On K-Theory of Hopf Algebras of Finite Type

by Bodo Pareigis

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This paper generalizes some results of the representation theory and K-theory of finite groups to Hopf algebras of finite type. The main aim is to find some generalization of Artin's theorem on the finiteness of the Grothendieck ring of representations of a finite group modulo the induced representations coming from cyclic subgroups. The group theoretic methods use heavily the properties of single group elements in the group ring which we don't have in general Hopf algebras any more. In particular the sum of the cyclic (as algebras generated by one element) sub Hopf algebras of a Hopf algebra is not any more all of the Hopf algebra. So induction from cyclic sub Hopf algebras cannot be handled with the present theory. Furthermore the fact that group rings over fields of characteristic zero are semisimple could not be removed. So our results apply mainly to semisimple Hopf algebras. By representations we mean ordinary modules over Hopf algebras. In [4] W. Haboush shows that for finite algebraic groups this leads to representations in the sense of algebraic groups. Since we admit also Hopf algebras which are not cocommutative this theory covers (by dualization) graded modules over the base ring with finite grading group. They are comodules over the corresponding group ring or modules over the dual of the group ring which is semisimple when the base ring is a field.

The first part covers a theory of Frobenius functors on a certain category of Hopf algebras. Since we admit Hopf algebras which are not cocommutative the Frobenius functos take values in not necessarily commutative rings. In particular the Grothendieck ring $G_o(H)$ of a Hopf algebra H will not in general be commutative.

In the second part we investigate the characters of representations which take values in the dual of a Hopf algebra H. In particular we determine the kernel and cokernel of the character map χ: $G_o^k(H) \longrightarrow H^*$ for a finite dimensional Hopf algebra H over a field k and discuss the usual orthogonality relations of characters.

The last part proves a generalization of Artin's theorem (Theorem 3.8) and gives some applications of this theorem.

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In an earlier version of this paper the orthogonality relations for characters where only discussed in case the Hopf algebra has a field as base ring. In many applications one wants the base ring to be rather arbitrary. So I tried to carry the discussion of characters and their orthogonality relations over Hopf algebras over arbitrary commutative rings up to the point where increased technical difficulties enter the situation. Several times you will find the restriction Pic(R) = 0. In view of the remark at the end of [7] the results should hold without this assumption. I hope to be able to work out the details in another context.

1. Frobenius functors and modules

All rings are assumed to be associative rings with unit, all ring homomorphisms preserve the units, all modules are unitary modules. If A is a ring then A-mod denotes the category of A left modules.

Let $f:A \to B$ be a ring homomorphism. Then f induces functors f_*: A-mod $\to$ B-mod by $f_*(M) = B\otimes_A M$ and f^*: B-mod $\to$ A-mod which sends a B module to the underlying abelian group considered as an A module via $f:A\to B$. f_* is the <u>induction functor</u>, f^* is the <u>restriction functor</u>.

<u>Lemma</u> 1.1: <u>Let</u> $f:A\to B$ <u>and</u> $g:B\to C$ <u>be ring homomorphisms</u>, <u>then</u>

1) $(gf)_* \simeq g_* f_*$

2) $(id_A)_* \simeq Id_{A\text{-mod}}$

3) $(gf)^* = f^* g^*$

4) $(id_A)^* = Id_{A\text{-mod}}$.

The proof is obvious.

Now let R be a commutative ring, let A, B, C, D be R-algebras and let $f:A\to B$ and $g:C\to D$ be R algebra homomorphisms.

By abuse of notation let f_* be $(f \otimes id_C)_*$: $A \otimes_R C$-mod $\to$ $B \otimes_R C$-mod and let f^* be $(f \otimes id_C)^*$: $B \otimes_R C$-mod $\to$ $A \otimes_R C$-mod. Similarly let g_* be $(id_A \otimes g)_*$ and g^* be $(id_A \otimes g)^*$.

Lemma 1.2: With the notations introduced above we have

1) $f_* g_* \simeq g_* f_*$: $A \otimes_R C$-mod $\to$ $B \otimes_R D$-mod

2) $f^* g^* = g^* f^*$: $B \otimes_R D$-mod $\to$ $A \otimes_R C$-mod

3) $f_* g^* \simeq g^* f_*$: $A \otimes_R D$-mod $\to$ $B \otimes_R C$-mod.

Proof: 1) and 2) depend on the fact that $(f \otimes id_D)(id_A \otimes g) = f \otimes g = (id_B \otimes g)(f \otimes id_C)$. 3) is a simple computation of tensor products.

Let R be a commutative ring. A Hopf algebra H over R is an R algebra H, which is also a coalgebra with structure maps $\Delta: H \to H \otimes_R H$ and $\varepsilon: H \to R$ being R algebra homomorphisms [12]. Let R-Hopf denote the category of Hopf algebras over R with antipode.

Let $H \in$ R-Hopf and $M, N \in$ H-mod. Then $M \otimes_R N \in$ H-mod by the map $H \xrightarrow{\Delta} H \otimes_R H \to End_R(M) \otimes_R End_R(N) \to End_R(M \otimes_R N)$. This defines an additive bifunctor:

$$\otimes_R : \text{(H-mod)} \times \text{(H-mod)} \to \text{H-mod}.$$

Let R_ε be the abelian group R with H module structure given by ε: $H \to R$.

Lemma 1.3: There are natural isomorphisms of H modules

$$(M \otimes_R N) \otimes_R P \simeq M \otimes_R (N \otimes_R P)$$

$$M \otimes_R R_\varepsilon \simeq M \simeq R_\varepsilon \otimes_R M.$$

If H is cocommutative then there is a natural isomorphism

$$M \otimes_R N \simeq N \otimes_R M.$$

Proof straightforward.

Proposition 1.4 (Frobenius Reciprocity): Let $j: H' \to H$ be in R-Hopf, $M \in$ H-mod, $N \in$ H'-mod. Then there are natural isomorphisms of H modules

$$j_*(j^*(M) \otimes_R N) \simeq M \otimes_R j_*(N)$$

$$j_*(N \otimes_R j^*(M)) \simeq j_*(N) \otimes_R M .$$

Proof: By symmetry we only have to investigate the first isomorphism. It is given by $\phi(h\otimes(m\otimes n)) = \sum_{(h)} h_{(2)}m\otimes(h_{(1)}\otimes n)$. This is clearly a natural transformation in M and N. The inverse map $\psi: M\otimes_R(H\otimes_{H'}N) \to H\otimes_{H'}(M\otimes_R N)$ is given by $\psi(m\otimes(h\otimes n)) = \sum_{(h)} h_{(1)}\otimes(S(h_{(2)})m\otimes n)$ where S is the antipode of H.

Corollary 1.5 [8, Lemma 6]: There is a natural isomorphism of H modules $M\otimes_R H \simeq M_\varepsilon\otimes_R H$ where $M_\varepsilon = (\eta\varepsilon)^*(M)$ with $\eta: R\to H$ the algebra structure map of H.

Proof: For $j: R\to H$ we get $M\otimes_R H \simeq M\otimes_R j_*(R_\varepsilon) \simeq j_*(j^*(M)\otimes_R R) \simeq M_\varepsilon\otimes_R H$.

Let A be an R algebra. Let $M_o(A,R)$ be the full subcategory of A-mod of finitely generated projective R modules. Let P(A) be the full subcategory of A-mod of finitely generated projective A modules. Let M(A) be the full subcategory of A-mod of Noetherian A-modules. Then we define

$$K_i(A) := K_i(P(A))$$

$$G_i^R(A) := K_i(M_o(A,R))$$

$$G_i(A) := K_i(M(A)) \qquad \text{for } i=0,1.$$

For the definition of K_i see [1]. The map $C_i(A)\colon K_i(A) \to G_i(A)$ with $C_i(A)[P] = [P]$ which is defined for a left Noetherian ring A is called the Cartan map.

Lemma 1.6: Let $H \in$ R-Hopf. The functor $\otimes_R$: H-mod $\times$ H-mod $\to$ H-mod induces functors

1) $\otimes_R\colon M_o(H,R) \times M_o(H,R) \to M_o(H,R)$

2) $\otimes_R\colon M_o(H,R) \times P(H) \to P(H)$

3) $\otimes_R\colon P(H) \times M_o(H,R) \to P(H)$.

If R is Noetherian and H is a finitely generated R-module then $\otimes_R$ induces functors

4) $\otimes_R\colon M_o(H,R) \times M(H) \to M(H)$

5) $\otimes_R\colon M(H) \times M_o(H,R) \to M(H)$.

<u>All these functors are exact in both variables.</u>

Proof: 1) The tensor product of two finitely generated projective R modules is a finitely generated projective R-module. Since short exact sequences of modules in $M_o(H,R)$ split, the tensor product is exact in both variables.

2) Let $M \in M_o(H,R)$ and $P \in P(H)$. Then there is $Q \in P(H)$ with $P \oplus Q \simeq \oplus H$. Hence $M \otimes_R P$ is a direct summand of $\oplus(M \otimes_R H) \simeq \oplus(M_\varepsilon \otimes_R H)$ by 1.5. Now M_ε is finitely generated and projective as an R-module hence there is an $N_\varepsilon \in M_o(H,R)$ with $M_\varepsilon \oplus N_\varepsilon \simeq \oplus R_\varepsilon$. Consequently $M_\varepsilon \otimes H$ is a direct summand of $\oplus(R_\varepsilon \otimes_R H) \simeq \oplus H$. All direct sums being finite we get that $M \otimes_R P$ is a direct summand of some $\oplus H$, hence $M \otimes_R P \in P(H)$. The functor is exact in $M_o(H,R)$ since all short exact sequences split over R. The functor is exact in P(H) since all short exact sequences split over H hence over R.

