR and DFR is the exponential.

Armed with these concepts, we can now state:

Lemma 2 *(Barlow and Proschan, 1966)*[11] *Assume that a distribution F with $F(0) = 0$ satisfies IFR (DFR). Then, $(n-i+1)(C_{i,n} - C_{i-i,n})$ is stochastically decreasing (increasing) in i for a fixed n.*

An application of this result yields:

Theorem 4 *Assume that F, the distribution of abilities, has increasing failure rate. Then, the optimal partition is the finest possible one: each status category contains an unique element.*

Proof. See Appendix. ■

Remark 1 *In the IFR case, total effort in the optimal partition with $n-1$ classes is given by :*

$$E_{total}^{(n-1)} = 2\sum_{i=1}^{n-1}(n-i)E(i,n)]$$

Recall that total effort in the optimal partition with two classes is given by

$$E_{total}^{2} = n(n-i^*)E(i^*,n)$$

where $i^* \in \arg\max_i[n(n-i^*)E(i^*,n)]$. We immediately obtain then that:

$$E_{total}^{2} > \frac{1}{2}E_{total}^{(n-1)}$$

[11]See also Boland et.al (2002) for recent developments in the area.

The approximation is very rough, and the coarse partition with only two classes yields for ,,well-behaved" distributions an even higher percentage of the optimal performance. For example, $E^2_{total} \geq \frac{3}{4}E^{(n-1)}_{total}$ *for the uniform distribution. Thus, if very fine partitions are, for some reason, costly (e.g., think about finely grading an exam versus awarding just "pass" and "fail" grades), one can achieve a substantial share of the optimal performance with a simple partition in two categories*[12]*.*

The next example shows that a coarse partition may be optimal if the IFR condition is not satisfied.

Example 2 *Let* $F(x) = x^{1/a}$, $a > 1$, *and note that* F *is concave. Since we know that the top category of an optimal partition always contains an unique element, the quest for optimality reduces for the case* $n = 3$ *to a comparison between the finest partition with three categories* ($r_i = i$, $i = 0, 1, 2, 3$) *and a partition with only two categories* ($r_0 = 0$, $r_1 = 2$, $r_2 = 3$) *. The expectations of the order statistics are given by:*

$$E(r, n) = \frac{n!\,(a + r - 1)!}{(r - 1)!\,(n + a)!}$$

Thus, the coarser partition is optimal if

$$\begin{aligned} E^{(2)}_{total} - E^{(3)}_{total} &= E(2, 3) - 4E(1, 3) > 0 \Leftrightarrow \\ \frac{6\,(a + 1)!}{(a + 3)!} &> 4\frac{6a!}{(a + 3)!} \Leftrightarrow a > 3. \end{aligned}$$

Suppose now that $n > 3$, *and suppose that the lowest category in an arbitrary partition with* k *categories (defined by a family* $\{r_i\}^k_{i=0}$) *contains more than one element. Dividing this category in two sub-categories with division point* r_d, $1 < r_d < r_1$, *yields:*

$$\begin{aligned} E^{(k)}_{total} - E^{(k+1)}_{total} &= r_d(n - r_1)E(r_1, n) - r_1(n - r_d)E(r_d, n) > 0 \Leftrightarrow \\ \frac{E(r_1, n)}{E(r_d, n)} &> \frac{r_1(n - r_d)}{r_d(n - r_1)} \end{aligned}$$

For any distribution G, *when* n *is large,* $E(r, n)$ *is approximated*[13] *by* $G^{-1}(r/(n + 1))$ *Using this approximation, or the explicit formulae of* $E(r, n)$ *for* $F = x^{1/a}$ *(see above), we obtain that, in this case,* $E(r, n) \approx [r/(n + 1)]^a$ *. This yields:*

$$\lim_{n\to\infty} \frac{E(r_1, n)}{E(r_d, n)} = (\frac{r_1}{r_d})^a > \lim_{n\to\infty} \frac{r_1(n - r_d)}{r_d(n - r_1)} = \frac{r_1}{r_d}$$

Thus, for large n *, it is not optimal to divide the lowest category, and this category will contain at least* r_1 *elements.*

[12] This is reminiscent of the "coarse matching" analyzed by McAfee (2002).

