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Abstract

We use a simple balanced budget contest to collect taxes on a private good in order to finance a pure public good. We show that—with an appropriately chosen structure of winning probabilities—this contest can provide the public good efficiently and without distorting private consumption. We provide extensions to multiple public goods and private taxation sources, asymmetric preferences, and show the mechanism's robustness across these settings. (JEL *C7, D7.* Keywords: *Taxation, Contests, Efficiency.*)

1 Introduction

The \$240 billion worldwide lottery industry is a thriving business by any standard.[1] Most governments regulate lottery activities to some extent and many participate in the generated revenues in some form or another. The lotteries in the 46 United States' jurisdictions which allowed such activities in 2009 represented a combined revenue of more than \$52.3 billion resulting in \$17.7 billion profits (Bloomberg, 11-Jan-2011). The resulting tax proceeds were and are essential for communities to balance budgets and finance public projects.

An example of a carefully designed lottery to directly finance public goods are the United Kingdom Premium Bonds.[2] A Premium Bond is a lottery bond issued by the UK government's National Savings and Investments scheme. The UK government promises to buy back the bond, on request, for its original price and pays interest on the bond (pegged at 1.5% in July 2010). But instead of the interest being paid into individual accounts, it is paid into a prize fund from which a monthly lottery distributes tax-free prizes, or premiums, to those bond-holders whose numbers are selected

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[1] LaFleur's World Lottery Almanac 2010, http://www.lafleurs.com.

[2] http://www.nsandi.com/products/pb.

randomly. In July 2010, the total estimated value of the UK premium bond prize pot was £52.3 million. The machine that generates the random numbers underlying the lottery is called ERNIE, for Electronic Random Number Indicator Equipment. In principle, this machine could generate random numbers according to any probability distribution specified and in particular it would be easy to generate the winning probabilities we design in the present paper.

This paper provides an answer to the question of why lotteries are so commonly used to finance public goods: they may induce efficient allocations. The mechanisms we design for this purpose are simple contests capable of collecting taxes on one or more private goods in order to finance any number of pure public goods. Since we employ contests which retain an element of luck to winning, there is a lottery feel to these mechanisms which is crucial for our results. Thus, we refer to these mechanisms as contests or lotteries synonymously throughout the paper. We show that—with appropriately chosen parameters—our contests can provide public goods efficiently and without distortion of private consumption. The tax parameters are individually rational and unanimously accepted if proposed.

The main idea behind our contest scheme is to carefully construct overinvestment incentives into a lottery. These overinvestments are siphoned into financing a public good which would not be provided in efficient quantity given private contributions. Our main point is that, in many cases, the lottery incentives can be designed such that the public good is provided efficiently. We show that any private good can be used to generate the required lottery. Our surprising result is that we can design the lottery such that *both* the public and the private goods are supplied efficiently.

Literature

To use lotteries for taxation purposes is not a new idea.[3] Nevertheless, we are only aware of a handful of papers developing ideas directly related to this paper, that is, to taxation through contests: Morgan (2000), Morgan and Sefton (2000), Moir (2004), Goeree, Maasland, Onderstal, and Turner (2005), and Gee (2010).[4] Contrary to the present analysis, these papers are mainly concerned with raising funds and not with designing a mechanism capable of achieving efficiency. From a technical modelling point of view, they employ separable (quasi-linear) utility, and the Tullock and all-pay-auction contest success functions, respectively.

Morgan (2000) is by far the closest paper to our analysis. Without considering a private good, he studies aspects of the 'simplified Tullock' case of our environment, that is, he does not design the winning probability aspect of the lottery but uses an exogenously fixed technology. His results

[3] "A Lottery is a Taxation, Upon all the Fools in Creation; And Heavn be praisd, It is easily raisd, Credulitys always in Fashion; For, Follys a Fund, Will never lose Ground; While Fools are so rife in the Nation." Henry Fielding, *The Lottery* (London: J. Watts, 1732), Scene 1, quoted in Clotfelter and Cook (1989, 219). Earlier still, according to `wikipedia`, Keno lottery slips from the Chinese Han Dynasty (205–187 B.C.) are believed to have helped financing the construction of the Great Wall of China.

[4] The idea that in some circumstances efficiency can be induced through a rank order tournament is due to Lazear and Rosen (1981). In these tournaments, prizes are allocated according to a relative ranking, hence ordinal information on performance is sufficient. This idea has found numerous applications and extensions, for instance in the work of Green and Stokey (1983), Nalebuff and Stiglitz (1983), Dixit (1987), Moldovanu and Sela (2001), or Siegel (2009). For a detailed survey of the contests literature see the comprehensive Konrad (2008).

have therefore a more negative flavour than ours. In his paper, the amount of public good provided depends on the size and structure of the lottery prize and on the return from the public good itself. In sum, Morgan's lottery tickets cost more than the prize sum needed to attract participation and he uses the surplus to finance the public good. He shows that if the prize sum is fixed in advance, a sufficiently large prize can provide the public good at close to the efficient level while if the prize is a percentage of contributions, the lottery does not perform better than voluntary contributions. Goeree, Maasland, Onderstal, and Turner (2005) introduce a general class of all-pay auctions, rank their revenues, and illustrate the extent to which they dominate winner-pay auctions and lotteries. Moreover, they identify the optimal fund-raising mechanism as being of the all-pay format they investigate. Moir and Childs (2005) and Moir (2006) discuss difficulties in the efficient design of one or more public goods lotteries in a comparatively conversational manner. The empirical relevance of the use of lottery money for financing public goods is testified to by, for instance, Landry and Price (2007). For a critical appraisal of tax lotteries see Hansen (2005).

The large literature on variants of Vickrey-Clarke-Groves mechanisms applied to taxation problems assumes incomplete information on preferences. An example tailored directly to public goods provision is Ledyard and Palfrey (1994). Modern exponents are Bierbrauer (2009), Bierbrauer and Hellwig (2009), or Hellwig (2007), but although fascinating and ingenious, this literature is not applicable to our classic perfect information setup.

As it is impossible to review the vast literature on optimal taxation here, we contend ourselves with pointing out the key advantages of our approach with respect to the classic remedies developed in the literature on efficient complete information taxation.[5] The textbook solution to avoiding the distortive effect of taxation is through lump-sum taxes. Implementing these, however, is politically often outright impossible. One difficulty with the Lindahl approach is the assumption that an individual expects to consume the amount of the (non-rival and non-excludable) public good, which solves the individual's utility maximisation problem. Thus, free riding on the public good is problematic given the efficient provision of the public good through the other players. Thus the Lindahl equilibrium is not a Nash equilibrium while ours is. A Pigouvian tax subsidises the public good such that an efficient allocation is obtained. Hence, this mechanism is not balanced budget. The idea of Coasian bargaining requires property rights for the public good which is not easily justified in many public goods contexts.

Following the model definition in section 2, we present the idea of tax contests through an illustrative example in section 3. Although highly stylised, this simple example conveys much of the intuition of the general results presented in section 4. The subsections extend these results to the asymmetric case and compare them with alternative schemes. Robustness checks and model extensions are examined in sections 5 and 6. All proofs are in the appendix.

[5] For recent and comprehensive surveys see Silvestre (2003) or Mankiw, Weinzierl, and Yagan (2009).

