Syntax without Abstract Objects
Nominalizing the Theory of Concatenation

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Nominalists deny the existence of abstract objects.[1] In view of the notoriously shadowy and intractable character of the latter, the formers' thesis certainly has a strong initial appeal to anyone sharing even the least bit of a naturalistic bent. But even though many philosophers nowadays do exhibit this inclination, the vices and virtues of nominalism remain a live issue in contemporary debates.

The reason for this is that proponents of nominalism with regard to some sort of alleged entity are usually seen to carry the burden of proof; it is expected that they demonstrate how one can actually get by without the entities in question. A philosopher who claims that, e.g., numbers do not exist, immediately has to face the fact that talk of numbers is ubiquitous in virtually any serious scientific endeavor. Since most nominalists show the broadly naturalistic tendencies I alluded to, this philosopher will normally be very reluctant to call for a grand revision of entrenched scientific practices on the grounds of her philosophical position. Accordingly, she will have to come up with methods of systematically *reconstructing* or *reinterpreting* portions of science seemingly committed to numbers in a way that eventually renders them ontologically innocuous, whilst retaining as many of the benefits as possible. Wether or not nominalists can fulfill this task satisfactorily is the question that keeps fuelling the debate despite the *prima facie* plausibility of nominalism.

There is no universally agreed-upon methodology that gets applied in such discussions. Sometimes nominalistic proposals are introduced in a very informal way that does not make it all too easy to assess their respective merits.[2] I believe, however, that it is possible to give rock solid arguments in favor or against certain nominalistic

[1]Of course this is a rather dubious sweeping statement. Versions of nominalism have been proposed that have no bearing whatsoever on the abstract-concrete distinction. I will not get into any of these in this paper. Instead I shall take the rejection of abstract objects as the defining criterion for being a nominalist.

[2]For a simple example that also served as an inspiration for this paper, cf. [Quine(1980), p.117], where it is 'shown' how a theory of (abstract) lengths might be reduced to a theory of physical objects.

reconstructions, once one moves into the realm of more formal investigations.

Ideally, we should be able to codify the platonistic orthodoxy as well as the nominalist's alternative ideas as two formal theories. Then the question of whether the reconstruction can succeed may be treated as a clear and distinct (meta-)logical problem: is the platonist's theory *relatively interpretable* in the nominalist's theory? I.e. can any platonistic arguments be sufficiently *simulated* in the nominalistic context, rendering platonism a harmless *façon de parler* – in a very rigorous sense.[3]

This paper is intended as a case study. I do not propose a general way of developing nominalistic theories, and I cannot show for arbitrary platonistic theories how they can be based on ontologically more parsimonious alternatives. I will, however, explore how to proceed with this in one particular field of enquiry: the theory of concatenation.

For describing formal languages and investigating the behavior of proofs, theories, and related subject matter, one needs a theory of syntax. In introductory textbooks and logic courses this role is usually played by a rather simple and informal fragment of set theory. More advanced texts on meta-mathematics, where precision is valued higher than accessibility and intuitiveness, almost uniformly rely on some formalized arithmetic for this task. Both these approaches seem philosophically unsatisfactory to me: the former mostly because it lacks rigor and too much is habitually left unsaid about the theoretical framework. But more importantly, both methods have in common the fundamental oddity that, in the end, one has to rely on heavy duty *mathematical* theories to prove facts about *linguistic* objects. Even more curiously, it turns out that from these perspectives, linguistic objects *really* are either sets or numbers.

This is where nominalistic worries enter. It would seem that terms, formulae, sentences, proofs etc. are quite concrete objects: chalk or ink traces on blackboards or sheets of paper. I find it baffling and certainly undesirable that we should be forced to make all sorts of assumptions about such elusive things as the natural numbers or even the set universe in order to find out facts about objects with which we seem to be so well acquainted.

So we are faced with a twofold diagnosis: (i) the standard syntax of formal languages takes a detour through specifically mathematical theories to talk about linguistic objects, and (ii) standard syntax takes a detour through the realm of abstract objects to

[3] Working with the notion of relative interpretability like this is, admittedly, only one of many possibilities of associating the broader philosophical issues with well-defined meta-mathematical ones. See, for instance, [Burgess and Rosen(1997)] for a different, yet somewhat related idea.

talk about very concrete things. What we need to rectify this situation is a theory that – on the one hand – is tailored to treat linguistic subject matter and – on the other hand – is also not susceptible to criticism on grounds of ontological parsimony.

