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ON GENERATORS AND COGENERATORS

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We consider the existence of generators and cogenerators in several categories. In many cases the non-existence of cogenerators results from the construction of arbitrarily high dimensional simple (restricted) Lie algebras. The existence of generators in the Hopf algebra categories results from the existence of generators in a certain coalgebra category and the construction of a free Hopf algebra on a coalgebra.

Introduction

The question of the existence of generators and cogenerators in a category is of interest in view of the special adjoint functor theorem. ISBELL has given an example (unpublished) which shows that the existence of a cogenerator is a necessary part of the hypothesis of the special adjoint functor theorem. This example also shows that the category of groups has no cogenerator. (Clearly the free group on one element is a generator in the category of groups.) It is well known that there exist generators and cogenerators in the categories of commutative groups, commutative Lie algebras (over a field) and commutative restricted Lie algebras, because all of these categories are module categories. By ISBELL's result when one drops the condition of commutativity for the category of commutative groups there is no longer a cogenerator. We have proved similar results for the categories of commutative Lie algebras and commutative restricted Lie algebras. The results are summarized in the list below where we have included some related categories.

Category		Generator	Cogenerator
Gr	Groups	Yes	No
L	Lie algebras	Yes	No
L_p	Restricted Lie algebras	Yes	No
L^{fin}	Finite dimensional Lie algebras	Yes	No
L_p^{fin}	Finite dimensional restricted Lie algebras	No	No
Gr^{fin}	Finite groups	No	No
F	Formal group schemes	Yes	No
CCH	Cocommutative Hopf algebras	Yes	No
A_{red}	Reduced affine algebraic group schemes	No	No
A	Affine algebraic group schemes in char $p > 0$	No	No
H	Hopf algebras	Yes	No
CH	Commutative Hopf algebras	Yes	?
CCCH	Commutative cocommutative Hopf algebras	Yes	?

Most of the negative results in the list were derived from results about groups and(restricted) Lie algebras, in particular from a construction of simple (restricted) Lie algebras of arbitrarily high (infinite) dimension. Most of the positive results are based on the construction of the free Hopf algebra on a coalgebra.

Unfortunately all the "no"'s in the list give examples of categories where the special adjoint functor theorem cannot be applied.

Cogenerators

Let $\underline{C}$ be a category with zero object and difference cokernels. We call an object $A \in \underline{C}$ simple if for any difference cokernel diagram

$$B \rightrightarrows A \xrightarrow{f} C$$

either f is a zero morphism or an isomorphism. $C \in \underline{C}$ is a <u>cogenerator</u> if $Mor_{\underline{C}}(-,C)$ is a faithful functor.

<u>LEMMA 1. Let $\underline{C}$ be a category with a zero object, kernel pairs, difference cokernels, and a cogenerator. Then every simple object $A \in \underline{C}$ admits a monomorphism into the cogenerator C.</u>

Proof: For the simple object 0 (zero object in $\underline{C}$) this is clear by definition [3, 1.7 Lemma 2]. Let $A \neq 0$ be simple. Then $Mor_{\underline{C}}(A,C)$ contains at least one non-zero morphism $f: A \longrightarrow C$, for if $Mor_{\underline{C}}(A,C) = \{0\}$ then the two different morphisms $0, 1_A: A \longrightarrow A$ are mapped by $Mor_{\underline{C}}(-,C)$ into the same morphism $\{0\} \longrightarrow \{0\}$ but $Mor_{\underline{C}}(-,C)$ is faithful.

Take a kernel pair

$$X \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} C$$

for f, then f can be factored through the difference cokernel $k: A \longrightarrow B$ of g and h:

$$\begin{array}{ccccc} X & \underset{h}{\overset{g}{\rightrightarrows}} & A & \xrightarrow{k} & B \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle j} & \\ & & C & & \end{array}$$

Since A is simple either $k = 0$ but then so is f or k is an isomorphism. By [3, 2.6 Lemma 2] the pair $(1_A, 1_A)$ is a kernel pair for k and for f [3, 2.6 Lemma 4]. Again by [3, 2.6 Lemma 2] f is a monomorphism.

<u>COROLLARY 2. Let $\underline{C}$ be a category as in the preceeding lemma which is locally small. Then $\underline{C}$ possesses only a set (not a proper class) of non-isomorphic simple objects.</u>

Proof: Two non-isomorphic simple objects define two different subobjects of the cogenerator C of $\underline{C}$. But C has only a set of different subobjects.

We need another categorical lemma to compare cogenerators in different categories.

