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## When Hopf Algebras Are Frobenius Algebras

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R. Larson and M. Sweedler recently proved that for free finitely generated Hopf algebras  $H$  over a principal ideal domain  $R$  the following are equivalent: (a)  $H$  has an antipode and (b)  $H$  has a nonsingular left integral. In this paper I give a generalization of this result which needs only a minor restriction, which, for example, always holds if  $\text{pic}(R) = 0$  for the base ring  $R$ . A finitely generated projective Hopf algebra  $H$  over  $R$  has an antipode if and only if  $H$  is a Frobenius algebra with a Frobenius homomorphism  $\psi$  such that  $\sum h_{(1)} \psi(h_{(2)}) = \psi(h) \cdot 1$  for all  $h \in H$ . We also show that the antipode is bijective and that the ideal of left integrals is a free rank 1,  $R$ -direct summand of  $H$ .

1. In Ref. [4], Larson and Sweedler proved the equivalence for a finite-dimensional Hopf algebra over a principal ideal domain to have a (necessarily unique) antipode and to have a nonsingular left integral. It is easy to see that this result implies that a finite-dimensional Hopf algebra over a principal ideal domain is a Frobenius algebra, which generalizes the well-known fact that a group ring of a finite group is Frobenius as well as the result of Berkson [1], that the restricted universal enveloping algebra of a finite-dimensional restricted Lie algebra is Frobenius. This result has consequences with respect to a cohomology theory of Hopf algebras which will be exhibited in a subsequent paper.

In this paper we want to generalize the main result of [4] to arbitrary commutative rings  $R$  and finitely generated projective Hopf algebras  $H$ . We need only a slight restriction on  $H$  or on  $R$ , viz.,  $\text{pic}(R) = 0$  to get the equivalence between the existence of an antipode and the fact that  $H$  is a Frobenius algebra with a Frobenius homomorphism  $\psi$ , such that  $\sum_{(h)} h_{(1)} \psi(h_{(2)}) = \psi(h) \cdot 1$  for all  $h \in H$ , where  $\sum_{(h)} h_{(1)} \otimes h_{(2)} = \Delta(h)$  is the Sweedler notation. We do not know whether the imposed restrictions on  $R$  or  $H$  are necessary for the above result.<sup>1</sup> In this context we also prove that the

<sup>1</sup> See footnote at the end.

antipode of a finitely generated projective Hopf algebra is bijective. This holds without any further restrictions.

Since integrals are also of interest in this general situation, we shall prove that in a Hopf algebra  $H$  which is Frobenius with a Frobenius homomorphism  $\psi$  such that  $\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h) \cdot 1$  for all  $h \in H$  the two-sided  $H$  ideal of left integrals is an  $R$ -free rank 1  $R$ -direct summand of  $H$ .

**2.** Let  $R$  be a commutative ring (associative with unit). All modules are assumed to be unitary  $R$  modules. All algebras are assumed to be associative  $R$  algebras with unit.

A *coalgebra*  $C$  is a module  $C$  together with homomorphisms  $\Delta : C \rightarrow C \otimes C$  (the tensor product is taken over  $R$ ), the diagonal, and  $\epsilon : C \rightarrow R$ , the counit or augmentation, such that

$$\begin{array}{ccc}
 C & \xrightarrow{\Delta} & C \otimes C \\
 \Delta \downarrow & & \downarrow \Delta \otimes 1 \\
 C \otimes C & \xrightarrow{1 \otimes \Delta} & C \otimes C \otimes C
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 C & \xrightarrow{\Delta} & C \otimes C \\
 \Delta \downarrow & \searrow 1 & \downarrow 1 \otimes \epsilon \\
 C \otimes C & \xrightarrow{\epsilon \otimes 1} & C
 \end{array}$$

commute where we identify  $C \otimes R$ ,  $C$ , and  $R \otimes C$ . We adopt the Sweedler notation  $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$  as explained in [7, p. 10].

A *Hopf algebra*  $H$  is an algebra  $H$  with structure maps  $\mu : H \otimes H \rightarrow H$  and  $\eta : R \rightarrow H$ , which is also a coalgebra with structure maps  $\Delta : H \rightarrow H \otimes H$  and  $\epsilon : H \rightarrow R$  such that  $\Delta$  and  $\epsilon$  are algebra homomorphisms. As in [7, Proposition 3.1.1] one shows that  $\mu$  and  $\eta$  are coalgebra homomorphisms.

