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A Non-Commutative Non-Cocommutative Hopf Algebra in "Nature"

BODO PAREIGIS

Mathematisches Institut der Universität München, Munich, Germany

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We show that there is a uniquely defined Hopf algebra H, such that H-Comod, the category of H-comodules, and K-Comp, the category of K-complexes, are isomorphic as monoidal categories, where the isomorphism is compatible with the obvious underlying functors, i.e.,

$$\begin{array}{ccc} H\text{-Comod} & \cong & K\text{-Comp} \\ & {\scriptstyle \mathscr{U}}\searrow \quad \swarrow{\scriptstyle \mathscr{U}} & \\ & K\text{-Mod} & \end{array}$$

commutes. The Hopf algebra H is defined as follows:

$$H = K\langle x, y, y^{-1}\rangle/(xy + yx, x^2) \qquad \text{(non-commuting variables)}$$

$$\Delta(x) = x \otimes 1 + y^{-1} \otimes x, \qquad s(x) = xy, \qquad \varepsilon(x) = 0,$$

$$\Delta(y) = y \otimes y \qquad s(y) = y^{-1}, \qquad \varepsilon(y) = 1.$$

H is a non-commutative, non-cocommutative Hopf algebra with antipode of order 4.

1

Let K be a commutative ring with unit. All algebras and coalgebras are defined over K and are (co-)associative with (co-)unit.

For an algebra A it is well known that the underlying functor $\mathscr{U}: A\text{-Mod} \to K\text{-Mod}$ determines the algebra A up to isomorphism. In fact $A \cong \mathrm{End}(\mathscr{U})$.

There is no such obvious description of a coalgebra C by the underlying functor $\mathscr{U}: C\text{-Comod} \to K\text{-Mod}$. Abstractly this follows from a remark in [4, Corollary 6.4]; in fact C is uniquely described up to isomorphism by $\mathscr{U}$.

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We need stronger results than those above. So we shall use the notation and results of [2–4].

Let $\mathscr{C}$ be a symmetric monoidal category (e.g., K-Mod with the usual tensor product over K or its dual).

We shall consider two monoids, B and C, and want to study the categories ${}_B\mathscr{C}$ and ${}_C\mathscr{C}$, when they carry themselves the structure of monoidal categories. It will turn out that this induces bimonoid structures on B, resp. C. Furthermore we study functors $\mathscr{F}: {}_B\mathscr{C} \to {}_C\mathscr{C}$ which are compatible with the underlying functors from ${}_B\mathscr{C}$, resp. ${}_C\mathscr{C}$, to $\mathscr{C}$ and preserve the monoidal structure. It will be shown that they induce bimonoid morphisms from C to B.

For the first propositions we need only a monoidal category $\mathscr{C}$, not necessarily symmetric.

Denote the underlying functor from ${}_B\mathscr{C}$ to $\mathscr{C}$ by $\mathscr{U}$ and the one from ${}_C\mathscr{C}$ to $\mathscr{C}$ by $\mathscr{V}$. Observe that ${}_B\mathscr{C}$, ${}_C\mathscr{C}$ and $\mathscr{C}$ carry in a natural way the structure of $\mathscr{C}$-categories and $\mathscr{U}$ and $\mathscr{V}$ are $\mathscr{C}$-functors. In [4, Corollary 6.4] we showed already how to obtain B from $\mathscr{U}: {}_B\mathscr{C} \to \mathscr{C}$ as $B^{\mathrm{op}} \cong [\mathscr{U}, \mathscr{U}]$. Now we want to study functors $\mathscr{F}: {}_B\mathscr{C} \to {}_C\mathscr{C}$.

PROPOSITION 1. *Let $(\mathscr{F}, \xi): {}_B\mathscr{C} \to {}_C\mathscr{C}$ be a $\mathscr{C}$-functor and $\varphi: \mathscr{V}\mathscr{F} \cong \mathscr{U}$ be a natural $\mathscr{C}$-isomorphism. Then there exists a unique $\mathscr{C}$-functor $\mathscr{G}: {}_B\mathscr{C} \to {}_C\mathscr{C}$ such that $\mathscr{V}\mathscr{G} = \mathscr{U}$ as $\mathscr{C}$-functors and $\varphi: \mathscr{F} \cong \mathscr{G}$ is a $\mathscr{C}$-isomorphism.*

Proof. Let $\xi: \mathscr{F}(M \otimes X) \cong \mathscr{F}(M) \otimes X$ be given with $\mathscr{F}$. Let (M, ν_M) be in ${}_B\mathscr{C}$. Define a C-structure on M by

$$\begin{aligned} \nu'_M: C \otimes M = C \otimes \mathscr{U}(M, \nu_M) &\xrightarrow{C \otimes \varphi^{-1}} C \otimes \mathscr{V}\mathscr{F}(M, \nu_M) \\ &\xrightarrow{\nu_{\mathscr{F}(M,\nu)}} \mathscr{V}\mathscr{F}(M, \nu_M) \xrightarrow{\varphi} \mathscr{U}(M, \nu_M) = M. \end{aligned}$$

It is easy to show that M becomes a C-object. So we define $\mathscr{G}: {}_B\mathscr{C} \to {}_C\mathscr{C}$ by $\mathscr{G}(M, \nu_M) := (M, \nu'_M)$. For $f \in {}_B\mathscr{C}$ we define $\mathscr{G}(f) := f$ which turns out to be in ${}_C\mathscr{C}$. $\mathscr{G}$ clearly is a functor. Furthermore the morphisms $\varphi(M, \nu_M)$: $\mathscr{V}\mathscr{F}(M, \nu_M) \to \mathscr{U}(M, \nu'_M)$ are C-morphisms by the definition of ν'_M; hence φ defines a natural isomorphism $\varphi: \mathscr{F} \cong \mathscr{G}$.

Using the hypothesis that $\varphi: \mathscr{V}\mathscr{F} \cong \mathscr{U}$ is a $\mathscr{C}$-isomorphism, it is easy to show that the induced $\varphi: \mathscr{F} \cong \mathscr{G}$ becomes again a $\mathscr{C}$-isomorphism, i.e., that

$$\begin{array}{ccc} \mathscr{F}(M \otimes X) & \overset{\xi}{\cong} & \mathscr{F}(M) \otimes X \\ \|\wr \varphi & & \|\wr \varphi \\ \mathscr{G}(M \otimes X) & = & \mathscr{G}(M) \otimes X \end{array}$$

commutes for $M \in {}_B\mathscr{C}$, $X \in \mathscr{C}$. Finally we have $\mathscr{V}\mathscr{G} = \mathscr{U}$ with id: $\mathscr{G}(M \otimes X) = \mathscr{G}(M) \otimes X$ as structure morphism.

If $\mathscr{V}\mathscr{G}' = \mathscr{U}$ and $\varphi: \mathscr{F} \cong \mathscr{G}'$ is also a $\mathscr{C}$-isomorphism, then one easily shows $\mathscr{G} = \mathscr{G}'$; hence $\mathscr{G}$ is unique.

PROPOSITION 2. *Under the hypotheses of Proposition* 1 *there is a unique monoid morphism* $g: C \to B$ *such that* $\mathscr{G}(M, v_M) = (M, C \otimes M \to^{g \otimes M} B \otimes M \to^{v_M} M)$.

Proof. For $(B, \mu) \in {}_B\mathscr{C}$ let $\mathscr{G}(B, \Delta) = (B, v'_B)$ with $v'_B: C \otimes B \to B$. Define $g: C \to B$ by $g(c) := v'_B(c \otimes 1_B) = c \cdot 1_B$ with $1_B \in B(I)$. Since $\mathscr{G}$ is a $\mathscr{C}$-functor and $v_M: B \otimes M \to M$ is a morphism in ${}_B\mathscr{C}$, the following commute:

$$\begin{array}{ccccc} C \otimes \mathscr{G}(B) \otimes M = & C \otimes \mathscr{G}(B \otimes M) & \xrightarrow{C \otimes \mathscr{G}(v_M)} & C \otimes \mathscr{G}(M) \\ \downarrow{\scriptstyle v'_B \otimes M} & \downarrow{\scriptstyle v'_B \otimes M} & & \downarrow{\scriptstyle v'_M} \\ \mathscr{G}(B) \otimes M = & \mathscr{G}(B \otimes M) & \xrightarrow{\mathscr{G}(v_M)} & \mathscr{G}(M) \end{array}$$

hence

$$\begin{array}{ccc} C \otimes B \otimes M & \xrightarrow{C \otimes v_M} & C \otimes M \\ \downarrow{\scriptstyle v'_B \otimes M} & & \downarrow{\scriptstyle v'_M} \\ B \otimes M & \xrightarrow{v_M} & M \end{array}$$

or

$$c \cdot (b \cdot m) = (c \cdot b) \cdot m.$$

Thus $g(1_C) = 1_C \cdot 1_B = 1_B$ and

$$\begin{aligned} g(c \cdot c') &= (c \cdot c') \cdot 1_B = c \cdot (c' \cdot 1_B) = c \cdot (1_B \cdot (c' \cdot 1_B)) \\ &= (c \cdot 1_B) \cdot (c' \cdot 1_B) = g(c) \cdot g(c'), \\ c \cdot m &= c \cdot (1_B \cdot m) = (c \cdot 1_B) \cdot m = g(c) \cdot m. \end{aligned}$$

Hence $g: C \to B$ is a monoid morphism which induces $\mathscr{G}$. If $g': C \to B$ is another monoid morphism with $c \cdot m = g'(c) \cdot m$, then $g'(c) = g'(c) \cdot 1_B = c \cdot 1_B = g(c)$; hence $g = g'$.