LEMMA 3. *Let* $\underline{F}: \underline{C} \longrightarrow \underline{D}$ *be a faithful functor with a right adjoint* $\underline{G}: \underline{D} \longrightarrow \underline{C}$. *If* $\underline{D}$ *possesses a cogenerator* D *then* $\underline{G}D$ *is a cogenerator in* $\underline{C}$.

Proof: We prove that $\mathrm{Mor}_{\underline{C}}(-,\underline{G}D)$ is a faithful functor. In fact $\mathrm{Mor}_{\underline{C}}(-,\underline{G}D) \cong \mathrm{Mor}_{\underline{D}}(\underline{F}-,D) = \mathrm{Mor}_{\underline{D}}(-,D)\underline{F}$ is a composite of two faithful functors.

Now we give examples of(restricted)Lie algebras of arbitrarily high cardinality. This will enable us to prove the non-existence of cogenerators in many categories. First we show that the usual concept of a simple (restricted) Lie algebra, i.e. without a proper (restricted) Lie ideal, coincides with the one given above for simple object in the category of (restricted) Lie algebras. If I is a proper (restricted) ideal of a (restricted) Lie algebra L, then I is a (restricted) Lie algebra and

$$I \underset{0}{\overset{i}{\rightrightarrows}} L \xrightarrow{f} L/I$$

is a difference cokernel diagram with $i: I \longrightarrow L$ the inclusion. Since I is a proper ideal, f is neither the zero homomorphism nor an isomorphism. If

$$L_1 \underset{g}{\overset{f}{\rightrightarrows}} L_2 \xrightarrow{h} L_3$$

is a difference cokernel diagram then $hf = hg$, so $h(f - g)(L_1) = 0$, so the (restricted) Lie ideal I generated by $(f - g)(L_1)$ is the kernel of h . If $k: L_2 \longrightarrow L_4$ is another Lie homomorphism with $kf = kg$, then $k(I) = 0$ as above, so k can be uniquely factored through $L_2 \longrightarrow L_2/I$. Consequently this is the difference cokernel of the pair f and g. If $h: L_2 \longrightarrow L_3$ is neither a zero homomorphism nor an isomorphism, then I must be a proper (restricted) ideal.

Let V be an arbitrary k-vector space. Let $\mathrm{End}^{o}(V)$ be the set of all k-endomorphisms of V with finite dimensional image. Then $End^o(V)$ is an ideal of $End(V)$, so it is an associative ring possibly without unit. There is an isomorphism of vector spaces $\Phi: V' \otimes V \longrightarrow End^o(V)$ given by $\Phi(\sum w_i' \otimes v_i)(v) = \sum w_i'(v)v_i$, where V' is the dual vector space of V. We identify $V' \otimes V$ and $End^o(V)$ by Φ. Since

$$\Phi(\sum r_i' \otimes s_i)\Phi(\sum w_j' \otimes v_j)(v) = \sum w_j'(v)r_i'(v_j)s_i = \Phi(\sum w_j' \otimes r_i'(v_j)s_i)(v)$$

we get that the product of $End^o(V)$ is carried over to

$$(\sum r_i' \otimes s_i)(\sum w_j' \otimes v_j) = \sum w_j' \otimes r_i'(v_j)s_i$$

by the isomorphism Φ. We define the <u>trace</u> of a map $\sum w_i' \otimes v_i$ in $End^o(V)$ to be $\sum w_i'(v_i)$. Clearly this coincides with the usual trace for finite dimensional vector spaces V.

Let $L(V)$ be the set of endomorphisms in $End^o(V)$ with trace 0. Since the trace map is a k-homomorphism, $L(V)$ has codimension 1 in $End^o(V)$, so for vector spaces V of $\dim V > 1$ the dimension of $L(V)$ has larger cardinality than the dimension of V.

<u>THEOREM 4.</u> *$L(V)$ is a simple Lie algebra if $\dim V = \infty$ or if the characteristic of k does not divide $\dim V$.*

Proof: We view $End^o(V)$ as a Lie algebra with the usual bracket. If $\sum r_i' \otimes s_i$ and $\sum w_j' \otimes v_j$ represent two elements in $L(V)$, then

$$tr(\sum r_i' \otimes w_j'(s_i)v_j - \sum w_j' \otimes r_i'(v_j)s_i) = \sum r_i'(w_j'(s_i)v_j) - \sum w_j'(r_i'(v_j)s_i) = 0$$

so $L(V)$ is a Lie subalgebra of $End^o(V)$.