[13] See David and Nagaraja (2003).

The observations in the above example can be generalized to a simple condition of "sufficient concavity" ensuring that the optimal partition must involve some coarseness.

Proposition 2 *Assume that the number of contestants $n > 2$ and the distribution of abilities F are such that*[14] $\frac{E(2,n)}{E(1,n)} > 2 + \frac{2}{n-2}$*. Then the finest partition cannot be optimal.*

Proof. Consider the finest partition with n status categories. Remove then the lowest division point, so that the new partition with only $n-1$ categories contains two elements in the lowest category. The change in total effort is given by

$$E_{total}^{(n-1)} - E_{total}^{(n)} = (n-2)E\left(2,n\right) - 2(n-1)E(1,n)$$

Thus, the coarser partition dominates the finest partition if:

$$E_{total}^{(n-1)} - E_{total}^{(n)} > 0 \Leftrightarrow \frac{E(2,n)}{E(1,n)} > 2 + \frac{2}{n-2}$$

Q.E.D. ■

In our example with $F\left(x\right) = x^{1/a}$ and $n = 3$, the above condition reduces to $E\left(2,3\right) > 4E\left(1,3\right)$, which is sufficient for the non-optimality of the finest partition.

4 Status Derived from Monetary Prizes

Until now we focused on the pure effect of status in contests: there were no other real prizes to drive efforts. We now consider contests where status is being indirectly (and solely) induced by the award of monetary prizes that differ in magnitude. In particular, we depart from the zero-sum world presented above.

Consider a partition with k categories determined by a family of division points $\{r_i\}_{i=0}^{k}$ where $r_0 = 0$ and $r_k = n$. Assume that a contestant ranked in the top category k (i.e., a contestant whose effort is among the top $r_k - r_{k-1}$) receives a prize of V_k , a contestant in the second highest category receives a prize of $V_{k-1} \leq V_k$, and so on till the lowest $V_1 \leq V_2 \leq ... \leq V_k$.

A player who is awarded the i-th highest monetary prize V_i perceives in fact a total prize (money + status) of :

$$v_i = V_i + r_{i-1} - (n - r_i).$$

[14] Recall that $E\left(i,n\right)/i$ is increasing in i if F is concave. Thus, for any concave distribution F we get: $\frac{E(2,n)}{E(1,n)} \geq 2$. For any F with DFR (which is a stronger than concavity), $(n-i+1)(C_{i,n} - C_{i-1,n})$ is stochastically increasing in i . In particular, this yields $\frac{E(2,n)}{E(1,n)} \geq 2 + \frac{1}{n-1}$.

In order to make the problem non-trivial, we add here two realistic assumptions: 1) The contest designer is financially constrained: the total amount of monetary prizes cannot exceed a given amount P. Otherwise, it is obvious that large enough monetary prizes can always swamp any status effects. 2) We impose individual rationality in the sense that, in any status category, the perceived total prize should be non-negative. Otherwise, contestants expecting to fall in low pay/low status categories will leave without competing.

By calculations similar to those performed for the case of pure status concerns, total effort in a symmetric equilibrium is given by

$$E_{total}^{(k)} = \sum_{i=1}^{k-1} (n - r_i)(r_{i+1} - r_{i-1})E(r_i, n) + \sum_{i=1}^{k-1} (n - r_i)E(r_i, n)(V_{i+1} - V_i)$$

