The Conics of Ludwig Kiepert: A Comprehensive Lesson in the Geometry of the Triangle

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1. Introduction

If a visitor from Mars desired to learn the geometry of the triangle but could stay in the earth's relatively dense atmosphere only long enough for a single lesson, earthling mathematicians would, no doubt, be hard-pressed to meet this request. In this paper, we believe that we have an optimum solution to the problem. The Kiepert conics, though seemingly unknown today, constitute a significant part of the geometry of the triangle and to study them one has to deal with many fundamental concepts related to this geometry such as the Euler line, Brocard axis, circumcircle, Brocard angle, and the Lemoine line in addition to well-known points including the centroid, circumcentre, orthocentre, and the isogonic centres. In the process, one comes into contact with not so well known, but no less important concepts, such as the Steiner point, the isodynamic points and the Spieker circle.

In this paper, we show how the Kiepert's conics are derived using both analytic and projective arguments and discuss their main properties, which we have drawn together from several sources. We have applied some modern technology, in this case computer graphics, to produce a series of pictures that should serve to increase the reader's appreciation for this interesting pair of conics. In addition, we have derived some results that we were unable to locate in the available literature.

2. Preliminaries

a. Coordinate Systems Two systems of specialized homogeneous coordinates are especially suited to this type of work, they are, the "trilinear" (or normal) system and the "barycentric" (or areal) system. In the trilinear case, the coordinates (x, y, z) of a point P in the plane of a given reference triangle ABC are proportional to the signed distances of P from the sides of the reference triangle, i.e.

$$x : y : z = d_a : d_b : d_c,$$

where, for example, d_a represents the *signed* distance of P from side BC. The sign of d_a is positive or negative accordingly as P and the unit point, the incentre $I = (r, r, r) = (1, 1, 1)$, r is the inradius, are on the same or opposite sides of BC. The actual distances d_a, d_b, d_c of a point P from the sides of ABC are related to the trilinear coordinates of P by the equations:

$$\frac{d_a}{x} = \frac{d_b}{y} = \frac{d_c}{z} = \frac{r(a+b+c)}{ax+by+cz},$$

see Sommerville [**32**, p. 157].

For the trilinear line coordinates $[u, v, w]$ of the line l: $ux + vy + wz = 0$, one has

$$u : v : w = ad_A : bd_B : cd_C,$$

where d_A represents the signed distance from vertex A to l. The signs of d_A and d_B, for example, are the same or different depending on whether or not A is in the half-plane determined by B and l.

For the barycentric type, the coordinates of P are proportional to the signed areas of the triangles PBC, PCA, PAB thus,

$$x : y : z = ad_a : bd_b : cd_c,$$

where the unit point of the system is now the centroid G.

Similarly, for the line case, one has

$$u : v : w = d_A : d_B : d_C.$$

Unless otherwise indicated, we shall use the trilinear system throughout. A curious application of these coordinates, the three jug-problem, is given by Coxeter and Greitzer [**11**, pp. 89–93].

b. Transformations We shall have occasion to use two special cases of so-called "Cremona" transformations, named after the Italian geometer Luigi Cremona (Pavia, 1830–Rome, 1903) who did considerable study on them, see Coolidge [**7**, pp. 287ff]. The Cremona transformations are birational transformations of the plane. The following quadratic type given by the systems of equations

$$(x', y', z') = \left(\frac{n_1}{x}, \frac{n_2}{y}, \frac{n_3}{z}\right) = (n_1yz, n_2zx, n_3xy), \tag{1}$$

where n_i, $i = 1, 2, 3$, are nonzero constants, is the only one of interest at this time.

These transformations induce involutions on the points in the plane not on the sides of the reference triangle and carry, in a one-to-one fashion, lines into conics that pass through the vertices of the reference triangle and vice versa. As mentioned earlier, we are interested in two special cases of (1).

Case 1. The quadratic transformation

$$P = (x, y, z) \mapsto P' = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) \tag{2}$$

is obtained from (1) by making the substitutions $n_1 = n_2 = n_3 = 1$. Geometrically, P' is obtained from P by reflecting the lines AP, BP, CP in the internal angle bisectors through A, B, C respectively: The reflected lines concur in the point P'. This point which we shall now denote by P^g, is called the *isogonal conjugate* of P and the transformation g defined by (2), the *isogonal* transformation. See Kimberling [**18**] for some familiar pairs of points related by this transformation.

A well-known pair of this sort is (K, G) where K denotes the symmedian point and G the centroid. The *symmedian point* of a triangle ABC is the point of concurrence of the *symmedians*, i.e., the reflections of the medians with respect to the angle bisectors. Thus, it is more or less by definition the isogonal conjugate of the centroid. But the symmedian point can also be defined as the point inside the triangle such that the sum $d_a^2 + d_b^2 + d_c^2$ is minimum. In fact, according to Mackay [21], the first appearance of this point in the mathematical literature was in the context of this very property. The symmedian point is usually referred to as the *Lemoine point* by French and British writers but is known as the *Grebe point* in Germany, see Johnson [14, p. 213]. Émile Michel Hyacinthe Lemoine (Quimper, 1840–Paris, 1912), one of the main promotors of the modern geometry of the triangle, had first been a teacher of mathematics and then, from 1870, Engineering Advisor at the Court of Commerce of Paris; Ernst Wilhelm Grebe (Michelbach near Marburg, 1804–Kassel, 1874) was a teacher of mathematics at the Gymnasium in Kassel. A considerable amount of research relating to the symmedian point seems to have been carried out during the 19th century.

Case 2. The quadratic transformation

$$P = (x, y, z) \mapsto P' = \left(\frac{1}{a^2 x}, \frac{1}{b^2 y}, \frac{1}{c^2 z} \right), \tag{3}$$

is obtained from (1) by making the substitutions $n_1 = 1/a^2, n_2 = 1/b^2, n_3 = 1/c^2$. Geometrically, P' is obtained from P by reflecting the points D, E, F (the intersections of the lines AP, BP, CP with the sides BC, CA, AB) in the midpoints of the side on which they lie: The lines AD', BE', CF' where D', E', F' denote the respective images of D, E, F concur in the point P'. This point, which we shall now denote by P^t, is called the *isotomic conjugate* of P and the transformation t defined by (3), the *isotomic* transformation.

