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Collective Production and Incentives*

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Abstract

We analyse incentive problems in collective production environments where contributors are compensated according to their observed and ranked efforts. This provides incentives to the contributors to choose first best efforts.

1 Introduction

We motivate the key aspects of our theoretical analysis with the following real world example of a relative performance compensation scheme. Lavy (2003) reports on an Israeli program where teachers were rewarded with cash bonuses for improvements in their students' performance on the high-school 'bagrut' matriculation exams. The bonus program was structured as a rank order tournament among teachers, in each subject separately. Thus, teachers were rewarded on the basis of their performance relative to other teachers of the same subjects. Relative performance was preferred over measurements based on absolute performance for two reasons: these awards would stay within budget and there were no obvious standards that could be used as a basis for absolute performance measures. The relative measurements were based on comparison of the achievements of each teacher's students with predicted values using regressions. Two measurements of students' achievements were used as indicators of teachers' performance: the passing rate and the average score on each matriculation exam. The total amount to be awarded in each tournament was predetermined and individual awards were determined on the basis of rank and a predetermined award scale.

There are comparable programs, for instance, in the USA: in Dade County, Florida, Denver, Colorado, and Dallas, Texas, in the mid-1990s; statewide programs in Iowa and Arizona in 2002; programs in Cincinnati, Philadelphia, and Coventry (Rhode Island); and the Milken Foundation TAP program. In the UK, the government recently concluded an agreement with

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the main teachers' unions on a new teachers' performance pay scheme to start in 2002/2003, with a budget of nearly £150 million. In New Zealand, the government completed a system wide program of performance-related pay for teachers in 2001. For references and a discussion of these programs see Lavy (2003).

This example, of course, does not fit our partnership and joint production setup well. But with a small modification we can think of this relative performance bonus scheme in our framework: let the budget available for the bonuses to teachers equal the (appropriately scaled) total sum of points scored by the students in the exams

$$\text{bonus budget} = \gamma \sum_{\text{students}} \#\ \text{points at bagrut exam.} \tag{1}$$

With this hypothetical extension, our setup is indeed motivated through the above example.

1.1 Related literature

We already remarked on Lavy (2003) who surveys an extensive experimental literature. Alchian and Demsetz (1972) and Holmström (1982) pose the original problem of unattainability of first best efforts in neoclassical partnerships when output is ex ante non-contractible and shared among partners. Legros and Matthews (1993) show that full efficiency can be obtained in some cases, for example, for partnerships with finite action spaces or with Leontief technologies. Nevertheless, they confirm and generalize Holmström (1982)'s result that full efficiency is unattainable for neoclassical partnerships, ie. the case which we study where the production and utility functions are smooth. They show that approximate efficiency can be achieved by mixed-strategy equilibria, where one partner takes an inefficient action with small probability. However, sustaining such equilibria depends crucially on the partners bearing *full* liability. If the partners are subject to limited liability, the mechanism does not work since it is impossible to impose the large fine on a partner which is necessary to prevent deviation. In our result, full efficiency is attainable even with limited liability, provided that the ranking technology gives a sufficiently high marginal probability of winning for symmetric efforts. Battaglini (2006) discusses the joint production of heterogenous goods. For multi-dimensional output he finds that implementing the efficient allocation is possible whenever the average dimensionality of the agents' strategy spaces is lower than the number of different goods produced. He confirms, however, that efficiency is unattainable in the standard case.

The classic reference on efficiency in tournaments is Lazear and Rosen (1981).[1] They compare rank order wage schemes to wages based on individual output and find that, for risk-neutral agents, both allocate resources efficiently. In contrast to our setup, their fixed prizes—and thus their efficiency result—arise from competitive markets while we do not require any market at all. Therefore our model credibly lends itself to the analysis of incentive problems even in single

[1] Kurshid and Sahai (1993) survey the measurements literature which lends support to the tournaments approach by arguing that ordinal statistics are inherently cheaper to produce than cardinal statistics.

partnerships regardless of the industry market structure.

