GOVERNANCE AND THE EFFICIENCY
OF ECONOMIC SYSTEMS
GESY

Discussion Paper No. 376

Continuous Time Contests

Christian Seel*
Phillip Strack**

* University of Bonn
** University of Bonn

March 2012

Financial support from the Deutsche Forschungsgemeinschaft through SFB/TR 15 is gratefully acknowledged.

Sonderforschungsbereich/Transregio 15 · www.sfbtr15.de
Universität Mannheim · Freie Universität Berlin · Humboldt-Universität zu Berlin · Ludwig-Maximilians-Universität München
Rheinische Friedrich-Wilhelms-Universität Bonn · Zentrum für Europäische Wirtschaftsforschung Mannheim

Speaker: Prof. Dr. Klaus M. Schmidt · Department of Economics · University of Munich · D-80539 Munich,
Phone: +49(89)2180 2250 · Fax: +49(89)2180 3510

CONTINUOUS TIME CONTESTS

Christian Seel* and Philipp Strack†

This paper introduces a contest model in which each player decides when to stop a privately observed Brownian motion with drift and incurs costs depending on his stopping time. The player who stops his process at the highest value wins a prize. Applications of the model include procurement contests and competitions for grants.

We prove existence and uniqueness of the Nash equilibrium outcome, even if players have to choose bounded stopping times. We derive the equilibrium distribution in closed form. If the noise vanishes, the equilibrium outcome converges to—and thus selects—the symmetric equilibrium outcome of an all-pay auction. For two players and constant costs, each players' profits increase if costs for both players increase, variance increases, or drift decreases. Intuitively, patience becomes a more important factor for contest success, which reduces informational rents.

Keywords: Contests, all-pay contests, silent timing games.

1. INTRODUCTION

Two types of informational assumptions are predominant in the literature on contests, races, and tournaments. Either there is no learning about the performance measure or standings throughout the competition at all or each player can observe the performance of all players at all points in time. The former category includes all-pay contests with complete information (Hillman and Samet, 1987; Siegel, 2009, 2010), Tullock contests (Tullock, 1980), silent timing games (Karlin, 1953; Park and Smith, 2008), and models with additive noise in the spirit of Lazear and Rosen (1981). The latter category contains wars of attrition (Maynard Smith, 1974; Bulow and Klemperer, 1999), races (Aoki, 1991; Hörner, 2004; Anderson and Cabral, 2007), and contest models with full observability such as Harris and Vickers (1987) and Moscarini and Smith (2007).

However, there are many applications in which players are unable to observe their rivals, but can base their (effort) decision on their own progress. For instance, in a procurement contest, each participant is well-informed about his own progress, but lacks knowledge

*Corresponding Author at University of Bonn, Hausdorff Center for Mathematics, Department of Economics, Lennéstr. 43, D-53113 Bonn, E-mail: cseel@uni-bonn.de

†University of Bonn, Hausdorff Center for Mathematics, Bonn Graduate School of Economics, Lennéstr. 43, D-53113 Bonn, E-mail: philipp.strack@uni-bonn.de

We are indebted to Dirk Bergemann, Paul Heidhues, Benny Moldovanu, and Tymon Tatur for their enduring support of this work. Furthermore, we would like to thank Mehmet Ekmekci, Eduardo Faingold, Johannes Hörner, Philippe Jehiel, Navin Kartik, Sebastian Kranz, Giuseppe Moscarini, Sven Rady, Ron Siegel, seminar audiences in Bonn, Cologne, Frankfurt, Maastricht, Mannheim, Yale, the Workshop on Stochastic Methods in Game Theory 2010 in Erice, the World Congress of the Econometric Society 2010 in Shanghai, and the Stony Brook Game Theory Festival 2011 for helpful suggestions. Philipp Strack is particularly grateful for an extended stay at Yale, during which a large part of this research was conducted. Financial support by Bonn Graduate School of Economics, Deutsche Forschungsgemeinschaft, Hausdorff Center for Mathematics, and SFB TR 15 is gratefully acknowledged.

about research progress and behavior of his competitors. Another example is a competition for grants. Here, applicants can continuously choose how much time they invest in writing a proposal depending on their previous progress. However, they do not see the proposals of their rivals. Similarly, in a job promotion contest within a large firm, contestants can decide on their effort level. While they take their past success into account, they might not be able to observe effort and success of their competitors. In an appropriate model for these applications, contestants should be allowed to base their research decision over time only on their own progress. We propose such a model in the present paper.

Formally, our model is an n-player contest in which each player i decides when to stop a privately observed Brownian motion (X_t^i) with drift μ and volatility σ. As long as a player exerts effort, i.e., does not stop the process, he incurs flow costs $c(X_t^i)$. The player who stops his process at the highest value wins a prize.

Hence, in contrast to a *silent timing game* (Karlin, 1953; Park and Smith, 2008) in which the sole determinant of contest success is the time at which players stop, here, contest success and players' strategies additionally depend on their luck during the game. For $\sigma = 0$, our model reduces to a silent timing game with one prize for the player who stops latest. Thus, our model presents a first step towards a more general class of random timing games.

We characterize the unique Nash equilibrium outcome of the game without any restriction on the stopping time. Moreover, this result remains true for stopping strategies that are bounded by a real number T. Hence, the equilibrium construction is also valid for a contest with a sufficiently high, but fixed deadline. As the noise level approaches zero, the equilibrium converges to, and thus selects, the symmetric equilibrium of an all-pay contest. For two players and constant costs, each participant's profits increase if productivity (drift) of both players decreases, volatility increases, or costs increase. Hence, participants prefer a contest design which impedes progress.

The formal analysis proceeds as follows. Proposition 1 and Theorem 1 establish existence and uniqueness of the equilibrium distribution. The existence proof first characterizes the equilibrium distribution $F(x)$ of values at the stopping time $X_\tau = x$ uniquely up to its endpoints. We then use a Skorokhod embedding approach (e.g., Skorokhod, 1961, 1965; for a survey, see Obłój, 2004) to show that there exists a stopping strategy which induces this distribution.

The equilibrium distribution is unique for stopping strategies with finite expectation and bounded time stopping strategies, i.e., strategies that have to stop almost surely before a deadline $T < \infty$. As an immediate consequence of the latter result, our equilibrium construction remains valid in a contest with a sufficiently high deadline. This is economically important, since most applications have a fixed deadline—for example, a procurement contest for a jet fighter is usually issued with a deadline in a few years. From a technical point of view, the bounded time result is a major contribution of the present paper, since none of the previously mentioned literature could obtain a similar result.

In the next step, we analyze the shape of the equilibrium distribution. As uncertainty vanishes, the distribution converges to the symmetric equilibrium distribution of an all-pay auction by Theorem 2. On the one hand, the model offers a microfoundation for the use of all-pay auctions to scrutinize environments containing little uncertainty; on the other hand, it gives an equilibrium selection criterion for the equilibria of the symmetric all-pay auction analyzed in Baye, Kovenock, and de Vries (1996). Furthermore, this result serves as a benchmark to discuss how our predictions differ from all-pay models if volatility is strictly positive.

In most of the remaining analysis, we restrict attention to the case of two players and constant costs. This allows us to derive a tractable closed-form solution for the profits of each player. More precisely, by Proposition 6, these profits depend only on the ratio $\frac{2\mu^2}{c\sigma^2}$. In particular, profits increase as costs c increase, volatility σ^2 increases, or drift μ decreases (Theorem 3). Hence, contestants prefer worse technologies for both players.

This economically novel comparative statics result has an intuitive explanation: an increase in the productivity of each player makes patience a more important factor for contest success compared to chance. Hence, in equilibrium, expected stopping times and expected costs increase. Since the equilibrium is symmetric, the winning probability of each player remains constant. Summing up, we obtain lower expected profits.

The solution method we develop is applicable beyond the current setting. It provides a new approach to analyze more general variants of other timing games as the ones discussed in Park and Smith (2008), and, possibly, other stochastic games without observability. Moreover, as we show in an extension, the construction is not restricted to Brownian motion with drift, but can be applied to other stochastic processes.

1.1. *Related Literature*

In more recent literature, a few other papers model private information in related settings. Hopenhayn and Squintani (2011) scrutinize a preemption game in which private information arrives according to a Poisson process. They prove existence and derive a closed-form solution for a class of equilibria. In these equilibria, information disclosure occurs later than in the corresponding preemption game with public information.

In a companion paper (Seel and Strack, 2009), we show how relative performance pay might induce gambling behavior, i.e., investments in gambles with negative expectation. For this purpose, we analyze a contest model without any costs, but with a (usually negative) drift and a bankruptcy constraint. The driving forces of both models differ substantially. As in most of the contest literature, here, contestants trade off higher costs versus a higher winning probability, whereas in Seel and Strack (2009) the trade-off is between winning probability and risk.

Taylor (1995) also analyzes a contest model with private information. However, in his T-period model, only the highest draw in a single period determines the winner. In equilibrium, a player stops whenever she has a draw above a deterministic, time-independent threshold value.

We proceed as follows. Section 2 sets up the model. In Section 3, we prove that an equilibrium exists and has a unique distribution. Section 4 discusses the relation to all-pay contests and derives the main comparative statics results. Section 5 concludes. Most proofs are relegated to the appendix.

2. THE MODEL

There are $n \geq 2$ agents indexed by $i \in \{1, 2, \ldots, n\} = N$ who face a stopping problem in continuous time. At each point in time $t \in \mathbb{R}_+$, agent i privately observes the realization of a stochastic process $(X_t^i)_{t\in\mathbb{R}_+}$ with

$$X_t^i = x_0 + \mu t + \sigma B_t^i \,.$$

The constant x_0 denotes the starting value of all processes; without loss of generality, we assume $x_0 = 0$. The drift $\mu \in \mathbb{R}_+$ is the common expected change of each process X_t^i per time, i.e., $\mathbb{E}(X_{t+\Delta}^i - X_t^i) = \mu\Delta$. The random terms σB_t^i are independent Brownian motions $(B_t^i)_{i\in N}$ scaled by $\sigma \in \mathbb{R}_+$.