The first step is easily taken. One can utilize the theory TC as proposed by [Grzegorczyk(2005)]. TC is formulated in the first-order language with two individual constants 'a' and 'b' as well as the two-place function symbol '$\circ$'[4], by taking (the universal closures of) the following formulae as axioms:

TC1. $(x \circ y) \circ z = x \circ (y \circ z)$

TC2. $x \circ y = u \circ v \rightarrow (x = u \wedge y = v) \vee \exists w((x \circ w = u \wedge w \circ v = y) \vee (u \circ w = x \wedge w \circ y = v))$

TC3. $x \circ y \neq a$

TC4. $x \circ y \neq b$

TC5. $a \neq b$

We give an informal interpretation to $\mathcal{L}_{TC}$ by specifying as the intended range of its variables the set of strings that can be formed out of just two different atomic symbols that the two individual constants are understood to respectively refer to. The circle is read as the operation of concatenating two strings or, in more everyday language, the operation of taking one string and immediately following it with another string on its right side.

It is plain to see that under this interpretation all five axioms turn out true. In the cases of TC1 and TC3-5, this is outright obvious: it should not be controversial that concatenation is associative: as far as the resulting string is concerned, it does not make a difference wether we first concatenate strings σ and τ and then follow the result by a third string ρ or we first concatenate τ with ρ and prefix the result of that by σ. As for axioms TC3-5 – they just give expression to the fact that we are dealing with two distinct primitive symbols, neither of which is itself a concatenate.

Only TC2 can look a little confusing at first sight. It is, however, quite straightforward what this formula states under the given informal reading: If a string is partitioned into two strings in different ways, then there will always exist an *interpolant*, i.e. a string that lies between the two 'borders'. In [Grzegorczyk and Zdanowski(2008)] TC2 is called the *editor axiom*, because it describes what happens when two editors

[4] Let us call this language $\mathcal{L}_{TC}$.

independently of each other are faced with the task of publishing the same text in two volumes. They either end up producing the same two volumes or one of them will distribute the text in such a way that some part of it is put at the end of volume 1, while the other has that very segment opening volume 2. With this illustration in mind, it should be fair enough to conclude that TC2, interpreted in the given manner, also states a truth about strings.

The characteristic ideas for TC actually originated in [Tarski(1935)], and related variants have been studied, e.g., by [Quine(1981)] and [Martin(1958)]; it is just the simple first-order formulation I use that is drawn from Grzegorczyk's paper. TC is mutually interpretable with Robinson's arithmetic Q[5] and, accordingly, the well-known applicability of the latter to the task of developing fundamental bits of syntax is directly inherited by the former.

Since the exact treatment of formal syntax in TC is not my primary concern here, I shall leave it at that. Let us just take this fact about the proof-theoretic strength of TC as evidence enough that part (i) of my diagnosis above – concerning the reliance on mathematical theories for investigating linguistic objects that permeates the logical literature – can more or less easily be answered by working with the non-mathematical TC, which delivers just enough resources.[6]

Let us turn to the really pressing issue: abstract objects. As i said before, there is good reason to think that the strings of symbols that we really encounter when dealing with formal languages are quite concrete. We now have to ask ourselves whether in moving to TC as our theory of syntax we have also remedied problem (ii)—the oddity that we normally posit a host of abstract objects in order to prove facts about those very tangible traces of chalk or ink. Is TC a nominalistically acceptable theory?

My answer to this question cannot draw on any general principle for determining the ontological commitments of formal theories and what they mean with regard to the debate about nominalism. Such a criterion would of course be very helpful, but it turns out to be surprisingly hard to pin down.[7]

I want to argue that TC is indeed a highly platonistic theory. For this I will only

[5] Cf. [Grzegorczyk(2005)], [Grzegorczyk and Zdanowski(2008)], [Švejdar(2009)], [Visser(2009)], and [Ganea(2009)] for this and a variety of related results about TC and some of its variants.