2 The model

There is a set $\mathcal{N}$ of $n > 1$ risk-neutral, identical individuals $i \in \mathcal{N}$ who each consume a public good, G, and can purchase quantities $x_i \in [0, \infty)$, $\mathbf{x} = (x_1, \ldots, x_n)$, of some private good. Consumption choices are not verifiable but a sales tax can be imposed and collected. Individual utility $u_i(\cdot)$, $i \in \mathcal{N}$, is assumed to be additively separable in consumption and money and is given by

$$u_i(x_i, G) = w + v(x_i, G) - px_i, \tag{1}$$

where w is monetary wealth and p is the unit price of the private good; we assume the latter to be unperturbed by taxation. The public good is produced at strictly monotonic and differentiable cost $C(G)$ and we assume v to be strictly quasi-concave in both arguments.

In order to overcome the free-riding problem associated with the provision of the public good, we introduce a contest tax scheme as follows. A sales tax of proportion $\alpha > 0$ is collected in order to form a prize pool

$$P = \alpha p \sum_{j=1}^{n} x_j. \tag{2}$$

The share $(1 - \beta)$ of this prize pool is used to finance the public good and the remaining share β is awarded as the winner's prize in a contest held on the purchased quantity of the private good. The participation utility in such a contest is

$$u_i(\mathbf{x}) = w + v(x_i, G) + \pi_i(x)\beta P - (1 + \alpha)px_i, \tag{3}$$

where $\pi_i(\mathbf{x})$ is player i's probability of winning the contest as a function of all players' private good consumption. Thus, the prize βP goes to the winner of the contest and the second prize, $(1 - \beta)P$, is enjoyed by all consumers. The amount of the public good, G, is implicitly given by $C(G) = (1 - \beta)P$. We assume that the noisy (partial) ranking $\pi(\mathbf{x}) = (\pi_1(\mathbf{x}), \ldots, \pi_n(\mathbf{x}))$ of the players' consumption expenditures is observe- and verifiable. We assume that $\pi_i(\mathbf{x})$ is strictly increasing in x_i, strictly decreasing in all other arguments, equal to ${}^1/_n$ for identical arguments, twice continuously differentiable, and zero for $x_i = 0$ with at least one $x_{j \neq i} > 0$, $j \in \mathcal{N}$. In order to simplify the exposition of the results in the main body of the paper, we assume the contest to be governed by the generalised Tullock contest success function, that is, by the winning probability

$$\pi_i(x) = \frac{x_i^r}{\sum_{j=1}^{n} x_j^r}, \quad r > 0, \tag{4}$$

with $\pi_i(\mathbf{x}) = 1/n$ for $x_1 = x_2 = \cdots = x_n = 0$.[6] For this specification, we interpret the exponent r as the 'power' of the contest success function defining the (marginal) increase in the (marginal) probability of winning from higher consumption of the private good. This formulation allows for a

[6] The Tullock success function has been axiomatised by Skaperdas (1996) and others who show that only variants of the Tullock contest success function satisfy a set of desiderata similar to our assumptions in section 5.1. Fu and Lu (2007) and Jia (2008) derive distribution-based foundations for the general Tullock formulation.

more compact presentation of our results than the general framework. Our results, however, hold for a wider range of contest specifications, some of which we explore in section 5.

The objective of the designer is to maximise total utility by choosing a rank order taxation contest $\langle \alpha^*, \beta^*, \pi(\mathbf{x})\rangle$ which allows for the efficient provision of the public good while simultaneously ensuring efficient private good consumption. When the specific generalised Tullock formulation is used, then we write the efficient scheme as $\langle \alpha^*, \beta^*, r\rangle$.

3 Example

This section presents an illustrative example based on a Tullock contest and Cobb-Douglas preferences where the public good is produced at linear cost, $C(G) = qG$ with $q > 0$.[7] The idea is that we levy a (sales) tax equal to a share α of the consumption expenditures on the private good x. In return, the consumer gets a (non-transferable) lottery ticket entitling to the participation in a prize draw. We assume that $v(x, G) = x^a G^b$ with $a, b > 0$, $a + b < 1$, implying that the social planner's problem is concave and its solution is given by

$$W^* = \max_{x,G} n \left(w + x^a G^b - px\right) - qG. \tag{5}$$

The efficient quantities x^* and G^* are positive and unique. They are found by solving the first-order conditions with respect to x and G

$$x^* = \left(\frac{a}{p}\right)^{\frac{1-b}{1-a-b}} \left(\frac{nb}{q}\right)^{\frac{b}{1-a-b}}, \quad G^* = \left(\frac{a}{p}\right)^{\frac{a}{1-a-b}} \left(\frac{nb}{q}\right)^{\frac{1-a}{1-a-b}}. \tag{6}$$

Assuming symmetric contributions, each consumer needs to contribute a monetary amount equal to $qg^* = qG^*/n$ in order to provide the efficient amount of the public good. Given this, it is individually rational to consume the amount x^* of the private good.

In a game with voluntary contributions, taking existence of a symmetric Nash equilibrium as given, player i solves

$$\max_{x_i, g_i} \; w + x_i^a (g_i + (n-1)g_j)^b - px_i - qg_i. \tag{7}$$

Denoting the symmetric equilibrium strategies by x^{eq} and g^{eq}, we find

$$\frac{x^{eq}}{x^*} = n^{\frac{b}{1-a-b}}, \quad \frac{g^{eq}}{g^*} = n^{\frac{1-a}{1-a-b}}, \tag{8}$$

implying the well-known distortion that is due to the positive external effect which player i's contribution has on the other $n - 1$ players. In order to counterbalance this problem, we introduce a

[7] A similar example works for additive separable preferences. Since the fully separable case is known to lend itself to efficient public goods provision (under conditions discussed, for instance, by Deaton (1981) or Bergstrom and Cornes (1983)), we use Cobb-Douglas preferences as leading example.

contest tax scheme, thereby changing player i's utility (3) to

$$\begin{aligned} \max_{x_i} \quad & u_i(\mathbf{x}) = w + x_i^a G^b + \frac{x_i^r}{\sum_{j=1}^n x_j^r} \beta P - (1+\alpha) p x_i \\ \text{s.t.} \quad & qG = (1-\beta)P, \text{ and } P = \alpha p \textstyle\sum_{j=1}^n x_j. \end{aligned} \tag{9}$$

Our central result is that there exists a contest tax scheme $\langle \alpha^*, \beta^*, r \rangle$ which induces a symmetric equilibrium where every player $i \in \mathcal{N}$ consumes the efficient quantity of the private good x^* while the public good is provided in the efficient amount G^*. Suppose that a symmetric equilibrium exists where the other players play the symmetric strategy $x_j > 0$. Then i's utility is

$$\begin{aligned} u_i(\mathbf{x}) = \quad & w + x_i^a \left(\frac{1-\beta}{q} \alpha p \left(x_i + (n-1) x_j \right) \right)^b + \\ & \frac{x_i^r}{x_i^r + (n-1) x_j^r} \beta \alpha p (x_i + (n-1) x_j) - (1+\alpha) p x_i. \end{aligned} \tag{10}$$

