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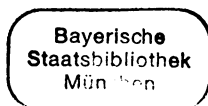
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A Non-Commutative Non-Cocommutative Hopf Algebra in "Nature"

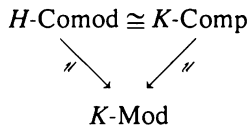
BODO PAREIGIS

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We show that there is a uniquely defined Hopf algebra H , such that H -Comod, the category of H -comodules, and K -Comp, the category of K -complexes, are isomorphic as monoidal categories, where the isomorphism is compatible with the obvious underlying functors, i.e.,



commutes. The Hopf algebra H is defined as follows:

$$\begin{aligned}
 H &= K\langle x, y, y^{-1} \rangle / (xy + yx, x^2) && \text{(non-commuting variables)} \\
 \Delta(x) &= x \otimes 1 + y^{-1} \otimes x, && s(x) = xy, \quad \varepsilon(x) = 0, \\
 \Delta(y) &= y \otimes y && s(y) = y^{-1}, \quad \varepsilon(y) = 1.
 \end{aligned}$$

H is a non-commutative, non-cocommutative Hopf algebra with antipode of order 4.

1

Let K be a commutative ring with unit. All algebras and coalgebras are defined over K and are (co-)associative with (co-)unit.

For an algebra A it is well known that the underlying functor $\mathcal{U}: A\text{-Mod} \rightarrow K\text{-Mod}$ determines the algebra A up to isomorphism. In fact $A \cong \text{End}(\mathcal{U})$.

There is no such obvious description of a coalgebra C by the underlying functor $\mathcal{U}: C\text{-Comod} \rightarrow K\text{-Mod}$. Abstractly this follows from a remark in [4, Corollary 6.4]; in fact C is uniquely described up to isomorphism by \mathcal{U} .

We need stronger results than those above. So we shall use the notation and results of [2-4].

Let \mathcal{C} be a symmetric monoidal category (e.g., $K\text{-Mod}$ with the usual tensor product over K or its dual).

We shall consider two monoids, B and C , and want to study the categories ${}_B\mathcal{C}$ and ${}_C\mathcal{C}$, when they carry themselves the structure of monoidal categories. It will turn out that this induces bimonoid structures on B , resp. C . Furthermore we study functors $\mathcal{F}: {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ which are compatible with the underlying functors from ${}_B\mathcal{C}$, resp. ${}_C\mathcal{C}$, to \mathcal{C} and preserve the monoidal structure. It will be shown that they induce bimonoid morphisms from C to B .

For the first propositions we need only a monoidal category \mathcal{C} , not necessarily symmetric.

Denote the underlying functor from ${}_B\mathcal{C}$ to \mathcal{C} by \mathcal{U} and the one from ${}_C\mathcal{C}$ to \mathcal{C} by \mathcal{V} . Observe that ${}_B\mathcal{C}$, ${}_C\mathcal{C}$ and \mathcal{C} carry in a natural way the structure of \mathcal{C} -categories and \mathcal{U} and \mathcal{V} are \mathcal{C} -functors. In [4, Corollary 6.4] we showed already how to obtain B from $\mathcal{U}: {}_B\mathcal{C} \rightarrow \mathcal{C}$ as $B^{\text{op}} \cong [\mathcal{U}, \mathcal{U}]$. Now we want to study functors $\mathcal{F}: {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$.

PROPOSITION 1. *Let $(\mathcal{F}, \xi): {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ be a \mathcal{C} -functor and $\varphi: \mathcal{V}\mathcal{F} \cong \mathcal{U}$ be a natural \mathcal{C} -isomorphism. Then there exists a unique \mathcal{C} -functor $\mathcal{G}: {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ such that $\mathcal{V}\mathcal{G} = \mathcal{U}$ as \mathcal{C} -functors and $\varphi: \mathcal{F} \cong \mathcal{G}$ is a \mathcal{C} -isomorphism.*

Proof. Let $\xi: \mathcal{F}(M \otimes X) \cong \mathcal{F}(M) \otimes X$ be given with \mathcal{F} . Let (M, v_M) be in ${}_B\mathcal{C}$. Define a C -structure on M by

$$v'_M: C \otimes M = C \otimes \mathcal{U}(M, v_M) \xrightarrow{C \otimes \varphi^{-1}} C \otimes \mathcal{V}\mathcal{F}(M, v_M) \\ \xrightarrow{v_{\mathcal{F}(M, v)}} \mathcal{V}\mathcal{F}(M, v_M) \xrightarrow{\varphi} \mathcal{U}(M, v_M) = M.$$

It is easy to show that M becomes a C -object. So we define $\mathcal{G}: {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ by $\mathcal{G}(M, v_M) := (M, v'_M)$. For $f \in {}_B\mathcal{C}$ we define $\mathcal{G}(f) := f$ which turns out to be in ${}_C\mathcal{C}$. \mathcal{G} clearly is a functor. Furthermore the morphisms $\varphi(M, v_M): \mathcal{V}\mathcal{F}(M, v_M) \rightarrow \mathcal{U}(M, v'_M)$ are C -morphisms by the definition of v'_M ; hence φ defines a natural isomorphism $\varphi: \mathcal{F} \cong \mathcal{G}$.

Using the hypothesis that $\varphi: \mathcal{V}\mathcal{F} \cong \mathcal{U}$ is a \mathcal{C} -isomorphism, it is easy to show that the induced $\varphi: \mathcal{F} \cong \mathcal{G}$ becomes again a \mathcal{C} -isomorphism, i.e., that

$$\begin{array}{ccc} \mathcal{F}(M \otimes X) & \stackrel{\xi}{\cong} & \mathcal{F}(M) \otimes X \\ \parallel \circ & & \parallel \circ \\ \mathcal{G}(M \otimes X) & = & \mathcal{G}(M) \otimes X \end{array}$$

commutes for $M \in {}_B\mathcal{C}$, $X \in \mathcal{C}$. Finally we have $\mathcal{V}\mathcal{G} = \mathcal{U}$ with $\text{id}: \mathcal{G}(M \otimes X) = \mathcal{G}(M) \otimes X$ as structure morphism.

If $\mathcal{F} \circ \mathcal{G}' = \mathcal{U}$ and $\varphi: \mathcal{F} \cong \mathcal{G}'$ is also a \mathcal{C} -isomorphism, then one easily shows $\mathcal{G} = \mathcal{G}'$; hence \mathcal{G} is unique.

PROPOSITION 2. *Under the hypotheses of Proposition 1 there is a unique monoid morphism $g: C \rightarrow B$ such that $\mathcal{G}(M, \nu_M) = (M, C \otimes M \xrightarrow{r^{\otimes M}} B \otimes M \xrightarrow{\nu_M} M)$.*

Proof. For $(B, \mu) \in {}_B\mathcal{C}$ let $\mathcal{G}(B, \Delta) = (B, \nu'_B)$ with $\nu'_B: C \otimes B \rightarrow B$. Define $g: C \rightarrow B$ by $g(c) := \nu'_B(c \otimes 1_B) = c \cdot 1_B$ with $1_B \in B(I)$. Since \mathcal{G} is a \mathcal{C} -functor and $\nu_M: B \otimes M \rightarrow M$ is a morphism in ${}_B\mathcal{C}$, the following commute:

$$\begin{array}{ccccc} C \otimes \mathcal{G}(B) \otimes M & = & C \otimes \mathcal{G}(B \otimes M) & \xrightarrow{C \otimes \nu'(v_M)} & C \otimes \mathcal{G}(M) \\ \downarrow \nu'_B \otimes M & & \downarrow \nu'_B \otimes M & & \downarrow \nu'_M \\ \mathcal{G}(B) \otimes M & = & \mathcal{G}(B \otimes M) & \xrightarrow{\nu(v_M)} & \mathcal{G}(M) \end{array}$$

hence

$$\begin{array}{ccc} C \otimes B \otimes M & \xrightarrow{C \otimes r_M} & C \otimes M \\ \downarrow \nu'_B \otimes M & & \downarrow \nu'_M \\ B \otimes M & \xrightarrow{r_M} & M \end{array}$$

or

$$c \cdot (b \cdot m) = (c \cdot b) \cdot m.$$

Thus $g(1_C) = 1_C \cdot 1_B = 1_B$ and

$$\begin{aligned} g(c \cdot c') &= (c \cdot c') \cdot 1_B = c \cdot (c' \cdot 1_B) = c \cdot (1_B \cdot (c' \cdot 1_B)) \\ &= (c \cdot 1_B) \cdot (c' \cdot 1_B) = g(c) \cdot g(c'), \\ c \cdot m &= c \cdot (1_B \cdot m) = (c \cdot 1_B) \cdot m = g(c) \cdot m. \end{aligned}$$

Hence $g: C \rightarrow B$ is a monoid morphism which induces \mathcal{G} . If $g': C \rightarrow B$ is another monoid morphism with $c \cdot m = g'(c) \cdot m$, then $g'(c) = g'(c) \cdot 1_B = c \cdot 1_B = g(c)$; hence $g = g'$.