[6] A fair question is why TC should not be a mathematical theory. Does it not look like just a bunch of fairly general algebraic principles? I confess that I cannot quickly produce a useful criterion as to what seperates a mathematical theory from a non-mathematical one. But I am content to respond to this by pointing out that TC was *designed* with the linguistic field of application in mind and hence maybe we need not count it as essentially mathematical.

[7] On close inspection, Quine's celebrated attempts at providing such a criterion all fail spectacularly.

offer one simple thought: The formula '$\forall xy \exists z(z = x \circ y)$' is easily proved in TC. Thus, if TC really is *the* theory of signs and their concatenates, then for any given pair of strings there exists a string that is the concatenate of them. This quite clearly includes the case of any string and itself. I claim: strings that behave like this cannot be concrete markings of chalk on a blackboard![8]

We certainly cannot produce an 'a'-trace on a board and follow it by itself to generate a new, longer trace. The only thing we could do is leave *another* mark of sufficiently similar shape behind the first one, but this is obviously not the same. From this I conclude that the subject matter of TC certainly cannot be the concrete objects that I would favor to talk about in syntax. TC is a theory of expression *types*—the abstract shapes that similar strings supposedly instantiate. So we definitely have not dealt with diagnosis (ii). We did not get rid of abstract objects so far!

Hence, what we need to look for is an alternative theory that can be read as being concerned with string *tokens*, and that is also capable of producing a nominalistic reconstruction of TC.

My proposal for a nominalistic theory of concatenation has to start by taking seriously the demand for a language without $\mathcal{L}_{TC}$'s tacit platonism. That means we need to lose the presupposition that there exists, e.g., *the* symbol denoted by 'a'. Rather we should think of a plethora of primitive tokens that all exhibit the a-shape but really are quite different from one another. We also have to avoid the use of a function constant to speak about concatenation. The reason for this is that in classical logic functions are always assumed to be *total*, meaning that the value of a function has to be defined for any choice of arguments. We already saw that token concatenation cannot be total, because there is no concatenate of any string token and itself.

Let $\mathcal{L}_{NTC}$ be defined by the following non-logical vocabulary (with its intended informal interpretation): one-place predicates 'A' and 'B' (for being an A-, resp. a B-token), a two-place predicate '$\approx$' (indicating shape-similarity of strings), and a three-place predicate 'C' (where '$Cxyz$' is to be read as expressing that x is a concatenate of y and z).

Now for the theory I call NTC. As I said before, the ideas underlying the non-logical axioms of TC should not be too controversial, even for nominalists. Once we express them in our non-platonistic language, they are definite candidates for being

[8]Note that the theorem is proved just by identity logic, existential, and universal generalization; we do not even use any of the non-logical axioms. TC's platonism thus seems to already be built into the classical background logic and its interaction with $\mathcal{L}_{TC}$, specifically the use of function terms.

true principles governing string tokens.

NTC1. $Cwxy \wedge Cw'yz \rightarrow (Cuwz \leftrightarrow Cuxw')$

NTC2. $Czxy \wedge Czuv \rightarrow (x = u \wedge y = v) \vee \exists w((Cuxw \wedge Cywv) \vee (Cxuw \wedge Cvwy))$

NTC3. $Cxyz \rightarrow \neg Ax$

NTC4. $Cxyz \rightarrow \neg Bx$

NTC5. $Ax \wedge By \rightarrow x \neq y$

Interestingly, once one drops the singular-type treatment of primitive symbols as well as the total-function understanding of concatenation, these axioms do not state that much anymore. All the existence presuppositions have succesfully been removed, as is seen by the fact that from the given axioms it is not provable that there are *any* A- or B-tokens or concatenates!

This makes the theory that would result from $NTC1-5$ quite uninteresting. It is, however, very helpful in understanding what is going on, since we are now forced to make explicit any existence claims we want to work with when trying to nominalistically simulate TC.[9]

I propose the following as plausible further axioms: (a) there are A- and B-tokens, (b) token concatenation is unique—if a concatenate of two strings exists, it is the only one, (c) two strings can be concatenated iff they do not share a common part, and (d) string parthood is transitive.