Taking the first-order condition w.r.t. x_i and replacing $x_i = x_j$ by x, respectively, we get

$$x^{a+b-1} \left(\frac{1-\beta}{q} \alpha p n \right)^b \left(a + \frac{b}{n} \right) + \frac{n-1}{n} r \beta \alpha p + \frac{\beta \alpha p}{n} - (1+\alpha) p = 0. \tag{11}$$

In order to isolate α in the above, we develop a second condition: Take $qG = (1-\beta)P$, from (9), insert the efficient quantities x^* and G^* from (6) to obtain the relation

$$\alpha(1-\beta) = \frac{b}{a}. \tag{12}$$

Solving for α, inserting this into (11) and simplifying leads to

$$x^{a+b-1} \left(\frac{pnb}{aq} \right)^b \left(a + \frac{b}{n} \right) + \frac{pb}{a(1-\beta)} \left(\frac{n-1}{n} r \beta + \frac{\beta}{n} - 1 \right) - p = 0. \tag{13}$$

After setting $x = x^*$ from (6), this simplifies to

$$\beta^* = \frac{1}{r}, \; \alpha^* = \frac{b}{a} \frac{r}{r-1}. \tag{14}$$

Note that 'limited liability' $\beta^* \in (0,1)$ is satisfied if $r > 1$ while $\alpha^* > 0$ is then assured. Thus, through the efficient mechanism $\langle \alpha^*, \beta^*, r \rangle$ the designer can induce a symmetric and efficient equilibrium candidate where each player consumes the efficient amount x^* of the private good. Notice that the designer has a degree of freedom in the choice of r. As apparent from (14), high values of r allow for lower contest prizes than low r.

Inserting x^*, G^*, α^* and β^* in (10), we get

$$u_i(\mathbf{x}) = (1-a-b) \left(\frac{a}{p} \right)^{\frac{a}{1-a-b}} \left(\frac{nb}{q} \right)^{\frac{b}{1-a-b}}. \tag{15}$$

The following figure confirms equilibrium existence for this candidate by plotting i's single-peaked utility arising from different consumption levels x_i under the contest tax scheme with α^* and β^*, assuming that every player $j \neq i$ chooses $x_j = x^*$.[8]

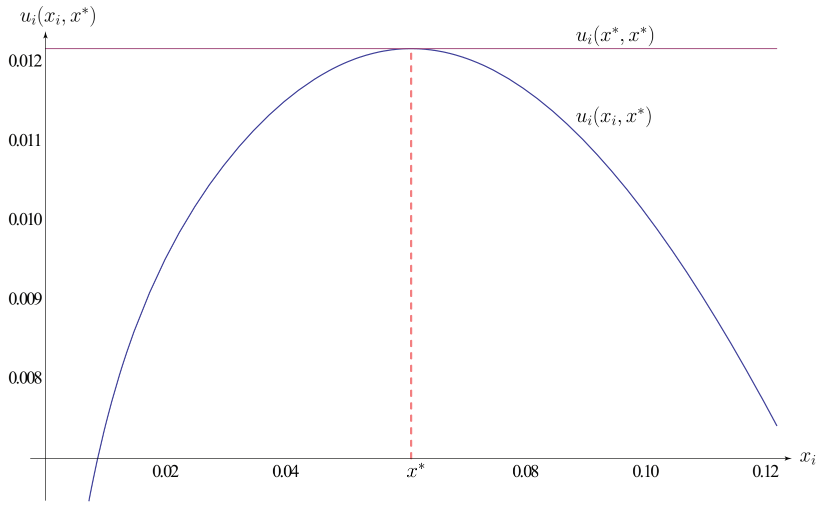

Figure 1: Private good consumption levels ensuring efficient public good provision.

4 Analysis

The social planner's problem is to maximise total utility net of cost,

$$W^* = \max_{x,G} n\left(w + v\left(x, G\right) - px\right) - C(G). \tag{16}$$

The efficient level of the private good, x_i^*, and the efficient total contribution to the public good G^* are then characterised by the first-order conditions

$$\left.\frac{\partial v(x,G)}{\partial x}\right|_{(x,G)=(x^*,G^*)} = p, \quad \left.\frac{\partial v(x,G)}{\partial G}\right|_{(x,G)=(x^*,G^*)} = \frac{C'(G)}{n}, \tag{17}$$

where the latter is known as the Samuelson condition which equates the sum of marginal utilities to the marginal cost of the public good. Samuelson (1954) shows that individual maximisation fails to attain these efficient levels. This well-known result is due to the positive externality created by each individual contribution to the public good. The basic idea of the contest tax scheme is to balance that positive externality with the negative externality inherent in the contest. If the scheme is appropriately designed, the two externalities cancel each other out. As discussed in section 4.2, several prominent alternative schemes fail to achieve this result.

[8] The parameters used for plotting the figure are $n = 2$, $p = q = 1$, $a = 3/4$, $b = 1/10$, $r = 2$, $x^* = .0601$, $g^* = 0.0081$, $\alpha^* = .73$, $\beta^* = 0.5$. We should also point out that there is a long-standing issue with the existence of symmetric pure strategy equilibria when $r > 1$ in standard contests with many players (see, eg., Schweinzer and Segev (2011)). As shown in proposition 2, however, existence conditions are not as restrictive in the present environment.

Provided that a symmetric pure strategy equilibrium $x_j > 0$ exists under the contest tax scheme (a fact that we establish for a large class of specifications in proposition 2), the individual utility maximisation problem (1) becomes

$$\begin{aligned} \max_{x_i} \quad & u_i(\mathbf{x}) = w + v(x_i, G) - (1+\alpha)px_i + \frac{x_i^r}{x_i^r + (n-1)x_j^r}\beta P \\ \text{s.t.:} \quad & C(G) = (1-\beta)P, \text{ and } P = \alpha p(x_i + (n-1)x_j). \end{aligned} \tag{18}$$

Our first main result gives a sufficient condition implying that the first-order condition of each player's best-reply problem is satisfied.

Proposition 1. *Using the tax contest scheme $\langle \alpha^*, \beta^*, r \rangle$, the private good is consumed at its efficient level and the public good can be provided efficiently for all $n \geq 2$ if $r > 1$. The corresponding equilibrium parameters α^* and β^* are*

$$\beta^* = \frac{1}{r} \quad \textit{and} \quad \alpha^* = \frac{C(G^*)/n}{px^*} \frac{1}{1-\beta^*} \tag{19}$$

The derived α^* has a nice interpretation: the first fraction is the per-person cost of the public good divided by the per-person cost of the private good. Notice that this is a sufficient condition. When β^* is significantly smaller than one, much less steep ranking technologies may still ensure efficient public good provision. Notice that β^* is independent of the tax rate.

Inserting x^*, G^*, α^* and β^* in the utility function of (18) we get

$$u_i(\mathbf{x}) = w + v(x^*, G^*) - px^* - \frac{C(G^*)}{n} = \frac{W^*}{n}. \tag{20}$$

Since (18) is not a convex optimisation problem, we need to be more specific for the derivation of a sufficient existence condition. The following proposition provides such a condition for a subclass of Cobb-Douglas preferences.