4) Since H is Noetherian, M(H) is the category of finitely generated H-modules. The tensor product of two finitely generated R-modules is again finitely generated as an R-module or as an H-module, which is the same in this context. $\otimes_R$ is exact in $M_o(H,R)$ since short exact sequences split over R. $\otimes_R$ is exact in M(H) since all modules in $M_o(H,R)$ are R-flat.

3) is symmetric to 2) and 5) is symmetric to 4).

Let R-hopf be the category of R-Hopf-algebras with antipodes which are finitely generated and projective as R-modules with Hopf algebra homomorphisms $H' \to H$, such that H is projective as an H' right (and/or left) module, as morphisms.

Let Frob be a category with (non-commutative) rings as objects and morphisms (i_*, i^*): $A \to B$ where $i_*: A \to B$ and $i^*: B \to A$ are additive maps such that

1) i^* is a ring homomorphism

2) $b i_*(a) = i_*(i^*(b)a)$ for all $a \in A$, $b \in B$

3) $i_*(a)b = i_*(a i^*(b))$ for all $a \in A$, $b \in B$.

The composition is $(i_*, i^*)(j_*, j^*) = (i_* j_*, j^* i^*)$. A functor G: C $\to$ Frob from a category C is called a Frobenius functor.

Proposition 1.7: G_o^R: R-hopf $\to$ Frob is a Frobenius functor.

Proof: $G_o^R(H) = K_o(M_o(H,R))$ is a ring by 1.6 with multiplication $[M] \cdot [N] = [M \otimes_R N]$. The unit element is $[R_\varepsilon]$ and the ring is associative by 1.3.

Let j: $H' \to H$ be in R-hopf. Then j_*: H'-mod $\to$ H-mod and j^*: H-mod $\to$ H'-mod induce functors

$j_*: M_o(H',R) \to M_o(H,R)$ and $j^*: M_o(H,R) \to M_o(H',R)$. In fact if $M \in M_o(H',R)$ then $j_*(M) = H \otimes_{H'} M$ is still finitely generated. Furthermore by a direct sum argument $j_*(M)$ is R-projective. Since j^* does not change the R module structure it restricts to $M_o(H,R)$.

Both functors are exact. This is trivial by definition for j^*. For j_* it is a consequence of the fact that H is H'-projective. Hence j_* and j^* induce homomorphisms of abelian groups $j_*: G_o^R(H') \to G_o^R(H)$ and $j^*: G_o^R(H) \to G_o^R(H')$. j^* is a ring homomorphism since j^* preserves $\otimes_R$ and $j^*(R_\varepsilon) \cong R_{\varepsilon'}$. For j_* Frobenius reciprocity (Prop. 1.4) gives $j_*(j^*[M]\cdot[N]) = [M]\cdot j_*[N]$ and $j_*([N]\cdot j^*[M]) = j_*[N]\cdot[M]$. This proves Prop. 1.7.

Similar definitions could have been made for mod-H the category of H right modules. Let H^{op} be the Hopf algebra with inverted multiplication and comultiplication of H. Then we may apply the original definitions to H^{op} instead of dealing with mod-H, except for the functor $\otimes_R$: mod-H $\times$ mod-H $\to$ mod-H. Since the comultiplication for H^{op} is the comultiplication for H with inverted factors, we have to invert the order of the factors for $\otimes_R$ if we switch over from mod-H to H^{op}-mod. Hence we get, by dealing with H right modules, G_o^R(mod-H) anti-isomorphic to $G_o^R(H^{op})$, where $G_o^R(H^{op})$ is defined for H^{op} left modules. Now the antipode S of H induces an isomorphism $H \simeq H^{op}$, hence $G_o^R(H^{op}) \simeq G_o^R(H)$. We have proved

<u>Proposition</u> 1.8: <u>The rings G_o^R(mod-H) and G_o^R(H-mod), one taken for H right modules, the second taken for H left modules, are anti-isomorphic to each other.</u>

If H is a cocommutative Hopf algebra then $G_o^R(H)$ is a commutative ring by 1.3, in this case it does not make any difference whether we define $G_o^R(H)$ for H left or right modules.

We remark that for any Hopf algebra homomorphism $j: H' \to H$ in R-Hopf we still get a ring homomorphism $j^*: G_o^R(H) \to G_o^R(H')$. This is true in particular for $\eta: R \to H$ and $\varepsilon: H \to R$. Since $\varepsilon\eta = id_R$ we get $\eta^*\varepsilon^* = id$ hence

<u>Lemma</u> 1.9: <u>For any $H \in$ R-Hopf $G_o^R(H)$ is an augmented $G_o^R(R) = K_o(R)$ algebra.</u>

Let F, G: $\mathcal{C} \to$ Frob be two Frobenius functors. Then a <u>Frobenius morphism</u> α is a ring homomorphism $\alpha(C)$: $F(C) \to G(C)$ for all $C \in \mathcal{C}$ such that for all j: $C \to D$ in $\mathcal{C}$ the diagrams

$$\begin{array}{ccc} F(C) & \xrightarrow{j_*} & F(D) \\ {\scriptstyle \alpha(C)}\downarrow & & \downarrow{\scriptstyle \alpha(D)} \\ G(C) & \xrightarrow{j_*} & G(D) \end{array} \quad \text{and} \quad \begin{array}{ccc} F(C) & \xleftarrow{j^*} & F(D) \\ {\scriptstyle \alpha(C)}\downarrow & & \downarrow{\scriptstyle \alpha(D)} \\ G(C) & \xleftarrow{j^*} & G(D) \end{array}$$

commute.

Let f: $R \to R'$ be a homomorphism of commutative rings, then the induction functor f_*: R-mod $\to$ R'-mod induces functors f_*: R-Hopf $\to$ R'-Hopf and f_*: R-hopf $\to$ R'-hopf. To prove that f_*(R-hopf) $\subseteq$ R'-hopf one uses direct sum arguments. The composition of $G_o^{R'}$: R'-hopf $\to$ Frob with f_*: R-hopf $\to$ R'-hopf defines a new Frobenius functor $G_o^{R'} f_*$.

<u>Proposition</u> 1.10. <u>Let</u> f: $R \to R'$ <u>be a homomorphism of commutative rings. Then</u> f_*: H-mod $\to$ $R' \otimes_R H$-mod <u>induces a Frobenius morphism</u> f_*: $G_o^R \to G_o^{R'} f_*$.

Proof: Although f_* occurs in many different situations it means always tensoring with R' over R. First we observe that f_*: H-mod $\to R'\otimes_R H$-mod restricts to $f_*: M_o(H,R) \to M_o(R'\otimes_R H, R')$ since f_* preserves finite generation and projectivity. Since short exact sequences in $M_o(H,R)$ are split over R we see that f_* is exact. Hence we get an additive map $f_*: G_o^R(H) \to G_o^{R'} f_*(H) = G_o^{R'}(R'\otimes_R H)$. Clearly $f_*(M\otimes_R N) \simeq f_*(M)\otimes_{R'} f_*(N)$ so f_* is a ring homomorphism. From Lemma 1.2 we get $f_* j_* \simeq j_* f_*$ and $f_* j^* \simeq j^* f_*$ which give the desired properties of a Frobenius morphism.

Let G: C $\to$ Frob be a Frobenius functor. Then a <u>Frobenius</u> G <u>left</u> <u>module</u> K consists of the following:

1) K assigns to each $C \in \mathcal{C}$ a G(C) left module K(C)

2) K assigns to each morphism j: $C \to C'$ in C a pair of additive maps $K(j) = (j_*, j^*)$ with $j_*: K(C) \to K(C')$ and $j^*: K(C') \to K(C)$ such that j^* is semi-linear with respect to $j^*: G(C') \to G(C)$ and such that

$$j_*(a)\cdot b' = j_*(a\cdot j^*(b')) \text{ for all } a\in G(C),\ b'\in K(C')$$

$$a'\cdot j_*(b) = j_*(j^*(a')\cdot b) \text{ for all } a'\in G(C'),\ b\in K(C).$$

3) $j \mapsto j_*$ and $j \mapsto j^*$ are covariant and contravariant functors respectively.