Let V be finite dimensional. If $\dim V \geq 3$ and the characteristic of k does not divide $\dim V$ and if I is a non-zero ideal in $L(V)$, then

$$[e_{ij},[e_{jk},[e_{ik},e_{rs}]]] = -\delta_{is}\delta_{kr}e_{ik} \quad \text{if } i \neq j \neq k.$$

So for $\sum a_{rs}e_{rs} \in I$, also $-a_{ki}e_{ik} \in I$. If $a_{ki} \neq 0$, then $e_{ik} \in I$. If all $a_{ki} = 0$ for $i \neq k$, then $\sum a_i e_{ii}$

is in I. But then

$$[e_{jk}, \sum a_i e_{ii}] = (a_j - a_k)e_{jk} .$$

Since the characteristic of k does not divide $\dim V$, we get again $e_{jk} \in I$ for some $j \neq k$. But $[L(V), e_{jk}] = L(V)$, so $L(V)$ is simple. For $\dim V = 2$ and $\operatorname{char} k \neq 2$ it is trivial to show that $L(V)$ will again be simple.

Now let V be infinite dimensional. Let I be a non-zero ideal in $L(V)$. Let $\sum w_i' \otimes v_i \in I$ be a non-zero element in $L(V)$. Let $\sum r_j' \otimes s_j$ be an arbitrary element in $L(V)$. Write $V = A \oplus B$, where A contains the v_i's and the s_j's and where $w_i'(B) = 0$ and $r_j'(B) = 0$ for all i and j. This can be done such that A is finite dimensional with $\dim A$ prime to the characteristic of k. Then we get a monomorphism $A' \otimes A \longrightarrow V' \otimes V$, where $A' \longrightarrow V'$ comes from the projection $V \longrightarrow A$. This is an algebra homomorphism, i.e. this map preserves the products defined above. It also preserves the trace. So this map defines an injection of Lie algebras $L(A) \longrightarrow L(V)$. Since A is finite dimensional, $L(A)$ is simple. By construction we have $\sum r_j' \otimes s_j$ and $\sum w_i' \otimes v_i$ in $L(A)$. So $I \cap L(A) \neq 0$ is an ideal of $L(A)$. Since $L(A)$ is simple, $I \supseteq L(A) \ni \sum r_j' \otimes s_j$. This proves that every element of $L(V)$ is in I, so $L(V)$ is simple.

COROLLARY 5. $L(V)$ *is a simple restricted Lie algebra if the characteristic of* k *is* $p > 0$ *and if* $\dim V = \infty$ *or* $p \nmid \dim V$.

Proof: We show that $\operatorname{tr}(f^p) = \operatorname{tr}(f)^p$. Let f be represented by $\sum w_i' \otimes v_i$. Then

$$\operatorname{tr}((\sum w_i' \otimes v_i)^p) =$$

$$\sum_{i_1} \cdots \sum_{i_p} w_{i_p}'(v_{i_1}) w_{i_1}'(v_{i_2}) \cdots w_{i_{p-1}}'(v_{i_p}) =$$

$$\sum_{(i_1,\ldots,i_p) \in T} p \cdot w_{i_p}'(v_{i_1}) \cdots w_{i_{p-1}}'(v_{i_p}) + \sum_{i=1}^{n} w_i'(v_i)^p$$

where T is a set of representatives for the orbits of

$$\{(i_1,\ldots,i_p) \mid i_j = 1,\ldots,n;\ \text{not all } i_j \text{ equal}\}$$

under the subgroup of the symmetric group S_p generated by the cycle $(1,\ldots,p)$. So $tr((\sum w_i' \otimes v_i)^p) = (tr(\sum w_i' \otimes v_i))^p$. In particular if $f \in L(V)$, then $f^p \in L(V)$ so $L(V)$ is a restricted Lie algebra which is simple even as a Lie algebra.

<u>COROLLARY 6.</u> a) <u>The category of Lie algebras over a field</u> k <u>does not contain a cogenerator</u>.
b) <u>The category of restricted Lie algebras over a field</u> k <u>of characteristic</u> $p > 0$ <u>does not contain a cogenerator</u>.

Proof: Both categories have a zero-object, kernel pairs, and difference cokernels [3, 3.2 Satz and 3.4 Korollar 3] and are locally small [3, 3.2 Korollar 2]. So corollary 2 applies. Since for any cardinal there is a simple (restricted) Lie algebra of larger cardinality than the given cardinal, there is more than a set of simple (restricted) Lie algebras. So there cannot be a cogenerator.

<u>COROLLARY 7. The category of finite dimensional (restricted) Lie algebras over a field</u> k <u>does not contain a cogenerator</u>.

Proof: A cogenerator would have a certain finite dimension but we have seen tnat for any n there are (restricted) simple Lie algebras of finite dimension bigger than n.