Moldovanu and Sela (2001) characterize the optimal prize structures in tournaments. They analyze an exogenously given, fixed budget for prizes and show that for convex cost functions, it is optimal to give positive prizes not only to the winner. Our analysis shows that convexity per se does not lead to the multiple prizes in partnerships. In order to get multiple prizes, the ranking technology has a crucial role as well. Cohen, Kaplan, and Sela (2004) also characterize optimal prize structures. However, in contrast to Moldovanu and Sela (2001), they allow an agent's reward to depend on his effort. They do not, however, address the problem of efficiency in partnerships or teams. Galanter and Palay (1991) discuss compensation in elite law firms and argue that promotion-to-partner schemes indeed constitute tournaments. The contests literature has flourished recently and is surveyed by Konrad (2004).

Lazear and Kandel (1992) show that the existence of peer pressure can weaken the free rider problem in teams and partnerships. Their concept of peer pressure among partners captures factors such as guilt, norms, and mutual monitoring which all serve as disciplinary devices. The difference to our approach is that their compensation scheme is not a tournament but consists of constant shares of output. Miller (1997) shows that whenever a single partner can observe and report on at least one other's actions, efficient efforts can be implemented. Strausz (1999) shows that when agents choose their efforts sequentially and observe the actions taken by their predecessors, there exists a sharing rule which implements efficient efforts. This sharing rule induces players to reveal a shirking partner by influencing final output in a particular way.

2 The model

There are two identical, risk neutral agents exerting unobservable individual efforts $e_i \in [\delta, \infty)$, $i \in \{1, 2\}$, for some positive δ.[2] However, some noisy ranking of efforts of the partners is assumed to be observable and verifiable. The technology that translates a partner's effort into his place in the ranking is described, for $x := e_i/e_j$, by[3]

$$\Gamma(e_i, e_j) = [f_i(x), f_j(x)] \tag{2}$$

where $f_k(\cdot)$ is the probability that partner $k \in \{i, j\}$ gets the first place in the ranking given his effort e_i and the rival partner's effort e_j, and $f_i + f_j = 1$. We make the following assumptions on $f(\cdot)$

A1 Symmetry: $f_i(x) = f_j(1/x)$, for $x \in [\delta, \infty)$;

A2 Responsiveness: $\dfrac{df_i(x)}{dx} > 0$, $\dfrac{df_j(x)}{dx} < 0$; $\lim_{x \to \delta} f_i(x) = 0$ and $\lim_{x \to \infty} f_i(x) = 1$;

[2] The natural effort choice set would be $[0, \infty)$ but we avoid zero effort for technical reasons. We generalize our results to more than two players in the appendix but the full intuition can be understood from the two players case.

[3] This class includes the Tullock success function $\frac{e_i^r}{e_i^r + e_j^r}$ where $f_i(x) = \frac{1}{1+x^{-r}}$.

A3 $f(\cdot)$ is twice continuously differentiable.

Assumption **A1** captures the symmetry of the two partners. Assumption **A2** captures the idea that the probability that one partner is ranked first in effort is dependent upon the relative performance of the two partners, measured by $x = e_i/e_j$. In particular, partner i's winning probability is increasing in x, but partner j's winning probability is decreasing in x.

If a partnership is formed, the output of the partnership is a function of the total efforts of the partners. Denote the production function as $y := y(\sum_i e_i)$. The production function is smooth and twice continuously differentiable, with $y(2\delta) = 0$, $y'(\cdot) > 0$ and $y''(\cdot) \leq 0$. A partner who receives a share s of the final output, given his own effort e_i and the other partner's effort e_j, gets utility

$$u_i(e_i, e_j) = sy(e_i + e_j) - C(e_i)$$

where $C : [\delta, \infty) \to \mathbb{R}$ is a cost function with $C(\delta) = 0$, $C'(\cdot) > 0$ and $C''(\cdot) > 0$.

The objective of a partner is to maximize his own expected utility. Our goal is to analyze whether it is possible to induce the players to exert the efficient level of efforts using a rank order compensation scheme.

2.1 Timing

At the first stage, an arbitrary partner initiates the partnership formation by making a proposal to the other partner, offering a sharing rule $(s, 1 - s)$. Without loss of generality let partner 1 be the proposer. Partner 2 then decides whether to accept the proposal or not. If he accepts, the partnership is set up, and the game proceeds to the next stage. If he rejects, the game ends and each player obtains his reservation utility which we normalize to zero. At the second stage, conditional on the formation of the partnership, the partners choose their efforts simultaneously to maximize their own expected utility. Some noisy ranking of efforts is observed and the final output is realized. The final output is distributed between the two partners. The partner who ranks first in efforts obtains the majority share s of the final output, and the other partner gets $1 - s$.