Proposition 2. *Consider the subclass of problems with Cobb-Douglas utility $v(x, G) = x^a G^b$, linear cost of the public good $C(G) = qG$ with $q > 0$, and $b = t(n)$ where t is an arbitrary positive continuous function with $t' < 0$ and $\lim_{n\to\infty} t(n) = 0$. For this subclass, there exists a threshold number of players $\tilde{n}$, such that for any $n > \tilde{n}$, existence of symmetric pure strategy equilibrium is ensured for any $r > 1$, irrespective of price p, marginal cost q, and $a \in (0, 1-b)$.*

Thus, an equilibrium exists in pure strategies for tax contests of this subclass provided that the number of participants is high enough. Participation in the lottery tax scheme is jointly rational, ie., consumers vote in favour of our tax proposal if the alternative is private provision of the public good. To see this, it is sufficient to recall that the tax contest implements the utility-maximising, efficient allocation while private contributions necessarily underprovide the public good.

4.1 Asymmetric Players

Let us now introduce asymmetries among players by individualising player i's utility function as $v_i(x_i, G)$. Denote the vector of private consumption by $\mathbf{x} = (x_1, \ldots, x_n)$ and the efficient quantities by $\mathbf{x}^* = (x_1^*, \ldots x_n^*)$. The efficient amounts of private consumption and public good are

$$\left.\frac{\partial v_i(x_i, G)}{\partial x_i}\right|_{(x_i,G)=(x_i^*,G^*)} = p, \quad \forall\, i \in \mathcal{N}, \text{ and } \quad \sum_{i=1}^{n} \left.\frac{\partial v_i(x_i, G)}{\partial G}\right|_{(\mathbf{x},G)=(\mathbf{x}^*,G^*)} = C'(G^*), \tag{21}$$

where the right-hand side expression is, again, the Samuelson condition. For this asymmetric model, we can characterise the optimal parameters of our tax contest scheme α^* and β^* as follows.

Proposition 3. *Suppose that players are asymmetric, with player i's utility function given by $v_i(x_i, G)$. Using the tax contest scheme $\langle \alpha^*, \beta^*, r\rangle$, the private good is consumed at its efficient level and the public good can be provided efficiently for all $n \geq 2$ if $r > 1$ and*

$$\beta^* = \frac{(n-1)\left(\sum_{i=1}^{n}(x_i^*)^r\right)^2}{r\sum_{i=1}^{n}(x_i^*)\sum_{i=1}^{n}\left((x_i^*)^{r-1}\sum_{j\neq i}(x_j^*)^r\right)} \quad \text{and} \quad \alpha^* = \frac{C(G^*)}{(1-\beta^*)p\sum_{i=1}^{n}(x_i^*)}. \tag{22}$$

Notice that β^* is still independent of the tax rate. Even for only two players we are unable to analytically solve for the efficient quantities (21) and, thus, we are unable to obtain an analytical solution of the optimal asymmetric tax contest scheme. Therefore, we give a numerical example for the case of two players. The utility functions are $v_1(x_1, G) = x_1^{a_1} G^{b_1}$ and $v_2(x_2, G) = x_2^{a_2} G^{b_2}$ with $(a_1, b_1, a_2, b_2, p) = (1/2, 1/6, 1/4, 1/8, 1)$. The cost function is $C(G) = G$. For $r = 3$ we get the optimal tax contest scheme $\alpha^* \approx 0.6235$ and $\beta^* \approx 0.3358$ while the efficient quantities are $G^* \approx 0.09$, $x_1^* \approx 0.1121$ and $x_2^* \approx 0.1054$. Figure 2 establishes existence by plotting each player's single-peaked utility for different consumption levels, given that the other player consumes his efficient quantity.

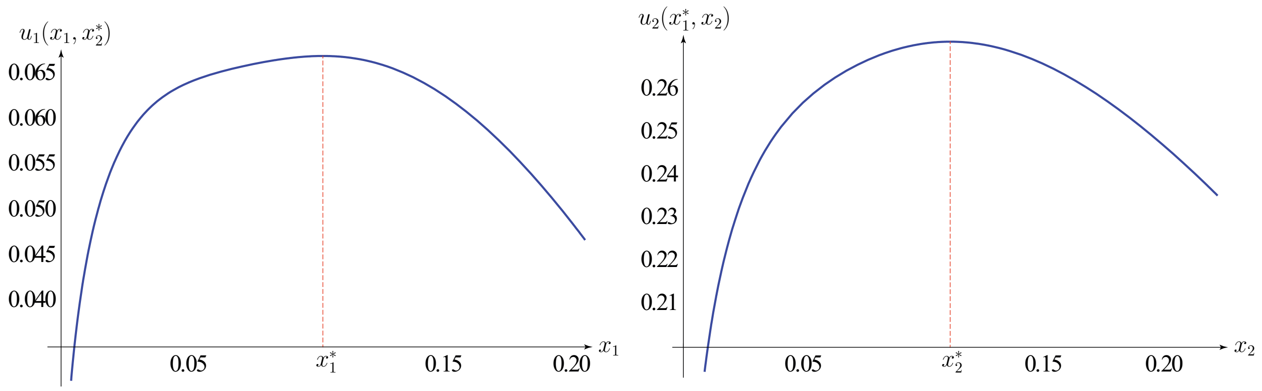

Figure 2: Best response problem in two player asymmetric model. Shown left is player 1's problem; player 2's is shown on the right.

The next result shows that only generalised Tullock contests (with $r \neq 1$) can possibly achieve efficiency.

Proposition 4. *Suppose that players are asymmetric, with player i's utility function given by $v_i(x_i, G)$. Using the tax contest scheme $\langle\alpha^*, \beta^*, r\rangle$, a simplified Tullock contest where $r = 1$ cannot induce efficiency.*

This extends a result in Morgan (2000) by showing that a simplified Tullock 'lottery' contest ($r = 1$) cannot achieve efficiency in a model with private contributions to a public good. This case is typically called *the* lottery contest because a player's winning probability is equal to his share of the total number of lottery tickets. Our analysis, however, shows that $r = 1$ is a knife-edge case. Many efficient contest mechanisms exist for the case of $r > 1$, that is, for mechanisms where the winning probabilities feature 'increasing returns' to scale.[9] For a more detailed discussion of Morgan's results see the following subsection.

4.2 Alternative Schemes

It is well known that voluntary contributions cannot provide the public good efficiently.[10] Moreover, as shown in a remarkable paper by Morgan (2000), even if voluntary contributions are combined with a (lottery) contest in which the players can win a prize financed by part of their contributions, efficiency cannot be attained. Morgan (2000) restricts attention to 'simplified' Tullock contests, i.e., the case of $r = 1$. Some recent results in the contest literature, for instance Gershkov, Li, and Schweinzer (2009) among others, suggest that by employing a 'generalised' Tullock contest with appropriately chosen power r, a designer may be able to induce exactly efficient contributions. We pursue this idea in a linear cost setting $C(G) = qG$.