Let  $C$  be a coalgebra. A  *$C$  right comodule* is a module  $M$  together with a homomorphism  $\chi : M \rightarrow M \otimes C$  such that

$$\begin{array}{ccc}
 M & \xrightarrow{\chi} & M \otimes C \\
 \chi \downarrow & & \downarrow 1 \otimes \Delta \\
 M \otimes C & \xrightarrow{\chi \otimes 1} & M \otimes C \otimes C
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 M & \xrightarrow{\chi} & M \otimes C \\
 & \searrow 1 & \downarrow 1 \otimes \epsilon \\
 & & M
 \end{array}$$

commute, where we identify  $M$  and  $M \otimes R$ . Here again we use the Sweedler notation  $\chi(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$ . Observe that the  $m_{(i)}$ 's for  $i \geq 1$  are elements in  $C$ , whereas the  $m_{(0)}$ 's are in  $M$ .

Let  $H$  be a Hopf algebra. An  *$H$  right Hopf module* is an  $H$  right module  $M$  which is also an  $H$  right comodule such that

$$\chi(mh) = \sum_{(m), (h)} m_{(0)}h_{(1)} \otimes m_{(1)}h_{(2)}.$$

Let  $H$  be a Hopf algebra. Then  $\text{hom}_R(H, H)$  is an associative  $R$  algebra with unit  $\eta\epsilon$ , if we define the multiplication by  $f * g := \mu(f \otimes g)\Delta$ , i.e.,  $f * g(h) = \sum_{(h)} f(h_{(1)})g(h_{(2)})$  [7, p. 70, exercise 1]. The *antipode*  $S$  of  $H$  is (if it exists) the two-sided inverse of the identity  $1_H$  of  $H$  in  $\text{hom}_R(H, H)$ .

**3. LEMMA 1.** *Let  $A, B$ , and  $C$  be  $R$  modules. If  $C$  is finitely generated and projective, then  $\lambda : \text{hom}(A, B \otimes C) \rightarrow \text{hom}(C^* \otimes A, B)$  with  $\lambda(f)(c^* \otimes a) := (1 \otimes c^*)f(a)$  is an isomorphism.*

*Proof.*  $\lambda$  defines a natural transformation

$$\text{hom}(A, B \otimes -) \rightarrow \text{hom}(- * \otimes A, B),$$

which is an isomorphism for  $C = R$ . By [6, 4.11. Lemma 2] the above lemma holds.

**PROPOSITION 1.** *Let  $C$  be a finitely generated projective  $R$  coalgebra and  $M$  be an  $R$  module. Then  $\chi : M \rightarrow M \otimes C$  defines a  $C$  right comodule if and only if  $\lambda(\chi) : C^* \otimes M \rightarrow M$  defines a  $C^*$  left module, where  $\lambda$  is defined as in Lemma 1.*

*Proof.*  $C^*$  is an  $R$  algebra by  $c^*d^*(c) = \sum_{(c)} c^*(c_{(1)})d^*(c_{(2)})$ . Assume  $\chi : M \rightarrow M \otimes C$  defines a  $C$  right comodule. Then

$$\begin{aligned} (c_1^*c_2^*)m &= \sum_{(m)} m_{(0)}(c_1^*c_2^*(m_{(1)})) \\ &= \sum_{(m)} m_{(0)}(c_1^*(m_{(1)})c_2^*(m_{(2)})) \\ &= \sum_{(m)} c_1^*m_{(0)}(c_2^*(m_{(1)})) \\ &= c_1^*(c_2^*m) \end{aligned}$$

and

$$\epsilon m = \sum_{(m)} m_{(0)}\epsilon(m_{(1)}) = m,$$

so  $M$  is a  $C^*$  left module.

Now let  $M$  be a  $C^*$  left module. The natural transformation

$$\rho : M \otimes C \otimes D \rightarrow \text{hom}(C^* \otimes D^*, M)$$

defined by  $\rho(m \otimes c \otimes d)(c^* \otimes d^*) = c^*(c)d^*(d)m$  is an isomorphism for finitely generated projective modules  $C$  and  $D$ , since it is an isomorphism for  $C = D = R$  [6, 4.11. Lemma 2]. So  $M \otimes C \otimes C \cong \text{hom}(C^* \otimes C^*, M)$  for our finitely generated projective coalgebra  $C$ .