2.2 (In-)Efficiency benchmark

Efficient actions are those which maximize the total welfare of the two partners absent of any incentive aspects

$$\max_{(e_i, e_j)} w(e_i, e_j) := y(e_i + e_j) - C(e_i) - C(e_j).$$

The first best effort level is determined by

$$y'(2e^*) = C'(e^*)$$

where $e_i^* = e_j^* = e^*$. Suppose the two partners fix the shares (s_i, s_j) ex ante, with $s_i + s_j = 1$. As shown by Holmström (1982), there is no sharing rule that achieves full efficiency and satisfies a balanced budget at the same time. Given the sharing rule (s_i, s_j), the partners choose their efforts to maximize

$$u_i(e_i, e_j) = s_i y(e_i + e_j) - C(e_i).$$

Conditional on the formation of the partnership, partner i's best response is given by

$$s_i y'(e_i + e_j) = C'(e_i),$$

where equilibrium efforts are dependent upon the share s_i received. The bigger the share received, the higher the effort. However, since $s_i + s_j = 1$, at least one of the partner always chooses suboptimal effort.

3 Example of efficient team production

In this section we use a specific example to illustrate that the proposed partnership tournament game achieves full efficiency . Let the production function be linear in total efforts

$$y = \alpha(e_i + e_j),\ \alpha > 0$$

and let costs functions be quadratic

$$C(e_i) = \frac{1}{2}e_i^2,\ i \in \{1, 2\}.$$

Let the technology which transforms partners' efforts into a ranking of efforts be described by the Tullock success function. Partner $i \in \{1, 2\}$ is ranked first with probability $f_i(e_i) = \frac{e_i}{e_i+e_j}$ if he exerts effort e_i and the other partner exerts effort e_j. The partner who is ranked first receives share s of the final outcome, and the partner who is ranked second receives share $1 - s$.

The efficient effort levels are given by $(e_1^*, e_2^*) = (\alpha, \alpha)$. In our tournament game, given the shares agreed on at the first stage, the partners choose their efforts non-cooperatively at the second stage. Thus partner $i \in \{1, 2\}$ chooses effort e_i to maximize

$$u_i(e_i, e_j) = \underbrace{\frac{e_i}{e_i + e_j}}_{\text{pr winning}} \underbrace{s\alpha(e_i + e_j)}_{\text{payoff if win}} + \underbrace{\frac{e_j}{e_i + e_j}}_{\text{pr losing}} \underbrace{(1 - s)\alpha(e_i + e_j)}_{\text{payoff if lose}} - \underbrace{\frac{1}{2}e_i^2}_{\text{cost}}. \tag{3}$$

The equilibrium efforts depend on s and are symmetric: $e_i(s) = e_j(s) = s\alpha$. We point out that equilibrium efforts are increasing in the share s. In the extreme case of $s = 1$, both partners exert the efficient level of efforts. The intuition is straightforward. As one partner increases effort, given the other partner's effort level, he increases the final output, and at the same time increases his probability of being ranked first. This implies that he has a higher probability of receiving the majority share of a bigger final outcome. The larger the majority share s, the higher the incentive for a partner to exert high effort. This incentive reaches its maximum when s takes its maximum value. This result is similar to—but in our case stronger than—the standard tournaments result with fixed prizes where incentives increase with the spread between the prizes.

Comparing a partner's objective function (3) with that of a social welfare maximizer

$$w(e_1, e_2) = \alpha(e_1 + e_2) - \frac{1}{2}e_1^2 - \frac{1}{2}e_2^2$$

we see that in (3), each partner's incentive to exert effort consists of two parts. The first one is that exerted effort increases total output $\alpha(e_i + e_j)$, which increases a partner's payoff no matter whether he is ranked first or second. This motive also exists in the social welfare maximization problem. In a partnership, an agent expects to receive only part of the output and thus does not internalize the positive externality of higher effort on the other partner. This is the usual incentive to free-ride leading to under-investment of effort in partnerships. In our game, however, a tournament is used to allocate the shares. Therefore, partners have an additional motive to exert effort, because higher effort increases the probability of getting a bigger share of the output and decreases the probability of getting the smaller share. With a Tullock success function and $s = 1$, this extra incentive brought about by the tournament exactly offsets the disincentive from profit sharing.