2.1. *Strategies*

A pure strategy of player i is a stopping time τ^i. As each player only observes his own process, the decision whether to stop at time t can only depend on the past realizations of his process $(X_s^i)_{s\leq t}$.[1] Mathematically, the agents' stopping decision until time t has to be $\mathcal{F}_t^i$-measurable, where $\mathcal{F}_t^i = \sigma(\{X_s^i : s < t\})$ is the sigma algebra induced by the possible observations of the process X_s^i before time t.

Although the unique equilibrium outcome of this paper can be obtained in pure strategies, we incorporate mixing to make the results more general. To do so, we allow for random stopping decisions. More precisely, each agent i can choose an $(\mathcal{F}_t^i)$-adapted increasing process $(\kappa_t^i)_{t\in\mathbb{R}_+}, \kappa_t^i \in [0,1]$ such that, for every t, $\mathbb{P}(\tau^i \leq t \,|\, \mathcal{F}_t^i) = \kappa_t^i$. A pure strategy is equivalent to a process κ_t^i that equals zero for all $t < \tau$ and one otherwise.

In the remainder of the paper, we require stopping times to be bounded by a real number $T < \infty$ such that $\tau^i < T$ almost surely. This restriction on the strategy space

[1] The equilibrium of the model would be the same if the stopping decision was reversible and stopped processes were constant.

makes it harder to derive the equilibrium—Proposition 2 shows that all results go through without it. We still think it is important to impose it, as most of the applications we have in mind, e.g., procurement contests and contests for grants have fixed deadlines. Hence, provided the deadline in the application is long enough—we give a condition in Lemma 10—the equilibrium we derive remains to hold for a contest with a deadline.

2.2. *Payoffs*

The player who stops his process at the highest value wins a prize $p > 0$. Ties are broken randomly. Until he stops, each player incurs flow costs $c : \mathbb{R} \to \mathbb{R}_{++}$ which depend on the current value of the process X_t, but not on the time t. The payoff π^i is thus

$$\pi^i = \frac{p}{k}\,\mathbf{1}_{\{X^i_{\tau^i} = \max_{j\in N} X^j_{\tau^j}\}} - \int_0^{\tau^i} c^i(X^i_t)dt\ ,$$

where $k = |\{i \in N : X^i_{\tau^i} = \max_{j\in N} X^j_{\tau^j}\}|$ is the number of agents who stop at the highest value. All agents maximize their expected profit $\mathbb{E}(\pi^i)$. We henceforth normalize p to 1, since agents only care about the trade-off between winning probability and cost-prize ratio. The cost function satisfies the following mild assumption:

ASSUMPTION 1 *For every $x \in \mathbb{R}$, the cost function $c : \mathbb{R} \to \mathbb{R}_{++}$ is continuous and bounded away from zero on $[x, \infty)$, i.e., there exists a $\underline{c} > 0$ such that, for all x, $c(x) \geq \underline{c}$.*

Figure 1 illustrates processes, stopping decisions, and corresponding costs in an example. Note that for the Brownian motion specification research progress need not be strictly increasing. This non-monotonicity allows us to capture organizational forgetting in the procurement example. The empirical literature shows that progress in firms is often non-monotone, since "organizations can forget the know-how gained through learning-by-doing due to labor turnover, periods of inactivity, and failure to institutionalize tacit knowledge" (Besanko, Doraszelski, Kryukov, and Satterthwaite, 2010, p.453); see, e.g., Benkard (2000) and Thompson (2001) for empirical evidence in the aircraft and ship-building industries. In the example of the grant competition, it also seems reasonable to include a small probability that the current quality of a proposal decreases, e.g., because a laptop is stolen or data are lost. Alternatively, an individual might just forget a detail which he had not yet written down; see Argote, Beckman, and Epple (1990) for related literature.

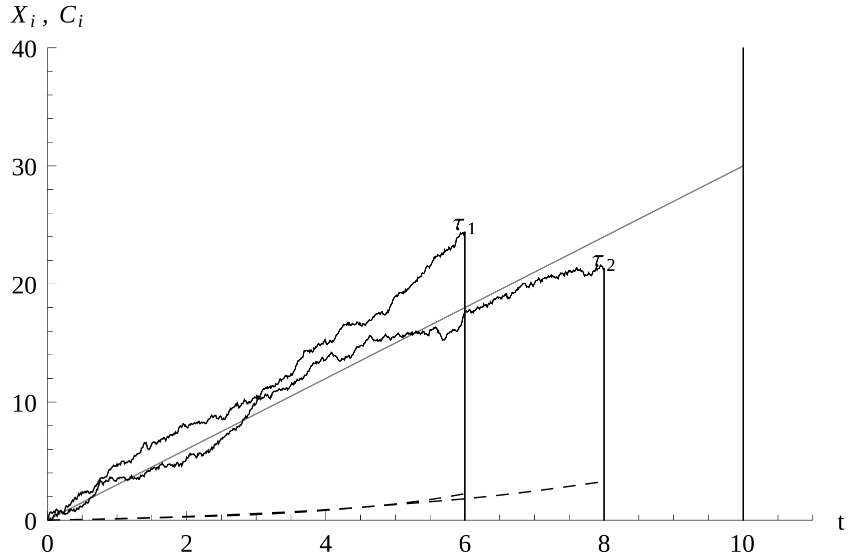

FIGURE 1.— An example for the game with two agents $i \in \{1, 2\}$. Time is depicted on the x-axis, while the value X_t^i (solid line) and total costs (dotted line) are depicted on the y-axis. The parameters are $c(X_t^i) = \frac{1}{10} \int_0^t \exp(\frac{1}{10} X_s^i) \mathrm{d}t$, $\mu = 3$, $\sigma = 1$, $T = 10$.

3. EQUILIBRIUM CONSTRUCTION

In this section, we first establish some necessary conditions on the distribution functions in equilibrium. In a second step, we prove existence and uniqueness of the Nash equilibrium outcome and determine the equilibrium distributions depending on the cost function.

Every strategy of agent i induces a (potentially non-smooth) cumulative distribution function (cdf) $F^i : \mathbb{R} \to [0, 1]$ of his stopped process $F^i(x) = \mathbb{P}(X^i_{\tau^i} \leq x)$. Denote the endpoints of the support of the equilibrium distribution of player i by

$$\begin{aligned} \underline{x}^i &= \inf\{x : F^i(x) > 0\} \\ \overline{x}^i &= \sup\{x : F^i(x) < 1\} . \end{aligned}$$

Let $\underline{x} = \max_{i \in N} \underline{x}^i$ and $\overline{x} = \max_{i \in N} \overline{x}^i$. In the next step, we establish a series of auxiliary results that are crucial to prove uniqueness of the equilibrium distribution.

LEMMA 1 *At least two players stop with positive probability on every interval $I = (a, b) \subset [\underline{x}, \overline{x}]$.*

LEMMA 2 *No player places a mass point in the interior of the state space, i.e., for all i, for all $x > \underline{x}$: $\mathbb{P}(X^i_{\tau^i} = x) = 0$. At least one player has no mass at the left endpoint, i.e., $F^i(\underline{x}) = 0$, for at least one player i.*

We omit the proof of Lemma 2, since it is simply a specialization of the standard logic in static game theory with a continuous state space; see, e.g., Burdett and Judd

(1983). Intuitively, in equilibrium, no player can place a mass point in the interior of the state space, since no other player would then stop slightly below the mass point. This contradicts Lemma 1.

Lemma 2 implies that the probability of a tie is zero. Thus, we can express the winning probability of player i if he stops at $X^i_{\tau^i} = x$, given the distributions of the other players, as

$$u^i(x) = \mathbb{P}(\max_{j \neq i} X^j_{\tau^j} \leq x) = \prod_{j \neq i} F^j(x)\,.$$

Lemma 3 *All players have the same right endpoint, $\overline{x}^i = \overline{x}$, for all i.*

Lemma 4 *All players have the same expected profit in equilibrium. Moreover, each player loses with certainty at $\underline{x}$, i.e., $u^i(\underline{x}) = 0$, for all i.*

Lemma 5 *All players have the same equilibrium distribution function, $F^i = F$, for all i.*

As players have symmetric distributions, we henceforth drop the superscript i. The previous lemmata imply that each player is indifferent between any stopping strategy on his support. By Itô's lemma, it follows from the indifference inside the support that, for every point $x \in (\underline{x}, \overline{x})$, the function $u(\cdot)$ must satisfy the second order ordinary differential equation (ODE)

$$c(x) = \mu u'(x) + \frac{\sigma^2}{2} u''(x)\,. \tag{1}$$

As (1) is a second order ODE, we need two boundary conditions to determine $u(\cdot)$ uniquely. One boundary condition is $u(\underline{x}) = 0$ from Lemma 4. We determine the other one in the following lemma:

Lemma 6 *In equilibrium, $u'(\underline{x}) = 0$.*

Imposing the two boundary conditions, the solution to equation (1) is unique. To calculate it, we define $\phi(x) = \exp(\frac{-2\mu x}{\sigma^2})$ as a solution of the homogeneous equation $0 = \mu u'(x) + \frac{\sigma^2}{2} u''(x)$. To solve the inhomogeneous equation, we apply the variation of the constants formula. We then use the two boundary conditions to calculate the unique solution candidate. Finally, we rearrange with Fubini's Theorem to get

$$u(x) = \begin{cases} 0 & \text{for } x < \underline{x} \\ \frac{1}{\mu} \int_{\underline{x}}^{x} c(z)(1 - \phi(x - z))\mathrm{d}z & \text{for } x \in [\underline{x}, \overline{x}] \\ 1 & \text{for } \overline{x} < x\,. \end{cases}$$

By symmetry of the equilibrium strategy, the function $F : \mathbb{R} \to [0,1]$ satisfies $F(x) = \sqrt[n-1]{u(x)}$. Consequently, the unique candidate for an equilibrium distribution is

$$F(x) = \begin{cases} 0 & \text{for all } x < \underline{x} \\ \sqrt[n-1]{\frac{1}{\mu}\int_{\underline{x}}^{x} c(z)(1-\phi(x-z))\mathrm{d}z} & \text{for all } x \in [\underline{x}, \overline{x}] \\ 1 & \text{for all } \overline{x} < x\,. \end{cases}$$

In the next step, we verify that F is a cumulative distribution function, i.e., that F is nondecreasing and that $\lim_{x\to\infty} F(x) = 1$.

