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CONTENT — СОДЕРЖАНИЕ

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ON SUBDIVISION OF SEMISIMPLICIAL SETS

§ 1 Introduction

The regular subdivision $\Delta' X$ of a semisimplicial set X can be easily defined in a purely combinatorial way; this has been done by Barratt [1] and Kan [7]. To investigate the geometrical meaning of this let us first consider especially the two degeneracy maps from a 2—simplex to an 1—simplex and their subdivisions. We find out that there can't be a natural homeomorphism between the geometric realizations of a semisimplicial p—simplex and its regular subdivision, although the underlying spaces are nothing but geometric p—simplices. As the category $\underline{S}$ of semisimplicial sets and semisimplicial maps is the completion of the category of semisimplicial simplices and semisimplicial maps with respect to colimits, such a natural homeomorphism would be necessary and sufficient for the existence of a natural homeomorphism between $|X|$ and $|\Delta' X|$, the geometric realizations of a semisimplicial set X and its regular subdivision $\Delta'X$. So we have

Theorem 1. *There exists no natural equivalence between the functors* $|?|:\underline{S}\to\underline{CW}$ and $|\Delta' ?|:\underline{S}\to\underline{\text{CW}}$ ("$\underline{\text{CW}}$" denotes the category of CW—complexes and continuous maps).

Now the question arises if there is any homeomorphism between $|X|$ and $|\Delta' X|$. The answer to this question is in my mind far away from being trivial—as many people believed a long time—and is first given in my paper [3] as a special case of a general result on a class of various subdivisions. Subsequent to this Puppe has found an explicit formula, which—as we proved in [4]—gives a homeomorphism in the regular case.

The question mentioned above has also suggested my paper [2] and my aim here is to outline the content of [2] and [3].

2 Standard division functors

Standard division functors are introduced in [2].

Definition 1. *A "standard division functor" is a pair* (U, u) *consisting of a functor*

$$U:\underline{\Delta}\to\underline{S}$$

and a family

$$u = (u_p / \ p \text{ non—negative integer})$$

such that the following conditions are satisfied:

(*i*) u_p *is a homeomorphism* $|U[p]| \to \Delta_p$ *for each non—negative integer p*

(*ii*) $$|\Delta\beta| \circ u_p = u_q \circ |U\beta|$$

for each injective map $\beta:[p]\to[q]$ *of* $\underline{\Delta}$. (Here the notation must be explained: $\underline{\Delta}$ denotes the category of non—empty finite ordered sets and weak order preserving maps and $[p]$ the set of the numbers 0, 1, 2, . . . , p, that means the ordered set of $p+1$ elements, for each non—negative integer p; the maps of $\underline{\Delta}$ are symbolized by small greek letters; we shall briefly write "$\beta \in \underline{\Delta}$" to indicate that β is a map of $\underline{\Delta}$. $\Delta : \underline{\Delta} \to \underline{S}$ means the functor which assigns to each object $[p]$ of $\underline{\Delta}$ the semisimplicial p—simplex $\Delta[p]$ and to each $\beta \in \underline{\Delta}$ the the semisimplicial map $\Delta\beta$; it is the simplest example for a standard division functor. Finally Δ_p denotes the geometric p-simplex.)

Given a standard division functor (U, u) one can identify each $|U[p]|$ with Δ_p by means of (*i*). Then we have two CW—structures on Δ_p, the one is induced by the simplices of $\Delta[p]$, the other by the simplices of $U[p]$; the same is to say that the underlying spaces of the CW—complexes $|\Delta[p]|$ and $|U[p]|$ coincide. From this point of view (*ii*) assures that $|U[p]|$ is a CW—subdivision of $|\Delta[p]|$, that means that each cell of $|U[p]|$ lies in a cell of $|\Delta[p]|$.