Let  $x = ((\chi \otimes 1)\chi - (1 \otimes \Delta)\chi)(m) \in M \otimes C \otimes C$ . Then  $\rho(x)(c_1^* \otimes c_2^*) = c_1^*(c_2^*m) - (c_1^*c_2^*)m = 0$  for all  $c_1^*, c_2^* \in C^*$ , so  $x = 0$ , i.e.,  $\chi$  is coassociative. Furthermore, we have  $(1 \otimes \epsilon)\chi(m) = \sum_{(m)} m_{(0)}\epsilon(m_{(1)}) = \epsilon m = m$ , since  $\epsilon$  is the unit element in  $C^*$ . So  $M$  is a  $C$  right comodule.

**PROPOSITION 2.** *Let  $H$  be a finitely generated projective Hopf algebra with antipode  $S$ . Then  $H^*$  is an  $H$  right Hopf module.*

*Proof.*  $H^*$  is an  $H^*$  left module, so it is an  $H$  right comodule. We have for  $g^*, h^* \in H^*$ , and  $h \in H$  and the comodule map  $\chi : H^* \rightarrow H^* \otimes H$  with  $\chi(h^*) = \sum_{(h)} h_{(0)}^* \otimes h_{(1)}^*$ .

$$g^*h^* = \sum_{(h^*)} h_{(0)}^*g^*(h_{(1)}^*) \tag{1}$$

and

$$g^*h^*(h) = \sum_{(h)} g^*(h_{(1)})h^*(h_{(2)}) = \sum_{(h^*)} h_{(0)}^*(h)g^*(h_{(1)}^*).$$

$H^*$  is also an  $H$  right module by  $h^* \cdot h = S(h) \circ h^*$ , where  $(h \circ h^*)(a) = h^*(ah)$  for all  $a \in H$ . For  $g^*, h^* \in H^*$ , and  $a, b \in H$  and  $\chi : H^* \rightarrow H^* \otimes H \cong \text{hom}(H^*, H^*)$ , we have

$$\begin{aligned} \chi(h^* \cdot a)(g^*)(b) &= (g^*(h^* \cdot a))(b) \\ &= \sum_{(b)} g^*(b_{(1)})h^*(b_{(2)}S(a)) \\ &= \sum_{(a)(b)} g^*(b_{(1)}\epsilon(a_{(2)}))h^*(b_{(2)}S(a_{(1)})) \\ &= \sum_{(a)(b)} (a_{(3)} \circ g^*)(b_{(1)}S(a_{(2)}))h^*(b_{(2)}S(a_{(1)})) \\ &= \sum_{(a)} ((a_{(2)} \circ g^*)h^*)(bS(a_{(1)})) \\ &= \sum_{(a)} (((a_{(2)} \circ g^*)h^*) \cdot a_{(1)})(b) \\ &= \sum_{(a)(h^*)} ((h_{(0)}^*(a_{(2)} \circ g^*)(h_{(1)}^*)) \cdot a_{(1)})(b) \\ &= \sum_{(a)(h^*)} ((h_{(0)}^* \cdot a_{(1)})(a_{(2)} \circ g^*)(h_{(1)}^*))(b) \\ &= \sum_{(a)(h^*)} ((h_{(0)}^* \cdot a_{(1)})g^*(h_{(1)}^*a_{(2)}))(b). \end{aligned}$$

This implies  $\chi(h^* \cdot a) = \sum_{(a)(h^*)} h_{(0)}^* \cdot a_{(1)} \otimes h_{(1)}^* a_{(2)}$  which proves the proposition.

Let  $M$  and  $N$  be  $H$  right Hopf modules over a Hopf algebra  $H$  with antipode  $S$ . We define an  $R$  module  $P(M) = \{m \in M \mid \chi(m) = m \otimes 1\}$ . Let  $f : M \rightarrow N$  be a module and comodule homomorphism. We define  $P(f)$  as restriction of  $f$  to  $P(M)$ . Then  $\chi(P(f)(m)) = \chi(f(m)) = (f \otimes 1)\chi(m) = f(m) \otimes 1 \in P(N)$ . Obviously  $P$  is a functor from the category of  $H$  right Hopf modules to the category of  $R$  modules.