To illustrate that full efficiency is achieved, we still need to show that it is optimal for partner 1 to propose the share $s = 1$ at the first stage and for partner 2 to accept. Given the equilibrium efforts $e(s)$, partner 1 chooses share s at the first stage to maximize

$$u_1(s) := u_1(e_1(s), e_2(s)) = \frac{1}{2}(2 - s)s\alpha^2 \tag{4}$$

subject to participation of the second player. Since choosing a minimal effort of δ generates nonnegative utility, this participation is ensured.[4] As $s = 1$ maximizes (4), the shares are chosen appropriately and the efficient equilibrium effort levels are implemented.

In this example, when a tournament is used as the share allocation mechanism, the optimal allocation rule is to give the entire outcome to the partner who ranks first in efforts. This is not a feature of the efficient mechanism in general. As we show in the next section, for a sufficiently precise ranking technology, the efficient mechanism shares output between the players such that each agent receives a positive prize.

[4] For the case of unlimited liability, the second player's participation constraint has to be examined separately.

4 Results

We now show that in the general setup, full efficiency is attainable for linear and concave production functions and a large class of ranking technologies. Recall the production technology

$$y = y(e_i + e_j), \quad \text{with} \quad y(2\delta) = 0, \quad y'(\cdot) > 0, \quad y''(\cdot) \leq 0.$$

Given the sharing rule s and partner j's effort of e_j, partner i's expected utility from exerting effort e_i is

$$u_i(e_i, e_j) = f_i\left(\frac{e_i}{e_j}\right) sy(e_i + e_j) + \left(1 - f_i\left(\frac{e_i}{e_j}\right)\right)(1 - s)y(e_i + e_j) - C(e_i).$$

Assuming the existence of interior solutions, this implies for $i = 1, 2$,

$$\begin{aligned} &f_i'\left(\frac{e_i}{e_j}\right)\frac{1}{e_j}(2s-1)y(e_i+e_j)+ \\ &\left(f_i\left(\frac{e_i}{e_j}\right)s + \left(1 - f_i\left(\frac{e_i}{e_j}\right)\right)(1-s)\right)y'(e_i+e_j) - C'(e_i) = 0. \end{aligned} \tag{5}$$

Given j's effort e_j, (5) implies that marginally increasing effort e_i has three effects: 1) a marginal increase of final output, 2) a marginal increase of partner i's winning probability, and 3) a marginal increase of effort cost.

When the symmetric Nash solution exists, $e_i = e_j = e$ and $f_i(1) = \frac{1}{2}$. Substituting these, we obtain from (5) that

$$\frac{f_i'(1)}{e}(2s-1)y(2e) + \frac{1}{2}y'(2e) = C'(e). \tag{6}$$

As equilibrium effort e is a function of s we write effort as $e(s)$ and the associated output as $y = y(2e(s))$. Intuitively, $f_i'(1)$ relates to the precision of the tournament's ranking technology. A high value of $f_i'(1)$ corresponds to a highly precise ranking technology. A high-precision technology involves a drastic change of the winning probability as the ratio e_i/e_j approaches 1. In the following lemma we begin the analysis of equilibrium effort choice.

Lemma 1. *Equilibrium effort e at the second stage is increasing in s.*

This corresponds to the standard tournament literature result that a partner's incentive is increasing in the spread between prizes. Here we have replaced the fixed prizes with a fixed sharing of the final output. It is natural that effort is increasing in the share s since a larger share means a bigger prize for the winner.

Lemma 2. *Denoting the first best, efficient efforts by e^*, the sharing rule s which satisfies*

$$f_i'(1)\frac{1}{e^*}(2s-1)y(2e^*) = \frac{1}{2}y'(2e^*) \tag{7}$$

elicits the efficient effort choice at the second stage.

