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Garbled Elections

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Garbled Elections

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Abstract

Majority rules are frequently used to decide whether or not a public good should be provided, but will typically fail to achieve an efficient provision. We provide a worst-case analysis of the majority rule with an optimally chosen majority threshold, assuming that voters have independent private valuations and are ex-ante symmetric (provision cost shares are included in the valuations). We show that if the population is large it can happen that the optimal majority rule is essentially no better than a random provision of the public good. But the optimal majority rule is worst-case asymptotically efficient in the large-population limit if (i) the voters' expected valuation is bounded away from 0, and (ii) an absolute bound for valuations is known.

1 Introduction

As observed by Buchanan and Tullock (1962, p. 132), making decisions via majority rule will typically fail to achieve an efficient allocation because the majority rule captures only the direction of each voter's preferences, and "ignores the varying intensities of preference among the separate voters." For example, a situation where two voters have a small willingness to pay for a smoking prohibition and a third has a large willingness to pay for avoiding the prohibition is indistinguishable from a situation where the first

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two voters' willingness to pay is large and the third voter's willingness to pay is small.

Incentives to reveal each agent's willingness to pay for a public good can be provided via a monetary transfer scheme. However, such a scheme cannot be implemented if some agents are budget constrained. Even without budget constraints, problems can arise. While implementation in dominant strategies is always possible (Vickrey, 1961, Clarke, 1971, Groves, 1973), such a scheme will typically run an ex-post deficit or budget surplus. When the scheme is altered such that the budget is balanced (d'Aspremont and Gérard-Varet, 1979), the design of the scheme depends in a delicate way on the beliefs about the agents' valuations, which raises the issue of robustness.

The problems associated with transfer schemes may contribute to the frequent use of majority rules, which are transfer-free and can be implemented in dominant strategies. We consider majority rules with an optimally chosen majority threshold (possibly different from 50%).[1] The fact that majority rules generally cause an efficiency loss, but have robustness properties that standard transfer-using schemes lack, creates a trade-off. In environments where the optimal majority rule causes a small efficiency loss, the trade-off may lean towards using the optimal majority rule, while it may lean towards using a more sophisticated transfer-using mechanism in environments where the optimal majority rule causes a large efficiency loss.[2]

The purpose of this paper is to measure the efficiency loss caused by the optimal majority rule. We consider environments where voters have quasi-linear risk-neutral preferences, have stochastically independent private valuations (provision cost shares are included in the valuations), and are ex-ante symmetric. Obviously, the voters who have a positive valuation are in conflict with the voters who have a negative valuation.[3] Full efficiency would be achieved by the rule that the public good is to be provided if and only if the sum of the agents' valuations is positive. The resulting ex-ante expected surplus is the *first best social surplus.* To measure the efficiency loss resulting from the optimal majority rule, we use *relative efficiency*, which is defined as the fraction of the first best social surplus that is achieved by the

[1] No transfer-free (incentive compatible) decision rule yields a higher expected social surplus than the optimal majority rule, in symmetric independent-private-values environments. We provide a proof of this result in Appendix B.

[2] However, if many independent decision problems are appropriately linked, efficiency can be achieved with a transfer-free scheme (Jackson and Sonnenschein, forthcoming).

[3] A different strand of the literature has focussed on environments where one alternative is best for all voters, so that a "good" voting rule is one that allows the voters to find the best alternative as often as possible (see, e.g., Young, 1995, Austin-Smith and Banks, 1996, and Feddersen and Pesendorfer, 1999).

optimal majority rule, where the surplus from a random provision is taken as a benchmark.[4]

We take the viewpoint of a mechanism designer who wishes to determine the *worst-case* relative efficiency within a *set* of environments, because she has no information whatsoever about which of these environments is likely to occur. This approach is analogous to Satterthwaite and Williams' (2002) worst-case analysis of double auctions, and is a reaction to the Wilson (1987) critique, which argues that a mechanism is successful if it operates well in a range of environments.[5] We assume that the mechanism designer has no information about the environment except, possibly, the population size, a *support restriction*, and two further parameters called *desirability* and *bias*.

The *support restriction* bounds the maximum possible absolute value of any agent's valuation. This restriction has bite because we normalize the agent's expected valuation, conditional on the event that she is in favor of provision, to 1. The mechanism designer may or may not be able to pin down a support restriction.

The *desirability* is the probability that any given agent wants the public good to be provided. The mechanism designer may know the desirability, or may know that it falls within a certain range.

The *bias*[6] relates the conditional expected valuation of an agent who prefers non-provision of the public good to the conditional expected valuation of an agent who prefers provision. For example, an environment where the agents who prefer non-provision "feel" on average twice as strongly as the agents who prefer provision has a bias of 2. The optimal majority threshold is a function of the bias. We assume that the mechanism designer knows the bias. Observe that the mechanism designer needs to know *nothing but* the bias to design an optimal majority rule.

We begin by showing that, for any given bias, desirability, support re-

[4]The terminology "relative efficiency" is adapted from Satterthwaite and Williams' (2002) term "relative inefficiency;" they define, in the context of exchange markets, *relative inefficiency* as "the fraction of the expected potential gains from trade [...] that the mechanism inefficiently fails to achieve in equilibrium." I.e., the no-trade outcome is taken as the benchmark. Rustichini et al. (1994) use the term "expected efficiency." A further related concept is Neeman's (2003) "effectiveness" of an auction, which is defined as the seller's expected revenue as a fraction of the expected surplus arising from an efficient allocation.

[5]In a similar vein, Neeman (2003) computes worst-case environments with respect to the seller's revenue from an English auction.

[6]The term "bias" is adapted from Barbera and Jackson (2006). In their model, each voter represents an entire population of individuals. They study weighted majority rules, where the optimal weights assigned to the various voters depend on the degree to which each vote reflects the utilities in the represented population.

striction, and population size, the worst-case relative efficiency is approximated if and only if each agent is with a high probability either nearly maximally (given the support restriction), positively or negatively, affected by the public good, or nearly unaffected. In other words, the worst-case relative efficiency is approximated if and only if the distribution of any agent's valuation is close, in the sense of the weak topology, to a distribution that puts weight only on the largest possible valuation, the smallest possible valuation, and the valuation 0; we call such distributions *garbling distributions*.

The intuition why the worst-case relative efficiency is approximated with a distribution close to a garbling distribution is that the votes of lightly affected agents count as much as the votes of strongly affected agents towards the collective decision, which allows the lightly affected individuals to "garble" the election result.

We provide an explicit formula for the worst-case relative efficiency, for any given desirability, bias, support restriction, and population size. The formula can be used by any mechanism designer with full or partial knowledge of these parameters to obtain a lower bound for the relative efficiency, that is, for the performance of the optimal majority rule.

Many majority elections involve a large number of voters. Hence, we are particularly interested in the efficiency properties of the optimal majority rule when the population is large. We show that it is of crucial importance to a mechanism designer whether or not (i) she knows the desirability is such that the voters' expected valuation is bounded away from 0, and whether or not (ii) she can pin down a support restriction.

We show that the optimal majority rule is *worst-case asymptotically efficient* if conditions (i) and (ii) are satisfied. Precisely, the smallest relative efficiency across any set of distributions satisfying (i) and (ii) converges to 1 in the large-population limit. In other words, conditions (i) and (ii) imply that the optimal majority rule will aggregate the agents' private information well if the population is large. How large the population must be to guarantee a particular level of relative efficiency can be computed explicitly.

The worst-case asymptotic efficiency result is in line with the common perception that majority rules have good information aggregation properties in large populations.[7]

[7] A number of papers, including Feddersen and Pesendorfer (1996, 1997) (see also Fey and Kim, 2002) for environments with common values and Ledyard and Palfrey (2002) for environments with private values, study the large-population properties of voting mechanisms in various settings and show that information is well aggregated if the population is large. Rather than following a worst-case approach, the approach taken in these papers is to *fix* the characteristics of an individual voter and establish limit-efficiency results under

We show that the optimal majority rule is *worst-case asymptotically garbled* if both conditions (i) and (ii) fail.[8] Precisely, for any given bias there exists a sequence of environments indexed by the population size such that the relative efficiency converges to 0 as the population grows. In other words, for any bias it can happen that the optimal majority rule is essentially no better than a random provision of the public good if the population is large. Hence, the optimal majority rule may totally fail to aggregate the voters' private information if the population is large.

The possible failure of information aggregation in environments with a large population may be surprising because it contrasts the common perception that majority rules have good information aggregation properties in large populations.

In Section 2 we describe the model. Section 3 presents the result that distributions close to garbling distributions approximate the worst-case relative efficiency. In Section 4 we provide the explicit formula for the worst-case relative efficiency and derive implications. Appendix A contains most proof details. In Appendix B we prove the optimality of the majority rule among all transfer-free decision rules.

2 The model

We consider $n \geq 3$ agents who have to decide collectively about whether or not to provide one unit of an indivisible public good such as a smoke prohibition or a bridge.

The willingness-to-pay or *valuation* of agent $i \in I = \{1, \dots, n\}$ for the public good is denoted v_i. In particular, agent i is in favor of provision if $v_i > 0$, and is against provision if $v_i < 0$. We assume that the cost of provision is included in $v_1, \dots, v_n$ via some cost sharing rule. We call v_i agent i's *type*.

We assume that agent i's type is a realization of a random variable $\tilde{v}_i$ with cumulative probability distribution (c.d.f.) F. The random variables $\tilde{v}_1, \dots, \tilde{v}_n$ are stochastically independent. We may assume that each agent is privately informed about her type, but the formal results, with the exception of Appendix B, continue to hold if the types are publicly known. We assume the c.d.f. F is such that no agent is ever indifferent between provision and

the assumption that sufficiently many identical voters exist.

[8] If one of the conditions (i) and (ii) is satisfied, while the other condition fails, then the optimal majority rule is neither worst-case asymptotically efficient nor worst-case asymptotically garbled.

non-provision,[9]

$$\Pr[\tilde{v}_i = 0] = 0. \qquad (1)$$

The probability

$$d_F \;\; = \;\; \Pr[\tilde{v}_i > 0]$$

that any given agent i favors the provision of the public good is called the *desirability* of the public good. To exclude trivial cases, we assume that the desirability is strictly between 0 and 1,

$$0 < d_F < 1. \qquad (2)$$

We assume that the first moment of F is finite,

$$E[\ |\tilde{v}_i|\ ] \;\; < \;\; \infty. \qquad (3)$$

For any $\gamma > 0$, if the agents' types were described by the random variables $\gamma\tilde{v}_1, \ldots, \gamma\tilde{v}_n$ instead of $\tilde{v}_1, \ldots, \tilde{v}_n$, the economy would remain unchanged; applying the factor γ can be interpreted as a change of the money unit. Hence, it is without loss of generality to restrict attention to distributions F such that[10]

$$E[\tilde{v}_i \mid \tilde{v}_i > 0] \;\; = \;\; 1. \qquad (4)$$

That is, we measure any agent's valuation in units of the expected strength of the impact of the public good on the agent, conditional on the event that the agent is in favor of provision. The conditional expectation

$$b_F \;\; = \;\; E[-\tilde{v}_i \mid \tilde{v}_i < 0]$$

is called the *bias* of the distribution F in favor of non-provision of the public good. If $b_F > 1$, then the agents who prefer non-provision "feel stronger" in expectation than the agents who prefer provision. Vice versa if $b_F < 1$.