PROPOSITION 3. *Let $f: B \rightarrow B$ induce the functor $\mathcal{F}: {}_B\mathcal{C} \rightarrow {}_B\mathcal{C}$ (with $\mathcal{U} \circ \mathcal{F} = \mathcal{U}$). If there is a \mathcal{C} -isomorphism $\varphi: \mathcal{F} \cong Id$, then $f: B \rightarrow B$ is an inner automorphism.*

Proof. Since φ is a \mathcal{C} -morphism, we get

$$\begin{array}{ccccc} \mathcal{F}(B \otimes M) & = & B \otimes M & \xrightarrow{\varphi(B) \otimes M} & B \otimes M \\ \downarrow \mathcal{F}(v_M) & & \downarrow \nu_M & & \downarrow \nu_M \\ \mathcal{F}(M) & = & M & \xrightarrow{\varphi(M)} & M \end{array}$$

commutes, hence $\varphi(M)(m) = \varphi(M) \nu_M(1_B \otimes m) = \nu_M(\varphi(B) \otimes M)(1_B \otimes m) = \varphi(B)(1_B) \cdot m$. Clearly φ^{-1} is also a \mathcal{C} -morphism; hence $\varphi^{-1}(M)(m) = \varphi^{-1}(B)(1_B) \cdot m$. Replace $m = \varphi(B)(1_B)$ to get $\varphi^{-1}(B)(1_B) \cdot \varphi(B)(1_B) = \varphi^{-1}(B)\varphi(B)(1_B) = \varphi^{-1}\varphi(B)(1_B) = 1_B$ and symmetrically $\varphi(B)(1_B) \cdot \varphi^{-1}(B)(1_B) = 1_B$. Now $\varphi(M): M \rightarrow M$ is a B -morphism, the second M carrying a given B -structure, the first M carrying the f -induced B -structure. Hence

$$\varphi(M)(f(b) \cdot m) = b \cdot \varphi(M)(m).$$

For $M = B$, $m = 1_B$, we get $\varphi(B)(f(b)) = b \cdot \varphi(B)(1_B)$ or $\varphi(B)(1_B) \cdot f(b) = b \cdot \varphi(B)(1_B)$. Since $\varphi(B)(1_B)$ is invertible, we get $f(b) = \varphi^{-1}(B)(1_B) \cdot b \cdot \varphi(B)(1_B)$.

PROPOSITION 4. *Let $\mathcal{F}: {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ and $\mathcal{G}: {}_C\mathcal{C} \rightarrow {}_B\mathcal{C}$ be a \mathcal{C} -equivalence with a natural \mathcal{C} -isomorphism $\varphi: \mathcal{V}\mathcal{F} \cong \mathcal{U}$. Then \mathcal{F} is \mathcal{C} -isomorphic to a \mathcal{C} -functor $\mathcal{F}': {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ which is induced by an isomorphism $f: C \rightarrow B$.*

Proof. First we observe that φ induces a \mathcal{C} -isomorphism $\mathcal{U}\mathcal{G} \cong \mathcal{V}\mathcal{F}\mathcal{G} \cong \mathcal{V}Id = \mathcal{V}$; hence the situation is symmetric in \mathcal{F} and \mathcal{G} . Replace \mathcal{F} by \mathcal{F}' and \mathcal{G} by \mathcal{G}' according to Propositions 1 and 2. Then clearly \mathcal{F}' and \mathcal{G}' are induced by $f: C \rightarrow B$, resp. $g: B \rightarrow C$, and are inverse \mathcal{C} -equivalences, \mathcal{C} -isomorphic to \mathcal{F} , resp. \mathcal{G} . Thus $\mathcal{G}'\mathcal{F}'$ and $\mathcal{F}'\mathcal{G}'$ are induced by fg , resp. gf . Since there are \mathcal{C} -isomorphisms of these functors with the corresponding identity functors, fg and gf are isomorphisms of monoids and so are f and g .

2

From now on we shall assume that \mathcal{C} is a symmetric monoidal category. To motivate the following considerations, let us assume that B is a bimonoid in \mathcal{C} , i.e., a monoid and a comonoid, such that comultiplication and counit are monoid-morphisms. Then the category ${}_B\mathcal{C}$ carries the structure of a monoidal category, the tensor product being defined as tensor product in \mathcal{C} with B -structure on $M \otimes N$ for $M, N \in {}_B\mathcal{C}$ defined by

$$\begin{aligned} B \otimes M \otimes N &\xrightarrow{\Delta \otimes M \otimes N} B \otimes B \otimes M \otimes N \\ &\xrightarrow{B \otimes \gamma \otimes N} B \otimes M \otimes B \otimes N \xrightarrow{L_M \otimes P_N} M \otimes N. \end{aligned}$$

It is easy to check that this again defines a B -object. Furthermore $I \in \mathcal{C}$ is a B -object by $\varepsilon\rho: B \otimes I \cong B \rightarrow I$. Thus ${}_B\mathcal{C}$ becomes a monoidal category, where we denote the tensor product by $\hat{\otimes}$, the neutral object by \hat{I} , and the induced natural transformations by $\hat{a}, \hat{\lambda}, \hat{\rho}$.

The underlying functor $\mathcal{U}: {}_B\mathcal{C} \rightarrow \mathcal{C}$ has the following properties

- (1) $\mathcal{U}(M \hat{\otimes} N) = \mathcal{U}(M) \otimes \mathcal{U}(N)$ for all $M, N \in {}_B\mathcal{C}$,
 $\mathcal{U}(f \hat{\otimes} g) = \mathcal{U}(f) \otimes \mathcal{U}(g)$ for all $f, g \in {}_B\mathcal{C}$;
- (2) $\mathcal{U}(I) = I$;
- (3) $\mathcal{U}(\hat{\alpha}) = \alpha$, $\mathcal{U}(\hat{\lambda}) = \lambda$, $\mathcal{U}(\hat{\rho}) = \rho$;
- (4) $\mathcal{U}(M \otimes X) = \mathcal{U}(M) \otimes X$ for all $M \in {}_B\mathcal{C}$, $X \in \mathcal{C}$,
 $\mathcal{U}(f \otimes h) = \mathcal{U}(f) \otimes h$ for all $f \in {}_B\mathcal{C}$, $h \in \mathcal{C}$;

(5) $(M \otimes X) \hat{\otimes} (N \otimes Y) \cong (M \hat{\otimes} N) \otimes X \otimes Y$ as B -objects functorially in $X, Y \in \mathcal{C}$, $M, N \in {}_B\mathcal{C}$. The isomorphism is $M \otimes \gamma \otimes Y$.

A monoidal category $(\mathcal{D}, \hat{\otimes}, \hat{I}, \hat{\alpha}, \hat{\lambda}, \hat{\rho})$ which is a \mathcal{C} -category will be called a \mathcal{C} -monoidal category if there are natural isomorphisms

$$\begin{aligned} \xi_L: (M \otimes X) \hat{\otimes} N &\cong (M \hat{\otimes} N) \otimes X, \\ \xi_R: M \hat{\otimes} (N \otimes X) &\cong (M \hat{\otimes} N) \otimes X \quad \text{for } M, N \in \mathcal{D}, \quad X \in \mathcal{C}, \end{aligned}$$

such that $-\hat{\otimes} N: \mathcal{D} \rightarrow \mathcal{D}$ and $M \hat{\otimes} -: \mathcal{D} \rightarrow \mathcal{D}$ together with ξ_L and ξ_R are \mathcal{C} -functors, and $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\rho}$ are \mathcal{C} -morphisms in all variables. Furthermore, all morphisms in this definition are assumed to be coherent.

Obviously the category ${}_B\mathcal{C}$ for any bimonoid B is a \mathcal{C} -monoidal category in a natural way. In particular, \mathcal{C} itself is \mathcal{C} -monoidal. Here we use the symmetry of \mathcal{C} .

