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CONTENTS OF VOLUME 98

NUMBER 1, APRIL 1984

NUMBER 2, MAY 1984

NUMBER 3, JUNE 1984

The Subject Index for Volume 98 will appear in the December 1984 issue as part of a cumulative index for the year 1984.

RADIATION RESEARCH **98**, 425–437 (1984)

Chord-Length Distributions and Related Quantities for Spheroids

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KELLERER, A. M. Chord-Length Distributions and Related Quantities for Spheroids. *Radiat. Res.* **98**, 425–437 (1984).

The chord-length distributions are derived that result when spheroids are randomly traversed by straight lines. The first part of the article applies generally to convex domains in three-dimensional or two-dimensional space; the relationships between the chord-length distributions and their moments for different types of randomness are summarized. Subsequently the chord-length distributions, the point-pair distance distributions, and the geometric reduction factors are derived by a suitable transformation from the distributions for the sphere. All integrals can be resolved and the resulting formulae are valid for both prolate and oblate spheroids. The moments of the chord-length distributions are obtained by the same transformation from those for the sphere. The solutions for ellipses are given in the Appendix and contain Legendre integrals.

INTRODUCTION

Chord-length distributions result when convex bodies are randomly intercepted by straight lines. These distributions and related concepts have applications in acoustics, microscopy, texture analysis, shielding or dose calculations, and microdosimetry. Early results have been obtained by Crofton (*1, 2*). Kingman (*3, 4*), Coleman (*5, 6*), and Enns and Ehlers (*7*) have made important recent contributions. Surveys are given by Kendall and Moran (*8*) and by Coleman (*9*). Weil (*10*) offers an excellent general overview with a comprehensive list of references. Simple analytical expressions of the chord-length distributions exist for the sphere and for the infinite slab, and, in the two-dimensional case, for the disc and for the rectangle (*4*). A formula containing one integration has been obtained for cylinders with convex cross sections (*11*). For circular cylinders the integral requires numerical integration (*12*). For parallelepipeds Coleman (*13*) has derived the explicit solutions.

The distance distributions of pairs of random points in convex bodies are linked to the chord-length distributions. They, too, have simple analytical expressions in the case of the sphere and the infinite slab; Borel (*14*) has considered the point-pair distance distributions for two-dimensional figures. A solution for general cylinders, including the case of circular cylinders and parallelepipeds (*15*), contains a somewhat simpler integral than the solution for the chord-length distributions.

The attention to parallelepipeds and circular cylinders stems from various applications. The case of spheroids has comparatively less pragmatic importance and has therefore rarely been treated. However, it is not without interest in applications of microdosimetry to radiobiology where one deals with cells, cell nuclei, or various cell

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organelles that have rounded but frequently nonspherical shape. Such structures are often adequately approximated as spheroids, i.e., as ellipsoids with two axes of equal length. Accordingly the objective of the present article is the derivation of the chord-length distributions, the point-pair distance distributions, and the moments of these distributions for spheroids. Allisy and Boutillon (*16*) have, within the context of microdosimetric computations for neutrons, utilized a transformation that links the chord-length distribution of the spheroid to that of the sphere. The present approach is somewhat different, but the idea is also to apply a suitable transformation to the solutions for a sphere. The results are not entirely new. Enns and Ehlers (*7*) have earlier given the chord-length distribution of prolate spheroids and their moments, although they have not reported the details of their derivation.

INTERRELATIONS AMONG THE VARIOUS QUANTITIES

The Distributions

The earlier article with solutions for cylinders (*15*) deals also with general properties of the chord-length distributions and point-pair distributions, and with their interrelations. This treatment applies equally to the present article. A brief but somewhat more complete restatement of essential relations and their synopsis in tabular form may nevertheless be useful.