<u>Remark</u>: A similar argument shows that the category of finite groups does not contain a cogenerator.

<u>PROPOSITION 8. The category of formal group schemes over a field</u> k <u>does not contain a cogenerator</u>.

Proof: The category of (restricted) Lie algebras allows a full faithful covariant functor into the category of formal group schemes over a field k which has a right adjoint functor [1]. By lemma 3 and corollary 6 the category of formal group schemes over k cannot contain a cogenerator.

A <u>Hopf algebra</u> will always be a Hopf algebra with antipode. Then again by [1] the category of formal group schemes over k is equivalent to the category of cocommutative Hopf algebras over k.

COROLLARY 9. *The category of cocommutative Hopf algebras over a field k does not contain a cogenerator.*

For the definition of <u>affine group schemes</u> over a field k we refer to [5]. An <u>affine algebraic group scheme</u> is then an affine group scheme, the affine k-algebra of which is finitely generated as an algebra.

PROPOSITION 10. a) *The category of reduced affine algebraic group schemes over an infinite field k does not contain a cogenerator.*
b) *The category of affine algebraic group schemes over a field k of characteristic $p > 0$ does not contain a cogenerator.*

Proof: a) Let Sl(m,k) denote the special linear group in $m \times m$-matrices over k. If Z is the center of Sl(m,k) then Z is finite [4, p.158 Thm.8.18] and if $m > 2$ Sl(m,k)/Z is a simple group [4, p.169 Thm.8.27]. Now suppose N is a closed normal subgroup of Sl(m,k) and $m > 2$. Let $q: Sl(m,k) \longrightarrow Sl(m,k)/Z$ be the natural projection. The simplicity of Sl(m,k)/Z implies that either $q(N) = \{e\}$ or $q(N) = Sl(m,k)/Z$. In the first case $N \subset Z$ so that N is finite and $\dim N = 0$. In the second case $NZ = Sl(m,k)$. Since N is closed and Sl(m,k) is connected it follows that $N = Sl(m,k)$.

Now suppose C is a cogenerator in the category of reduced affine algebraic group schemes. Then C has some "algebraic" dimension n. Since $\dim Sl(m,k) = m^2 - 1$ we can choose m so that $\dim Sl(m,k) > \dim C$. As in the proof of lemma 1 there is a non-zero morphism $f: Sl(m,k) \longrightarrow C$. Let $N = \mathrm{Ker}\, f$. Then $\dim Sl(m,k) = \dim N + \dim \mathrm{Im}\, f$. Since

f is non-zero N must not be equal to $Sl(m,k)$ so that N must have dimension 0 as shown above. Thus $\dim Sl(m,k) = \dim \operatorname{Im} f$. But $\dim Sl(m,k) > \dim C$ so that $\dim Sl(m,k) > \dim \operatorname{Im} f$ a contradiction. Thus no such C exists.

b) The functor which assigns to each finite dimensional restricted Lie algebra the dual of its restricted universal enveloping algebra viewed as an affine algebra of an affine algebraic group scheme is faithful and has as a right adjoint functor the functor which assigns to each affine algebraic group scheme its restricted Lie algebra which is finite dimensional [5]. So we can apply lemma 3 and corollary 7 to obtain the claimed result.

PROPOSITION **11.** The category of Hopf algebras over a field k does not contain a cogenerator.

We have the functor $\underline{U}$ from (restricted) Lie algebras to Hopf algebras where $\underline{U}(L)$ is the (restricted) enveloping algebra of L. If H is any Hopf algebra then the primitive elements $\underline{P}(H) = \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\}$ of H form a(restricted) Lie algebra under the bracket (and associative p^{th} power map, if $\operatorname{char} k = p > 0$). If H_1 and H_2 are Hopf algebras and $f\colon H_1 \longrightarrow H_2$ a Hopf algebra map then $f(\underline{P}(H_1)) \subset \underline{P}(H_2)$, because f is a coalgebra map. Also

$$f|_{\underline{P}(H_1)}\colon \underline{P}(H_1) \longrightarrow \underline{P}(H_2)$$

is a (restricted) Lie algebra map. Thus the functor $\underline{P}$ from Hopf algebras to (restricted) Lie algebras is a right adjoint to $\underline{U}$. By the(restricted) Birkhoff-Witt Theorem the natural map $L \longrightarrow \underline{U}(L)$ is injective so that $\underline{U}$ is a faithful functor. Thus by lemma 3 and corollary 6 the category of Hopf algebras cannot contain a cogenerator.