Let \mathcal{D} and \mathcal{E} be \mathcal{C} -monoidal categories. A monoidal functor $\mathcal{F}: \mathcal{D} \rightarrow \mathcal{E}$ with $\delta: \mathcal{F}(M \hat{\otimes} N) \cong \mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N)$, $\zeta: \mathcal{F}(\hat{I}) \cong \tilde{I}$, which is also a \mathcal{C} -functor with $\xi: \mathcal{F}(M \otimes X) \cong \mathcal{F}(M) \otimes X$, will be called a \mathcal{C} -monoidal functor, if

$$\begin{array}{ccccc} \mathcal{F}(M \hat{\otimes} (N \otimes X)) & \xrightarrow{\delta} & \mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N \otimes X) & \xrightarrow{\mathcal{F}(M) \tilde{\otimes} \xi} & \mathcal{F}(M) \tilde{\otimes} (\mathcal{F}(N) \otimes X) \\ \downarrow \mathcal{F}(\xi_R) & & & & \downarrow \xi_R \\ \mathcal{F}((M \hat{\otimes} N) \otimes X) & \xrightarrow{\xi} & \mathcal{F}(M \otimes N) \otimes X & \xrightarrow{\delta \otimes X} & (\mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N)) \otimes X \\ \\ \mathcal{F}((M \otimes X) \hat{\otimes} N) & \xrightarrow{\delta} & \mathcal{F}(M \otimes X) \tilde{\otimes} \mathcal{F}(N) & \xrightarrow{\xi \tilde{\otimes} \mathcal{F}(N)} & (\mathcal{F}(M) \otimes X) \tilde{\otimes} \mathcal{F}(N) \\ \downarrow \mathcal{F}(\xi_L) & & & & \downarrow \xi_L \\ \mathcal{F}((M \hat{\otimes} N) \otimes X) & \xrightarrow{\xi} & \mathcal{F}(M \hat{\otimes} N) \otimes X & \xrightarrow{\delta \otimes X} & (\mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N)) \otimes X \end{array}$$

commute. In particular all morphisms in this definition are assumed to be coherent.

A \mathcal{C} -monoidal natural transformation $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ will just be a monoidal and a \mathcal{C} -transformation.

Again it is clear that $\mathcal{U}: {}_B\mathcal{C} \rightarrow \mathcal{C}$ for any bimonoid B is a \mathcal{C} -monoidal functor. If $f: C \rightarrow B$ is a morphism of bimonoids, then the induced functor $\mathcal{F}: {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ is a \mathcal{C} -monoidal functor, as can be easily checked, with $\xi = id$, $\delta = id$, $\zeta = id$.

Now we want to invert our considerations and obtain from \mathcal{C} -monoidal structures certain bimonoid structures.

PROPOSITION 5. *Let B be a monoid. Assume that ${}_B\mathcal{C}$ has the structure of a \mathcal{C} -monoidal category $({}_B\mathcal{C}, \hat{\otimes}, \hat{I}, \hat{\alpha}, \hat{\lambda}, \hat{\rho}, \otimes, \beta, \sigma, \xi_L, \xi_R)$ and that $\mathcal{U}: {}_B\mathcal{C} \rightarrow \mathcal{C}$ is a \mathcal{C} -monoidal functor $(\mathcal{U}, \delta, \zeta, \xi)$, where $({}_B\mathcal{C}, \otimes, \beta, \sigma)$ is the ordinary \mathcal{C} -structure on ${}_B\mathcal{C}$ and (\mathcal{U}, ξ) is the ordinary \mathcal{C} -structure on \mathcal{U} . Then there exists a unique \mathcal{C} -monoidal structure $({}_B\mathcal{C}, \tilde{\otimes}, \tilde{I}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho}, \otimes, \beta, \sigma, \tilde{\xi}_L, \tilde{\xi}_R)$ on ${}_B\mathcal{C}$ such that $(Id, \delta, \zeta, \xi): ({}_B\mathcal{C}, \hat{\otimes}) \rightarrow ({}_B\mathcal{C}, \tilde{\otimes})$ and $(\mathcal{U}, id, id, id): {}_B\mathcal{C} \rightarrow \mathcal{C}$ are \mathcal{C} -monoidal functors.*

When we have proved Proposition 5 we can reduce arbitrary \mathcal{C} -monoidal structures on ${}_B\mathcal{C}$ and \mathcal{U} to isomorphic \mathcal{C} -monoidal structures on ${}_B\mathcal{C}$ and \mathcal{U} with $(\mathcal{U}, id, id, id)$ being the \mathcal{C} -monoidal functor, and this can be done in only one way.

Proof. We first show that the isomorphism $\delta: \mathcal{U}(M \hat{\otimes} N) \cong \mathcal{U}(M) \otimes \mathcal{U}(N)$ induces a unique B -structure on $M \otimes N$ for $M, N \in {}_B\mathcal{C}$, natural in both variables, such that $\delta: M \hat{\otimes} N \cong M \otimes N$ is a natural isomorphism of functors $\hat{\otimes}$ and \otimes from ${}_B\mathcal{C} \times {}_B\mathcal{C} \rightarrow {}_B\mathcal{C}$. Define the B -structure by the commutative diagram

$$\begin{array}{ccc} B \otimes (M \otimes N) & \xrightarrow{\gamma_{M \otimes N}} & M \otimes N \\ \parallel \scriptstyle B \otimes \delta & & \parallel \scriptstyle \delta \\ B \otimes \mathcal{U}(M \hat{\otimes} N) & \xrightarrow{\gamma_{M \otimes N}} & \mathcal{U}(M \hat{\otimes} N) \end{array}$$

where we use $\mathcal{U}(M) = M$ and $\mathcal{U}(N) = N$. This defines clearly a B -structure on $M \otimes N$; it is natural in M and N in ${}_B\mathcal{C}$ and δ becomes the desired isomorphism. Clearly $\gamma_{M \otimes N}$ is the only morphism making δ a natural isomorphism of functors to ${}_B\mathcal{C}$. Similarly I carries a B -structure uniquely such that $\zeta: \hat{I} \cong I$ is an isomorphism in ${}_B\mathcal{C}$.

Since $\hat{\alpha}, \hat{\lambda}, \hat{\rho}$ are natural isomorphisms in ${}_B\mathcal{C}$ and by the commutativity of the coherence diagrams for monoidal functors $\mathcal{U}, \alpha, \lambda$ and ρ will also be natural transformations in ${}_B\mathcal{C}$, $({}_B\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is again a monoidal category. ${}_B\mathcal{C}$ is also a \mathcal{C} -category with

$$\begin{aligned} (\beta: M \otimes (X \otimes Y) \cong (M \otimes X) \otimes Y) &:= \alpha, \\ (\sigma: M \otimes I \cong M) &:= \rho. \end{aligned}$$

The natural isomorphisms

$$\begin{aligned}
 (\xi_L: (M \otimes X) \otimes N &\cong (M \otimes N) \otimes X) := \alpha(M \otimes \gamma) \alpha^{-1}, \\
 (\xi_R: M \otimes (N \otimes X) &\cong (M \otimes N) \otimes X) := \alpha
 \end{aligned}$$

make ${}_B\mathcal{C}$ a \mathcal{C} -monoidal category.

Consider the functor $\text{Id}: {}_B\mathcal{C} \rightarrow {}_B\mathcal{C}$, the first copy of ${}_B\mathcal{C}$ carrying the \mathcal{C} -monoidal structure $\hat{\otimes}$, the second copy with the new tensor product \otimes . Then

$$\delta: M \hat{\otimes} N \cong M \otimes N$$

is a natural isomorphism by definition. $\zeta: \hat{I} \cong I$ is a B -isomorphism. Furthermore $(\xi: M \otimes X \cong M \otimes X) = id$ makes Id a \mathcal{C} -functor. It is now easy to check that Id is a \mathcal{C} -monoidal functor, since $(\mathcal{U}, \delta, \zeta, \xi)$ was.

Now consider $\mathcal{U}: {}_B\mathcal{C} \rightarrow \mathcal{C}$, where ${}_B\mathcal{C}$ carries the new \mathcal{C} -monoidal structure \otimes . Then $(\delta: \mathcal{U}(M \otimes N) \cong \mathcal{U}(M) \otimes \mathcal{U}(N)) := id$, $(\zeta: \mathcal{U}(I) \cong I) := id$ and $(\xi: \mathcal{U}(M \otimes X) \cong \mathcal{U}(M) \otimes X) := id$ form a \mathcal{C} -monoidal functor.