PROPOSITION 3. *Let* $f: B \to B$ *induce the functor* $\mathscr{F}: {}_B\mathscr{C} \to {}_B\mathscr{C}$ *(with* $\mathscr{U}\mathscr{F} = \mathscr{U}$*). If there is a* $\mathscr{C}$*-isomorphism* $\varphi: \mathscr{F} \cong Id$*, then* $f: B \to B$ *is an inner automorphism.*

Proof. Since φ is a $\mathscr{C}$-morphism, we get

$$\begin{array}{ccccc} \mathscr{F}(B \otimes M) = & B \otimes M & \xrightarrow{\varphi(B) \otimes M} & B \otimes M \\ \downarrow{\scriptstyle \mathscr{F}(v_M)} & \downarrow{\scriptstyle v_M} & & \downarrow{\scriptstyle v_M} \\ \mathscr{F}(M) = & M & \xrightarrow{\varphi(M)} & M \end{array}$$

commutes, hence $\varphi(M)(m)=\varphi(M)\,\nu_M(1_B\otimes m)=\nu_M(\varphi(B)\otimes M)(1_B\otimes m)=\varphi(B)(1_B)\cdot m$. Clearly φ^{-1} is also a $\mathscr{C}$-morphism; hence $\varphi^{-1}(M)(m)=\varphi^{-1}(B)(1_B)\cdot m$. Replace $m=\varphi(B)(1_B)$ to get $\varphi^{-1}(B)(1_B)\cdot\varphi(B)(1_B)=\varphi^{-1}(B)\,\varphi(B)(1_B)=\varphi^{-1}\varphi(B)(1_B)=1_B$ and symmetrically $\varphi(B)(1_B)\cdot\varphi^{-1}(B)(1_B)=1_B$. Now $\varphi(M)\colon M\to M$ is a B-morphism, the second M carrying a given B-structure, the first M carrying the f-induced B-structure. Hence

$$\varphi(M)(f(b)\cdot m)=b\cdot\varphi(M)(m).$$

For $M=B$, $m=1_B$, we get $\varphi(B)(f(b))=b\cdot\varphi(B)(1_B)$ or $\varphi(B)(1_B)\cdot f(b)=b\cdot\varphi(B)(1_B)$. Since $\varphi(B)(1_B)$ is invertible, we get $f(b)=\varphi^{-1}(B)(1_B)\cdot b\cdot\varphi(B)(1_B)$.

PROPOSITION 4. *Let* $\mathscr{F}\colon {}_B\mathscr{C}\to{}_C\mathscr{C}$ *and* $\mathscr{G}\colon {}_C\mathscr{C}\to{}_B\mathscr{C}$ *be a* $\mathscr{C}$*-equivalence with a natural* $\mathscr{C}$*-isomorphism* $\varphi\colon\mathscr{V}\mathscr{F}\cong\mathscr{U}$. *Then* $\mathscr{F}$ *is* $\mathscr{C}$*-isomorphic to a* $\mathscr{C}$*-functor* $\mathscr{F}\colon {}_B\mathscr{C}\to{}_C\mathscr{C}$ *which is induced by an isomorphism* $f\colon C\to B$.

Proof. First we observe that φ induces a $\mathscr{C}$-isomorphism $\mathscr{U}\mathscr{G}\cong\mathscr{V}\mathscr{F}\mathscr{G}\cong\mathscr{V}\,Id=\mathscr{V}$; hence the situation is symmetric in $\mathscr{F}$ and $\mathscr{G}$. Replace $\mathscr{F}$ by $\mathscr{F}'$ and $\mathscr{G}$ by $\mathscr{G}'$ according to Propositions 1 and 2. Then clearly $\mathscr{F}'$ and $\mathscr{G}'$ are induced by $f\colon C\to B$, resp. $g\colon B\to C$, and are inverse $\mathscr{C}$-equivalences, $\mathscr{C}$-isomorphic to $\mathscr{F}$, resp. $\mathscr{G}$. Thus $\mathscr{G}'\mathscr{F}'$ and $\mathscr{F}'\mathscr{G}'$ are induced by fg, resp. gf. Since there are $\mathscr{C}$-isomorphisms of these functors with the corresponding identity functors, fg and gf are isomorphisms of monoids and so are f and g.

2

From now on we shall assume that $\mathscr{C}$ is a symmetric monoidal category. To motivate the following considerations, let us assume that B is a bimonoid in $\mathscr{C}$, i.e., a monoid and a comonoid, such that comultiplication and counit are monoid-morphisms. Then the category ${}_B\mathscr{C}$ carries the structure of a monoidal category, the tensor product being defined as tensor product in $\mathscr{C}$ with B-structure on $M\otimes N$ for $M, N\in{}_B\mathscr{C}$ defined by

$$B\otimes M\otimes N\xrightarrow{\Delta\otimes M\otimes N}B\otimes B\otimes M\otimes N$$
$$\xrightarrow{B\otimes\gamma\otimes N}B\otimes M\otimes B\otimes N\xrightarrow{\nu_M\otimes\nu_N}M\otimes N.$$

It is easy to check that this again defines a B-object. Furthermore $I\in\mathscr{C}$ is a B-object by $\varepsilon\rho\colon B\otimes I\cong B\to I$. Thus ${}_B\mathscr{C}$ becomes a monoidal category, where we denote the tensor product by $\hat{\otimes}$, the neutral object by $\hat{I}$, and the induced natural transformations by $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\rho}$.

The underlying functor $\mathcal{U}: {}_B\mathcal{C} \to \mathcal{C}$ has the following properties

(1) $\mathcal{U}(M \hat{\otimes} N) = \mathcal{U}(M) \otimes \mathcal{U}(N)$ for all $M, N \in {}_B\mathcal{C}$,
$\mathcal{U}(f \hat{\otimes} g) = \mathcal{U}(f) \otimes \mathcal{U}(g)$ for all $f, g \in {}_B\mathcal{C}$;

(2) $\mathcal{U}(\hat{I}) = I$;

(3) $\mathcal{U}(\hat{\alpha}) = \alpha$, $\mathcal{U}(\hat{\lambda}) = \lambda$, $\mathcal{U}(\hat{\rho}) = \rho$;

(4) $\mathcal{U}(M \otimes X) = \mathcal{U}(M) \otimes X$ for all $M \in {}_B\mathcal{C}$, $X \in \mathcal{C}$,
$\mathcal{U}(f \otimes h) = \mathcal{U}(f) \otimes h$ for all $f \in {}_B\mathcal{C}$, $h \in \mathcal{C}$;

(5) $(M \otimes X) \hat{\otimes} (N \otimes Y) \cong (M \hat{\otimes} N) \otimes X \otimes Y$ as B-objects functorially in $X, Y \in \mathcal{C}$, $M, N \in {}_B\mathcal{C}$. The isomorphism is $M \otimes \gamma \otimes Y$.

A monoidal category $(\mathcal{D}, \hat{\otimes}, \hat{I}, \hat{\alpha}, \hat{\lambda}, \hat{\rho})$ which is a $\mathcal{C}$-category will be called a *$\mathcal{C}$-monoidal category* if there are natural isomorphisms

$$\xi_L: (M \otimes X) \hat{\otimes} N \cong (M \hat{\otimes} N) \otimes X,$$
$$\xi_R: M \hat{\otimes} (N \otimes X) \cong (M \hat{\otimes} N) \otimes X \qquad \text{for } M, N \in \mathcal{D}, \quad X \in \mathcal{C},$$

such that $- \hat{\otimes} N: \mathcal{D} \to \mathcal{D}$ and $M \hat{\otimes} -: \mathcal{D} \to \mathcal{D}$ together with ξ_L and ξ_R are $\mathcal{C}$-functors, and $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\rho}$ are $\mathcal{C}$-morphisms in all variables. Furthermore, all morphisms in this definition are assumed to be coherent.

Obviously the category ${}_B\mathcal{C}$ for any bimonoid B is a $\mathcal{C}$-monoidal category in a natural way. In particular, $\mathcal{C}$ itself is $\mathcal{C}$-monoidal. Here we use the symmetry of $\mathcal{C}$.

