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# When Hopf Algebras Are Frobenius Algebras

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R. Larson and M. Sweedler recently proved that for free finitely generated Hopf algebras $H$ over a principal ideal domain $R$ the following are equivalent: (a) $H$ has an antipode and (b) $H$ has a nonsingular left integral. In this paper I give a generalization of this result which needs only a minor restriction, which, for example, always holds if $\mathrm{pic}(R) = 0$ for the base ring $R$. A finitely generated projective Hopf algebra $H$ over $R$ has an antipode if and only if $H$ is a Frobenius algebra with a Frobenius homomorphism $\psi$ such that $\sum h_{(1)} \psi(h_{(2)}) = \psi(h) \cdot 1$ for all $h \in H$. We also show that the antipode is bijective and that the ideal of left integrals is a free rank 1, $R$-direct summand of $H$.

**1.** In Ref. [4], Larson and Sweedler proved the equivalence for a finite-dimensional Hopf algebra over a principal ideal domain to have a (necessarily unique) antipode and to have a nonsingular left integral. It is easy to see that this result implies that a finite-dimensional Hopf algebra over a principal ideal domain is a Frobenius algebra, which generalizes the well-known fact that a group ring of a finite group is Frobenius as well as the result of Berkson [1], that the restricted universal enveloping algebra of a finite-dimensional restricted Lie algebra is Frobenius. This result has consequences with respect to a cohomology theory of Hopf algebras which will be exhibited in a subsequent paper.

In this paper we want to generalize the main result of [4] to arbitrary commutative rings $R$ and finitely generated projective Hopf algebras $H$. We need only a slight restriction on $H$ or on $R$, viz., $\mathrm{pic}(R) = 0$ to get the equivalence between the existence of an antipode and the fact that $H$ is a Frobenius algebra with a Frobenius homomorphism $\psi$, such that $\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h) \cdot 1$ for all $h \in H$, where $\sum_{(h)} h_{(1)} \otimes h_{(2)} = \Delta(h)$ is the Sweedler notation. We do not know whether the imposed restrictions on $R$ or $H$ are necessary for the above result.[1] In this context we also prove that the

[1] See footnote at the end.

antipode of a finitely generated projective Hopf algebra is bijective. This holds without any further restrictions.

Since integrals are also of interest in this general situation, we shall prove that in a Hopf algebra $H$ which is Frobenius with a Frobenius homomorphism $\psi$ such that $\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h) \cdot 1$ for all $h \in H$ the two-sided $H$ ideal of left integrals is an $R$-free rank 1 $R$-direct summand of $H$.

**2.** Let $R$ be a commutative ring (associative with unit). All modules are assumed to be unitary $R$ modules. All algebras are assumed to be associative $R$ algebras with unit.

A *coalgebra* $C$ is a module $C$ together with homomorphisms $\Delta : C \to C \otimes C$ (the tensor product is taken over $R$), the diagonal, and $\epsilon : C \to R$, the counit or augmentation, such that

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes 1} \\ C \otimes C & \xrightarrow{1\otimes\Delta} & C \otimes C \otimes C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & \searrow^{1} & \downarrow{\scriptstyle 1\otimes\epsilon} \\ C \otimes C & \xrightarrow{\epsilon\otimes 1} & C \end{array}$$

commute where we identify $C \otimes R$, $C$, and $R \otimes C$. We adopt the Sweedler notation $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$ as explained in [7, p. 10].

A *Hopf algebra* $H$ is an algebra $H$ with structure maps $\mu : H \otimes H \to H$ and $\eta : R \to H$, which is also a coalgebra with structure maps $\Delta : H \to H \otimes H$ and $\epsilon : H \to R$ such that $\Delta$ and $\epsilon$ are algebra homomorphisms. As in [7, Proposition 3.1.1] one shows that $\mu$ and $\eta$ are coalgebra homomorphisms.