A familiar pair of points related by this transformation is formed by the Gergonne point and the Nagel point. The *Gergonne point* of a triangle is the point of concurrence of the line segments connecting the vertices with the points of contact of the incircle with the opposite sides. The definition of the *Nagel point* is similar except now we consider the points of contact of the three excircles. A discussion of both of these points is given in Johnson [14, pp. 184–185]; their trilinear coordinates can be found in Kimberling [18]. Joseph Diaz Gergonne (Nancy, 1771–Montpellier, 1859) founded, in 1810, the first only wholly mathematical journal, the *Annales de mathématiques pures et appliquées*; he held the chair of astronomy at the University of Montpellier, where he also acted as rector. Christian Heinrich (von) Nagel (Stuttgart, 1803–Ulm, 1882, nobled in 1875) was Professor for Mathematics and Science at the Gymnasium in Ulm and director of a so-called "Realschule". He devoted a lot of his activities to a modernization of the school system.

3. The Hyperbola

The following problem was proposed in 1868 by Lemoine [19].

> *Construire un triangle, connaissant les sommets des triangles équilatéraux construits sur les côtes.* (Construct a triangle, given the peaks of the equilateral triangles constructed on the sides.)

A solution by Friedrich Wilhelm August Ludwig Kiepert (Breslau, 1846–Hannover, 1934) was published in 1869 [17]. At the time, Kiepert was a doctoral student at the University of Berlin under Weierstraß. He later moved to Hannover as Professor of Higher Mathematics and became Dean in 1901. He wrote a textbook on calculus that had been used frequently in German universities up to the 1920s. His later mathematical work was concerned mainly with actuarial theory. For more information on this mathematician, see Volk [37].

Kiepert's solution contains a remark that we shall cast in the form of a theorem. In order to avoid degenerate cases, we fix a triangle ABC that is assumed, for the remainder of this paper, to be scalene with $\alpha > \beta > \gamma$, where α, β, γ denote the measures of the angles at the vertices A, B, C respectively. In addition, this will be the reference triangle when computations with coordinates are carried out.

THEOREM 1. *If the three triangles $A'BC$, $AB'C$ and ABC', constructed on the sides of the given triangle ABC as bases, are similar isosceles and similarly situated, then the lines AA', BB', CC' concur at a point P. The locus of P as the base angle varies between $-\pi/2$ and $\pi/2$ is the conic*

$$\boldsymbol{\Gamma}\colon \frac{\sin(\beta-\gamma)}{x} + \frac{\sin(\gamma-\alpha)}{y} + \frac{\sin(\alpha-\beta)}{z} = 0, \tag{4}$$

or, equivalently,

$$\boldsymbol{\Gamma}\colon \frac{bc(b^2-c^2)}{x} + \frac{ca(c^2-a^2)}{y} + \frac{ab(a^2-b^2)}{z} = 0. \tag{5}$$

Proof. We denote the measure of the base angles of the similar, isosceles triangles by ϕ (noting that the orientation of these triangles is counterclockwise when $\phi < 0$ and clockwise when $\phi > 0$) and immediately obtain the following representations for the vertices of the triangle $A'B'C'$,

$$\begin{aligned} A' &= (-\sin\phi, \sin(\gamma+\phi), \sin(\beta+\phi)), \\ B' &= (\sin(\gamma+\phi), -\sin\phi, \sin(\alpha+\phi)), \\ C' &= (\sin(\beta+\phi), \sin(\alpha+\phi), -\sin\phi). \end{aligned}$$

Now, recall that the trilinear (or barycentric) coordinates $[u, v, w]$ of the line $ux + vy + wz = 0$ connecting the two given points can be taken as the cross product of the two vectors in $\mathbb{R}^3$ whose components are the trilinear (barycentric) coordinates of these points. Since $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$, we readily derive the representations

$$\begin{aligned} AA' &= [0, -\sin(\beta+\phi), \sin(\gamma+\phi)], \\ BB' &= [\sin(\alpha+\phi), 0, -\sin(\gamma+\phi)], \\ CC' &= [-\sin(\alpha+\phi), \sin(\beta+\phi), 0]. \end{aligned}$$

The point P, shown in FIGURE 1, is easily seen to have the coordinates

$$\begin{aligned} (x, y, z) = (&\sin(\beta+\phi)\sin(\gamma+\phi), \sin(\gamma+\phi)\sin(\alpha+\phi), \\ &\sin(\alpha+\phi)\sin(\beta+\phi)), \end{aligned} \tag{6}$$

which implies

$$x\sin(\alpha+\phi) = y\sin(\beta+\phi) = z\sin(\gamma+\phi).$$

It now follows that

$$(x\sin\alpha - y\sin\beta)\cos\phi + (x\cos\alpha - y\cos\beta)\sin\phi = 0$$

and

$$(x\sin\alpha - z\sin\gamma)\cos\phi + (x\cos\alpha - z\cos\gamma)\sin\phi = 0,$$

a system of homogeneous equations linear in the variables $\cos\phi, \sin\phi$. Since each triple (x, y, z) gives a nontrivial solution, the determinant must vanish, hence

$$(x\sin\alpha - y\sin\beta)(x\cos\alpha - z\cos\gamma) = (x\sin\alpha - z\sin\gamma)(x\cos\alpha - y\cos\beta),$$

which completes the proof.

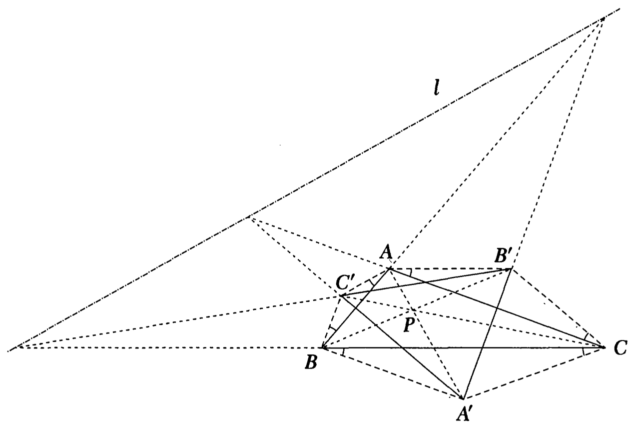

FIGURE 1

In order to determine the conic type, we inspect the line at infinity. If we consider the trilinear line coordinates as described in Section 2a, it is clear that, for a line far distant from the reference triangle, the distances d_A, d_B, d_C become almost equal, consequently, in the limit, we obtain $[a, b, c]$ as the line coordinates of the line at infinity. Furthermore, an examination of the intersection of this line with $\mathbf{\Gamma}$ reveals that the relevant discriminant reduces to the expression $E = (a^2 + b^2 + c^2)^2 - 3(a^2b^2 + b^2c^2 + c^2a^2)$. Since $E > 0$, see Bottema et al. [**2**, p. 11], the conic under consideration meets the line at infinity in two distinct points; thus, it is a hyperbola, *Kiepert's hyperbola*. The significance of the line l in Figure 1 will be explained later, in Theorem 2.

