L 10

CONCEPTUAL BASIS FOR CALCULATIONS OF ABSORBED-DOSE DISTRIBUTIONS

Recommendations of the
NATIONAL COUNCIL ON RADIATION PROTECTION AND MEASUREMENTS

Issued March 31, 1991

National Council on Radiation Protection and Measurements
7910 WOODMONT AVENUE / Bethesda, MD 20814

Contents

Appendix C
A Compilation of Geometric Reduction Factors for Standard Geometries

In the following, typical geometries are considered and the resulting geometric-reduction factors are given. The point-pair distance distributions are not listed but can readily be obtained from the geometric-reduction factor:

$$p_{AB}(x) = 4\pi\, x^2 U_{AB}(x)/V_B \,. \tag{C.1}$$

C.1 The Autologous Case (A = B)

As source region A and receptor B coincide in the autologous case, the simplified notation $U(x)$ can be used instead of $U_{AB}(x)$. Outside the specified intervals for x, all functions $U(x)$ or $U_{AB}(x)$ are zero.

a) *Infinite slab of height h*:

$$U(x) = \begin{cases} 1 - \dfrac{x}{2h}, & 0 \le x \le h \\[2ex] \dfrac{h}{2x}, & x \ge h. \end{cases} \tag{C.2}$$

b) *Sphere of radius r*:

$$U(x) = 1 - \frac{3x}{4r} + \frac{x^3}{16r^3}, \; 0 \le x \le 2r. \tag{C.3}$$

c) *Spherical shell with outer radius R and inner radius r.*

For $r \leq R/3$:

$$U(x) = \frac{1}{R^3 - r^3} \begin{cases} R^3 - r^3 - 3(R^2 + r^2)\,x/4 + x^3/8, & 0 \leq x \leq 2r \\ R^3 - 2r^3 - 3R^2 x/4 + x^3/16, & 2r \leq x \leq R - r \\ -r^3 + 3(R^2 - r^2)^2/8x + 3r^2 x/4 - x^3/16, & R - r < x \leq R + r \\ R^3 - 3R^2 x/4 + x^3/16, & R + r < x \leq 2R. \end{cases} \tag{C.4}$$

For $r \geq R/3$:

$$U(x) = \frac{1}{R^3 - r^3} \begin{cases} R^3 - r^3 - 3(R^2 + r^2)x/4 + x^3/8, & 0 \leq x \leq R - r \\ 3(R^2 - r^2)^2/8x, & R - r \leq x \leq 2r \\ -r^3 + 3(R^2 - r^2)^2/8x + 3r^2 x/4 - x^3/16, & 2r \leq x \leq R + r \\ R^3 - 3R^2 x/4 + x^3/16, & R + r \leq x \leq 2R. \end{cases} \tag{C.5}$$

Limit of thin spherical shell with radius r and thickness δ ($\delta \ll r$):

$$U(x) = \begin{cases} 1 - \dfrac{x}{2\delta}, & 0 \leq x \leq \delta \\ \dfrac{\delta}{2x}, & \delta < x \leq 2r. \end{cases} \tag{C.6}$$

d) *Spheroid with two axes d and the third axis e·d (Kellerer, 1984)*

$$U(x) = 1 - \frac{3x}{2d}\frac{c_1}{e} + \frac{x^3}{2d^3}\frac{c_2}{e} \tag{C.7}$$

$$+ \frac{3}{8}\frac{\epsilon}{(e^{-1} - e)}\left[\sqrt{\left|\frac{d^2}{x^2} - 1\right|}\left(\frac{x^2}{2d^2} + 1\right) + \left(\frac{x^3}{2d^3} - \frac{2x}{d}\right)\mathrm{ci}\left(\frac{d}{x}\right)\right],$$

$$\text{for } 0 < x \leq \mathrm{Max}\,(d, e{\cdot}d).$$

The expression in the first line of Equation (C.7) applies for $x \leq \mathrm{Min}(d, e{\cdot}d)$. In the case of $e<1$, the expression in the second line applies for $x > e{\cdot}d$; in the case of $e>1$, the sum of the expressions in the first and in the second lines is used for $x>d$.