Limited liability restricts the partners' possible shares to $s \in [0, 1]$. The next lemma establishes a threshold precision for the ranking technology for efficiency to obtain under limited liability.

Lemma 3. *Under unlimited liability, there always exists a share s^* such that (7) is satisfied. Under limited liability with $s \in [0, 1]$, there exists an s^* that satisfies (7) if $f_i'(1) \geq \frac{1}{4}$.*

We now show that with unlimited liability, first best can always be implemented.

Proposition 1. *Under unlimited liability, full efficiency is obtained. At the first stage, partner 1 proposes a sharing rule $(s^*, 1 - s^*)$ and at the second stage, each partner exerts first best efforts.*

Notice that when $f_i'(1)$ is sufficiently low, the equilibrium share which induces efficient efforts may exceed 1. Denote by $\tilde{s}$ the solution to (7).

Proposition 2. *Under limited liability, if $\tilde{s} \in [0, 1]$, then full efficiency can always be obtained. If $\tilde{s} \notin [0, 1]$, then player 1 proposes shares $(s^*, 1 - s^*) = (1, 0)$ and the agents choose suboptimal efforts.*

Since for a sufficiently precise ranking technology it is always the case that $\tilde{s} \in [0, 1]$, there is a threshold precision above which efficiency is guaranteed.

Corollary 1. *If the ranking technology is sufficiently precise, that is if $f_i'(1) \geq \frac{1}{4}$, then full efficiency can always be obtained.*

The above propositions show that for the class of production functions studied, as long as the ranking technology is such that the marginal winning probability for symmetric efforts is sufficiently large, full efficiency can always be achieved, even under limited liability. There is no necessity for a budget breaker. The only requirement is the observability of some noisy ranking of efforts. This result does not depend on whether or not one can deduce the other partner's effort after output is observed. The efficiency result is robust to production functions of other forms, as long as the concept of symmetric equilibrium can be applied.

The condition on the marginal winning probability for symmetric efforts is critical for limited liability. In the symmetric equilibrium we consider here, a partner is only willing to increase his efforts if doing so significantly increases his probability of winning a bigger share of the final output. We emphasize that $f'(1) < {}^1\!/_4$ is a necessary but not sufficient condition for inefficiency under limited liability. When inefficiency occurs, it depends on the curvature of the production function and the tournament ranking technology. In the example of section 3, if we replace the Tullock success function with the more general function (2) and leave the linear production function unchanged, $f'(1) = {}^1\!/_4$ is exactly the critical value between full efficiency and inefficiency. If the production function is strictly concave, the difference between

$y(x)$ and its linear approximation $y'(x)x$ is positive and the required threshold on the marginal probability of winning decreases. We emphasize that for $f_i'(1) > {}^1/_4$, the player coming out second also gets a positive share of output. In the appendix we show that our main findings extend to the case of $n > 2$ partners.

Conclusion

Possible extensions to the current setup are to more general ranking technologies, more general production functions, and a fully general characterization of the n-players case. The case of risk-averse partners is another possibility for future research. We expect, however, that all these generalizations will leave our principle result in place that first best efficiency is attainable through observable rankings of effort for a wide range of parameters.

Appendix

More than two partners

In this subsection we show that our full efficiency result is not an artifact of two member partnerships, where one can deduce the effort level of the other partner from observing the output. We prove that given the formation of a partnership of n members, the mechanism which allocates the entire final outcome among the partners achieves first best efforts.[5] Moreover, we show that this efficient sharing rule will be proposed at the first stage of the game.

At the first stage, partner 1 proposes a sharing rule of $(s_1, s_2, \cdots, s_n)$. Suppose that the production function takes form

$$y(e) = \alpha \sum_i e_i \tag{8}$$

and the winning probability technology is described by the Tullock success function which specifies the probability of partner i coming out first in the ranking of effort with probability

$$p_i^1(e) = \frac{e_i}{e_1 + e_2 + \ldots + e_n}, \; i \in \{1, \ldots, n\},$$

which is also the probability that partner i wins the share s_1 of the final outcome. Denote $e_{-i} := (e_1, ..., e_{i-1}, e_{i+1}, ..., e_n)$. Then, partner i wins share s_2 with probability

$$p_i^2(e) = p_2^1(e) \cdot \frac{e_1}{e_{-2}} + \ldots + p_n^1(e) \cdot \frac{e_1}{e_{-n}}.$$

Probabilities $p_i^3, \ldots, p_i^n$ are given similarly.[6] Then, given that a partnership of n partners is set

[5] Assume that, if any of the n partners fails to participate in the mechanism, the partnership is not formed and the game ends.