Let $\mathcal{D}$ and $\mathcal{E}$ be $\mathcal{C}$-monoidal categories. A monoidal functor $\mathcal{F}: \mathcal{D} \to \mathcal{E}$ with $\delta: \mathcal{F}(M \hat{\otimes} N) \cong \mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N)$, $\zeta: \mathcal{F}(\hat{I}) \cong \tilde{I}$, which is also a $\mathcal{C}$-functor with $\xi: \mathcal{F}(M \otimes X) \cong \mathcal{F}(M) \otimes X$, will be called a *$\mathcal{C}$-monoidal functor*, if

$$\begin{array}{ccccc}
\mathcal{F}(M \hat{\otimes} (N \otimes X)) & \xrightarrow{\delta} & \mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N \otimes X) & \xrightarrow{\mathcal{F}(M) \tilde{\otimes} \xi} & \mathcal{F}(M) \tilde{\otimes} (\mathcal{F}(N) \otimes X) \\
\downarrow{\scriptstyle \mathcal{F}(\xi_R)} & & & & \downarrow{\scriptstyle \xi_R} \\
\mathcal{F}((M \hat{\otimes} N) \otimes X) & \xrightarrow{\xi} & \mathcal{F}(M \otimes N) \otimes X & \xrightarrow{\delta \otimes X} & (\mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N)) \otimes X
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{F}((M \otimes X) \hat{\otimes} N) & \xrightarrow{\delta} & \mathcal{F}(M \otimes X) \tilde{\otimes} \mathcal{F}(N) & \xrightarrow{\xi \tilde{\otimes} \mathcal{F}(N)} & (\mathcal{F}(M) \otimes X) \tilde{\otimes} \mathcal{F}(N) \\
\downarrow{\scriptstyle \mathcal{F}(\xi_L)} & & & & \downarrow{\scriptstyle \xi_L} \\
\mathcal{F}((M \hat{\otimes} N) \otimes X) & \xrightarrow{\xi} & \mathcal{F}(M \hat{\otimes} N) \otimes X & \xrightarrow{\delta \otimes X} & (\mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N)) \otimes X
\end{array}$$

commute. In particular all morphisms in this definition are assumed to be coherent.

A *$\mathcal{C}$-monoidal* natural transformation $\varphi: \mathcal{F} \to \mathcal{G}$ will just be a monoidal and a $\mathcal{C}$-transformation.

Again it is clear that $\mathscr{U}: {}_B\mathscr{C} \to \mathscr{C}$ for any bimonoid B is a $\mathscr{C}$-monoidal functor. If $f: C \to B$ is a morphism of bimonoids, then the induced functor $\mathscr{F}: {}_B\mathscr{C} \to {}_C\mathscr{C}$ is a $\mathscr{C}$-monoidal functor, as can be easily checked, with $\xi = id$, $\delta = id$, $\zeta = id$.

Now we want to invert our considerations and obtain from $\mathscr{C}$-monoidal structures certain bimonoid structures.

PROPOSITION 5. *Let B be a monoid. Assume that ${}_B\mathscr{C}$ has the structure of a $\mathscr{C}$-monoidal category $({}_B\mathscr{C}, \hat{\otimes}, \hat{I}, \hat{\alpha}, \hat{\lambda}, \hat{\rho}, \otimes, \beta, \sigma, \xi_L, \xi_R)$ and that $\mathscr{U}: {}_B\mathscr{C} \to \mathscr{C}$ is a $\mathscr{C}$-monoidal functor $(\mathscr{U}, \delta, \zeta, \xi)$, where $({}_B\mathscr{C}, \otimes, \beta, \sigma)$ is the ordinary $\mathscr{C}$-structure on ${}_B\mathscr{C}$ and $(\mathscr{U}, \xi)$ is the ordinary $\mathscr{C}$-structure on $\mathscr{U}$. Then there exists a unique $\mathscr{C}$-monoidal structure $({}_B\mathscr{C}, \tilde{\otimes}, \tilde{I}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\rho}, \otimes, \beta, \sigma, \tilde{\xi}_L, \tilde{\xi}_R)$ on ${}_B\mathscr{C}$ such that $(Id, \delta, \zeta, \xi): ({}_B\mathscr{C}, \hat{\otimes}) \to ({}_B\mathscr{C}, \tilde{\otimes})$ and $(\mathscr{U}, id, id, id): {}_B\mathscr{C} \to \mathscr{C}$ are $\mathscr{C}$-monoidal functors.*

When we have proved Proposition 5 we can reduce arbitrary $\mathscr{C}$-monoidal structures on ${}_B\mathscr{C}$ and $\mathscr{U}$ to isomorphic $\mathscr{C}$-monoidal structures on ${}_B\mathscr{C}$ and $\mathscr{U}$ with $(\mathscr{U}, id, id, id)$ being the $\mathscr{C}$-monoidal functor, and this can be done in only one way.

Proof. We first show that the isomorphism $\delta: \mathscr{U}(M \hat{\otimes} N) \cong \mathscr{U}(M) \otimes \mathscr{U}(N)$ induces a unique B-structure on $M \otimes N$ for $M, N \in {}_B\mathscr{C}$, natural in both variables, such that $\delta: M \hat{\otimes} N \cong M \otimes N$ is a natural isomorphism of functors $\hat{\otimes}$ and $\otimes$ from ${}_B\mathscr{C} \times {}_B\mathscr{C} \to {}_B\mathscr{C}$. Define the B-structure by the commutative diagram

$$\begin{array}{ccc} B \otimes (M \otimes N) & \xrightarrow{\gamma_{M \otimes N}} & M \otimes N \\ \| \wr \; B \otimes \delta & & \| \wr \; \delta \\ B \otimes \mathscr{U}(M \hat{\otimes} N) & \xrightarrow{\gamma_{M \hat{\otimes} N}} & \mathscr{U}(M \hat{\otimes} N) \end{array}$$

where we use $\mathscr{U}(M) = M$ and $\mathscr{U}(N) = N$. This defines clearly a B-structure on $M \otimes N$; it is natural in M and N in ${}_B\mathscr{C}$ and δ becomes the desired isomorphism. Clearly $\gamma_{M \otimes N}$ is the only morphism making δ a natural isomorphism of functors to ${}_B\mathscr{C}$. Similarly I carries a B-structure uniquely such that $\zeta: \hat{I} \cong I$ is an isomorphism in ${}_B\mathscr{C}$.

Since $\hat{\alpha}, \hat{\lambda}, \hat{\rho}$ are natural isomorphisms in ${}_B\mathscr{C}$ and by the commutativity of the coherence diagrams for monoidal functors $\mathscr{U}$, α, λ and ρ will also be natural transformations in ${}_B\mathscr{C}$, $({}_B\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ is again a monoidal category. ${}_B\mathscr{C}$ is also a $\mathscr{C}$-category with

$$(\beta: M \otimes (X \otimes Y) \cong (M \otimes X) \otimes Y) := \alpha,$$

$$(\sigma: M \otimes I \cong M) := \rho.$$

The natural isomorphisms

$$(\xi_L\colon (M \otimes X) \otimes N \cong (M \otimes N) \otimes X) := \alpha(M \otimes \gamma)\,\alpha^{-1},$$
$$(\xi_R\colon M \otimes (N \otimes X) \cong (M \otimes N) \otimes X) := \alpha$$

make ${}_B\mathscr{C}$ a $\mathscr{C}$-monoidal category.

Consider the functor Id: ${}_B\mathscr{C} \to {}_B\mathscr{C}$, the first copy of ${}_B\mathscr{C}$ carrying the $\mathscr{C}$-monoidal structure $\hat{\otimes}$, the second copy with the new tensor product $\otimes$. Then

$$\delta\colon M \hat{\otimes} N \cong M \otimes N$$

is a natural isomorphism by definition. $\zeta\colon \hat{I} \cong I$ is a B-isomorphism. Furthermore $(\xi\colon M \otimes X \cong M \otimes X) = id$ makes Id a $\mathscr{C}$-functor. It is now easy to check that Id is a $\mathscr{C}$-monoidal functor, since $(\mathscr{U}, \delta, \zeta, \xi)$ was.

Now consider $\mathscr{U}\colon {}_B\mathscr{C} \to \mathscr{C}$, where ${}_B\mathscr{C}$ carries the new $\mathscr{C}$-monoidal structure $\otimes$. Then $(\delta\colon \mathscr{U}(M \otimes N) \cong \mathscr{U}(M) \otimes \mathscr{U}(N)) := id$, $(\zeta\colon \mathscr{U}(I) \cong I) := id$ and $(\xi\colon \mathscr{U}(M \otimes X) \cong \mathscr{U}(M) \otimes X := id$ form a $\mathscr{C}$-monoidal functor.