LEMMA 2. *Let  $M$  be a Hopf module over a Hopf algebra  $H$  with antipode  $S$ . Then  $M \cong P(M) \otimes H$  as right Hopf modules. Furthermore,  $P(M)$  is an  $R$  direct summand of  $M$ .*

Proof. The natural injection  $P(M) \rightarrow M$  has a retraction  $M \ni m \mapsto \sum_{(m)} m_{(0)} S(m_{(1)}) \in P(M)$  for

$$\begin{aligned} \chi \left( \sum_{(m)} m_{(0)} S(m_{(1)}) \right) &= \sum_{(m)} m_{(0)} S(m_{(3)}) \otimes m_{(1)} S(m_{(2)}) \\ &= \sum_{(m)} m_{(0)} S(m_{(1)}) \otimes 1. \end{aligned}$$

Now  $\alpha : P(M) \otimes H \rightarrow M$  and  $\beta : M \rightarrow P(M) \otimes H$ , defined by  $\alpha(m \otimes h) = mh$  and  $\beta(m) = \sum_{(m)} m_{(0)} S(m_{(1)}) \otimes m_{(2)}$ , are inverse  $R$  homomorphisms of each other.  $\alpha$  being an  $H$  module homomorphism,  $\beta$  is an  $H$  module homomorphism. Furthermore,  $\chi\beta(m) = (\beta \otimes 1)\chi(m)$  implies that  $\beta$  and, consequently, also  $\alpha$  is a comodule homomorphism.

PROPOSITION 3. *Let  $H$  be a finitely generated projective Hopf algebra with antipode  $S$ . Then  $P(H^*)$  is a finitely generated projective rank 1  $R$  module.*

Proof. For each prime ideal  $\mathfrak{p}$  in  $R$  the isomorphism  $H^* \cong P(H^*) \otimes H$  implies  $H_{\mathfrak{p}}^* \cong P(H^*)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} H_{\mathfrak{p}}$ . Now  $H_{\mathfrak{p}}^* \cong H_{\mathfrak{p}}$  as free finite-dimensional  $R_{\mathfrak{p}}$  modules. So  $\dim(P(H^*)_{\mathfrak{p}}) = 1$  for all prime ideals  $\mathfrak{p}$  in  $R$ . Thus,  $P(H^*)$  has rank 1. Furthermore, it is finitely generated projective as a direct summand of  $H^*$ .

4. An  $R$  algebra  $H$  is a *Frobenius algebra* if  $H$  is a finitely generated projective  $R$  module and if there is an isomorphism  $\Phi : {}_H H \cong {}_H H^*$ , where we consider  $H^*$  as an  $H$  left module via  $h \circ h^*(a) = h^*(ah)$  for all  $a, h \in H, h^* \in H^*$  [2].  $\Phi$  is called a *Frobenius isomorphism*.  $\Phi(1) =: \psi$  is a free generator of  $H^*$  as an  $H$  left module called *Frobenius homomorphism*. By [3, p. 220, (4)],  $\psi$  is also a free generator of  $H^*$  as an  $H$  right module, where  $h^* \circ h(a) := h^*(ha)$  for all  $h^* \in H^*$ , and  $h, a \in H$ . This is a consequence of the proof that the

conditions for a Frobenius algebra are independent of the choice of the sides, i.e.,  ${}_H H \cong {}_H H^*$  implies  $H_H \cong H_H^*$  (if  $H$  is finitely generated and projective).  $\psi$  is unique up to multiplication with an invertible element of  $H$  [5, Satz 1].

If a Frobenius algebra  $H$  has an augmentation  $\epsilon$ , then the element  $N$  with  $N \circ \psi = \epsilon$  is called a *left norm* of  $H$ . A left norm  $N$  is also unique up to multiplication with an invertible element of  $H$  from the right side. We have  $((hN) \circ \psi)(a) = \psi(ahN) = (N \circ \psi)(ah) = \epsilon(ah) = \epsilon(a)\epsilon(h) = (N \circ \psi)(a)\epsilon(h) = \epsilon(h)N \circ \psi(a)$  for all  $a, h \in H$ . This implies

$$hN = \epsilon(h)N \quad \text{for all } h \in H.$$

An element  $a \in H$  of an augmented algebra  $H$  with  $ha = \epsilon(h)a$  for all  $h \in H$  is called a *left integral* of the augmented algebra  $H$ . So a left norm is in particular a left integral.