In order to express these principles in the formal language, I will rely on the following definitions:

- $x \sqsubseteq_{\text{ini}} y :\leftrightarrow x = y \vee \exists z Cyxz$
- $x \sqsubseteq_{\text{end}} y :\leftrightarrow x = y \vee \exists z Cyzx$
- $x \sqsubseteq y :\leftrightarrow x \sqsubseteq_{\text{ini}} y \vee x \sqsubseteq_{\text{end}} y \vee \exists uvw(Cuxv \wedge Cywu)$

Here are the resulting new axioms:

NTC6. $\exists x Ax$

[9] The type perspective on strings could directly be restored by adding as additional postulates the existence of exactly one A- and exactly one B-object as well as the functionality of 'C'. In this manner, a very direct simulation of TC could obviously be reached, retaining its vicious platonism – but we definitely want to avoid this, so we have to find another way.

NTC7. $\exists x Bx$

NTC8. $Cxyz \wedge Cwyz \to x = w$

NTC9. $\neg\exists z(z \sqsubseteq x \wedge z \sqsubseteq y) \leftrightarrow \exists z Czxy$

NTC10. $x \sqsubseteq y \to (y \sqsubseteq z \to x \sqsubseteq z)$

Four of these I expect to be uncontroversial under the intended interpretation of $\mathcal{L}_{NTC}$. Only NTC9 might raise some suspicion, especially the implication from left to right. Are there not, e.g. on this page, lots of pairs of strings that do not have any substrings in common, yet still there is no concatenate of them? For instance, take the very first word and the last one as such a couple.

This is a fair point and it forces me to be more specific about a topic that until now I have carefully avoided: which objects specifically are the variables of $\mathcal{L}_{NTC}$ supposed to be ranging over under the intended interpretation?

The remark I just made about NTC9 dictates that a very narrow understanding of string tokens as actually physically present, uninterrupted inscriptions on a certain piece of paper is not too congenial with the axioms I proposed. To incorporate this, we would have to be more restrictive and impose further conditions on the existence of concatenates. For instance, we might say that strings need to be positioned directly adjacent for there being a concatenate of them. Hence we would be pressed to further expand the language and include such topological vocabulary as 'next to', as well as axioms governing this relation. Seeing as this would apparently make the resulting theory even more complicated, I prefer to avoid any such move.

Hence it is desirable that we find a wider understanding of inscription tokens that can still reasonably be counted as nominalistic, yet does not present us with such obvious counterexamples to NTC9. My suggestion here draws on the following illustration: take a typestter's letter case or a box of building bricks inscribed with letters and make sure there are some A's and B's among those. The latter we treat as the atomic string tokens. We now have two ways of specifying the range of the variables of $\mathcal{L}_{NTC}$ that are in line with NTC9 being true: either we treat any configuration of bricks, however scattered, as an inscription token, or we declare any *possible* concrete horizontal, gap free lining-up of bricks to be such a token.

Admittedly, hardcore nominalists will probably not be too happy to accept something as a concrete object that consists of two letters that are positioned on distinct planets, as we would, according to the first idea. They are probably even less inclined to countenance merely *possible* objects. Be that as it may, I will be satisfied if at least on some conception of nominalism that is not entirely artificial, the given interpretation of our formal language can be counted as including *only* concrete objects among the values of the variables. And in regard of the fact that the simple atomic tokens I specified, out of which all the others have to be built up, are very tangible, I shall claim that this is indeed the case![10] Thus, NTC9 is approved and will be used as an axiom in our nominalistic theory.

So far, I have not said anything about the relation of shape-similarity and its interaction with the other primitives of the language. This relation will be needed later on, when we attempt a nominalistic reconstruction of TC, because we are going to – roughly speaking – distinguish tokens only up to shape-similarity in order to obtain types.

Clearly, shape-similarity must be an equivalence relation. Also, we will demand that all the A- and B-tokens are only similar to A- and B-tokens respectively. This seems quite straightforward, but we still have to say more: (e) concatenates of respectively similar inscriptions must be similar themselves, (f) for any pair of strings there is a pair of respectively similar strings that that do not have a common part, and (g) concatenates that are similar to each other can be partitioned into respectively similar substrings.