To gain more insight into the angles that actually produce the points at infinity, we note that these points correspond to angles ϕ satisfying the equation

$$\frac{\sin\alpha}{\sin(\alpha+\phi)} + \frac{\sin\beta}{(\sin\beta+\phi)} + \frac{\sin\gamma}{\sin(\gamma+\phi)} = 0.$$

By means of the usual formulae and theorems ($\sin\alpha = a/2R, \dots, R = abc/4\Delta$, R the circumradius, Δ the area of the reference triangle, law of cosines, Heron's formula for the area of a triangle) this equation can be transformed into

$$\sin(2\phi + \omega) = -2\sin\omega,$$

where the angle ω is the Brocard angle, determined by the property that there is a (unique) point τ_1, *Brocard's first point*, such that $\sphericalangle\tau_1 CA = \sphericalangle\tau_1 AB = \sphericalangle\tau_1 BC = \omega$, see Kimberling [**18**]. Pierre René Jean-Baptiste Henri Brocard (Vignot 1845–1922 Barle-Duc) is, like Lemoine, regarded one of the fathers of the modern geometry of the triangle. He was not a professional mathematician but, rather, served as an army officer for engineering in Algier and Montpellier. His widespread results form the basis for what is now known as *Brocard geometry*.

Since $0 < \sin\omega = 2\Delta/\sqrt{a^2b^2 + b^2c^2 + c^2a^2} < 1/2$ for a scalene triangle, this equation has two solutions in the range required, namely

$$\max(-\pi/2, -\alpha) < \phi_1 < -\beta < -\phi_2 < -\gamma.$$

The following table indicates some special points on the hyperbola corresponding to certain specific values of ϕ:

Measures of ϕ	Points
0	centroid (G)
$\frac{\pi}{2}$	orthocentre (H)
$-\alpha$, if $\alpha \leq 90^\circ$; $180^\circ - \alpha$, if $\alpha > 90^\circ$	A
$-\beta$	B
$-\gamma$	C
$\frac{\pi}{3}$	Fermat point (F_1) (first isogonic centre)
$-\frac{\pi}{3}$	second isogonic centre (F_2)
ω	τ^g
$-\omega$	Brocard's third point (τ_3)

Since some of the cases above may not be familiar, we include the following explanatory remarks:

(i) The *Fermat point* is the point inside the triangle (provided no angle exceeds 120°) such that the sum $\overline{AP} + \overline{BP} + \overline{CP}$, $P \in ABC$, is minimum, see Johnson [**14**, p. 221]. Nicholas D. Kazarinoff presents an interesting alternative treatment of this point using an elementary idea of statics [**15**, pp. 117–118].

(ii) In addition to Brocard's first point, there is *Brocard's second point*, the unique point τ_2 such that $\sphericalangle AB\tau_2 = \sphericalangle BC\tau_2 = \sphericalangle CA\tau_2 = \omega$, see Lemoine [**20**], and Kimberling [**18**]. The point τ in the table is the midpoint of the segment $\tau_1\tau_2$, called *Brocard midpoint* by Kimberling [**18**]. An added significance for τ will be given later when various properties of the hyperbola are discussed.

(iii) The point τ_3 in the table, *Brocard's third point* will be discussed more explicitly in Section 5. The triangle $A'B'C'$ corresponding to $\phi = -\omega$ is *Brocard's first triangle*, see [**14**, pp. 277–280]. We refer to this triangle again when the parabola is discussed. Other points important in Brocard geometry occur on the hyperbola by taking other measures of ϕ that involve ω. We shall not dwell on these here, instead, we refer the interested reader to the paper of M'Cay [**23**].

At this point we make a reference to a property relating to the case when $\phi = \pi/3$ that is generally attributed to Napoleon Bonaparte (1769–1821) [**9**, p. 23], which reads

The circumcircles of the triangles $A'BC$, $AB'C$ and ABC' meet at the Fermat point F_1 and their centres form a fourth equilateral triangle.

In fact, this would appear to be the starting point for the whole story. Equilateral triangles being erected on the faces of an arbitrary triangle appeared first in the context of Napoleon's Theorem. For more information, see the recent papers by Schmidt [**30**] and Wetzel [**38**], where the theorem is traced up to 1825. One can thus assume that it was known to Lemoine and, most likely, served as the basis for his initial question, which was answered by Kiepert.

We now describe the course of the hyperbola more precisely. Assume the triangle ABC to be acute-angled with $\alpha > \beta > (\pi/3)\gamma > \omega$. Let us start out with $\phi = -\pi/2$, in which case the point P that traces out the hyperbola is at the orthocentre. The first notable value of ϕ is $\phi = -\alpha$ whereby $P = A$. Then P moves to infinity, passes through B when $\phi = -\beta$, through F_2 when $\phi = -\pi/3$ and back again to infinity. The remaining values of interest are $\phi = -\gamma$ $(P = C)$, $\phi = -\omega$ $(P = \tau_3)$, $\phi = 0$ $(P = G)$, $\phi = \omega$ $(P = \tau^g)$, $\phi = \pi/3$ $(P = F_1)$ and finally $\phi = \pi/2$ when P returns to the orthocentre, see FIGURE 2. The course changes somewhat when ABC is obtuse-angled, see FIGURE 3.