The same argument shows that there is no cogenerator in the category of cocommutative Hopf algebras. In fact the same functors have been used for the proof of proposition 8.

Generators

Both the categories of Lie algebras and restricted Lie algebras contain generators [3, 3.4 Satz 3]. A generator for the category of Lie algebras is the one dimensional abelian Lie algebra; this is the free Lie algebra on one element. This is also a generator in the category of finite dimensional Lie algebras.

We now show there is no generator in the category of finite dimensional restricted Lie algebras (or even finite dimensional abelian restricted Lie algebras).

Let X_n denote the abelian restricted Lie algebra with basis $x_1,\ldots,x_n$ where $x_i^{[p]} = x_{i+1}$ for $i = 1,\ldots,n-1$ and $x_n^{[p]} = 0$.

LEMMA 12. *If* $L \neq \{0\}$ *is a restricted Lie subalgebra of* X_n *then there is* $1 \leq t \leq n$ *such that* L *is the span of* $x_t,\ldots,x_n$.

Proof: Choose an element $x = a_t x_t + \ldots + a_n x_n$ in L with $a_t \neq 0$ and t minimal. We show that L contains $x_t,\ldots,x_n$. By the minimality of t in the choice of x this shows that L is spanned by $x_t,\ldots,x_n$. To show $x_t,\ldots,x_n \in L$ it suffices to prove $x_t \in L$ since L is closed under p^{th} powers.

Replacing x by x/a_t shows we may assume $a_t = 1$ and $x = x_t + \ldots + a_n x_n$. Replacing x by $x - a_{t+1}x^{[p]}$ shows that we may assume $a_{t+1} = 0$, then replacing x by $x - a_{t+2}x^{[p]^2}$ shows that we may assume $0 = a_{t+1} = a_{t+2}$. Continuing in this manner shows that $x_t \in L$.

PROPOSITION 13. *There is no generator in the category of finite dimensional (abelian) restricted Lie algebras.*

Proof: Suppose G is a generator. Choose $n > \dim G$.

Consider the restricted Lie algebra map $f: X_n \longrightarrow X_1$ with

$$f(x_i) = \begin{cases} x_1 & \text{for } i = 1 \\ 0 & \text{for } i = 2,\dots,n. \end{cases}$$

Let g be the zero map from X_n to X_1. Since G is a generator there is a restricted Lie algebra map $h: G \longrightarrow X_n$ with $fh \neq gh$. Thus $\operatorname{Im} h$ does not lie in the span of $x_2,\dots,x_n$. Since $\operatorname{Im} h$ is a restricted Lie subalgebra of X_n it follows from lemma 12 that $\operatorname{Im} h$ is all of X_n. This contradicts the fact that $n > \dim G$.

Remark: A similar type of proof shows that there is no generator in the category of finite groups. A generator G has to map non-trivially into the cyclic group of order p for each prime p. Thus G has at least p elements which is a contradiction.

PROPOSITION 14. There is no generator in the category of (reduced) affine algebraic group schemes over an algebraicly closed field.

Proof: If G is a (reduced) affine algebraic group scheme over an algebraicly closed field k, then G is a finite union of connected components each of which has at least one rational point [5, 2 Thm.6.4]. So G has a finite constant quotient group scheme $p: G \longrightarrow F$ "of connected components". Given any finite constant group scheme H and map of algebraic group schemes $f: G \longrightarrow H$ the map f factors uniquely through $p: G \longrightarrow F$. So the constant group scheme functor from the category of finite groups to the category of (reduced) affine algebraic group schemes has the left adjoint functor "group of connected components" and clearly is faithful. Thus we can apply the dual of lemma 3 and the fact that there is no generator in the category of finite groups to get the result.

We now show that there is a generator in the category of Hopf algebras. The same techniques show that there is a generator in the category of commutative Hopf algebras, the category of cocommutative Hopf algebras, and the category of commutative cocommutative Hopf algebras

Five categories. Let k be the ground field and let $\underline{H}$ be the category whose objects are Hopf algebras over k. For A,B objects in $\underline{H}$ we let $\underline{H}(A,B)$ be all Hopf algebra morphisms from A to B.

Let $\underline{C}$ be the category whose objects are coalgebras over k. For A,B in $\underline{C}$ let $\underline{C}(A,B)$ be all coalgebra maps from A to B.

$\underline{C}_1$ is a full subcategory of $\underline{C}$. An object A of $\underline{C}$ is in $\underline{C}_1$ if the coalgebra A contains at least one subcoalgebra which is isomorphic to the coalgebra k. For A,B in $\underline{C}_1$ let $\underline{C}_1(A,B)$ be all coalgebra maps from A to B.