Let $C$ be a coalgebra. A $C$ *right comodule* is a module $M$ together with a homomorphism $\chi : M \to M \otimes C$ such that

$$\begin{array}{ccc} M & \xrightarrow{\chi} & M \otimes C \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ M \otimes C & \xrightarrow{\chi\otimes 1} & M \otimes C \otimes C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \xrightarrow{\chi} & M \otimes C \\ & \searrow_{1} & \downarrow{\scriptstyle 1\otimes\epsilon} \\ & & M \end{array}$$

commute, where we identify $M$ and $M \otimes R$. Here again we use the Sweedler notation $\chi(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$. Observe that the $m_{(i)}$'s for $i \geqslant 1$ are elements in $C$, whereas the $m_{(0)}$'s are in $M$.

Let $H$ be a Hopf algebra. An $H$ *right Hopf module* is an $H$ right module $M$ which is also an $H$ right comodule such that

$$\chi(mh) = \sum_{(m),(h)} m_{(0)}h_{(1)} \otimes m_{(1)}h_{(2)} .$$

Let $H$ be a Hopf algebra. Then $\hom_R(H, H)$ is an associative $R$ algebra with unit $\eta\epsilon$, if we define the multiplication by $f * g :=\mu(f \otimes g)\Delta$, i.e., $f * g(h) = \sum_{(h)} f(h_{(1)})\, g(h_{(2)})$ [7, p. 70, exercise 1]. The *antipode* $S$ of $H$ is (if it exists) the two-sided inverse of the identity $1_H$ of $H$ in $\hom_R(H, H)$.

3. Lemma 1. *Let $A$, $B$, and $C$ be $R$ modules. If $C$ is finitely generated and projective, then $\lambda : \hom(A, B \otimes C) \to \hom(C^* \otimes A, B)$ with $\lambda(f)(c^* \otimes a) := (1 \otimes c^*)f(a)$ is an isomorphism.*

*Proof.* $\lambda$ defines a natural transformation

$$\hom(A, B \otimes -) \to \hom(-^* \otimes A, B),$$

which is an isomorphism for $C = R$. By [6, 4.11. Lemma 2] the above lemma holds.

Proposition 1. *Let $C$ be a finitely generated projective $R$ coalgebra and $M$ be an $R$ module. Then $\chi : M \to M \otimes C$ defines a $C$ right comodule if and only if $\lambda(\chi) : C^* \otimes M \to M$ defines a $C^*$ left module, where $\lambda$ is defined as in Lemma* 1.

*Proof.* $C^*$ is an $R$ algebra by $c^*d^*(c) = \sum_{(c)} c^*(c_{(1)})\, d^*(c_{(2)})$. Assume $\chi : M \to M \otimes C$ defines a $C$ right comodule. Then

$$\begin{aligned}(c_1{}^*c_2{}^*)m &= \sum_{(m)} m_{(0)}(c_1{}^*c_2{}^*(m_{(1)}))\\ &= \sum_{(m)} m_{(0)}(c_1{}^*(m_{(1)})\, c_2{}^*(m_{(2)}))\\ &= \sum_{(m)} c_1{}^*m_{(0)}(c_2{}^*(m_{(1)}))\\ &= c_1{}^*(c_2{}^*m)\end{aligned}$$

and

$$\epsilon m = \sum_{(m)} m_{(0)}\epsilon(m_{(1)}) = m,$$

so $M$ is a $C^*$ left module.

Now let $M$ be a $C^*$ left module. The natural transformation

$$\rho : M \otimes C \otimes D \to \hom(C^* \otimes D^*, M)$$

defined by $\rho(m \otimes c \otimes d)(c^* \otimes d^*) = c^*(c)\, d^*(d)m$ is an isomorphism for finitely generated projective modules $C$ and $D$, since it is an isomorphism for $C = D = R$ [6, 4.11. Lemma 2]. So $M \otimes C \otimes C \cong \hom(C^* \otimes C^*, M)$ for our finitely generated projective coalgebra $C$.