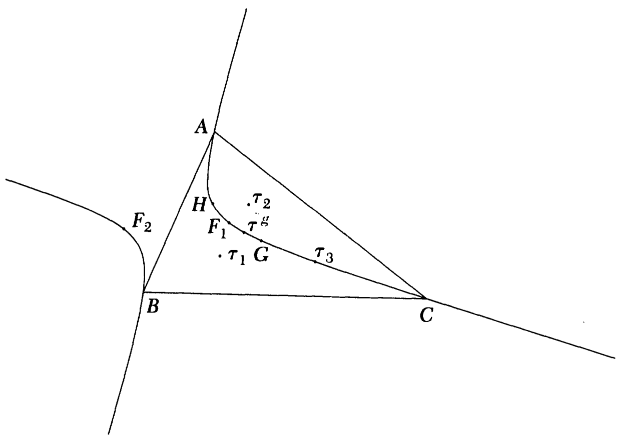

FIGURE 2

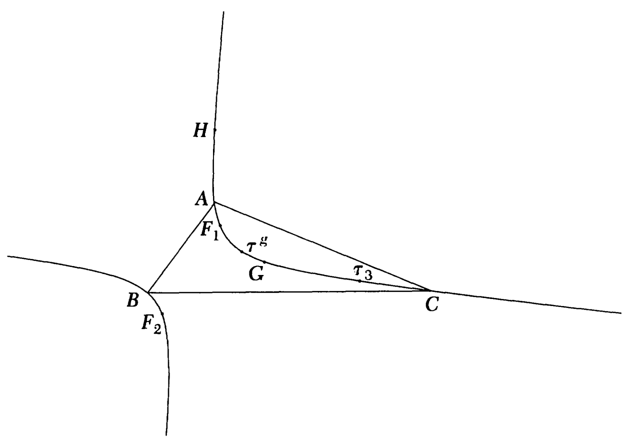

FIGURE 3

A projective derivation of Γ is possible by considering the points A' and B' as elements of the point ranges represented by the perpendicular bisectors of BC and CA respectively. Since the triangles $A'BC$ and $AB'C$ are similar, any four positions of A' have the same cross ratio as the four corresponding position of B'. The point $P = AA' \cap BB'$ is thus the intersection of corresponding elements of two projectively related pencils centred at A and B and hence its locus is a conic through A and B [**36**, pp. 109ff]. This derivation was considered by, among others, Frederick G. Maskell and Jordi Dou [**22**].

Kiepert's hyperbola has a number of interesting properties that serve to emphasize its importance in the geometry of the triangle. We now summarize those that would seem to be most accessible to the general reader.

(i) It is rectangular (asymptotes are perpendicular) and its centre lies on the nine-point circle. This is an immediate consequence of the following theorem that, although attributed to Karl Wilhelm Feuerbach (1800–1834) by Cooledge [6, p. 123], can not be found in Feuerbach's book of 1822 [13]. The claim on the centre had been proved earlier in 1821, by Charles-Julien Brianchon (1783–1864) and Jean-Victor Poncelet (1788–1867) [4, THEOREM VII].

> *The locus of the centres of all conics through the vertices and orthocentre of a triangle, which conics, when not degenerate, are rectangular hyperbolas, is a circle through the middle points of the sides, the points half-way from the orthocentre to the vertices, and the feet of the altitudes.*

Moreover, the centre of the hyperbola is midway between the isogonic centres of ABC, see FIGURE 4.

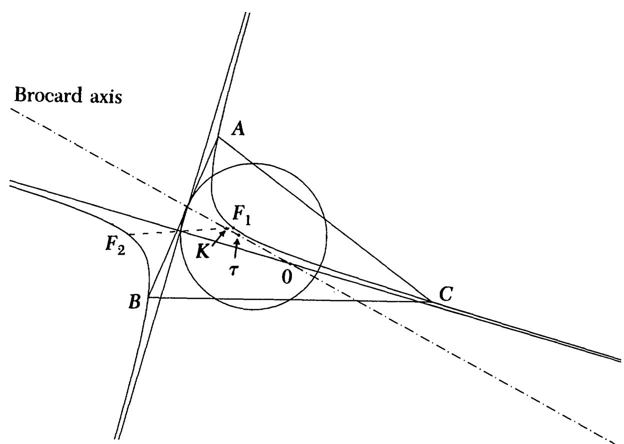

FIGURE 4

(ii) The image Γ^g of Kiepert's hyperbola, under the isogonal transformation, has the equation

$$\Gamma^g\colon \sin(\beta-\gamma)x+\sin(\gamma-\alpha)y+\sin(\alpha-\beta)z=0, \tag{7}$$

or, equivalently,

$$\Gamma^g\colon bc(b^2-c^2)x+ca(c^2-a^2)y+ab(a^2-b^2)z=0, \tag{8}$$

which represents the *Brocard axis* of ABC defined as the line connecting the symmedian point to the circumcentre. This line is perpendicular to the *Lemoine line*, i.e., the axis of the perspectivity of the triangle ABC and its *tangential triangle* $A_tB_tC_t$ formed by the tangents to the circumcircle at the vertices. The name "Lemoine line" is justified by the fact that the Lemoine point—see Section 2b, Case 1—is the centre of this perspectivity. The Brocard axis contains, in addition to the symmedian point and the circumcentre, the isodynamic points F_1^g, F_2^g (which also are the common points of the circles of Apollonius), the Brocard midpoint τ discussed previously, and at least seven more noteworthy points of the reference triangle, see Kimberling [18].

(iii) One may also make use of the isogonal transformation when looking for the asymptotes. To this end, recall first the notion of the Wallace-Simson line of a point on the circumcircle. The feet of the perpendicular lines from a point P to the sides of a triangle are collinear if, and only if, P belongs to the circumcircle of the triangle in which case the resulting line is called the *Wallace-Simson line* of P, see [11]. The geometric description given for the isogonal transformation shows that the isogonal transform of the circumcircle is the line at infinity and the Wallace-Simson line of any point on the circumcircle of the reference triangle passes through the isogonal conjugate of its diametral point. Furthermore, a tedious but straightforward, computation shows that the asymptotes of any circumscribed equilateral hyperbola are the Wallace-Simson lines of the isogonal conjugates of its points at infinity, i.e., the intersection points of the isogonal transform of the hyperbola with the circumcircle. From this we conclude that the Wallace-Simson lines of the intersections P, Q of the Brocard axis with the circumcircle of ABC are the asymptotes of Kiepert's hyperbola, see FIGURE 4. Further treatments of the asymptotes and the centre are given by Mineur [**25**] and Rigby [**29**].