Suppose that, in order to produce an amount g_i of the public good, the individual can contribute the monetary amount $(1+\gamma)qg_i$. With this contribution, he also enters a lottery where the winner is paid a prize equal to γqG where $G = (g_1 + \cdots + g_n)$ is the quantity of the pubic good. Suppose further that the lottery is governed by a generalised Tullock contest where winning probabilities are increasing in own contributions to the public good. Under such a scheme, player i maximises

$$u_i(\mathbf{x}, g_i) = w + v\left(x_i, \sum_{j=1}^n g_j\right) - px_i - (1+\gamma)qg_i + \frac{g_i^r}{\sum_{j=1}^n g_j^r}\gamma q \sum_{j=1}^n g_j. \tag{23}$$

Consider the first-order condition w.r.t. g_i. In symmetric equilibrium with $g = g_1 = \cdots = g_n$ and $G = ng$, this simplifies to

$$\frac{\partial v\left(x_i, G\right)}{\partial g_i} = \frac{q}{n}\left(n - (n-1)(r-1)\gamma\right). \tag{24}$$

Since $\partial v/\partial g_i = (\partial v/\partial G)(\partial G/\partial g_i)$ and $\partial G/\partial g_i = 1$, efficiency requires that $\gamma^* = 1/(r-1)$, which—from (17)—equalises the left-hand side to $C'(G)/n = q/n$. Feasibility of γ^* requires that

[9] This, coincidentally, seems to correspond to reality: Many examples exist—for instance in the German or Austrian Klassenlotterien—where purchasing a higher 'class' ticket increases winning chances (or prizes) disproportionately. For a detailed description of these institutions see Schoenbein (2008).

[10] For details see the short discussion in section 3.

$r > 1$, which implies that 'pari-mutuel raffles' with $r = 1$ cannot achieve efficiency.[11]

Since the maximisation problem (23) is not convex, satisfying the first-order condition is not sufficient for a global maximum. Assume that the second derivative of v exists. Then, evaluating the second derivative of $u_i(\mathbf{x}, g_i)$ w.r.t. g_i at $g = g_1 = \cdots = g_n = G^*/n$ and $\gamma = \gamma^*$, we get

$$\frac{(n-2)(n-1)qr}{G^* n} + \left.\frac{\partial^2 v(x,G)}{(\partial G)^2}\right|_{(\mathbf{x},G)=(\mathbf{x}^*,G^*)}. \tag{25}$$

Existence requires a *local* maximum, i.e., (25) must be negative. Since the left-hand term is strictly positive for $n > 2$, this might be impossible. In fact, in our Cobb-Douglas example, (25) is positive for $n > 2$, implying a *local minimum*. Adding a sales tax to such a contest scheme with voluntary contributions cannot implement efficiency either. This is due to the fact that a sales tax distorts the consumption of the private good. Our contest tax scheme counterbalances this effect by increasing the marginal utility of additional consumption by letting the players take part in a contest where the winning probability is increasing in their private consumption amounts.

Morgan (2000) also analyses 'fixed-prize raffles,' where a fixed prize R is awarded to the contest winner. That prize is financed by the players' contributions, such that the amount of the public good is $G = \sum_{i=1}^{n} x_i - R$ with cost of $C(G) = G$. Applying Morgan's fixed-prize scheme to our model with a private good we confirm Morgan's finding that $r = 1$ cannot induce efficiency.

5 Robust lottery specification

In order to simplify the exposition in the main part of the paper we employ there the widely used Tullock contest success function to determine winning probabilities. This, of course, is only one of many possible contest or lottery specifications. In the following we demonstrate our main results for a more general form of noisy ranking based on ratios as well as for contests based on differences of consumption quantities.

5.1 Contest with ratio-based ranking

Let us now assume that there exists a noisy and partial but verifiable ranking of private good consumption decisions

$$\Gamma(\tilde{x}) = \left[\pi_1\left(\tilde{x}_1\right), \ldots, \pi_n\left(\tilde{x}_n\right)\right] \tag{26}$$

where $x_{ij} = x_i/x_j$ and $\tilde{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})$, where the i^{th} element is 1. Thus (26) ranks the players on the basis of ratios between consumption pairs such that $\pi_i(\tilde{x}_i)$ is player i's probability of being ranked first given the consumption vector $x = (x_1, \ldots, x_n)$. Player i's utility becomes

$$u_i(\mathbf{x}) = w + v(x_i, G) + \pi_i(\tilde{x}_i)\beta P - (1+\alpha)px_i. \tag{27}$$

[11] Morgan's model is different from ours but the results are comparable. In particular, Morgan (2000) employs an asymmetric model with quasilinear utility and cost function $C(G) = G$. The share p of the lottery ticket revenue is the winner's prize in the lottery and the remaining share $1-p$ is used to finance the public good. In that model, we find that the first-order condition requires $p = 1/r$ which, similar to our result, is not feasible with $r = 1$.

We make the following assumptions on $\pi_i(\cdot)$:

A1 Symmetry: For any two players $l \neq m$ and for any two consumption vectors, $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ with $x_k = x'_k$ for $k \notin \{l, m\}$ and $x_l = x'_m$ and $x_m = x'_l$, we have

$$\pi_l(\tilde{x}_l) = \pi_m(\tilde{x}'_m).$$

Moreover, for any player i, let the elements of a consumption ratio vector $\tilde{x}'_i$ be arbitrary permutations of those in $\tilde{x}_i$ except for the element at the ith position. For these we require

$$\pi_i(\tilde{x}_i) = \pi_i(\tilde{x}'_i).$$

A2 Responsiveness: For any $l \in \{1, \ldots, n\}$ and $l \neq i$, $\dfrac{\partial \pi_i(\tilde{x}_i)}{\partial x_{il}} > 0$.

A3 $\pi(\cdot)$ is twice continuously differentiable.

This class includes the Tullock success function used in the main body of the paper. **A1** says that every opponent of player i affects the winning probability of i in a similar way. Thus, if players l and m exchange their consumption levels, this does not affect the winning probability of player $i \notin \{l, m\}$. The interpretation of **A2** is that the probability of being ranked first should react positively to increased consumption. **A3** is technical (and excludes the case of the all-pay auction). **A1** also implies that in symmetric equilibrium, where $x_1 = \cdots = x_n$, the slope of π_i with respect to any ratio x_{ij} is the same for all $i, j \in \mathcal{N}$, and each ratio is equal to 1. We simply denote this slope by $\pi'(1)$.

Provided that a symmetric pure strategy equilibrium $x_j > 0$ exists under the contest tax scheme, the individual utility maximisation problem (1) becomes

$$\begin{aligned} \max_{x_i} \quad & u_i(\mathbf{x}) = w + v(x_i, G) - (1+\alpha)px_i + \pi_i(\tilde{x}_i)\beta P \\ \text{s.t.:} \quad & C(G) = (1-\beta)P, \text{ and } P = \alpha p(x_i + (n-1)x_j). \end{aligned} \tag{28}$$

The next proposition gives a sufficient condition such that the first-order condition of each player's best-reply problem is satisfied.