**PROPOSITION 4.** *Let  $H$  be a finitely generated projective Hopf algebra with antipode  $S$ . Then  $S$  is bijective.*

*Proof.*  $S$  is injective: Let  $\theta : H^* \cong P(H^*) \otimes H$  be the isomorphism of  $H$  right modules defined by Proposition 2 and Lemma 2. Let  $S(h) = 0$  and  $p \otimes a \in P(H^*) \otimes H$ , then

$$\theta^{-1}(p \otimes ah) = \theta^{-1}(p \otimes a) \cdot h = S(h) \circ \theta^{-1}(p \otimes a) = 0.$$

So  $p \otimes ah = 0$  for all  $p \otimes a \in P(H^*) \otimes H$  and the  $R$  endomorphism of  $P(H^*) \otimes H$ , defined by the multiplication with  $h$ , is the zero endomorphism.

Let  $\mathfrak{m}$  be a maximal ideal of  $R$ . If we localize with respect to  $\mathfrak{m}$ , we get an  $R_{\mathfrak{m}}$  Hopf algebra  $H_{\mathfrak{m}}$  with antipode  $S_{\mathfrak{m}}$ . Furthermore,  $(H^*)_{\mathfrak{m}} \cong (H_{\mathfrak{m}})^*$ , where the second  $*$  means dualization with respect to  $R_{\mathfrak{m}}$ . Also,  $P(M)_{\mathfrak{m}} \cong P(M_{\mathfrak{m}})$  for an  $H$  Hopf module  $M$  since  $P(M)$  is the kernel of  $\chi - 1_M \otimes \eta$  and localization is an exact functor. So  $(P(H^*) \otimes H)_{\mathfrak{m}} \cong P((H_{\mathfrak{m}})^*) \otimes_{R_{\mathfrak{m}}} H_{\mathfrak{m}}$ . As in Proposition 3,  $P((H_{\mathfrak{m}})^*)$  is a free  $R_{\mathfrak{m}}$  module on one generator so  $P((H_{\mathfrak{m}})^*) \otimes_{R_{\mathfrak{m}}} H_{\mathfrak{m}} \cong H_{\mathfrak{m}}$ . The multiplication of  $H_{\mathfrak{m}}$  with  $h$  on the right is a zero morphism for all maximal ideals  $\mathfrak{m}$  of  $R$ . So multiplication of  $H$  with  $h$  on the right must be zero which implies  $h = 0$ . So  $S$  is injective.

$S$  is surjective: Let  $0 \rightarrow H \xrightarrow{S} H \rightarrow Q \rightarrow 0$  be an  $R$  exact sequence. Then for all maximal ideals  $\mathfrak{m} \subseteq R$ , we have  $0 \rightarrow H_{\mathfrak{m}} \rightarrow H_{\mathfrak{m}} \rightarrow Q_{\mathfrak{m}} \rightarrow 0$  is  $R_{\mathfrak{m}}$  exact. So  $H_{\mathfrak{m}}/\mathfrak{m}H_{\mathfrak{m}} \xrightarrow{T} H_{\mathfrak{m}}/\mathfrak{m}H_{\mathfrak{m}} \rightarrow Q_{\mathfrak{m}}/\mathfrak{m}Q_{\mathfrak{m}} \rightarrow 0$  is  $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$  exact, where  $T$  is the antipode of the  $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$  Hopf algebra  $H_{\mathfrak{m}}/\mathfrak{m}H_{\mathfrak{m}}$ . Since this Hopf algebra is finitely generated,  $T$  is injective so that  $T$  is bijective and  $Q_{\mathfrak{m}}/\mathfrak{m}Q_{\mathfrak{m}} = 0$ . Since  $Q$  and  $Q_{\mathfrak{m}}$  are finitely generated,  $Q_{\mathfrak{m}} = 0$  for all maximal ideals  $\mathfrak{m} \subseteq R$ . So  $Q = 0$  and  $S$  is surjective.