There are different types of randomness for the intercept of a convex body, K, by straight lines. Following the recent terminology of Coleman (*13*) one can distinguish three main types: *isotropic uniform randomness* results when the body is exposed to a uniform isotropic fluence of infinite straight lines; *weighted randomness* results when a uniformly distributed random point is chosen in K and is traversed by a straight line with uniform random direction; *two-point randomness* is obtained when a straight line traverses two random points that are independently and uniformly distributed in K. The subscripts μ, ν, λ, respectively, are used for these three cases.[1] For example, $f_\mu(x)$ designates the chord-length density under isotropic uniform randomness, and $F_\mu(x)$ designates the sum distribution. For convenience all sum distributions are summed from the left (see Eq. (14)). The letters μ, ν, and λ are also used for the moments; the order is given by an index. For example, μ_1 is the mean value and μ_k is the moment of order k of $f_\mu(x)$:

$$\mu_k = \int_0^\infty x^k f_\mu(x)dx. \tag{1}$$

The probability densities of the chord lengths for the three different types of randomness are related (see (*4, 12*)):

$$f_\mu(x) = ax^{-1}f_\nu(x) = bx^{-m}f_\lambda(x). \tag{2}$$

In three-dimensional space, R_3,

$$a = 4V/S, \qquad b = 12V^2/\pi S \qquad \text{and} \qquad m = 4 \tag{3}$$

V: volume; S: surface of K. In two-dimensional space, R_2,

[1] $f_\nu(x)$ was formerly designated by $f_1(x)$, and the term *interior* randomness was used instead of *weighted* randomness.

TABLE I

Relations between Chord-Length Distributions for Different Types of Randomness[a]

	μ		ν		i		p		λ
μ	$f_\mu(x)$	=	$ax^{-1}f_\nu(x)$	=	$-af'_i(x)$	=	$aU''(x)$	=	$bx^{-n-1}f_\lambda(x)$
ν	$xf_\mu(x)/a$	=	$f_\nu(x)$	=	$xf'_i(x)$	=	$xU''(x)$	=	$x^{-n}f_\lambda(x)/c$
i	$F_\mu(x)/a$			=	$f_i(x)$	=	$-U'(x)$		
p	$\int_x^\infty F_\mu(x)/a$			=	$F_i(x)$	=	$U(x)$		
λ	$x^{n+1}f_\mu(x)/b$	=	$cx^n f_\nu(x)$	=	$cxf'_i(x)$	=	$cx^{n+1}U''(x)$	=	$f_\lambda(x)$

[a] For simplicity a number of relations are omitted; they can be obtained from the inverse relations. The geometric reduction factor, $U(x)$, is used instead of $f_p(x)$; $s(x)$ is the proximity function.
In R_3: $n = 3$, $a = 4V/S$, $b = 12\ V^2/\pi S$, $f_p(x) = 4\pi x^2 U(x)/V = s(x)/V$, $c = a/b$.
In R_2: $n = 2$, $a = \pi A/C$, $b = 3A^2/C$, $f_p(x) = 2\pi x U(x)/A = s(x)/A$, $c = a/b$.

$$a = \pi A/C \quad \text{and} \quad b = 3A^2/C \quad \text{and} \quad m = 3 \tag{4}$$

A: area; C: perimeter of K. The constant a is equal to the mean chord length μ_1 (see Eqs. (12), (13) and Table II).

Other types of randomness involve the selection of random points on the surface of the body (*17*). They will not be considered here, since they stand in no known relation to μ-, ν-, or λ-randomness.

A further concept of importance in radiation dosimetry has been termed *i*-randomness (internal source randomness) (*11*); it refers to the distribution of the length, x, of randomly oriented *rays* from random points in K. The length, x, of the ray is the distance from the random point to the intersection of the ray with the surface of S. The probability density is designated by $f_i(x)$. It is related to the chord-length distribution (*11*)

$$f_i(x) = F_\mu(x)/a. \tag{5}$$

$F_i(x)$ is equal to the *geometric reduction factor,* $U(x)$, that is important in dose calculations with internal emitters (see (*18, 19*)) and that is closely linked to the point-pair distance distribution.[2] This latter distribution results if pairs of points are chosen independently and uniformly in the body, and if their distances are considered. The probability density of these distances is designated by $f_p(x)$; it is related to $U(x)$ or $F_i(x)$:

$$f_p(x) = 4\pi x^2 U(x)/V = 4\pi x^2 F_i(x)/V \text{ in } R_3 \tag{6}$$

$$f_p(x) = 2\pi x U(x)/A = 2\pi x F_i(x)/A \text{ in } R_2. \tag{7}$$

Table I gives a synopsis of the various interrelations that result from Eqs. (2)–(7). For simplicity the quantity $U(x)$ is utilized instead of the density $f_p(x)$. The proximity function, $s(x)$, (see (*15*)) equals $f_p(x) \cdot V$ in R_3 and $f_p(x) \cdot A$ in R_2.