The agents' expected valuation for the public good is given by

$$E[\tilde{v}_i] \;\; = \;\; d_F - (1 - d_F)b_F. \qquad (5)$$

Any pair (F, n) satisfying (1), (2), (3), and (4) is called an *environment*.

[9] This assumption avoids ambiguities in the definition of the optimal majority rule. It is obviously satisfied for any continuous distribution.

[10] Alternatively, we could have normalized the first moment to 1, but (4) is notationally more convenient.

Optimal majority rule

We analyze the collective decision problem from an *ex-ante* perspective, supposing that no agent does yet know her type. Under the assumption that preferences are quasi-linear with respect to monetary transfers, the Pareto efficient rule is to provide the public good if and only if the sum of the agents' valuations is positive, yielding the *first-best social surplus*[11]

$$W^*_{F,n} \quad = \quad E\left[\left[\sum_{i=1}^{n} \tilde{v}_i\right]_+\right]. \tag{6}$$

Assuming that the agents are risk neutral, $W^*_{F,n}$ equals n times the ex ante expected utility of any agent if the Pareto efficient rule is used.

A useful benchmark decision rule is random provision of the public good, where the public good is provided with probability 1/2 independently of the agents' valuations. Using (5), the *social surplus from a random provision* is

$$W^{\text{random}}_{F,n} \quad = \quad \frac{n}{2}(d_F - (1-d_F)b_F). \tag{7}$$

Observe that $W^{\text{random}}_{F,n}$ equals n times the ex ante expected utility of any agent under the random provision rule.

An α*-majority rule* ($\alpha \in (0,1)$) stipulates that the public good is provided if and only if the number of agents who favor provision is at least αn. This corresponds to an election where each agent may vote in favor or against provision, the fraction of votes required for provision is α, and each agent uses the weakly dominant strategy of voting in favor of provision if and only if she is in fact in favor. The social surplus arising from the α-majority rule is

$$\sum_{\alpha n \le k \le n} (k - (n-k)b_F)\ d_F^k (1-d_F)^{n-k} \begin{pmatrix} n \\ k \end{pmatrix}. \tag{8}$$

Observe that the expression (8) equals n times the ex ante expected utility of any agent under the α-majority rule.

We say that α *is optimal in environment* (F, n) if it is a maximizer of (8). Clearly, α is optimal if and only if the following conditions hold:

$$\begin{array}{llll} \text{if} & k-(n-k)b_F > 0 & \text{then} & \alpha n \le k, \\ \text{if} & k-(n-k)b_F < 0 & \text{then} & \alpha n > k. \end{array} \tag{9}$$

[11] We use the shortcut $[x]_+ = \max\{0, x\}$ for any $x \in \mathbb{R}$.

This yields the following result.[12]

Remark 1 *Consider any F satisfying (1), (2), (3), and (4), and any $\alpha \in (0,1)$. Then α is optimal in the environment (F,n) for all n if and only if*

$$\alpha = \frac{b_F}{1+b_F}. \tag{10}$$

The α-majority rule with α satisfying (10) is called the *optimal majority rule* in environment (F,n). Observe that the 50%-majority rule is optimal for all n if and only if $b_F = 1$.

Observe that the mechanism designer needs minimal knowledge about the environment (F,n) in order to implement the optimal majority rule—knowing the bias b_F is enough. In particular, the mechanism designer does not need to know the population size n, nor does she need to know the desirability d_F or any finer knowledge of the distribution F.

Using (8) and (9), one sees that the *social surplus arising from the optimal majority rule* is

$$W_{F,n}^{\text{opt maj}} \quad = \quad n\beta(b_F, d_F, n), \tag{11}$$

where

$$\beta(b,d,n) \quad \stackrel{\text{def}}{=} \quad \frac{1}{n}\sum_{k=0}^{n} [k-(n-k)b]_+ \; d^k(1-d)^{n-k} \begin{pmatrix} n \\ k \end{pmatrix}, \tag{12}$$

for any $b > 0$, $d \in (0,1)$, and $n \geq 3$.

From (11) we obtain a useful representation of the social surplus arising from the optimal majority rule. In any environment (F,n),

$$W_{F,n}^{\text{opt maj}} \quad = \quad E\left[\left[\sum_{i=1}^{n} \tilde{b}_i\right]_+\right], \tag{13}$$

where $\tilde{b}_1, \ldots, \tilde{b}_n$ denote stochastically independent random variables, each taking the value 1 with probability d_F and taking the value $-b_F$ with probability $1-d_F$.

[12] The "if" part is a special case of Barbera and Jackson (2006, Corollary 1). The case $b_F = 1$ is treated by Taylor (1969), Schofield (1971), and Curtis (1972), who build on Rae (1969).

Relative efficiency

The optimal majority rule will typically fail to achieve the first best expected surplus. To measure the efficiency loss caused by the optimal majority rule in general environments, we define for any environment (F, n) the *relative efficiency* $\rho(F, n)$, which captures the fraction of the first best social surplus that is realized by the optimal majority rule, using the social surplus from a random provision as a benchmark,

$$\rho(F, n) \quad = \quad \frac{W_{F,n}^{\text{opt maj}} - W_{F,n}^{\text{random}}}{W_{F,n}^{*} - W_{F,n}^{\text{random}}}. \tag{14}$$

Observe that $\rho(F, n)$ is a number in $[0, 1]$, where $\rho(F, n) = 1$ if and only if the optimal majority rule achieves the first best social surplus, and $\rho(F, n) = 0$ if and only if the optimal majority rule is as bad as a random provision of the public good (however, the results below imply that $\rho(F, n) > 0$ for all environments (F, n)).

Remark 2 shows that the first best is achieved in environments where the distribution F is concentrated on two points. By (1) and (2), this will be a positive and a negative point. In such environments, an agent's voting behavior in the optimal majority rule fully reveals her preferences.

Remark 2 *Consider any F satisfying (1) and (2) such that supp$F$ contains exactly two points. Then $\rho(F, n) = 1$ for all n.*

Worst case relative efficiency

In the following, we take the viewpoint of a mechanism designer who wishes to determine the set of possible relative efficiency levels across a set of environments. To begin with, let us assume that the mechanism designer knows the bias $b > 0$ (which is needed to determine the optimal majority rule), the desirability $d \in (0, 1)$, and an upper bound for the absolute value of any agent's valuation $s > \max\{1, b\}$. That is, every c.d.f. in

$$\mathcal{F}(b, d, s) = \{F \mid (1), (2), (3), (4),\ b_F = b,\ d_F = d,\ \text{supp}F \subseteq [-s, s]\}$$

is considered possible. Observe that, in view of the normalization (4), the *support restriction* s amounts to excluding the possibility that the impact of the public good on an agent is arbitrarily stronger than the *expected* strength of the impact.[13] We are particularly interested in the *worst-case*

[13] Neeman (2003) uses an analogous restriction in his analysis of the worst-case performance of English auctions. He restricts the ratio between the expected valuation of a

relative efficiency

$$\underline{\rho}(b, d, s, n) = \inf_{F \in \mathcal{F}(b,d,s)} \rho(F, n) \tag{15}$$

for any population size $n \geq 3$. Furthermore, we are interested in determining the c.d.f.s in $\mathcal{F}(b, d, s)$ such that the worst-case relative efficiency is (approximately) attained.

We also consider the possibility that the mechanism designer has more limited information. For example, the mechanism designer may not be able to pin down a support restriction s; that is, she may consider every c.d.f. in

$$\mathcal{F}(b, d, \infty) \quad = \quad \cup_{s > \max\{1,b\}} \mathcal{F}(b, d, s)$$

possible. Or the mechanism designer may consider an entire set $K \subseteq (0, 1)$ of desirability levels possible; that is, she may consider every c.d.f. in

$$\mathcal{F}(b, K, s) \quad = \quad \cup_{d \in K} \mathcal{F}(b, d, s)$$

possible, where $s > \max\{1, b\}$ or $s = \infty$. For example, $K = [0.7, 0.8]$ captures a situation where it is known that every agent prefers provision of the public good with a probability between 0.7 and 0.8.

In many applications the number of voters is large. Hence, we put particular emphasis on this case. Consider any set $\mathcal{D}$ of c.d.f.s F satisfying (1), (2), (3), and (4). The optimal majority rule is called *worst-case asymptotically efficient* in $\mathcal{D}$ if

$$\lim_{n \to \infty} \inf_{F \in \mathcal{D}} \rho(F, n) \quad = \quad 1.$$

This captures a class of c.d.f.s where the optimal majority rule is guaranteed to perform well if the population is sufficiently large.

The optimal majority rule is called *worst-case asymptotically garbled* in $\mathcal{D}$ if there exists a sequence $(F_n)_{n \geq 3}$ in $\mathcal{D}$ such that

$$\lim_{n \to \infty} \rho(F_n, n) \quad = \quad 0.$$

This captures a class of c.d.f.s where the mechanism designer cannot exclude the possibility that the optimal majority rule is essentially no more useful than a random decision if the population is large.

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bidder and the largest possible valuation. This is equivalent to the existence of a largest possible valuation if the expected valuation is normalized to a fixed number.

Garbling distributions

The valuation of an agent has a *garbling distribution* if she is with some probability maximally, positively or negatively, affected by the public good, and is with the remaining probability not affected at all. The *garbling distribution with parameters* $d \in (0,1)$*,* $b > 0$*, and* $s > \max\{1, b\}$ is the c.d.f. $F^*_{b,d,s,0}$ given by

$$F^*_{b,d,s,0}(v) \quad = \quad b\frac{1-d}{s}\;\mathbf{1}_{v\geq -s} \;+\; \left(\frac{s-b}{s} + d\frac{b-1}{s}\right)\;\mathbf{1}_{v\geq 0} \;+\; \frac{d}{s}\mathbf{1}_{v\geq s}$$

for all $v \in [-s, s]$. Garbling distributions violate assumption (1), because they put weight on the point 0. Hence, they are not directly involved in the computation of (15). Nevertheless, garbling distributions will play a crucial role in our results.