The fact that $\delta = id$ for the new $\mathscr{C}$-monoidal structure requires that the new tensor product be $\otimes$ with a suitable B-structure. This B-structure is unique by the requirement that (Id, δ, ζ, ξ) be a $\mathscr{C}$-monoidal functor, in particular that δ be a B-isomorphism. Similarly the requirement $(\zeta\colon \mathscr{U}(I) \cong I) = id$ implies I with a unique B-structure as the only possible neutral object in ${}_B\mathscr{C}$. α, λ, ρ are imposed by the fact that $(\mathscr{U}, id, id)$ be monoidal. The $\mathscr{C}$-structure on ${}_B\mathscr{C}$ was to be retained anyway. Finally, ξ_L and ξ_R on ${}_B\mathscr{C}$ and $\mathscr{C}$ have to be the same morphisms.

PROPOSITION 6. *Let B be a monoid. Let $({}_B\mathscr{C}, \otimes, I, \alpha, \lambda, \rho, \otimes, \beta, \sigma, \xi_L, \xi_R)$ be a $\mathscr{C}$-monoidal category such that $(\mathscr{U}, id, id, id)\colon {}_B\mathscr{C} \to \mathscr{C}$ is a $\mathscr{C}$-monoidal functor. Then there is a unique bimonoid structure on B which induces the $\mathscr{C}$-monoidal structures on ${}_B\mathscr{C}$ and $\mathscr{U}$ as described in the beginning of this section.*

Proof. Observe that by Proposition 5 $(\mathscr{U}, id, id, id)$ implies that the tensor product on ${}_B\mathscr{C}$ has to be $\otimes$ with a suitable B-structure and that α, λ, ρ, β, σ, ξ_L and ξ_R coincide in ${}_B\mathscr{C}$ and $\mathscr{C}$. Henceforth we shall omit these structure maps and say ${}_B\mathscr{C}$ is $\mathscr{C}$-monoidal with $\mathscr{U}\colon {}_B\mathscr{C} \to \mathscr{C}$ $\mathscr{C}$-monoidal.

Now define

$$\varepsilon := (B \cong B \otimes I \xrightarrow{\gamma_I} I) \qquad \text{or} \qquad \varepsilon(b) = b \cdot 1_I = b \cdot 1,$$

$$\Delta := (B \cong B \otimes (I \otimes I) \xrightarrow{B \otimes (\eta \otimes I)} B \otimes (B \otimes B) \xrightarrow{\gamma_{B \otimes B}} B \otimes B)),$$

or

$$\Delta(b) = b \cdot (1 \otimes 1) =: b_{(1)} \otimes b_{(2)} \qquad \text{for all } b \in B(X).$$

LEMMA 7. *The B-structure $\gamma_{M \otimes N}$ on $M \otimes N$ is given by*

$$B \otimes (M \otimes N) \xrightarrow{\Delta \otimes (M \otimes N)} (B \otimes B) \otimes (M \otimes N)$$
$$\cong (B \otimes M) \otimes (B \otimes N) \xrightarrow{\gamma_B \otimes \gamma_N} M \otimes N$$

or

$$b \cdot (m \otimes n) = b_{(1)} \cdot m \otimes b_{(2)} \cdot n.$$

Proof. The diagram

$$\begin{array}{ccc} B \otimes B \otimes M & \xrightarrow{\mu \otimes M} & B \otimes M \\ \downarrow{\scriptstyle B \otimes \gamma_M} & & \downarrow{\scriptstyle \gamma_M} \\ B \otimes M & \xrightarrow{\gamma_M} & M \end{array}$$

commutes; hence γ_M is a B-morphism where $B \otimes M$ carries the B-structure just on the left factor via $\mu: B \otimes B \to B$. γ_N is a B-morphism, too; hence $\gamma_M \otimes \gamma_N$ is a B-morphism and the following commutes in $\mathscr{C}$:

$$\begin{array}{ccccc} B \otimes (B \otimes B)) \otimes (M \otimes N) \cong & B \otimes ((B \otimes M) \otimes (B \otimes N)) & \longrightarrow & B \otimes (M \otimes N) \\ \downarrow{\scriptstyle \gamma_{B \otimes B} \otimes (M \otimes N)} & \downarrow{\scriptstyle \gamma_{(B \otimes M) \otimes (B \otimes N)}} & & \downarrow{\scriptstyle \gamma_{M \otimes N}} \\ (B \otimes B) \otimes (M \otimes N) \quad \cong & (B \otimes M) \otimes (B \otimes N) & \longrightarrow & M \otimes N \end{array}$$

where the horizontal arrows are $B \otimes (\gamma_M \otimes \gamma_N)$, resp. $\gamma_M \otimes \gamma_N$.

Elementwise we get $a \cdot (b \cdot m \otimes c \cdot n) = (a \cdot (b \otimes c)) \cdot (m \otimes n)$ for all $a \in B(X)$, $b \in B(Y)$, $c \in C(Z)$, $m \in M(U)$, $n \in N(V)$, where $(b \otimes c) \cdot (m \otimes n) = b \cdot m \otimes c \cdot n$. Now $b \cdot (m \otimes n) = b \cdot (1 \cdot m \otimes 1 \cdot n) = (b \cdot (1 \otimes 1)) \cdot (m \otimes n) = (b_{(1)} \otimes b_{(2)}) \cdot (m \otimes n) = b_{(1)} \cdot m \otimes b_{(2)} \cdot n$.

LEMMA 8. *$\Delta: B \to B \otimes B$ is a monoid homomorphism.*

Proof. $\Delta(a \cdot b) = (a \cdot b)_{(1)} \otimes (a \cdot b)_{(2)} = (a \cdot b) \cdot (1 \otimes 1) = a \cdot (b \cdot (1 \otimes 1)) = a \cdot (b_{(1)} \cdot 1 \otimes b_{(2)} \cdot 1) = a \cdot (b_{(1)} \otimes b_{(2)}) = a_{(1)} \cdot b_{(1)} \otimes a_{(2)} \cdot b_{(2)} = \Delta(a) \cdot \Delta(b)$.

$$\Delta(1) = 1 \cdot (1 \otimes 1) = 1 \otimes 1.$$

LEMMA 9. *$\varepsilon: A \to I$ is a monoid homomorphism.*

Proof. $\varepsilon(a \cdot b) = (a \cdot b) \cdot 1 = a \cdot (b \cdot 1) = a \cdot \varepsilon(b) = a \cdot (1 \cdot \varepsilon(b)) =^{(*)} (a \cdot 1) \cdot \varepsilon(b) = \varepsilon(a) \cdot \varepsilon(b)$, where (*) holds, since any multiplication with $x \in I(X)$ can be pulled by any morphism in $\mathscr{C}$.

$$\varepsilon(1_B) = 1_B \cdot 1 = 1.$$

LEMMA 10. *Δ is coassociative.*

Proof. We use the fact that α is a B-morphism; hence

$$\begin{aligned}
\alpha(1 \otimes \Delta)\Delta(b) &= \alpha(b_{(1)} \otimes (b_{(2)(1)} \otimes b_{(2)(2)})) \\
&= \alpha(b_{(1)} \cdot 1 \otimes b_{(2)} \cdot (1 \otimes 1)) \\
&= \alpha(b \cdot (1 \otimes (1 \otimes 1))) \\
&= b \cdot \alpha(1 \otimes (1 \otimes 1)) \\
&= b \cdot ((1 \otimes 1) \otimes 1) \\
&= ((b_{(1)(1)} \otimes b_{(1)(2)}) \otimes b_{(2)}) \\
&= (\Delta \otimes 1)\Delta(b).
\end{aligned}$$

LEMMA 11. *(B, Δ, ε) is a comonoid.*

Proof. Since λ and ρ a B-morphisms, we get

$$\begin{aligned}
b &= b \cdot 1_B = b \cdot \lambda(1 \otimes 1_B) = \lambda(b \cdot (1 \otimes 1_B)) \\
&= \lambda(b_{(1)} \cdot 1 \otimes b_{(2)} \cdot 1_B) = \lambda(\varepsilon(b_{(1)}) \otimes b_{(2)}) \\
&= \varepsilon(b_{(1)})\, b_{(2)} = (\varepsilon \otimes 1)\Delta(b), \\
b &= b \cdot 1_B = b \cdot \rho(1_B \otimes 1) = \rho(b \cdot (1_B \otimes 1)) \\
&= \rho(b_{(1)} \cdot 1_B \otimes b_{(2)} \cdot 1) = \rho(b_{(1)} \otimes \varepsilon(b_{(2)})) \\
&= b_{(1)} \cdot \varepsilon(b_{(2)}) = (1 \otimes \varepsilon)\Delta(b).
\end{aligned}$$

Thus we have proved that B is a bimonoid in $\mathscr{C}$ and that the $\mathscr{C}$-monoidal structure on ${}_B\mathscr{C}$ is induced by the bimonoid structure of B, i.e.,

$$b \cdot (m \otimes n) = b_{(1)} \cdot m \otimes b_{(2)} \cdot n, \qquad b \in B(X), \quad m \otimes n \in (M \otimes N)(Y), \tag{1}$$

$$b \cdot x = \varepsilon(b)\, x, \qquad b \in B(X), \quad x \in I(Y). \tag{2}$$

Now Δ is unique with (1), just take $m \otimes n := 1_B \otimes 1_B$. The uniqueness of ε

for a comonoid is shown in the same way as the uniqueness of the unit $\eta = 1$ in a monoid: $1' = 1' \cdot 1 = 1$.