Let $x = ((\chi \otimes 1)\chi - (1 \otimes \Delta)\chi)(m) \in M \otimes C \otimes C$. Then $\rho(x)(c_1{}^* \otimes c_2{}^*) = c_1{}^*(c_2{}^*m) - (c_1{}^*c_2{}^*)m = 0$ for all $c_1{}^*$, $c_2{}^* \in C^*$, so $x = 0$, i.e., $\chi$ is coassociative. Furthermore, we have $(1 \otimes \epsilon)\chi(m) = \sum_{(m)} m_{(0)}\epsilon(m_{(1)}) = \epsilon m = m$, since $\epsilon$ is the unit element in $C^*$. So $M$ is a $C$ right comodule.

PROPOSITION 2. *Let $H$ be a finitely generated projective Hopf algebra with antipode $S$. Then $H^*$ is an $H$ right Hopf module.*

*Proof.* $H^*$ is an $H^*$ left module, so it is an $H$ right comodule. We have for $g^*$, $h^* \in H^*$, and $h \in H$ and the comodule map $\chi : H^* \to H^* \otimes H$ with $\chi(h^*) = \sum_{(h)} h^*_{(0)} \otimes h^*_{(1)}$.

$$g^*h^* = \sum_{(h^*)} h^*_{(0)} g^*(h^*_{(1)}) \tag{1}$$

and

$$g^*h^*(h) = \sum_{(h)} g^*(h_{(1)})\, h^*(h_{(2)}) = \sum_{(h^*)} h^*_{(0)}(h) g^*(h^*_{(1)}).$$

$H^*$ is also an $H$ right module by $h^* \cdot h = S(h) \circ h^*$, where $(h \circ h^*)(a) = h^*(ah)$ for all $a \in H$. For $g^*$, $h^* \in H^*$, and $a, b \in H$ and $\chi : H^* \to H^* \otimes H \simeq \hom(H^*, H^*)$, we have

$$\begin{aligned}
\chi(h^* \cdot a)(g^*)(b) &= (g^*(h^* \cdot a))(b) \\
&= \sum_{(b)} g^*(b_{(1)})\, h^*(b_{(2)}S(a)) \\
&= \sum_{(a)(b)} g^*(b_{(1)}\epsilon(a_{(2)}))\, h^*(b_{(2)}S(a_{(1)})) \\
&= \sum_{(a)(b)} (a_{(3)} \circ g^*)(b_{(1)}S(a_{(2)}))\, h^*(b_{(2)}S(a_{(1)})) \\
&= \sum_{(a)} ((a_{(2)} \circ g^*)\, h^*)(bS(a_{(1)})) \\
&= \sum_{(a)} (((a_{(2)} \circ g^*)\, h^*) \cdot a_{(1)})(b) \\
&= \sum_{(a)(h^*)} ((h^*_{(0)}(a_{(2)} \circ g^*)(h^*_{(1)})) \cdot a_{(1)})(b) \\
&= \sum_{(a)(h^*)} ((h^*_{(0)} \cdot a_{(1)})(a_{(2)} \circ g^*)(h^*_{(1)}))(b) \\
&= \sum_{(a)(h^*)} ((h^*_{(0)} \cdot a_{(1)}) g^*(h^*_{(1)}a_{(2)}))(b).
\end{aligned}$$

This implies $\chi(h^* \cdot a) = \sum_{(a)(h^*)} h^*_{(0)} \cdot a_{(1)} \otimes h^*_{(1)}a_{(2)}$ which proves the proposition.

Let $M$ and $N$ be $H$ right Hopf modules over a Hopf algebra $H$ with antipode $S$. We define an $R$ module $P(M) = \{m \in M \mid \chi(m) = m \otimes 1\}$. Let $f: M \to N$ be a module and comodule homomorphism. We define $P(f)$ as restriction of $f$ to $P(M)$. Then $\chi(P(f)(m)) = \chi(f(m)) = (f \otimes 1)\chi(m) = f(m) \otimes 1 \in P(N)$. Obviously $P$ is a functor from the category of $H$ right Hopf modules to the category of $R$ modules.