Each garbling distribution is constructed such that it can be approximated arbitrarily closely by c.d.f.s in $\mathcal{F}(b, d, s)$. For all $t \in (0, \min\{1, b\})$, define a c.d.f. $F^*_{b,d,s,t}$ by the formula

$$\begin{aligned} F^*_{b,d,s,t}(v) \quad &= \quad (1-d)\frac{b-t}{s-t}\;\mathbf{1}_{v\geq -s} + (1-d)\frac{s-b}{s-t}\;\mathbf{1}_{v\geq -t} \\ &\quad +d\frac{s-1}{s-t}\;\mathbf{1}_{v\geq t} + d\frac{1-t}{s-t}\;\mathbf{1}_{v\geq s} \quad (v \in [-s, s]). \end{aligned} \tag{16}$$

Observe that $F^*_{b,d,s,t}$ concentrates its weight on the four points $-s$, $-t$, t, and s. The total probability weight on $\{t, s\}$ equals d. The expectations conditional on $\{t, s\}$ and $\{-s, -t\}$ are 1 and $-b$, respectively. Hence,

$$F^*_{b,d,s,t} \in \mathcal{F}(b, d, s). \tag{17}$$

Moreover, using the weak topology of c.d.f.s on $[-s, s]$,[14]

$$F^*_{b,d,s,0} \quad = \quad \lim_{t\to 0} F^*_{b,d,s,t}. \tag{18}$$

That is, each garbling distribution is approximated by a c.d.f. such that an agent with the approximating c.d.f. is either maximally affected by the public good, or very lightly affected.

The fact that an agent with a garbling distribution is with positive probability indifferent about the public good means that the social surplus that

[14]A sequence of c.d.f.s converges weakly if it converges pointwise at all points where the limit c.d.f. is continuous. The weak topology is metrizable. In particular, any sequentially continuous function is continuous, and any sequentially compact set is compact. For more details see Billingsley (1968).

arises from the optimal majority rule is not well-defined. However, the definition of the first best social surplus (6) extends straightforwardly to garbling distributions. The probability that x^- agents have the valuation $-s$, that x^0 agents have the valuation 0, and that x^+ agents have the valuation s, equals

$$p(x^-, x^0, x^+, s) \quad = \quad \left(b\frac{1-d}{s}\right)^{x^-} \left(\frac{s-b}{s} + d\frac{b-1}{s}\right)^{x^0} \left(\frac{d}{s}\right)^{x^+} \frac{n!}{x^-!x^0!x^+!}. \tag{19}$$

Hence,

$$W^*_{F^*_{b,d,s,0},n} \quad = \sum_{\substack{x^+, x^0, x^- = 0,1,2,\ldots, \\ x^+ + x^0 + x^- = n}} s[x^+ - x^-]_+ \, p(x^-, x^0, x^+, s). \tag{20}$$

We now have all the tools that we need.

3 Garbling distributions and the worst case

In this section, we show that the worst-case relative efficiency is approximated if and only if each agent has a c.d.f. that is close, in the sense of the weak topology, to a garbling distribution.[15]

Proposition 1 *Let $b > 0$, $d \in (0,1)$, $s > \max\{1, b\}$, and $n \geq 3$. Given any sequence $(F^m)_{m=1,2,\ldots}$ in $\mathcal{F}(b,d,s)$, the condition*

$$\lim_{m\to\infty} \rho(F^m, n) = \underline{\rho}(b,d,s,n) \tag{21}$$

holds if and only if

$$\lim_{m\to\infty} F^m = F^*_{b,d,s,0}. \tag{22}$$

In the light of (17) and (18), Proposition 1 shows that the worst-case relative efficiency is approximated if each agent has a c.d.f. $F^*_{b,d,s,t}$ with a small t, that is, if each agent is either maximally or very lightly affected by the public

[15]While we focus on the worst-case relative efficiency, *any* relative efficiency between the worst-case relative efficiency $\underline{\rho}(b,d,s,n)$ and 1 is attained by some c.d.f. in $\mathcal{F}(b,d,s)$. By Remark 2, there exists a c.d.f. in $\mathcal{F}(b,d,s)$ such that the relative efficiency equals 1. By continuity, all intermediate relative efficiency levels can be obtained as well.

good, with probabilities determined by b and d. Vice versa, the worst-case relative efficiency is approximated *only if* each agent is with a high probability either nearly maximally affected or nearly unaffected. However, the result that the worst-case relative efficiency is approximated only if each agent's c.d.f. approximates a garbling distribution does not mean that other c.d.f.s lead to a high relative efficiency.

The intuition why the worst-case relative efficiency is approximated with a c.d.f. $F^*_{b,d,s,t}$ with a small t is as follows. Whether a lightly affected agent has a positive valuation t or a negative valuation $-t$ is essentially irrelevant for the social surplus. However, the vote of a lightly affected agent counts as much as the vote of a strongly affected agent. Hence, the votes of the lightly affected individuals tend to "garble" the election result; the greatest amount of surplus is lost if the non-lightly affected individuals are maximally affected.

Garbling distributions have a surprisingly simple structure. Nevertheless, the plausibility of c.d.f.s that approximate a garbling distribution may be put into question by the observation that lightly affected individuals will not participate in the election if there are positive participation costs. Adding participation costs to the model is nontrivial.[16] However, it seems reasonable to conjecture that an appropriately extended concept of worst-case relative efficiency would depend *continuously* on the participation costs. I.e., small participation costs would lead to a small change of the worst-case relative efficiency.

The starting point of the proof of Proposition 1 (for details see Appendix A) is the observation that the social surplus from a random provision (7) and the surplus that arises from the optimal majority rule (11) are constant on $\mathcal{F}(b, d, s)$. Hence, determining the worst-case relative efficiency amounts to maximizing the first best social surplus in $\mathcal{F}(b, d, s)$. However, a maximizer does not exist because the constraint set $\mathcal{F}(b, d, s)$ is not compact (using the weak topology). To obtain a compact constraint set, we include c.d.f.s that violate assumption (1). The extended constraint set includes the topological closure of $\mathcal{F}(b, d, s)$. We embed the resulting maximization problem into the vector space of signed Borel measures and apply a version of Kuhn and Tucker's theorem for infinite dimensional spaces to show that the unique solution of the maximization problem is given by $F^*_{b,d,s,0}$ (this step presents the greatest technical difficulties). By (17) and (18), $F^*_{b,d,s,0}$ belongs to the

[16]Börgers (2004) shows that participation costs introduce a new source of inefficiency, because the voters' participation decisions will generally be inefficient in a majority election with voluntary participation. Krasa and Polborn (2006) generalize the model of Börgers and show that subsidizing participation may increase welfare.

topological closure of $\mathcal{F}(b,d,s)$. From this, the equivalence of (21) and (22) is straightforward.

The following example illustrates the fact that the garbling distribution $F^*_{b,d,s,0}$ is the unique maximizer of the first best social surplus in the topological closure of $\mathcal{F}(b,d,s)$. We consider c.d.f.s that concentrate their weight on the set $\{-s,0,x\}$, where $x \leq s$, and show that the first best social surplus is uniquely maximized if $x = s$.

For any $p \in [d/s, d]$, consider the c.d.f. F_p that is constructed from $F^*_{b,d,s,0}$ by replacing the atom at s by an atom of weight p at the point $x = d/p$; i.e.,

$$F_p(v) \;\; = \;\; b\frac{1-d}{s}\,\mathbf{1}_{v\geq -s} \; + \; (1-b\frac{1-d}{s}-p)\,\mathbf{1}_{v\geq 0} \; + \; p\,\mathbf{1}_{v\geq x}.$$

By construction, F_p can be approximated arbitrarily closely using c.d.f.s in $\mathcal{F}(b,d,s)$. Hence, F_p belongs to the topological closure of $\mathcal{F}(b,d,s)$,

We claim that the first best social surplus $W^*_{F_p,n}$ is maximized if and only if $p = d/s$, that is, if and only if $F_p = F^*_{b,d,s,0}$.

By construction, conditional on the event that every agent has one of the valuations 0 or x, a change of p leads, in expectation, to no change of the social surplus. Only in the event that at least one agent has the valuation $-s$ can the realized social surplus depend on p. For concreteness, consider the case $n = 3$. Then the crucial event $\mathcal{E}$ is that one agent has the valuation $-s$ and two agents have the valuation x. If $x \leq s/2$, the social surplus arising from event $\mathcal{E}$ is 0; if $x > s/2$, the social surplus arising from event $\mathcal{E}$ is

$$3b\frac{1-d}{s}p^2(2x-s) \;\; = \;\; 3b(1-d)p(2\frac{d}{s}-p).$$

This expression is uniquely maximized at $p = d/s$, or $F_p = F^*_{b,d,s,0}$.[17]

4 Properties of the worst-case relative efficiency

In this section, we provide an explicit formula for the worst-case relative efficiency and draw several implications. Proposition 2 provides an explicit formula for the worst-case relative efficiency. The proof relies on Proposition 1 (for details see Appendix A).

[17] If $n = 2$, the first-best surplus arising from F_p is independent of p. This can be used to show that (21) does not imply (22) if $n = 2$. For this reason we are assuming $n \geq 3$.

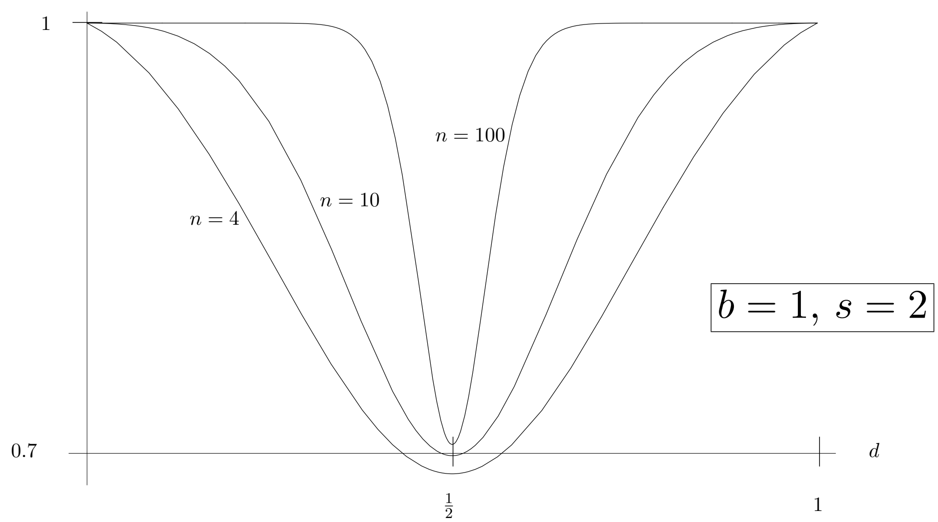

Figure 1: Illustration of formula (23) with bias $b = 1$, support restriction $s = 2$, and population sizes $n = 4, 10, 100$: the worst-case relative efficiency $\underline{\rho}(b, d, s, n)$ as a function of the desirability d.