LEMMA 7 *F is a cumulative distribution function.*

PROOF: By construction of F, $F(\underline{x}) = 0$. Clearly, F is increasing on $(\underline{x}, \overline{x})$, as the derivative with respect to x,

$$F'(x) = \frac{F(x)^{2-n}}{(n-1)}(\frac{2}{\sigma^2}\int_{\underline{x}}^{x} c(z)\phi(x-z)\mathrm{d}z)\,,$$

is greater than zero for all $x > \underline{x}$. It remains to show that there exists an $x > \underline{x}$ such that $F(x) = 1$.

$$\begin{aligned} F(x)^{n-1} &= \frac{1}{\mu}\int_{\underline{x}}^{x} c(z)(1-\phi(x-z))\mathrm{d}z \\ &\geq \frac{1}{\mu}\underline{c}\left(x - \underline{x} - \frac{\sigma^2}{2\mu}(1-\phi(x-\underline{x}))\right) \\ &\geq \frac{1}{\mu}\underline{c}(x - \underline{x} - \frac{\sigma^2}{2\mu}) \end{aligned}$$

By Assumption 1, $\underline{c} > 0$. Continuity of F implies that there exists a unique point $\overline{x} > \underline{x}$ such that $F(\overline{x}) = 1$. *Q.E.D.*

The next lemma derives a necessary condition for a distribution F to be the outcome of a strategy τ.

LEMMA 8 *If $\tau \leq T < \infty$ is a bounded stopping time that induces the continuous distribution $F(\cdot)$, i.e., $F(z) = \mathbb{P}(X_\tau \leq z)$, then $1 = \int_{\underline{x}}^{\overline{x}} \phi(x)F'(x)\mathrm{d}x$.*

PROOF: Observe that $(\phi(X_t))_{t\in\mathbb{R}_+}$ is a martingale. Hence, by Doob's optional stopping theorem, for any bounded stopping time τ,

$$1 = \phi(X_0) = \mathbb{E}[\phi(X_\tau)] = \int_{\underline{x}}^{\overline{x}} \phi(x)F'(x)\mathrm{d}x\,.$$

Q.E.D.

We use the necessary condition from Lemma 8 to prove that the equilibrium distribution is unique.

PROPOSITION 1 *There exists a unique pair* $(\underline{x}, \overline{x}) \in \mathbb{R}^2$ *such that the distribution*

$$F(x) = \begin{cases} 0 & \text{for all } x \leq \underline{x} \\ \sqrt[n-1]{\frac{1}{\mu}\int_{\underline{x}}^{x} c(z)(1-\phi(x-z))dz} & \text{for all } x \in (\underline{x}, \overline{x}) \\ 1 & \text{for all } x \geq \overline{x} \end{cases}$$

is the unique candidate for an equilibrium distribution.

PROOF: **Uniqueness:** As F is absolutely continuous, the right endpoint $\overline{x}$ satisfies (a) : $1 = \int_{\underline{x}}^{\overline{x}} F'(x; \underline{x}, \overline{x})\mathrm{d}x$. By Lemma 7, there exists a unique $\overline{x}$ for every $\underline{x}$ such that (a) is satisfied. Lemma 8 states that any feasible distribution satisfies (b) : $1 = \int_{\underline{x}}^{\overline{x}} F'(x; \underline{x})\phi(x)\mathrm{d}x$. We show that intersection of the set of solutions to equation (a) and to equation (b) consists of a single point. Since $F'(x; \underline{x}, \overline{x})$ is independent of $\overline{x}$, we henceforth drop the dependency in our notation. By the implicit function theorem, the set of solutions to (a) satisfies

$$(2) \qquad \frac{\partial \overline{x}}{\partial \underline{x}} = -\frac{-\overbrace{F'(\underline{x}; \underline{x})}^{=0} + \int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x; \underline{x})\mathrm{d}x}{F'(\overline{x}; \underline{x})} = -\frac{\int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x; \underline{x})\mathrm{d}x}{F'(\overline{x}; \underline{x})}.$$

Applying the implicit function theorem to (b) gives us

$$\begin{aligned} (3) \qquad \frac{\partial \overline{x}}{\partial \underline{x}} &= -\frac{-\overbrace{F'(\underline{x}; \underline{x})}^{=0} + \int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x; \underline{x})\phi(x)\mathrm{d}x}{F'(\overline{x}; \underline{x})\phi(\overline{x})} \\ &= -\frac{\int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x; \underline{x}) \overbrace{\phi(x - \overline{x})}^{>1} \mathrm{d}x}{F'(\overline{x}; \underline{x})} \\ &< -\frac{\int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x; \underline{x})\mathrm{d}x}{F'(\overline{x}; \underline{x})}. \end{aligned}$$

The last inequality follows from $\frac{\partial}{\partial \underline{x}} F'(x; \underline{x}) \geq 0$. As (2) > (3), the solution sets to equation (a) and (b) intersect at most once. Thus, in equilibrium, both the left and the right endpoint are unique.

Existence: We have shown in Lemma 7 that, for every $\underline{x}$, there exists a unique $\overline{x}$ such that $F(\cdot; \underline{x}, \overline{x})$ is a distribution function. Furthermore, rewriting the last inequality in the proof of Lemma 7, we get $\overline{x} \leq \underline{x} + \frac{\mu}{\underline{c}}\frac{\sigma^2}{2\mu}$ and thus $\underline{x} \to -\infty \Rightarrow \overline{x} \to -\infty$. Consider a left endpoint $\underline{x} \geq 0$ and a right endpoint $\overline{x}$ such that (a) is satisfied. Then, $\phi(x) < 1$ for all $x \in [\underline{x}, \overline{x}]$ and hence $\mathbb{E}(\phi(x)) < 1$ for $x \sim F(\cdot, \underline{x}, \overline{x})$. Now consider a left endpoint $\underline{x} < 0$ and a right endpoint $\overline{x} < 0$ such that (a) is satisfied. Then, $\phi(x) > 1$ for all $x \in [\underline{x}, \overline{x}]$ and hence $\mathbb{E}(\phi(x)) > 1$ for $x \sim F(\cdot, \underline{x}, \overline{x})$. Equation (b) is equivalent to $\mathbb{E}(\phi(x)) = 1$. By continuity and the intermediate value theorem, there exists an $\underline{x}, \overline{x}$ such (a) and (b) are satisfied. *Q.E.D.*

Hence, each equilibrium strategy induces the distribution F. The next lemma shows that this condition is also sufficient.

LEMMA 9 *Every strategy that induces the unique distribution F from Proposition 1 is an equilibrium strategy.*

PROOF: Define $\Psi(\cdot)$ as the unique solution to (1) with the boundary conditions $\Psi(\underline{x}) = 0$ and $\Psi'(\underline{x}) = 0$. By construction, the process $\Psi(X_t^i) - \int_0^t c(X_s^i)\mathrm{d}s$ is a martingale and $\Psi(x) = u(x)$ for all $x \in [\underline{x}, \overline{x}]$. As $\Psi'(x) < 0$ for $x < \underline{x}$ and $\Psi'(x) > 0$ for $x > \overline{x}$, $\Psi(x) > u(x)$ for all $x \notin [\underline{x}, \overline{x}]$. For every stopping time S, we use Itô's Lemma to calculate the expected value

$$\begin{aligned}\mathbb{E}[u(X_S) - \int_0^S c(X_t)\mathrm{d}t] &\leq \mathbb{E}[\Psi(X_S) - \int_0^S c(X_t)\mathrm{d}t] \\ &= \Psi(X_0) = u(X_0) = \mathbb{E}(u(X_\tau))\,.\end{aligned}$$

The last equality results, as each agent is indifferent to stop immediately with the expected payoff $u(X_0)$ or to play the equilibrium strategy with the expected payoff $\mathbb{E}(u(X_\tau))$. The inequality implies that any possible deviation strategy obtains a weakly lower payoff than the equilibrium candidate. *Q.E.D.*

The intuition is simple. By construction of F, all agents are indifferent between all stopping strategies, which stop inside the support $[\underline{x}, \overline{x}]$. As every agent wins with probability one at the right endpoint, it is strictly optimal to stop there. The condition $F'(\underline{x}) = 0$ ensures that it is also optimal to stop at the left endpoint.

So far, we have verified that a bounded stopping time $\tau \leq T < \infty$ is an equilibrium strategy if and only if it induces the distribution $F(\cdot)$, i.e., $F(z) = \mathbb{P}(X_\tau \leq z)$. To show that the game has a Nash equilibrium, the existence of a bounded stopping time inducing $F(\cdot)$ remains to be established. The problem of finding a stopping time τ such that a Brownian motion stopped at τ has a given centered probability distribution F, i.e., $F \sim B_\tau$, is known in the probability literature as the Skorokhod embedding problem (SEP). Since its initial formulation in Skorokhod (1961, 1965), many solutions have been derived; for a survey article, see Obłój (2004). In a recent mathematical paper, Ankirchner and Strack (2011) find conditions guaranteeing for a given $T \in \mathbb{R}_+$ the existence of a stopping time τ that is *bounded* by T and embeds a given distribution in Brownian motion with drift. They provide an analytical construction for such a stopping time (see p.221).[2]

In addition to the assumption stated in the next lemma, Ankirchner and Strack (2011) assume that the condition in Lemma 8 holds, which we have already imposed. They define $g(x) = F^{-1}(\Phi(x))$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(\frac{z^2}{2}) \mathrm{d}z$ is the density function of the normal distribution.