The fact that $\delta = id$ for the new \mathcal{C} -monoidal structure requires that the new tensor product be \otimes with a suitable B -structure. This B -structure is unique by the requirement that $(\text{Id}, \delta, \zeta, \xi)$ be a \mathcal{C} -monoidal functor, in particular that δ be a B -isomorphism. Similarly the requirement $(\zeta: \mathcal{U}(I) \cong I) = id$ implies I with a unique B -structure as the only possible neutral object in ${}_B\mathcal{C}$. α, λ, ρ are imposed by the fact that (\mathcal{U}, id, id) be monoidal. The \mathcal{C} -structure on ${}_B\mathcal{C}$ was to be retained anyway. Finally, ξ_L and ξ_R on ${}_B\mathcal{C}$ and \mathcal{C} have to be the same morphisms.

PROPOSITION 6. *Let B be a monoid. Let $({}_B\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \otimes, \beta, \sigma, \xi_L, \xi_R)$ be a \mathcal{C} -monoidal category such that $(\mathcal{U}, id, id, id): {}_B\mathcal{C} \rightarrow \mathcal{C}$ is a \mathcal{C} -monoidal functor. Then there is a unique bimonoid structure on B which induces the \mathcal{C} -monoidal structures on ${}_B\mathcal{C}$ and \mathcal{U} as described in the beginning of this section.*

Proof. Observe that by Proposition 5 $(\mathcal{U}, id, id, id)$ implies that the tensor product on ${}_B\mathcal{C}$ has to be \otimes with a suitable B -structure and that $\alpha, \lambda, \rho, \beta, \sigma, \xi_L$ and ξ_R coincide in ${}_B\mathcal{C}$ and \mathcal{C} . Henceforth we shall omit these structure maps and say ${}_B\mathcal{C}$ is \mathcal{C} -monoidal with $\mathcal{U}: {}_B\mathcal{C} \rightarrow \mathcal{C}$ \mathcal{C} -monoidal.

Now define

$$\begin{aligned}
 \varepsilon &:= (B \cong B \otimes I \xrightarrow{\gamma} I) \quad \text{or} \quad \varepsilon(b) = b \cdot 1_I = b \cdot 1, \\
 \Delta &:= (B \cong B \otimes (I \otimes I) \xrightarrow{B \otimes (\eta \otimes I)} B \otimes (B \otimes B) \xrightarrow{\gamma \otimes \beta} B \otimes B),
 \end{aligned}$$

or

$$\Delta(b) = b \cdot (1 \otimes 1) =: b_{(1)} \otimes b_{(2)} \quad \text{for all } b \in B(X).$$

LEMMA 7. *The B-structure $\gamma_{M \otimes N}$ on $M \otimes N$ is given by*

$$\begin{aligned} B \otimes (M \otimes N) &\xrightarrow{\Delta \otimes (M \otimes N)} (B \otimes B) \otimes (M \otimes N) \\ &\cong (B \otimes M) \otimes (B \otimes N) \xrightarrow{\gamma_B \otimes \gamma_N} M \otimes N \end{aligned}$$

or

$$b \cdot (m \otimes n) = b_{(1)} \cdot m \otimes b_{(2)} \cdot n.$$

Proof. The diagram

$$\begin{array}{ccc} B \otimes B \otimes M & \xrightarrow{\mu \otimes M} & B \otimes M \\ \downarrow B \otimes \gamma_M & & \downarrow \gamma_M \\ B \otimes M & \xrightarrow{\gamma_M} & M \end{array}$$

commutes; hence γ_M is a B -morphism where $B \otimes M$ carries the B -structure just on the left factor via $\mu: B \otimes B \rightarrow B$. γ_N is a B -morphism, too; hence $\gamma_M \otimes \gamma_N$ is a B -morphism and the following commutes in \mathcal{C} :

$$\begin{array}{ccccc} B \otimes (B \otimes B) \otimes (M \otimes N) & \cong & B \otimes ((B \otimes M) \otimes (B \otimes N)) & \longrightarrow & B \otimes (M \otimes N) \\ \downarrow \gamma_{B \otimes B} \otimes (M \otimes N) & & \downarrow \gamma_{(B \otimes M) \otimes (B \otimes N)} & & \downarrow \gamma_{M \otimes N} \\ (B \otimes B) \otimes (M \otimes N) & \cong & (B \otimes M) \otimes (B \otimes N) & \longrightarrow & M \otimes N \end{array}$$

where the horizontal arrows are $B \otimes (\gamma_M \otimes \gamma_N)$, resp. $\gamma_M \otimes \gamma_N$.

Elementwise we get $a \cdot (b \cdot m \otimes c \cdot n) = (a \cdot (b \otimes c)) \cdot (m \otimes n)$ for all $a \in B(X)$, $b \in B(Y)$, $c \in C(Z)$, $m \in M(U)$, $n \in N(V)$, where $(b \otimes c) \cdot (m \otimes n) = b \cdot m \otimes c \cdot n$. Now $b \cdot (m \otimes n) = b \cdot (1 \cdot m \otimes 1 \cdot n) = (b \cdot (1 \otimes 1)) \cdot (m \otimes n) = (b_{(1)} \otimes b_{(2)}) \cdot (m \otimes n) = b_{(1)} \cdot m \otimes b_{(2)} \cdot n$.

LEMMA 8. $\Delta: B \rightarrow B \otimes B$ is a monoid homomorphism.

Proof. $\Delta(a \cdot b) = (a \cdot b)_{(1)} \otimes (a \cdot b)_{(2)} = (a \cdot b) \cdot (1 \otimes 1) = a \cdot (b \cdot (1 \otimes 1)) = a \cdot (b_{(1)} \cdot 1 \otimes b_{(2)} \cdot 1) = a \cdot (b_{(1)} \otimes b_{(2)}) = a_{(1)} \cdot b_{(1)} \otimes a_{(2)} \cdot b_{(2)} = \Delta(a) \cdot \Delta(b)$.

$$\Delta(1) = 1 \cdot (1 \otimes 1) = 1 \otimes 1.$$

LEMMA 9. $\varepsilon: A \rightarrow I$ is a monoid homomorphism.

Proof. $\varepsilon(a \cdot b) = (a \cdot b) \cdot 1 = a \cdot (b \cdot 1) = a \cdot \varepsilon(b) = a \cdot (1 \cdot \varepsilon(b)) = (a \cdot 1) \cdot \varepsilon(b) = \varepsilon(a) \cdot \varepsilon(b)$, where (*) holds, since any multiplication with $x \in I(X)$ can be pulled by any morphism in \mathcal{C} .

$$\varepsilon(1_B) = 1_B \cdot 1 = 1.$$

LEMMA 10. Δ is coassociative.

Proof. We use the fact that α is a B -morphism; hence

$$\begin{aligned} \alpha(1 \otimes \Delta) \Delta(b) &= \alpha(b_{(1)} \otimes (b_{(2)(1)} \otimes b_{(2)(2)})) \\ &= \alpha(b_{(1)} \cdot 1 \otimes b_{(2)} \cdot (1 \otimes 1)) \\ &= \alpha(b \cdot (1 \otimes (1 \otimes 1))) \\ &= b \cdot \alpha(1 \otimes (1 \otimes 1)) \\ &= b \cdot ((1 \otimes 1) \otimes 1) \\ &= ((b_{(1)(1)} \otimes b_{(1)(2)}) \otimes b_{(2)}) \\ &= (\Delta \otimes 1) \Delta(b). \end{aligned}$$

LEMMA 11. (B, Δ, ε) is a comonoid.

Proof. Since λ and ρ are B -morphisms, we get

$$\begin{aligned} b &= b \cdot 1_B = b \cdot \lambda(1 \otimes 1_B) = \lambda(b \cdot (1 \otimes 1_B)) \\ &= \lambda(b_{(1)} \cdot 1 \otimes b_{(2)} \cdot 1_B) = \lambda(\varepsilon(b_{(1)}) \otimes b_{(2)}) \\ &= \varepsilon(b_{(1)}) b_{(2)} = (\varepsilon \otimes 1) \Delta(b), \\ b &= b \cdot 1_B = b \cdot \rho(1_B \otimes 1) = \rho(b \cdot (1_B \otimes 1)) \\ &= \rho(b_{(1)} \cdot 1_B \otimes b_{(2)} \cdot 1) = \rho(b_{(1)} \otimes \varepsilon(b_{(2)})) \\ &= b_{(1)} \cdot \varepsilon(b_{(2)}) = (1 \otimes \varepsilon) \Delta(b). \end{aligned}$$

Thus we have proved that B is a bimonoid in \mathcal{C} and that the \mathcal{C} -monoidal structure on ${}_B\mathcal{C}$ is induced by the bimonoid structure of B , i.e.,

$$b \cdot (m \otimes n) = b_{(1)} \cdot m \otimes b_{(2)} \cdot n, \quad (1)$$

$$b \in B(X), \quad m \otimes n \in (M \otimes N)(Y),$$

$$b \cdot x = \varepsilon(b) x, \quad b \in B(X), \quad x \in I(Y). \quad (2)$$

Now Δ is unique with (1), just take $m \otimes n := 1_B \otimes 1_B$. The uniqueness of ε

for a comonoid is shown in the same way as the uniqueness of the unit $\eta = 1$ in a monoid: $1' = 1' \cdot 1 = 1$.