Proposition 2 *For all $b > 0$, $d \in (0, 1)$, $s > \max\{1, b\}$, and $n \geq 3$, the worst-case relative efficiency is given by*

$$\underline{\rho}(b, d, s, n) \quad = \quad \frac{\beta(b, d, n) - \frac{1}{2}(d - (1 - d)b)}{\frac{1}{n} W^*_{F^*_{b,d,s,0}, n} - \frac{1}{2}(d - (1 - d)b)}. \tag{23}$$

Formula (23) is directly useful to a mechanism designer who knows the bias, the desirability, the maximum possible support, and the population size, but has no additional information about the environment. For example, the worst-case relative efficiency among environments with desirability $d = 1/2$, bias $b = 1$, support restriction $s = 2$, and a population size of $n = 100$ is approximately 71% (cf. Figure 1). This number drops to 32% if the support restriction is $s = 10$ (cf. Figure 2).

Proposition 2 can also be used to deal with situations where the mechanism designer has limited information about the desirability, the maximum possible support, or the population size. For example, the mechanism designer may not be able to pin down a support restriction. Then the following result is relevant.

Corollary 1 *For any bias $b > 0$, desirability $d \in (0, 1)$, and population size $n \geq 3$, the worst-case relative efficiency without support restriction is given*

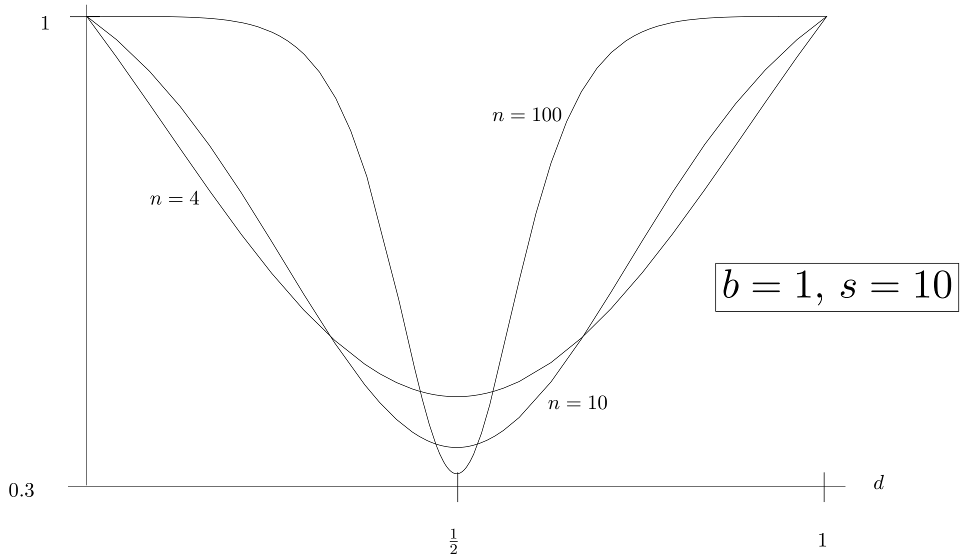

Figure 2: Illustration of formula (23) with bias $b = 1$, support restriction $s = 10$, and population sizes $n = 4, 10, 100$: the worst-case relative efficiency $\underline{\rho}(b, d, s, n)$ as a function of the desirability d. Observe that the worst-case relative efficiency is not monotonic in the population size n.

by

$$\lim_{s\to\infty} \underline{\rho}(b, d, s, n) \quad = \quad \frac{\beta(b, d, n) - \frac{1}{2}(d - (1-d)b)}{\frac{1}{2}(d + (1-d)b)}. \tag{24}$$

To prove Corollary 1, it is by Proposition 2 sufficient to show that

$$\lim_{s\to\infty} W^*_{F^*_{b,d,s,0},n} \quad = \quad nd.$$

Using (19) and (20) one sees that, as s gets large, the only contribution to the first best social surplus that is not of the order s^k with $k \leq -1$ comes from the constellation where one agent has valuation s and all other agents have valuation 0, that is, $x^+ = 1$ and $x^0 = n - 1$. Because each agent may be the one with valuation s, the probability of this constellation approaches n times d/s as s gets large, which yields the expected contribution nd.

Environments with a large population size are of special interest. Many real elections involve a large number of voters. It seems especially worthwhile to examine the validity of the common perception that majority rules yield asymptotically efficient decisions in large-population environments.

Corollary 2 *For all $b > 0$, $d \in (0,1)$, and $s > \max\{1, b\}$, the large-population limit of the worst-case relative efficiency is given by*

$$\lim_{n\to\infty} \underline{\rho}(b,d,s,n) \quad = \quad \begin{cases} \sqrt{\frac{1+b}{2s}} & \textit{if } d-(1-d)b = 0, \\ 1 & \textit{if } d-(1-d)b \neq 0. \end{cases} \tag{25}$$

Corollary 2 reveals the role of the agents' expected valuation for the worst-case relative efficiency. If the expected valuation $d-(1-d)b \neq 0$ (cf. (5)), then the limit in (25) equals 1, which corresponds to full efficiency. If the expected valuation equals 0, the limit in (25) is smaller than 1. Accordingly, convergence to 1 in (25) will be slow if the expected valuation is close to 0 (cf. Figure 1 and Figure 2 at $d \approx 1/2$). We can interpret this result as follows. In environments where the expected valuation is close to 0 it is most uncertain ex ante whether provision or non-provision of the public good will be the efficient decision. Hence, in these environments the optimal majority rule has the "most difficult task," and the most can go wrong.

The proof of Corollary 2 (for details see Appendix A) relies on the fact that the garbling distribution $F^*_{b,d,s,0}$ is *independent* of the population size n. This makes it possible to apply the law of large numbers if $d-(1-d)b \neq 0$, and the central limit theorem if $d-(1-d)b = 0$.

Next we give a result parallel to Corollary 2 for cases where the mechanism designer cannot pin down a support restriction.[18]

Corollary 3 *For all $b > 0$ and $d \in (0,1)$, the large-population limit of the worst-case relative efficiency without support restriction is given by*

$$\lim_{n\to\infty} \lim_{s\to\infty} \underline{\rho}(b,d,s,n) \quad = \quad \frac{\mid d-(1-d)b \mid}{d+(1-d)b} < 1. \tag{26}$$

The expression (26) is minimized if the expected valuation $d-(1-d)b = 0$ (cf. Figure 3). This is analogous to Corollary 2. In contrast to Corollary 2, however, the limit (26) is always smaller than 1. For example, if $b = 1$ and $d = 0.55$, then a relative efficiency close to 10% is possible if the population is large. This number becomes 1% if $d = 0.505$. In contrast, if d is close to 0 or 1, then (26) is close to 1. This reflects the fact that it is pretty clear ex ante whether or not the public good should be provided, so that the optimal majority rule has an "easy task." For example, if $b = 1$ and $d \geq 0.75$, then the optimal majority rule has a relative efficiency of at least 50%.

[18] It is worthwhile to note the additional result (a proof of which is sketched in Appendix A) that (24) is weakly decreasing in n. Hence, (26) provides a tight lower bound for the relative efficiency among all environments with bias b and desirability d.

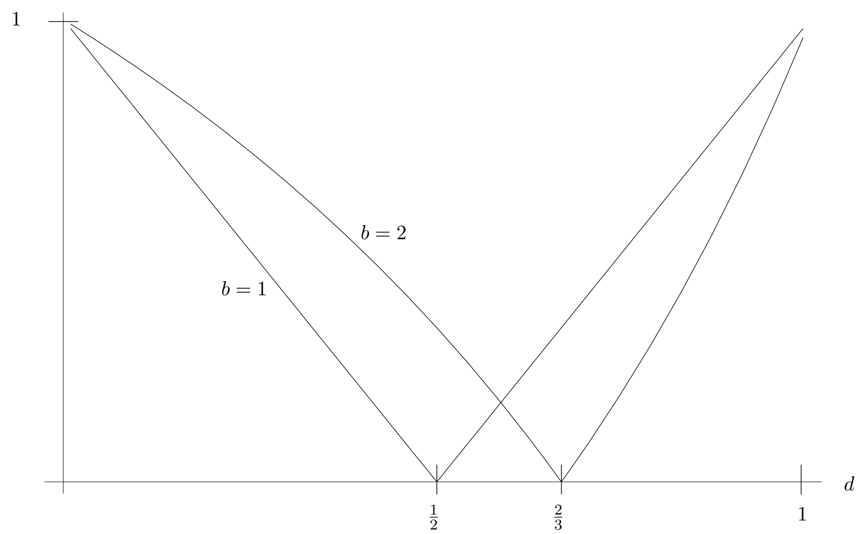

Figure 3: Illustration of formula (26): the large-population limit of the worst-case relative efficiency without support restriction, as a function of the desirability d at bias $b = 1$ and bias $b = 2$.

The proof of Corollary 3 relies on the law of large numbers. For details see Appendix A.

We conclude with two results pertaining the worst-case asymptotic properties of the optimal majority rule when the mechanism designer has only partial knowledge of the desirability.

Proposition 3 says that the optimal majority rule is guaranteed to perform approximately efficiently in environments with a large population, if and only if (i) the mechanism designer knows that the desirability is bounded away from $b/(1+b)$ (that is, the expected valuation is bounded away from 0) and (ii) there exists a support restriction.[19]

Proposition 3 *Let $b > 0$, $K \subseteq (0,1)$ closed, and $s > \max\{1, b\}$ or $s = \infty$. The optimal majority rule is worst-case asymptotically efficient in $\mathcal{F}(b, K, s)$ if and only if*

$$\frac{b}{1+b} \notin K \quad \textit{and} \quad s < \infty.$$

The proof uses Corollary 2, Corollary 3, and standard uniform-convergence arguments. For details see Appendix A.

[19] A set $K \subseteq \mathbb{R}$ is closed if for any sequence (x_m) in K such that $x_m \to x \in \mathbb{R}$ we have $x \in K$.

Proposition 4 says that the optimal majority rule can lead to a relative efficiency arbitrarily close to 0 in large-population environments if conditions (i) and (ii) fail.