LEMMA 2. *Let $M$ be a Hopf module over a Hopf algebra $H$ with antipode $S$. Then $M \cong P(M) \otimes H$ as right Hopf modules. Furthermore, $P(M)$ is an $R$ direct summand of $M$.*

*Proof.* The natural injection $P(M) \to M$ has a retraction $M \ni m \mapsto \sum_{(m)} m_{(0)}S(m_{(1)}) \in P(M)$ for

$$\chi\left(\sum_{(m)} m_{(0)}S(m_{(1)})\right) = \sum_{(m)} m_{(0)}S(m_{(3)}) \otimes m_{(1)}S(m_{(2)})$$

$$= \sum_{(m)} m_{(0)}S(m_{(1)}) \otimes 1.$$

Now $\alpha : P(M) \otimes H \to M$ and $\beta : M \to P(M) \otimes H$, defined by $\alpha(m \otimes h) = mh$ and $\beta(m) = \sum_{(m)} m_{(0)}S(m_{(1)}) \otimes m_{(2)}$, are inverse $R$ homomorphisms of each other. $\alpha$ being an $H$ module homomorphism, $\beta$ is an $H$ module homomorphism. Furthermore, $\chi\beta(m) = (\beta \otimes 1)\chi(m)$ implies that $\beta$ and, consequently, also $\alpha$ is a comodule homomorphism.

PROPOSITION 3. *Let $H$ be a finitely generated projective Hopf algebra with antipode $S$. Then $P(H^*)$ is a finitely generated projective rank 1 $R$ module.*

*Proof.* For each prime ideal $\mathfrak{p}$ in $R$ the isomorphism $H^* \cong P(H^*) \otimes H$ implies $H_\mathfrak{p}{}^* \cong P(H^*)_\mathfrak{p} \otimes_{R_\mathfrak{p}} H_\mathfrak{p}$. Now $H_\mathfrak{p}{}^* \cong H_\mathfrak{p}$ as free finite-dimensional $R_\mathfrak{p}$ modules. So $\dim(P(H^*)_\mathfrak{p}) = 1$ for all prime ideals $\mathfrak{p}$ in $R$. Thus, $P(H^*)$ has rank 1. Furthermore, it is finitely generated projective as a direct summand of $H^*$.

**4.** An $R$ algebra $H$ is a *Frobenius algebra* if $H$ is a finitely generated projective $R$ module and if there is an isomorphism $\Phi : {}_HH \cong {}_HH^*$, where we consider $H^*$ as an $H$ left module via $h \circ h^*(a) = h^*(ah)$ for all $a, h \in H$, $h^* \in H^*$ [2]. $\Phi$ is called a *Frobenius isomorphism.* $\Phi(1) =: \psi$ is a free generator of $H^*$ as an $H$ left module called *Frobenius homomorphism.* By [3, p. 220, (4)], $\psi$ is also a free generator of $H^*$ as an $H$ right module, where $h^* \circ h(a) := h^*(ha)$ for all $h^* \in H^*$, and $h, a \in H$. This is a consequence of the proof that the conditions for a Frobenius algebra are independent of the choice of the sides, i.e., ${}_H H \cong {}_H H^*$ implies $H_H \cong H_H{}^*$ (if $H$ is finitely generated and projective). $\psi$ is unique up to multiplication with an invertible element of $H$ [5, Satz 1].

If a Frobenius algebra $H$ has an augmentation $\epsilon$, then the element $N$ with $N \circ \psi = \epsilon$ is called a *left norm* of $H$. A left norm $N$ is also unique up to multiplication with an invertible element of $H$ from the right side. We have $((hN) \circ \psi)(a) = \psi(ahN) = (N \circ \psi)(ah) = \epsilon(ah) = \epsilon(a)\epsilon(h) = (N \circ \psi)(a)\epsilon(h) = (\epsilon(h)N \circ \psi)(a)$ for all $a, h \in H$. This implies

$$hN = \epsilon(h)N \qquad \text{for all} \quad h \in H.$$

An element $a \in H$ of an augmented algebra $H$ with $ha = \epsilon(h)a$ for all $h \in H$ is called a *left integral* of the augmented algebra $H$. So a left norm is in particular a left integral.