Thus far we have reduced any given $\mathscr{C}$-monoidal structures on ${}_B\mathscr{C}$ and $\mathscr{U}$ (with standard $\mathscr{C}$-structure) in a unique way to $\mathscr{C}$-monoidal structures induced by a bimonoid structure on B. Now we want to do the same reduction for a $\mathscr{C}$-monoidal functor $\mathscr{F}: {}_B\mathscr{C} \to {}_C\mathscr{C}$ with a $\mathscr{C}$-monoidal isomorphism $\varphi: \mathscr{U}\mathscr{F} \cong \mathscr{U}$.

PROPOSITION 12. *Let* $({}_B\mathscr{C}, \hat{\otimes})$ *and* $({}_C\mathscr{C}, \tilde{\otimes})$ *be* $\mathscr{C}$*-monoidal categories and* $(\mathscr{U}, \delta_{\mathscr{U}}, \zeta_{\mathscr{U}}, \xi_{\mathscr{U}}): {}_B\mathscr{C} \to \mathscr{C}$ *and* $(\mathscr{V}, \delta_{\mathscr{V}}, \zeta_{\mathscr{V}}, \xi_{\mathscr{V}}): {}_C\mathscr{C} \to \mathscr{C}$ *be* $\mathscr{C}$*-monoidal functors. Let* $(\mathscr{F}, \delta_{\mathscr{F}}, \zeta_{\mathscr{F}}, \xi_{\mathscr{F}}): {}_B\mathscr{C} \to {}_C\mathscr{C}$ *be a* $\mathscr{C}$*-monoidal functor. Let* $({}_B\mathscr{C}, \otimes)$ *and* $({}_C\mathscr{C}, \otimes)$ *be the* $\mathscr{C}$*-monoidal categories with their structures induced by the bimonoid structures on B and C. Then there is a unique* $\mathscr{C}$*-monoidal functor* $(\mathscr{F}', \delta', \zeta', \xi')$ *which makes the diagram*

$$\begin{array}{ccc} ({}_B\mathscr{C}, \hat{\otimes}) & \xrightarrow{\mathscr{F}} & ({}_C\mathscr{C}, \tilde{\otimes}) \\ \downarrow {\scriptstyle (Id, \delta_{\mathscr{U}}, \zeta_{\mathscr{U}}, \xi_{\mathscr{U}})} & & \downarrow {\scriptstyle (Id, \delta_{\mathscr{V}}, \zeta_{\mathscr{V}}, \xi_{\mathscr{V}})} \\ ({}_B\mathscr{C}, \otimes) & \xrightarrow{\mathscr{F}'} & ({}_C\mathscr{C}, \otimes) \end{array}$$

commutative.

Proof. Since $(Id, \delta_{\mathscr{U}}, \zeta_{\mathscr{U}}, \xi_{\mathscr{U}})$ is invertible as a $\mathscr{C}$-monoidal functor, $\mathscr{F}'$ is to be the composition of $\mathscr{C}$-monoidal functors; hence $\mathscr{F}' = (\mathscr{F}, \delta_{\mathscr{V}} \cdot \delta_{\mathscr{F}} \delta_{\mathscr{U}}^{-1}, \zeta_{\mathscr{V}} \cdot \zeta_{\mathscr{F}} \zeta_{\mathscr{U}}^{-1}, \xi_{\mathscr{V}} \cdot \xi_{\mathscr{F}} \xi_{\mathscr{U}}^{-1})$.

COROLLARY 13. *Under the hypotheses of Proposition* 12 *let* $\varphi: \mathscr{V}\mathscr{F} \cong \mathscr{U}$ *be a* $\mathscr{C}$*-monoidal isomorphism. Then* $\varphi: \mathscr{V}\mathscr{F}' \cong \mathscr{U}$ *is also* $\mathscr{C}$*-monoidal.*

Proof. The first isomorphism is meant to be

$$\varphi: (\mathscr{V}, \delta_{\mathscr{V}}, \zeta_{\mathscr{V}}, \xi_{\mathscr{V}}) \circ (\mathscr{F}, \delta_{\mathscr{F}}, \zeta_{\mathscr{F}}, \xi_{\mathscr{F}}) \cong (\mathscr{U}, (\delta_{\mathscr{U}}, \zeta_{\mathscr{U}}, \xi_{\mathscr{U}});$$

the second is

$$\varphi: (\mathscr{V}, id, id, id) \circ (\mathscr{F}, \delta_{\mathscr{V}} \cdot \delta_{\mathscr{F}} \mathscr{F}(\delta_{\mathscr{U}}^{-1}), \zeta_{\mathscr{V}} \cdot \zeta_{\mathscr{F}} \mathscr{F}(\zeta_{\mathscr{U}}^{-1}), \xi_{\mathscr{V}} \cdot \xi_{\mathscr{F}} \mathscr{F}(\xi_{\mathscr{U}}^{-1}))$$
$$\cong (\mathscr{U}, id, id, id).$$

Since we do not change the $\mathscr{C}$-structure, we only have to check that φ respects the change of monoidal structures:

$$\begin{array}{ccccc} \mathscr{V}\mathscr{F}(M \hat{\otimes} N) & \xrightarrow{\mathscr{V}(\delta_{\mathscr{F}})} & \mathscr{V}(\mathscr{F}(M) \tilde{\otimes} \mathscr{F}(N)) & \xrightarrow{\delta_{\mathscr{V}}} & \mathscr{V}\mathscr{F}(M) \otimes \mathscr{V}\mathscr{F}(N) \\ \downarrow \varphi & & & & \downarrow \varphi \\ \mathscr{U}(M \hat{\otimes} N) & & \xrightarrow{\delta_{\mathscr{U}}} & & \mathscr{U}(M) \otimes \mathscr{U}(N) \end{array}$$

commutes; hence

$$\begin{array}{ccccc} \mathscr{V}\mathscr{F}(M\otimes N) & \xrightarrow{\mathscr{F}(\delta_{\mathscr{U}}^{-1})} & \mathscr{V}\mathscr{F}(M\hat{\otimes} N) & \xrightarrow{\delta_{\mathscr{V}}\cdot\delta_{\mathscr{F}}} & \mathscr{V}\mathscr{F}(M)\otimes\mathscr{V}\mathscr{F}(N)\\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi}\\ \mathscr{U}(M\otimes N) & \xrightarrow{\delta_{\mathscr{U}}^{-1}} & \mathscr{U}(M\hat{\otimes} N) & \xrightarrow{\delta_{\mathscr{U}}} & \mathscr{U}(M)\otimes\mathscr{U}(N) \end{array}$$

commutes, where we omit the application of the underlying functors $\mathscr{U}$, $\mathscr{V}$. Furthermore

$$\begin{array}{ccc} \mathscr{U}\mathscr{F}(I) & \xrightarrow{\zeta_{\mathscr{F}}} & \mathscr{V}(\hat{I})\\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\zeta_{\mathscr{V}}}\\ \mathscr{U}(\hat{I}) & \xrightarrow{\zeta_{\mathscr{U}}} & I \end{array}$$

commutes; hence

$$\begin{array}{ccccc} \mathscr{V}\mathscr{F}(I) & \xrightarrow{\mathscr{F}(\zeta_{\mathscr{U}}^{-1})} & \mathscr{V}\mathscr{F}(\hat{I}) & \xrightarrow{\zeta_{\mathscr{F}}} & \mathscr{V}(\hat{I})\\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\zeta_{\mathscr{V}}}\\ \mathscr{U}(I) & \xrightarrow{\zeta_{\mathscr{U}}^{-1}} & \mathscr{U}(\hat{I}) & \xrightarrow{\zeta_{\mathscr{U}}} & I \end{array}$$

commutes. We have now reduced the general $\mathscr{C}$-monoidal situation

$$\begin{array}{ccc} {}_B\mathscr{C} & \xrightarrow{\mathscr{F}} & {}_C\mathscr{C}\\ & {\scriptstyle\mathscr{U}}\searrow \quad \swarrow{\scriptstyle\mathscr{V}} & \\ & \mathscr{C} & \end{array}$$

with $\varphi: \mathscr{V}\mathscr{F} \cong \mathscr{U}$ to the special situation, where ${}_B\mathscr{C}$ and ${}_C\mathscr{C}$ carry $\mathscr{C}$-monoidal structures induced by bimonoids B and C. Now we want to change $\mathscr{F}$ to an isomorphic $\mathscr{C}$-monoidal functor $\mathscr{G}$ as in Proposition 1.