LEMMA 10 (Ankirchner and Strack, 2011, Theorem 2) *Suppose that $g(\cdot)$ is Lipschitz-continuous with Lipschitz constant $\sqrt{T}$. Then the distribution F can be embedded in $X_t = \mu t + B_t$, with a stopping time that stops almost surely before T.*

The lemma enables us to prove the main result of this section:

THEOREM 1 *The game has a Nash equilibrium. In equilibrium, the strategy of each player induces the distribution F from Proposition 1.*

The proof in the appendix verifies Lipschitz continuity of the function g, which makes Lemma 10 applicable. Thus, a Nash equilibrium in bounded time stopping strategies exists and, by Proposition 1, the equilibrium distribution F is unique.

In the next proposition, we show that existence and uniqueness of the equilibrium distribution F continue to hold if we do not restrict agents to bounded stopping times. For this purpose, define the payoff if an agent never stops to be $-\infty$.

PROPOSITION 2 *The distribution F is also the unique equilibrium distribution if agents are not restricted to bounded stopping times.*

In summary, we have proven existence and uniqueness of the equilibrium distribution for both finite and bounded stopping times. The uniqueness result differs from related models such as symmetric all-pay auctions or symmetric silent timing games, in which there are usually multiple equilibria. In some of these equilibria only a subset of players is active, i.e., submits a positive bid in the auction or does not stop directly in the silent timing game. Here, the variance leads all players to be active in the game. Hence, we obtain a unique equilibrium distribution.

REMARK 1 *While the equilibrium distribution is unique, the equilibrium strategy is not. In fact, there are uncountably many finite stopping times that solve the Skorokhod embedding problem; see Proposition 4.1 in Obłój (2004).*

[2] Ankirchner and Strack (2011) use a construction of the stopping time introduced for Brownian motion without drift in Bass (1983) and for the case with drift in Ankirchner, Heyne, and Imkeller (2008).

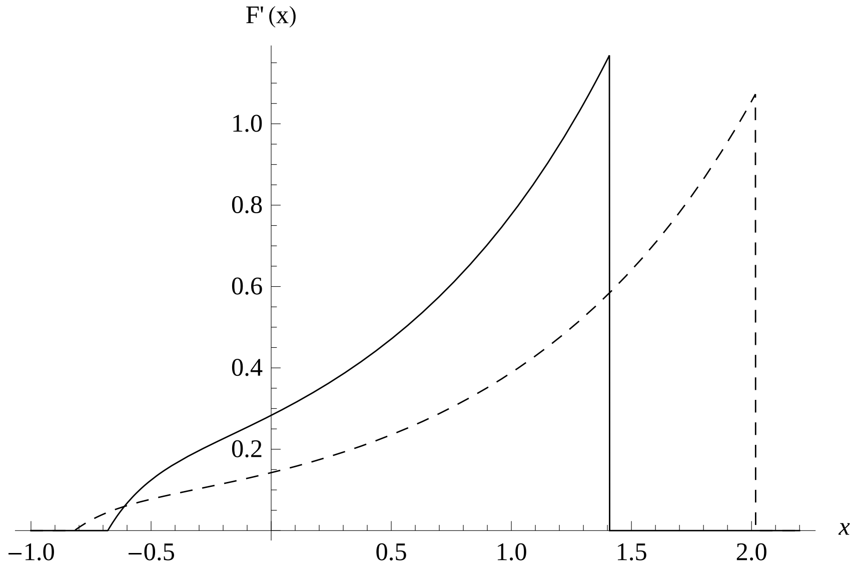

FIGURE 2.— The density function $F'(\cdot)$ for the parameters $n = 2$, $\mu = 3$, $\sigma = 1$ and the cost-functions $c(x) = \exp(x)$ solid line and $c(x) = \frac{1}{2}\exp(x)$ dashed line.

A major conceptual innovation compared to the literature is the bounded time requirement $\tau < T$, which makes the equilibrium derivation applicable to contests with a fixed deadline. To see that this result is not trivial to obtain, note that for any fixed time horizon T, there exists a positive probability that X_t does not leave any interval $[a, b]$ with $a < X_0 < b$. Hence, a simple construction through a mixture over cutoff strategies of the form

$$\tau_{a,b} = \inf\{t : \mathbb{R}_+ : X_t \notin [a, b]\}\,,$$

cannot be used to implement F.

In tug-of-war models with full observability (Harris and Vickers, 1987; Moscarini and Smith, 2007; Gul and Pesendorfer, 2011), it is not possible to obtain the same equilibrium for finite time strategies and bounded time strategies. Intuitively, for any fixed deadline, in these models there is a positive probability that no player has a sufficient lead until the deadline, which detains a similar result.

4. EQUILIBRIUM ANALYSIS

4.1. *Convergence to the All-pay Auction*

This subsection considers the relationship between the literature on all-pay contests and our model for vanishing noise. We first establish an auxiliary result about the left endpoint:

Lemma 11 *If the noise vanishes, the left endpoint of the equilibrium distribution converges to zero, i.e.,* $\lim_{\sigma\to 0} \underline{x} = 0$.

Proof: For any bounded stopping time, for any $\sigma > 0$, feasibility implies that $\underline{x} \leq 0$. By contradiction, assume there exists a constant ϵ such that $\underline{x} \leq \epsilon < 0$ for some sequence $(\sigma_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \sigma_k = 0$. Then F' is bounded away from zero by

$$\begin{aligned} F'(x) &= \frac{F(x)^{2-n}}{n-1}\frac{2}{\sigma^2}\int_{\underline{x}}^{x} c(z)\phi(x-z)\mathrm{d}z \\ &\geq \frac{1}{n-1}\frac{2}{\sigma^2}\int_{\epsilon}^{x} c(z)\phi(x-z)\mathrm{d}z \\ &\geq \frac{\underline{c}}{\mu(n-1)}(1-\phi(x-\epsilon))\,. \end{aligned}$$

For every point $x < 0$, $\lim_{\sigma_k\to 0}\phi(x) = \infty$. Thus, $\lim_{\sigma_k\to 0}\int_{\underline{x}}^{0} F'(x)\phi(x)\mathrm{d}x > 1$, which contradicts feasibility, because $\int_{\underline{x}}^{0} F'(x)\phi(x)\mathrm{d}x \leq \int_{\underline{x}}^{\overline{x}} F'(x)\phi(x)\mathrm{d}x = 1$. *Q.E.D.*

Taking the limit $\sigma \to 0$, the equilibrium distribution converges to

$$\lim_{\sigma\to 0} F(x) = \sqrt[n-1]{\frac{1}{\mu}\int_0^x c(z)\mathrm{d}z}.$$

In a static n-player all-pay auction, the symmetric equilibrium distribution is

$$F(x) = \sqrt[n-1]{\frac{x}{v}},$$

where x is the total outlay of a participant and v is her valuation; see, e.g., Hillman and Samet (1987). In our case, the total outlay depends on the flow costs at each point, the speed of research μ, and the stopping time τ. More precisely, it is $\int_0^x \frac{c(z)}{\mu}\mathrm{d}z$. The valuation v in the analysis of Hillman and Samet (1987) coincides with the prize p—which we have normalized to one—in our contest. This yields us the following proposition:

Theorem 2 *For vanishing noise, the equilibrium distribution converges to the symmetric equilibrium distribution of an all-pay auction.*

Thus, our model supports the use of all-pay auctions to analyze contests in which variance is negligible. Figure 3 illustrates the similarity to the all-pay auction equilibrium if variance σ and costs $c(\cdot)$ are small in comparison to the drift μ.

Moreover, the symmetric all-pay auction has multiple equilibria—for a full characterization see Baye, Kovenock, and de Vries (1996). This paper offers an equilibrium selection criterion in favor of the symmetric equilibrium. Intuitively, all other equilibria of the symmetric all-pay auction include mass points at zero for some players, which is not possible in our model for any positive σ by Lemma 2.

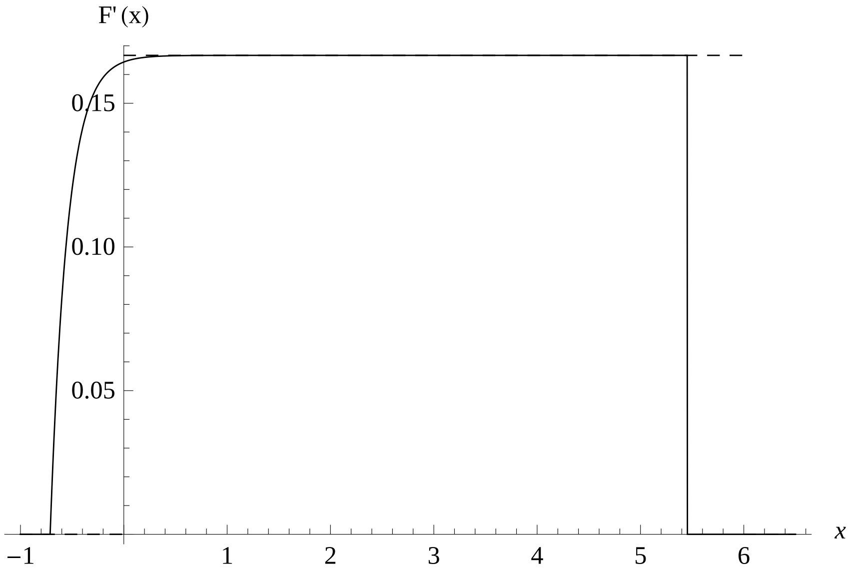

FIGURE 3.— This figure shows the density function $F'(\cdot)$ with support $[-0.71, 5.45]$ for the parameters $n = 2$, $\mu = 3$, $\sigma = 1$ and the cost-functions $c(x) = \frac{1}{2}$ (solid line) and for the same parameters the equilibrium density of the all-pay auction with support $[0, 6]$ (dashed line).