Thus far we have reduced any given \mathcal{C} -monoidal structures on ${}_B\mathcal{C}$ and \mathcal{U} (with standard \mathcal{C} -structure) in a unique way to \mathcal{C} -monoidal structures induced by a bimonoid structure on B . Now we want to do the same reduction for a \mathcal{C} -monoidal functor $\mathcal{F}: {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ with a \mathcal{C} -monoidal isomorphism $\varphi: \mathcal{U}\mathcal{F} \cong \mathcal{U}$.

PROPOSITION 12. *Let $({}_B\mathcal{C}, \hat{\otimes})$ and $({}_C\mathcal{C}, \tilde{\otimes})$ be \mathcal{C} -monoidal categories and $(\mathcal{U}, \delta_{\mathcal{U}}, \zeta_{\mathcal{U}}, \xi_{\mathcal{U}}): {}_B\mathcal{C} \rightarrow \mathcal{C}$ and $(\mathcal{V}, \delta_{\mathcal{V}}, \zeta_{\mathcal{V}}, \xi_{\mathcal{V}}): {}_C\mathcal{C} \rightarrow \mathcal{C}$ be \mathcal{C} -monoidal functors. Let $(\mathcal{F}, \delta_{\mathcal{F}}, \zeta_{\mathcal{F}}, \xi_{\mathcal{F}}): {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ be a \mathcal{C} -monoidal functor. Let $({}_B\mathcal{C}, \otimes)$ and $({}_C\mathcal{C}, \otimes)$ be the \mathcal{C} -monoidal categories with their structures induced by the bimonoid structures on B and C . Then there is a unique \mathcal{C} -monoidal functor $(\mathcal{F}', \delta', \zeta', \xi')$ which makes the diagram*

$$\begin{array}{ccc} ({}_B\mathcal{C}, \hat{\otimes}) & \xrightarrow{\mathcal{F}} & ({}_C\mathcal{C}, \tilde{\otimes}) \\ \downarrow (Id, \delta_{\mathcal{U}}, \zeta_{\mathcal{U}}, \xi_{\mathcal{U}}) & & \downarrow (Id, \delta_{\mathcal{V}}, \zeta_{\mathcal{V}}, \xi_{\mathcal{V}}) \\ ({}_B\mathcal{C}, \otimes) & \xrightarrow{\mathcal{F}'} & ({}_C\mathcal{C}, \otimes) \end{array}$$

commutative.

Proof. Since $(Id, \delta_{\mathcal{U}}, \zeta_{\mathcal{U}}, \xi_{\mathcal{U}})$ is invertible as a \mathcal{C} -monoidal functor, \mathcal{F}' is to be the composition of \mathcal{C} -monoidal functors; hence $\mathcal{F}' = (\mathcal{F}, \delta_{\mathcal{F}}, \zeta_{\mathcal{F}}, \xi_{\mathcal{F}}) \circ (\mathcal{U}, \delta_{\mathcal{U}}, \zeta_{\mathcal{U}}, \xi_{\mathcal{U}})$.

COROLLARY 13. *Under the hypotheses of Proposition 12 let $\varphi: \mathcal{V}\mathcal{F} \cong \mathcal{U}$ be a \mathcal{C} -monoidal isomorphism. Then $\varphi: \mathcal{V}\mathcal{F}' \cong \mathcal{U}$ is also \mathcal{C} -monoidal.*

Proof. The first isomorphism is meant to be

$$\varphi: (\mathcal{V}, \delta_{\mathcal{V}}, \zeta_{\mathcal{V}}, \xi_{\mathcal{V}}) \circ (\mathcal{F}, \delta_{\mathcal{F}}, \zeta_{\mathcal{F}}, \xi_{\mathcal{F}}) \cong (\mathcal{U}, \delta_{\mathcal{U}}, \zeta_{\mathcal{U}}, \xi_{\mathcal{U}});$$

the second is

$$\begin{aligned} \varphi: (\mathcal{V}, id, id, id) \circ (\mathcal{F}, \delta_{\mathcal{F}}, \zeta_{\mathcal{F}}, \xi_{\mathcal{F}}) &\cong (\mathcal{U}, \delta_{\mathcal{U}}, \zeta_{\mathcal{U}}, \xi_{\mathcal{U}}) \\ &\cong (\mathcal{U}, id, id, id). \end{aligned}$$

Since we do not change the \mathcal{C} -structure, we only have to check that φ respects the change of monoidal structures:

$$\begin{array}{ccc} \mathcal{V}\mathcal{F}(M \hat{\otimes} N) & \xrightarrow{\tau(\delta_{\mathcal{F}})} & \mathcal{V}(\mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N)) \xrightarrow{\delta_{\mathcal{V}}} \mathcal{V}\mathcal{F}(M) \otimes \mathcal{V}\mathcal{F}(N) \\ \downarrow \varphi & & \downarrow \varphi \\ \mathcal{U}(M \otimes N) & \xrightarrow{\delta_{\mathcal{U}}} & \mathcal{U}(M) \otimes \mathcal{U}(N) \end{array}$$

commutes; hence

$$\begin{array}{ccccc}
 \mathcal{V}\mathcal{F}(M \otimes N) & \xrightarrow{\mathcal{F}(\delta_N^{-1})} & \mathcal{V}\mathcal{F}(M \hat{\otimes} N) & \xrightarrow{\delta_T \cdot \delta_{\mathcal{F}}} & \mathcal{V}\mathcal{F}(M) \otimes \mathcal{V}\mathcal{F}(N) \\
 \downarrow \varphi & & \downarrow \varphi & & \downarrow \varphi \\
 \mathcal{U}(M \otimes N) & \xrightarrow{\delta_N^{-1}} & \mathcal{U}(M \hat{\otimes} N) & \xrightarrow{\delta_{\mathcal{U}}} & \mathcal{U}(M) \otimes \mathcal{U}(N)
 \end{array}$$

commutes, where we omit the application of the underlying functors \mathcal{U}, \mathcal{V} . Furthermore

$$\begin{array}{ccc}
 \mathcal{U}\mathcal{F}(I) & \xrightarrow{\zeta_{\mathcal{F}}} & \mathcal{V}(\tilde{I}) \\
 \downarrow \varphi & & \downarrow \zeta_{\mathcal{V}} \\
 \mathcal{U}(\tilde{I}) & \xrightarrow{\zeta_{\mathcal{U}}} & I
 \end{array}$$

commutes; hence

$$\begin{array}{ccccc}
 \mathcal{V}\mathcal{F}(I) & \xrightarrow{\mathcal{F}(\zeta_N^{-1})} & \mathcal{V}\mathcal{F}(\tilde{I}) & \xrightarrow{\zeta_{\mathcal{F}}} & \mathcal{V}(\tilde{I}) \\
 \downarrow \varphi & & \downarrow \varphi & & \downarrow \zeta_{\mathcal{V}} \\
 \mathcal{U}(I) & \xrightarrow{\zeta_N^{-1}} & \mathcal{U}(\tilde{I}) & \xrightarrow{\zeta_{\mathcal{U}}} & I
 \end{array}$$

commutes. We have now reduced the general \mathcal{C} -monoidal situation

$$\begin{array}{ccc}
 {}_B\mathcal{C} & \xrightarrow{\mathcal{F}} & {}_C\mathcal{C} \\
 \searrow \mathcal{U} & & \swarrow \mathcal{V} \\
 & \mathcal{C} &
 \end{array}$$

with $\varphi: \mathcal{V}\mathcal{F} \cong \mathcal{U}$ to the special situation, where ${}_B\mathcal{C}$ and ${}_C\mathcal{C}$ carry \mathcal{C} -monoidal structures induced by bimonoids B and C . Now we want to change \mathcal{F} to an isomorphic \mathcal{C} -monoidal functor \mathcal{G} as in Proposition 1.