Let K and K' be Frobenius G left modules. A G-*homomorphism* $\beta: K \to K'$ consists of a family of G(C)-homomorphisms $\beta(C): K(C) \to K(C')$ for all $C \in \mathcal{C}$ such that for all $j: C \to D$ in $\mathcal{C}$ the diagrams

$$\begin{array}{ccc} K(C) & \xrightarrow{j_*} & K(D) \\ {\scriptstyle \beta(C)}\downarrow & & \downarrow{\scriptstyle \beta(D)} \\ K'(C) & \xrightarrow{j_*} & K'(D) \end{array} \quad \text{and} \quad \begin{array}{ccc} K(C) & \xleftarrow{j^*} & K(D) \\ {\scriptstyle \beta(C)}\downarrow & & \downarrow{\scriptstyle \beta(D)} \\ K'(C) & \xleftarrow{j^*} & K'(D) \end{array}$$

commute. One easily checks

Proposition 1.11: *The category* G-mod *of Frobenius* G *left modules is an abelian category where*

$$(K \oplus K')(C) \simeq K(C) \oplus K'(C)$$

$$\mathrm{Ker}(\beta)(C) \simeq \mathrm{Ker}(\beta(C))$$

$$\mathrm{Cok}(\beta)(C) \simeq \mathrm{Cok}(\beta(C)).$$

Proposition 1.12: K_i *and* G_i^R *for* $i=0,1$ *are* G_o^R *left and right modules on* R-hopf.

Proof: The module structure for $i=0$ is a consequence of Lemma 1.6. $[M]\cdot[N,\alpha] = [M\otimes_R N, M\otimes_R \alpha]$ induces a module structure on $K_1(H)$ and $G_1^R(H)$. Now let $j: H'\to H$ be in R-hopf. Then we get exact functors $j_*: M_o(H',R) \to M_o(H,R)$, $j^*: M_o(H,R) \to M_o(H',R)$, $j_*: P(H') \to P(H)$ and $j^*: P(H) \to P(H')$ since H is finitely generated and projective as an H'-module. This gives additive maps j_* and j^* on K_i and G_i^R. By Frobenius reciprocity and the fact that j^* preserves tensor products over R, we get property 2 for Frobenius modules. Property 3 is trivial.

Proposition 1.13: *Let $f: R\to R'$ be a homomorphism of commutative rings. Then $f_*: K_R \to K_{R'}, f_*$ is a G_o^R-homomorphism for K_R any of the four G_o^R left or right modules K_i and G_i^R for* $i=0,1$.

Proof: This follows from Lemma 1.2. similar to the proof of Prop. 1.10.

Proposition 1.14: *Let R be a Noetherian commutative ring. Then G_i for* $i=0,1$ *are G_o^R left and right modules on* R-hopf. *Furthermore $\phi_i: K_i(H) \to G_i^R(H)$ with* $[P] \mapsto [P]$ *and*

$[P,\alpha] \to [P,\alpha]$ *and* ψ_i: $G_i^R(H) \to G_i(H)$ *with* $[M] \mapsto [M]$ *and* $[M,\alpha] \mapsto [M,\alpha]$ *are* G_o^R-*homomorphisms*.

Proof is similar to the proof of Prop. 1.12. Clearly ϕ_i and ψ_i are G_o^R-homomorphisms commuting with induction and restriction with respect to j: $H' \to H$ in R-hopf.

Corollary 1.15: *Let* R *be a Noetherian commutative ring. Then the Cartan map* $K_o(H) \to G_o(H)$ *is a* G_o^R-*homomorphism of Frobenius* G_o^R *modules on* R-hopf. *In particular the kernel and the cokernel of the Cartan map are* G_o^R-*modules*.

2. Characters

Let k be a commutative ring and H be a Hopf algebra in k-hopf. Then $H^* = \mathrm{Hom}_k(H,k)$ is again a Hopf algebra in k-hopf.

We define a map $\chi\colon G_o^k(H) \longrightarrow H^*$ by $\chi[M](h) =$ trace $(\hat{h}\colon M \longrightarrow M)$ where $\hat{h}(m) = hm$ and where we use the definition and the properties of the trace as given in [14, A II p.78]. We call χ the character map.

Proposition 2.1: The character map $\chi\colon G_o^k(H) \longrightarrow H^*$ is a ring homomorphism.

Proof: Let $M,N \in M_o(H,k)$ and $h \in H$, $m \in M$, $n \in N$. Then $h\cdot(m \otimes n) = \sum_{(h)} h_{(1)}m \otimes h_{(2)}n$. Let $\hat{h}_{(1)}$ correspond to $\sum m_{(1)i} \otimes m^*_{(1)i}$ under $\mathrm{Hom}_k(M,M) \cong M \otimes M^*$ and let $\hat{h}_{(2)}$ correspond to $\sum n_{(2)j} \otimes n^*_{(2)j}$. Then $\mathrm{trace}(\hat{h}\colon M\otimes N \longrightarrow M \otimes N) =$ $\mathrm{trace}(\sum_{(h)} \hat{h}_{(1)} \otimes \hat{h}_{(2)}) = \sum_{(h),i,j} (m^*_{(1)i} \otimes n^*_{(2)j})(m_{(1)i} \otimes n_{(2)j}) =$ $\sum_{(h)} \sum_i m^*_{(1)i}(m_{(1)i}) \cdot \sum_j n^*_{(2)j}(n_{(2)j}) = \sum_{(h)} \mathrm{trace}(\hat{h}_{(1)}) \cdot \mathrm{trace}(\hat{h}_{(2)})$.

This proves $\chi[M\otimes_k N](h) = \sum_{(h)} \chi[M](h_{(1)})\cdot\chi[N](h_{(2)}) =$

$= (\chi[M]*\chi[N])(h)$, where $*$ is the multiplication in H^*.

Furthermore we have $\chi[k_\varepsilon](h) = \varepsilon(h)$ hence $\chi[k_\varepsilon] = \varepsilon$.

An element $f\in H^*$ is a <u>class function</u> if

$f(\sum_{(h)} h_{(2)}xS(h_{(1)})) = \varepsilon(h)f(x)$ for all $h,x\in H$.

<u>Lemma</u> 2.2: $f\in H^*$ <u>is a class function if and only if</u> $f(hx) = f(xh)$ <u>for all</u> $h,x\in H$.

Proof: If f is a class function then $f(xh) = \sum_{(h)} \varepsilon(h_{(2)})f(xh_{(1)}) =$

$= \sum_{(h)} f(h_{(3)}xh_{(1)}S(h_{(2)})) = f(hx)$. If $f(xh) = f(hx)$ for all $h,x\in H$ then $f(\sum_{(h)} h_{(2)}xS(h_{(1)})) = f(\sum_{(h)} xS(h_{(1)})h_{(2)}) = \varepsilon(h)f(x)$.

Since the character of a module $\chi[M]$ is a trace it satisfies $\chi[M](xh) = \chi[M](hx)$, hence $\chi[M]$ is a class function.

A similar relation as in Lemma 2.2 can be derived for the elements of a Hopf algebra themselves. We call a Hopf algebra H <u>cyclic</u> with generating element g if H as a k-algebra is generated by g, i.e., if $H\cong k[x]/(p(x))$ as k-algebras.

Lemma 2.3: *Let H' be a sub Hopf algebra of H. Then* $xg = gx$ *for all* $g \in H'$ *if and only if* $\sum_{(g)} g_{(1)} x S(g_{(2)}) = \varepsilon(g)x$ *for all* $g \in H'$.

Proof: Since $g_{(1)}$ and $g_{(2)}$ are elements of H' if $g \in H'$ it is clear that $xg = gx$ for all $g \in H'$ implies $\sum_{(g)} g_{(1)} x S(g_{(2)}) = \varepsilon(g)x$ for all $g \in H'$. Conversely we have $gx = \sum_{(g)} g_{(1)} x S(g_{(2)}) g_{(3)} = \sum_{(g)} x\varepsilon(g_{(1)}) g_{(2)} = xg$.

Let R be a commutative ring with Pic(R) = 0.Then we know from [7, Corollary 1] that any Hopf algebra H in R-hopf is a Frobenius algebra. In view of [9, Satz 10] there is an element $\sum a_i \otimes b_i \in H \otimes H$ with $h \cdot \sum a_i \otimes b_i = \sum a_i \otimes b_i h$ for all $h \in H$. Such an element will be called *Casimir element* for H. From [9, Satz 10] we also know that the Casimir element and the Frobenius homomorphism ψ can be picked in such a way that $\sum \psi(a_i) b_i = 1 = \sum a_i \psi(b_i)$. Then $\sum a_i \otimes b_i$ is called a *dual basis* for H.

In [7] a <u>left integral</u> in H was defined to be an element $a \in H$ with $ha = \varepsilon(h)a$ for all $h \in H$. A <u>left norm</u> is the element $N \in H$ with $N \circ \psi = \varepsilon$. By [7, Thm. 3] the left norm generates the two-sided ideal of left integrals in H as a free R-module.

<u>Lemma</u> 2.4: a) <u>Let $\sum a_i \otimes b_i$ be a Casimir element of</u> H. <u>Then</u> $\sum a_i \varepsilon(b_i)$ <u>is a left integral in</u> H.

b) $a \in H$ <u>is a left integral if and only if</u> $\sum_{(a)} a_{(1)} \otimes S(a_{(2)})$ <u>is a Casimir element</u>.