[2] In spite of their numerical identity for convex bodies, the concepts of $F_i(x)$ and $U(x)$ are distinguished. $U(x)$ has earlier been designated by $\phi(x)$ and $\Omega(x)$ (*7, 18*); it is defined as the probability that a shift by x in random direction from a random point in K leads again to a point in K. This definition, as the definition of $f_p(x)$, applies equally to convex and nonconvex bodies. The chord-length distributions, on the other hand, pertain only to convex bodies; they admit different extensions to nonconvex bodies.

TABLE II

Relations between the Moments of the Chord-Length Distributions

					In R_3[a]				In R_2[b]			
μ		ν		i		p		λ		p		λ
μ_{-1}	=	$a\nu_{-2}$					=	$b\lambda_{-5}$			=	$b\lambda_{-4}$
1	=	$a\nu_{-1}$					=	$b\lambda_{-4}$			=	$b\lambda_{-3}$
μ_1	=	a					=	$b\lambda_{-3}$			=	$b\lambda_{-2}$
μ_2	=	$a\nu_1$	=	$2ai_1$	=	$b/6p_{-2}$	=	$b\lambda_{-2}$	=	$b/3p_{-1}$	=	$b\lambda_{-1}$
μ_3	=	$a\nu_2$	=	$3ai_2$	=	$b/2p_{-1}$	=	$b\lambda_{-1}$	=	b		
μ_4	=	$a\nu_3$	=	$4ai_3$	=	b			=	$2bp_1$	=	$b\lambda_1$
μ_5	=	$a\nu_4$	=	$5ai_4$	=	$5b/3p_1$	=	$b\lambda_1$	=	$10b/3p_2$	=	$b\lambda_2$
μ_k	=	$a\nu_{k-1}$	=	kai_{k-1}	=	$b\dfrac{k(k-1)}{12}p_{k-4}$	=	$b\lambda_{k-4}$	=	$b\dfrac{k(k-1)}{6}p_{k-3}$	=	$b\lambda_{k-3}$

[a] In R_3: $a = 4V/S$, $b = 12\,V^2/\pi S$.
[b] In R_2: $a = \pi A/C$, $b = 3A^2/C$.

The Moments

Equations (2)–(4) yield the relations between the moments of the chord-length distributions:

$$\mu_k = a\nu_{k-1} = b\lambda_{k-m}, \qquad (k \geqslant -1).^3 \tag{8}$$

From Eqs. (5)–(7) one obtains by partial integration

$$\mu_k = kai_{k-1}, \qquad (k \geqslant 1) \tag{9}$$

and

$$p_k = \frac{4\pi}{V(k+3)}\, i_{k+3} = \frac{12}{(k+3)(k+4)}\,\lambda_k, \qquad (k \geqslant -2) \text{ in } R_3 \tag{10}$$

$$p_k = \frac{2\pi}{A(k+2)}\, i_{k+2} = \frac{6}{(k+2)(k+3)}\,\lambda_k, \qquad (k \geqslant -1) \text{ in } R_2. \tag{11}$$

Table II gives a synopsis of these relations and contains several notable identities for moments that depend only on integral parameters of K. In R_3 one has for any convex body

$$\begin{aligned} \mu_1 &= a = 4V/S & \mu_4 &= b = 12V^2/\pi S \\ \nu_{-1} &= 1/a = S/4V & \nu_3 &= b/a = 3V/\pi \\ & & i_3 &= b/4a = 3V/4\pi \\ \lambda_{-4} &= 1/b = \pi S/12V^2 & \lambda_{-3} &= a/b = \pi/3V. \end{aligned} \tag{12}$$

[3] The case $k = -1$ does not apply to figures with corners or to bodies with edges; μ_{-1}, ν_{-2}, and λ_{-4} (in R_2) or λ_{-5} (in R_3) are then infinite.

in R_2 one has

$$\begin{aligned} \mu_1 &= a = \pi A/C & \mu_3 &= b = 3A^2/C \\ \nu_{-1} &= 1/a = C/\pi A & \nu_2 &= b/a = 3A/\pi \\ & & i_2 &= b/3a = A/\pi \\ \lambda_{-3} &= 1/b = C/3A^2 & \lambda_{-2} &= a/b = \pi/3A. \end{aligned} \tag{13}$$

No such general relation is known for moments of the point-pair distance distributions.