PROPOSITION 14. *Under the hypotheses of Corollary 13 the functor \mathcal{G} induced in Proposition 1 is a \mathcal{C} -monoidal functor $(\mathcal{G}, id, id, id): ({}_B\mathcal{C}, \otimes) \rightarrow ({}_C\mathcal{C}, \otimes)$ and $\varphi: \mathcal{F} \cong \mathcal{G}$ and $\mathcal{V}\mathcal{G} = \mathcal{U}$ are \mathcal{C} -monoidal transformations.*

Proof. By Proposition 1 we know already that \mathcal{F} is a \mathcal{C} -functor with $\xi_r = (id: \mathcal{F}(M \otimes X) = \mathcal{F}(M) \otimes X)$. Furthermore the diagram

$$\begin{array}{ccccc}
 \mathcal{V} \mathcal{F}(M \otimes N) & \xrightarrow{id} & \mathcal{V}(\mathcal{F}(M) \otimes \mathcal{F}(N)) & \xrightarrow{id} & \mathcal{V} \mathcal{F}(M) \otimes \mathcal{V} \mathcal{F}(N) \\
 \parallel \scriptstyle \omega^{-1} & (1) & \parallel \scriptstyle (\omega \otimes \omega)^{-1} & (2) & \parallel \scriptstyle (\omega \otimes \omega)^{-1} \\
 \mathcal{V} \mathcal{F}'(M \otimes N) & \xrightarrow{\delta'_{\mathcal{F}}} & \mathcal{V}(\mathcal{F}'(M) \otimes \mathcal{F}'(N)) & \xrightarrow{id} & \mathcal{V} \mathcal{F}'(M) \otimes \mathcal{V} \mathcal{F}'(N) \\
 \parallel \scriptstyle \omega & & (3) & & \parallel \scriptstyle \omega \otimes \omega \\
 \mathcal{U}(M \otimes N) & \xrightarrow{id} & & & \mathcal{U}(M) \otimes \mathcal{U}(N)
 \end{array}$$

commutes at (3) because $\varphi: \mathcal{V} \mathcal{F}' \cong \mathcal{U}$ is \mathcal{C} -monoidal, and clearly at (2) and at (1) because the outer diagram commutes. So the only possible δ for \mathcal{F} is $id: \mathcal{F}(M \otimes N) \rightarrow \mathcal{F}(M) \otimes \mathcal{F}(N)$ and it makes $\varphi: \mathcal{F}' \cong \mathcal{F}$ a monoidal transformation. \mathcal{F} together with $id: \mathcal{F}(M \otimes N) \rightarrow \mathcal{F}(M) \otimes \mathcal{F}(N)$ clearly is monoidal because in both categories ${}_B \mathcal{C}$ and ${}_C \mathcal{C}$ we have the same morphisms α, λ, ρ and because ξ and δ are identities. The isomorphism $\zeta: \mathcal{F}(I) \cong I$ will also be the identity because

$$\begin{array}{ccc}
 \mathcal{V} \mathcal{F}(I) & \xrightarrow{id} & I \\
 \parallel \scriptstyle \omega^{-1} & & \parallel \\
 \mathcal{V} \mathcal{F}'(I) & \xrightarrow{\zeta_{\mathcal{F}'}} & I \\
 \parallel \scriptstyle \omega & & \parallel \\
 \mathcal{U}(I) & = & I
 \end{array}$$

commutes. Thus we have that \mathcal{F} is a monoidal \mathcal{C} -functor with structure morphisms $(\delta, \zeta, \xi) = (id, id, id)$. Furthermore $\varphi: \mathcal{F}' \cong \mathcal{F}$ is a monoidal \mathcal{C} -transformation. \mathcal{F} is \mathcal{C} -monoidal since in both categories ${}_B \mathcal{C}$ and ${}_C \mathcal{C}$ the morphisms ξ_R are just α and $\xi_L = \alpha(M \otimes \gamma) \alpha^{-1}$. Finally the identity $\mathcal{V} \mathcal{F} = \mathcal{U}$ is also \mathcal{C} -monoidal because all structure morphisms are identities.

THEOREM 15. *Let B and C be monoids in \mathcal{C} . Let the \mathcal{C} -categories ${}_B \mathcal{C}$ and ${}_C \mathcal{C}$ carry the structure of \mathcal{C} -monoidal categories such that the underlying \mathcal{C} -functors $\mathcal{U}: {}_B \mathcal{C} \rightarrow \mathcal{C}$ and $\mathcal{V}: {}_C \mathcal{C} \rightarrow \mathcal{C}$ are \mathcal{C} -monoidal. Let $\mathcal{F}: {}_B \mathcal{C} \rightarrow {}_C \mathcal{C}$ be a \mathcal{C} -monoidal functor and $\varphi: \mathcal{V} \mathcal{F} \cong \mathcal{U}$ be a \mathcal{C} -monoidal natural isomorphism. Then there are unique bimonoid structures on B and C and a unique bimonoid morphism $g: C \rightarrow B$ such that the induced \mathcal{C} -monoidal structures on ${}_B \mathcal{C}$, resp. ${}_C \mathcal{C}$, are isomorphic to the original ones by the identity functor and the induced \mathcal{C} -monoidal functor $\mathcal{G}: {}_B \mathcal{C} \rightarrow {}_C \mathcal{C}$ is \mathcal{C} -monoidally isomorphic to \mathcal{F} via φ .*

Proof. By Propositions 1 and 2 there is a unique monoid morphism $g: C \rightarrow B$ such that the induced \mathcal{C} -functor $\mathcal{G}: {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ satisfies $\mathcal{V}\mathcal{G} = \mathcal{U}$ and $\varphi: \mathcal{F} \cong \mathcal{G}$ a \mathcal{C} -isomorphism. By Propositions 5 and 6 the \mathcal{C} -monoidal structure of ${}_B\mathcal{C}$, resp. ${}_C\mathcal{C}$, is isomorphic to a \mathcal{C} -monoidal structure induced by a unique bimonoid structure on B , resp. C , via the identity functor and by Corollary 13 the given functor $\mathcal{F}: {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ and isomorphism $\varphi: \mathcal{V}\mathcal{F} \cong \mathcal{U}$ are \mathcal{C} -monoidal in a unique way also with respect to the bimonoid-induced \mathcal{C} -monoidal structures on ${}_B\mathcal{C}$, resp. ${}_C\mathcal{C}$. By Proposition 14 the \mathcal{C} -functor $\mathcal{G}: {}_B\mathcal{C} \rightarrow {}_C\mathcal{C}$ induced by the monoid morphism $g: B \rightarrow C$ is \mathcal{C} -monoidal: $(\mathcal{G}, id, id, id)$ and $\varphi: \mathcal{F} \cong \mathcal{G}$ is also \mathcal{C} -monoidal. Also $\mathcal{V}\mathcal{G} = \mathcal{U}$ is already a \mathcal{C} -monoidal equality.

So the only thing to prove is that g is a bimonoid morphism. Now $\delta: \mathcal{G}(M \otimes N) \rightarrow \mathcal{G}(M) \otimes \mathcal{G}(N)$ is not only the identity but also a \mathcal{C} -morphism and so is $\zeta: \mathcal{G}(I) \rightarrow I$. Hence we get

$$\begin{aligned} \delta(\Delta_B g(c) \cdot m \otimes n) &= \delta(g(c) \cdot m \otimes n) = \delta(c \cdot m \otimes n) \\ &= c \cdot \delta(m \otimes n) = \Delta_c(c) \cdot \delta(m \otimes n) \\ &= (g \otimes g) \Delta_c(c) \cdot \delta(m \otimes n) \end{aligned}$$

and

$$\begin{aligned} \zeta(\varepsilon_B g(c)) &= \zeta(g(c) \cdot 1_I) = \zeta(c \cdot 1_I) = c \cdot \zeta(1_I) \\ &= \varepsilon_c(c) \cdot \zeta(1_I). \end{aligned}$$

Observe $\delta = id$ and $\zeta = id$ and set $m \otimes n = 1_B \otimes 1_B$ to get

$$\begin{aligned} \Delta_B g(c) &= \Delta_B g(c) \cdot 1_B \otimes 1_B = (g \otimes g) \Delta_c(c) \cdot 1_B \otimes 1_B \\ &= (g \otimes g) \Delta_c(c), \\ \varepsilon_B g(c) &= \varepsilon_c(c). \end{aligned}$$

COROLLARY 16. *Under the hypotheses of Theorem 14 let \mathcal{F} be a \mathcal{C} -monoidal equivalence. Then $g: C \rightarrow B$ is a bimonoid isomorphism.*

Proof. This is simply a consequence of Proposition 4.