Proof: a) $h \cdot \sum a_i \varepsilon(b_i) = \sum a_i \varepsilon(b_i h) = \varepsilon(h) \cdot \sum a_i \varepsilon(b_i)$.

b) Let $a \in H$ be a left integral. Then $ha = \varepsilon(h)a$. Hence

$$\sum_{(a)(h)} h_{(1)} a_{(1)} \otimes S(a_{(2)}) S(h_{(2)}) = \varepsilon(h) \cdot \sum_{(a)} a_{(1)} \otimes S(a_{(2)}).$$

This implies $\sum_{(a)} a_{(1)} \otimes S(a_{(2)})h = \sum_{(h)} \varepsilon(h_{(1)}) \sum_{(a)} a_{(1)} \otimes S(a_{(2)}) h_{(2)} =$

$$= \sum_{(a)(h)} h_{(1)} a_{(1)} \otimes S(a_{(2)}) S(h_{(2)}) h_{(3)} = h \cdot \sum_{(a)} a_{(1)} \otimes S(a_{(2)}).$$

For the converse we know from a) that $\sum_{(a)} a_{(1)} \varepsilon(S(a_{(2)})) = a$ is a left integral.

We call an element $a \in H$ <u>cocommutative</u> if $\sum_{(a)} a_{(1)} \otimes a_{(2)} = \sum_{(a)} a_{(2)} \otimes a_{(1)}$. A Frobenius algebra H with

Frobenius homomorphism ψ is called <u>symmetric</u> if $\psi(ab)=\psi(ba)$ for all $a,b\in H$, which means that the Nakayama automorphism is the identity [9]. Since $\psi(ab) = \sum_{(\psi)} \psi_{(1)}(a)\psi_{(2)}(b)$, a Hopf algebra is symmetric if and only if $\psi\in H^*$ is cocommutative. Since a left norm is a Frobenius homomorphism for H^* we may say that H is <u>cosymmetric</u> if a left norm of H is cocommutative.

<u>Proposition</u> 2.5: <u>Let a left norm</u> N <u>for</u> H <u>be cocommutative</u>. <u>Then</u> $\sum_{(N)} N_{(1)} \otimes S(N_{(2)})$ <u>is a dual basis for</u> H.

Proof: Lemma 2.4 shows that $\sum N_{(1)} \otimes S(N_{(2)})$ is a Casimir element. By [7, Proof of Thm. 2] we have

$S(h) = \sum_{(N)} N_{(1)}\psi(hN_{(2)})$. Then $\sum_{(N)} N_{(1)}\psi(S(N_{(2)})) =$

$$\sum_{(N)} N_{(1)}\psi(\sum_{(N')} N'_{(1)}\psi(N_{(2)}N'_{(2)})) = \sum_{(N)(N')} N_{(1)}\psi(N_{(2)}\psi(N'_{(1)})N'_{(2)}) =$$

$$\sum_{(N)(N')} N_{(1)}\psi([\psi(N'_{(1)})N'_{(2)}]^* N_{(2)}) = S((\sum_{(N')} \psi(N'_{(1)})N'_{(2)})^*)$$

where h^* is the image of h under the Nakayama automorphism and $N'=N$. Now $1=S(1) = \sum_{(N)} N_{(1)}\psi(N_{(2)}) = \sum_{(N)} \psi(N_{(1)})N_{(2)}$ by cocommutativity of N, hence $\sum_{(N)} N_{(1)}\psi(S(N_{(2)})) = 1$. Similarly we get $\sum_{(N)} \psi(N_{(1)})S(N_{(2)}) = \sum_{(N)(N')} \psi(N_{(1)})N'_{(1)}\psi(N_{(2)}N'_{(2)}) =$

$\sum_{(N')(N)} N'_{(1)}\psi(\psi(N_{(1)})N_{(2)}N'_{(2)}) = 1.$

This proof uses only $\sum_{(N)} \psi(N_{(1)})N_{(2)} = 1$ which is a consequence of the cocommutativity of N. We do not know whether $\sum_{(N)} \psi(N_{(1)})N_{(2)} = 1$ is always fulfilled.

We come back to the discussion of the character map. Let $H \in$ k-hopf where k is a commutative ring. Let $M,N \in M_o(H,k)$, $M^* = \mathrm{Hom}_k(M,k)$ and $a,b \in H$. Then we have

<u>Lemma</u> 2.6: a) $\chi[\mathrm{Hom}_k(M,N)] = \chi[M^* \otimes_k N]$ in $(H \otimes_k H)^*$.

b) $\chi[M \otimes_k N](a \otimes b) = \chi[M](a) \cdot \chi[N](b)$.

c) $\chi[M^*](a) = \chi[M](a)$.

Proof: a) $\mathrm{Hom}_k(M,N)$ and $M^* \otimes_k N$ are isomorphic $H \otimes_k H^{op}$-modules.

b) Let a correspond to $\sum m_i \otimes m_i^*$ and b to $\sum n_j \otimes n_j^*$. Then $\chi[M \otimes_k N](a \otimes b) = \sum (m_i^* \otimes n_j^*)(m_i \otimes n_j) = \sum m_i^*(m_i)n_j^*(n_j)$ $= \chi[M](a) \cdot \chi[N](b)$.

c) The representation of H in M^* uses the transposed matrix of the representation of H in M, if M is a free k-module.

Proposition 2.7 (First Orthogonality Relation): Let $H \in$ k-hopf. Let $\sum a_i \otimes b_i$ be a Casimir element for H. Let M,N be two nonisomorphic simple H-modules. Then

$$\sum \chi[N](a_i)\cdot\chi[M](b_i) = 0.$$

Proof: Let T: $\mathrm{Hom}_k(M,N) \to \mathrm{Hom}_k(M,N)$ be defined by $T(f)(m) = \sum a_i f(b_i m)$. This is the trace map Tr of [8]. $T(f) \in \mathrm{Hom}_H(M,N)$ because of $\sum a_i f(b_i hm) = h\sum a_i f(b_i m)$. Since $\mathrm{Hom}_H(M,N) = 0$ we get $T = 0$. Hence $0 = \chi[\mathrm{Hom}_k(M,N)](T) = \sum \chi[M](b_i)\cdot\chi[N](a_i)$.

In the special case of the Casimir element N this relation is equivalent to $(\chi[N]*(\chi[M]S))(N) = 0$. This is the first relation of [6, Thm. 2.7] expressed for the dual Hopf algebra. We assume now that k is a field.

Proposition 2.8: Let $H \in$ k-hopf. Let $\sum a_i \otimes b_i$ be a Casimir element for H, such that $0 \neq \sum a_i b_i \in k$. Let M be a simple

H-module with $K = \mathrm{Hom}_H(M,M)$. *Then*

$$\sum \chi[M](a_i)\cdot\chi[M](b_i) = \sum a_i b_i \cdot \dim_k K.$$

Proof: The map T in the proof of Prop. 2.7 restricted to $\mathrm{Hom}_H(M,M)$ is multiplication by $\sum a_i b_i \neq 0$, hence T has a section. So $\mathrm{Hom}_k(M,M) \cong U \oplus \mathrm{Hom}_H(M,M)$ and T is multiplication with $\sum a_i b_i$ on $\mathrm{Hom}_H(M,M) = K$ and zero on U. Hence $\sum \chi[M](a_i)\cdot\chi[M](b_i) = \chi[\mathrm{Hom}_k(M,M)](T) = \sum a_i b_i \cdot \dim_k K$.

The field k is called a *splitting field* for H if every simple H-module remains simple under any base field extension (absolutely irreducible representation). This is equivalent to $\dim_k \mathrm{Hom}_H(M,M) = 1$ for every simple H-module M.

Corollary 2.9 (*Second Orthogonality Relation*): *Let* $H \in$ k-hopf *be a semisimple Hopf algebra over a splitting field* k. *Let* M *be a simple H-module. Let* $\sum a_i \otimes b_i$ *be the Casimir element* $\varepsilon(N)^{-1} \sum_{(N)} N_{(1)} \otimes S(N_{(2)})$. *Then*

$$\sum \chi[M](a_i)\cdot\chi[M](b_i) = 1.$$

Proof: By [8, Cor. 6] $\varepsilon(N)$ is invertible since H is semisimple. Since $\dim_k \mathrm{Hom}_H(M,M)=1$ and $\varepsilon(N)^{-1} \sum_{(N)} N_{(1)} S(N_{(2)})=1$ the result follows from Prop. 2.8.

This relation is the second relation of [6, Thm. 2.7] expressed for the dual Hopf algebra.

Proposition 2.10: Let $H \in$ k-hopf. Let $\sum a_i \otimes b_i$ be a Casimir element for H such that $\sum \chi[M](a_i)\cdot\chi[M](b_i) \neq 0$ for each simple H-module M. Then H is semisimple.

Proof: k is a simple H-module. Hence

$0 \neq \sum \chi[k_\varepsilon](a_i)\cdot\chi[k_\varepsilon](b_i) = \chi[\mathrm{Hom}_k(k_\varepsilon,k_\varepsilon)](T) = \chi[k_\varepsilon](\sum a_i b_i) =$

$\varepsilon(\sum a_i b_i)$. Since $\varepsilon(\sum a_i b_i) = \varepsilon(\sum a_i \varepsilon(b_i)) = \varepsilon(\alpha\cdot N)$ we get

$\varepsilon(N) \neq 0$ and by [8, Cor. 6] that H is semisimple.

We want to investigate more closely the image and the kernel of the character map χ. Since the image has close connection with the center Z(H) of H we shall first study some properties of Z(H).