In Equation (C.7), $\epsilon = \sqrt{|e^2 - 1|}$ is the eccentricity; then

$$c_1 = \frac{1}{2} + \frac{e^2}{2\epsilon}\,\mathrm{ci}\left(\frac{1}{e}\right), \qquad c_2 = \frac{1}{4e^2} + \frac{3}{4}c_1, \tag{C.8}$$

and

$$\mathrm{ci}(x) = \begin{cases} \cos^{-1}(x), & \text{for } 0 \le x \le 1, \\ \cosh^{-1}(x) = \ell n(x + \sqrt{x^2-1}), & \text{for } x>1. \end{cases} \quad \text{(C.9)}$$

The auxiliary function ci(x) is introduced to permit one common formula for the oblate and the prolate spheroid (Kellerer, 1984). Throughout Appendix C, $f^{-1}(x)$ is defined as the inverse function of f(x).

e) *Right cylinder with arbitrary cross section of width d and height h (Kellerer, 1981):*

$$U(x) = \frac{1}{x}\int_{z_1}^{z_2}\left(1 - \frac{z}{h}\right) U_c(\sqrt{x^2-z^2})\,dz, \; x \le \sqrt{h^2+d^2}\,, \quad \text{(C.10)}$$

$$z_1 = \sqrt{\mathrm{Max}(o, x^2-d^2)}\,;\; z_2 = \mathrm{Min}(x,h)\,. \quad \text{(C.11)}$$

$U_c(x)$ is the geometric-reduction factor of the cross section ($x \le d$) as follows:

Circular cross section:

$$U_c(x) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{x}{d}\right) - \frac{x}{d}\sqrt{1 - \frac{x^2}{d^2}}\right], \qquad x \le d. \quad \text{(C.12)}$$

Square cross section (see Coleman, 1969):

$$U_c(x) = \frac{1}{\pi}\begin{cases} \dfrac{x^2}{d^2} - \dfrac{4x}{d} + \pi\,, & x \le d \\ \pi - 2 - 4\cos^{-1}\left(\dfrac{d}{x}\right) + 4\sqrt{\dfrac{x^2}{d^2} - 1} - \dfrac{x^2}{d^2}\,, & d < x \le d\sqrt{2}\,. \end{cases} \quad \text{(C.13)}$$

Integrating the expression, one obtains for the unit cube the relation derived by Piefke (1978):

$$U(x) = \begin{cases} 1 - \dfrac{3}{2}x + \dfrac{2}{\pi}x^2 - \dfrac{1}{4\pi}x^3, & o \le x \le 1 \\ \dfrac{6\pi-1}{4\pi x} - 2 + \dfrac{3x}{2\pi} + \dfrac{x^3}{2\pi} + \dfrac{6}{\pi}x\cos^{-1}\left(\dfrac{1}{x}\right) - \dfrac{2}{\pi x}(2x^2+1)\sqrt{x^2-1}, & 1 < x \le \sqrt{2} \\ \dfrac{6\pi-5}{4\pi x} + 1 - \dfrac{3(1+\pi)x}{2\pi} - \dfrac{x^3}{4\pi} + \dfrac{2}{\pi x}(x^2+1)\sqrt{x^2-2} - \dfrac{6}{\pi x}A(x), & \sqrt{2} < x \le \sqrt{3}, \end{cases}$$

(C.14)

where

$$A(x) = \tan^{-1}(\sqrt{x^2-2}) + 2x\tan^{-1}(x^2-1-x\sqrt{x^2-2}) - x^2\tan^{-1}\left(\frac{1}{\sqrt{x^2-2}}\right).$$

Figure C.1 shows geometric-reduction factors and point-pair distance distributions for the autologous case according to some of the preceding equations.

C.2 The Heterologous Case (A ≠ B)

a) *Two concentric spheres* A *and* B *with radii R and r, respectively:*

$$U_{AB}(x) = \begin{cases} \mathrm{Min}(1, r^3/R^3), & x<|R-r| \\ \frac{3}{4R^3}\left[\frac{2}{3}(R^3+r^3) - \frac{1}{4}(R^2-r^2)^2\frac{1}{x} - \frac{1}{2}(R^2+r^2)x + \frac{1}{12}x^3\right], & |R-r|<x<R+r. \end{cases} \quad (C.15)$$

b) *A spherical surface,* A, *of radius R and a concentric sphere,* B, *of radius r:*

$$U_{AB}(x) = \begin{cases} \frac{1}{2} + \frac{r^2-R^2}{4Rx} - \frac{x}{4R}, & |R-r|<x<R+r \\ 1, & R<r \text{ and } x<r-R. \end{cases} \quad (C.16)$$

c) *A spherical surface,* A, *of radius R and a concentric shell,* B, *of inner radius* r_1 and outer radius r_2:

The geometric-reduction factor is equal to the preceding solution with $r=r_2$ minus the solution with $r=r_1$.