The resulting axioms – much clearer than their natural language renditions – are:

NTC11. $x \approx x$

NTC12. $x \approx y \to y \approx x$

NTC13. $x \approx y \to (y \approx z \to x \approx z)$

NTC14. $Ax \to (Ay \leftrightarrow x \approx y)$

NTC15. $Bx \to (By \leftrightarrow x \approx y)$

NTC16. $x \approx y \land z \approx w \to (Cuxz \land Cvyw \to u \approx v)$

NTC17. $\exists x'y'(x \approx x' \land y \approx y' \land \neg\exists z(z \sqsubseteq x' \land z \sqsubseteq y'))$

NTC18. $Czxy \land z \approx z' \to \exists x'y'(x \approx x' \land y \approx y' \land Cz'x'y')$

Once again, I am convinced that the bulk of this list is quite obviously true under the given informal interpretation. But again there is one formula among these that is

[10]This is actually not that great a stretch of the imagination. Basic theories of mereology characteristically state the existence of arbitrarily scattered objects, much like NTC9 on the given reading. They enjoy a reputation as paradigmatically nominalistic theories nonetheless!

bound to raise some eyebrows: NTC17. Remember our box of bricks? Under real life conditions, there will be a finite number of bricks in the box. So let us consider some configuration c that uses up all the bricks there are. Now instantiate both x and y in NTC17 with this c. The formula then states that there are strings x' and y' that are both similar in shape to c and additionally have no part in common! But this is quite clearly impossible, because by assumption all the bricks that were available have been used to form c, and hence any other brick configuration would share a common atomic part with c.[11]

We see that the finite-brick-box understanding of inscription tokens is still not appropriate to all the principles I have formulated. To accept NTC17 as stating a truth, we must accordingly help ourselves to another idealization: let the intended range of $\mathcal{L}_{NTC}$'s variables be the set of all the configurations of a box of *infinitely many* bricks, containing just A's and B's.

Now all the material is on the table: let NTC be the theory in $\mathcal{L}_{NTC}$ that is axiomatized by (the universal closures of) NTC1-18!

I have already claimed that TC is a useful theory of syntax, having only one shortcoming: its platonistic ontology. I now want to offer NTC as a nominalistic alternative, thereby removing any nasty ontological commitments. To achieve the goal of my paper I thus have to argue that NTC is a nominalistic theory worthy of our attention and that it allows for a *reconstruction* of the platonistic arguments of TC without its platonism.

This aim can be broken down into three questions that need to be answered affirmatively: (a) Is NTC a nominalistic theory? (b) Is NTC consistent? (For if it were not, it certainly would not be worth any attention.) (c) Is TC *interpretable* in NTC? The latter two questions are as precise as one can demand – at least as soon as I specify an applicable definition of interpretability. (a) on the other hand is more vague. Let us turn to this first.

As I said before, it is notoriously hard to come up with a good criterion to decide wether or not a theory is committed to abstract objects. All that I can offer here are certain considerations of plausibility. In the case of TC it seemed quite plausible to call the incorporated treatment of strings a platonistic one. As a reason for this, I pointed out a feature that the entities of any interpretation that would render TC a sound theory must share. This very feature I then declared to be characteristic – at least in any *linguistic* context – of abstract rather than concrete objects. I concluded that the

[11] If this argument is rendered more rigorous in a set-theoretical setting, it delivers a proof that the theory NTC to be introduced has no finite models.

theory was platonistic because it did not admit of *any* purely concrete interpretation.

How do things look with NTC? In accord with what I just decreed for platonistic theories, I shall be happy to call NTC nominalistically acceptable, if at least *some* interpretation of its language can be laid out that renders the axioms true and contains only concrete objects. I already talked about more or less clearly nominalistic interpretations of the language of NTC under which some of the given axioms turn out false. This led me to specify a certain intuitive interpretation as the *intended* one that we should connect with NTC.

Regarding this – the infinite-brick-box understanding of inscription tokens – I have pointed out its two features that might be most troubling to a nominalist: the commitment to either strangely scattered objects or *possibilia* on the one hand and its infinity assumption on the other.