PROPOSITION 14. *Under the hypotheses of Corollary* 13 *the functor* $\mathscr{G}$ *induced in Proposition* 1 *is a* $\mathscr{C}$*-monoidal functor* $(\mathscr{G}, id, id, id)$: $({}_B\mathscr{C}, \otimes) \to ({}_C\mathscr{C}, \otimes)$ *and* $\varphi: \mathscr{F} \cong \mathscr{G}$ *and* $\mathscr{V}\mathscr{G} = \mathscr{U}$ *are* $\mathscr{C}$*-monoidal transformations.*

Proof. By Proposition 1 we know already that $\mathcal{G}$ is a $\mathcal{C}$-functor with $\xi_{\mathcal{G}} = (id\colon \mathcal{G}(M \otimes X) = \mathcal{G}(M) \otimes X)$. Furthermore the diagram

$$\begin{array}{ccccc}
\mathcal{V}\mathcal{G}(M \otimes N) & \xrightarrow{id} & \mathcal{V}(\mathcal{G}(M) \otimes \mathcal{G}(N)) & \xrightarrow{id} & \mathcal{V}\mathcal{G}(M) \otimes \mathcal{V}\mathcal{G}(N) \\
\| \wr\ \varphi^{-1} & (1) & \| \wr\ (\varphi \otimes \varphi)^{-1} & (2) & \| \wr\ (\varphi \otimes \varphi)^{-1} \\
\mathcal{V}\mathcal{F}'(M \otimes N) & \xrightarrow{\delta_{\mathcal{F}'}} & \mathcal{V}(\mathcal{F}'(M) \otimes \mathcal{F}'(N)) & \xrightarrow{id} & \mathcal{V}\mathcal{F}'(M) \otimes \mathcal{V}\mathcal{F}'(N) \\
\| \wr\ \varphi & & (3) & & \| \wr\ \varphi \otimes \varphi \\
\mathcal{U}(M \otimes N) & & \xrightarrow{\qquad id \qquad} & & \mathcal{U}(M) \otimes \mathcal{U}(N)
\end{array}$$

commutes at (3) because $\varphi\colon \mathcal{V}\mathcal{F}' \cong \mathcal{U}$ is $\mathcal{C}$-monoidal, and clearly at (2) and at (1) because the outer diagram commutes. So the only possible δ for $\mathcal{G}$ is $id\colon \mathcal{G}(M \otimes N) \to \mathcal{G}(M) \otimes \mathcal{G}(N)$ and it makes $\varphi\colon \mathcal{F}' \cong \mathcal{G}$ a monoidal transformation. $\mathcal{G}$ together with $id\colon \mathcal{G}(M \otimes N) \to \mathcal{G}(M) \otimes \mathcal{G}(N)$ clearly is monoidal because in both categories ${}_B\mathcal{C}$ and ${}_C\mathcal{C}$ we have the same morphisms α, λ, ρ and because ξ and δ are identities. The isomorphism $\zeta\colon \mathcal{G}(I) \cong I$ will also be the identity because

$$\begin{array}{ccc}
\mathcal{V}\mathcal{G}(I) & \xrightarrow{id} & I \\
\| \wr\ \varphi^{-1} & & \| \\
\mathcal{V}\mathcal{F}'(I) & \xrightarrow{\zeta_{\mathcal{F}'}} & I \\
\| \wr\ \varphi & & \| \\
\mathcal{U}(I) & = & I
\end{array}$$

commutes. Thus we have that $\mathcal{G}$ is a monoidal $\mathcal{C}$-functor with structure morphisms $(\delta, \zeta, \xi) = (id, id, id)$. Furthermore $\varphi\colon \mathcal{F}' \cong \mathcal{G}$ is a monoidal $\mathcal{C}$-transformation. $\mathcal{G}$ is $\mathcal{C}$-monoidal since in both categories ${}_B\mathcal{C}$ and ${}_C\mathcal{C}$ the morphisms ξ_R are just α and $\xi_L = \alpha(M \otimes \gamma)\,\alpha^{-1}$. Finally the identity $\mathcal{V}\mathcal{G} = \mathcal{U}$ is also $\mathcal{C}$-monoidal because all structure morphisms are identities.

THEOREM 15. *Let B and C be monoids in $\mathcal{C}$. Let the $\mathcal{C}$-categories ${}_B\mathcal{C}$ and ${}_C\mathcal{C}$ carry the structure of $\mathcal{C}$-monoidal categories such that the underlying $\mathcal{C}$-functors $\mathcal{U}\colon {}_B\mathcal{C} \to \mathcal{C}$ and $\mathcal{V}\colon {}_C\mathcal{C} \to \mathcal{C}$ are $\mathcal{C}$-monoidal. Let $\mathcal{F}\colon {}_B\mathcal{C} \to {}_C\mathcal{C}$ be a $\mathcal{C}$-monoidal functor and $\varphi\colon \mathcal{V}\mathcal{F} \cong \mathcal{U}$ be a $\mathcal{C}$-monoidal natural isomorphism. Then there are unique bimonoid structures on B and C and a unique bimonoid morphism $g\colon C \to B$ such that the induced $\mathcal{C}$-monoidal structures on ${}_B\mathcal{C}$, resp. ${}_C\mathcal{C}$, are isomorphic to the original ones by the identity functor and the induced $\mathcal{C}$-monoidal functor $\mathcal{G}\colon {}_B\mathcal{C} \to {}_C\mathcal{C}$ is $\mathcal{C}$-monoidally isomorphic to $\mathcal{F}$ via φ.*

Proof. By Propositions 1 and 2 there is a unique monoid morphism $g: C \to B$ such that the induced $\mathscr{C}$-functor $\mathscr{G}: {}_B\mathscr{C} \to {}_C\mathscr{C}$ satisfies $\mathscr{V}\mathscr{G} = \mathscr{U}$ and $\varphi: \mathscr{F} \cong \mathscr{G}$ a $\mathscr{C}$-isomorphism. By Propositions 5 and 6 the $\mathscr{C}$-monoidal structure of ${}_B\mathscr{C}$, resp. ${}_C\mathscr{C}$, is isomorphic to a $\mathscr{C}$-monoidal structure induced by a unique bimonoid structure on B, resp. C, via the identity functor and by Corollary 13 the given functor $\mathscr{F}: {}_B\mathscr{C} \to {}_C\mathscr{C}$ and isomorphism $\varphi: \mathscr{V}\mathscr{F} \cong \mathscr{U}$ are $\mathscr{C}$-monoidal in a unique way also with respect to the bimonoid-induced $\mathscr{C}$-monoidal structures on ${}_B\mathscr{C}$, resp. ${}_C\mathscr{C}$. By Proposition 14 the $\mathscr{C}$-functor $\mathscr{G}: {}_B\mathscr{C} \to {}_C\mathscr{C}$ induced by the monoid morphism $g: {}_B\mathscr{C} \to {}_C\mathscr{C}$ is $\mathscr{C}$--monoidal: $(\mathscr{G}, id, id, id)$ and $\varphi: \mathscr{F} \cong \mathscr{G}$ is also $\mathscr{C}$-monoidal. Also $\mathscr{V}\mathscr{G} = \mathscr{U}$ is already a $\mathscr{C}$-monoidal equality.

So the only thing to prove is that g is a bimonoid morphism. Now $\delta: \mathscr{G}(M \otimes N) \to \mathscr{G}(M) \otimes \mathscr{G}(N)$ is not only the identity but also a $\mathscr{C}$-morphism and so is $\zeta: \mathscr{G}(I) \to I$. Hence we get

$$\begin{aligned}\delta(\Delta_B g(c) \cdot m \otimes n) &= \delta(g(c) \cdot m \otimes n) = \delta(c \cdot m \otimes n)\\ &= c \cdot \delta(m \otimes n) = \Delta_c(c) \cdot \delta(m \otimes n)\\ &= (g \otimes g)\Delta_c(c) \cdot \delta(m \otimes n)\end{aligned}$$

and

$$\begin{aligned}\zeta(\varepsilon_B g(c)) &= \zeta(g(c) \cdot 1_I) = \zeta(c \cdot 1_I) = c \cdot \zeta(1_I)\\ &= \varepsilon_c(c) \cdot \zeta(1_I).\end{aligned}$$

Observe $\delta = id$ and $\zeta = id$ and set $m \otimes n = 1_B \otimes 1_B$ to get

$$\begin{aligned}\Delta_B g(c) &= \Delta_B g(c) \cdot 1_B \otimes 1_B = (g \otimes g)\Delta_c(c) \cdot 1_B \otimes 1_B\\ &= (g \otimes g)\Delta_c(c),\\ \varepsilon_B g(c) &= \varepsilon_c(c).\end{aligned}$$

COROLLARY 16. *Under the hypotheses of Theorem* 14 *let* $\mathscr{F}$ *be a* $\mathscr{C}$-*monoidal equivalence. Then* $g: C \to B$ *is a bimonoid isomorphism.*

Proof. This is simply a consequence of Proposition 4.