(iv) Since the coordinates of the nine-point centre are

$$(\cos(\beta-\gamma), \cos(\gamma-\alpha), \cos(\alpha-\beta)),$$

see [**32**, p. 159], it follows from the equation

$$\begin{pmatrix} 0 & \sin(\alpha-\beta) & \sin(\gamma-\alpha) \\ \sin(\alpha-\beta) & 0 & \sin(\beta-\gamma) \\ \sin(\gamma-\alpha) & \sin(\beta-\gamma) & 0 \end{pmatrix} \begin{pmatrix} \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) \end{pmatrix} = \begin{pmatrix} \sin(\gamma-\beta) \\ \sin(\alpha-\gamma) \\ \sin(\beta-\alpha) \end{pmatrix}$$

that the polar of the nine-point centre with respect to Kiepert's hyperbola is the Brocard axis. For the basic properties of poles and polars, see Somerville [**32**, pp. 26–28].

(v) Another connection of Kiepert's hyperbola with the nine-point circle is found in [**5**, p. 459]. If one considers the triangle formed by the tangents to Γ at the vertices A, B, C, then the orthocentre of this triangle is the centre of the nine-point circle. Casey attributes this property to Brocard.

(vi) A more recent "rediscovery" of Kiepert's hyperbola is given in the following problem of Bottema and van Hoorn [**3**].

Let P be a point in the plane of a nonequilateral triangle ABC and let π be the trilinear polar (or harmonical) of P with respect to ABC. Show that the locus of the points P, such that π is perpendicular to the Euler line of ABC, is a rectangular hyperbola passing through the vertices of ABC, through its centroid and through its orthocentre.

Here, the *trilinear polar* of a point P is the axis of perspectivity of the triangles ABC and DEF (D, E, F are again the intersections of the lines AP, BP, CP with the sides BC, CA, AB). If $P=(p,q,r)$, the trilinear polar is the line with coordinates $[qr, rp, pq]$, see [**10**, p. 185]. While this is an interesting and somewhat different aspect of Kiepert's hyperbola, the result is not new. A reference to this particular property may be found in M'Cay [**23**].

(vii) Another recent rediscovery of the hyperbola is given by Courcouf [**8**] as an application of areal coordinates to the geometry of the triangle. Here as in the case of the previous problem, see [**24**], the name Kiepert is not associated with this conic. This is further evidence that the Kiepert conics are not well known today.

(viii) A subtle comment on the centre of Γ is given by Thébault [**34**]. Let angular bisectors of the reference triangle meet BC, CA, AB in the points A_1, B_1, C_1. Let $A_1^{\#}$ be the harmonic conjugate of A_1 with respect to B and C, and let $B_1^{\#}$ and $C_1^{\#}$ be defined in a similar manner. Thébault's result is that the circles $A_1B_1C_1$, $A_1B_1^{\#}C_1^{\#}$, $B_1C_1^{\#}A_1^{\#}$, $C_1A_1^{\#}B_1^{\#}$ meet at the centre of Γ.

(ix) A novel treatment of the hyperbola using a complex-number approach is given by Kelly and Merriell **[16]**. In the notation of our Theorem 1, the authors show that the perpendiculars from A to $B'C'$, B to $C'A'$, C to $A'B'$ concur and the locus of the point of concurrence with the trilinear coordinates $(1/\cos(\alpha-\phi), 1/\cos(\beta-\phi), 1/\cos(\gamma-\phi))$, as ϕ varies, is Kiepert's hyperbola, see FIGURE 5.

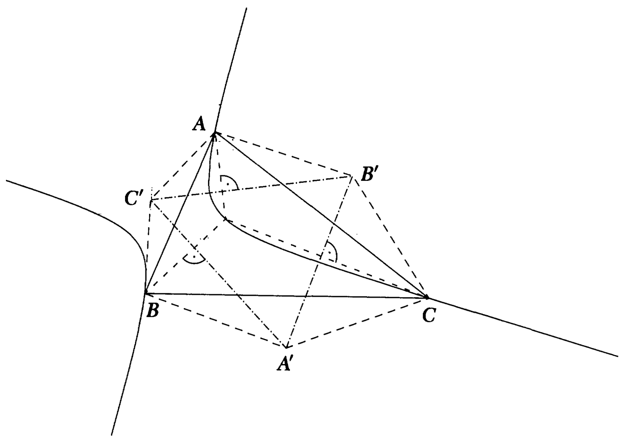

FIGURE 5

(x) Vanderghen **[35]** notes that Kiepert's hyperbola is the cevian transform (isotomic conjugate) of the tangent to $\mathbf{\Gamma}$ at the centroid G. To see this consider the alternative form of $\mathbf{\Gamma}$ given by (5). Since the coordinates of G are (bc, ca, ab), the line coordinates of the tangent at this point are given by

$$\begin{pmatrix} 0 & ab(a^2-b^2) & ca(c^2-a^2) \\ ab(a^2-b^2) & 0 & bc(b^2-c^2) \\ ca(c^2-a^2) & bc(b^2-c^2) & 0 \end{pmatrix}\begin{pmatrix} bc \\ ca \\ ab \end{pmatrix} = \begin{pmatrix} a(c^2-b^2) \\ b(a^2-c^2) \\ c(b^2-a^2) \end{pmatrix}; \qquad (9)$$

these are the line coordinates of the isotomic conjugate of $\mathbf{\Gamma}$.

(xii) If a *point conic* F is given by the equation $\sum_{i,j} a_{ij}x_i x_j = 0$; $i, j = 1, 2, 3$, then it is an easy exercise to verify that the corresponding *line* form f of F is defined by the equation $\sum_{i,j} A_{ij}u_i u_j = 0$, where (A_{ij}) is the adjoint of (a_{ij}). Thus for the hyperbola, the line form is

$$\gamma\colon p^2u^2 + q^2v^2 + r^2w^2 - 2pquv - 2qrvw - 2rpwu = 0,$$

where $(p, q, r) = (bc(b^2-c^2), ca(c^2-a^2), ab(a^2-b^2))$ and $[u, v, w]$ is a tangent to $\mathbf{\Gamma}$.