[6] Important are not so much the exact probabilities of coming out second, third etc, but the symmetry of the ranking probabilities between agents and the existence of a symmetric equilibrium for any sharing rule.

up, an allocation rule that assigns the entire output to the partner who is ranked first in effort elicits first best efforts.

Proposition 3. *Under the sharing rule* $(1, 0, \ldots, 0)$*, agents choose efforts efficiently.*

In the next proposition, we show that for production functions (8), if some sharing rules elicit first best effort levels at the second stage, they are among partner 1's choice set of sharing rules in the first stage. This implies that the sharing rule $(1, 0, \cdots, 0)$ stipulated in proposition 3 is part of a subgame perfect equilibrium. That this sharing rule is indeed chosen is shown in the following proposition.[7]

Proposition 4. *Suppose the ranking technology is such that, in symmetric equilibrium, each partner is ranked at each place with equal probability* $1/n$*. If there exist shares* $\hat{s} = (s_1, s_2, \cdots, s_n)$ *which elicit first best efforts at the second stage, then such* $\hat{s}$ *also maximizes partner 1's expected payoff at the first stage.*

As in the case of $n = 2$, if the production function is strictly concave, full efficiency is easier to obtain than in the above linear case. A full characterization of the case of arbitrary ranking technologies is difficult for partnerships with more than two partners since the generalization of the ranking technology poses conceptual and technical problems.

Proofs

Proof of lemma 1. From the implicit function theorem, it follows that

$$\frac{de(s)}{ds} = \frac{\frac{2f_i'(1)}{e(s)} y(2e(s))}{C'' - y''(2e(s)) + f_i'(1)\frac{1}{e(s)}(2s-1)\left(\frac{y(2e(s))}{e(s)} - 2y'(2e)\right)}.$$

If (5) is the foc leading to an equilibrium, then an additional derivative wrt e_i must be negative. This derivative equals

$$\begin{aligned}\frac{d^2 u_i(e_i, e_j)}{de_i^2} &= f_i''\left(\frac{e_i}{e_j}\right)\frac{1}{e_j^2}(2s-1)y(e_i+e_j) + 2f_i'\left(\frac{e_i}{e_j}\right)\frac{1}{e_j}(2s-1)y'(e_i+e_j) \\ &+ \left(f_i\left(\frac{e_i}{e_j}\right)s + \left(1 - f_i\left(\frac{e_i}{e_j}\right)\right)(1-s)\right)y''(e_i+e_j) - C''(e_i)\end{aligned}$$

At the point of symmetric efforts $e = e_i = e_j$ we have

$$f_i''(1)\frac{1}{e^2}(2s-1)y(2e) + 2f_i'(1)\frac{1}{e}(2s-1)y'(2e) + \frac{1}{2}y''(2e) - C''(e) < 0 \tag{9}$$

We will now show that

$$f_i''(1) = -f_i'(1). \tag{10}$$

[7] As shown in the proof, proposition 4 holds for more general production functions than (8).

We know that $f_i(x)+f_j(x)=1$ for any $x\in(0,\infty)$. Differentiating this expression wrt x gives

$$f_i'(x)+f_j'(x)=0. \tag{11}$$

We know from assumption **A1**, that for any $x\in(0,\infty)$, $f_i(x)=f_j(^1/_x)$. Differentiating this expression gives

$$f_i'(x)=-\frac{1}{x^2}f_j'(\frac{1}{x}). \tag{12}$$

Plugging (12) into (11), we obtain

$$f_i'(x)-\frac{1}{x^2}f_i'\left(\frac{1}{x}\right)=0.$$

Differentiating this identity wrt x, we get

$$f_i''(x)-\left((-2)x^{-3}f_i'\left(\frac{1}{x}\right)-\frac{1}{x^2}f_i''\left(\frac{1}{x}\right)\right)=0.$$