Be that as it may, I still want to claim that one can think of the infinite-brick-box interpretation as nominalistic. The reason for this is that *the rejection of abstract objects* is the defining characteristic of nominalism, as I have emphasized from the outset. The indicated worries arising from NTC's intended interpretation certainly are *congenial* to nominalism, but they cannot really be *based* on the mere rejection of abstract objects. Wether the tokens we talk about in NTC are spatio-temporally scattered or merely possible objects, they still are built up entirely out of concrete bricks and as such there is not much reason to take them to be in any way abstract. Moreover, nominalism and *finitism* are quite independent doctrines: nothing forces one to deny the existence of infinitely many concrete things only because one does not accept abstract objects!

I think these considerations suffice to make the following claim look not too far-fetched: NTC is a nominalistic theory.

Moving on to questions that are amenable to definite solutions: We shall first investigate (b), the matter of the consistency of NTC.

Theorem 1. *NTC is consistent.*

Proof Sketch. Consider the structure $\mathfrak{M} := \langle M, A^{\mathfrak{M}}, B^{\mathfrak{M}}, \approx^{\mathfrak{M}}, C^{\mathfrak{M}} \rangle$, where M is the set of injective finite sequences of natural numbers, $A^{\mathfrak{M}} := \{\{\langle 0, n\rangle\} | n \text{ is even}\}$, $B^{\mathfrak{M}} := \{\{\langle 0, n\rangle\} | n \text{ is odd}\}$, $\approx^{\mathfrak{M}} := \{\langle f, g\rangle | f, g \in M \mathbin{/\!\!\!\wedge} Lg(f)$[12]$= Lg(g) \mathbin{/\!\!\!\wedge} \forall n < Lg(f) [f(n) \text{ is even} \Leftrightarrow g(n) \text{ is even }]\}$, and $C^{\mathfrak{M}} := \{\langle f, g, h\rangle | f, g, h \in M \mathbin{/\!\!\!\wedge} f = g \cup \{\langle Lg(g) + n, m\rangle | \langle n, m\rangle \in h\}\}$.

[12]Where f is a finite sequence, by '$Lg(f)$' I mean the *length* of f.

It is rather to tedious to work through all the axioms, but in the end it turns out that $\mathfrak{M}$ is a model of NTC. (Details in [Kozian(2010), p.81-87])[13] $\square$

This just leaves one question to be answered: what are the prospects of a reconstruction within NTC of what we do in TC? More importantly, what does such a reconstruction look like? To develop my answer to this in the rigorous way it deserves, I shall start off with the notion of interpretability I want to use.

Definition. *Interpretations between theories.*[14]

- Let $\mathcal{L}, \mathcal{L}'$ be first-order languages, T a theory in $\mathcal{L}$, S a theory in $\mathcal{L}'$, Δ an $\mathcal{L}$-formula in just the free variable 'v_0', such that $T \vdash \exists v_0 \Delta$, and finally let I be a function that maps:
  - every n-place $\mathcal{L}'$-predicate P to an $\mathcal{L}$-formula $\varphi_P(v_1, \ldots, v_n)$, with at most '$v_1$', …,'$v_n$' occuring free.
  - every n-place $\mathcal{L}'$-function-symbol f to an $\mathcal{L}$-formula $\varphi_f(v_1, \ldots, v_n, v_0)$, with at most '$v_0$', …,'$v_n$' occuring free.
  - every $\mathcal{L}'$-constant c to an $\mathcal{L}$-formula $\varphi_c(v_0)$, with at most 'v_0' occuring free.
  - each variable v_n to the formula $\ulcorner(v_0 = v_{n+1})\urcorner$.
- Let I_t be extended to all $\mathcal{L}'$-terms as follows:
  - $I_t(\sigma) = I(\sigma)$ for atomic $\mathcal{L}'$-terms σ
  - $I_t(\ulcorner f(\sigma_1, \ldots \sigma_n)\urcorner) = \ulcorner \forall v_{i+1} \ldots v_{i+n}(\Delta(v_{i+1}/v_0) \wedge \cdots \wedge \Delta(v_{i+n}/v_0) \wedge I_t(\sigma_1)(v_{i+1}/v_0) \wedge \cdots \wedge I_t(\sigma_n)(v_{i+n}/v_0) \rightarrow \varphi_f(v_{i+1}/v_1, \ldots, v_{i+n}/v_n, v_0/v_0))\urcorner$ for n-place function terms f of $\mathcal{L}'$, where i is the least number greater than the arities of all $\mathcal{L}'$-predicates, such that $i > j$ for each j with v_j occurring in either of $I_t(\sigma_1), I_t(\sigma_2), \ldots, I_t(\sigma_n)$, or $I(f)$.