4. The Parabola

In order to introduce the second conic, we state and prove the following:

THEOREM 2. *The envelope of the axis of the triangles ABC and A′B′C′ is the parabola*

$$\mathbf{\Sigma}\colon \frac{\sin\alpha(\sin^2\beta-\sin^2\gamma)}{u} + \frac{\sin\beta(\sin^2\gamma-\sin^2\alpha)}{v} + \frac{\sin\gamma(\sin^2\alpha-\sin^2\beta)}{w} = 0, \qquad (10)$$

or, equivalently,

$$\Sigma: \frac{a(b^2-c^2)}{u} + \frac{b(c^2-a^2)}{v} + \frac{c(a^2-b^2)}{w} = 0, \tag{11}$$

where $[u, v, w]$ is a tangent to the parabola.

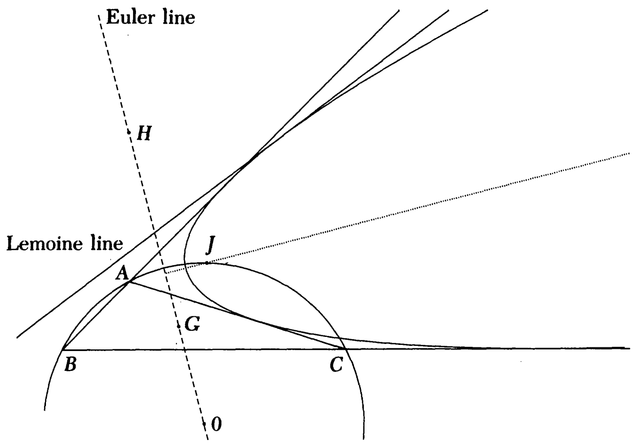

FIGURE 6

Proof. Since the triangles ABC and $A'B'C'$ are perspective from the point P, they are (by Desargues's theorem) perspective from a line. This line has the trilinear representation

$$l = \left[\frac{1}{\sin\beta\sin\gamma + \sin\alpha\sin 2\phi}, \frac{1}{\sin\gamma\sin\alpha + \sin\beta\sin 2\phi}, \frac{1}{\sin\alpha\sin\beta + \sin\gamma\sin 2\phi}\right], \tag{12}$$

see proof of THEOREM 1, or, equivalently,

$$u : v : w = \frac{1}{\sin\beta\sin\gamma + \sin\alpha\sin 2\phi} : \frac{1}{\sin\gamma\sin\alpha + \sin\beta\sin 2\phi} : \frac{1}{\sin\alpha\sin\beta + \sin\gamma\sin 2\phi}. \tag{13}$$

We now have

$$u(\sin\beta\sin\gamma + \sin\alpha\sin 2\phi) = v(\sin\gamma\sin\alpha + \sin\beta\sin 2\phi) = w(\sin\alpha\sin\beta + \sin\gamma\sin 2\phi),$$

hence

$$\sin 2\phi = \frac{(v\sin\alpha - u\sin\beta)\sin\gamma}{u\sin\alpha - v\sin\beta} = \frac{(w\sin\alpha - u\sin\gamma)\sin\beta}{u\sin\alpha - w\sin\gamma}, \tag{14}$$

from which we obtain the desired result.

It is obvious that the envelope (11) represents a parabola since the line at infinity $[a, b, c]$ is one of its tangents. Furthermore, this conic (*Kiepert's parabola*) is inscribed in the triangle ABC and has for a fifth tangent the Lemoine line

$x/a + y/b + z/c = 0$. See the important work of Jean Baptiste Joseph Neuberg (Luxembourg 1840–Liège 1926, Professor of Geometry at the Athénée and Lecturer at the École des Mines of Liège) [**26**, **27**, **28**]. Using the fact that $a = 2R\sin\alpha$, for example, where R denotes the circumradius of ABC, it follows, from (12), that the line at infinity corresponds to the case $\phi = 0$. However, it appears that this is the only one of the five given tangents that can be obtained by a specific value of ϕ. In addition, the presence of the term $\sin 2\phi$ indicates that, as ϕ varies from 0 to $\pi/2$, one part of the parabola is traversed twice while the other part is not obtained. According to Neuberg, this parabola was first studied in 1884 by the Senior Teacher at the Gymnasium of Recklinghausen, Germany, August Artzt, in a so-called "school programm" [**1**].

An alternate approach will lead to a projective derivation of the parabola Σ. Consider now a triangle $A^\star B^\star C^\star$ homothetic with the tangential triangle $A_tB_tC_t$ with respect to the circumcentre of ABC as shown in Figure 7. Since the tangents to the circumcircle at the vertices have the line coordinates $[0, c, b], [c, 0, a], [b, a, 0]$, the vertices of the tangential triangle have the trilinear representation $A_t = (-a, b, c)$, $B_t = (a, -b, c)$, $C_t = (a, b, -c)$. Recall that the coordinates of the circumcentre O are $(\cos\alpha, \cos\beta, \cos\gamma)$, thus, with respect to a real parameter μ, an arbitrary point A^* has the coordinates

$$A^* = \left(\mu\cos^2\alpha - a^2, \mu\cos\alpha\cos\beta + ab, \mu\cos\alpha\cos\gamma + ac\right).$$

Now we compute the line coordinates of the parallels through A^* to the corresponding tangents and obtain for the remaining vertices under consideration

$$B^* = \left(\mu\cos\beta\cos\alpha + ba, \mu\cos^2\beta - b^2, \mu\cos\beta\cos\gamma + bc\right),$$
$$C^* = \left(\mu\cos\gamma\cos\alpha + ca, \mu\cos\gamma\cos\beta + cb, \mu\cos^2\gamma - c^2\right).$$

Since the lines AA^*, BB^*, CC^* concur at the point

$$\left((\mu\cos\beta\cos\gamma + bc)^{-1}, (\mu\cos\gamma\cos\alpha + ca)^{-1}, (\mu\cos\beta\cos\alpha + ba)^{-1}\right),$$

the triangles ABC and A^*, B^*, C^* are perspective from a line, i.e., the points $N_a = BC \cap B^\star C^\star$, $N_b = CA \cap C^\star A^\star$, $N_c = AB \cap A^\star B^\star$ belong to one line that we denote by $N_aN_bN_c$ in the sequel. The assignment $N_a \mapsto C^\star$ is a perspectivity from the line BC to the line OC_t, and the assignment $C^\star \mapsto N_b$ is another perspectivity; hence the composite assignment is a projectivity and so, the lines $N_aN_bN_c$ envelope a conic; see [**36**, pp. 109].