3

Let K be a commutative ring and let $\mathscr{C} = K$-Mod be the monoidal category of K-modules with the usual tensor product over K. Consider the category K-Comp of complexes of K-modules

$$\mathscr{A} = (\cdots \to A_i \xrightarrow{\partial_i} A_{i+1} \to \cdots), \qquad \partial^2 = 0$$

and complexes homomorphisms $(f: \mathfrak{A} \to \mathfrak{B}) = (f_i : A_i \to B_i \mid i \in \mathbf{Z}, \partial_i f_i = f_{i+1}\partial_i)$.

K-Comp is a $\mathscr{C}$-category where

$$\mathfrak{A} \otimes X := (\cdots \to A_i \otimes X \xrightarrow{\partial_i \otimes X} A_{i+1} \otimes X \to \cdots),$$
$$f \otimes X := (f_i \otimes X).$$

The isomorphism $\beta: \mathfrak{A} \otimes (X \otimes Y) \cong (\mathfrak{A} \otimes X) \otimes Y$ is induced by $\alpha: A_i \otimes (X \otimes Y) \cong (A_i \otimes X) \otimes Y$ and $\sigma: \mathfrak{A} \otimes K \cong \mathfrak{A}$ by $\rho: A_i \otimes K \cong A_i$. Clearly all these definitions are functorial in all variables and coherent (in the sense of [1] or [5]).

K-Comp is also monoidal with the usual tensor product of complexes (take tensor products separately of all components and then make the double complex into a single complex by adding diagonally with the usual sign shift). To be more precise

$$\mathfrak{A} \otimes \mathfrak{B} := \left(\cdots \bigoplus_{j+k=i} (A_j \otimes B_k) \xrightarrow{\delta_i} \bigoplus_{j+k=i} (A_i \otimes B_k) \to \cdots\right),$$

where

$$\delta_i = \bigoplus_{j+k=i} ((-1)^k \partial_j \otimes B_k + A_j \otimes \partial'_k)$$

with $\partial_j: A_j \to A_{j+1}$ and $\partial'_k: B_k \to B_{k+1}$. This is wellknown to be natural in both variables and associativity $\mathfrak{A} \otimes (\mathfrak{B} \otimes \mathscr{C}) \cong (\mathfrak{A} \otimes \mathfrak{B}) \otimes \mathscr{C}$ induced by α can easily be checked. The neutral element in K-Comp is

$$\mathscr{K} = (\cdots \to 0 \to K \to 0 \to \cdots)$$

with K at position zero. The isomorphisms $\mathfrak{A} \otimes \mathscr{K} \cong \mathfrak{A} \cong \mathscr{K} \otimes \mathfrak{A}$ are induced by λ and ρ and thus coherent with α. K-Comp is even $\mathscr{C}$-monoidal with structure morphisms.

$$\xi_L : (\mathfrak{A} \otimes X) \otimes \mathfrak{B} \cong (\mathfrak{A} \otimes \mathfrak{B}) \otimes X,$$
$$\xi_R : \mathfrak{A} \otimes (\mathfrak{B} \otimes X) \cong (\mathfrak{A} \otimes \mathfrak{B}) \otimes X$$

induced by those in K-Mod. Again coherence is clear from coherence in K-Mod.

Now consider the functor $\mathscr{V}: K\text{-Comp} \to K\text{-Mod}$ given by

$$\mathscr{V}(\mathfrak{A}) := \bigoplus A_i,$$

$$\mathscr{V}(f) := \bigoplus f_i.$$

It is a $\mathscr{C}$-functor by the natural isomorphism

$$\mathscr{V}^{\cdot}(\mathscr{A} \otimes X) = \oplus (A_i \otimes X) \cong \left(\oplus A_i\right) \otimes X = \mathscr{V}^{\cdot}(\mathscr{A}) \otimes X.$$

Furthermore, it is monoidal by the natural isomorphisms

$$\mathscr{V}^{\cdot}(\mathscr{A} \otimes \mathscr{B}) = \oplus_i \left(\bigoplus_{j+k=i} (A_j \otimes B_k)\right) \cong \left(\bigoplus_j A_j\right) \otimes \left(\bigoplus_k B_k\right)$$
$$= \mathscr{V}^{\cdot}(\mathscr{A}) \otimes \mathscr{V}^{\cdot}(\mathscr{B}),$$
$$\mathscr{V}^{\cdot}(\mathscr{K}) = \left(\oplus 0\right) \oplus K \oplus \left(\oplus 0\right) \cong K.$$

Actually $\mathscr{V}^{\cdot}$ is $\mathscr{C}$-monoidal as can be easily checked.

Thus K-Comp is a $\mathscr{C}$-monoidal category with $\mathscr{C} = K$-Mod and $\mathscr{V}^{\cdot}: K$-Comp $\rightarrow K$-Mod is a $\mathscr{C}$-monoidal functor.

Now we define a K-bialgebra B by

$$B = K\langle s, t, t^{-1}\rangle/(s^2, st + ts),$$

where $K\langle s, t, t^{-1}\rangle$ denotes adjoining two variables s, t, which do not commute with each other, but with all of K, and adjoining an inverse of t. We factor out the two-sided ideal generated by s^2 and $st + ts$. For the diagonal we take

$$\Delta(t) = t \otimes t, \qquad \Delta(s) = s \otimes 1 + t^{-1} \otimes s.$$

The augmentation is defined by

$$\varepsilon(t) = 1, \qquad \varepsilon(s) = 0.$$

LEMMA 17. *B is a bialgebra.*

Proof. B has obviously the K-basis $\{t^i \mid i \in \mathbf{Z}\} \cup \{t^i s \mid i \in \mathbf{Z}\}$. If Δ is to be multiplicative, $\Delta(t^i) = t^i \otimes t^i$ and $\Delta(t^i s) = t^i s \otimes t^i + t^{i-1} \otimes t^i s$ must hold. Then Δ can be expanded by linearity and it is trivial to see that Δ is an algebra morphism. Δ is associative because it is on t and s. $\varepsilon(t^i) = 1$ and $\varepsilon(t^i s) = 0$ defines again an algebra homomorphism and (B, Δ, ε) forms a coalgebra. Thus B is a bialgebra.

Observe now that the category B-Comod of B-comodules for any bialgebra B is $\mathscr{C}$-monoidal for $\mathscr{C} = K$-Mod essentially in the same way as it is $\mathscr{C}$-monoidal for $\mathscr{C} = (K\text{-mod})^{\text{op}}$ (${}_B\mathscr{C}$ in Section 2 was $\mathscr{C}$-monoidal). Also the underlying functor $\mathscr{U}: B$-Comod $\rightarrow K$-Mod is $\mathscr{C}$-monoidal.

THEOREM 18. *There is a $\mathscr{C}$-monoidal equivalence $\mathscr{F}$: B-Comod → K-Comp and a $\mathscr{C}$-monoidal isomorphism $\varphi: \mathscr{V}\mathscr{F} \cong \mathscr{U}$.*

COROLLARY 19. *The bialgebra B is uniquely determined up to isomorphism by Theorem* 17.

Proof of Corollary 19 is a simple application of Corollary 16.

Proof of Theorem 18. Let M be a B-comodule with structure map λ_M: $M \to B \otimes M$. With respect to the basis $\{t^i, st^i\}$ we can write

$$\lambda_M(m) = \sum_i t^i \otimes m_i + \sum_i t^i s \otimes m'_i. \tag{3}$$

Now apply $(1 \otimes \lambda)\lambda = (\Delta \otimes 1)\lambda$ to get

$$\sum t^i \otimes \lambda(m_i) + \sum t^i s \otimes \lambda(m'_i)$$

$$= \sum t^i \otimes t^i \otimes m_i + \sum t^i s \otimes t^i \otimes m'_i$$

$$+ \sum t^{i-1} \otimes t^i s \otimes m'_i;$$

hence by comparison of the coefficients

$$\lambda(m_i) = t^i \otimes m_i + t^{i+1} s \otimes m'_{i+1}, \tag{4}$$

$$\lambda(m'_i) = t^i \otimes m'_i. \tag{5}$$

If we apply $(\varepsilon \otimes 1)\lambda(m) = m$ to (3) we get

$$m = \sum m_i. \tag{6}$$

Now define $M_i := \{m \in M \mid \lambda(m) = t^i \otimes m + t^{i+1} s \otimes m'\}$ and $\partial: M_i \to M_{i+1}$ by $\partial(m) = m'$ in $\lambda(m) = t^i \otimes m + t^{i+1} s \otimes m'$. Clearly M_i is a K-module and ∂ is linear. To see that $m' \in M_{i+1}$, observe (4) and (5) which give $\lambda(m') = t^{i+1} \otimes m'$. Furthermore $\partial\partial(m) = 0$ for $m \in M_i$ again by (4) and (5). By (6) we get $M = \sum M_i$. Now if $\sum m_i = 0$ with $m_i \in M_i$, then $0 = \lambda(\sum_i m_i) = \sum_i t^i \otimes m_i + t^{i+1} s \otimes m'_i$ and hence $m_i = 0$. So $M = \bigoplus M_i$.