Each standard division functor (U, u) can be extended uniquely to a continuous functor from $\underline{S}$ to itself, which we denote—by abuse of notation—also by "U". Such a so—called "*division functor*" has the following properties:

Proposition 1. U *preserves the fundamental group and the homology groups up to natural equivalence.*

Proposition 2. U *preserves coverings.*

A semisimplicial map f is said to be a "*weak homotopy equivalence*" if f induces an isomorphism of the fundamental groups and $\tilde{f}$, the universal covering of f, induces isomorphisms of the homology groups. This definition is justified by the fact that the geometric realization of a semisimplicial weak homotopy equivalence is indeed a homotopy equivalence. From the propositions 1 and 2 now it follows at once

Proposition 3. U *preserves weak homotopy equivalences.*

Much deeper is the following result:

Theorem 2. *If X is a semisimplicial set, then the CW—complexes* $|X|$ *and* $|UX|$ *have the same homotopy type;* more precisely: *if* $[?]:\underline{CW}\to\underline{CWh}$ *denotes the projection onto the homotopy category of CW—complexes, then the functors* $[|U?|]$ *and* $[|?|]$ *are naturally equivalent.*

To obtain this result we need an interesting device, which is explained in the following section.

§ 3 Non—degenerate semisimplicial sets

Definition 2. *A semisimplicial set X is "non—degenerate" if no non—degenerate simplex of X has a degenerate face. If X and Y are non—degenerate semisimplicial sets, a semisimplicial map $f: X \to Y$ is "non—degenerate" if f maps non—degenerate simplices of X on non—degenerate simplices of Y.*

Non—degenerate semisimplicial sets and non—degenerate semisimplicial maps from a subcategory $\underline{P}$ of $\underline{S}$ and one can prove:

Proposition 4. *$\underline{P}$ is a reflective subcategory of $\underline{S}$.*
That means that the embedding functor $\mathrm{E}: \underline{P} \to \underline{S}$ is left adjoint to a functor $\underline{S} \to \underline{P}$, the reflector R. To prove this one has to define to each semisimplicial set X a non—degenerate semisimplicial set $\mathrm{R}X$ and a semisimplicial map $X: \mathrm{R}X \to X$ such that for each semisimplicial map $f: Y \to X$ with Y non—degenerate there exists a unique non—degenerate semisimplicial map $f': Y \to \mathrm{R}X$ with $f = \mathrm{r}X \circ f'$ [1].

Having done this, a simple straightforward computation yields

Proposition 5. $\mathrm{r}X$ *induces in a natural way isomorphisms of the fundamental groups and all homology groups,* and

Proposition 6. R *preserves coverings.*
From this two propositions it follows at once

Theorem 3. $\mathrm{r}X$ *is a weak homotopy equivalence* [2].

In our context the meaning of the category $\underline{P}$ is due to

Proposition 7. *If X is a non—degenerate semisimplicial set, then the spaces $|X|$ and $|\mathrm{U}X|$ are homeomorphic;* more precisely: *the functors $|\mathrm{E}?|: \underline{P} \to \underline{CW}$ and $|\mathrm{U}\,\mathrm{E}?|: \underline{P} \to \underline{CW}$ are naturally equivalent.*

We omit the proof of this proposition; now theorem 2 is an easy consequence of theorem 3, proposition 3 and proposition 7.

§ 4 Examples

1) The "regular" or "barycentric" subdivision of the semisimplicial simplices induces a standard division functor. It was already descrided by Kan [7]; he uses the symbol "Δ'" for the functor $\underline{\Delta} \to \underline{S}$, but the symbol "*Sd*" for the extended division functor, which we according to our conventions denote also by "Δ'"; we mentioned it in § 1.

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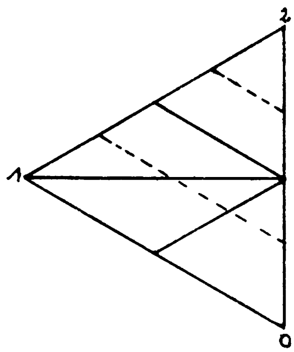
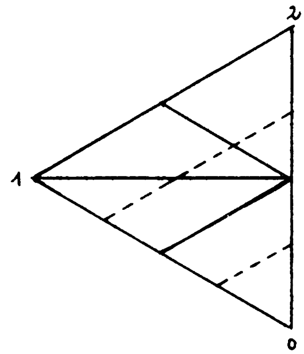
[1] For this situation the terminology "coreflective" seems to become standard; but obeying the demand for logical consistency we use "reflective" in accordance with the book of Mitchell [9].