Therefore, the designer's problem is as follows:

$$\begin{aligned} \max_{k,\{r_i\}_{i=1}^k,\{V_i\}_{i=1}^k} E_{total}^{(k)} &= \sum_{i=1}^{k-1} (n - r_i)(r_{i+1} - r_{i-1})E(r_i, n) + \sum_{i=1}^{k-1} (n - r_i)E(r_i, n)(V_{i+1} - V_i) \\ \text{subject to} &: \ 1) \ \ 1 \le k \le n \\ &: \ 2) \ \ \sum_{i=1}^{k} (r_i - r_{i-1})V_i = P \\ &: \ 3) \ \ V_i \ge (n - r_i) - r_{i-1}, \ \ i = 1, 2, .., k \\ &: \ 4) \ \ V_k \ge V_{k-1} \ge ... \ge V_1 \end{aligned}$$

Theorem 5 *If $P > n$, (i.e., if the available budget is as least as large as the number of contestants), the optimal solution to the designer's problem has the following structure: The designer induces a partition with two status categories such that the contestant with the highest effort receives a monetary prize $V_2 = P - (n - 1)$,while all other contestants receive a monetary prize $V_1 = 1$. If $P \le n$, it is optimal to restrict entry to the contest until the condition above holds.*

Proof. See Appendix. ■

The above result is reminiscent of the optimality of a unique "first" prize in Moldovanu and Sela's (2001) contest model, and contrasts the results we obtained for pure status prizes where the structure of the optimal partition depended on the form of the distribution of abilities. The point is that, with pure status prizes, some of the prizes are negative (in such an environment the Moldovanu and Sela (2001) result does not hold), whereas here all prizes are positive by the individual rationality constraint. Since status is purely driven by monetary prizes, the optimality of a unique first prize naturally translates here into a partition in two status classes, with a singleton in the top category.

5 Conclusion

We have studied a contest model where heterogeneous agents who care about relative standing are ranked according to output, and are then partitioned into status categories. Our main results describe the structure of the optimal partition into status classes from the point of view of a designer that maximizes total output. The model explains ubiquitous phenomena such as top status classes that contain a unique individual, and the proliferation of status classes in organizations where high-skilled individuals are not rare. We also studied the interplay between pure status and monetary prizes.

As already mentioned in the introduction, in most real-life situations status is only partly determined by measurable differences in monetary compensation. Social, cultural and other economic considerations that may be connected to a concern for relative position in a future interaction are also important determinants. Modeling a specific situation requires a simple combination of the two variants displayed here, and the corresponding results will be driven by the relative strengths of the monetary versus the less tangible parts.

Finally, note that, in principle, a analysis analogous to ours is possible for other agents' utility functions, or other designer's goals. In particular, for given, fixed utility functions, the equilibrium analysis is not affected by the designer goal, and this can be modified to conform the requirements of various applications.

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7 Appendix

Proof of Theorem 1:

Proof. Let a partition with k categories be given by $\{\ (0, r_1], (r_1, r_2], ...(r_{i-1}, r_i], ..., (r_{k-1}, n]\ \}$. Assuming a symmetric equilibrium in strictly increasing strategies[15], the optimization problem of

[15]It can be shown that there is a unique symmetric equilibrium.

a player with ability c is

$$\max_s \left\{ \begin{array}{c} [1-F_{r_1,n-1}(s)][-(n-r_1)] \\ +\sum_{i=2}^{k-1}\left[F_{r_{i-1},n-1}(s)-F_{r_i,n-1}(s)\right][r_{i-1}-(n-r_i)] \\ +F_{r_{k-1},n-1}(s)r_{k-1}-\frac{\beta(s)}{c} \end{array} \right\}$$

where the first term is the utility of being in the lowest category, the second term is the utility of being in categories 2 till $(k-1)$, and the third term is the utility of being in the highest category.