The Distributions for the Sphere

The solutions for the sphere will be required subsequently; they can also serve to illustrate the various distributions. One obtains, with the diameter d and with $0 \leqslant x \leqslant d$:

$$\begin{aligned} f_\mu(x) &= 2x/d^2; & F_\mu(x) &= 1 - (x/d)^2 \\ f_\nu(x) &= 3x^2/d^3; & F_\nu(x) &= 1 - (x/d)^3 \\ f_\lambda(x) &= 6x^5/d^6; & F_\lambda(x) &= 1 - (x/d)^6 \\ f_i(x) &= \frac{3}{2d} - \frac{3x^2}{2d^3}; & F_i(x) &= U(x) = 1 - \frac{3}{2}\frac{x}{d} + \frac{1}{2}\left(\frac{x}{d}\right)^3 \\ f_p(x) &= \frac{24x^2}{d^3}\left(1 - \frac{3x}{2d} + \frac{x^3}{2d^3}\right); & F_p(x) &= 1 - 8\left(\frac{x}{d}\right)^3 + 9\left(\frac{x}{d}\right)^4 - 2\left(\frac{x}{d}\right)^6. \end{aligned} \tag{14}$$

The distributions are represented in Fig. 1. The moments will be listed subsequently in Eqs. (32)–(36) for spheroids; the solutions for the sphere result when all parameters e_k are set equal to unity.

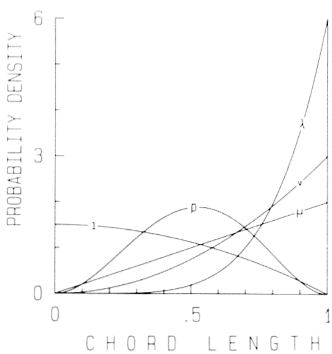

FIG. 1. Chord-length distributions for the sphere (see Eq. (14)): μ-randomness (uniform isotropic distribution of straight lines); ν-randomness (straight lines through random point in sphere); λ-randomness (straight lines through two random points in sphere); i-randomness (ray originating in random point); p: point-pair distance distribution.

SOLUTION FOR SPHEROIDS

Principle of the Solution and Derivation of the Transformation Kernel

The solutions will first be formulated in terms of the two distributions $f_\lambda(x)$ and $f_p(x)$ that result from the random choice of pairs of points in K. The related functions $s(x)$, $U(x)$, $f_\mu(x)$, $f_\nu(x)$, or $f_i(x)$ can then readily be obtained.

A unidirectional compression or expansion by the factor e transforms the sphere into an oblate ($e < 1$) or prolate ($e > 1$) spheroid, i.e., this transformation, T, establishes a one-to-one relation between the points of the sphere and their image points in the spheroid. Two independently, uniformly distributed random points in the sphere have images that are also independently, uniformly distributed in the spheroid. This will be utilized in the solution, i.e., the distributions $f_\lambda(x)$ and $f_p(x)$ in the spheroid will be obtained by applying a suitable transformation to the corresponding distributions in the sphere.

Consider two points in the sphere that are separated by the distance u. The distance, x, between their image points can then, depending on orientation, have any value between u and ue. The point pairs in the sphere are randomly oriented and the distribution, $h_u(x)$, of resulting distances is therefore obtained by considering a spherical surface of radius u, and by asking for the distribution in distance from the center that results after the points of the surface are subjected to the transformation T. Figure 2 indicates this schematically for an oblate spheroid; it will be noted that the result applies equally to prolate spheroids.

Let $H_u(x)$ be the sum distribution that belongs to $h_u(x)$, i.e., the probability that the transformation changes the distance u into a distance larger than x. It is apparent that $H_u(x)$ is equal to that fraction of the surface of the hemisphere in Fig. 2, that lies to the right of the broken line for $e > 1$ and to the left for $e < 1$. These fractions are $1 - z/u$ and z/u:

$$H_u(x) = \begin{cases} z/u = \sqrt{(1 - x^2/u^2)/(1 - e^2)}, & ue \leqslant x \leqslant u \quad \text{for} \quad e < 1 \\ 1 - z/u = 1 - \sqrt{(x^2/u^2 - 1)/(e^2 - 1)}, & u \leqslant x \leqslant ue \quad \text{for} \quad e > 1 \end{cases} \tag{15}$$

$$h_u(x) = -\frac{dH_u(x)}{dx} = \frac{x}{u\sqrt{(1 - e^2)(u^2 - x^2)}}. \tag{16}$$

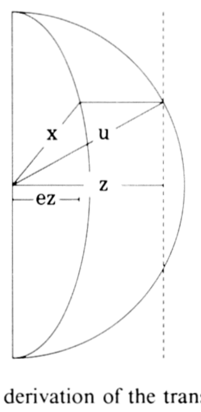

FIG. 2. Diagram for the derivation of the transformation kernel $H_u(x)$.