Proposition 4 *Let $b > 0$, $K \subseteq (0,1)$ closed, and $s > \max\{1, b\}$ or $s = \infty$. The optimal majority rule is worst-case asymptotically garbled in $\mathcal{F}(b, K, s)$ if and only if*

$$\frac{b}{1+b} \in K \quad \textit{and} \quad s = \infty.$$

The proof uses Corollary 2, Corollary 3, and standard uniform-convergence arguments. For details see Appendix A.

Proposition 4 shows that if the population is large it can happen that the optimal majority rule is essentially no better than a random provision of the public good. Hence, there is no guarantee that the optimal majority rule has good information aggregation properties in large populations.

Appendix A

Lemma 1 *For all $n \geq 3$ and $s > 0$, the functional defined by formula (6),*

$$W^*_{\cdot,n}: \quad \{F \mid F \text{ c.d.f., } \mathit{supp} F \subseteq [-s, s]\} \to \mathbb{R}, \qquad F \mapsto W^*_{F,n},$$

is continuous with respect to the weak topology on the set of c.d.f.s on $[-s, s]$.

Proof. We use the "$\Rightarrow$" sign to denote weak convergence. Because the weak topology is metrizable, it is sufficient to show sequential continuity. Consider a sequence of c.d.f.s (F_m) such that $F_m \Rightarrow F$ for some c.d.f. F. By Billingsley (1968, Theorem 3.2),

$$\underbrace{F_m \times \ldots \times F_m}_{n \text{ times}} \Rightarrow F \times \ldots \times F,$$

where we use the topology of weak convergence of c.d.f.s on $[-s, s]^n$. The map

$$\mathbb{R}^n \to \mathbb{R}, \quad (v_1, \ldots, v_n) \mapsto \left[\sum_i v_i\right]_+$$

is continuous. Hence, using (6) and Billingsley (1968, Theorem 2.1 (ii)),

$$\begin{aligned} W^*_{F_m,n} &= \int_{[-s,s]^n} \left[\sum_i v_i\right]_+ \mathrm{d}F_m(v_1) \cdots \mathrm{d}F_m(v_n) \\ &\to \int_{[-s,s]^n} \left[\sum_i v_i\right]_+ \mathrm{d}F(v_1) \cdots \mathrm{d}F(v_n) \;=\; W^*_{F,n}. \end{aligned}$$

QED

For all $b > 0$, $d \in (0,1)$, $s > \max\{1, b\}$, and $n \geq 3$, consider the optimization problem

$$(b,d,s,n) \qquad \max_{\substack{F \text{ c.d.f.,} \\ \mathrm{supp} F \subseteq [-s,s]}} \qquad W^*_{F,n}$$

$$\text{s.t.} \qquad \int_{-s}^{s} \mathbf{1}_{v>0} v \mathrm{d}F(v) = d, \tag{27}$$

$$\int_{-s}^{s} \mathbf{1}_{v<0} v \mathrm{d}F(v) = -b(1-d), \tag{28}$$

$$\int_{-s}^{s} \mathbf{1}_{v>0} \mathrm{d}F(v) \leq d, \tag{29}$$

$$\int_{-s}^{s} \mathbf{1}_{v<0} \mathrm{d}F(v) \leq 1-d. \tag{30}$$

The constraint set of problem (b,d,s,n) is denoted $\mathcal{C}(b,d,s)$. Observe that $\mathcal{C}(b,d,s)$ includes c.d.f.s that violate (1) by putting a positive weight on the point 0.

Let $\overline{\mathcal{F}}(b,d,s)$ denote the topological closure of $\mathcal{F}(b,d,s)$.

Lemma 2 *The sets $\mathcal{C}(b,d,s)$ and $\overline{\mathcal{F}}(b,d,s)$ are compact, and*

$$\mathcal{F}(b,d,s) \subseteq \mathcal{C}(b,d,s). \tag{31}$$

Proof. Because the map

$$\mathbb{R} \to \mathbb{R}, \ v \to \mathbf{1}_{v>0} v$$

is continuous, the constraint (27) defines a closed set of c.d.f.s (Billingsley (1968, Theorem 2.1 (ii)). Similarly, (28) defines a closed set. The left-hand side of constraint (29) corresponds to the probability of the set $(0,s]$, which is open in $[-s,s]$. Hence, (29) defines a closed set by Billingsley (1968, Theorem 2.1 (iv)). Similarly, (30) defines a closed set. Hence, $\mathcal{C}(b,d,s)$ is closed. By Helly's selection theorem, the set of all c.d.f.s on $[-s,s]$ is compact. As a closed subset of a compact set, $\mathcal{C}(b,d,s)$ is compact.

It is straightforward to check (31). Compactness of $\overline{\mathcal{F}}(b,d,s)$ follows from (31). *QED*

Lemma 3 *Problem (b,d,s,n) has a solution F^*.*

Proof. By Lemma 2, the constraint set of problem (b, d, s, n) is compact. By Lemma 1, the objective function of problem (b, d, s, n) is continuous. Hence, a solution exists by Weierstrass' theorem. *QED*

Lemma 4 *Any solution F^* of problem (b, d, s, n) is such that suppF^* contains at least three elements.*

Proof. Suppose that F^* puts all probability weight on two points x and $y \geq x$. By (27), $y > 0$. By (28), $x < 0$. By (29) and (30), $\Pr[\tilde{v}_i = y] = d$ and $\Pr[\tilde{v}_i = x] = 1 - d$, when $\tilde{v}_i$ denotes a random variable with c.d.f. F^*. Hence, $y = 1$ and $x = -b$ by (27) and (28) with $F = F^*$. By Remark 2,

$$W^{\text{opt maj}}_{F^*,n} \quad = \quad W^*_{F^*,n}. \tag{32}$$

Let F be a c.d.f. that puts the weight $1 - d$ on the point $-b$, the weight $d(s-1)/(s-b/n)$ on the point b/n, and the weight $d(1-b/n)/(s-b/n)$ on the point s. Then $F \in \mathcal{F}(b, d, s)$. Moreover, $W^{\text{opt maj}}_{F,n} < W^*_{F,n}$ because, with positive probability, $n-1$ agents have valuation b/n and one agent has valuation $-b$, in which case $(n-1)b/n - b < 0$.

Using that $F^* \in \mathcal{F}(b, d, s)$ and that $W^{\text{opt maj}}_{\cdot,n}$ is constant on $\mathcal{F}(b, d, s)$ by (11),

$$W^*_{F^*,n} \overset{(32)}{=} W^{\text{opt maj}}_{F^*,n} = W^{\text{opt maj}}_{F,n} < W^*_{F,n},$$

which contradicts the optimality of F^*. *QED*

It is useful to embed the constraint set of problem (b, d, s, n) into the vector space of signed Borel measures on $[-s, s]$,[20] and to extend the function $W^*_{\cdot,n}$ by defining

$$W^*_{F,n} \quad = \quad \int_{-s}^{s} \cdots \int_{-s}^{s} \left[\sum_{i=1}^{n} v_i\right]_+ \mathrm{d}F(v_1) \cdots \mathrm{d}F(v_n), \tag{33}$$

for any signed Borel measure F on $[-s, s]$. Let Θ denote the set of positive Borel measures on $[-s, s]$. Consider the relaxed problem

$$(b, d, s, n)' \qquad \max_{F \in \Theta} \qquad W^*_{F,n}$$

[20] A signed Borel measure on $[-s, s]$ is a real-valued and countably additive function on the set of Borel sets in $[-s, s]$. As with probability measures, we represent any signed measure by a function F on $[-s, s]$, where $F(x)$ denotes the measure of the set $[-s, x]$.

A signed Borel measure that takes only non-negative values is called positive.

$$\text{s.t.} \quad \int_{-s}^{s} \mathbf{1}_{v>0} v \mathrm{d}F(v) \leq d, \tag{34}$$

$$\int_{-s}^{s} \mathbf{1}_{v<0} v \mathrm{d}F(v) \leq -b(1-d), \tag{35}$$

$$(29), (30),$$

$$\int_{-s}^{s} \mathrm{d}F(v) \leq 1. \tag{36}$$

Lemma 5 *Any solution F^* of problem (b, d, s, n) is a solution of problem $(b, d, s, n)'$.*

Proof. Let H be a solution of $(b, d, s, n)'$. Because the constraint set of $(b, d, s, n)'$ includes the constraint set of (b, d, s, n), it is sufficient to show that

$$W^*_{F^*,n} \geq W^*_{H,n}. \tag{37}$$

Construct a c.d.f. F from H by first adding weight at the point 0 until the total weight of the measure equals 1, and then redistributing weight from $[0, s)$ to the point s until constraint (27) is satisfied, and redistributing weight from $[-s, 0)$ to the point 0 until constraint (28) is satisfied. Then $W^*_{F,n} \geq W^*_{H,n}$ and F belongs to the constraint set of problem (b, d, s, n). By optimality of F^*, we have $W^*_{F^*,n} \geq W^*_{F,n}$, showing (37). *QED*

Because Θ is convex, we can apply a version of Kuhn and Tucker's theorem to obtain necessary conditions for any solution to $(b, d, s, n)'$. By Lemma 5, these conditions apply to any solution F^* of problem (b, d, s, n).

For any signed Borel measure H on $[-s, s]$, the Gateaux differential of $W^*_{\cdot,n}$ at F^* with increment H, is defined as[21]

$$\delta W^*(F^*, H) \quad = \quad \lim_{\gamma \to 0} \frac{W^*_{F^*+\gamma H,n} - W^*_{F^*,n}}{\gamma}. \tag{38}$$

Lemma 6 *Let F^* be a solution to problem (b, d, s, n). For all signed Borel measures H,*

$$\delta W^*(F^*, H) \quad = \quad n \int_{[-s,s]^n} \left[\sum_{j=1}^{n} v_j \right]_+ dH(v_1) \prod_{k=2}^{n} dF^*(v_k).$$

[21]We drop the argument n when writing the Gateaux differential.

Proof. For any signed Borel measure H on $[-s,s]$,

$$\begin{aligned}
& W^*_{F^*+\gamma H,n} \\
\stackrel{(33)}{=} & \int_{[-s,s]^n} \left[\sum_{j=1}^n v_j\right]_+ \prod_{k=1}^n (\mathrm{d}F^*(v_k)+\gamma \mathrm{d}H(v_k)) \\
= & \int_{[-s,s]^n} \left[\sum_{j=1}^n v_j\right]_+ \left(\prod_{k=1}^n \mathrm{d}F^*(v_k) + \gamma \sum_{i=1}^n \mathrm{d}H(v_i) \prod_{k\neq i} \mathrm{d}F^*(v_k) + o(\gamma)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\delta W^*(F^*,H) &= \sum_{i=1}^n \int_{[-s,s]^n} \left[\sum_{j=1}^n v_j\right]_+ \mathrm{d}H(v_i) \prod_{k\neq i} \mathrm{d}F^*(v_k) \\
&= n \int_{[-s,s]^n} \left[\sum_{j=1}^n v_j\right]_+ \mathrm{d}H(v_i) \prod_{k\neq i} \mathrm{d}F^*(v_k),
\end{aligned}$$

for any $i=1,\ldots,n$. In particular, we can take $i=1$. *QED*

For any c.d.f. F^*, let G^* denote the c.d.f. for $-\sum_{i=2}^n \tilde{v}_i$, given that each $\tilde{v}_i$ has the c.d.f. F^*.