Proposition 5. *Using the tax contest scheme $\langle \alpha^*, \beta^*, \pi(\mathbf{x}) \rangle$, the private good is consumed at its efficient level and the public good can be provided efficiently for all $n \geq 2$ if $\pi'(1) > {}^1/_4$. The corresponding equilibrium parameters α^* and β^* are given by*

$$\beta^* = \frac{1}{n^2 \pi'(1)} \quad \text{and} \quad \alpha^* = \frac{C(G^*)/n}{px^*} \frac{1}{1-\beta^*}. \tag{29}$$

Again, β^* is independent of the tax rate. It is easily verified that these results correspond to our results for the Tullock contest, where

$$\pi_i(\tilde{x}_i) = \left(\sum_{j=1}^{n} x_{ij}^{-r} \right)^{-1} = \frac{x_i^r}{\sum_{j=1}^{n} x_j^r}, \quad r > 0. \tag{30}$$

5.2 Contests with difference-based ranking

In this extension we present a simple two-players version of the example of section 3 based on the difference form success function introduced by Hirshleifer (1989).[12] The social planner's problem is unchanged and thus the efficient quantities x^* and G^* are still given by (6). Using a contest tax scheme, player i's utility (3) is now

$$\begin{aligned} \max_{x_i} \quad & u_i(\mathbf{x}) = w + x_i^a G^b + \frac{1}{1+\exp(r(x_j - x_i))}\beta P - (1+\alpha)px_i \\ \text{s.t.} \quad & qG = (1-\beta)P, \text{ and } P = \alpha p(x_i + x_j) \end{aligned} \tag{31}$$

where $r > 0$ can be interpreted as a precision parameter similar to the Tullock model. Working through steps similar to those outlined in the example section and setting $x = x^*$ for $n = 2$ from (6), we obtain the same optimality condition (12) as in the example section. Here this results in

$$\beta^* = \frac{1}{rx^*}, \alpha^* = \frac{b}{a}\frac{rx^*}{rx^* - 1}. \tag{32}$$

Existence can be confirmed by a graph which is almost identical to figure 3 for the same parametrisation as stated in footnote 8.

6 Extensions

6.1 Larger number of goods

The proposed tax contest scheme also works for a larger number of goods as the following example illustrates. Consider a symmetric model with two private goods (with prices p_1 and p_2) and two public goods (produced according to cost functions $C_1(G_1) = q_1 G_1$ and $C_2(G_2) = q_2 G_2$). Assume that $v(x_1, x_2, G_1, G_2) = x_1^{a_1} x_2^{a_2} G_1^{b_1} G_2^{b_2}$.[13] The social planner's problem is then to

$$\max_{x_1, x_2, G_1, G_2} n\left(w + x_1^{a_1} x_2^{a_2} G_1^{b_1} G_2^{b_2} - p_1 x_1 - p_2 x_2\right) - q_1 G_1 - q_2 G_2. \tag{33}$$

Denoting $S = 1 - a_1 - a_2 - b_1 - b_2$, the efficient quantities are

$$\begin{aligned} x_1^* &= n^{\frac{b_1+b_2}{S}} \left(\frac{a_1}{p_1}\right)^{\frac{S+a_1}{S}} \left(\frac{a_2}{p_2}\right)^{\frac{a_2}{S}} \left(\frac{b_1}{q_1}\right)^{\frac{b_1}{S}} \left(\frac{b_2}{q_2}\right)^{\frac{b_2}{S}}, \\ x_2^* &= n^{\frac{b_1+b_2}{S}} \left(\frac{a_1}{p_1}\right)^{\frac{a_1}{S}} \left(\frac{a_2}{p_2}\right)^{\frac{S+a_2}{S}} \left(\frac{b_1}{q_1}\right)^{\frac{b_1}{S}} \left(\frac{b_2}{q_2}\right)^{\frac{b_2}{S}}, \\ G_1^* &= n^{\frac{S+b_1+b_2}{S}} \left(\frac{a_1}{p_1}\right)^{\frac{a_1}{S}} \left(\frac{a_2}{p_2}\right)^{\frac{a_2}{S}} \left(\frac{b_1}{q_1}\right)^{\frac{S+b_1}{S}} \left(\frac{b_2}{q_2}\right)^{\frac{b_2}{S}}, \\ G_2^* &= n^{\frac{S+b_1+b_2}{S}} \left(\frac{a_1}{p_1}\right)^{\frac{a_1}{S}} \left(\frac{a_2}{p_2}\right)^{\frac{a_2}{S}} \left(\frac{b_1}{q_1}\right)^{\frac{b_1}{S}} \left(\frac{b_2}{q_2}\right)^{\frac{S+b_2}{S}}. \end{aligned} \tag{34}$$

[12] This type of difference-form success function can be extended to $n > 2$ players but then—depending on the precise formulation used—the difference to the ratio-form may become blurred. Hence we use the most distant 2-players case from our specification in section 3 for our robustness argument.

[13] Assume $a_1, a_2, b_1, b_2 \in (0, 1)$, $a_1 + a_2 + b_1 + b_2 < 1$.

Under the tax contest scheme, the respective private goods are taxed with tax rates α_1 and α_2. The prize pool P equals total tax revenues. The shares γ_1 and γ_2 of the prize pool are used to finance the two public goods, respectively. The remaining share, $1-\gamma_1-\gamma_2$ is paid to the contest winner. Each player's total consumption enters the contest. Assuming that the other players play the symmetric strategies $x_{1j}>0$ and $x_{2j}>0$, player i maximises

$$\begin{aligned} u_i(x_{1i},x_{2i}) = \;& w + x_1^{a_1} x_2^{a_2} G_1^{b_1} G_2^{b_2} - (1+\alpha_1)p_1 x_{1i} - (1+\alpha_2)p_2 x_{2i} \\ & + \frac{(x_{1i}+x_{2i})^r}{(x_{1i}+x_{2i})^r + (n-1)(x_{1j}+x_{2j})^r}(1-\gamma_1-\gamma_2)P \\ \text{s.t.}\;\; & q_1 G_1 = \gamma_1 P, q_2 G_2 = \gamma_2 P, \\ \text{and}\;\; & P = \alpha_1 p_1 (x_{1i} + (n-1)x_{1j}) + \alpha_2 p_2 (x_{2i} + (n-1)x_{2j}). \end{aligned} \tag{35}$$

Using steps similar to those above, we find the following optimal contest specification

$$\begin{aligned} \alpha_1^* &= \frac{r}{r-1}\frac{(b_1+b_2)p_2}{a_1p_2+a_2p_1}, \quad \alpha_2^* = \frac{r}{r-1}\frac{(b_1+b_2)p_1}{a_1p_2+a_2p_1}, \\ \gamma_1^* &= \frac{r-1}{r}\frac{b_1}{b_1+b_2}, \quad \gamma_2^* = \frac{r-1}{r}\frac{b_2}{b_1+b_2}. \end{aligned} \tag{36}$$

Checking the example $(a_1,a_2,b_1,b_2,p_1,p_2,q_1,q_2,n,r) = (1/8,1/8,1/100,1/100,1,1,1,1,10,3)$ ensures that the set of equilibria for this scheme is non-empty and open.

7 Concluding remarks

We show that a simple (lottery) tax contest can implement both efficient private and public good consumption. Many desirable generalisations of the model are left for future work: In reality, which share of total private goods consumption would have to go into the lottery? Is the resulting wealth redistribution one we would like to see? Is it realistic to assume that prices are unaffected by the introduction of the contest? What is the effect of risk attitudes? These questions are to a large extent empirical and may well have policy implications. At any rate we do not feel qualified to answer these questions now. What we do provide, however, are firm results showing that an incentive mechanism along the lines we indicate can *in principle* provide public goods efficiently without infracting upon private consumption or recurring to the coercive powers of the state.