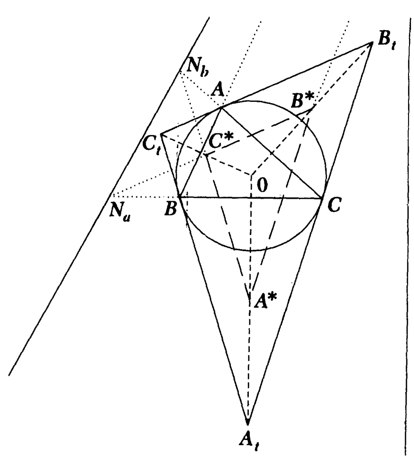

FIGURE 7

To see that this conic is actually Kiepert's parabola, we show how to identify the five specific tangents listed above. The Lemoine line is an obvious tangent since it is just the axis of perspectivity of the triangles ABC and $A_tB_tC_t$. Next, consider the case when $A, A^\star, C^\star$ are collinear, see FIGURE 8. Now $C^\star A^\star \cap CA = N_b = A$ and $A^\star B^\star \cap AB = N_c$ is always a point on AB, thus the side AB is a tangent to the conic. Similar arguments show that BC and CA are tangents also. Finally, when $A^\star, B^\star, C^\star$ are themselves on the line at infinity, i.e., $\mu = -4R^2$, the line $N_aN_bN_c$ is the tangent at infinity.

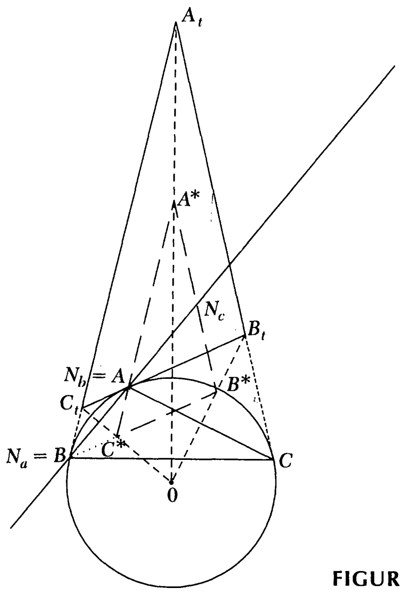

FIGURE 8

As in the case of the hyperbola, we list some properties of the parabola that serve to demonstrate that this conic also plays an important role in the geometry of the triangle.

(i) The Euler line, see [**18**], of the triangle ABC is the directrix of Kiepert's parabola. To prove this, we shall need the following, see [**31**, p. 70].

LEMMA. *The directrices of all parabolas inscribed in a triangle pass through the orthocentre.*

Since this reference may not be well known today, we sketch a proof. The foot of the perpendicular from the focus of a parabola to any tangent belongs to the tangent at the vertex. Thus, the three feet of the perpendiculars from the focus to the sides of a tangential triangle are collinear showing that the focus belongs to the circumcircle of the triangle and that the tangent at the vertex is its Wallace-Simson line. The latter bisects the segment joining the focus to the orthocentre and, consequently, the orthocentre belongs to the directrix, see also [**31**, pp. 48ff].

To see (i), first note that the above lemma implies that the directrix of Kiepert's parabola Σ contains the orthocentre H of the triangle ABC. Second, consider the tangent to Σ that corresponds to $\phi = -\gamma$ where A' is on AC and B' is on BC. Then, the circumcentre O of ABC is the orthocentre of $A'B'C$ and Σ is also inscribed in this triangle. Now, by the lemma, the point O belongs to the directrix of Σ. The directrix thus contains the points O and H and hence, is the Euler line e of ABC.

(ii) The coordinates of the focus J of Kiepert's parabola are given by

$$J = \left(\frac{1}{\sin(\beta-\gamma)}, \frac{1}{\sin(\gamma-\alpha)}, \frac{1}{\sin(\alpha-\beta)} \right).$$

To see this note that the pole of a line $[u, v, w]$ with respect to Σ is given by

$$\big[(v \sin^2 \gamma \sin(\alpha - \beta) + w \sin^2 \beta \sin(\gamma - \alpha)),$$
$$(w \sin^2 \alpha \sin(\beta - \gamma) + u \sin^2 \gamma \sin(\alpha - \beta)),$$
$$(u \sin^2 \beta \sin(\gamma - \alpha) + v \sin^2 \alpha \sin(\beta - \gamma))\big].$$

Since $e = [\sin 2\alpha \sin(\beta - \gamma), \sin 2\beta \sin(\gamma - \alpha), \sin 2\gamma \sin(\alpha - \beta)]$, the coordinates of J are the claimed ones.

In the sketch of the proof of the lemma above it has been mentioned that the focus of any parabola inscribed a triangle belongs to the circumcircle of this triangle. In our case, it's an easy exercise to verify that the given coordinates for J satisfy the equation

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0,$$

which is that of the circumcircle of ABC.

(iii) If a conic is inscribed in a triangle, then the joins of the vertices of this triangle and the points of contact are concurrent in what may be termed the *Brianchon point* of the conic with respect to the triangle, see [**36**, p. 111]. Since a conic inscribed in the triangle ABC has an equation of the form $f/u + g/v + h/w = 0$, its Brianchon point is easily seen to have coordinates $(1/f, 1/g, 1/h)$. Thus, for Kiepert's parabola, it is, from (11), $(1/a(b^2 - c^2), 1/b(c^2 - a^2), 1/c(a^2 - b^2))$, which is the *Steiner point* S of the triangle, see [**18**]. This is the point of concurrence of the three lines drawn through the vertices of a triangle parallel to the corresponding sides of Brocard's first triangle. In addition, the Steiner point is on the circumcircle of ABC, see [**14**, pp. 281ff] and FIGURE 9.