For $R>r_2$, i.e., for A outside B:

$$U_{AB}(x) = \begin{cases} \frac{1}{2} + \frac{r_2^2-R^2}{4Rx} - \frac{x}{4R}, & \begin{cases} R-r_2\le x\le R-r_1 \\ R+r_1\le x\le R+r_2 \end{cases} \\ \frac{r_2^2-r_1^2}{4Rx}, & R-r_1<x<R+r_1. \end{cases} \quad (C.17)$$

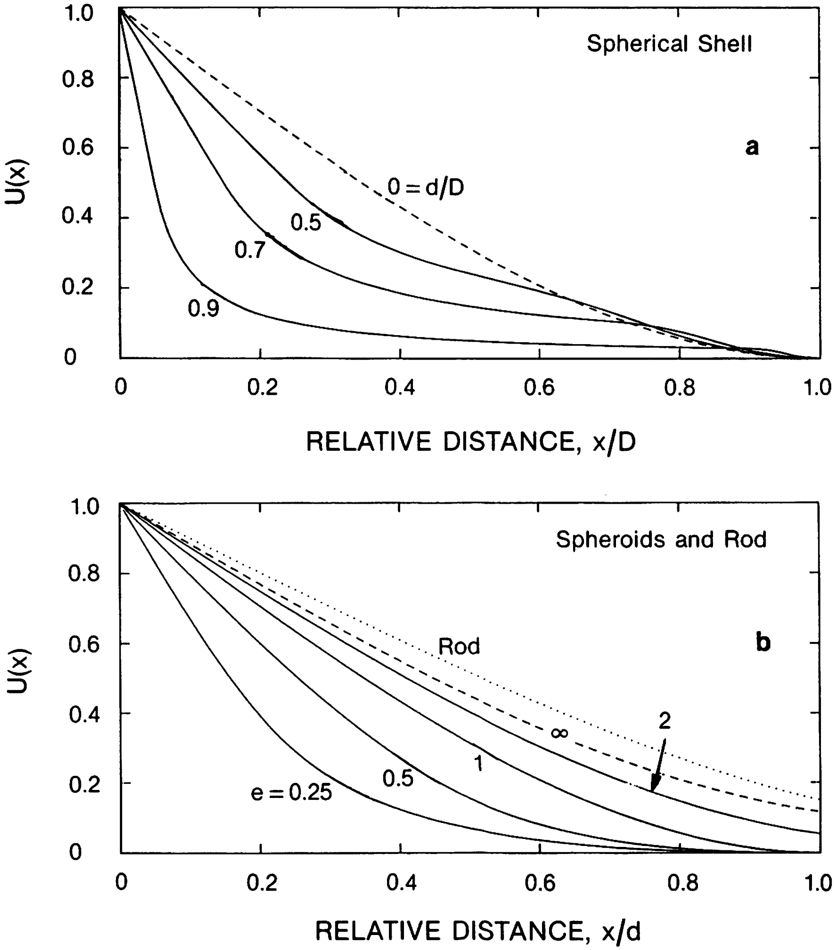
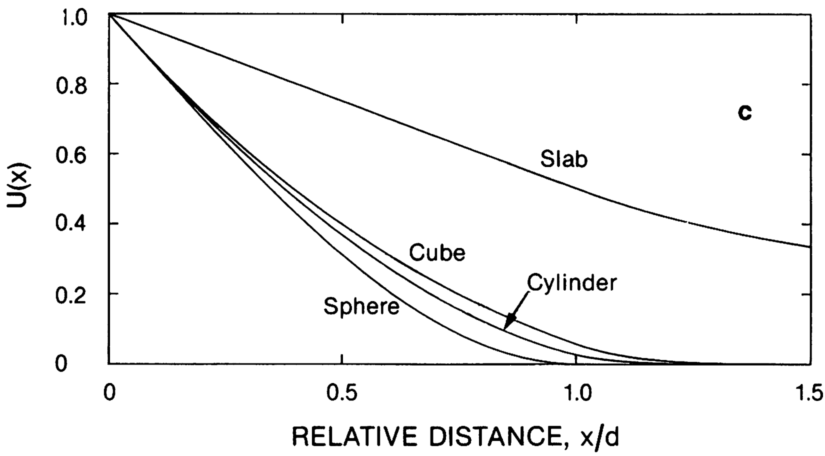