**THEOREM 1.** *Let  $H$  be a finitely generated projective Hopf algebra with*

antipode  $S$ . Let  $P(H^*) \cong R$  as  $R$  modules. Then  $H$  is a Frobenius algebra with a Frobenius homomorphism  $\psi$  such that

$$\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h)1 \quad \text{for all } h \in H.$$

*Proof.* By Proposition 2 and Lemma 2 there exists an isomorphism  $H^* \cong P(H^*) \otimes H$  of  $H$  right modules where  $(h^* \cdot h)(a) = h^*(aS(h))$  or  $h^* \cdot h = S(h) \circ h^*$  is the definition of the module structure on  $H^*$ . Since  $P(H^*) \cong R$ , let  $\theta : H^* \cong H$  be an isomorphism of  $H$  right modules. Define  $\Phi = \theta^{-1}S^{-1} : H \cong H^*$ , where we use Proposition 4. Then  $\Phi(ha) = \theta^{-1}S^{-1}(ha) = \theta^{-1}(S^{-1}(a)S^{-1}(h)) = \theta^{-1}(S^{-1}(a)) \cdot S^{-1}(h) = \Phi(a) \cdot S^{-1}(h) = h \circ \Phi(a)$ , so  $H$  is a Frobenius algebra. Before we prove the formula on the Frobenius homomorphism we prove the following:

LEMMA 3. Let  $H$  be a finitely generated projective Hopf algebra with antipode  $S$  with  $P(H^*) \cong R$ . Let  $\Phi : H \cong H^*$  be the Frobenius isomorphism constructed above and let  $\psi = \Phi(1)$  be a Frobenius homomorphism. Then  $\psi \in P(H^*)$  and  $\psi$  is a left integral in  $H^*$ .

*Proof.*  $\Phi(1) = \psi$  implies  $S\theta(\psi) = 1$  and also  $\theta(\psi) = 1$ .  $\theta$  is a comodule homomorphism so  $\sum \theta(\psi_{(0)}) \otimes \psi_{(1)} = (\theta \otimes 1)\chi(\psi) = \Delta(\theta(\psi)) = \Delta(1) = 1 \otimes 1 = \theta(\psi) \otimes 1$ .  $\theta \otimes 1$  being an isomorphism this implies  $\chi(\psi) = \psi \otimes 1$  so  $\psi \in P(H^*)$ . Now

$$\begin{aligned} h^* \left( \sum_{(h)} h_{(1)}\psi(h_{(2)}) \right) &= \sum_{(h)} h^*(h_{(1)}) \psi(h_{(2)}) \\ &= (h^*\psi)(h) \\ &= \sum_{(\psi)} \psi_{(0)}(h) h^*(\psi_{(1)}) \quad \text{(by (1))} \\ &= \psi(h) h^*(1) \\ &= h^*(\psi(h)1) \end{aligned}$$

for all  $h \in H$  and  $h^* \in H^*$ . This implies

$$\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h)1 \quad \text{for all } h \in H.$$

This also means  $(h^*\psi)(h) = h^*(1)\psi(h)$ , which proves that  $\psi$  is a left integral in  $H^*$ .

COROLLARY 1. Let  $R$  be a commutative ring with  $\text{pic}(R) = 0$ . Then each finitely generated projective Hopf algebra with antipode is a Frobenius algebra.

5. LEMMA 4. *Let  $P$  be a finitely generated projective  $R$  module and let  $f : P \rightarrow P$  be an epimorphism. Then  $f$  is an isomorphism.*

*Proof.* The sequences

$$\begin{aligned} 0 \rightarrow A \rightarrow P \xrightarrow{f} P \rightarrow 0, \\ 0 \rightarrow A_{\mathfrak{m}} \rightarrow P_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} P_{\mathfrak{m}} \rightarrow 0, \\ 0 \rightarrow A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} \rightarrow P_{\mathfrak{m}}/\mathfrak{m}P_{\mathfrak{m}} \xrightarrow{\tilde{f}} P_{\mathfrak{m}}/\mathfrak{m}P_{\mathfrak{m}} \rightarrow 0 \end{aligned}$$

are all split exact. But  $\tilde{f}$  is an isomorphism by reasons of dimension so  $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} = 0$ .  $A_{\mathfrak{m}}$  is finitely generated so that  $A_{\mathfrak{m}} = 0$  for all maximal ideals  $\mathfrak{m} \subseteq R$ , so that  $A = 0$  and  $f$  is an isomorphism.

THEOREM 2. *Let  $H$  be a Hopf algebra and a Frobenius algebra with a Frobenius homomorphism  $\psi$  such that  $\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h)1$  for all  $h \in H$ . Then  $H$  has an antipode  $S$ .*

*Proof.* We define  $S : H \rightarrow H$  by  $S(h) = \sum_{(N)} N_{(1)}\psi(hN_{(2)})$ , where  $N$  is a left norm, i.e.,  $N \circ \psi = \epsilon$ . Then

$$\begin{aligned} \sum_{(h)} h_{(1)}S(h_{(2)}) &= \sum_{(h)(N)} h_{(1)}N_{(1)}\psi(h_{(2)}N_{(2)}) \\ &= \psi(hN)1 \\ &= \epsilon(h)1 \\ &= \eta\epsilon(h) \end{aligned}$$

for all  $h \in H$  by definition of  $N$ . So  $1 * S = \eta\epsilon$ .