Proposition 2.11: Let $H \in$ R-hopf be free and let Pic(R)=0. Let $\sum r_i \otimes \ell_i$ be a dual basis for H with the r_i's a basis for H. Let $\beta_{i,j,k}$ be defined by $\sum_k \beta_{i,j,k} r_k = \sum_{(r_j)} (r_j)_{(1)} r_i S((r_j)_{(2)})$. Then $\sum_i \alpha_i r_i \in Z(H)$ if and only if $\sum_i \alpha_i \beta_{i,j,k} = \varepsilon(r_j)\alpha_k$ for all j and k.

Proof: It is clear that the required dual-basis always exists. By Lemma 2.3 we have $x \in Z(H)$ if and only if

$\sum_{(g)} g_{(1)} x S(g_{(2)}) = \varepsilon(g)x$ for all $g \in H$ if and only if

$\sum_{(r_j)} (r_j)_{(1)} x S((r_j)_{(2)}) = \varepsilon(r_j)x$ for all r_j if and only if

$\sum_{i,(r_j)} \alpha_i (r_j)_{(1)} r_i S((r_j)_{(2)}) = \sum_k \varepsilon(r_j)\alpha_k r_k$ for all j if and only if $\sum_i \alpha_i \beta_{i,j,k} = \varepsilon(r_j)\alpha_k$ for all j and k.

If H is a group algebra RG and $g_1, \ldots, g_n$ are the elements of G, let $\phi(i,j)=k$ be defined by $g_j g_i g_j^{-1} = g_k$. For fixed i the function takes images just over the indices of of the elements of the conjugacy class of r_i. Let $\psi(k,j)=i$ be defined by $g_j^{-1} g_k g_j = g_i$. Then $x = \sum_i \alpha_i g_i \in Z(H)$ if and only if $\alpha_k = \alpha_{\psi(k,j)}$ for all j and k. This may be seen from $\beta_{i,j,k} = \delta_{\phi(i,j),k} = \delta_{i,\psi(k,j)}$ and Prop. 2.11, which is the well-known result that $Z(RG) = Rh_1 \oplus \ldots \oplus Rh_m$ where $h_i = \sum(g_j | g_j$ in the i-th conjugacy class of G).

Proposition 2.12: *Let* H $\in$ k-hopf' *such that* k *is a splitting field for* H. *Then* $G_o(H)$ *is a free abelian group with* rank $(G_o(H)) = \dim (Z(H)/\text{rad } Z(H))$.

Proof: Since H is Artinian $G_o(H)$ is free with basis $\{[M]\}$ where M are simple H-modules. Each $x \in Z(H)$ induces a map in $Hom_H(M,M) = k$, since k is a splitting field, so $\hat{x}:M \to M$ is multiplication by some $\alpha_M \in k$. Hence we get a map $\phi: Z(H) \ni x \mapsto (\alpha_{M_1}, \ldots, \alpha_{M_r}) \in k \oplus \ldots \oplus k$ which is an algebra homomorphism. $x \in Ker(\phi)$ if and only if $xM_i = 0$ for all simple non-isomorphic modules M_i if and only if x is in each maximal left ideal of H if and only if $x \in rad\ Z(H)$, hence $Ker(\phi) = rad\ Z(H)$. By the Frobenius-Schur theorem [2, Thm. 10.10] ϕ is onto. Hence $\dim (Z(H)/rad\ Z(H)) = r =$ $= rank\ (G_o(H))$.

<u>Theorem</u> 2.13: *Let* $H \in$ k-hopf *be a semisimple Hopf algebra with antipode* S *such that* $S^2 = id$ *and such that* k *is a splitting field for* H. *Let* Z(H) *be the center of* H *and let* $\{M_i\}$ *be a set of representatives of the isomorphism classes of simple* H-*modules*. *Then the elements* $\chi[M_i]|_{Z(H)}$ *defined by* $G_o(H) \xrightarrow{\chi} H^* \to Z^*$ *form a basis for* Z^*.

Proof: Let $\sum_i \beta_i \chi[M_i](x) = 0$ for all $x \in Z(H)$. Since $x \in Z(H)$ operates on M_i by multiplication by $\alpha_i \in k$, we get $\chi[M_i](x) = \alpha_i \dim(M_i)$. By [6, Theorem 2.8] $\dim (M_i)$ is invertible in k. The map $\phi: Z(H) \to k \oplus \ldots \oplus k$ constructed in the proof of the preceding proposition is bijective hence there is an $x \in Z(H)$ with $\phi(x) = (0,\ldots,1,\ldots,0)$. Hence $0 = \sum_i \beta_i \chi[M_i](x) = \beta_i \cdot 1 \cdot \dim(M_i)$. So the $\chi[M_i]|_{Z(H)}$ are linearly independent. Since rank $(G_o(H)) = \dim Z(H)^*$ by proposition 2.12, the $\chi[M_i]|_{Z(H)}$ form a basis.

Lemma 2.14: Let $H \in$ k-hopf be semisimple with splitting field k. Let Cf(H) be the set of all class functions in H^*. Then $Cf(H) = k \cdot \chi(G_o(H))$.

Proof: Clearly the class functions form a subspace of H^*. Let $f \in Cf(H)$. Since H is semisimple, decompose f with respect to the simple subalgebras of H as $f = f_1 + \ldots + f_n$. The f_i's are again class functions by Lemma 2.2, hence $f_i(ab) = f_i(ba)$ for $a,b \in Hom_k(M_i,M_i)$. Hence $f_i = \alpha_i \chi[M_i]$.

Corollary 2.15: Under the hypotheses of Theorem 2.13 we have $Cf(H) \oplus Z(H)^{\perp} = H^*$ and $Cf(H)|_{Z(H)} = Z(H)^*$.

Proof: $\dim (k\chi(G_o(H))) = \text{rank } (G_o(H)) = \dim (Z(H)^*)$ implies that the restriction map $Cf(H) = k \cdot \chi(G_o(H)) \to Z^*$ is an isomorphism since it is onto by Theorem 2.13. Hence $Cf(H) \oplus Z^{\perp} = H^*$.

Let $H \in$ R-Hopf for a commutative ring R. Let $f:R\to K$ be a ring-homomorphism, where K is a field. We say that K is a p-splitting field for H if $p \nmid \dim_K(\mathrm{Hom}_{K\otimes H}(M,M))$ for every simple $K\otimes_R H$-module M.

Theorem 2.16: Let k be a field of characteristic $p\neq 0$. Let $H \in$ k-hopf such that k is a p-splitting field for H. Then the sequence

$$0 \to G_o(H) \xrightarrow{p} G_o(H) \xrightarrow{\chi} H^*$$

is exact.

Proof: Clearly $\chi p=0$ where the map p is multiplication by p. Let $x \in G_o(H)$ with $x = \sum t_i[M_i] = [M] - [N]$ with semisimple H-modules M,N. Let $[M] = \sum m_i[M_i]$ and $[N] = \sum n_i[M_i]$ with $m_i,n_i \in \mathbb{N}$ and M_i simple H-modules. We want to show that if $\chi[M] = \chi[N]$ then $p \mid m_i-n_i$. Since M and N are semisimple modules, they are semisimple modules over $H/\mathrm{rad}(H) = \overline{H}$ which is a semisimple ring. Let $\overline{H}_i = \mathrm{Hom}_{D_i}(M_i,M_i)$ with $D_i = \mathrm{Hom}_H(M_i,M_i)$ be the simple component of $\overline{H}$ with respect to M_i. Since $\overline{H}_i \cong M_n(D_i)$ the full n×n-matrix ring over D_i, let $e_i \in \bar{H}_i$ correspond to the matrix (α_{ij}) in $M_n(D_i)$ with $\alpha_{ij}=0$ for $i \neq 1$ or $j \neq 1$ and $\alpha_{11}=1$. Then

$$\chi[M](e_i) = \chi(m_i[M_i])(e_i) = m_i \cdot \chi[M_i](e_i) = m_i \cdot \dim_k (D_i).$$

Hence $0 = \chi(x)(e_i) = (m_i - n_i) \dim_k(D_i) = t_i \cdot \dim_k(D_i) \mod (p)$.

Since $p \nmid \dim_k(D_i)$ we get $t_i \in (p) \subseteq \mathbb{Z}$, hence $x \in p \cdot G_o(H)$.

3. Finiteness Theorems

We first study some homological properties before we go back to K-theory.

Theorem 3.1: *Let G $\in$ k-hopf be semisimple and let k be a field. Let H be a sub-Hopf-algebra of G and let G be projective as an H-module. Then H is semisimple.*

Proof: By [8, Cor. 6] $\hat{H}^0(G,k) = 0$. Hence 1=0 for the cup-product. In [8, Cor 10] we remarked that $i(G,H)(1) = 1$. Hence 1=0 also for the cup-product of $\hat{H}^n(H,-)$. Hence $\hat{H}^0(H,k)=0$ and by [8, Cor. 6] H is semisimple.

This theorem gives rise to the question under which conditions the radicals of Hopf-algebras G and their sub-Hopf-algebras H satisfy the relation rad $(G) \cap H$ = rad (H).

This is certainly not always the case. Let $k = \mathbb{Z}/2\mathbb{Z}$, $G = S_3$ the symmetric group on 3 letters and $H = \{(1),(12)\} \subseteq G$. Then rad $(kH) = k\cdot((1)+(12))$ since this is the augmentation ideal and $((1)+(12))^2 = 0$. If $(1)+(12) \in$ rad (kG) then $[((1)+(12))(132)]^2 \in$ rad (kG) and is idempotent and different from zero. This is a contradiction, hence kH $\cap$ rad (kG) $\neq$ rad (kH).