PROPOSITION 4. *Let $H$ be a finitely generated projective Hopf algebra with antipode $S$. Then $S$ is bijective.*

*Proof.* $S$ is injective: Let $\theta : H^* \cong P(H^*) \otimes H$ be the isomorphism of $H$ right modules defined by Proposition 2 and Lemma 2. Let $S(h) = 0$ and $p \otimes a \in P(H^*) \otimes H$, then

$$\theta^{-1}(p \otimes ah) = \theta^{-1}(p \otimes a) \cdot h = S(h) \circ \theta^{-1}(p \otimes a) = 0.$$

So $p \otimes ah = 0$ for all $p \otimes a \in P(H^*) \otimes H$ and the $R$ endomorphism of $P(H^*) \otimes H$, defined by the multiplication with $h$, is the zero endomorphism.

Let $\mathfrak{m}$ be a maximal ideal of $R$. If we localize with respect to $\mathfrak{m}$, we get an $R_\mathfrak{m}$ Hopf algebra $H_\mathfrak{m}$ with antipode $S_\mathfrak{m}$. Furthermore, $(H^*)_\mathfrak{m} \cong (H_\mathfrak{m})^*$, where the second $*$ means dualization with respect to $R_\mathfrak{m}$. Also, $P(M)_\mathfrak{m} \cong P(M_\mathfrak{m})$ for an $H$ Hopf module $M$ since $P(M)$ is the kernel of $\chi - 1_M \otimes \eta$ and localization is an exact functor. So $(P(H^*) \otimes H)_\mathfrak{m} \cong P((H_\mathfrak{m})^*) \otimes_{R_\mathfrak{m}} H_\mathfrak{m}$. As in Proposition 3, $P((H_\mathfrak{m})^*)$ is a free $R_\mathfrak{m}$ module on one generator so $P((H_\mathfrak{m})^*) \otimes_{R_\mathfrak{m}} H_\mathfrak{m} \cong H_\mathfrak{m}$. The multiplication of $H_\mathfrak{m}$ with $h$ on the right is a zero morphism for all maximal ideals $\mathfrak{m}$ of $R$. So multiplication of $H$ with $h$ on the right must be zero which implies $h = 0$. So $S$ is injective.

$S$ is surjective: Let $0 \to H \xrightarrow{S} H \to Q \to 0$ be an $R$ exact sequence. Then for all maximal ideals $\mathfrak{m} \subseteq R$, we have $0 \to H_\mathfrak{m} \to H_\mathfrak{m} \to Q_\mathfrak{m} \to 0$ is $R_\mathfrak{m}$ exact. So $H_\mathfrak{m}/\mathfrak{m}H_\mathfrak{m} \xrightarrow{T} H_\mathfrak{m}/\mathfrak{m}H_\mathfrak{m} \to Q_\mathfrak{m}/\mathfrak{m}Q_\mathfrak{m} \to 0$ is $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$ exact, where $T$ is the antipode of the $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$ Hopf algebra $H_\mathfrak{m}/\mathfrak{m}H_\mathfrak{m}$. Since this Hopf algebra is finitely generated, $T$ is injective so that $T$ is bijective and $Q_\mathfrak{m}/\mathfrak{m}Q_\mathfrak{m} = 0$. Since $Q$ and $Q_\mathfrak{m}$ are finitely generated, $Q_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m} \subseteq R$. So $Q = 0$ and $S$ is surjective.

THEOREM 1. *Let $H$ be a finitely generated projective Hopf algebra with* *antipode S. Let $P(H^*) \cong R$ as R modules. Then H is a Frobenius algebra with a Frobenius homomorphism $\psi$ such that*

$$\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h)1 \qquad \textit{for all} \quad h \in H.$$