Lemma 7 *Let F^* be a solution of problem (b,d,s,n). Then there exist nonnegative numbers l^+, l^-, p^+, p^-, p such that*

$$\forall v \in [-s,s]: \ \eta(v) \leq 0 \tag{39}$$

and

$$\int_{-s}^{s} \eta(v) dF^*(v) = 0, \tag{40}$$

where

$$\eta(v) \stackrel{def}{=} n\int_{I\!R} [v-w]_+ dG^*(w) - \left((l^+v+p^+)\mathbf{1}_{v>0} + (l^-v+p^-)\mathbf{1}_{v<0} + p\right).$$

Proof. By Lemma 5, F^* is a solution to $(b,d,s,n)'$. Moreover, there exists H (namely, $H=-F^*$) such that $F^*+H \in \Theta$ and all inequality constraints of $(b,d,s,n)'$ are satisfied with strict inequality at $F=F^*+H$. Hence, by Kuhn and Tucker's theorem (see Luenberger, 1969, p. 249),[22]

[22] Luenberger's version of the theorem does not directly apply because of the additional constraint $F \in \Theta$. However, because Θ is convex, the proof of the theorem can be easily adapted to encompass the additional constraint. One only has to add the condition $x_0 + h \in \Theta$ throughout the proof, where x_0 is a solution of the optimization problem, and h is an increment.

there exist nonnegative numbers (Lagrange multipliers) l^+, l^-, p^+, p^-, p such that, for all H with $F^* + H \in \Theta$,

$$\delta W^*(F^*, H) - \int_{-s}^{s} \left((l^+ v + p^+)\mathbf{1}_{v>0} + (l^- v + p^-)\mathbf{1}_{v<0} + p\right) \mathrm{d}H(v) \leq 0. \quad (41)$$

Using $H = F^*$ and $H = -F^*$ in (41), we obtain opposite inequalities. Hence,

$$\delta W^*(F^*, F^*) - \int_{-s}^{s} \left((l^+ v + p^+)\mathbf{1}_{v>0} + (l^- v + p^-)\mathbf{1}_{v<0} + p\right) \mathrm{d}F^*(v) = 0. \quad (42)$$

Using Lemma 6,

$$\begin{aligned} \delta W^*(F^*, F^*) &= n \int_{[-s,s]^n} \left[\sum_{j=1}^{n} v_j\right]_+ \prod_{k=1}^{n} \mathrm{d}F^*(v_k) \\ &= n \int_{-s}^{s} \int_{\mathbb{R}} [v_1 - w]_+ \, \mathrm{d}G^*(w) \mathrm{d}F^*(v_1). \quad (43) \end{aligned}$$

From (42) and (43) we obtain (40).

Letting $H = \mathbf{1}_v$ ($v \in [-s, s]$) in (41) and using Lemma 6, we obtain (39). *QED*

Lemma 8 *Let G be an arbitrary c.d.f. on $\mathbb{R}$, and let $x, y, z \in \mathbb{R}$ such that $x < y < z$ and*

$$\int_{\mathbb{R}} \frac{[z-w]_+ - [y-w]_+}{z-y} dG(w) \leq \int_{\mathbb{R}} \frac{[y-w]_+ - [x-w]_+}{y-x} dG(w).$$

Then $suppG \cap (x, z) = \emptyset$.

Proof. Straightforward by partitioning the area of integration as follows,

$$\mathbb{R} = (-\infty, x] \cup (x, y] \cup (y, z) \cup [z, \infty).$$

QED

Lemma 9 *For any solution F^* of problem (b, d, s, n), if $suppF^* \cap (0, s) \neq \emptyset$, then $suppG^* \cap (0, s) = \emptyset$.*

Proof. Suppose that supp$F^* \cap (0,s) \neq \emptyset$. By (39) and (40), there exists $y \in (0,s)$ such that $\eta(y) = 0$. Choose x, z such that $0 < x < y$ and $y < z < s$. By (39), $\eta(y) - \eta(x) \geq 0$, hence

$$n \int_{\mathbb{R}} ([y-w]_+ - [x-w]_+) \, \mathrm{d}G^*(w) \quad \geq \quad l^+(y-x).$$

Similarly, $\eta(z) - \eta(y) \leq 0$, hence

$$n \int_{\mathbb{R}} ([z-w]_+ - [y-w]_+) \, \mathrm{d}G^*(w) \quad \leq \quad l^+(z-y).$$

The claim follows from Lemma 8. *QED*

Lemma 10 *For any solution F^* of problem (b,d,s,n), if supp$F^* \cap (-s,0) \neq \emptyset$, then supp$G^* \cap (-s,0) = \emptyset$.*

Proof. Analogous to Lemma 9.

Lemma 11 *For any solution F^* of problem (b,d,s,n), supp$F^* \subseteq \{-s,0,s\}$.*

Proof. We show supp$F^* \cap (0,s) = \emptyset$ (the proof of supp$F^* \cap (-s,0) = \emptyset$ is analogous). Suppose that

$$\exists x \in \mathrm{supp} F^* \cap (0,s). \tag{44}$$

First consider the case $0 \in \mathrm{supp}F^*$. By definition of G^*,

$$-\mathrm{supp} F^* \subseteq \mathrm{supp} G^*. \tag{45}$$

Using (44) and (45) we find

$$-x \in (-\mathrm{supp}F^*) \cap (-s,0) \subseteq \mathrm{supp}G^* \cap (-s,0).$$

Hence, supp$F^* \cap (-s,0) = \emptyset$ by Lemma 10. Thus, using (28) with $F = F^*$,

$$\mathrm{supp}F^* \cap [-s,0) = \{-s\}.$$

Hence, using the definition of G^* and the assumption $n \geq 3$,

$$-s + x + \underbrace{0 + \ldots + 0}_{n-3 \text{ times}} \in (-\mathrm{supp}G^*) \cap (-s,0) = -(\mathrm{supp}G^* \cap (0,s)),$$

which contradicts Lemma 9 because of (44).

Now consider the case $0 \notin \text{supp}F^*$. Because the constraint (28) holds for $F = F^*$,

$$\exists y \in \text{supp}F^* \cap (-s, 0). \tag{46}$$

Using (44) and Lemma 9, $\text{supp}G^* \cap (0, s) = \emptyset$. Using (46) and Lemma 10, $\text{supp}G^* \cap (-s, 0) = \emptyset$. Hence,

$$\text{supp}G^* \cap [-s, s] \subseteq \{-s, 0, s\}. \tag{47}$$

For $k = 2, \ldots, n$, define $\gamma(k) = (n - k)y + (k - 1)x \in -\text{supp}G^*$.

We claim that

$$\exists \hat{k} \in \{2, \ldots, n - 1\} : \ \gamma(\hat{k}) = 0. \tag{48}$$

If $\gamma(2) \neq 0$, then $\gamma(2) \leq -s$ by (47). On the other hand, $\gamma(n) > 0$, and

$$\gamma(k) = \gamma(2) + (k - 2)\underbrace{(x - y)}_{\in(0,2s)}.$$

Hence, there exists $\hat{k}$ such that $\gamma(\hat{k}) \in (-s, s)$, implying (48) by (47).

By Lemma 4, there exists $x' \neq x$ such that $x' \in \text{supp}F^* \cap (0, s]$, or $y' \neq y$ such that $y' \in \text{supp}F^* \cap [-s, 0)$. If an x' exists, then $|x - x'| < s$, hence

$$-\text{supp}G^* \ni (n - \hat{k})y + (\hat{k} - 2)x + x' = \gamma(\hat{k}) + (x' - x) = x' - x,$$

contradicting (47). Similarly, if an y' exists then $|y - y'| < s$, hence

$$-\text{supp}G^* \ni (n - \hat{k} - 1)y + y' + (\hat{k} - 1)x = \gamma(\hat{k}) + (y' - y) = y' - y,$$

contradicting (47). *QED*

Lemma 12 *The unique solution of problem* (b, d, s, n) *is given by*

$$\arg \max_{F \in \mathcal{C}(b,d,s)} W^*_{F,n} \quad = \quad \{F^*_{b,d,s,0}\}.$$

Proof. Straightforward from Lemma 11 and the constraints of problem (b, d, s, n). *QED*

Proof of Proposition 1.

“(22) $\Rightarrow$ (21).” Suppose that (22) holds. Using Lemma 1 and Lemma 12,

$$\lim_{m\to\infty} W^*_{F^m,n} = W^*_{F^*_{b,d,s,0},n} > W^*_{F,n}, \tag{49}$$

for all $F \in \mathcal{F}(b,d,s)$. Hence,

$$\lim_{m\to\infty} \rho(F^m,n) \overset{(7),(11),(14)}{=} \frac{n\beta(b,d,n) - \frac{n}{2}(d-(1-d)b)}{\lim_{m\to\infty} W^*_{F^m,n} - \frac{n}{2}(d-(1-d)b)} \overset{(49)}{<} \rho(F,n).$$

This implies (21), by definition of $\underline{\rho}$.

“(21) $\Rightarrow$ (22).” Let (F^m) be a sequence in $\mathcal{F}(b,d,s)$ that satisfies (21).