Therefore, for $x=1$, we obtain the required equality (10). If we plug this identity back into (9), we get

$$C''-\frac{1}{2}y''(2e(s))+\frac{f_i'(1)}{e(s)}(2s-1)\left(\frac{y(2e(s))}{e(s)}-2y'(2e)\right)>0.$$

Since $C''>0$, $y''(\cdot)\leq 0$ and $y(x)\geq y'(x)x$, we are done. □

Proof of lemma 2. Given s that satisfies (7), at stage 2, partner i chooses effort such that (6) is satisfied. Substituting (7) into (6), one obtains

$$y'(2e)=C'(e)$$

which determines the fully efficient effort level. □

Proof of lemma 3. Rewrite equation (7) as

$$4f_i'(1)(2s-1)y(2e^*)=2y'(2e^*)e^* \tag{13}$$

Since $y(\cdot)$ is concave function, for any $x\in[\delta,\infty)$ holds that $y(x)\geq y'(x)x$. Therefore, whenever $4f_i'(1)\geq 1$, there exists $s^*\in[0,1]$ that solves (13). □

Proof of proposition 1. Expecting the symmetric equilibrium effort levels $e_1(s)=e_2(s)=e(s)$ that are determined by equation (6), in choosing the optimal share s, partner 1's expected utility is

$$\begin{aligned}u_1(s)=u_1(e(s),e(s))&=f_1(1)sy(2e(s))+(1-f_1(1))(1-s)y(2e(s))-C(e(s))\\&=\frac{1}{2}y(2e(s))-C(e(s))\end{aligned}$$

subject to partner 2's participation constraint which we will verify later for the derived equilibrium. Solving partner 1's utility maximization problem gives us the following first order condition

$$\frac{d}{ds}\, u_1(s) = (y'(2e(s)) - C'(e(s)))\, \frac{de(s)}{ds} = 0.$$

Partner 1 chooses $\hat{s}$ such that

$$y'(2e(\hat{s})) = C'(e(\hat{s})).$$

Therefore the sharing rule which implements efficient efforts maximizes player 1's utility. It is now easily verified that partner 2's participation constraint holds because in symmetric equilibrium both players expect the same utilities and by offering $s = {}^1\!/_2$, the proposer can ensure non-negative utility. □

Proof of proposition 2. If $\tilde{s} \in [0, 1]$, then the proof is exactly as the proof of the previous proposition. If there is no $\tilde{s} \in [0, 1]$ which solves (7), meaning $\tilde{s} > 1$. Since $de(s)/ds > 0$ limited liability equilibrium efforts are necessarily lower than the efficient levels e^*. Therefore we have, for the optimal sharing rule s^*,

$$\frac{d}{ds}\, u_1(s^*) = (y'(2e(s^*)) - C'(e(s^*)))\, \frac{de(s^*)}{ds} > 0,$$

where $de(s^*)/ds > 0$ by lemma 1 and $y'(2e) > C'(e)$ for any $e < e^*$ from our curvature assumptions on production and cost functions. This implies that the optimal $s^* = 1$. □

Proof of proposition 3. Given the allocation rule, partner i chooses effort e_i receives share 1 with probability $p_i(e_i, e_{-i})$ and receives a share of 0 otherwise. He chooses his effort e_i to maximize

$$u_i(e_i, e_{-i}) = \frac{e_i}{\sum_{j=1}^{n} e_j} y \left(\sum_{i=1}^{n} e_i \right) - C(e_i) = \alpha e_i - C(e_i).$$

The optimal choice of e_i is determined by the first order condition

$$\alpha = C'(e_i)$$

which implies that the equilibrium effort level is equal to the first best effort level. □

Proof of proposition 4. Since in symmetric equilibrium, each player expects a payoff of ${}^1\!/_n s_1 + {}^1\!/_n s_2 + \ldots {}^1\!/_n s_n$, partner 1 faces the following maximization problem at the first stage

$$u_1(s) = u_1(ne(s)) = \frac{1}{n} y(ne(s)) - C(e(s)).$$

Partner 1 chooses the first best s which satisfies the first order condition

$$(y'(ne(s)) - C'(e(s)))\frac{de(s)}{ds} = 0.$$

□

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