4.2. *Comparative Statics and Rent Dispersion*

Theorem 2 has linked all-pay contests with complete information to our model for the case of vanishing noise. In the following, we scrutinize how predictions differ for positive noise. In a symmetric all-pay contest with complete information, agents make zero profits in equilibrium. This does not hold true in our model for any positive level of variance σ:

PROPOSITION 3 *In equilibrium, all agents make strictly positive expected profits.*

PROOF: In equilibrium, agents are indifferent between stopping immediately and the equilibrium strategy. Their expected profit is thus $u(0)$, which is strictly positive as $\underline{x} < 0$. *Q.E.D.*

Intuitively, agents generate informational rents through the private information about their research progress. A similar result is known in the literature on all-pay contests with incomplete information, see, e.g., Hillman and Riley (1989), Amann and Leininger (1996), Krishna and Morgan (1997), and Moldovanu and Sela (2001). In these models, participants take a draw from a distribution prior to the contest, which determines their effort cost or valuation. The outcome of the draw is private information. In contrast to this, private information about one's progress arrives continuously over time in our model.

We now derive comparative statics in the number of players for constant costs. Define the support length as $\Delta = \overline{x} - \underline{x}$.

LEMMA 12 *If the number of players n increases and $c(x) = c$, the support length Δ remains constant and both endpoints increase.*

PROOF: If $c(x) = c$, $F(\overline{x}) - F(\underline{x})$ clearly depends only on Δ. Hence, for $F(\overline{x}) - F(\underline{x}) = 1$, Δ has to be constant. As F gets more concave if n increases, by feasibility $\underline{x} \nearrow$ and $\overline{x} \nearrow$. *Q.E.D.*

PROPOSITION 4 *If the number of agents n increases and $c(x) = c$, the expected profit of each agent decreases.*

PROOF: The function $u(x)$ depends only on $x - \underline{x}$. As n increases, $\underline{x}$ increases by Lemma 12. Thus, the expected value of stopping immediately, $u(0)$, which is an optimal strategy in both cases, decreases as n increases. *Q.E.D.*

Hence, in accordance with most other models, each player is worse off if the number of contestants increases.

4.3. *The Special Case of Two Players and Constant Costs*

We henceforth restrict attention to the case $n = 2$ and $c(x) = c$ to get more explicit results. For this purpose, we require additional notation. In particular, we denote by $W_0 : [-\frac{1}{e}, \infty) \to \mathbb{R}_+$ the principal branch of the Lambert W-function. This branch is implicitly defined on $[-\frac{1}{e}, \infty)$ as the unique solution of $x = W(x)\exp(W(x))$, $W \geq -1$. Define $h : \mathbb{R}_+ \to [0, 1]$ by

$$h(y) = \exp(-y - 1 - W_0(-\exp(-1 - y)))\,.$$

The next proposition pins down the left and right endpoints of the support.

PROPOSITION 5 *The left and right endpoint are*

$$\begin{aligned} \underline{x} &= \frac{\sigma^2}{2\mu}\left(2\log(1 - h(\frac{2\mu^2}{c\sigma^2})) - \log(\frac{4\mu^2}{c\sigma^2})\right) \\ \overline{x} &= \frac{\sigma^2}{2\mu}\left(2\log(1 - h(\frac{2\mu^2}{c\sigma^2})) - \log(\frac{4\mu^2}{c\sigma^2}) - \log(h(\frac{2\mu^2}{c\sigma^2}))\right)\,. \end{aligned}$$

For an illustration how the endpoints change depending on the parameters, see Figure 4. The next proposition derives a closed-form solution of the profits π of each player.

PROPOSITION 6 *The equilibrium profit of each player depends only on the ratio $y = \frac{2\mu^2}{c\sigma^2}$. It is given by*

$$\pi = \frac{(1 - h(y))^2}{2y^2} - \frac{2\log(1 - h(y)) - \log(2y) - 1}{y}\,.$$

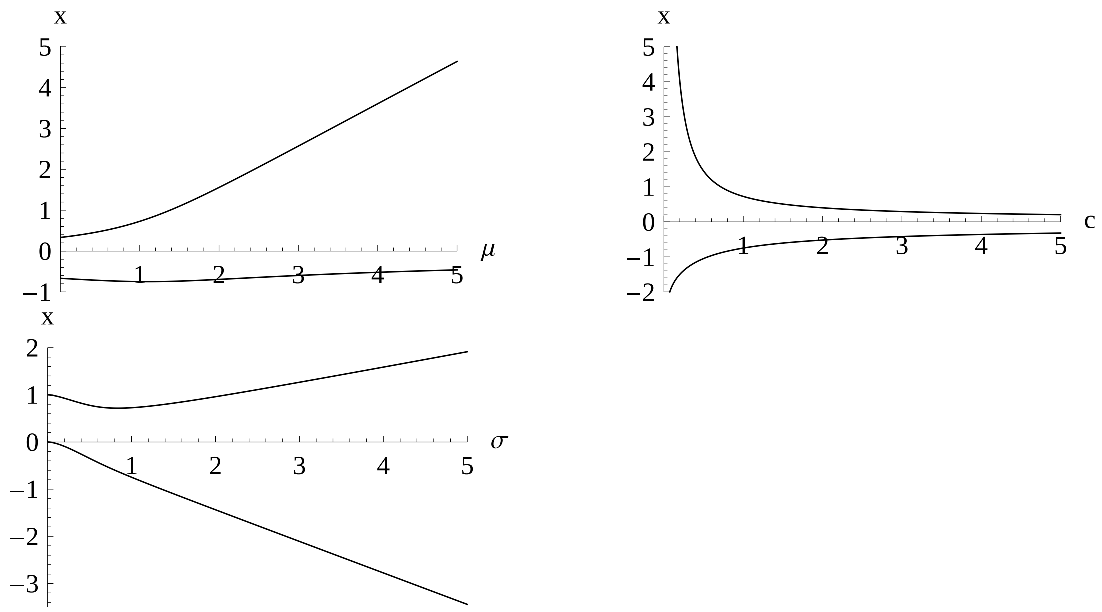

FIGURE 4.— This figure shows the left endpoint $\underline{x}$ and the right endpoint $\overline{x}$ for $n = 2, \sigma = 1, \mu = 1$ and constant cost $c = 1$ varying the productivity μ in the first figure, the costs c in the second figure and the variance σ in the third.

Given the previous proposition, it is simple to establish the main comparative static result of this paper.

THEOREM 3 *The equilibrium profit of each player increases if costs increase, variance increases, or drift decreases.*

To get an intuition for the result, we decompose the term $\frac{2\mu^2}{c\sigma^2}$, which determines the equilibrium profit of the players, into two parts:

$$\frac{2\mu^2}{c\sigma^2} = \underbrace{\frac{\mu}{c}}_{\text{Productivity}} \times \underbrace{\frac{2\mu}{\sigma^2}}_{\text{Signal to noise ratio}} .$$

The first term $\frac{\mu}{c}$ is a deterministic measure of productivity per time. On the other hand, the second term $\frac{2\mu}{\sigma^2}$ measures the impact of randomness (σ^2) on the final outcome. If both firms increase their productivity, patience becomes more critical to winning than chance. Hence, the expected stopping time in equilibrium increases. As each firms' winning probability remains $\frac{1}{2}$ by symmetry, profits decrease. In summary, participants prefer to have worse—more costly, more random, or less productive—technologies.

Even for a perfectly uninformative signal, however, agents cannot extract the full surplus:

PROPOSITION 7 *The equilibrium profit of each agent is bounded from above by* $4/9$.

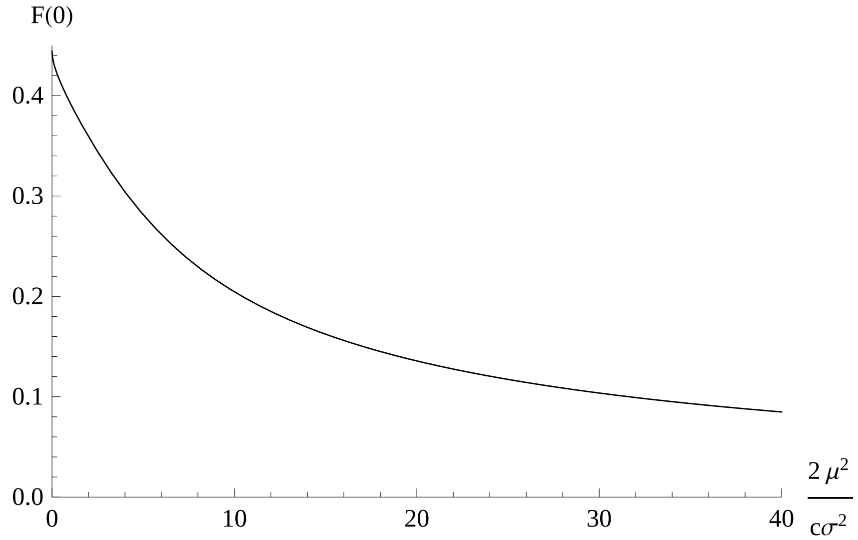

FIGURE 5.— This figure shows the equilibrium profit $F(0)$ of the agents on the y-Axes for $n = 2$ constant cost-functions $c(x) = c \in \mathbb{R}_+$ with $y = \frac{2\mu^2}{c\sigma^2}$ on the x-Axes.

PROOF: The agents profit is decreasing in $y = \frac{2\mu^2}{c\sigma^2} \geq 0$ by Theorem 3. Hence, profits are bounded from above by $\lim_{y\to 0} u(0)$. By l'Hôpital's rule,

$$\lim_{y\to 0} \frac{(1-h(y))^2}{2y^2} - \frac{2\log(1-h(y)) - \log(2y) - 1}{y} = \frac{4}{9}\,.$$

Q.E.D.