In general one proves easily rad(H) $\supseteq$ H $\cap$ rad(G) for a sub-Hopf-algebra H of G $\in$ hopf-k. The relation

$\mathrm{rad}(H) = H \cap \mathrm{rad}(G)$ holds in each of the following cases.

1) G is semisimple and H-projective by above theorem,

2) G is commutative,

3) G is local,

4) H is a normal sub-Hopf-algebra of G and H is local (a proof of this can be given similar to a proof in [5, 5.17 Satz])

5) G is a group algebra and H is the group algebra of a normal subgroup [5, 17.3 Hauptsatz].

Proposition 3.2: *Let R be a commutative ring with* $\mathrm{Pic}(R)=0$. *Let* $H \in$ *R-hopf. Let M be an H-module. Let* $I=\mathrm{annil}_R(M)$. *Let* N_H *be the left norm of H. If I and* $\varepsilon(N_H)$ *are relatively prime in R, then* $\mathrm{pd}_H(M) \leq \mathrm{pd}_R(M)$.

Proof: First we show that M is (H,R)-projective. $IM=0$ implies $I\cdot\mathrm{Hom}_R(A,M)=0$ for all $A \in$ H-mod. Hence

$I\cdot H^n(H,\mathrm{Hom}_R(A,M)) = I\cdot\mathrm{Ext}^n_{(H,R)}(A,M) = 0$ [8, Prop. 2].

Also $\varepsilon(N_H)\cdot H^n(H,\mathrm{Hom}_R(A,M)) = \varepsilon(N_H)\cdot\mathrm{Ext}^n_{(H,R)}(A,M) = 0$ by [8, Cor. 2]. Since I and $\varepsilon(N_H)$ are relatively prime, we get $\mathrm{Ext}^n_{(H,R)}(A,M) = 0$ for all $A \in$ H-mod. Hence M is (H,R)-projective.

Now M is a direct summand of $H\otimes_R M$ by $H\otimes_R M \ni h\otimes m \to hm \in M$ and $M \ni m \mapsto 1\otimes m \in H\otimes_R M$ since M is (H,R)-projective. Hence $pd_H(M) \leq pd_H(H\otimes_R M)$. If $0\to P_n \to \dots \to P_o \to M \to 0$ is an R-projective resolution then $0\to H\otimes_R P_n \to \dots \to H\otimes_R P_o \to H\otimes_R M \to 0$ is an H-projective resolution. Hence $pd_H(H\otimes_R M) \leq pd_R(M)$.

Now we come back to the study of Frobenius functors. Let C be a category and let C be a class of morphisms in C. Let F be a Frobenius functor and let K be a Frobenius F module. Define

$$F_C(H) := \sum(i_* F(H') \mid i:H'\to H,\ i\in C)$$

$$K_C(H) := \sum(i_* K(H') \mid i:H'\to H,\ i\in C)$$

$$F^C(H) := \bigcap(\mathrm{Ker}(i^*:F(H)\to F(H')) \mid i: H'\to H,\ i\in C)$$

$$K^C(H) := \bigcap(\mathrm{Ker}(i^*:K(H)\to K(H')) \mid i: H'\to H,\ i\in C).$$

We say that a subgroup B of a group A has _exponent_ n in A if $na\in B$ for all $a\in A$. A has _exponent_ n if $nA=0$.

Lemma 3.3: _Let_ K _be a left Frobenius_ F _module over a Frobenius functor_ F _on_ C. _Let_ C _be a class of morphisms in_ C. _Then_

1) $F(H)K_C(H) + F_C(H)K(H) \subseteq K_C(H)$

$F(H)K^C(H) + F^C(H)K(H) \subseteq K^C(H)$

2) $F_C(H)K^C(H) = 0 = F^C(H)K_C(H)$.

3) If $f:K \to L$ is an F-homomorphism then

$f(H)(K_C(H)) \subseteq L_C(H)$,

$f(H)(K^C(H)) \subseteq L^C(H)$.

4) If for each morphism $j:H' \to H$ in C,

$j^*(K_C(H)) \subseteq K_C(H')$ resp. $j_*(K^C(H')) \subseteq K^C(H)$

then K_C resp. K^C is a Frobenius F-module.

The proof is similar to the proof of [1, XI.2.4 Prop.] and is left to the reader.

Lemma 3.4: Let $F: C \to$ Frob be a Frobenius functor. Let $C \subseteq C$ be a class of morphisms. Let $F_C(H)$ have exponent n in F(H).

a) Let K be a Frobenius F left module. Then $K_C(H)$ has exponent n in K(H) and $K^C(H)$ has exponent n.

b) *Let* K(H') *be a torsion module (have exponent* r) *for all* H' *with a morphism* $i:H'\to H$ *in* C, *then* K(H) *is a torsion module (has exponent* nr).

c) *Let* $f:K\to L$ *be a morphism of Frobenius*-F-*left*-*modules*. *If* K(H) *is torsion free and* Ker(f(H')) *is a torsion module for all* H' *with a morphism* $i:H'\to H$ *in* C *then* f(H) *is a monomorphism*. *If* L(H) *is torsionfree and* Im(f(H')) *is a torsion module for all* H' *with a morphism* $i:H'\to H$ *in* C *then* f(H) *is the zero map*.

Proof: a) $K(H)/K_C(H)$ is a $F(H)/F_C(H)$-module. So is $K^C(H)$.

b) K(H) is an extension of $K_C(H)$ and $K(H)/K_C(H)$, both of which are torsion modules. $K(H)/K_C(H)$ has exponent n, $K_C(H)$ has exponent r.

c) Since Ker(f(H)) and Im(f(H)) are both torsion submodules of torsionfree modules, both are equal to zero.

If C is a class of morphisms in hopf-R, then the smallest exponent $e_C(H)$ of $(G_o^R)_C(H)$ in $G_o^R(H)$ is called the *induction exponent* of H with respect to C. Then for any Frobenius G_o^R module K and any $H \in$ hopf-R, $K_C(H)$ in K(H) and $K^C(H)$ have exponent $e_C(H)$.

Lemma 3.5: Let $f:R\to R'$ be a homomorphism of commutative rings. Let C be a class of morphisms in hopf-R. Let $C' := f_*(C) = \{i\otimes_R R' \mid i \in C\} \subseteq$ hopf-R'. Let $H \in$ hopf-R. Then

$$e_{C'}(H\otimes_R R') \text{ divides } e_C(H).$$

Proof: By Prop. 1.10 $f_*: G_o^R \to G_o^{R'} f_*$ is a Frobenius-morphism. Since $C'=f_*(C)$ and $H\otimes_R R' = f_*(H)$ this lemma follows from Lemma 3.4.a.

Proposition 3.6: Let R be a regular commutative domain with field of fractions L and let $p \in \mathrm{Spec}(R)$. Let $H \in$ hopf-R. Then

a) $e_{C\otimes R/p}(H\otimes_R R/p)$ divides $e_{C\otimes L}(H\otimes_R L)$

b) $e_C(H)$ divides $e_{C\otimes L}(H\otimes_R L)^2$.

Proof: We may identify $G_o^R(H) = G_o(H)$, $G_o^L(H\otimes_R L) = G_o(H\otimes_R L)$, and $G_o^{R/p}(H\otimes_R R/p) = G_o(H\otimes_R R/p)$ by [11, Thm. 1.2]. By [11, Thm. 1.9] we get the commutative triangle

$$\begin{array}{ccc} G_o(H) & \xrightarrow{\alpha} & G_o(H\otimes_R L) \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & G_o(H\otimes_R R/p) & \end{array}$$

where α and β are derived from $R\to L$ and $R\to R/p$ resp., hence they are Frobenius morphisms. Since γ is derived from the surjectivity of α, it is also a Frobenius homomorphism. This proves a).

By [11, Thm. 1.7] there is a short exact sequence

$$\coprod_{p\in \mathrm{Spec}(R)} G_o(H\otimes_R R/p) \xrightarrow{f} G_o(H) \xrightarrow{g} G_o(H\otimes_R L) \to 0$$

where f is induced by restrictions and g is induced by inductions. Hence both commute with restriction and induction homomorphisms arising from maps in hopf-R by Lemma 1.2. $G_o(H') \to G_o(H'\otimes_R L)$ and hence $(G_o^R)_C(H) \to (G_o^L)_{C\otimes L}(H\otimes_R L)$ are surjective. Let $e = e_{C\otimes L}(H\otimes_R L)$. So there is $a\in(G_o^R)_C(H)$ with $g(a) = e\cdot 1$, hence $e\cdot 1-a \in \mathrm{Ker}(g)$. Let $b \in G_o(H\otimes_R R/p)$ with $f(b) = e\cdot 1-a$. By a) we get $e\cdot b \in (G_o^{R/p})_{C\otimes R/p}(H\otimes_R R/p)$. Hence $f(e\cdot b) = e^2\cdot 1-e\cdot a \in (G_o^R)_C(H)$. Consequently $e^2\cdot 1 \in (G_o^R)_C(H)$.