[2] In some papers Giever [5] and Hu [6] have studied the space $|\mathrm{R}X|$; they denoted it by "*PX*" and called it "geometric realization of X", but they had not defined the semisimplicial set $\mathrm{R}X$ explicitly. Then Kodama [8] has constructed the map $|\mathrm{r}X|$—he denoted it by "*pX*"—and a homotopy inverse to it, but by using the fact that $|\Delta' X|$ can be interpreted as CW—subdivision of $|X|$, which was not proved at that time.

2) The "natural" subdivision. We denote the corresponding functor $\underline{\Delta}' \to \underline{S}$ by "$\Delta^{\vee}$". Its effect on [2] and the two degeneracy maps $\sigma_e : [2] \to [1]$ $(e=0,1)$ can be illustrated by the following pictures:



$|\Delta^{\vee} \sigma_0|$

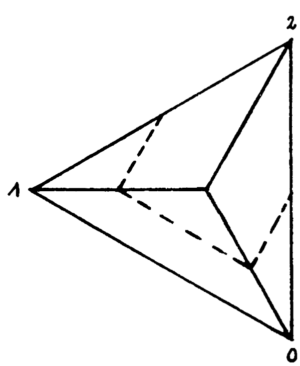


$|\Delta^{\vee} \sigma_1|$

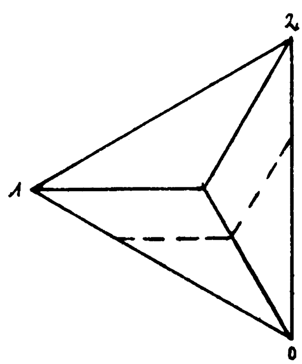
(The fully traced line segments indicate the cell structure of $\Delta_2 = |\Delta^{\nu}[2]|$; $|\Delta^{\nu}\sigma_0|$ resp. $|\Delta^{\nu}\sigma_1|$ identifies the dotted line segments and just so their parallels to a point.)

In this case the condition (ii) of definition 1 is satisfied for all $\beta \in \underline{\Delta}$, not only for the injective ones. The name "natural" is justified by the fact that for all semisimplicial sets X the CW-complexes $|\Delta' X|$ and $|X|$ are naturally homeomorphic. The disadvantage of this functor is that we know no method to approximate continuous maps by semisimplicial maps by means of it. The approximation constructed by Kan [7] can't be transferred.

3) The "r-skeleton-preserving" subdivision (r non-negative integer) has got its name from the fact that for all semisimplicial sets X there is a natural semisimplicial isomorphism between $U_r(X^r)$ and X^r, where U_r denotes the corresponding division functor and X^r the r-skeleton of X. The effect of U_1 on [2] and the two degeneracy maps $\sigma_e : [2] \to [1]$ $(e=0,1)$ can be illustrated by the following pictures:



$|U_1 \sigma_0|$



$|U_1 \sigma_1|$

(Again the fully traced line segments indicate the cell structure, here that of $\Delta_2 = |U_1[2]|$; $|U_1\sigma_0|$ resp. $|U_1\sigma_1|$ identifies the dotted line segments and just so their parallels to a point; moreover $|U_1\sigma_1|$ identifies the whole left upper triangle to one point.)

§ 5 Natural transformations

One can ask now if the natural equivalence in the category $\underline{CWh}$ of theorem 2 is induced by a natural transformation in $\underline{CW}$. We are not able to give a general answer to this question. Here we list the partial results we have obtained.