The expected equilibrium effort $\mathbb{E}(\tau^i)$ is bounded from below by

$$\begin{aligned} & \frac{4}{9} \geq \mathbb{E}(F(X^i_{\tau^i}) - c\tau^i) = \frac{1}{2} - c\mathbb{E}(\tau^i) \\ \Leftrightarrow \quad & \mathbb{E}(\tau^i) \geq \frac{1}{18c}\,. \end{aligned}$$

4.4. *Extension to Other Stochastic Processes*

Up to now, we have derived our results for Brownian motion with drift. However, it is straightforward to extend them to any process Z_t such that Z_t is the result of a strictly monotone transformation $\rho : \mathbb{R} \to \mathbb{R}$ of Brownian motion with drift X_t, i.e., $Z_t = \rho(X_t)$. First, observe that the success process Z_t is only payoff relevant as an ordinal variable, as agent i receives a prize if

$$Z^i_{\tau^i} > \max_{j\neq i} Z^j_{\tau^j} \Leftrightarrow \rho^{-1}(Z^i_{\tau^i}) > \max_{j\neq i} \rho^{-1}(Z^j_{\tau^j}) \Leftrightarrow X^i_{\tau^i} > \max_{j\neq i} X^j_{\tau^j}\,.$$

As costs $c(\cdot)$ depend on the value of the process Z_t, we define the cost function $\tilde{c}(x) = c(\rho(x))$ for the process X_t. Let $\tilde{F}$ be the equilibrium distribution for the game with a

Brownian motion with drift X_t and cost $\tilde{c}$, i.e., $X^i_{\tau^i} \sim \tilde{F}$. The equilibrium distribution of the original game is now given by $F(x) = \tilde{F}(\rho^{-1}(x))$, i.e., $Z^i_{\tau^i} \sim F$. For example, this generalization includes geometric Brownian motion with drift ($Z_t = \exp(X_t)$), which is relevant in many financial applications.

5. CONCLUSION AND DISCUSSION

In this paper, we have introduced a model of contests in continuous time. Our informational assumptions, which differ from most of the contest literature, seem appropriate to model applications such as procurement contests and contests for grants. Under mild assumptions on the cost function, a Nash equilibrium outcome exists and is unique for both bounded and finite time stopping strategies. If the research progress contains little uncertainty, the equilibrium is close to the outcome of the symmetric equilibrium of a static all-pay auction. Thus, our model provides an equilibrium selection result for the symmetric all-pay auction. If the research outcome is uncertain, each player prefers higher research costs, worse technologies, and higher uncertainty for all players. This economically novel feature results from chance becoming more important for contest success than patience.

From a technical perspective, we have introduced a method to construct equilibria in continuous time games that are independent of the time horizon. Furthermore, we have introduced a constructive method to calculate a lower bound on the time horizon that ensures the existence of such equilibria. These methodological contributions are economically relevant, since most real world applications have a fixed deadline. They should prove useful to future research on random timing games and other stochastic games without observability.

6. APPENDIX

Proof of Lemma 1: As players have to use bounded time stopping strategies, each player i stops with positive probability on every subinterval of $[\underline{x}^i, \overline{x}^i]$. Hence, it suffices to show that at least two players have $\overline{x}$ as their right endpoint. Assume, by contradiction, only player i has $\overline{x}$ as his right endpoint. Denote $\overline{x}^{-i} = \max_{j \neq i} \overline{x}^j$. Then, for any $\epsilon > 0$, at $\overline{x}^{-i} + \epsilon$, player i strictly prefers to stop, which yields him the maximal possible winning probability of 1 without any additional costs. This contradicts the optimality of a strategy, which stops at $\overline{x}^i > \overline{x}^{-i} + \epsilon$. *Q.E.D.*

Remark 2 *We write $\tau^i_{(a,b)}(x)$ shorthand for the continuation strategy $\inf\{t : X^i_t \notin (a,b) | X^i_s = x\}$ in the next three proofs. Clearly, $\tau^i_{(a,b)}(x)$ is not a bounded time continuation strategy, but we use it to bound the payoffs. Moreover, for sufficiently large time horizon T, the payoff from stopping at $\min\{\tau^i_{(a,b)}(x), T\}$ is arbitrarily close to that of $\tau^i_{(a,b)}(x)$.*

Proof of Lemma 3: Assume $\overline{x}^j > \overline{x}^i$. For at least two players j, j', the payoff from continuation with $\tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)$ is weakly higher than from stopping at $X^j_t = \overline{x}^i$ by Lemma 1. By Lemma 2, at least one of these players—denote it j—wins with probability zero at $\underline{x}^j$. Note that $u^i(\overline{x}^i) = \prod_{h\neq i} F^h(\overline{x}^i) < \prod_{h\neq j} F^h(\overline{x}^i) = u^j(\overline{x}^i)$, because $F^i(\overline{x}^i) = 1 > F^j(\overline{x}^i)$.

Optimality of $\tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)$ implies non-negative continuation payoffs,

$$u^j(\overline{x}^i) \leq \mathbb{P}(X^j_{\tau^j} = \overline{x}^j | \tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)) u^j(\overline{x}^j) - \mathbb{E}(c(\tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)))\,.$$

On the other hand, optimality of stopping at $\overline{x}^i$ for player i implies

$$u^i(\overline{x}^i) < u^j(\overline{x}^i) \leq \mathbb{P}(X^i_{\tau^i} = \overline{x}^j | \tau^i_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)) u^i(\overline{x}^j) - \mathbb{E}(c(\tau^i_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)))\,.$$

Hence, at $X^i_t = \overline{x}^i$, for a sufficiently long time horizon T, player i can profitably deviate by stopping at $\min\{\tau^i_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i), T\}$. This contradicts the equilibrium assumption. *Q.E.D.*

Proof of Lemma 4: To prove the first statement, we distinguish two cases.
(i) If at least two players have $F^i(\underline{x}) = 0$, then $u^i(\underline{x}) = 0\ \forall i$. Assume there exists a player j who makes less profit than a player i, where $\pi^i \leq \mathbb{P}(X^i_{\tau^i} = \overline{x} | \tau^i_{(\underline{x}^i,\overline{x})}(0)) - \mathbb{E}(c(\tau^i_{(\underline{x}^i,\overline{x})}(0)))$. If player j deviates to the strategy $\min\{\tau^j_{(\underline{x}^i,\overline{x})}(0), T\}$, he gets a profit arbitrarily close to π^i; this contradicts optimality of player j's strategy.
(ii) If only one player has $F^i(\underline{x}) = 0$, then $u^i(\underline{x}) > 0$. We now consider the case in which this player i makes a weakly higher payoff than the remaining players, who make the same payoff each—otherwise the argument in the first part of the proof leads to a contradiction. For any interval $I \in [\underline{x}, \overline{x}]$ in which player i stops with positive probability, by Lemma 1, there exists another player j who also stops in the interval. In particular, for $x \in I$, for any $\epsilon > 0$, we get

$$\mathbb{P}(X^i_{\tau^i} = \overline{x} | \tau^i_{(\underline{x},\overline{x})}(x)) + \mathbb{P}(X^i_{\tau^i} = \underline{x} | \tau^i_{(\underline{x},\overline{x})}(x)) u^i(\underline{x}) - \mathbb{E}(c(\tau^i_{(\underline{x},\overline{x})}(x))) < u^i(x) + \epsilon$$

and $\mathbb{P}(X^j_{\tau^j} = \overline{x} | \tau^j_{(\underline{x},\overline{x})}(x)) - \mathbb{E}(c(\tau^j_{(\underline{x},\overline{x})}(x))) \geq u^j(x)\ \forall j \neq i$.

For $\epsilon \to 0$, the two equations imply that $u^i(x) > u^j(x)$, for all $j \neq i$, for all x in the support of player i. Hence, $F^i(x) \leq F^j(x)\ \forall j$, for all x on the support of player i, and, by monotonicity of F^j, on $[\underline{x}, \overline{x}]$. Thus, the distribution of player i stochastically dominates that of all other players. This contradicts feasibility, since all players start at the same value and stopping times have to be bounded.
The second statement of the lemma follows immediately from the proof of (ii). *Q.E.D.*

Proof of Lemma 5: Recall that all players have the same profit, and $u^i(\underline{x}) = 0\ \forall i$. Each player stops on any interval $I \subset [\underline{x}, \overline{x}]$ with positive probability, since stopping times are bounded. By contradiction, assume $u^i(x) > u^j(x)$ for some players i, j and some value

x. As it is weakly optimal for player i to continue at x with $\tau^i_{(\underline{x},\overline{x})}(x)$, this strategy is strictly optimal for player j. At x, player j thus has a continuation strategy whose expected payoff is arbitrarily close to $u^i(x)$, which contradicts $u^i(x) > u^j(x)$. Hence, $u^i(x) = u^j(x)$ holds globally, for all i, j. This implies $F^i(x) = F$, for all i.

Q.E.D.