3

Let K be a commutative ring and let $\mathcal{C} = K\text{-Mod}$ be the monoidal category of K -modules with the usual tensor product over K . Consider the category $K\text{-Comp}$ of complexes of K -modules

$$\mathcal{C} = (\dots \rightarrow A_i \xrightarrow{\partial_i} A_{i+1} \rightarrow \dots), \quad \partial^2 = 0$$

and complexes homomorphisms $(f: \mathcal{A} \rightarrow \mathcal{B}) = (f_i: A_i \rightarrow B_i \mid i \in \mathbf{Z}, \partial_i f_i = f_{i+1} \partial'_{i+1})$.

$K\text{-Comp}$ is a \mathcal{C} -category where

$$\begin{aligned} \mathcal{A} \otimes X &:= (\dots \rightarrow A_i \otimes X \xrightarrow{\partial_i \otimes X} A_{i+1} \otimes X \rightarrow \dots), \\ f \otimes X &:= (f_i \otimes X). \end{aligned}$$

The isomorphism $\beta: \mathcal{A} \otimes (X \otimes Y) \cong (\mathcal{A} \otimes X) \otimes Y$ is induced by $\alpha: A_i \otimes (X \otimes Y) \cong (A_i \otimes X) \otimes Y$ and $\sigma: \mathcal{A} \otimes K \cong \mathcal{A}$ by $\rho: A_i \otimes K \cong A_i$. Clearly all these definitions are functorial in all variables and coherent (in the sense of [1] or [5]).

$K\text{-Comp}$ is also monoidal with the usual tensor product of complexes (take tensor products separately of all components and then make the double complex into a single complex by adding diagonally with the usual sign shift). To be more precise

$$\mathcal{A} \otimes \mathcal{B} := \left(\dots \bigoplus_{j+k=i} (A_j \otimes B_k) \xrightarrow{\delta_i} \bigoplus_{j+k=i} (A_i \otimes B_k) \rightarrow \dots \right),$$

where

$$\delta_i = \bigoplus_{j+k=i} ((-1)^k \partial_j \otimes B_k + A_j \otimes \partial'_k)$$

with $\partial_j: A_j \rightarrow A_{j+1}$ and $\partial'_k: B_k \rightarrow B_{k+1}$. This is wellknown to be natural in both variables and associativity $\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \cong (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}$ induced by α can easily be checked. The neutral element in $K\text{-Comp}$ is

$$\mathcal{K} = (\dots \rightarrow 0 \rightarrow K \rightarrow 0 \rightarrow \dots)$$

with K at position zero. The isomorphisms $\mathcal{A} \otimes \mathcal{K} \cong \mathcal{A} \cong \mathcal{K} \otimes \mathcal{A}$ are induced by λ and ρ and thus coherent with α . $K\text{-Comp}$ is even \mathcal{C} -monoidal with structure morphisms.

$$\begin{aligned} \xi_L &: (\mathcal{A} \otimes X) \otimes \mathcal{B} \cong (\mathcal{A} \otimes \mathcal{B}) \otimes X, \\ \xi_R &: \mathcal{A} \otimes (\mathcal{B} \otimes X) \cong (\mathcal{A} \otimes \mathcal{B}) \otimes X \end{aligned}$$

induced by those in $K\text{-Mod}$. Again coherence is clear from coherence in $K\text{-Mod}$.

Now consider the functor $\mathcal{V}: K\text{-Comp} \rightarrow K\text{-Mod}$ given by

$$\begin{aligned} \mathcal{V}(\mathcal{A}) &:= \bigoplus A_i, \\ \mathcal{V}(f) &:= \bigoplus f_i. \end{aligned}$$

It is a \mathcal{C} -functor by the natural isomorphism

$$\mathcal{V}(\mathcal{A} \otimes X) = \bigoplus (A_i \otimes X) \cong \left(\bigoplus A_i \right) \otimes X = \mathcal{V}(\mathcal{A}) \otimes X.$$

Furthermore, it is monoidal by the natural isomorphisms

$$\begin{aligned} \mathcal{V}(\mathcal{A} \otimes \mathcal{B}) &= \bigoplus_i \left(\bigoplus_{j+k=i} (A_j \otimes B_k) \right) \cong \left(\bigoplus_j A_j \right) \otimes \left(\bigoplus_k B_k \right) \\ &= \mathcal{V}(\mathcal{A}) \otimes \mathcal{V}(\mathcal{B}), \\ \mathcal{V}(\mathcal{K}) &= \left(\bigoplus 0 \right) \oplus K \oplus \left(\bigoplus 0 \right) \cong K. \end{aligned}$$

Actually \mathcal{V} is \mathcal{C} -monoidal as can be easily checked.

Thus $K\text{-Comp}$ is a \mathcal{C} -monoidal category with $\mathcal{C} = K\text{-Mod}$ and $\mathcal{V}: K\text{-Comp} \rightarrow K\text{-Mod}$ is a \mathcal{C} -monoidal functor.

Now we define a K -bialgebra B by

$$B = K\langle s, t, t^{-1} \rangle / (s^2, st + ts),$$

where $K\langle s, t, t^{-1} \rangle$ denotes adjoining two variables s, t , which do not commute with each other, but with all of K , and adjoining an inverse of t . We factor out the two-sided ideal generated by s^2 and $st + ts$. For the diagonal we take

$$\Delta(t) = t \otimes t, \quad \Delta(s) = s \otimes 1 + t^{-1} \otimes s.$$

The augmentation is defined by

$$\varepsilon(t) = 1, \quad \varepsilon(s) = 0.$$

LEMMA 17. *B is a bialgebra.*

Proof. B has obviously the K -basis $\{t^i \mid i \in \mathbf{Z}\} \cup \{t^i s \mid i \in \mathbf{Z}\}$. If Δ is to be multiplicative, $\Delta(t^i) = t^i \otimes t^i$ and $\Delta(t^i s) = t^i s \otimes t^i + t^{i-1} \otimes t^i s$ must hold. Then Δ can be expanded by linearity and it is trivial to see that Δ is an algebra morphism. Δ is associative because it is on t and s . $\varepsilon(t^i) = 1$ and $\varepsilon(t^i s) = 0$ defines again an algebra homomorphism and (B, Δ, ε) forms a coalgebra. Thus B is a bialgebra.

Observe now that the category $B\text{-Comod}$ of B -comodules for any bialgebra B is \mathcal{C} -monoidal for $\mathcal{C} = K\text{-Mod}$ essentially in the same way as it is \mathcal{C} -monoidal for $\mathcal{C} = (K\text{-mod})^{\text{op}}$ (${}_B\mathcal{C}$ in Section 2 was \mathcal{C} -monoidal). Also the underlying functor $\mathcal{V}: B\text{-Comod} \rightarrow K\text{-Mod}$ is \mathcal{C} -monoidal.

THEOREM 18. *There is a \mathcal{C} -monoidal equivalence $\mathcal{F}: B\text{-Comod} \rightarrow K\text{-Comp}$ and a \mathcal{C} -monoidal isomorphism $\varphi: \mathcal{V}\mathcal{F} \cong \mathcal{U}$.*

COROLLARY 19. *The bialgebra B is uniquely determined up to isomorphism by Theorem 17.*

Proof of Corollary 19 is a simple application of Corollary 16.

Proof of Theorem 18. Let M be a B -comodule with structure map $\lambda_M: M \rightarrow B \otimes M$. With respect to the basis $\{t^i, st^i\}$ we can write

$$\lambda_M(m) = \sum_i t^i \otimes m_i + \sum_i t^i s \otimes m'_i. \tag{3}$$

Now apply $(1 \otimes \lambda)\lambda = (\Delta \otimes 1)\lambda$ to get

$$\begin{aligned} & \sum t^i \otimes \lambda(m_i) + \sum t^i s \otimes \lambda(m'_i) \\ &= \sum t^i \otimes t^i \otimes m_i + \sum t^i s \otimes t^i \otimes m'_i \\ & \quad + \sum t^{i-1} \otimes t^i s \otimes m'_i; \end{aligned}$$

hence by comparison of the coefficients

$$\lambda(m_i) = t^i \otimes m_i + t^{i+1} s \otimes m'_{i+1}, \tag{4}$$

$$\lambda(m'_i) = t^i \otimes m'_i. \tag{5}$$

If we apply $(\varepsilon \otimes 1)\lambda(m) = m$ to (3) we get

$$m = \sum m_i. \tag{6}$$

Now define $M_i := \{m \in M \mid \lambda(m) = t^i \otimes m + t^{i+1} s \otimes m'\}$ and $\partial: M_i \rightarrow M_{i+1}$ by $\partial(m) = m'$ in $\lambda(m) = t^i \otimes m + t^{i+1} s \otimes m'$. Clearly M_i is a K -module and ∂ is linear. To see that $m' \in M_{i+1}$, observe (4) and (5) which give $\lambda(m') = t^{i+1} \otimes m'$. Furthermore $\partial\partial(m) = 0$ for $m \in M_i$ again by (4) and (5). By (6) we get $M = \sum M_i$. Now if $\sum m_i = 0$ with $m_i \in M_i$, then $0 = \lambda(\sum_i m_i) = \sum_i t^i \otimes m_i + t^{i+1} s \otimes m'_i$ and hence $m_i = 0$. So $M = \bigoplus M_i$.