*Proof.* By Proposition 2 and Lemma 2 there exists an isomorphism $H^* \cong P(H^*) \otimes H$ of $H$ right modules where $(h^* \cdot h)(a) = h^*(aS(h))$ or $h^* \cdot h = S(h) \circ h^*$ is the definition of the module structure on $H^*$. Since $P(H^*) \cong R$, let $\theta : H^* \cong H$ be an isomorphism of $H$ right modules. Define $\Phi = \theta^{-1}S^{-1} : H \cong H^*$, where we use Proposition 4. Then $\Phi(ha) = \theta^{-1}S^{-1}(ha) = \theta^{-1}(S^{-1}(a)S^{-1}(h)) = \theta^{-1}(S^{-1}(a)) \cdot S^{-1}(h) = \Phi(a) \cdot S^{-1}(h) = h \circ \Phi(a)$, so $H$ is a Frobenius algebra. Before we prove the formula on the Frobenius homomorphism we prove the following:

LEMMA 3. *Let H be a finitely generated projective Hopf algebra with antipode S with $P(H^*) \cong R$. Let $\Phi : H \cong H^*$ be the Frobenius isomorphism constructed above and let $\psi = \Phi(1)$ be a Frobenius homomorphism. Then $\psi \in P(H^*)$ and $\psi$ is a left integral in $H^*$.*

*Proof.* $\Phi(1) = \psi$ implies $S\theta(\psi) = 1$ and also $\theta(\psi) = 1$. $\theta$ is a comodule homomorphism so $\sum \theta(\psi_{(0)}) \otimes \psi_{(1)} = (\theta \otimes 1)\chi(\psi) = \Delta(\theta(\psi)) = \Delta(1) = 1 \otimes 1 = \theta(\psi) \otimes 1$. $\theta \otimes 1$ being an isomorphism this implies $\chi(\psi) = \psi \otimes 1$ so $\psi \in P(H^*)$. Now

$$\begin{aligned} h^*\Big(\sum_{(h)} h_{(1)}\psi(h_{(2)})\Big) &= \sum_{(h)} h^*(h_{(1)})\, \psi(h_{(2)}) \\ &= (h^*\psi)(h) \\ &= \sum_{(\psi)} \psi_{(0)}(h)\, h^*(\psi_{(1)}) \qquad \text{(by (1))} \\ &= \psi(h)\, h^*(1) \\ &= h^*(\psi(h)1) \end{aligned}$$

for all $h \in H$ and $h^* \in H^*$. This implies

$$\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h)1 \qquad \text{for all} \quad h \in H.$$

This also means $(h^*\psi)(h) = h^*(1)\psi(h)$, which proves that $\psi$ is a left integral in $H^*$.

COROLLARY 1. *Let R be a commutative ring with* pic$(R) = 0$. *Then each finitely generated projective Hopf algebra with antipode is a Frobenius algebra.*

5. Lemma 4. *Let $P$ be a finitely generated projective $R$ module and let $f: P \to P$ be an epimorphism. Then $f$ is an isomorphism.*

*Proof.* The sequences

$$0 \to A \to P \xrightarrow{f} P \to 0,$$
$$0 \to A_{\mathfrak{m}} \to P_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} P_{\mathfrak{m}} \to 0,$$
$$0 \to A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} \to P_{\mathfrak{m}}/\mathfrak{m}P_{\mathfrak{m}} \xrightarrow{\tilde{f}} P_{\mathfrak{m}}/\mathfrak{m}P_{\mathfrak{m}} \to 0$$

are all split exact. But $\tilde{f}$ is an isomorphism by reasons of dimension so $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} = 0$. $A_{\mathfrak{m}}$ is finitely generated so that $A_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subseteq R$, so that $A = 0$ and $f$ is an isomorphism.

Theorem 2. *Let $H$ be a Hopf algebra and a Frobenius algebra with a Frobenius homomorphism $\psi$ such that $\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h)1$ for all $h \in H$. Then $H$ has an antipode $S$.*

*Proof.* We define $S: H \to H$ by $S(h) = \sum_{(N)} N_{(1)}\psi(hN_{(2)})$, where $N$ is a left norm, i.e., $N \circ \psi = \epsilon$. Then

$$\begin{aligned}\sum_{(h)} h_{(1)}S(h_{(2)}) &= \sum_{(h)(N)} h_{(1)}N_{(1)}\psi(h_{(2)}N_{(2)}) \\ &= \psi(hN)1 \\ &= \epsilon(h)1 \\ &= \eta\epsilon(h)\end{aligned}$$

for all $h \in H$ by definition of $N$. So $1 * S = \eta\epsilon$.