The solution of the resulting differential equation with boundary condition $\beta(0)=0$ is

$$\beta(c)=\int_0^c x\left\{f_{r_1,n-1}(x)(n-r_1)+\sum_{i=2}^{k-1}\left[f_{r_{i-1},n-1}(x)-f_{r_i,n-1}(x)\right](r_{i-1}+r_i-n)+f_{r_{k-1},n-1}(x)r_{k-1}\right\}dx \tag{2}$$

Thus, total effort is given by:

$$E_{total}=n\int_0^1 \beta(c)f(c)dc \tag{3}$$

The above integral can be calculated by inserting formula 2 in 3 and by integrating by parts the constituent terms, who all have the form $b\int_0^1\left[\int_0^c xf_{r,n-1}(x)dx\right]f(c)dc$ where b is a constant. Note that :

$$\begin{aligned} & \int_0^1\left[\int_0^c xf_{r,n-1}(x)dx\right]f(c)dc \\ = & \left[F(c)\int_0^c xf_{r,n-1}(x)dx\right]_0^1-\int_0^1 F(c)cf_{r,n-1}(c)dc \\ = & \int_0^1 cf_{r,n-1}(c)dc-\int_0^1 cF(c)f_{r,n-1}(c)dc \\ = & \int_0^1 c\left[1-F(c)\right]f_{r,n-1}(c)dc \\ = & E(r,n-1)-\frac{r}{n}E(r+1,n) \\ = & \frac{n-r}{r}E(r,n) \end{aligned}$$

The last equality follows by a well known identity among order statistics (see David and Nagaraja, 2003, page 44). Assembling all terms in equation 3, and recalling that $r_0=0$, and $r_k=n$ finally yields:

$$\begin{aligned} E_{total}^{(k)} &= \left\{ \begin{array}{c} (n-r_1)^2E(r_1,n) \\ +\sum_{i=2}^{k-1}(r_{i-1}+r_i-n)\left[(n-r_{i-1})E(r_{i-1},n)-(n-r_i)E(r_i,n)\right] \\ +r_{k-1}(n-r_{k-1})E(r_{k-1},n) \end{array} \right\} \\ &= \sum_{i=1}^{k-1}(r_{i+1}-r_{i-1})(n-r_i)E(r_i,n) \end{aligned}$$

Q.E.D. ■

Proof of Theorem 2:

Proof. Consider a partition $\{r_i\}_{i=0}^{k}$ for a given number of contestants m. Total effort is given by

$$\begin{aligned} E_{total} &= \sum_{i=1}^{k-1}(r_{i+1}-r_{i-1})(m-r_i)E(r_i,m) \\ &= r_2(m-r_1)E(r_1,m)+\sum_{i=2}^{k-1}(r_{i+1}-r_{i-1})(m-r_i)E(r_i,m) \end{aligned}$$

Assume now that a designer faced with $m+1$ contestants expands by one the size of the lowest status category: thus, consider the new partition $\{r_i'\}_{i=0}^{k}$ where $r_0'=0, r_1'=r_1+1, r_2'=r_2+1, ..., r_{k-1}'=r_{k-1}+1, r_k'=m+1$.

Total effort for this new partition is given by

$$\begin{aligned} E_{total}^{'} &= \sum_{i=1}^{k-1}(r_{i+1}'-r_{i-1}')(m+1-r_i')E(r_i',m+1) \\ &= (r_2+1)(m-r_1)E(r_1+1,m+1)+\sum_{i=2}^{k-1}(r_{i+1}-r_{i-1})(m-r_i)E(r_i+1,m+1) \end{aligned}$$

We obtain:

$$\begin{aligned} & E_{total}^{'}-E_{total} \\ = \ & (m-r_1)E(r_1+1,m+1)+\sum_{i=1}^{k-1}(r_{i+1}-r_{i-1})(m-r_i)[E(r_i+1,m+1)-E(r_i,m)]\geq 0 \end{aligned}$$

The last inequality holds since, for all i,m, $C_{i+1,m+1}$ stochastically dominates $C_{i,m}$[16]. The claim follows now by starting from an optimal partition for m contestants, and expanding the size of the lowest category as above. Further eventual optimization of the partition for $m+1$ contestants must weakly increase the total effort even further, thus yielding the wished result. Q.E.D. ■