Thus $M \in B\text{-Comod}$ defines a complex

$$\dots \rightarrow M_i \xrightarrow{\partial} M_{i+1} \rightarrow \dots$$

in $K\text{-Comp}$.

If $f: M \rightarrow N$ is a comodule morphism then for $m \in M_i$ we get

$$\lambda f(m) = (1 \otimes f) \lambda(m) = t^i \otimes f(m) + t^{i+1} s \otimes f(\partial m);$$

hence $f(m) \in N_i$, $f_i = f|_{M_i}: M_i \rightarrow N_i$ and $\partial f(m) = f \partial(m)$, so f defines a complex homomorphism. Altogether we have thus obtained a functor $\mathcal{F}: B\text{-Comod} \rightarrow K\text{-Comp}$.

For the underlying functors we have

$$\mathcal{U} \circ \mathcal{F}(M) = \bigoplus M_i = \mathcal{U}(M);$$

$$\mathcal{U} \circ \mathcal{F}(f) = \bigoplus f_i = \mathcal{U}(f);$$

hence $\mathcal{U} \circ \mathcal{F} = \mathcal{U}$.

For the \mathcal{C} -structures we get

$$\begin{aligned} \mathcal{F}(M \otimes X) &= (\dots \rightarrow (M \otimes X)_i \xrightarrow{\partial} (M \otimes X)_{i+1} \rightarrow \dots) \\ &\cong (\dots \rightarrow M_i \otimes X \xrightarrow{\partial \otimes X} M_{i+1} \otimes X \rightarrow \dots) \\ &= \mathcal{F}(M) \otimes X. \end{aligned}$$

To see $M_i \otimes X \cong (M \otimes X)_i$ consider $M_i \otimes X \subseteq M \otimes X$ by $M \otimes X = \bigoplus (M_i \otimes X)$. Then $M_i \otimes X \subseteq (M \otimes X)_i$ and $\bigoplus (M_i \otimes X) = \bigoplus (M \otimes X)_i$; hence $M_i \otimes X = (M \otimes X)_i$ under this identification. The isomorphism $\xi: \mathcal{F}(M \otimes X) \cong \mathcal{F}(M) \otimes X$ is functorial and satisfies the coherence conditions. So \mathcal{F} is a \mathcal{C} -functor. Also the identity $\mathcal{U} \circ \mathcal{F} = \mathcal{U}$ is compatible with the \mathcal{C} -structure:

$$\mathcal{U} \circ \mathcal{F}(M \otimes X) \cong \mathcal{U}(\mathcal{F}(M) \otimes X) \cong \mathcal{U} \circ \mathcal{F}(M) \otimes X$$

is the identity; hence $\mathcal{U} \circ \mathcal{F}$ and \mathcal{U} are equal as \mathcal{C} -functors.

For the monoidal structures we get

$$\mathcal{F}(M \otimes N) = (\dots \rightarrow (M \otimes N)_i \xrightarrow{\partial} (M \otimes N)_{i+1} \rightarrow \dots).$$

To study $(M \otimes N)_i$ observe that every element in $M \otimes N$ can be written as a sum $\sum_{j,k} m_j \otimes n_k$ with $m_j \in M_j$, $n_k \in N_k$. Then $\lambda(\sum m_j \otimes n_k) = \sum (t^j \cdot t^k \otimes m_j \otimes n_k + t^{j+1} s \otimes m_j \otimes \partial(n_k) + t^{j+1} s t^k \otimes \partial(m_j) \otimes n_k + t^{j+1} s t^{k+1} s \otimes \partial(m_j) \otimes \partial(n_k)) = \sum_i t^i \otimes (\sum_{j+k=i} m_j \otimes n_k) + \sum_i t^{i+1} s \otimes (\sum_{j+k=i} m_j \otimes \partial(n_k) + (-1)^k \partial(m_j) \otimes n_k)$. Hence $(M \otimes N)_i = \bigoplus_{i=j+k} (M_j \otimes N_k)$ and

$$\partial_{M \otimes N, i} = \bigoplus (M_j \otimes \partial'_k + (-1)^k \partial_j \otimes N_k),$$

i.e., $\delta: \mathcal{F}(M \otimes N) \cong \mathcal{F}(M) \otimes \mathcal{F}(N)$. Furthermore $\zeta: \mathcal{F}(K) = \mathcal{K}$. Both satisfy the coherence conditions, so \mathcal{F} is a monoidal functor. One also checks easily that

$$\gamma \circ \mathcal{F}(M \otimes N) \cong \gamma(\mathcal{F}(M) \otimes \mathcal{F}(N)) \cong \gamma \mathcal{F}(M) \otimes \gamma \mathcal{F}(N)$$

is the identity, so $\gamma \circ \mathcal{F} = \mathcal{U}$ as monoidal functors.

Finally \mathcal{F} is \mathcal{C} -monoidal since all the morphisms for coherence are naturally defined in $K\text{-mod}$ and coherent there.

Now we construct an equivalence inverse for \mathcal{F} . Let $\mathcal{O} \in K\text{-Comp}$. We define $\mathcal{G}(\mathcal{O}) := \bigoplus A_i$. To get the comodule structure on $\bigoplus A_i$, define for $a_i \in A_i$

$$\lambda(a_i) := t^i \otimes a_i + t^{i+1}s \otimes \partial(a_i) \in B \otimes \left(\bigoplus A_i \right).$$

This defines a B -comodule structure on $\mathcal{G}(\mathcal{O})$ by easy computation. For a complex homomorphism f define $\mathcal{G}(f) := \bigoplus f_i$ and verify it is a comodule homomorphism. So $\mathcal{G}: K\text{-Comp} \rightarrow B\text{-Comod}$ is a functor. Then it is easy to check $\mathcal{G} \circ \mathcal{F} \cong Id$ and $\mathcal{F} \circ \mathcal{G} \cong Id$. It is tedious but straightforward to check that \mathcal{G} again is \mathcal{C} -monoidal and that the isomorphisms $\mathcal{G} \circ \mathcal{F} \cong Id$ and $\mathcal{F} \circ \mathcal{G} \cong Id$ are \mathcal{C} -monoidal, thus \mathcal{F} is a \mathcal{C} -monoidal equivalence and $\gamma \circ \mathcal{F} = \mathcal{U}$ as \mathcal{C} -monoidal functors.

COROLLARY 20. *The bialgebra B defined by Theorem 18 has an antipode of order 4 (2 in characteristic 2).*

Proof. The antipode S is given by $S(t) = t^{-1}$ and $S(s) = st$. Check that this indeed defines an antipode if continued as an algebra antimorphism. We have then $S^2(t) = t$ and $S^2(s) = t^{-1}S(s) = -s$ and $S^4 = id$.

We remark that there is an additional structure on both $K\text{-Comp}$ and $B\text{-Comod}$. Both categories are symmetric. This is surprising since one should think that B must be commutative in this case. But the symmetry we shall describe does not coincide with the symmetry in $K\text{-Mod}$ by the underlying functor.

The symmetry in $K\text{-Comp}$ is given by $M_i \otimes N_j \cong N_j \otimes M_i$, $m_i \otimes n_j \mapsto (-1)^{ij} n_j \otimes m_i$. In $B\text{-Comod}$ the symmetry can be described in this way. Define a linear map $\Psi: B \otimes B \rightarrow K$ by $t^i \otimes t^j \mapsto (-1)^{ij}$ and $\Psi(t^i s \otimes t^j) = \Psi(t^j s \otimes t^i s) = \Psi(t^i \otimes t^j s) = 0$. We shall not investigate its meaning for B , but in a certain sense it is induced by the multiplication on \mathbf{Z} through $\mathbf{Z} \ni i \mapsto t^i \in B$. Then the symmetry $\gamma: M \otimes N \cong N \otimes M$ is given by $\gamma(m \otimes n) = \sum \Psi(m_{(0)} \otimes n_{(0)}) \cdot n_{(1)} \otimes m_{(1)}$. This is a comodule map with $\gamma^2 = id$, functorial and coherent in $B\text{-Comod}$ and the functor $\mathcal{F}: K\text{-Comp} \rightarrow B\text{-Comod}$ is compatible with the two symmetries.

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