Now  $\text{hom}_R(H, H)$  is an associative  $R$  algebra with multiplication  $f * g = \mu(f \otimes g)\Delta$ .  $\text{Hom}_R(H, H)$  is also finitely generated and projective since  $H$  is. The map

$$\text{hom}_R(H, H) \ni f \mapsto f * S \in \text{hom}_R(H, H)$$

is an  $R$  epimorphism for  $(f * 1) * S = f * (1 * S) = f * \eta\epsilon = f$ . By the preceding lemma,  $- * S$  is an isomorphism with inverse map  $- * 1$ . So  $S * 1 = \eta\epsilon * S * 1 = \eta\epsilon$ , i.e.,  $S$  is an antipode.

THEOREM 3. *Let  $H$  be a Hopf algebra and a Frobenius algebra. Then the two-sided ideal of left integrals  $h \in H$  (with  $ah = \epsilon(a)h$  for all  $a \in H$ ) is a free  $R$  direct summand of  $H$  of rank 1 with basis  $\{N\}$ , a left norm of  $H$ .*

*Proof.* Let  $h$  be a left integral, then  $\psi(ah) = \psi(\epsilon(a)h) = \psi(h)\epsilon(a) = \psi(h)\psi(aN) = \psi(a\psi(h)N)$  for all  $a \in H$ , so that  $h \circ \psi = \psi(h)N \circ \psi$  or  $h = \psi(h)N$ .

Furthermore,  $\epsilon$  is an epimorphism since  $\epsilon\eta = 1_R$ . So  $\epsilon^* : R \rightarrow H^*$  is a monomorphism. Also  $\rho : R \rightarrow H^* \cong H$  is a monomorphism. But  $\epsilon^*(r)(h) = r\epsilon^*(1)(h) = r\epsilon(h) = \psi(hrN)$ , so that  $\rho(r) = rN$ . Since  $\rho$  is injective,  $R \ni r \mapsto rN \in H$  is injective. Thus,  $RN$  is free of rank 1.

Finally,  $H \ni h \mapsto \psi(h)N \in RN$  is a retraction for the inclusion  $RN \subseteq H$ , in which case  $RN$  splits off as a direct summand.

By Theorem 1,  $\psi$  is a left integral and  $H \ni h \mapsto h \circ \psi \in H^*$  as well as  $H \ni h \mapsto \psi \circ h \in H^*$  are isomorphisms, so that  $\psi$  is a nonsingular left integral in  $H^*$ . Now let  $\text{pic}(R) = 0$ . If  $H$  is a finitely generated projective Hopf algebra with an antipode, then so is  $H^*$ . Furthermore,  $H^{**} \cong H$  as Hopf algebras with antipode and also  $H$  has a nonsingular left integral. This implies the main result of (see remark on page 588) [4] for the case that  $R$  is a principal ideal domain, for then  $\text{pic}(R) = 0$  holds.

*Note added in proof.* P. Gabriel gave an example of a finitely generated projective Hopf algebra with antipode which is not a Frobenius algebra, showing that  $\text{pic}(R) = 0$  is a necessary condition for Corollary 1. We have however the following

*Theorem.* Let  $H$  be a finitely generated projective Hopf algebra over a commutative ring  $R$ .  $H$  has an antipode if and only if  $H$  is a quasi Frobenius algebra and  $H^*$  has a finite set of generators  $\psi_1, \dots, \psi_n$  as an  $H$  left module ( $h \circ h^*(a) = h^*(ah)$ ) such that for all  $k = 1, \dots, n$

$$\sum_{(h)} h_{(1)}\psi_k(h_{(2)}) = \psi_k(h)1 \quad \text{for all } h \in H.$$

Here a quasi Frobenius algebra is taken in the sense of B. Müller, Quasi-Frobenius-Erweiterungen, *Math. Zeitschr.* **85**, 345–368 (1964). This theorem guarantees that the cohomology theory of Hopf algebras can be developed without the restrictive condition  $\text{pic}(R) = 0$ .

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