Proof of Theorem 4:

Proof. Suppose that, in an optimal partition with k categories, the j-th $(1\leq j\leq k)$ category contains more than one element. Divide the j-th category into two sub-categories and denote by r_d the dividing point, $r_{j-1}<r_d<r_j$. Letting $E(0,n)\equiv 0$, the difference in total effort between the

[16] This result is not completely trivial. For example, note that $C_{i,m+1}$ is actually stochastically dominated by $C_{i,m}$. See Shaked and Shanthikumar (1994) for more details.

new and the initial partition is given by:

$$E^{(k+1)}_{total} - E^{(k)}_{total} = \left\{ \begin{array}{c} (r_j - r_{j-1})(n - r_d)E(r_d, n) \\ -(r_j - r_d)(n - r_{j-1})E\left(r_{j-1}, n\right) \\ -(r_d - r_{j-1})(n - r_j)E(r_j, n) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} (r_j - r_d)\left[(n - r_d)\,E\left(r_d, n\right) - (n - r_{j-1})\,E\left(r_{j-1}, n\right)\right] \\ -\left(r_d - r_{j-1}\right)\left[(n - r_j)\,E\left(r_j, n\right) - (n - r_d)\,E\left(r_d, n\right)\right] \end{array} \right\}$$

Let $t = r_j - r_{j-1}$, $r_d = r_{j-1} + 1$. Then,

$$\begin{aligned} & E^{(k+1)}_{total} - E^{(k)}_{total} \\ = & \left\{ \begin{array}{c} (t-1)\left[(n - r_d)\,E\left(r_d, n\right) - (n - (r_d - 1))\,E\left(r_{j-1}, n\right)\right] \\ -\left[(n - (r_d + t - 1))\,E\left(r_d + t - 1, n\right) - (n - r_d)\,E\left(r_d, n\right)\right] \end{array} \right\} \\ = & \left\{ \begin{array}{c} (t-1)\left[(n - r_d)\,E\left(r_d, n\right) - (n - (r_d - 1))\,E\left(r_d - 1, n\right)\right] \\ -\left[(n - (r_d + t - 1))\,E\left(r_d + t - 1, n\right) - (n - (r_d + t - 2))\,E\left(r_d + t - 2, n\right)\right] \\ -\left[(n - (r_d + t - 2))\,E\left(r_d + t - 2, n\right) - (n - (r_d + t - 3))\,E\left(r_d + t - 3, n\right)\right] \\ -.... \\ -\left[(n - (r_d + 1))\,E\left(r_d + 1, n\right) - (n - r_d)\,E\left(r_d, n\right)\right] \end{array} \right\} \end{aligned}$$

Note that

$$\begin{aligned} & (n - r)\,E\,(r, n) - (n - (r - 1))\,E\,(r - 1, n) \\ = \; & (n - r + 1)\left[E\,(r, n) - E\,(r - 1, n)\right] - E\,(r, n) \end{aligned}$$

By Barlow and Proschan's Lemma about IFR distributions, and by the fact that $-E\,(r, n)$ is decreasing in r, it immediately follows that $[(n - r)\,E\,(r, n) - (n - (r - 1))\,E\,(r - 1, n)]$ is decreasing in r. Therefore $E^{(k+1)}_{total} - E^{(k)}_{total} \geq 0$. This contradicts the assumption that the initial partition was optimal. Therefore, each category in the optimal partition must contain a unique element. Q.E.D. ■

Proof of Theorem 5

Proof. The designer's problem is:

$$\begin{aligned}
\max_{k,\{r_i\}_{i=1}^k,\{V_i\}_{i=1}^k} E_{total}^{(k)} &= \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n)+\sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) \\
\text{subject to} &: \ 1) \ 1 \leq k \leq n \\
&: \ 2) \ \sum_{i=1}^{k}(r_i-r_{i-1})V_i = P \\
&: \ 3) \ V_i \geq (n-r_i)-r_{i-1}, \ i=1,2,..,k \\
&: \ 4) \ V_k \geq V_{k-1} \geq .... \geq V_1
\end{aligned}$$

Assume first that a given partition with k status categories is fixed. We derive the optimal allocation of money prizes consistent with such a partition. Subsequently, we find the optimal partition.