Fig. C.1. Geometric-reduction factors, U(x), or the point-pair distance distributions, p(x), for the autologous case.

a: Geometric-reduction factors for spherical shells according to Equations (C.4) and (C.5). The distance, x, is given relative to the outer diameter, D. The ratio of inner diameter, d, to outer diameter of the shell is indicated on the curves. The dashed curve corresponds to the sphere [Equation (C.3)].

b: Geometric-reduction factors for spheroids according to Equation (C.7). The distance, x, is given relative to the two equal diameters, d, of the spheroid. The ratio e of the third diameter to d is indicated on the curves. The dashed curve represents the limit of an infinitely extended prolate spheroid, while the dotted curve represents, for comparison, an infinitely long circular rod of diameter d [Equations (C.10) and (C.12)].



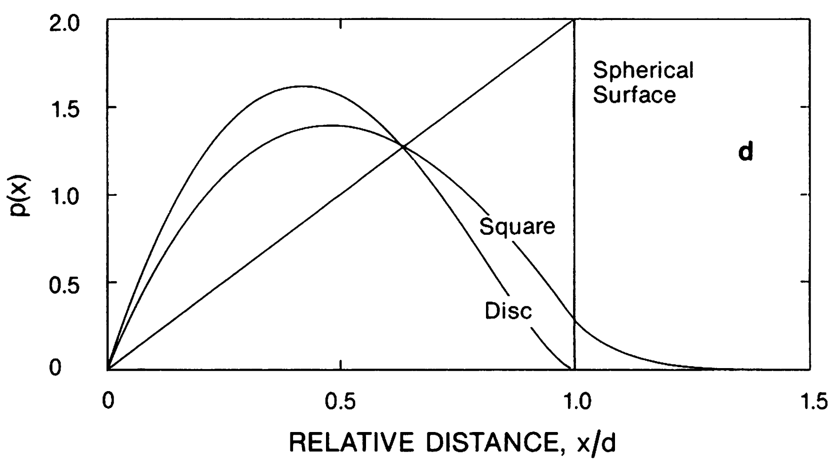


Fig. C.1. *continued*

c: Geometric-reduction factors for the sphere of diameter d [Equation (C.3)], the circular right cylinder of diameter and height d [Equations (C.10) and (C.12)], the cube of side length d [Equation (C.14)], and the infinite slab of height d [Equation (C.2)]. The distance, x, is given relative to d.

d: Point-pair distance densities for a disc of diameter d [Equation (C.12)], a square of side length d [Equation (C.13)], and a spherical surface of diameter d [Equation (C.6)]. The distance, x, is given relative to d.

For $R < r_1$, *i.e.*, for A in the interior void of the shell for $R \geq (r_2 - r_1)/2$:

$$U_{AB}(x) = \begin{cases} \frac{1}{2} - \frac{r_1^2 - R^2}{4Rx} + \frac{x}{4R}, & r_1 - R \leq x \leq r_2 - R \\ \frac{r_2^2 - r_1^2}{4Rx}, & r_2 - R \leq x \leq R + r_1 \\ \frac{1}{2} + \frac{r_2^2 - R^2}{4Rx} - \frac{x}{4R}, & R + r_1 < x \leq R + r_2. \end{cases} \tag{C.18}$$

For $R \leq (r_2 - r_1)/2$

$$U_{AB}(x) = \begin{cases} \frac{1}{2} - \frac{r_1^2 - R^2}{4Rx} + \frac{x}{4R}, & r_1 - R \leq x \leq r_1 + R \\ 1, & r_1 + R < x \leq r_2 - R \\ \frac{1}{2} + \frac{r_2^2 - R^2}{4Rx} - \frac{x}{4R}, & r_2 - R < x \leq r_2 + R. \end{cases} \tag{C.19}$$