Appendix

Lemma 1 (Local Maxima in Tullock-Cobb-Douglas)**.** *Consider the Tullock-Cobb-Douglas example. Look at* $u_i''(\mathbf{x})$ *(where* $u_i(\mathbf{x})$ *is given in* (9)*). Evaluate this derivative at* $x_1 = \cdots = x_n = x^*$, $G = G^*$, $\alpha = \alpha^*$, $\beta = \beta^*$. *Iff* $b < a(1-a)$, *then there exists an* $r > 1$ *such that for every* $n \geq 2$ *there is a local maximum of* i*'s best-reply problem at* $x_i = x^*$.

The proof of lemma 1 is a straightforward computation and is therefore omitted.

Proof of proposition 1. The proof is covered by the proof of proposition 5 since the Tullock contest success function is a special case of the ratio-based ranking. In particular,

$$\begin{aligned} \pi_i(\tilde{x}_i) &= \left(\sum_{j=1}^{n} x_{ij}^{-r}\right)^{-1} = \frac{x_i^r}{\sum_{j=1}^{n} x_j^r}, \quad r > 0, \\ \frac{\partial \pi_i(\tilde{x}_i)}{\partial x_{ij}} &= -\left(\sum_{j=1}^{n} x_{ij}^{-r}\right)^{-2} (-r) x_{ij}^{-r-1} \end{aligned} \tag{37}$$

and $\pi'(1) = r/n^2$. □

Proof of proposition 2. Throughout this proof we ignore the individual wealth level, w, since its only effect on the analysis is to increase each utility level by the same constant. Denote by $u_i(x, y)$ player i's utility if consuming x, while the other players choose the symmetric strategy y. Define $\Delta u_k = u_i(x^*, x^*) - u_i(kx^*, x^*)$ where $k \geq 0$. Thus, by varying k we can evaluate any feasible deviation from the equilibrium candidate. Whereas $u_i(x^*, x^*)$ is given by (15), we can determine $u_i(kx^*, x^*)$ as follows. Recalling (10), setting $x_i = kx$, $x_j = x$, and simplifying, we get

$$u_i(kx, x) = k^a \left(\frac{(1-\beta)p\alpha(k+n-1)}{q}\right)^b x^{a+b} - px\left((1+\alpha)k \; - k^r \frac{k+n-1}{k^r+n-1}\alpha\beta\right). \tag{38}$$

Replacing α and β with the optimal levels given in (14) yields

$$u_i(kx, x) = k^a \left(\frac{bp(k+n-1)}{aq}\right)^b x^{a+b} - px\left(k + \frac{b}{a(r-1)}\left(kr - k^r \frac{k+n-1}{k^r+n-1}\right)\right). \tag{39}$$

Setting $x = x^*$ from (6) gives

$$\begin{gathered} u_i(kx^*, x^*) = \left(\frac{a}{p}\right)^{\frac{a}{1-a-b}} \left(\frac{bn}{q}\right)^{\dfrac{b}{1-a-b}} \\ \left(k(1-a-b) - \frac{b}{r-1}\left(k - \frac{k^r(k-1+n)}{(k^r-1+n)}\right) - k + k^a\left(\frac{k-1+n}{n}\right)^b\right). \end{gathered} \tag{40}$$

Now it is straightforward to obtain

$$\begin{gathered} \Delta u_k = \left(\frac{a}{p}\right)^{\dfrac{a}{1-a-b}} \left(\frac{bn}{q}\right)^{\dfrac{b}{1-a-b}} \\ \left((1-k)(1-a-b) + \frac{b}{r-1}\left(k - \frac{k^r(k-1+n)}{(k^r-1+n)}\right) + k - k^a\left(\frac{k-1+n}{n}\right)^b\right). \end{gathered} \tag{41}$$

Inserting $b = t(n)$, we get

$$\Delta u_k|_{b=t(n)} = \overbrace{\left(\frac{a}{p}\right)^{\frac{a}{1-a-t(n)}}}^{=(A)} \overbrace{\left(\frac{nt(n)}{q}\right)^{\frac{t(n)}{1-a-t(n)}}}^{=(B)}$$
$$\underbrace{\left((1-k)(1-a-t(n)) + k - k^a \left(\frac{k+n-1}{n}\right)^{t(n)} + \frac{t(n)}{r-1}\frac{(k-k^r)(n-1)}{k^r+n-1}\right)}_{=(C)} \tag{42}$$

By definition $\Delta u_k = 0$ for $k = 1$ and $\Delta u_k = u_i(x^*, x^*) > 0$ for $k = 0$.[14] In order to prove existence of the equilibrium for sufficiently large n, it suffices to show that, for all n above some threshold $\tilde{n}$, $\Delta u_k|_{b=t(n)} > 0$ for all k with $k > 0$ and $k \neq 1$.

By assumption, $t(n)$ vanishes for large n. Since parts (A) and (B) in (42) are always positive—with $t(n)$ vanishing in (A), and (B) converging to 1—the sign of (42) is determined by part (C). There, in the first of four terms, $t(n)$ vanishes and $(1-k)(1-a)$ remains as n becomes large. The second term, k, is independent of n. The third term converges to k^a and the last term vanishes. Note that (42) is continuous in n. Thus, we find

$$\lim_{n\to\infty} \Delta u_k|_{b=t(n)} = \left(\frac{a}{p}\right)^{\frac{a}{1-a}} (1 - a + ka - k^a)\,, \tag{43}$$

where $(1 - a + ka - k^a)$ is the limit of part (C) in (42). This term determines the sign of (43). We need to show that this term is positive for all $k > 0$ and $k \neq 1$, i.e.,

$$1 - a + ka - k^a > 0, \quad \forall k > 0, k \neq 1. \tag{44}$$

It is easily verified that $1 - a + ka - k^a$ is strictly convex with minimum value zero at $k = 1$. Thus, condition (44) is satisfied for any $a \in (0, 1)$, all $k > 0$, and $k \neq 1$. This implies that the sign of (43) is positive. By continuity of (42), this, in turn, implies existence of our equilibrium for sufficiently large n. Note that this finding does neither depend on the choice of r (as long as $r > 1$), nor on the prices p and q, nor on the size of a, as long as a is feasible, i.e., $a \in (0, 1 - t(n))$. □

Proof of proposition 3. The utility maximisation problem is identical with (18), except for the use of v_i instead of v and $\sum_{j\neq i} x_j$ (resp. $\sum_{j\neq i} x_j^r$) instead of $(n-1)x_j$ (resp. $(n-1)x_j^r$). The first derivative of player i's expected utility is

$$u_i'(\mathbf{x}) = \frac{\partial v_i}{\partial x_i} + \frac{\partial v_i}{\partial G}\frac{\partial G}{\partial x_i} - (1+\alpha)p + \alpha p \beta \left(\frac{r x_i^{r-1} \sum_{j\neq i} x_j^r}{\left(\sum_{j=1}^n x_j^r\right)^2} \sum_{j=1}^n x_j + \frac{x_i^r}{\sum_{j=1}^n x_j^r}\right). \tag{45}$$

We set this equal to zero and evaluate it at the efficient levels as follows. First, replace x_i by x_i^* for

[14] Note that in the present setting, x^* and G^* are unique and positive and the corresponding welfare, W^*, see (5), is positive as well (since welfare is zero if either x or G are zero). Thus, $u_i(x^*, x^*)$ is positive since it is equal to W^*/n, see (15).