Let A be an algebra and B be a subalgebra of A. Let $V\in A$-mod be a simple A-module and $W\in B$-mod be a simple B-module. Then let $(V:W)$ be the number of composition factors isomorphic to W in V considered as a B-module. Let $(A\otimes_B W:V)$ be the number of composition factors V in $A\otimes_B W$.

Lemma 3.7: *Let k be a field. Let H,G∈k-hopf be semisimple and let H be a sub Hopf algebra of G. Let V∈G-mod and W∈H-mod be simple. Then*

$$(G\otimes_H W:V)\cdot \dim_k\ (\mathrm{Hom}_G(V,V)) = (V:W)\cdot \dim_k\ (\mathrm{Hom}_H(W,W)).$$

Proof: $(G\otimes_H W:V)\cdot\dim_k(\mathrm{Hom}_G(V,V)) = \dim_k(\mathrm{Hom}_G(G\otimes_H W,V)) = \dim_k(\mathrm{Hom}_H(W,V)) = (V:W)\cdot\dim_k(\mathrm{Hom}_H(W,W))$.

Theorem 3.8: *Let k be a field of characteristic p>0. Let G ∈ k-hopf be semisimple. Let C be a class of injective morphisms in k-hopf such that* $G = \sum_{i\in C} i(H)$ *and such that k is a splitting field for all H with* $(i:H\to G) \in C$. *Then* $G_o(G)/(G_o)_C(G)$ *is finite and has no elements of order p.*

Proof: $\coprod_{i\in C} G_o(H) \xrightarrow{f} G_o(G) \to G_o(G)/(G_o)_C(G) \to 0$ is an exact sequence by definition. Tensoring with $\mathbb{Z}/p\mathbb{Z}$ keeps it exact. Let A be the representing matrix of f with respect to the sets of non-isomorphic simple G-modules $\{S_1,\dots,S_n\}$ and non-isomorphic simple H-modules $\{T_{1_H},\dots,T_{n_H}\}$ for all $i:H\to G$ in C. If $\mathbb{Z}/p\mathbb{Z}\otimes A$ is an epimorphism, then $\mathbb{Z}/p\mathbb{Z}\otimes G_o(G)/(G_o)_C(G)$ is zero, hence $G_o(G)/(G_o)_C(G)$ is finite with order of all elements prime to p.

Assume that k is a splitting field for G. For $i:H\to G$ we have $i_*[T_{i_H}] = \sum_{j=1}^{n} a_{i_H,j}[S_j]$. By Lemma 3.7 and by Theorem 3.1 we get

$$i^*[S_j] = \sum_{i_H} a_{i_H,j}[T_{i_H}].$$

We want to show that n rows of A are linearly independent modulo p. Given $b_j \in \mathbb{Z}$ such that $\sum_{j=1}^{n} b_j a_{i_H,j} = 0 \bmod(p)$. Then $\sum_j b_j i^*[S_j] = 0 \bmod(p)$. Hence $\chi(\sum b_j i^*[S_j]) = 0$ by Theorem 2.16. Since $\sum b_j \chi(i^*[S_j])(h) = \sum b_j\ [S_j](h) = 0$ for all $h\in H$ and also for all $i:H\to G$ in C we get $\sum b_j \chi[S_j] = 0$. By Theorem 2.16 $\sum b_j[S_j] = p\cdot\sum c_j[S_j]$ and thus $b_j = pc_j$. Hence $\mathbb{Z}/p\mathbb{Z} \otimes A$ has rank n and $\mathbb{Z}/p\mathbb{Z} \otimes G_o(G)$ has dimension n, so $\mathbb{Z}/p\mathbb{Z} \otimes A$ is onto.

We have proved the theorem in case k is a splitting field for G. If this is not the case, let L be a splitting field for G. Then we get a commutative diagram

$$\begin{array}{ccccccc}
\coprod G_o(H) & \longrightarrow & G_o(G) & \longrightarrow & G_o(G)/(G_o)_C(G) & \longrightarrow & 0 \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
\coprod G_o(L\otimes_k H) & \longrightarrow & G_o(L\otimes_k G) & \longrightarrow & G_o(L\otimes_k G)/(G_o)_{L\otimes C}(L\otimes_k G) & \longrightarrow & 0
\end{array}$$

where α is an isomorphism since every irreducible H-module is absolutely irreducible (k is a splitting field). Since $G \simeq \Pi\,(\mathrm{Hom}_G(S_i^{n_i},S_i^{n_i})$ and $G_o(G) \simeq \Pi G_o(\mathrm{Hom}_G(S_i^{n_i},S_i^{n_i}))$ tensoring with L will map components of $G_o(G)$ to components of $G_o(L\otimes_k G)$, hence β is a monomorphism. By the 5-Lemma we get that γ is a monomorphism. The theorem has already been proved for $G_o(L\otimes_k G)/(G_o)_{L\otimes C}(L\otimes_k G)$, so it holds also for $G_o(G)/(G_o)_C(G)$.

<u>Corollary</u> 3.9: <u>Let</u> k <u>be a field of characteristic</u> p>0. <u>Let</u> $G \in$ k-hopf <u>be semisimple with antipode</u> S <u>such that</u> $S^2=id$. <u>Let</u> C <u>be a class of sub Hopf algebras</u> H <u>of</u> G <u>in</u> k-hopf <u>such that</u> $\sum_{H\in C} H \supseteq Z(G)$, <u>the center of</u> G, <u>and</u> k <u>is a splitting field for all</u> $H\in C$. <u>Then</u> $G_o(G)/(G_o)_C(G)$ <u>is finite and has no elements of order</u> p.

Proof: In the proof of Theorem 3.8 we used $G = \sum_{i\in C} i(H)$ only for the conclusion that $\sum b_j\chi[S_j](h)=0$ for all $h\in H$ and $i\in C$ implies $\sum b_j\chi[S_j]=0$. In view of Theorem 2.13 this may already be done if one only knows the $\chi[S_j]$ on Z(H).

In K-theory of finite groups one tries a reduction of $G_o(G)$ to $G_o(H)$ where H are group rings of cyclic subgroups of G. In that situation one gets $G=\sum H$ for free from the structure of group rings. In the general case we cannot use this reduction as the following example shows.

Let $k = \mathbb{Z}/p\mathbb{Z}$, μ_p be the functor $\mu_p(R)=\{r \in R \mid r^p=1\}$ viewed as multiplicative group with $R \in k\text{-Alg}$. Then $\mu_p \times \mu_p$ has the affine Hopf algebra $k[x,y]/(x^p-1,y^p-1)$ with $\Delta(x) = x \otimes x$, $\Delta(y) = y \otimes y$. The dual $G = \text{Hom}_k(k[x,y]/(x^p-1,y^p-1),k)$ is the Hopf algebra $k[x,y]/(x^p-x,y^p-y)$ with $\Delta(x) = x \otimes 1 + 1 \otimes x, \Delta(y) = y \otimes 1 + 1 \otimes y$ which is semisimple as an algebra. G is the universal p-enveloping algebra of the two dimensional abelian p-Lie-algebra $kx \oplus ky$ with $x^{[p]}=x$, $y^{[p]}=y$. G has p^2 non-isomorphic simple modules $k[x,y]/(x-\alpha,y-\beta) = S_{\alpha,\beta}$ with $(\alpha,\beta) \in k \times k$, since it has p^2 different maximal ideals corresponding to the p^2 elements of $k\text{-Alg}(G,\bar{k})$ where $\bar{k}$ is the algebraic closure of k. Hence $G_o(G)$ is a free abelian group of rank p^2.

Let us look at the sub Hopf algebras of G which are generated as an algebra by one element (cyclic sub Hopf algebras). Their duals represent subfunctors of $\mu_p \times \mu_p$ which are semigroups. Since $\mu_p \times \mu_p(R) = \mu_p(R) \times \mu_p(R)$ is a p-group, the semigroup subfunctors are group subfunctors. They are infinitesimal (local) height one algebraic groups, hence by [13, II.7.4.2] their bialgebras are universal p-algebras of sub p-Lie-algebras of $kx \oplus ky$. Consequently

the cyclic sub Hopf algebras of G are $H_\infty = k[x]/(x^p - x), \Delta(x) = x\otimes 1 + 1\otimes x$ and $H_\alpha = k[\alpha x+y]/((\alpha x+y)^p - (\alpha x+y)), \Delta(\alpha x+y) = (\alpha x+y)\otimes 1 + 1\otimes(\alpha x+y)$ for all $\alpha \in k$. So we get p+1 copies of the bialgebra of μ_p.
A simple induction proof shows

$$\dim (H_\infty + \sum_{\alpha\in k} H_\alpha) \leqq \frac{p^2-1}{2} < p^2 = \dim G$$

hence $G \neq \sum_{\alpha\in k} H_\alpha + H_\infty$. So Theorem 3.8 or Corollary 3.9 cannot be applied in this situation.