Special cases:

An outer surface of the shell ($R = r_2$):

$$U_{AB}(x) = \begin{cases} \frac{1}{2} - \frac{x}{4r_2}, & \begin{cases} 0 < x < r_2 - r_1 \\ r_1 + r_2 \leq x \leq 2r_2 \end{cases} \\ \frac{r_2^2 - r_1^2}{4r_2 x}, & r_2 - r_1 < x < r_1 + r_2; \end{cases} \tag{C.20}$$

An inner surface of the shell ($R = r_1$) and $R \geq r_2/3$:

$$U_{AB}(x) = \begin{cases} \frac{1}{2} + \frac{x}{4R}, & 0 \leq x \leq r_2 - r_1 \\ \frac{r_2^2 - r_1^2}{4r_1 x}, & r_2 - r_1 < x \leq 2r_1 \\ \frac{1}{2} + \frac{r_2^2 - r_1^2}{4r_1 x} - \frac{x}{4r_1}, & 2r_1 < x \leq r_1 + r_2. \end{cases} \tag{C.21}$$

d) *Two concentric shells* A *and* B *of infinitesimal thickness and with radii R and r:*

$$U_{AB}(x) = \frac{r\delta}{2Rx}, \qquad |R-r|<x<R+r, \tag{C.22}$$

where δ is the (infinitesimal) thickness of the shell B.

Note: Solutions b) to d) apply also if A is part of the spherical surface, e.g., a sector, a spherical cap, a ring within the shell or a point.

e) *A plane* A *(or part of a plane) and a parallel plate* B *of thickness h at distance a:*

$$U_{AB}(x) = \begin{cases} \frac{1}{2} - \frac{a}{2x}, & a \le x \le a+h \\ \frac{h}{2x}, & a+h<x. \end{cases} \tag{C.23}$$

f) *Two parallel infinite plates* A *and* B *of thickness H and h, respectively, and separated by distance a (H<h)*

$$U_{AB}(x) = \begin{cases} \frac{1}{4} - \frac{a}{4x}, & a \le x \le a+H \\ \frac{x-a-H/2}{2x}, & a+H<x\le a+h \\ \frac{h}{2x} - \frac{(H+h+a-x)^2}{4Hx}, & a+h<x<a+H+h \\ \frac{h}{2x}, & x>a+H+h \end{cases} \tag{C.24}$$

$$U_{BA}(x) = U_{AB}(x) \bullet \frac{H}{h} \tag{C.25}$$

The same solution applies if A is a section of the infinite plate, i.e., a right cylinder (finite plate).

g) *A point* A *and a right cylinder* B*:*

$$U_{AB}(x) = \frac{1}{2x}\int_{z_1}^{z_2} U_{AC}(\sqrt{x^2-z^2})\, dz\ , \quad \sqrt{y_1^2+b^2} <x< \sqrt{(b+h)^2 + y_2^2} \tag{C.26}$$

$$z_1 = \mathrm{Max}(b, \sqrt{\mathrm{Max}(0,x^2-y_2^2)}\,)\ , \ z_2 = \mathrm{Min}\,(b+h, \sqrt{x^2-y_1^2})\,.$$

$U_{AC}(x)$ is the two-dimensional geometric-reduction factor for the cross section C of the cylinder relative to the projection of point A, P(A), into the plane which corresponds to C; y_1 and y_2 are the minimum and maximum distances in this plane between P(A) and C; h is the height of the cylinder; b is the projected distance on the cylinder axis between A and P(A). Point A is assumed to lie below the cylinder bottom. If A lies between the two planes through the faces of the cylinder, one obtains the solution as a sum by suitable subdivision of B into two cylinders.