all $i \in \mathcal{N}$. Second, replace $\dfrac{\partial v_i}{\partial x_i}$ with the efficient level p (see (21)). Third, denote $B = (1-\beta)P$, and recall that $P = (1-\beta)\alpha p \sum_{j=1}^{n} x_j$ and note that $\frac{\partial P}{\partial x_i} = (1-\beta)\alpha p$. Under the tax contest scheme $C(G) = (1-\beta)P = B$ and we obtain

$$C(G) = B \iff G = C^{-1}(B) \Rightarrow \frac{\partial G}{\partial x_i} = \frac{\partial C^{-1}(B)}{\partial B}\frac{\partial B}{\partial x_i} = \frac{1}{C'(G)}(1-\beta)\alpha p. \qquad (46)$$

Thus, we replace $\dfrac{\partial G}{\partial x_i}$ in (46) with $((1-\beta)\alpha p)/C'(G)$ and get

$$\frac{\partial v_i}{\partial G}\frac{1}{C'(G)}(1-\beta)\alpha p - \alpha p + \alpha p\beta \left(\frac{r(x_i^*)^{r-1}\sum_{j\neq i}(x_j^*)^r}{\left(\sum_{j=1}^{n}(x_j^*)^r\right)^2}\sum_{j=1}^{n} x_j^* + \frac{(x_i^*)^r}{\sum_{j=1}^{n}(x_j^*)^r} \right) = 0. \qquad (47)$$

Adding the first-order conditions for all n players, we get

$$\sum_{i=1}^{n}\frac{\partial v_i}{\partial G}\frac{1}{C'(G)}(1-\beta)\alpha p - n\alpha p + \alpha p\beta \left(\frac{\sum_{i=1}^{n} r(x_i^*)^{r-1}\sum_{j\neq i}(x_j^*)^r}{\left(\sum_{j=1}^{n}(x_j^*)^r\right)^2}\sum_{j=1}^{n} x_j^* + 1 \right) = 0. \qquad (48)$$

We replace $\frac{\partial v_i(x_i,G)}{\partial G}$ with the efficient level $C'(G)$ (see (21)). After straightforward simplification we get β^* as given in (22). By the tax contest scheme, α^* follows immediately from the condition $C(G^*) = (1-\beta^*)P$ where $P = \alpha^* p \sum_{j=1}^{n} x_j^*$. □

Proof of proposition 4. By the proof of proposition 3, for any $r > 0$, the first-order condition for efficiency in equilibrium is given by (22). With $r = 1$, (22) simplifies to $\beta^* = 1$ which is not feasible. □

Proof of proposition 5. The proof proceeds as follows. We suppose existence of a symmetric efficient equilibrium that is characterised by the first-order condition of player i's best-reply problem, given that the players $j \neq i$ choose $x_j = x^*$. First, we derive β^*. Second, we derive the equilibrium tax rate, α^*. Third, we derive conditions ensuring feasibility of the equilibrium parameters α^*, β^*, and P.

1) Suppose a symmetric equilibrium $x_j > 0$ exists. Then $x_{ik} = x_{il} = \dfrac{x_i}{x_j}$ for all $k, l \in \mathcal{N} \setminus i$. Thus, $\pi_i(\tilde{x}_i) = \pi_i\left(\frac{x_i}{x_j}, \ldots, 1, \ldots, \frac{x_i}{x_j}\right)$, with "1" at the i'th position. Then i's utility is

$$u_i(\mathbf{x}) = w + v(x_i, G) - (1+\alpha)px_i + \pi_i\left(\frac{x_i}{x_j}, \ldots, 1, \ldots, \frac{x_i}{x_j}\right)\beta\alpha p(x_i + (n-1)x_j), \qquad (49)$$

Since, by **A1**, the derivative of π_i w.r.t. any ratio x_i/x_j is the same, the first-order condition can be written as

$$
\begin{aligned}
u_i'(\mathbf{x}) \;&= \tfrac{\partial v}{\partial x_i} + \tfrac{\partial v}{\partial G}\tfrac{\partial G}{\partial x_i} - (1+\alpha)p + (n-1)\frac{\partial \pi_i}{\partial x_{ij}}\tfrac{1}{x_j}\beta\alpha p(x_i + (n-1)x_j) \\
&+\pi_i\left(\tfrac{x_i}{x_j},\ldots,1,\ldots,\tfrac{x_i}{x_j}\right)\beta\alpha p = 0.
\end{aligned} \tag{50}
$$

Now we evaluate (50) at the efficient levels $x_1 = \cdots = x_n = x^*$ and $G = G^*$ as follows. Using (17), we replace $\frac{\partial v(x_i,G)}{\partial x_i}$ in (50) with the efficient level p as given in (17). Denote the total public goods expenditure by $B = (1-\beta)\alpha p(x_i + (n-1)x_j)$ and note that $\frac{\partial B}{\partial x_i} = (1-\beta)\alpha p$. We get

$$
C(G) = B \iff G = C^{-1}(B) \;\Rightarrow\; \frac{\partial G}{\partial x_i} = \frac{\partial C^{-1}(B)}{\partial B}\frac{\partial B}{\partial x_i} = \frac{1}{C'(G)}(1-\beta)\alpha p. \tag{51}
$$

Next, we replace $\frac{\partial v(x_i,G)}{\partial G}$ in (50) with the efficient level $\frac{C'(G)}{n}$ (as given in (17)) and replace $\frac{\partial G}{\partial x_i}$ with the term derived in (51). Finally, set $x^* = x_1 = \cdots = x_n$. Then (50) becomes

$$
p + \tfrac{1-\beta}{n}\alpha p - (1+\alpha)p + \left((n-1)\tfrac{\partial \pi_i}{\partial x_{ij}}\right)\tfrac{1}{x^*}\beta\alpha p n x^* + \pi_i((1,\ldots,1))\beta\alpha p = 0 \tag{52}
$$

As mentioned earlier in the text, **A1** implies that in symmetric equilibrium $\dfrac{\partial \pi_i}{\partial x_{ij}}$ is the same for all ratios and players (and denoted by $\pi'(1)$) since all ratios x_i/x_j are equal to one. Moreover, again by **A1**, $\pi_i(1,\ldots,1) = 1/n$. Applying this to (52) and simplifying leads to the left part of (29).

2) Take $C(G) = (1-\beta)P$ and evaluate at $x_1 = \cdots = x_n = x^*$, $G = G^*$, $\alpha = \alpha^*$, and $\beta = \beta^*$, and solve for α^*.

3) In a symmetric and efficient equilibrium, each player consumes $x^* > 0$. We only have to ensure that the corresponding α, β and P are feasible. In 1) and 2) we derived the values of α and β that are consistent with equilibrium existence. Feasibility requires $\alpha^* > 0$ and $\beta^* \in (0,1)$. By **A2** and $n \geq 2$, $\beta^* > 0$. Moreover, $\beta^* < 1$ if and only if $\pi'(1) > \frac{1}{n^2}$. This condition is satisfied for all $n \geq 2$ if it holds for $n = 2$. Thus, $\pi'(1) > \frac{1}{4}$ ensures feasibility of β^* for all $n \geq 2$. Given this, $\alpha^* > 0$ since each factor in the right part of (29) is positive. Finally, $P = \alpha^* p n x^* > 0$. □

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