An object of the category $\underline{D}$ is a coalgebra A together with a coalgebra antimorphism $S_A: A \longrightarrow A$. (S_A is a coalgebra antimorphism means that $\varepsilon S_A = \varepsilon$ and $(S_A \otimes S_A)\Delta = T\Delta S_A$, where T is the twist map which interchanges the right and the left tensorands.) We do not require that S_A has order 2 or is surjective or injective. For A,B in $\underline{D}$ we let $\underline{D}(A,B)$ consist of all coalgebra maps f where $fS_A = S_Bf$.

$\underline{D}_1$ is a full subcategory of $\underline{D}$. An object A of $\underline{D}$ is in $\underline{D}_1$ if the coalgebra A contains at least one subcoalgebra which is isomorphic to the coalgebra k. We do not require that this subcoalgebra be stable under S_A or in any other way be related to S_A. For A,B in $\underline{D}_1$ let $\underline{D}_1(A,B)$ be all coalgebra maps f from A to B where $fS_A = S_Bf$.

Let $\underline{U}$ be the functor from $\underline{D}$ to $\underline{C}$ where for any object A in $\underline{D}$, $\underline{U}(A)$ is the coalgebra with S_A forgotten. For $f \in \underline{D}(A,B)$ we let $\underline{U}(f)$ be f viewed solely as a coalgebra map. Note that $\underline{U}$ carries the category $\underline{D}_1$ to $\underline{C}_1$. We let $\underline{U}_1$ be the functor from $\underline{D}_1$ to $\underline{C}_1$ which is the restriction of $\underline{U}$.

LEMMA 15. *$\underline{U}$ has a left adjoint $\underline{A}$ which carries $\underline{C}_1$ to $\underline{D}_1$. If we let $\underline{A}_1$ be the functor from $\underline{C}_1$ to $\underline{D}_1$ which is the restriction of $\underline{A}$ then $\underline{A}_1$ is a left adjoint to $\underline{U}_1$.*

Proof: For a coalgebra A let $\bar{A}$ be a vector space which is isomorphic to A by the isomorphism $A \longrightarrow \bar{A}$, $a \longmapsto \bar{a}$. Let $\bar{A}$ have the opposite coalgebra structure to A so that for $\bar{a} \in \bar{A}$

$$\Delta(\bar{a}) = \sum_{(a)} \overline{a_{(2)}} \otimes \overline{a_{(1)}}$$

$$\varepsilon(\bar{a}) = \varepsilon(a).$$

Then $\bar{\ }: A \longrightarrow \bar{A}$ is a coalgebra antimorphism.

We define $\underline{A}(A)$ to be the coalgebra which is the direct sum $A \oplus \bar{A} \oplus A \oplus \bar{A} \oplus A \oplus \bar{A} \oplus \ldots$. $S_{\underline{A}(A)}$ is the coalgebra antimorphism given by

$$(a_o, \overline{a_1}, a_2, \overline{a_3}, \ldots) \longmapsto (0, \overline{a_o}, a_1, \overline{a_2}, a_3, \ldots).$$

Suppose B is an object in $\underline{D}$ and $f \in \underline{D}(\underline{A}(A), B)$. Let $f' \in \underline{C}(A, \underline{U}(B))$ be given by $f'(a) = f(a, 0, 0, \ldots)$. If $g \in \underline{C}(A, \underline{U}(B))$ then define g^i as follows:

$$g^o = g\colon A \longrightarrow \underline{U}(B)$$

$$g^i = S_B{}^i \cdot g\colon A \longrightarrow \underline{U}(B) \quad \text{for } 1 < i \text{ and } i \text{ even,}$$

$$g^i = S_B{}^i \cdot g \cdot (\bar{\ })^{-1}\colon \bar{A} \longrightarrow \underline{U}(B) \quad \text{for } 0 < i \text{ and } i \text{ odd.}$$

Each g^i is a coalgebra map and the g^i's induce

$$\tilde{g} = g^o \oplus g^1 \oplus \ldots\colon \underline{A}(A) \longrightarrow B$$

and $\tilde{g} \in \underline{D}(\underline{A}(A), B)$.

One easily checks that the correspondences

$$\begin{array}{ccc} \underline{D}(\underline{A}(A),B) & \longleftrightarrow & \underline{C}(A,\underline{U}(B)) \\ f & \longmapsto & f' \\ \tilde{g} & \longleftarrow & g \end{array}$$

are inverse to each other and functorial in A and B. Thus $\underline{A}$ is left adjoint to $\underline{U}$.

The remaining claims are easily verified.