For the circular area of radius r relative to the point at distance a from the center, one has:

$$U_{AC}(x) = \frac{1}{\pi} \cos^{-1}\left[\mathrm{Max}\left(-1, \frac{x^2+a^2-r^2}{2ax}\right)\right], \quad \mathrm{Max}\,(a-r,o) \leq x \leq a+r. \tag{C.27}$$

h) *A point* A *and a planar domain* B *of infinitesimal thickness* δ:

Let h be the distance of A from the plane C of B, and let $U_2(y)$ be the 2-dimensional geometric-reduction factor ($y_1<y<y_2$) of B relative to the projection of A in C. Then:

$$U_{AB}(x) = \frac{\delta}{2x} U_2(\sqrt{x^2-h^2}), \quad \sqrt{h^2+y_1^2} < x < \sqrt{h^2+y_2^2}\,. \tag{C.28}$$

For a disc of radius r and a projected distance a of A from its center, one obtains

$$U_{AB}(x) = \frac{\delta}{2\pi x} \cos^{-1}\left[\mathrm{Max}\left(-1, \frac{x^2+a^2-r^2-h^2}{2a\sqrt{x^2-h^2}}\right)\right]. \tag{C.29}$$

For a rectangle with side lengths a and b, we restrict to a special case for A: The straight lines which come out from the sides divide the plane into 8 rectangles with infinite content; the projection of A, P(A), is assumed to lie in the left lowest rectangle with infinite content; s is the distance of P(A) to the next rectangle line parallel to a, t is analogously related to b. (Different positions of A can be reduced to the described special case.) For the two dimensional geometrical factor, one has:

$$U_2(y) = \frac{1}{2\pi}\left[\sin^{-1}\left(\frac{\mathrm{Min}\,(a+s, \sqrt{y^2-t^2})}{y}\right) - \sin^{-1}\left(\frac{\mathrm{Max}(s, \sqrt{\mathrm{Max}(0, y^2-(b+t)^2)})}{y}\right)\right]; \tag{C.30}$$

therefore,

$$U_{AB}(x) = \frac{\delta}{4\pi x}\left[\sin^{-1}\left(\frac{\text{Min}\,(a+s,\ \sqrt{x^2-h^2-t^2})}{\sqrt{x^2-h^2}}\right)\right. \tag{C.31}$$

$$\left. - \sin^{-1}\left(\frac{\text{Max}(s,\ \sqrt{\text{Max}(0,x^2-h^2-(b+t)^2)})}{\sqrt{x^2-h^2}}\right)\right]$$

The diagrams in Figure (C.2) give geometric-reduction factors and point-pair distance distributions for the heterologous case according to some of the preceding equations.