The transformation will be utilized to obtain the distributions and the moments for spheroids from the solutions for the sphere.

Evaluation of the Transformation Formula

If the distributions for the sphere are marked by a star, one has the following formulae for the spheroid:

$$f_\lambda(x) = \int_{u_1}^{u_2} h_u(x) f_\lambda^*(u)\,du \tag{17}$$

and

$$f_p(x) = \int_{u_1}^{u_2} h_u(x) f_p^*(u)\,du \tag{18}$$

with:

$$\begin{array}{llllll} 0 \leqslant x \leqslant d, & u_1 = x & \text{and} & u_2 = \mathrm{Min}(d, x/e), & \text{for} & e < 1 \\ 0 < x < ed, & u_1 = x/e & \text{and} & u_2 = \mathrm{Min}(d, x), & \text{for} & e > 1. \end{array} \tag{19}$$

Derivation of the chord-length distribution. Equation (17) for the chord-length distribution will be evaluated first. The distribution, $f_\mu(x)$, for uniform isotropic randomness has more pragmatic importance than $f_\lambda(x)$; utilizing Eq. (2) one can rewrite Eq. (17) in terms of $f_\mu(x)$ and $f_\mu^*(x)$:

$$\begin{aligned} f_\mu(x) &= \frac{e^2\pi d^2}{Sx^4} \int_{u_1}^{u_2} h_u(x) u^4 f_\mu^*(\mathrm{u})\,du \\ &= \frac{2e^2\pi}{Sx^3\epsilon} \int_{u_1}^{u_2} \frac{u^4}{|u^2 - x^2|^{.5}}\,du \end{aligned} \tag{20}$$

$\epsilon = |1 - e^2|^{.5}$ is the linear excentricity; S is the surface area of the spheroid.

To obtain a common formulation, in terms of real functions, for the oblate and the prolate spheroid, it is practical to use a function that merges the inverse cosine and the inverse hyperbolic cosine:

$$ci(x) = \begin{cases} \cos^{-1}(x), & \text{for} \quad 0 \leqslant x \leqslant 1 \\ \mathrm{Cos}^{-1}(x) = \ln(x + (x^2 - 1)^{.5}), & \text{for} \quad x > 1. \end{cases} \tag{21}$$

Furthermore one can use the two constants

$$c_1 = \frac{1}{2} + \frac{e^2}{2\epsilon}\, ci\left(\frac{1}{e}\right) \quad \text{and} \quad c_2 = \frac{1}{4e^2} + \frac{3}{4} c_1. \tag{22}$$

The surface of the spheroid is $S = c_1 \cdot \pi \cdot d^2$, and the integration of Eq. (20) gives the chord-length distribution for the spheroid[4]

$$f_\mu(x) = \frac{2x}{c_1 d^2}\left[c_2 + \frac{\epsilon}{4(e^{-2} - 1)}\left\{\left|\frac{d^2}{x^2} - 1\right|^{.5}\left(\frac{d^3}{x^3} + \frac{3d}{2x}\right) + \frac{3}{2}\, ci\left(\frac{d}{x}\right)\right\}\right]. \tag{23}$$

[4] The corresponding distribution, $f_\nu(x) = (3c_1x/2ed)f_\mu(x)$, for weighted randomness agrees with the expression given by Enns and Ehlers (*7*, Eq. (31)) for the prolate spheroid, although the identity of the two expressions is not readily evident.

The first term, c_2, in the square bracket applies only for $0 < x < ed$, the second term for $ed < x < d$ or $d < x < ed$. The distributions are illustrated in Fig. 3.