Consider the functor $\underline{V}$ from $\underline{H}$ to $\underline{D}$. If A is a Hopf algebra then the antipode S_A is a coalgebra antimorphism of the underlying coalgebra structure [2]. We let $\underline{V}(A)$ be the underlying coalgebra of A with the antimorphism S_A. If A,B are in $\underline{H}$ and $f \in \underline{H}(A,B)$ then $\underline{V}(f)$ is f considered as a coalgebra map which preserves the antimorphism structure. Note that $\underline{V}$ actually carries $\underline{H}$ to $\underline{D}_1$ since any Hopf algebra A contains a copy of k in the form of $k1_A$. Let $\underline{V}_1$ be $\underline{V}$ with its range restricted to $\underline{D}_1$.

PROPOSITION 16. *$\underline{V}$ has a left adjoint $\underline{B}$. If we let $\underline{B}_1$ be the restriction of $\underline{B}$ to $\underline{D}_1$ then $\underline{B}_1$ is a left adjoint to $\underline{V}_1$.*

Proof: Suppose A is an object in $\underline{D}$. Let $\underline{F}A$ be the free algebra on A over k. By the universal property of $\underline{F}A$ there are unique algebra maps $\varepsilon_F: \underline{F}A \longrightarrow k$ and $\Delta_F: \underline{F}A \longrightarrow \underline{F}A \otimes \underline{F}A$ making commute

$$\begin{array}{ccc} A & \xrightarrow{\iota} & \underline{F}A \\ & \varepsilon\searrow \quad \swarrow\varepsilon_F & \\ & k & \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\iota} & \underline{F}A \\ \Delta\downarrow & & \downarrow\Delta_F \\ A \otimes A & \xrightarrow{\iota \otimes \iota} & \underline{F}A \otimes \underline{F}A \end{array}$$

where $\iota: A \longrightarrow \underline{F}A$ is the natural inclusion. Thus $\underline{F}A$ becomes the free bialgebra on A in that given any bialgebra L and coalgebra map $\xi: A \longrightarrow L$ there is a unique bialgebra map $\Xi: \underline{F}A \longrightarrow L$ where $\Xi\iota = \xi$ [6, p.62 Exercise 1]. By the universal property of $\underline{F}A$ there is a unique algebra antimorphism $S_F: \underline{F}A \longrightarrow \underline{F}A$ making commute

$$\begin{array}{ccc} A & \xrightarrow{\iota} & \underline{F}A \\ S_A\downarrow & & \downarrow S_F \\ A & \xrightarrow{\iota} & \underline{F}A \end{array}.$$

S_F is not necessarily an antipode for $\underline{F}A$ and we must factor out the necessary relations to make S_F an antipode.

Let $L: \underline{F}A \longrightarrow \underline{F}A$, $L(h) = \sum_{(h)} S_F(h_{(1)})h_{(2)} - \varepsilon(h)$ and

$R\colon \underline{F}A \longrightarrow \underline{F}A, \quad R(h) = \sum_{(h)} h_{(1)}S_F(h_{(2)}) - \varepsilon(h).$

A bit of calculation shows that

$$\Delta L(h) = \sum_{(h)} L(h_{(2)}) \otimes S_F(h_{(1)})h_{(3)} + 1 \otimes L(h),$$

$$\Delta R(h) = \sum_{(h)} h_{(1)}S_F(h_{(3)}) \otimes R(h_{(2)}) + R(h) \otimes 1,$$

$$S_F L(h) = RS_F(h),$$

$$S_F R(h) = LS_F(h),$$

for all $h \in \underline{F}A$. Thus by [6, p.88 Exercise] if we let J be the ideal generated by $\operatorname{Im} R + \operatorname{Im} L$ we have that J is a biideal and $\underline{F}A/J$ is a quotient bialgebra of $\underline{F}A$. From the last two relations we see that $S_F(J) \subset J$ so that $\underline{F}A/J$ has a coalgebra and algebra antimorphism $S_{F/J}$ induced by S_F. It is clear from the relations which generate J that $S_{F/J}$ is an antipode for $\underline{F}A/J$. Thus $\underline{F}A/J$ is a Hopf algebra with antipode $S_{F/J}$. Let $\pi\colon \underline{F}A \longrightarrow \underline{F}A/J$ be the natural projection. The following results are easily verified:

1. The diagram

$$\begin{array}{ccccc} A & \xrightarrow{\iota} & \underline{F}A & \xrightarrow{\pi} & \underline{F}A/J \\ {\scriptstyle S_A}\downarrow & & \downarrow{\scriptstyle S_F} & & \downarrow{\scriptstyle S_{F/J}} \\ A & \xrightarrow{\iota} & \underline{F}A & \xrightarrow{\pi} & \underline{F}A/J \end{array}$$

commutes.