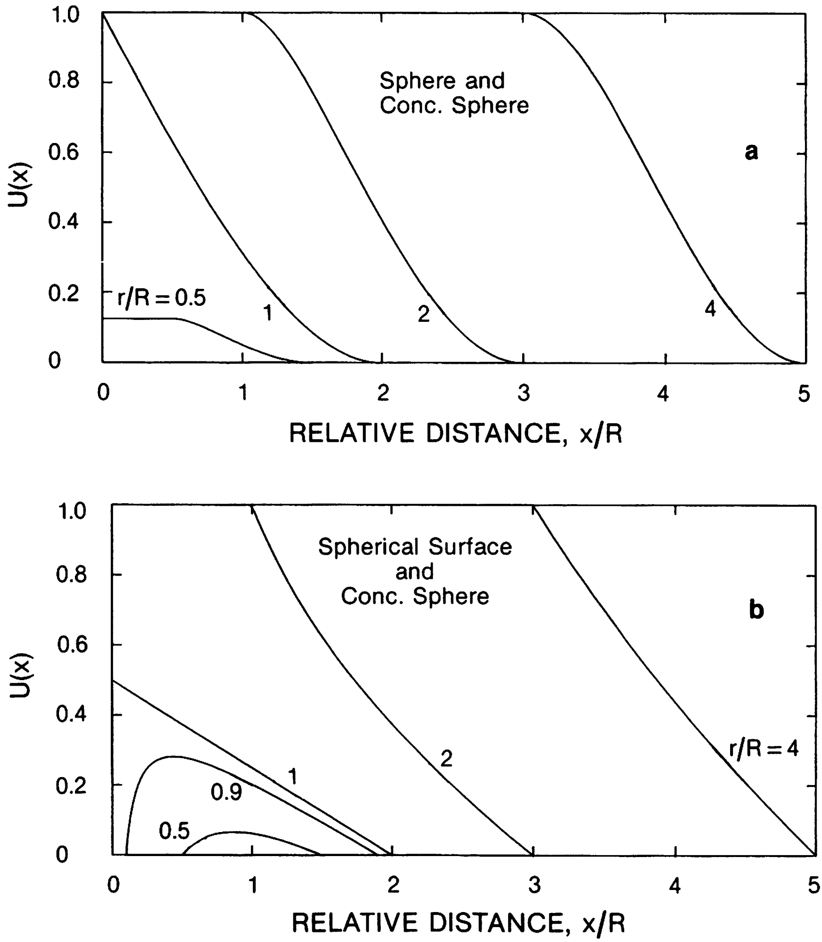


Fig. C.2. Geometric reduction factors, U(x), or the point-pair distance distributions, p(x), for the heterologous case.

a: Geometric-reduction factors for a sphere, A, of radius R, and a concentric sphere, B, of radius r according to Equation (C.15). The ratio r/R is indicated on the curves. The distance, x, is given relative to R.

b: Geometric-reduction factors for a spherical surface, A, of radius R and a concentric sphere, B, of radius r according to Equation (C.16). The ratio r/R is indicated on the curves. The distance, x, is given relative to R.

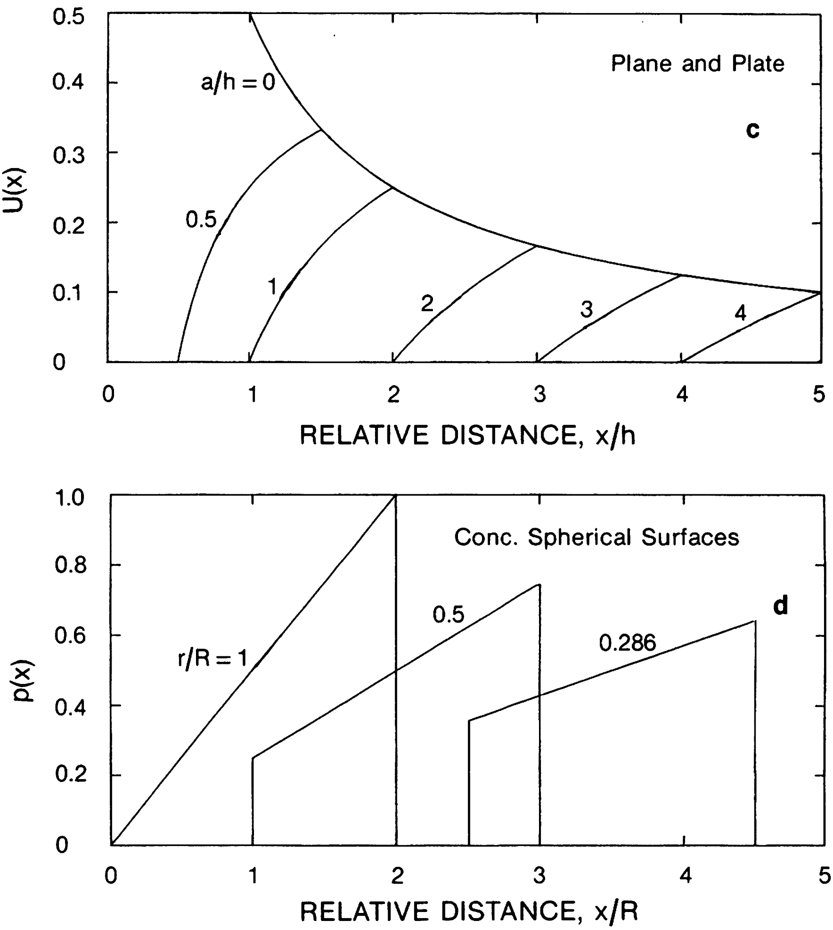


Fig. C.2. *continued*

c: Geometric-reduction factors for a plane, A, separated by distance a from an infinite plate, B, of height h according to Equation (C.23). The ratio a/h is indicated on the curves. The distance, x, is given relative to h.

d: Point-pair distance distributions between a spherical surface of radius R and a concentric spherical surface of radius r according to Equation (C.22). The ratio r/R is indicated on the curves. The distance, x, is given relative to r.