Derivation of the point-pair distance distributions. Inserting the expression from Eq. (14) into Eq. (18) one obtains

$$f_p(x) = \frac{24x}{d^3\epsilon}\int_{u_1}^{u_2} \frac{\left(u - \frac{3}{2}\frac{u^2}{d} + \frac{u^4}{2d^3}\right)}{|u^2 - x^2|^{.5}}\,du. \tag{24}$$

The integration yields

$$f_p(x) = \frac{24x^2}{d^3e}\left[1 - \frac{3x}{2d}\frac{c_1}{e} + \frac{x^3}{2d^3}\frac{c_2}{e}\right.$$
$$\left. + \frac{3}{8}\frac{\epsilon}{(e^{-1} - e)} \times \left\{\left|\frac{d^2}{x^2} - 1\right|^{.5}\left(\frac{x^2}{2d^2} + 1\right) + \left(\frac{x^3}{2d^3} - \frac{2x}{d}\right)ci\left(\frac{d}{x}\right)\right\}\right]. \tag{25}$$

The expression in the first line of the equation applies for $0 < x < ed$, the expression in the second line for $ed < x < d$ or $d < x < ed$.

The distributions are represented in Fig. 4. The related quantity $U(x) = F_i(x) = f_p(x)d^3e/24x^2$ is equal to the expression in the square brackets in Eq. (25).

The Moments

A direct derivation of the moments will make it unnecessary to integrate the complicated distributions.

If u is the distance of two random points in the sphere, or the length of the chord defined by these random points, then the distribution of the corresponding distances, x, in the spheroid is $h_u(x)$ and the expectation of x^k for this fixed value u is

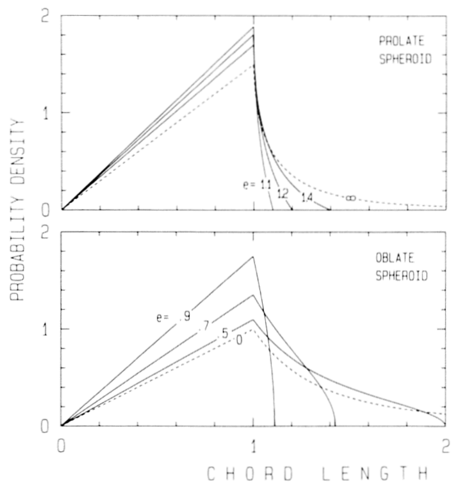

FIG. 3. Chord-length distributions (μ-randomness) for spheroids of elongation e and of unit length smaller axis ($d = 1$ for $e > 1$; $d = 1/e$ for $e < 1$).

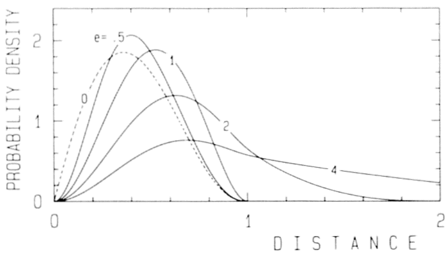

FIG. 4. Point-pair–distance distributions for the spheroid of elongation e and with two equal axes of unit length ($d = 1$).

$$\langle x_u^k \rangle = \int x^k h_u(x) dx$$
$$= \frac{1}{\epsilon} \int \frac{x^{k+1}}{u|u^2 - x^2|^{.5}} dx \tag{26}$$
$$= e_k u^k$$

with

$$e_k = \frac{\epsilon}{1 - e^2} \int_e^1 \frac{z^{k+1}}{|1 - z^2|^{.5}} dz, \qquad (e_k = 1 \text{ for } e = 1) \tag{27}$$

Evaluation of these integrals gives

$$\begin{aligned} e_{-5} &= \frac{2}{3e} + \frac{1}{3e^3} & e_{-1} &= \frac{1}{\epsilon} ci(e) \\ e_{-4} &= \frac{1}{2e^2} + \frac{1}{2\epsilon} ci\left(\frac{1}{e}\right) & e_0 &= 1 \\ e_{-3} &= \frac{1}{e} & e_1 &= \frac{e}{2} + \frac{1}{2\epsilon} ci(e) \\ e_{-2} &= \frac{1}{\epsilon} ci\left(\frac{1}{e}\right) & e_2 &= \frac{2}{3} + \frac{e^2}{3}. \end{aligned} \tag{28}$$