[13] If one prefers a consistency result relative to some humble arithmetic theory over the set machinery of model theory, then the idea indicated here is easily transformed into a proof that EA interprets NTC.

[14] This is not the standard definition, as that would make the enterprise look hopeless from the get-go. I could not find a treatment in the literature that does not take a detour through definitional extensions as did [Tarski et al.(1953), p.20-30], is capable of dealing with the absence of function terms in the target language, and treats identity as liberally as the context dictates. That is why I developped this variant that to some might seem excessively general and formal. There is, however, a certain affinity between this and the version in [Visser(1998), p.314f.]

- Now let $* : Form_{\mathcal{L}'} \to Form_{\mathcal{L}}$ be defined as follows, depending on a given Δ and I:
  - Let $\sigma_1, \ldots, \sigma_n$ be terms of $\mathcal{L}'$ and ψ be an atomic $\mathcal{L}'$-formula $\ulcorner P(\sigma_1, \ldots \sigma_n)\urcorner$. Then $\psi^* = \ulcorner \forall v_{l+1} \ldots v_{l+n}(\Delta(v_{l+1}/v_0) \wedge \cdots \wedge \Delta(v_{l+n}\ /v_0) \wedge I_t(\sigma_1)(v_{l+1}/v_0) \wedge \cdots \wedge I_t(\sigma_n)(v_{l+n}/v_0) \to \varphi_P(v_{l+1} \ldots v_{l+n}))\urcorner$ where l is the least number greater than the arities of all $\mathcal{L}'$-predicates, with $l > j$ for each j, such that v_j occurs in either of $I_t(\sigma_1), I_t(\sigma_2), \ldots, I_t(\sigma_n)$, or $I(P)$.
  - for complex formulae we demand: $\ulcorner\neg\psi\urcorner^* = \ulcorner\neg\psi^*\urcorner$, $\ulcorner(\psi \to \chi)\urcorner^* = \ulcorner(\psi^* \to \chi^*)\urcorner$, and $\ulcorner\forall v_n\psi\urcorner^* = \ulcorner\forall v_{n+1}(\Delta(v_{n+1}/v_0) \to \psi^*)\urcorner$, resp. $\ulcorner\exists v_n\psi\urcorner^* = \ulcorner\exists v_{n+1}(\Delta(v_{n+1}/v_0) \wedge \psi^*)\urcorner$
- $*$ is an *interpretation of S in T* if and only if: (i) for each n-place function symbol f of $\mathcal{L}'$ such that $\varphi_f(v_1, \ldots, v_n, v_0) = I(f)$: $T \vdash \forall v_1 \ldots v_n(\Delta(v_1/v_0) \wedge \cdots \wedge \Delta(v_n/v_0) \to \exists v_{i+1}(\Delta(v_{i+1}/v_0) \wedge \varphi_f(v_1, \ldots, v_n, v_{i+1}) \wedge \forall v_{i+2}(\Delta(v_{i+2}/v_0) \wedge \varphi_f(v_1, \ldots, v_n, v_{i+2}) \to (v_i = v_{i+1})^*)))$,[15] where i is a suitable number $> n$ such that for all j with v_j occurring in either Δ or $I(f)$: $i > j$, (ii) for all $\mathcal{L}'$-constants c: if $\varphi_c(v_0) = I(c)$, then $T \vdash \exists v_{i+1}(\Delta(v_{i+1}/v_0) \wedge \varphi_c(v_{i+1}) \wedge \forall v_{i+2}(\Delta(v_{i+2}/v_0) \wedge \varphi_c(v_{i+2}) \to (v_i = v_{i+1})^*))$, with $i > j$ for each j such that v_j occurs in either Δ or $I(c)$, and (iii) for each axiom φ of S – including the purely logical ones – we have: $T \vdash \varphi^*$.
- S is *interpretable* in T (T *interprets* S) if and only if there is an interpretation of S in T.