Proof of Lemma 6: By definition, $u(x) = 0$, for all $x \leq \underline{x}$. Hence, the left derivative $\partial_- u(\underline{x})$ is zero. It remains to prove that the right derivative $\partial_+ u(\underline{x})$ is also zero. For a given $u : \mathbb{R} \to \mathbb{R}_+$, let $\Psi : \mathbb{R} \to \mathbb{R}$ be the unique function that satisfies the second order ordinary differential equation $c(x) = \mu\Psi'(x) + \frac{\sigma^2}{2}\Psi''(x)$ with boundary conditions $\Psi(\underline{x}) = \partial_+ u(\underline{x})$ and $\Psi'(\underline{x}) = \partial_+ u(\underline{x})$. As $\Psi'(\underline{x}) > 0$, there exists a point $\hat{x} < \underline{x}$ such that $\Psi(\hat{x}) < 0 = u(\hat{x})$. Consider the strategy S that stops when either the point $\hat{x}$ or $\overline{x}$ is reached or at 1,

$$S = \min\{1, \inf\{t \in \mathbb{R}_+ : X^i_t \notin [\hat{x}, \overline{x}]\}\}.$$

As $u(\hat{x}) > \Psi(\hat{x})$, it follows that $\mathbb{E}(u(X_S)) > \mathbb{E}(\Psi(X_S))$. Thus,

$$\mathbb{E}(u(X_S) - \int_0^S c(X^i_t)\mathrm{d}t) > \mathbb{E}(\Psi(X_S) - \int_0^S c(X^i_t)\mathrm{d}t)\,.$$

Note that, by Itô's lemma, the process $\Psi(X^i_t) - \int_0^t c(X^i_s)\mathrm{d}s$ is a martingale. By Doob's optional sampling theorem, agent i is indifferent between the equilibrium strategy τ and the bounded time strategy S, i.e.,

$$\begin{aligned}\mathbb{E}(\Psi(X_S) - \int_0^S c(X^i_t)\mathrm{d}t) &= \mathbb{E}(\Psi(X_\tau) - \int_0^\tau c(X^i_t)\mathrm{d}t) \\ &= \mathbb{E}(u(X_\tau) - \int_0^\tau c(X^i_t)\mathrm{d}t)\,.\end{aligned}$$

The last step follows because $u(x)$ and $\Psi(x)$ coincide for all $x \in (\underline{x}, \overline{x})$. Consequently, the strategy S is a profitable deviation, which contradicts the equilibrium assumption. *Q.E.D.*

Proof of Theorem 1: The function Φ is Lipschitz continuous with constant $\frac{1}{\sqrt{2\pi}}$. Consequently, it suffices to prove Lipschitz continuity of F^{-1} to get the Lipschitz continuity of $F^{-1} \circ \Phi$. The density $f(\cdot)$ is

$$\begin{aligned}f(x) &= \frac{F(x)^{-n+2}}{n-1}\frac{2}{\sigma^2}\int_{\underline{x}}^x c(z)\phi(x-z)\mathrm{d}z \\ &= \frac{F(x)^{-n+2}}{n-1}\frac{2}{\sigma^2}\left(\int_{\underline{x}}^x c(z)\mathrm{d}z - \mu F(x)^{n-1}\right)\end{aligned}$$

As $f(x) > 0$ for all $x > \underline{x}$, it suffices to show Lipschitz continuity of F^{-1} at 0. We substitute $x = F^{-1}(y)$ to get

$$(f \circ F^{-1})(y) \geq \frac{1}{n-1}\frac{2}{\sigma^2}\left(y^{2-n}\underbrace{(\min_{z\in[\underline{x},\overline{x}]} c(z))}_{=\underline{c}}(F^{-1}(y) - F^{-1}(0)) - \mu y\right).$$

Rearranging with respect to $F^{-1}(y) - F^{-1}(0)$ gives

$$\begin{aligned} F^{-1}(y) - F^{-1}(0) &\leq \left(\frac{(n-1)\sigma^2}{2}(f \circ F^{-1})(y) + \mu y\right)\frac{y^{n-2}}{\underline{c}} \\ &\leq \left(\frac{(n-1)\sigma^2}{2}f(\overline{x}) + \mu\right)\frac{y^{n-2}}{\underline{c}}. \end{aligned}$$

This proves the Lipschitz continuity of $F^{-1}(\cdot)$ for $n > 2$. Note that for two agents $n = 2$ the function $F^{-1}(\cdot)$ is not Lipschitz continuous as $f(\underline{x}) = 0$. However, we show in the following paragraph that $F^{-1} \circ \Phi$ is Lipschitz continuous for $n = 2$.

$$\begin{aligned} F(x) &= \int_{\underline{x}}^{x} \frac{c(z)}{\mu}(1 - \phi(x - z)\mathrm{d}z \\ &\leq \underbrace{\left(\sup_{z\in[\underline{x},\overline{x}]} \frac{c(z)}{\mu}\right)}_{=\overline{c}}\left(x - \underline{x} - \frac{\sigma^2}{2\mu}(1 - \phi(x - \underline{x}))\right) \end{aligned}$$

A second order Taylor expansion around $\underline{x}$ yields that, for an open ball around $\underline{x}$ and $\underline{x} < x$, we have the following upper bound

$$x - \underline{x} - \frac{\sigma^2}{2\mu}(1 - \phi(x - \underline{x})) \leq \frac{2\mu}{\sigma^2}(1 - \phi(x - \underline{x}))^2.$$

For an open ball around $\underline{x}$, we get an upper bound on $F(x) \leq \frac{2\overline{c}}{\sigma^2}(1 - \phi(x - \underline{x}))^2$ and hence the following estimate

$$1 - \phi(x - \underline{x}) \geq \sqrt{\frac{\sigma^2}{2\overline{c}}F(x)}.$$

We use this estimate to obtain a lower bound on $f(\cdot)$ depending only on $F(\cdot)$

$$\begin{aligned} f(x) &= \frac{2}{\sigma^2}\left(\int_{\underline{x}}^{x} c(z)\phi(x - z)\mathrm{d}z\right) \geq \frac{2\underline{c}}{\sigma^2}\left(\frac{\sigma^2}{2\mu}(1 - \phi(x - \underline{x}))\right) \\ &\geq \frac{\underline{c}}{\mu}\sqrt{\frac{\sigma^2}{2\overline{c}}F(x)}. \end{aligned}$$

Consequently, there exists an $\epsilon > \underline{x}$ such that, for all $x \in [\underline{x}, \epsilon)$, we have an upper bound on $\frac{(\phi\circ\Phi^{-1}\circ F)(x)}{f(x)}$. Taking the limit $x \to \underline{x}$ yields

$$\lim_{x\to\underline{x}} \frac{(\phi\circ\Phi^{-1}\circ F)(x)}{f(x)} \leq \lim_{x\to\underline{x}} \frac{(\phi\circ\Phi^{-1}\circ F)(x)}{\frac{\underline{c}}{\mu}\sqrt{\frac{\sigma^2}{2\overline{c}}F(x)}} \leq \sqrt{\frac{2\overline{c}\mu^2}{\underline{c}^2\sigma^2}} \lim_{y\to 0}\frac{(\phi\circ\Phi^{-1})(y)}{\sqrt{y}} = 0\,.$$

Q.E.D.

Proof of Proposition 2: The existence proof proceeds in three steps. First, we show that any optimal strategy has finite expectation. In the second step, we prove that the support of any equilibrium distribution induced by a finite stopping time is bounded. These conditions allow us to apply Doob's theorem in Step 3. Thus, Lemma 8 and 9, which guarantee equilibrium existence, extend to arbitrary stopping times.

Step 1: As c is bounded from below $c(\cdot) \geq \underline{c}$, the payoff of agent i if he does not stop before time t is bounded from above by

$$\mathbf{1}_{\{X^i_{\tau^i} \geq \max_{j\neq i} X^j_{\tau^j}\}} - \int_0^t c(X^i_s)\mathrm{d}s \leq 1 - t\underline{c}\,.$$

Furthermore,

$$\mathbb{E}(\mathbf{1}_{\{X^i_{\tau^i} \geq \max_{j\neq i} X^j_{\tau^j}\}} - \int_0^{\tau^i} c(X^i_s)\mathrm{d}s) \leq 1 - \mathbb{E}(\tau^i)\underline{c}\,.$$

Thus, to satisfy $1 - \mathbb{E}(\tau^i)\underline{c} > 0$, any optimal strategy must have finite expectation.

Step 2: In the next step, we show that the support of any equilibrium distribution induced by a finite stopping time is bounded. Note that $\overline{x} \neq \infty$, since for positive drift ($\mu > 0$), the expected stopping time would be infinite otherwise. Hence, it remains to show that $\underline{x} \neq -\infty$. As the optimal strategy is Markovian, for any point x in the support, it is optimal to continue until $X^i_\tau \in \{\underline{x}, \overline{x}\}$. Suppose the left endpoint of the equilibrium distribution of an agent equals minus infinity, i.e. $\underline{x} = -\infty$. The expected stopping time for the strategy that stops only if the right endpoint is reached equals

$$\mathbb{E}(\tau_{(-\infty,\overline{x})}(x) - t) = \frac{\overline{x} - x}{\mu}\,.$$

As $\tau_{(-\infty,\overline{x})}$ is an optimal continuation strategy, $\underline{c}\mathbb{E}(\tau) \leq 1$. Hence,

$$1 \geq \underline{c}\mathbb{E}(\tau) = \frac{\underline{c}}{\mu}(\overline{x} - x) \Leftrightarrow x \geq \overline{x} - \frac{\mu}{\underline{c}}\,.$$

Consequently, $\tau_{(\infty,\overline{x})}$ is not an optimal continuation strategy for $x < \overline{x} - \frac{\mu}{\underline{c}}$. This contradicts the assumption $\underline{x} = -\infty$.

Step 3: As the support of the equilibrium distribution is bounded, for any equilibrium stopping time τ, the process $X_{\min\{t,\tau\}}$ is uniformly integrable. Thus, the conditions for Doob's optional sampling theorem ($\mathbb{P}(\tau < \infty) = 1$, $\mathbb{E}[|X_\tau|] < \infty$, and uniform integrability) hold. Hence, Lemma 8 and 9 follow by the same argument as in the text.

The proof for uniqueness of the equilibrium distribution for finite expected stopping times works similar as in bounded time case, with slight modifications in the proofs of Lemma 1, 2, and 5. The details are available upon request.

Q.E.D.