Accordingly one has the relation between the moments for the spheroid and the moments, λ_k^*, of the sphere

$$\lambda_k = \int_0^d \langle x_u^k \rangle f_\lambda^*(u) du = e_k \lambda_k^*, \qquad k \geqslant -5. \tag{29}$$

One has (see Eqs. (13), (14)):

$$\lambda_k^* = 6/(k + 6) d^k, \qquad k \geqslant -5 \tag{30}$$

$$b = 1/\lambda_{-4} = d^4/3e_{-4} \qquad \text{and} \qquad b/a = 1/\lambda_{-3} = d^3/2e_{-3}. \tag{31}$$

Therefore one obtains, according to the relations in Table II,

$$\mu_k = \frac{e_{k-4}}{e_{-4}} \frac{2}{(k + 2)} d^k, \qquad k \geqslant -1 \tag{32}$$

$$\nu_k = \frac{e_{k-3}}{e_{-3}} \frac{3}{(k+3)} d^k, \qquad k \geqslant -2 \tag{33}$$

$$i_k = \frac{e_{k-3}}{e_{-3}} \frac{3}{(k+1)(k+3)} d^k, \qquad k \geqslant 0 \tag{34}$$

$$\lambda_k = e_k \frac{6}{(k+6)} d^k, \qquad k \geqslant -5 \tag{35}$$

$$p_k = e_k \frac{3 \cdot 4 \cdot 6}{(k+3)(k+4)(k+6)} d^k, \qquad k \geqslant -2 \tag{36}$$

For the sphere all e_k's are equal to 1.

Figure 5 gives, as numerical example, the mean values μ_1, ν_1 $(=2i_1)$, and x_1 $(=5/3p_1)$ for spheroids of unit length smaller axis.

APPENDIX: SOLUTION FOR ELLIPSES

The solution for ellipses is analogous to that for spheroids; however, the integrals cannot be solved analytically. It will be sufficient to cite the results; all notations correspond to those for the three-dimensional case.

Distributions for the Circle

For the circle of diameter d and with the abbreviation $X = x/d$ one has

$$f_\mu(x) = \frac{1}{d} X/\sqrt{1 - X^2}; \qquad F_\mu(x) = \sqrt{1 - X^2} \tag{A.1}$$

$$f_\nu(x) = \frac{4}{\pi d} X^2/\sqrt{1 - X^2}; \qquad F_\nu(x) = \frac{2}{\pi}[\cos^{-1}(X) + X\sqrt{1 - X^2}] \tag{A.2}$$

$$f_i(x) = \frac{4}{\pi d}\sqrt{1 - X^2}; \qquad F_i(x) = \frac{2}{\pi}[\cos^{-1}(X) - X\sqrt{1 - X^2}] \tag{A.3}$$

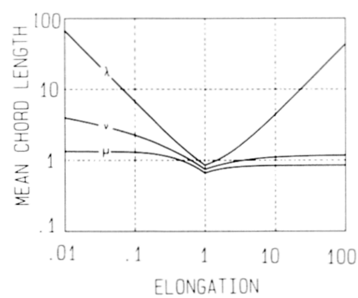

FIG. 5. The mean values μ_1, ν_1 $(=2i_1)$, and λ_1 $(=5/3p_1)$ for the different types of randomness in the spheroid. The smaller axis is taken to be of unit length ($d = 1$ for $e > 1$; $d = 1/e$ for $e < 1$).

$$f_\lambda(x) = \frac{16}{3\pi d} X^4/\sqrt{1 - X^2}; \qquad F_\lambda(x) = \frac{2}{\pi}\left[\left(X + \frac{2}{3}X^3\right) \times \sqrt{1 - X^2} - \sin^{-1}(X)\right] \quad (A.4)$$

$$f_p(x) = \frac{16}{\pi d} X[\cos^{-1}(X) - X\sqrt{1 - X^2}]; \qquad F_p(x) = \frac{2}{\pi}\left[\left(X + \frac{1}{2}X^3\right) \times \sqrt{1 - X^2} + (1 - 4X^2)\cos^{-1}(X)\right]. \quad (A.5)$$

Transformation Kernel

It is sufficient to consider the case $e > 1$. One obtains

$$H_u(x) = \frac{2}{\pi}\cos^{-1}\left[\left(\frac{x^2/u^2 - 1}{e^2 - 1}\right)^{.5}\right], \qquad u \leqslant x \leqslant eu \quad (A.6)$$

and

$$h_u(x) = \frac{2x}{\pi[(e^2u^2 - x^2)(x^2 - u^2)]^{.5}}, \qquad u \leqslant x \leqslant eu. \quad (A.7)$$