2. $\pi\iota\colon A \longrightarrow \underline{F}A/J$ is a coalgebra map.

3. If L is a Hopf algebra and $\Xi\colon \underline{F}A \longrightarrow L$ is a bialgebra map then there exists a unique Hopf algebra map $\equiv\colon \underline{F}A/J \longrightarrow L$ where $\equiv\pi = \Xi$.

4. If L is a Hopf algebra and $\lambda\colon A \longrightarrow L$ is a coalgebra map where $\lambda S_A = S_L\lambda$ then there is a unique Hopf algebra map $\Lambda\colon \underline{F}A/J \longrightarrow L$ with $\Lambda(\pi\iota) = \lambda$.

Of course 4. shows that if we define $\underline{B}(A)$ to be $\underline{F}A/J$ then $\underline{B}$ is a left adjoint to $\underline{V}$. The remaining claims are easily verified.

THEOREM 17. *The category* $\underline{C}_1$ *has a generator* G.

Proof: The isomorphism classes of finite dimensional coalgebras over k form a set. This is true because if V is an n-dimensional vector space over k then a coalgebra structure on V consists of an element of $\mathrm{Hom}_k(V,k)$ and an element of $\mathrm{Hom}_k(V,V \otimes V)$ satisfying certain conditions. Thus the set of coalgebra structures on V is a certain set $C_n \subset \mathrm{Hom}_k(V,k) \times \mathrm{Hom}_k(V,V \otimes V)$. I_n, the isomorphism classes of n-dimensional coalgebras, is a certain set of equivalence classes of C_n. Thus the set of isomorphism classes of finite dimensional coalgebras over k is the disjoint union $I = \dot{\bigcup}_{n=0}^{\infty} I_n$.

For each $x \in I$ let C_x be a coalgebra in the isomorphism class of x. Let G be the coalgebra $\bigoplus_{x \in I} C_x$. Since the isomorphism class of k is in I we have that G has a subcoalgebra which is isomorphic to k. Thus G is in the category $\underline{C}_1$.

Let A be any coalgebra in the category $\underline{C}_1$. Let B be a subcoalgebra of A which is isomorphic to k. Given any $a \in A$ let C be the subcoalgebra of A which is generated by a [6, p.45 Definition]. By [6, p.46 Thm.2.2.1] C is finite dimensional and there is a unique element $z \in I$ which is the isomorphism class of C.

Fix a coalgebra isomorphism $\psi_z: C_z \longrightarrow C$. For $x \in I$ where $x \neq z$ let $\psi_x: C_x \longrightarrow A$ be the composite

$$C_x \xrightarrow{\varepsilon_x} k \cong B \longrightarrow A .$$

Then $\psi = \bigoplus_{x \in I} \psi_x: G \longrightarrow A$ is a coalgebra map with $a \in C \subset \mathrm{Im}\ \psi$. This proves that G is a generator for $\underline{C}_1$.

COROLLARY 18. *The categories* $\underline{D}_1$ *and* $\underline{H}$ *have generators.*

Proof: The functor $\underline{U}_1$ from $\underline{D}_1$ to $\underline{C}_1$ is faithful. Thus $\underline{A}_1(G)$ is a generator for $\underline{D}_1$. The functor $\underline{V}_1$ from $\underline{H}$ to $\underline{D}_1$ is faithful. Thus $\underline{B}_1\underline{A}_1(G)$ is a generator for $\underline{H}$.

COROLLARY 19. <u>The categories of commutative Hopf algebras, of cocommutative Hopf algebras, and of commutative cocommutative Hopf algebras have generators</u>.

Proof: To prove that there is a generator in the category of cocommutative Hopf algebras, proceed as above but replace the term "coalgebra" by the term "cocommutative coalgebra" throughout.

To prove that there is a generator in the category of commutative Hopf algebras let $\underline{H}$ be the category of commutative Hopf algebras and let $\underline{C}_1$, $\underline{C}$, $\underline{D}_1$, and $\underline{D}$ be as before. In the proof that $\underline{V}$ has a left adjoint B replace $\underline{F}A$ the free algebra on A by $\underline{S}A$ the symmetric algebra on A. The rest is unchanged.

To prove that there is a generator in the category of commutative cocommutative Hopf algebras, proceed as if proving that there is a generator in the category of commutative Hopf algebras but replace the term "coalgebra" by the term "cocommutative coalgebra" throughout.

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