Proposition 8. *Each natural transformation between $|U\,?|$ and $|\,?|$ induces natural equivalence in the homotopy category $\underline{CWh}$.*

Proposition 9. *The natural transformations between $|U\,?|$ and $|\,?|$ and between $|U\Delta\,?|$ and $|\Delta\,?|$ correspond in an one-to-one fashion.*

So it suffices to consider natural transformations between the functors $|U\Delta\,?|$ and $|\Delta\,?|$. We know almost nothing about natural transformations $|\Delta\,?| \to |U\Delta\,?|$, therefore let us deal with natural transformations $|U\Delta\,?| \to |\Delta\,?|$. Such a natural transformation can be given by a sequence $t_0, t_1, t_2, \ldots$ of maps $t_i : \Delta_i \to \Delta_i$ such that certain commutativities hold. Then one can prove:

Proposition 10. *Each natural transformation $t_0, t_1, t_2, \ldots$ is uniquely determined by t_1.*

The essential device for proving this is the following almost trivial

Lemma. *Let V be a topological space, $f, g : V \to \Delta_n$ continuous maps, i_0, i_1 distinct elements of $[n-1]$ and*

$$|\Delta\sigma_{i_e}| \circ f = |\Delta\sigma_{i_e}| \circ g, \quad \textit{for } e = 0,1$$

($\sigma_{i_e} : [n] \to [n-1]$ denotes the i_e*—th* degeneracy map). *Then holds*: $f = g$.

This lemma also yields

Proposition 11. *Each triple t_0, t_1, t_2 with*

$$|\Delta\beta| \circ t_m = t_n \circ |U\beta|$$

for $0 \leqslant m, n \leqslant 2$ and $\beta \in \underline{\Delta}$ such that $|\Delta\beta| \circ t_m$ and $t_n \circ |U\beta|$ are defined can be extended to a natural transformation $|U\Delta?| \to |\Delta?|$.

We can also give necessary and sufficient conditions that a continuous map t_1 induces a natural transformation. But we do not know if for each standard division functor (U, u) there exists a map t_1, which satisfies these conditions. This may at most depend on certain hypotheses about $U\sigma_e$ in case dim $\sigma_e = 2$ $(e = 0,1)$. In our example the existence of such natural transformations can be asily established; there are even natural transformations $U \to \Delta$. To end this section we mention that there is an infinite number of natural transformations $|U\Delta\,?| \to |\Delta\,?|$, if there is one.

§ 6 Standard homotopies

Now we turn to the main problem: We want to show that under certain further assumptions on a given division functor U the CW—complexes $|UX|$ and $|X|$ are homeomorphic X being any semisimplicial set. Such a homeomorphism will be constructed inductively, so we arrive at the problem to continue a given map of the boundary of Δ_n onto itself over the whole geometric simplex Δ_n such that the interior of Δ_n is mapped homeomorphically onto itself: We have to stuff holes. To this end we need

Definition 3. *Let V be a topological space. A homotopy $h_t : V \to V$ is "stuffing" if $h_0 = \mathrm{id}\,V$ and h_t is a homeomorphism for all $t<1$*

By means of a stuffing homotopy one can stuff holes:

Proposition 12. *Given a stuffing homotopy $h_t : S^n \to S^n$ there exists an extension $h : B^{n+1} \to B^{n+1}$ of h_1 such that the interior of B^{n+1} is mapped homeomorphically onto itself. Moreover there is a stuffing homotopy $H_t : B^{n+1} \to B^{n+1}$ such that $H_1 = h$ and $H_t v = h_t v$ for all $v \in S^n$ and $t \in [0,1]$.* ("S^n" denotes the n-sphere and "B^{n+1}" the (n+1)-ball.)

Now let be given a fixed division functor U; we describe the additional condition:

Definition 4. *A "standard homotopy (for* U" *is a family $(l_t \beta / \beta \in \underline{\Delta})$ of stuffing homotopies $l_t \beta : \Delta_{\dim \beta} \to \Delta_{\dim \beta}\, \beta$ such that the following conditions are satisfied*:

(10) $l_t(id) = id$,

(11) $|\Delta \beta| \circ l_t(\alpha\beta) = l_t \alpha \circ |\Delta\beta|$ *for injective* $\beta \in \underline{\Delta}$,

(12) $l_t(\alpha\beta) \circ (l_t \beta)^{-1}$ *single-valued* (and therefore a continuous map)?