The Distribution for Ellipses

Using Eqs. (A.1), (A.4), and (A.7) one obtains

$$f_\mu(x) = \frac{2e}{Cx^2}\int_{u_1}^{u_2} \frac{u^4 du}{[(u^2 - x^2/e^2)(x^2 - u^2)(d^2 - u^2)]^{.5}}, \qquad 0 < x < ed \quad (A.8)$$

with $u_1 = x/e$ and $u_2 = \text{Min}(x, d)$. C is the perimeter of the ellipse:

$$C = 2ed\int_0^{\pi/2} [1 - (e^2 - 1)\sin^2 z]^{.5} dz \approx \frac{(1 + e)\pi}{2}\,\frac{64 - 3\lambda^4}{64 - 16\lambda^2}\, d,$$

$$\text{with} \qquad \lambda = \frac{e - 1}{e + 1}. \quad (A.9)$$

For the point-pair distance distribution one finds:

$$f_p(x) = \frac{32x}{d^2\pi^2}\int_{u_1}^{u_2} \frac{u\left[\cos^{-1}\left(\frac{u}{d}\right) - \frac{u}{d}\left(1 - \frac{u^2}{d^2}\right)^{.5}\right]du}{[(e^2u^2 - x^2)(x^2 - u^2)]^{.5}}, \qquad 0 \leqslant x \leqslant ed. \quad (A.10)$$

Eqs. (A.8) and (A.10) can not be solved in closed form, but can be expressed in terms of standard Legendre functions.

The Moments

From Eq. (A.1) one obtains the moments for the circle

$$\mu_k^* = j_k d^k, \qquad k \geqslant -1$$

with

$$j_k = \int_0^1 \frac{z^{k+1}}{(1-z^2)^{.5}}\,dz = \frac{\sqrt{\pi}}{2}\cdot\Gamma\left(\frac{k}{2}+1\right)\Big/\Gamma\left(\frac{k}{2}+\frac{3}{2}\right)$$

$$= \frac{\pi}{2}, 1, \frac{\pi}{4}, \frac{2}{3}, \frac{3\pi}{16}, \frac{8}{15}, \frac{5\pi}{32}, \frac{16}{35}, \cdots \quad \text{for} \quad k = -1, 0, 1, 2 \cdots \quad \text{(A.11)}$$

and from Eq. (8) with $b = 3\pi d^3/16$

$$\lambda_k^* = (16\, j_{k+3}/3\pi)d^k. \quad \text{(A.12)}$$

In analogy to Eq. (26) one obtains

$$\langle x_u^k \rangle = e_k u^k \quad \text{with} \quad e_k = \frac{2}{\pi}\int_1^e \frac{z^{k+1}}{[(e^2 - z^2)(z^2 - 1)]^{.5}}\,dz. \quad \text{(A.13)}$$

Therefore the moments for the ellipse are

$$\lambda_k = e_k(16\, j_{k+3}/3\pi)d^k, \qquad k \geqslant -4 \quad \text{(A.14)}$$

and with $b = 1/\lambda_{-3}$ and $a/b = \lambda_2$ (see Eq. (11)) and the relations in Table II:

$$\mu_k = \frac{\lambda_{k-3}}{\lambda_{-3}} = \frac{e_{k-3}}{e_{-3}}\, j_k d^k, \qquad k \geqslant -1$$

$$\nu_k = \frac{\lambda_{k-2}}{\lambda_{-2}} = \frac{e_{k-2}}{e_{-2}}\frac{4\, j_{k+1}}{\pi}\, d^k, \qquad k \geqslant -2$$

$$i_k = \frac{\lambda_{k-2}}{(k+1)\lambda_{-2}} = \frac{e_{k-2}}{e_{-2}}\frac{4\, j_{k+1}}{(k+1)\pi}\, d^k, \qquad k \geqslant 0$$

$$p_k = \frac{6\lambda_k}{(k+2)(k+3)} = e_k \frac{32\, j_{k+3}}{(k+2)(k+3)\pi}\, d^k, \qquad k \geqslant -1 \quad \text{(A.15)}$$

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