Proof of Proposition 5 and 6: Rearranging the density condition $1 = F(\overline{x}) = \frac{c}{\mu}[\Delta - \frac{\sigma^2}{2\mu}(1 - \phi(\Delta))]$ yields

$$\exp(\frac{-2\mu\Delta}{\sigma^2}) = -\frac{2\mu}{\sigma^2}[\Delta - (\frac{\mu}{c} + \frac{\sigma^2}{2\mu})].$$

The solution to the transcendental algebraic equation $e^{-a\Delta} = b(\Delta - d)$ is $\Delta = d + \frac{1}{a}W_0(\frac{ae^{-ad}}{b})$, where $W_0 : [-\frac{1}{e}, \infty) \to \mathbb{R}_+$ is the principal branch of the Lambert W-function. This branch is implicitly defined on $[-\frac{1}{e}, \infty)$ as the unique solution of $x = W(x)\exp(W(x))$, $W \geq -1$. Hence,

$$\Delta = \frac{\mu}{c} + \frac{\sigma^2}{2\mu}[1 + W_0(-\exp(-1 - \frac{2\mu^2}{c\sigma^2}))]$$

and

$$\begin{aligned} \phi(\Delta) &= \exp(\frac{-2\mu^2}{c\sigma^2} - 1 - W_0(-\exp(-1 - \frac{2\mu^2}{c\sigma^2}))) \\ &= \exp(-1 - y - W_0(-\exp(-1 - y))) \\ &= h(y)\,. \end{aligned}$$

Note that $\phi(\Delta)$ only depends on $y = \frac{2\mu^2}{c\sigma^2}$. Moreover, $h(y)$ is strictly decreasing in y, as $W_0(\cdot)$ and $\exp(\cdot)$ are strictly increasing functions. For constant costs, the feasibility condition from Lemma 8 reduces to

$$\begin{aligned} 1 &= \int_{\underline{x}}^{\overline{x}} F'(x)\phi(x)\mathrm{d}x \\ &= \frac{c\sigma^2}{2\mu^2}[\frac{1}{2}\phi(\underline{x}) + \frac{1}{2}\phi(2\overline{x} - \underline{x}) - \phi(\overline{x})]\,. \end{aligned}$$

Dividing by $\phi(\underline{x})$ gives

$$\begin{aligned}
\phi(-\underline{x}) &= \frac{c\sigma^2}{2\mu^2}[\frac{1}{2} + \frac{1}{2}\phi(\Delta)^2 - \phi(\Delta)] \\
&= \frac{1}{y}[\frac{1}{2} + \frac{1}{2}h(y)^2 - h(y)] \\
&= \frac{1}{2y}(1 - h(y))^2 \\
&= g(y)
\end{aligned}$$

Note that $g : \mathbb{R}_+ \to [0,1]$ is strictly decreasing in y. We calculate $\underline{x}$ as

$$\underline{x} = -\phi^{-1}(\phi(-\underline{x})) = -\frac{\sigma^2}{2\mu}\log(\frac{2\mu^2}{c\sigma^2[\frac{1}{2} + \frac{1}{2}\phi(\Delta)^2 - \phi(\Delta)]}) .$$

Simple algebraic transformations yield the expression for $\underline{x}$ and $\overline{x}$ (inserting Δ) in Proposition 5.

We plug in $\underline{x}$ to get:

$$\begin{aligned}
F(0) &= \frac{c}{\mu}[-\underline{x} - \frac{\sigma^2}{2\mu}(1 - \phi(-\underline{x}))] \\
&= \frac{c\sigma^2}{2\mu^2}[\log(\frac{\frac{2\mu^2}{c\sigma^2}}{\frac{1}{2} + \frac{1}{2}\phi(\Delta)^2 - \phi(\Delta)}) + \frac{\frac{1}{2} + \frac{1}{2}\phi(\Delta)^2 - \phi(\Delta)}{\frac{2\mu^2}{c\sigma^2}} - 1] \\
&= \frac{1}{y}[\log(\frac{y}{\frac{1}{2} + \frac{1}{2}h(y)^2 - h(y)}) + \frac{\frac{1}{2} + \frac{1}{2}h(y)^2 - h(y)}{y} - 1] \\
&= \frac{1}{y}[g(y) - \log(g(y)) - 1]
\end{aligned}$$

Hence, the value of $F(0)$ depends on the value of the fraction $y = \frac{2\mu^2}{c\sigma^2}$ in the above way, which completes the proof of Proposition 6. *Q.E.D.*

PROOF OF THEOREM 3: By Proposition 6, it suffices to show that the profit $F(0)$ is increasing in y. Consider the following expression from the previous proof:

$$F(0) = \frac{1}{y}[g(y) - \log(g(y)) - 1]$$

The function $x - \log(x)$ is increasing in x. Hence, $g(y) - \log(g(y)) - 1$ is decreasing in y, because $g(y)$ is decreasing in y. As $\frac{1}{y}$ is also decreasing in y, the product $\frac{1}{y}[g(y) - \log(g(y)) - 1]$ is decreasing in y. *Q.E.D.*

REFERENCES

Amann, E., and W. Leininger (1996): "Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case," *Games and Economic Behavior*, 14, 1–18.

Anderson, A., and L. Cabral (2007): "Go for broke or play it safe? Dynamic competition with choice of variance," *RAND Journal of Economics*, 38, 593–609.

Ankirchner, S., G. Heyne, and P. Imkeller (2008): "A BSDE approach to the Skorokhod embedding problem for the Brownian motion with drift," *Stochastics and Dynamics*, 8, 35–46.

Ankirchner, S., and P. Strack (2011): "Skorokhod Embeddings In Bounded Time," *Stochastics and Dynamics*, 11, 215–226.

Aoki, R. (1991): "R&D Competition for Product Innovation," *American Economic Review*, 81, 252–256.

Argote, L., S. Beckman, and D. Epple (1990): "The Persistence and Transfer of Learning in an Industrial Setting," *Management Science*, 36, 140–154.

Bass, R. F. (1983): "Skorokhod imbedding via stochastic integrals," in *Seminar on probability, XVII*, vol. 986 of *Lecture Notes in Math.*, pp. 221–224. Springer, Berlin.

Baye, M., D. Kovenock, and C. de Vries (1996): "The all-pay auction with complete information," *Economic Theory*, 8, 291–305.

Benkard, C. L. (2000): "Learning and Forgetting: The Dynamics of Aircraft Production," *American Economic Review*, 90, 1034–1054.

Besanko, D., U. Doraszelski, Y. Kryukov, and M. Satterthwaite (2010): "Learning-by-doing, Organizational Forgetting, and Industry Dynamics," *Econometrica*, 78, 453–508.

Bulow, J., and P. Klemperer (1999): "The Generalized War of Attrition," *American Economic Review*, 89, 175–189.

Burdett, K., and K. L. Judd (1983): "Equilibrium Price Dispersion," *Econometrica*, 51, 955–969.

Gul, F., and W. Pesendorfer (2011): "The War of Information," *Review of Economic Studies*, forthcoming.

Harris, C., and J. Vickers (1987): "Racing with Uncertainty," *Review of Economic Studies*, 54, 1–21.

Hillman, A., and J. Riley (1989): "Politically contestable rents and transfers," *Economics and Politics*, 1, 17–39.

Hillman, A., and D. Samet (1987): "Dissipation of contestable rents by small numbers of contenders," *Public Choice*, 54, 63–82.

Hopenhayn, H., and F. Squintani (2011): "Preemption Games with Private Information," *Review of Economic Studies*, 78, 667–692.

Hörner, J. (2004): "A Perpetual Race to Stay Ahead," *Review of Economic Studies*, 71, 1065–1088.

Karlin, S. (1953): "Reduction of certain classes of games to integral equations," in: H. Kuhn & A. Tucker (Eds.), 'Contributions to the Theory of Games, II.', Vol.28 of *Annals of Mathematical Studies*, Princeton University Press, Princeton, pp.125-128.

Krishna, V., and J. Morgan (1997): "An Analysis of the War of Attrition and the All-Pay Auction," *Journal of Economic Theory*, 72, 343–362.

Lazear, E. P., and S. Rosen (1981): "Rank-Order Tournaments as Optimal Labor Contracts," *Journal of Political Economy*, 89, 841–864.

Maynard Smith, J. (1974): "The theory of games and the evolution of animal conflicts," *Journal of Theoretical Biology*, 47, 209–221.

Moldovanu, B., and A. Sela (2001): "The Optimal Allocation of Prizes in Contests," *American Economic Review*, 91, 542–558.

Moscarini, G., and L. Smith (2007): "Optimal Dynamic Contests," *Working Paper*.

Obłój, J. (2004): "The Skorokhod embedding problem and its offspring," *Probability Surveys*, 1, 321–392.

Park, A., and L. Smith (2008): "Caller Number Five and related timing games," *Theoretical Economics*, 3, 231–256.

Seel, C., and P. Strack (2009): "Gambling in Dynamic Contests," *Working Paper.*

Siegel, R. (2009): "All-Pay Contests," *Econometrica*, 77, 71–92.

——— (2010): "Asymmetric Contests with Conditional Investments," *American Economic Review*, 100, 2230–2260.

Skorokhod, A. V. (1961): "Issledovaniya po teorii sluchainykh protsessov (Stokhasticheskie differentsialnye uravneniya i predelnye teoremy dlya protsessov Markova)," *Izdat Kiev University.*

Skorokhod, A. V. (1965): *Studies in the theory of random processes*, Translated from the Russian by Scripta Technica, Inc. Addison-Wesley Publishing Co., Inc., Reading, Mass.

Taylor, C. (1995): "Digging for Golden Carrots: An Analysis of Research Tournaments," *American Economic Review*, 85, 872–890.

Thompson, P. (2001): "How Much Did the Liberty Shipbuilders Forget? New Evidence for an Old Case Study," *Journal of Political Economy*, 109, 103–137.

Tullock, G. (1980): *Efficient rent seeking, in: J.Buchanan, R.Tollison and G.Tullock, (eds.), Towards a Theory of Rent-Seeking Society.* Texas A&M University Press.