For this example let us compute $G_o(G)/(G_o)_C(G)$ with $C = \{H_\infty, H_o, H_1, \ldots, H_{p-1}\}$. The simple H_α-modules are $T_{\alpha,\beta} = k[\alpha x+y]/(\alpha x+y-\beta)$ and the simple H_∞-modules are $T_{\infty,\beta} = k[x]/(x-\beta)$ for $\beta\in k$. Then

$$i_*([T_{\alpha,\beta}]) = [G\otimes_{H_\alpha} T_{\alpha,\beta}] = \sum_{\gamma=o}^{p-1} [S_{\gamma,\beta-\alpha\gamma}]$$

and

$$i_*([T_{\infty,\beta}]) = \sum_{\gamma=o}^{p-1} [S_{\beta,\gamma}].$$

Then the following relation holds:

$$p\cdot[S_{\gamma,\delta}] = \sum_{\alpha=o}^{p-1} i_*[T_{\alpha,\delta+\alpha\gamma}] - \sum_{\substack{\varepsilon=o\\ \varepsilon\neq\gamma}}^{p-1} i_*[T_{\infty,\varepsilon}].$$

This proves that $G_o(G)/(G_o)_C(G)$ is a $\mathbb{Z}_p$-module. The representing matrix of $\amalg G_o(H_i) \to G_o(G)$ modulo p has rank $\leq p^2-1$ since each $i_*([T_{\alpha,\beta}])$ and $i_*([T_{\infty,\beta}])$ induces p unit entries into the matrix. If we take the sum of all the columns we get the zero column modulo p. Since $G_o(G)$ has rank p^2, we get $G_o(G)/(G_o)_C(G) \neq 0$.

This example leaves the open question to find a more general hypothesis for G and C such that $G_o(G)/(G_o)_C(G)$ is finite but may have elements of order p. A generalization of Theorem 3.9 away from the semi-simplicity of G is the following

Corollary 3.10: Let R be a regular commutative domain, K its quotient field, $p \in \mathrm{Spec}(R)$. Let $G \in$ R-hopf with $\varepsilon(N_G) \neq 0$ where N_G is a norm of G. Let C be a class of injective morphisms $i: H \to G$ in R-hopf, such that $G = \sum_{i \in C} i(H)$, K is a splitting field for all H with $i: H \to G$ in C and all such H are commutative. Then $G_o^R(G)/(G_o^R)_C(G)$ and $G_o^{R/p}(G \otimes_R R/p)/(G_o^{R/p})_{C \otimes R/p}(G \otimes_R R/p)$ are finite. If the characteristic of K is p>0, then neither of both groups contains elements of order p.

Remark: As in Corollary 3.9 the condition $G = \sum_{i \in C} i(H)$

may be replaced by $S^2 = id_G$ and $\sum_{i \in C} i(H) \supseteq Z(G)$.

Proof: $N_G \otimes 1$ is the norm of $G \otimes_R K$ and $\varepsilon(N_G \otimes 1) = \varepsilon(N_G) \otimes 1 \neq 0$ hence $G \otimes_R K$ is semisimple. Then $G_o^K(G \otimes_R K)/(G_o^K)_{C \otimes K}(G \otimes_R K)$ is finite (with no elements of order p if char(K)=p). Thus we may apply Proposition 3.6 to get the result.

Lemma 3.11: Let k be a field and H $\in$ k-hopf and H commutative. Then the Cartan-map H_o: $K_o(H) \to G_o(H)$ is a monomorphism with finite cokernel.

Proof: Since H is Artinian, H is a product of local rings I_r with maximal ideals m_r. $K_o(H)$ has basis $\{[I_r]\}$, $G_o(H)$ has basis $\{[I_r/m_r]\}$. I_r has only copies of I_r/m_r as composition factors. Hence $H_o(H) = H_o(I_1) \oplus \ldots \oplus H_o(I_n)$ where the $H_o(I_r)$ are non-zero 1×1-matrices.

Proposition 3.12: Let R be a regular commutative local domain, K its quotient field, m its maximal ideal. Let G$\in$R-hopf with $\varepsilon(N_G) \neq 0$ where N_G is a norm of G. Let C be a class of injective morphisms i:H→G in R-hopf, such that $G = \sum_{i \in C} i(H)$, K is a splitting field for all H with i:H→G in C and all such H are commutative. Then

$H_o(G\otimes_R R/m): K_o(G\otimes_R R/m) \to G_o(G\otimes_R R/m)$ is a monomorphism with finite cokernel.

Proof: Both $K_o(G\otimes_R/m)$ and $G_o(G\otimes_R R/m)$ are $G_o^{R/m}(G\otimes_R R/m)$ Frobenius modules and $H_o(G\otimes_R R/m)$ is a $G_o^{R/m}$ homomorphism. By Lemma 3.4.c for the claim that $H_o(G\otimes_R R/m)$ is a monomorphism and by Lemma 3.4.b for the claim that the cokernel of $H_o(G\otimes_R R/m)$ is finite it is enough to prove the theorem for commutative Hopf algebras H and to show that $G_o^{R/m}(G\otimes_R R/m)/(G_o^{R/m})_{C\otimes R/m}(G\otimes_R R/m)$ is finite. The first assertion is Lemma 3.11, the second is Corollary 3.10.

Corollary 3.13: Let the assumptions be the same as in Proposition 3.12. Let P,Q be finitely generated projective G-modules with $P\otimes_R K \simeq Q\otimes_R K$ over $G\otimes_R K$. Then $P \simeq Q$.

Proof: Apply [11, Theorem 1.10] to Proposition 3.12.

Corollary 3.14: Let R be a Dedekind domain with quotient field K. Let $G \in$ R-hopf with $\varepsilon(N_G) \neq 0$. Let C be a class of injective morphisms $i: H \to G$ in R-hopf such that

$G = \sum_{i \in C} i(H)$, K is a splitting field for all H with $i: H \to G$

in C and all such H are commutative. Let P and Q be finitely generated projective G-modules with $P\otimes_R K \approx Q\otimes_R K$ as $G\otimes_R K$-modules. Let $\mathfrak{a}$ be a non-zero ideal of R. Then there is a short exact sequence

$$0 \to P \to Q \to X \to 0$$

of G-modules with $(\mathrm{ann}_R(X), \mathfrak{a}) = R$.

Proof: By localization we get $P \approx Q$ from Corollary 3.13. The assertion then follows from a theorem of Roiter [11, Theorem 3.1].

Theorem 3.15: Let R be a Dedekind domain with Pic(R)=0 with quotient field K. Let $G \in$ R-hopf with $\varepsilon(N_G) \neq 0$. Let C be a class of injective morphisms $i:H\to G$ in R-hopf such that $G = \sum_{i\in C} i(H)$, K is a splitting field for all H with $i:H\to G$ in C and all such H are commutative. Let P be a finitely generated projective G-module with $P\otimes_R K$ free on m generators as a $G\otimes_R K$-module. Then P is isomorphic to $G \oplus \ldots \oplus G \oplus I$ (m-1 copies of G), where I is an ideal of G. If $\mathfrak{a}$ is a non-zero ideal of R then I may be chosen such that $(\mathrm{ann}_R(G/I), \mathfrak{a}) = R$.

Proof: The proof of [11, Theorem 3.3] may be verbally taken over if one observes that the necessary generalization of the crucial lemma of Rim [11, Lemma 3.4] is given by Proposition 3.2.

Corollary 3.16: *Under the assumptions of Theorem* 3.15 *let the Jordan-Zassenhaus theorem hold for* R. *Then* $K_o(G)$ *and* $G_o(G)$ *are finitely generated abelian groups and the maps* $K_o(G) \to G_o(G \otimes_R K)$ *and* $G_o(G) \to G_o(G \otimes_R K)$ *have finite kernels*.

Proof: The proof of these facts is essentially the same as the proofs given for group ring in [11, Theorem 3.8 and Theorem 4.1] using the theory developed above.

Bibliography

[1] Bass, H.: Algebraic K-Theory. Benjamin. New York. 1968.

[2] Burrow, M.: Representation Theory of Finite Groups. Academic Press. New York and London. 1965.

[3] Curtis, Ch. and Reiner, I.: Representation Theory of Finite Groups and Associative Algebras. Interscience 1962.

[4] Haboush, W.: On Representations of Finite Group Schemes. Mimeographed notes. State University of New York at Albany. 1972.

[5] Huppert, B.: Endliche Gruppen. Springer. Berlin - Heidelberg - NewYork 1967.

[6] Larson, R.: Characters of Hopf Algebras. J. Algebra 17 (352-368) 1971.

[7] Pareigis,.B.: When Hopf Algebras are Frobenius Algebras. J. Algebra 18 (588-596) 1971.

[8] Pareigis, B.: On the Cohomology of Modules over Hopf Algebras. J. Algebra 22 (161-182) 1972.

[9] Pareigis, B.: Einige Bemerkungen über Frobenius-Erweiterungen. Math. Annalen 153 (1-13) 1964.

[10] Swan, R.: Algebraic K-Theory. Lcture notes in math. 76, Springer, Berlin 1968.

[11] Swan, R. and Evans, G.: K-Theory of Finite Groups and Orders. Lecture notes in math. 149. Springer, Berlin 1970.

[12] Sweedler, M.: Hopf Algebras. Benjamin. New York. 1969.

[13] Demazure, M. et Gabriel, P.: Groupes Algébriques, Tome I. North Holland Co. Amsterdam 1970.

[14] Bourbaki, N.: Eléments de Mathématique, Algèbre, Chap. 1-3. Herman, Paris 1970.

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