Just a quick review of what just happened. The aim is to give a systematic characterization of translations between possibly very different formal languages that preserve logical structure except for relativizing quantifiers and maybe reinterpreting identity. If one of those can be specified that also preserves theoremhood with respect to theories S and T – which is easily seen to be guaranteed by preservation of theoremhood of the axioms –, then we have found what we call an interpretation of one theory in another and we will claim that we can reconstruct S in T, or that S can be reduced to T.

A special feature of my way of handling this notion is made necessary by the fact that TC speaks of singular types, where NTC has multiple tokens of the same shape.

[15] Note the variable shift produced by $*$. Hence the occurrence of different variables in the untranslated identity statement.

Hence we cannot reconstruct the behavior of identity in TC by using identity in NTC. We do need the translation of 'there is exactly one a' to be a theorem, but it need only look something like this: 'there is exactly one *a up to translated identity*'.

Let us now approach the main result of this paper: We ultimately want to prove the interpretability of TC in NTC, thereby securing the advertised nominalization of the theory of concatenation. Let $\Delta := \ulcorner(v_0 = v_0)\urcorner$ and I be a function such that

- $I(`=\text{'}) = \ulcorner(v_1 \approx v_2)\urcorner$
- $I(`\circ\text{'}) = \ulcorner\forall v_3 v_4 v_5 (v_3 \approx v_1 \wedge v_4 \approx v_2 \wedge C v_5 v_3 v_4 \rightarrow v_5 \approx v_0)\urcorner$
- $I(`a\text{'}) = \ulcorner A v_0\urcorner$
- $I(`b\text{'}) = \ulcorner B v_0\urcorner$
- $I(`v_n\text{'}) = \ulcorner(v_0 = v_{n+1})\urcorner$

To obtain the interpretability theorem, we have to show that the function $*$ that results from these according to the definition above, is an interpretation of TC in NTC. The proof of this requires quite a lengthy series of lemmas. I have proved those in full detail in [Kozian(2010), p.70-81]. I will not get into any of this here. Suffice it to say that one ultimately has to make sure that all the axioms of TC are mapped to NTC-theorems – and this includes the purely logical ones because their respective translations do not universally turn out to be logical axioms of NTC. The process is – again – tedious, albeit not entirely without interest. Space is limited here, however, whence we immediately proceed to the statement of

Theorem 2. *TC is interpretable in NTC.*

The time has come to take stock of what was achieved in this paper. I started by voicing a characteristically nominalistic qualm that can arise out of observing a certain entrenched scientific practice: investigations regarding formal languages are usually conducted in an ontologically highly suspicious environment. Not only do people work with mathematical surrogates for essentially linguistic objects, they also seem to rely heavily on strong ontological claims. If standard syntax theories are supposed to be *true*, then abstract objects must exist!

I subsequently considered an alternative theory for dealing with formal syntax that seemingly removed the purely mathematical subject matter but retained a lot of the desired proof theoretical resources. This theory – TC – has the great disadvantage, however, that it is committed to countenance clearly abstract objects just as well, so that the *nominalistic* worry could not yet be calmed by this move.

Following this assessment, I set out to develop yet another alternative theory for doing syntax, one that could reasonably be claimed not to be ontologically committed to abstract objects, yet would still allow us a *reconstruction* of what happens in TC. This theory I named NTC.

I then argued that in spite of all the shortcomings that NTC might exhibit, it actually satisfies those requirements. There is some charitable understanding of what NTC claims that would render it a nominalistic theory. I also gave a somewhat idiosyncratic clarification of what I have in mind when I speak of 'reconstructing' one theory in another, and finally I hinted at a proof that such a reconstruction of TC in NTC is possible. I have also made it clear that NTC is a consistent and hence non-trivial theory.

For all these reasons I can now come to a closing of the paper and declare the theory of concatenation successfully nominalized, stripped of its unsavory ontological ballast!

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