(13) $|\Delta \beta| \circ l_1(\alpha\beta) = l_1 \alpha \circ |\mathrm{U}\beta|$;

from (13) it follows that (for all suitable degeneracy maps $\sigma_i \in \underline{\Delta}$) $l_1(\beta\sigma_i) \circ l_1(\sigma_i)^{-1}$ maps each line segment parallel to the line segment between the i-th and (i+1)-st vertex of $\Delta_n = |\Delta[n]|$ on such a line segment. We demand further:

(14) *for each such line segment this map is weakly monotone.*

We do not know if there exists a standard homotopy for each division functor U. If there is a standard homotopy for a given U, then it follows at once that the sequence

$$l_1([0] \to [0]),\ l_1([1] \to [0]),\ l_1([2] \to [0]), \ldots$$

represents a natural transformation $|\mathrm{U}\ ?| \to |\ ?|$, which we call the "*corresponding natural transformation*".

§ 7 The main theorem

Main theorem. *Let U be a division functor with standard homotopy. Then X being any semisimplicial set there is a homeomorphism* $|UX| \to |X|$ *which is homotopic to the map* $|UX| \to |X|$ *deduced from the corresponding natural transformation.*

Here we can only give the idea of the proof. To do this we need a more explicit description of the spaces $|UX|$ and $|X|$ for a given semisimplicial set X. They are quotient spaces of $FX = \Sigma_p X_p \times \Delta_p$, where X_p denotes the set of p-simplices of X provided with the discrete topology and Σ the topological sum. We obtain $|UX|$ by taking the equivalence relation which is generated by

$$(x\beta, v) \sim (x, |U\beta| v)$$

and $|X|$ by taking that which is generated by

$$(x\beta, v) \sim (x, |\Delta\beta| v)$$

for $x \in X_q$, $\beta : [p] \to [q]$ in $\underline{\Delta}$ (as X is a semisimplicial set, β induces a map from X_q to X_p and $x\beta$ denotes the image of x under this map).
Therefore a continuous map $|UX| \to |X|$ can be constructed if there is given a family $(h_x / x \in X)$ of continuous maps $h_x : \Delta_{\dim x} \to \Delta_{\dim x}$ with $|\Delta\beta| \circ h_{x\beta} = h_x \circ |U\beta|$ for all $x \in X$ and all $\beta \in \underline{\Delta}$ such that $x\beta$ is defined. It is easy to show that a map $|UX| \to |X|$ constructed in such a way is a homeomorphism iff h_x maps the interior of Δ_p homeomorphically onto itself for each non-degenerate $x \in X_p$. In order to establish the first part of the main theorem one has to construct such a family $(h_x / x \in X)$ and it is obvious that the given standard homotopy plays an essential part in this construction.

§ 8 Examples of standard homotopies

We indicate how a standard homotopy for the regular subdivision can be given. Let be $\beta \in \underline{\Delta}$ and $p = \dim \beta$. $l_1 \beta$ maps the cells of $|\Delta'[p]| = \Delta_p$—they are simplices—linearly and $l_t \beta$ is the linear connection between the identity and $l_1 \beta$. So it suffices to show as $l_1 \beta$ maps the vertices of $|\Delta'[p]|$. Any vertex b of $|\Delta'[p]|$ corresponds to a O-simplex of $\Delta'[p]$ that is an injective map $\mu : [m] \to [p]$ of $\underline{\Delta}$. As $p = \dim \beta$ the composition $\beta\mu$ is defined; $\beta\mu$ can be uniquely decomposed in an injective and a surjective part; let us denote the latter by ϱ. We define a right inverse $\hat{\rho}$ to ϱ by setting

$$\hat{\rho}(i) = max\, \varrho^{-1}(i).$$

Then we take

$$l_1 \beta(b) = |\mu \hat{\rho}|$$

(we interpret $\mu\hat{\rho}$ to be a O-simplex of $\Delta'[p]$ and denote by $|\mu\hat{\rho}|$ the corresponding vertex of $|\Delta'[p]|$, which is obviously a point of Δ_p),

For the other examples of standard division functors we have given there exist standard homotopies, too; in the case of natural subdivision it is easy to see that one can take the identity.

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