Thus $M \in B$-Comod defines a complex

$$\cdots \to M_i \xrightarrow{\partial} M_{i+1} \to \cdots$$

in K-Comp.

If $f\colon M \to N$ is a comodule morphism then for $m \in M_i$ we get

$$\lambda f(m) = (1 \otimes f)\,\lambda(m) = t^i \otimes f(m) \;+\; t^{i+1}s \otimes f(\partial m);$$

hence $f(m) \in N_i$, $f_i = f|_{M_i}\colon M_i \to N_i$ and $\partial f(m) = f\partial(m)$, so f defines a complex homomorphism. Altogether we have thus obtained a functor $\mathscr{F}\colon B$-Comod $\to K$-Comp.

For the underlying functors we have

$$\mathscr{V}\mathscr{F}(M) = \bigoplus M_i = \mathscr{U}(M);$$

$$\mathscr{V}\mathscr{F}(f) = \bigoplus f_i = \mathscr{U}(f);$$

hence $\mathscr{V}\mathscr{F} = \mathscr{U}$.

For the $\mathscr{C}$-structures we get

$$\begin{aligned} \mathscr{F}(M \otimes X) &= (\cdots \to (M \otimes X)_i \xrightarrow{\partial} (M \otimes X)_{i+1} \to \cdots) \\ &\cong (\cdots \to M_i \otimes X \xrightarrow{\partial \otimes X} M_{i+1} \otimes X \to \cdots) \\ &= \mathscr{F}(M) \otimes X. \end{aligned}$$

To see $M_i \otimes X \cong (M \otimes X)_i$ consider $M_i \otimes X \subseteq M \otimes X$ by $M \otimes X = \bigoplus (M_i \otimes X)$. Then $M_i \otimes X \subseteq (M \otimes X)_i$ and $\bigoplus (M_i \otimes X) = \bigoplus (M \otimes X)_i$; hence $M_i \otimes X = (M \otimes X)_i$ under this identification. The isomorphism $\xi\colon \mathscr{F}(M \otimes X) \cong \mathscr{F}(M) \otimes X$ is functorial and satisfies the coherence conditions. So $\mathscr{F}$ is a $\mathscr{C}$-functor. Also the identity $\mathscr{V}\mathscr{F} = \mathscr{U}$ is compatible with the $\mathscr{C}$-structure:

$$\mathscr{V}\mathscr{F}(M \otimes X) \cong \mathscr{V}(\mathscr{F}(M) \otimes X) \cong \mathscr{V}\mathscr{F}(M) \otimes X$$

is the identity; hence $\mathscr{V}\mathscr{F}$ and $\mathscr{U}$ are equal as $\mathscr{C}$-functors.

For the monoidal structures we get

$$\mathscr{F}(M \otimes N) = (\cdots \to (M \otimes N)_i \xrightarrow{\partial} (M \otimes N)_{i+1} \to \cdots).$$

To study $(M \otimes N)_i$ observe that every element in $M \otimes N$ can be written as a sum $\sum_{j,k} m_j \otimes n_k$ with $m_j \in M_j$, $n_k \in N_k$. Then $\lambda(\sum m_j \otimes n_k) = \sum(t^j \cdot t^k \otimes m_j \otimes n_k + t^j t^{k+1}s \otimes m_j \otimes \partial(n_k) + t^{j+1}st^k \otimes \partial(m_j) \otimes n_k + t^{j+1}st^{k+1}s \otimes \partial(m_j) \otimes \partial(n_k)) = \sum_i t^i \otimes (\sum_{j+k=i} m_j \otimes n_k) + \sum_i t^{i+1}s \otimes (\sum_{j+k=i} m_j \otimes \partial(n_k) + (-1)^k\, \partial(m_j) \otimes n_k)$. Hence $(M \otimes N)_i = \bigoplus_{i=j+k} (M_j \otimes N_k)$ and

$$\partial_{M \otimes N, i} = \bigoplus (M_j \otimes \partial'_k + (-1)^k\, \partial_j \otimes N_k),$$

i.e., $\delta: \mathscr{F}(M \otimes N) \cong \mathscr{F}(M) \otimes \mathscr{F}(N)$. Furthermore $\zeta: \mathscr{F}(K) = \mathscr{K}$. Both satisfy the coherence conditions, so $\mathscr{F}$ is a monoidal functor. One also checks easily that

$$\mathscr{V}\mathscr{F}(M \otimes N) \cong \mathscr{V}(\mathscr{F}(M) \otimes \mathscr{F}(N)) \cong \mathscr{V}\mathscr{F}(M) \otimes \mathscr{V}\mathscr{F}(N)$$

is the identity, so $\mathscr{V}\mathscr{F} = \mathscr{U}$ as monoidal functors.

Finally $\mathscr{F}$ is $\mathscr{C}$-monoidal since all the morphisms for coherence are naturally defined in K-mod and coherent there.

Now we construct an equivalence inverse for $\mathscr{F}$. Let $\mathcal{A} \in K$-Comp. We define $\mathscr{G}(\mathcal{A}) := \bigoplus A_i$. To get the comodule structure on $\bigoplus A_i$, define for $a_i \in A_i$

$$\lambda(a_i) := t^i \otimes a_i + t^{i+1}s \otimes \partial(a_i) \in B \otimes \left(\bigoplus A_i\right).$$

This defines a B-comodule structure on $\mathscr{G}(\mathcal{A})$ by easy computation. For a complex homomorphism f define $\mathscr{G}(f) := \bigoplus f_i$ and verify it is a comodule homomorphism. So $\mathscr{G}: K\text{-Comp} \to B\text{-Comod}$ is a functor. Then it is easy to check $\mathscr{G}\mathscr{F} \cong Id$ and $\mathscr{F}\mathscr{G} \cong Id$. It is tedious but straigthforward to check that $\mathscr{G}$ again is $\mathscr{C}$-monoidal and that the isomorphisms $\mathscr{G}\mathscr{F} \cong Id$ and $\mathscr{F}\mathscr{G} \cong Id$ are $\mathscr{C}$-monoidal, thus $\mathscr{F}$ is a $\mathscr{C}$-monoidal equivalence and $\mathscr{V}\mathscr{F} = \mathscr{U}$ as $\mathscr{C}$-monoidal functors.

COROLLARY 20. *The bialgebra B defined by Theorem* 18 *has an antipode of order* 4 (2 *in characteristic* 2).

Proof. The antipode S is given by $S(t) = t^{-1}$ and $S(s) = st$. Check that this indeed defines an antipode if continued as an algebra antimorphism. We have then $S^2(t) = t$ and $S^2(s) = t^{-1}S(s) = -s$ and $S^4 = id$.

We remark that there is an additional structure on both K-Comp and B-Comod. Both categories are symmetric. This is surprising since one should think that B must be commutative in this case. But the symmetry we shall describe does not coincide with the symmetry in K-Mod by the underlying functor.

The symmetry in K-Comp is given by $M_i \otimes N_j \cong N_j \otimes M_i$, $m_i \otimes n_j \mapsto (-1)^{ij} n_j \otimes m_i$. In B-Comod the symmetry can be described in this way. Define a linear map $\Psi: B \otimes B \to K$ by $t^i \otimes t^j \mapsto (-1)^{ij}$ and $\Psi(t^i s \otimes t^j) = \Psi(t^i s \otimes t^j s) = \Psi(t^i \otimes t^j s) = 0$. We shall not investigate its meaning for B, but in a certain sense it is induced by the multiplication on $\mathbf{Z}$ through $\mathbf{Z} \ni i \mapsto t^i \in B$. Then the symmetry $\gamma: M \otimes N \cong N \otimes M$ is given by $\gamma(m \otimes n) = \sum \Psi(m_{(0)} \otimes n_{(0)}) \cdot n_{(1)} \otimes m_{(1)}$. This is a comodule map with $\gamma^2 = id$, functorial and coherent in B-Comod and the functor $\mathscr{F}: K\text{-Comp} \to B\text{-Comod}$ is compatible with the two symmetries.

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