Because $W^{\text{opt maj}}_{\cdot,n}$ is constant on $\mathcal{F}(b,d,s)$ by (13),

$$\lim_{m\to\infty} W^*_{F^m,n} = \sup_{F\in\mathcal{F}(b,d,s)} W^*_{F,n}. \tag{50}$$

Observe that

$$\sup_{F\in\mathcal{F}(b,d,s)} W^*_{F,n} \overset{(17)}{\geq} \limsup_{t\to 0} W^*_{F^*_{b,d,s,t},n} \overset{\text{Lemma 1}}{=} W^*_{F^*_{b,d,s,0},n}. \tag{51}$$

By (31) and Lemma 12,

$$\sup_{F\in\mathcal{F}(b,d,s)} W^*_{F,n} \leq W^*_{F^*_{b,d,s,0},n}. \tag{52}$$

From (50), (51), and (52),

$$\lim_{m\to\infty} W^*_{F^m,n} = W^*_{F^*_{b,d,s,0},n}. \tag{53}$$

Suppose that $F^m \not\to F^*_{b,d,s,0}$. Then there exists a subsequence (F^{m_k}) and an open neighborhood N of $F^*_{b,d,s,0}$ such that

$$\forall k:\ F^{m_k} \notin N.$$

Using Lemma 2, $\overline{\mathcal{F}}(b,d,s)\backslash N$ is compact. Hence, by the Bolzano Weierstrass theorem, (F^{m_k}) has a subsequence $(F^{m_{k_l}})$ that converges to some

$$F^* \in \overline{\mathcal{F}}(b,d,s) \setminus N. \tag{54}$$

Using Lemma 1,

$$W^*_{F^*,n} = \lim_{l\to\infty} W^*_{F^{m_{k_l}},n} \overset{(53)}{=} W^*_{F^*_{b,d,s,0},n}. \tag{55}$$

By (54) and Lemma 2, $F^* \in \mathcal{C}(b,d,s)$. Hence, $F^* = F^*_{b,d,s,0}$ by (55) and Lemma 12, in contradiction with (54). Hence, $F^m \Rightarrow F^*_{b,d,s,0}$, showing (22). *QED*

Proof of Proposition 2. From (18) and Proposition 1,

$$\begin{aligned} \underline{\rho}(b,d,s,n) &= \lim_{t\to 0} \rho(F^*_{b,d,s,t}, n) \\ &\overset{(11),\ (17)}{=} \frac{n\beta(b,d,n) - \frac{n}{2}(d-(1-d)b)}{\lim_{t\to 0} W^*_{F^*_{b,d,s,t},n} - \frac{n}{2}(d-(1-d)b)}. \end{aligned}$$

Hence, the result follows from Lemma 1. *QED*

Proof of Corollary 2. From (12) one sees that

$$\beta(b,d,n) \quad = \quad \frac{1}{n} E\left[\left[\sum_{i=1}^{n} \tilde{b}_i\right]_+\right], \tag{56}$$

where $\tilde{b}_1, \ldots, \tilde{b}_n$ denote stochastically independent random variables, each taking the value 1 with probability d and taking the value $-b$ with probability $1-d$.

By the law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \tilde{b}_i \quad = \quad E[\tilde{b}_i] = d-(1-d)b. \tag{57}$$

By (56) and (57),

$$\lim_{n\to\infty} \beta(b,d,n) \quad = \quad [d-(1-d)b]_+ .$$

Thus,

$$\lim_{n\to\infty} \beta(b,d,n) - \frac{1}{2}(d-(1-d)b) \quad = \quad \frac{1}{2} \mid d-(1-d)b \mid . \tag{58}$$

Let $\tilde{v}_i$ denote a random variable with c.d.f. $F^*_{b,d,s,0}$. By the law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \tilde{v}_i \quad = \quad E[\tilde{v}_i] = d-(1-d)b. \tag{59}$$

From (6) and (59),

$$\lim_{n\to\infty} \frac{1}{n} W^*_{F^*_{b,d,s,0},n} - \frac{1}{2}(d-(1-d)b) \quad = \quad \frac{1}{2} \mid d-(1-d)b \mid . \tag{60}$$

Suppose that $d-(1-d)b \neq 0$. Then (25) follows by dividing (58) through (60).

Suppose that $d-(1-d)b = 0$. Then $E[\tilde{v}_i] = E[\tilde{b}_i] = 0$. By the central limit theorem, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{b}_i \quad \Rightarrow \quad \tilde{n}_1,$$

where $\Rightarrow$ denotes weak convergence, and $\tilde{n}_1$ denotes a normally distributed random variable with mean 0 and variance $VAR[\tilde{b}_i] = b$. Hence, using (12),

$$\sqrt{n}\;\beta(b,d,n) \quad \to \quad E\left[[\tilde{n}_1]_+\right]. \tag{61}$$

Similarly, using (6),

$$\frac{1}{\sqrt{n}} W^*_{F^*_{b,d,s,0},n} \quad \to \quad E\left[[\tilde{n}_2]_+\right], \tag{62}$$

where $\tilde{n}_2$ denotes a normally distributed random variable with mean 0 and variance

$$VAR[\tilde{v}_i] = E[\tilde{v}_i^2] = s(b(1-d)+d) = \frac{2sb}{1+b} =: \sigma^2_{b,s}.$$

Using again $d-(1-d)b = 0$, we can conclude from (61) and (62) that

$$\begin{aligned}
\lim_{n\to\infty} \underline{\rho}(b,d,s,n) \quad &= \quad \frac{E\left[[\tilde{n}_1]_+\right]}{E\left[[\tilde{n}_2]_+\right]} \\
&= \quad \frac{\frac{1}{\sqrt{b}} \int_0^\infty t e^{-\frac{t^2}{2b}} \mathrm{d}t}{\frac{1}{\sigma_{b,s}} \int_0^\infty t e^{-\frac{t^2}{2\sigma^2_{b,s}}} \mathrm{d}t} \quad = \quad \frac{\frac{1}{\sqrt{b}}(-b)\; e^{-\frac{t^2}{2b}}\Big|_0^\infty}{\frac{1}{\sigma_{b,s}}(-\sigma^2_{b,s})\; e^{-\frac{t^2}{2\sigma^2_{b,s}}}\Big|_0^\infty} \\
&= \quad \sqrt{\frac{1+b}{2s}}.
\end{aligned}$$

QED

Proof of Corollary 3. In the proof of Corollary 2 we have seen that

$$\lim_{n\to\infty} \beta(b,d,n) \quad = \quad [d-(1-d)b]_+ \, .$$

Using this in (24) we obtain the result. *QED*

Proof of Proposition 3. "if": By (25), the sequence of continuous functions $(\underline{\rho}(b, \cdot, s, n))_{n=3,4,\ldots}$ converges pointwise to 1 on K, as $n \to \infty$. Because K is compact, the convergence is uniform, that is,

$$\lim_{n\to\infty} \sup_{d\in K} \mid \underline{\rho}(b, d, s, n) - 1 \mid \quad = \quad 0.$$

This implies

$$\lim_{n\to\infty} \inf_{d\in K} \underline{\rho}(b, d, s, n) \quad = \quad 1,$$

as was to be shown.

"only if": Suppose that $s = \infty$. Then worst-case asymptotic efficiency cannot hold, by (26).

Suppose that $b/(1+b) \in K$. Defining $d = b/(1+b)$, we have $d-(1-d)b = 0$, hence worst-case asymptotic efficiency cannot hold, by (25). *QED*

Proof of Proposition 4. "if": follows from (26).

"only if": Suppose that $s < \infty$. Define

$$\rho^* \quad = \quad \sqrt{\frac{1+b}{2s}}.$$

Extend $\underline{\rho}$ continuously to points with $d = 0$ and $d = 1$, that is,

$$\underline{\rho}(b, 0, s, n) = 1, \qquad \underline{\rho}(b, 1, s, n) = 1.$$

By (25), the sequence of continuous functions $(\min\{\underline{\rho}(b, \cdot, s, n), \rho^*\})_{n=3,4,\ldots}$ converges pointwise to ρ^* on $[0, 1]$, as $n \to \infty$. Because $[0, 1]$ is compact, the convergence is uniform, that is,

$$\lim_{n\to\infty} \sup_{d\in[0,1]} \mid \min\{\underline{\rho}(b, d, s, n), \rho^*\} - \rho^* \mid \quad = \quad 0.$$

This implies

$$\liminf_{n\to\infty} \inf_{d\in[0,1]} \underline{\rho}(b, d, s, n) \quad \geq \quad \rho^*,$$

showing that the optimal majority rule is not worst-case asymptotically garbled in $\mathcal{F}(b, (0, 1), s)$.

Now suppose that $b/(1+b) \notin K$. Then a uniform-convergence argument using (26) shows that the optimal majority rule is not worst-case asymptotically garbled in $\mathcal{F}(b, K, \infty)$. *QED*

Proof of the claim made in footnote 18. Let $\alpha = b/(1+b)$. By (24), it is sufficient to show that

$$\beta(b, d, n) \quad \geq \quad \beta(b, d, n+1). \tag{63}$$

Using Remark 1 and (11), one sees that (63) is equivalent to

$$\frac{W^{\text{opt maj}}_{F,n}}{n} \quad \geq \quad \frac{W^{\text{opt maj}}_{F,n+1}}{n+1}, \tag{64}$$

for any F with bias b and desirability d.

Suppose that αn is an integer. Let $k = \alpha n$.

Call agent i a Y-agent if $v_i > 0$, and call her an N-agent otherwise.

Consider the following rule *. If the number of Y-agents in $\{1, \ldots, n\}$ is greater than or equal to $k+1$, the public good is provided. If the number of Y-agents in $\{1, \ldots, n\}$ is smaller than or equal to $k-1$, it is not provided. If the number of Y-agents in $\{1, \ldots, n\}$ equals k, agent $n+1$ decides about the provision.

Observe that rule * is identical to the optimal majority rule with $n+1$ agents, because

$$k < \alpha(n+1) < k+1.$$

We claim that

> ** the expectation of the joint surplus of the agents 1 to n according to rule * is identical to the expectation of the joint surplus of the agents 1 to n according to the optimal majority rule in the environment without agent $n+1$,

thus showing that (64) holds with "=".

Let $\tilde{k}$ denote the random variable for the number of Y-agents in $\{1, \ldots, n\}$. According to rule *, conditional on the event $\tilde{k} \neq k$, agent $n+1$ has no impact on the collective decision, so that the joint surplus of the agents 1 to n is the same as in the optimal majority rule in an environment with n agents.

Conditional on the event $\tilde{k} = k$, the *expectation* of the joint surplus of the agents 1 to n according to the optimal majority rule in an environment with n agents is

$$k \cdot 1 + (n-k)(-b) \quad = \quad 0.$$

According to rule *, the expectation of the joint surplus of the agents 1 to n conditional on the event $\tilde{k} = k$ is still 0. While agent $n+1$ makes the decision, this has no impact on the conditional expectation because the valuation of agent $n+1$ is stochastically independent of the valuations of the agents 1 to n. This completes the proof of **.

Suppose that αn is not an integer. Then there exists an integer k such that

$$k < \alpha n < k+1.$$

There are two cases. If $\alpha(n+1) \leq k+1$, the rule * is, as before, identical to the optimal majority rule with $n+1$ agents. However, ** does not hold. Conditional on the event $\tilde{k} = k$, the expectation of the joint surplus of the agents 1 to n according to the optimal majority rule in an environment with n agents is 0 (because the public good is not provided). But according to rule * the public good is provided whenever agent $n+1$ is a Y-agent. Conditional on the event that $\tilde{k} = k$ and agent $n+1$ is a Y-agent, the expectation of the joint surplus of the agents 1 to n according to rule * is

$$k \cdot 1 + (n-k)(-b) \quad < \quad 0.$$

Hence, (64) holds with "$>$".