Note that $\frac{dE_{total}^{(k)}}{dV_1} < 0$, and therefore $V_1 = n-r_1$. Since $n-r_1 > (n-r_{i+1})-r_i$ for all $k-1 \geq i \geq 1$ the maximization problem reduces to:

$$\begin{aligned}
&\max_{\{V_i\}_{i=1}^k} \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n)+\sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) \\
&\text{subject to: } \sum_{i=1}^{k}(r_i-r_{i-1})V_i = P \\
: \quad & V_k \geq V_{k-1} \geq ....V_1 = n-r_1
\end{aligned}$$

Assuming that all the constraints $V_k \geq .... \geq V_1 = n-r_1$ are binding, the Lagrangian is

$$\begin{aligned}
L &= \sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n)+\sum_{i=1}^{k-1}(n-r_i)E(r_i,n)(V_{i+1}-V_i) - \\
& \alpha_0(\sum_{i=1}^{k}(r_i-r_{i-1})V_i - P)+\sum_{i=1}^{k}\alpha_i(V_i-(n-r_1))
\end{aligned}$$

The first order conditions are

$$\frac{dL}{dV_i} = [(n-r_{i-1})E\left(r_{i-1},n\right)-(n-r_i)E\left(r_i,n\right)]-\alpha_0(r_i-r_{i-1})-\alpha_i = 0, \quad i=1,....,k$$

The solution of this problem is:

$$\begin{aligned}
V_{k-1} &= ... = V_1 = (n-r_1); \\
V_k &= \frac{P-r_{k-1}(n-r_1)}{n-r_{k-1}} \\
\alpha_0 &= E(r_{k-1},n); \\
\alpha_i &= [(n-r_{i-1})E\left(r_{i-1},n\right)-(n-r_i)E\left(r_i,n\right)]-\alpha_0(r_i-r_{i-1}), \ i=1,..,k
\end{aligned}$$

Note that :

$$\begin{aligned}\alpha_i &= [(n-r_{i-1})E\left(r_{i-1},n\right)-(n-r_i)E\left(r_i,n\right)]-\alpha_0(r_i-r_{i-1}) \\ &< (r_i-r_{i-1})(E(r_i,n)-E(r_{k-1},n))\leq 0\end{aligned}$$

That is, our assumption that all the constraints $V_{k-1}\geq....\geq V_1=n-r_1$ are binding ($V_k\geq n-r_1$ is not binding) was correct. Now, at the optimal solution, total effort is given by

$$E_{total}^{(k)}=\sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n)+E(r_{k-1},n)(P-n(n-r_1))$$

For a partition with $k=2$ with division point r_1', the above formula yields:

$$E_{total}^{(2)}=PE(r_1',n)$$

which is maximized for $r_1'=n-1$. Noting that $\sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})=n(n-r_1)$, and that for any k, $r_{k-1}\leq n-1$, we obtain that

$$\begin{aligned}&E_{total}^{(2)}-E_{total}^{(k)} \\ =\ & PE(n-1,n)-\left(\sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})E(r_i,n)+E(r_{k-1},n)(P-n(n-r_1))\right) \\ &=P[E(n-1,n)-E(r_{k-1},n)]-\sum_{i=1}^{k-1}(n-r_i)(r_{i+1}-r_{i-1})[E(r_i,n)-E(r_{k-1},n)]\geq 0\end{aligned}$$

Thus, a partition with two status categories where the top category contains a unique element is optimal. Q.E.D. ■