Now $\hom_R(H, H)$ is an associative $R$ algebra with multiplication $f * g = \mu(f \otimes g)\Delta$. $\mathrm{Hom}_R(H, H)$ is also finitely generated and projective since $H$ is. The map

$$\hom_R(H, H) \ni f \mapsto f * S \in \hom_R(H, H)$$

is an $R$ epimorphism for $(f * 1) * S = f * (1 * S) = f * \eta\epsilon = f$. By the preceeding lemma, $- * S$ is an isomorphism with inverse map $- * 1$. So $S * 1 = \eta\epsilon * S * 1 = \eta\epsilon$, i.e., $S$ is an antipode.

Theorem 3. *Let $H$ be a Hopf algebra and a Frobenius algebra. Then the two-sided ideal of left integrals $h \in H$ (with $ah = \epsilon(a)h$ for all $a \in H$) is a free $R$ direct summand of $H$ of rank 1 with basis $\{N\}$, a left* norm *of $H$.*

*Proof.* Let $h$ be a left integral, then $\psi(ah) = \psi(\epsilon(a)h) = \psi(h)\epsilon(a) = \psi(h)\psi(aN) = \psi(a\psi(h)N)$ for all $a \in H$, so that $h \circ \psi = \psi(h)N \circ \psi$ or $h = \psi(h)N$.

Furthermore, $\epsilon$ is an epimorphism since $\epsilon\eta = 1_R$. So $\epsilon^* : R \to H^*$ is a monomorphism. Also $\rho : R \to H^* \cong H$ is a monomorphism. But $\epsilon^*(r)(h) = r\epsilon^*(1)(h) = r\epsilon(h) = \psi(hrN)$, so that $\rho(r) = rN$. Since $\rho$ is injective, $R \ni r \mapsto rN \in H$ is injective. Thus, $RN$ is free of rank 1.

Finally, $H \ni h \mapsto \psi(h)N \in RN$ is a retraction for the inclusion $RN \subseteq H$, in which case $RN$ splits off as a direct summand.

By Theorem 1, $\psi$ is a left integral and $H \ni h \mapsto h \circ \psi \in H^*$ as well as $H \ni h \mapsto \psi \circ h \in H^*$ are isomorphisms, so that $\psi$ is a nonsingular left integral in $H^*$. Now let $\mathrm{pic}(R) = 0$. If $H$ is a finitely generated projective Hopf algebra with an antipode, then so is $H^*$. Furthermore, $H^{**} \cong H$ as Hopf algebras with antipode and also $H$ has a nonsingular left integral. This implies the main result of (see remark on page 588) [4] for the case that $R$ is a principal ideal domain, for then $\mathrm{pic}(R) = 0$ holds.

*Note added in proof.* P. Gabriel gave an example of a finitely generated projective Hopf algebra with antipode which is not a Frobenius algebra, showing that $\mathrm{pic}(R) = 0$ is a necessary condition for Corollary 1. We have however the following

*Theorem.* Let $H$ be a finitely generated projective Hopf algebra over a commutative ring $R$. $H$ has an antipode if and only if $H$ is a quasi Frobenius algebra and $H^*$ has a finite set of generators $\psi_1, \ldots, \psi_n$ as an $H$ left module $(h \circ h^*(a) = h^*(ah))$ such that for all $k = 1, \ldots, n$

$$\sum_{(h)} h_{(1)}\psi_k(h_{(2)}) = \psi_k(h)1 \qquad \text{for all } h \in H.$$

Here a quasi Frobenius algebra is taken in the sense of B. Müller, Quasi-Frobenius-Erweiterungen, *Math. Zeitschr.* **85**, 345–368 (1964). This theorem guarantees that the cohomology theory of Hopf algebras can be developed without the restrictive condition $\mathrm{pic}(R) = 0$.

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