If $\alpha(n+1) > k+1$, the rule * is not identical to the optimal majority rule with $n+1$ agents. In this case, the joint surplus of the agents 1 to n according to the optimal majority rule with $n+1$ agents is smaller than according to rule *, which in turn is smaller than their joint surplus according to the optimal majority rule in an environment with n agents. Again, (64) holds with "$>$". *QED*

Appendix B

In this section, we assume that each agent is privately informed about her valuation.

A *(transfer-free) decision rule* is a function

$$\phi : \Omega^n \rightarrow [0,1],$$

where $\Omega = \text{supp}F \setminus \{0\}$ denotes each agent's type space, and $\phi(v_1, \ldots, v_n)$ denotes the probability that the public good is provided if the realized type profile is $(v_1, \ldots, v_n)$. A *transfer-free mechanism* M is a game form that assigns collective decisions to action profiles,

$$M : \ A_1 \times \ldots \times A_n \rightarrow [0,1],$$

where A_i denotes player i's action space in M. By the revelation principle, a decision rule ϕ is a Bayesian Nash equilibrium outcome of some transfer-free mechanism if and only if it is *incentive compatible*,

$$\forall i, v_i, v_i' : \; E[\phi(v_i, \tilde{v}_{-i})]v_i \geq E[\phi(v_i', \tilde{v}_{-i})]v_i. \tag{65}$$

The following result characterizes incentive compatible decision rules. The proof is immediate from (65).

Lemma 13 *A decision rule ϕ is incentive compatible if and only if there exist numbers $q_1^+, \ldots, q_n^+ \in [0,1]$ and $q_1^-, \ldots, q_n^- \in [0,1]$ such that,*

$$\forall i, v_i : \; E[\phi(v_i, \tilde{v}_{-i})] = \begin{cases} q_i^+ & \text{if } v_i > 0, \\ q_i^- & \text{if } v_i < 0, \end{cases}$$

and $q_i^+ \geq q_i^-$ for $i = 1, \ldots, n$.

Our goal is to find an incentive compatible decision rule that maximizes the social surplus

$$W_{F,n}(\phi) = E\left[\sum_{i=1}^{n} \tilde{v}_i \phi(\tilde{v})\right].$$

A decision rule ϕ is *second best* in environment (F, n) if it maximizes $W_{F,n}(\phi)$ among all incentive compatible decision rules.

For all $\alpha \in (0,1)$, the *α-majority rule* is given by

$$\phi_{\alpha\text{-maj}}(v_1, \ldots, v_n) = \begin{cases} 1 & \text{if } |\{i | v_i > 0\}| \geq \alpha \, n, \\ 0 & \text{otherwise.} \end{cases}$$

It is useful to define the partition of the set of vectors in $\mathbb{R}^n$ with nonzero components into orthants,

$$\mathcal{Q}(n) = \{I_1 \times \ldots \times I_n \mid \forall i : I_i = (0, \infty) \text{ or } I_i = (-\infty, 0)\}.$$

Each element of $\mathcal{Q}(n)$ corresponds to a set of vectors that is characterized by a constant sign for each agent's type (if $n = 2$ then $\mathcal{Q}(n)$ consists of four quadrants).

A decision rule ϕ is called a *voting rule* if ϕ is constant on each element of $\mathcal{Q}(n)$.[23]

[23]From Lemma 13 it is clear that not every voting rule is incentive compatible. Moreover, it can be shown that not every incentive compatible decision rule is a voting rule.

Lemma 14 below shows that any level of social surplus that can be obtained with an incentive compatible decision rule can also be obtained with an incentive compatible voting rule. The idea of the proof is to compute, for an arbitrary incentive compatible decision rule ϕ, the average probability of providing the public good on each element of $\mathcal{Q}(n)$. We construct a voting rule $\hat{\phi}$ via these averages, and show that $\hat{\phi}$ is incentive compatible and yields the same social surplus as ϕ.

Lemma 14 *For any incentive compatible decision rule ϕ there exists an incentive compatible voting rule $\hat{\phi}$ such that $W_{F,n}(\phi) = W_{F,n}(\hat{\phi})$.*

Proof. Let $q_1^+, \ldots, q_n^+, q_1^-, \ldots, q_n^-$ denote the parameters associated with ϕ according to Lemma 13. Then,

$$\begin{aligned} W_{F,n}(\phi) &= \sum_{i=1}^{n} E\left[\tilde{v}_i \phi(\tilde{v}_1, \ldots, \tilde{v}_n)\right] \\ &= \sum_{i=1}^{n} E\left[\tilde{v}_i E\left[\phi(\tilde{v}_i, \tilde{\mathbf{v}}_{-\mathbf{i}}) \mid \tilde{v}_i\right]\right] \\ &= \sum_{i=1}^{n} E\left[\tilde{v}_i (q_i^+ \mathbf{1}_{\tilde{v}_i > 0} + q_i^- \mathbf{1}_{\tilde{v}_i < 0})\right] \\ &= \sum_{i=1}^{n} \left(q_i^+ E\left[\tilde{v}_i \mathbf{1}_{\tilde{v}_i > 0}\right] + q_i^- E\left[\tilde{v}_i \mathbf{1}_{\tilde{v}_i < 0}\right]\right). \end{aligned} \tag{66}$$

Let $\tilde{\mathbf{v}} = (\tilde{v}_1, \ldots, \tilde{v}_n)$. Define a rule $\hat{\phi}$ by $\hat{\phi}(\mathbf{v}) = E[\phi(\tilde{\mathbf{v}}) \mid \tilde{\mathbf{v}} \in Q]$ for all $\mathbf{v} \in Q \cap \mathcal{S}$ and all $Q \in \mathcal{Q}(n)$. Hence, $\hat{\phi}$ is a voting rule.

Because $q_i^+ = E[\phi(v_i, \tilde{\mathbf{v}}_{-\mathbf{i}})]$ for every $v_i > 0$ by Lemma 13,

$$q_i^+ = E[\phi(\tilde{v}_i, \tilde{\mathbf{v}}_{-\mathbf{i}}) \mid \tilde{v}_i > 0].$$

Hence, for all $v_i > 0$,

$$\begin{aligned} q_i^+ &= \sum_{Q = I_1 \times \ldots \times I_n \in \mathcal{Q}(n),\ v_i \in I_i} E[\phi(\tilde{\mathbf{v}}) \mid \tilde{\mathbf{v}} \in Q] \cdot \Pr[\tilde{\mathbf{v}} \in Q \mid \tilde{v}_i > 0] \\ &= \sum_{Q = I_1 \times \ldots \times I_n \in \mathcal{Q}(n),\ v_i \in I_i} E[\hat{\phi}(v_i, \tilde{\mathbf{v}}_{-\mathbf{i}}) \mid \tilde{\mathbf{v}} \in Q] \cdot \Pr[\tilde{\mathbf{v}} \in Q \mid \tilde{v}_i > 0] \\ &= E[\hat{\phi}(v_i, \tilde{\mathbf{v}}_{-\mathbf{i}}) \mid \tilde{v}_i > 0] \\ &= E[\hat{\phi}(v_i, \tilde{\mathbf{v}}_{-\mathbf{i}})]. \end{aligned}$$

Similarly, for all $v_i < 0$, we find $q_i^- = E[\hat{\phi}(v_i, \tilde{\mathbf{v}}_{-\mathbf{i}})]$. Thus, $\hat{\phi}$ is incentive compatible by Lemma 13. Taking the steps leading to (66) in reverse order while replacing ϕ with $\hat{\phi}$, we obtain $W_{F,n}(\hat{\phi}) = W_{F,n}(\phi)$. *QED*

Proposition 5 shows that some α-majority rule is always second best, and characterizes the optimal α.[24]

Proposition 5 *For any environment* (F, n)*, an* α*-majority rule with* $\alpha = b_F/(1+b_F)$ *is second best.*

To prove this result, we can, by Lemma 14, focus on voting rules. We determine an optimal voting rule, ignoring incentive compatibility (65) at first. The resulting rule is the α-majority rule.

Proof of Proposition 5. Let ϕ be a second-best decision rule. By Lemma 14, we can assume w.l.o.g. that ϕ is a voting rule. We use the shortcut $\phi(Q) = \phi(v)$ for any $v \in Q$ and $Q \in \mathcal{Q}(n)$. Then,

$$\begin{aligned} W_{F,n}(\phi) &= \sum_{Q \in \mathcal{Q}(n)} \phi(Q) E[\sum_i \tilde{v}_i \mid \tilde{\mathbf{v}} \in Q] \Pr[\tilde{\mathbf{v}} \in Q] \\ &= \sum_{Q \in \mathcal{Q}(n)} \phi(Q) \sum_i E[\tilde{v}_i \mid \tilde{\mathbf{v}} \in Q] \Pr[\tilde{\mathbf{v}} \in Q] \\ &= \sum_{Q = I_1 \times \ldots \times I_n \in \mathcal{Q}(n)} \phi(Q) \Pr[\tilde{\mathbf{v}} \in Q] \sum_i E[\tilde{v}_i \mid \tilde{v}_i \in I_i]. \end{aligned}$$

This expression is maximized if, for all $Q = I_1 \times \ldots \times I_n \in \mathcal{Q}(n)$,

$$\phi(Q) = \begin{cases} 1 & \text{if } \sum_i E[\tilde{v}_i \mid \tilde{v}_i \in I_i] > 0, \\ 0 & \text{if } \sum_i E[\tilde{v}_i \mid \tilde{v}_i \in I_i] < 0. \end{cases} \tag{67}$$

Defining $k(Q) = |\{i | I_i = (0, \infty)\}|$,

$$\sum_i E[\tilde{v}_i \mid \tilde{v}_i \in I_i] = k(Q) - b_F(n - k(Q)) = (1+b_F)n(\frac{k(Q)}{n} - \frac{b_F}{1+b_F}). \tag{68}$$

By (67) and (68), the α-majority rule with $\alpha = b_F/(1+b_F)$ is second best. *QED*

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[24] The proof is similar to the proof of Barbera and Jackson (2006, Theorem 1). Barbera and Jackson do not consider arbitrary transfer-free decision rules, but focus on the class of *weighted majority rules* in general asymmetric environments and compute optimal weights. Our arguments can be extended to asymmetric environments to show that the optimal weighted majority rule is a second best transfer-free decision rules.

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