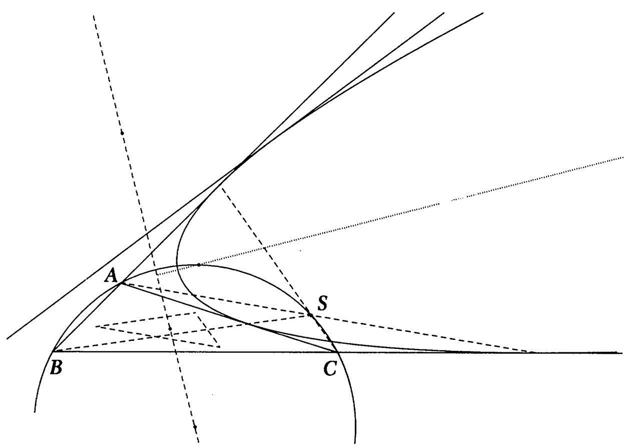

FIGURE 9

(iv) On the basis of property (xi) of the hyperbola, the point form of the parabola has equation

$$\Sigma\colon f^2x^2 + g^2y^2 + h^2z^2 - 2fgxy - 2ghyz - 2hfzx = 0,$$

Artzt also studied other parabolas associated with the triangle. Of particular relevance at this time is a trio referred to by Casey as the Artzt's parabolas (second group). Consider the configuration of THEOREM 1. Since the line $A'B'$, for example, is the join of two projectively related points, it envelopes a conic. This conic is a parabola such that the internal and external bisectors of $\measuredangle BCA$ are tangents as are the perpendicular bisectors of BC and CA. Similar arguments hold for the lines $B'C'$ and $C'A'$.

5. Results Not Found in the Available Literature

Here we present some material, in the form of theorems, which we believe to be new.

THEOREM 3. *The centre of the circle inscribed in the triangle DEF, where D, E, F are the midpoints of the sides BC, CA, AB respectively of the given triangle ABC, lies on Kiepert's hyperbola.*

Proof. Since the triangle DEF is homothetic to the triangle ABC with factor $-1/2$, the radius ρ of the given circle, also known as the *Spieker circle*, see [14, p. 226], is $r/2 = \Delta/2s$, where $s = (a + b + c)/2$ and Δ denotes the area of the triangle ABC. Consequently, the distance d_a of the centre V of the Spieker circle from the side BC of the triangle ABC is given by the equation

$$d_a = \frac{h_a}{2} - \rho = \Delta\left(\frac{1}{a} - \frac{1}{2s}\right),$$

where h_a denotes the altitude to the side BC. The coordinates of V are now easily seen to be

$$V = \left(\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}\right),$$

which satisfy the equation (5) of Kiepert's hyperbola, see FIGURE 10.

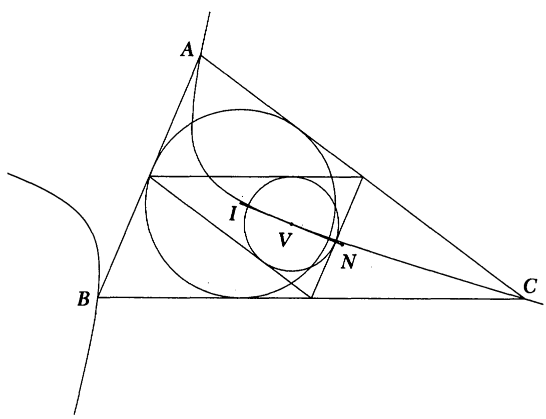

FIGURE 10

Remark. By means of barycentric coordinates one can show that V is midway between the incentre I and the Nagel point N, see Section 2b. For further properties of the Spieker circle see [14, pp. 226ff] and [18].

In M'Cay [23], it is given that the point D, the centre of homology of the triangle ABC and Brocard's first triangle, lies on the hyperbola. It is, by its very definition, nothing but the point P of concurrence in the sense of our THEOREM 1 corresponding to $\phi = -\omega$. We have been able to link this result with the following already mentioned fact:

THEOREM 4. *Brocard's third point lies on Kiepert's hyperbola.*

Proof. The barycentric coordinates of this point, which we denote as τ_3, are given in [5, p. 66] as $(1/a^2, 1/b^2, 1/c^2)$, which implies that the trilinear coordinates are $\tau_3 = (1/a^3, 1/b^3, 1/c^3)$. It now becomes a trivial exercise to verify that these satisfy equation (5).

THEOREM 5. *The points D and τ_3 are one and the same.*

Proof. From (6), the coordinates of the point on Kiepert's hyperbola corresponding to $\phi = -\omega$ are

$$\left(\frac{1}{\sin(\alpha-\omega)}, \frac{1}{\sin(\beta-\omega)}, \frac{1}{\sin(\gamma-\omega)}\right),$$

which can also be written in the form

$$\left(\frac{1}{(\sin\alpha\cot\omega-\cos\alpha)}, \frac{1}{(\sin\beta\cot\omega-\cos\beta)}, \frac{1}{(\sin\gamma\cot\omega-\cos\gamma)}\right).$$

But $\cot\omega = (a^2+b^2+c^2)/4\Delta$, see [**14**, pp. 264ff], and $4\Delta = 2bc\sin\alpha$, thus $(1/\sin\omega(\sin\alpha\cot\omega-\cos\alpha)) = (bc/a^2\sin\omega)$ and similarly for the other two coordinates. We now have

$$\left(\frac{bc}{a^2\sin\omega}, \frac{ca}{b^2\sin\omega}, \frac{ab}{c^2\sin\omega}\right) = \left(\frac{1}{a^3}, \frac{1}{b^3}, \frac{1}{c^3}\right)$$

as the coordinates of τ_3.

We actually discovered that τ_3 was on the hyperbola before seeing the information in Casey. Since we believe that this was accomplished by a rather pretty argument, we supply some details. By comparing equations (5) and (11) it is easy to see that the two are related by the elliptic polarity $\rho x_i = \sum_j a_{ij} u_j$; $i, j = 1,2,3$; $\rho \neq 0$, where $a_{11} = b^2c^2$, $a_{22} = c^2a^2$, $a_{33} = a^2b^2$ and $a_{ij} = 0$ when $i \neq j$, which maps the points of the hyperbola to the tangent lines of the parabola. Brocard's third point corresponds to the Lemoine line $[bc, ca, ab]$ under this transformation. We note further that $\tau_3 = K^t$, the isotomic conjugate of the symmedian point. The reader may wish to use this idea to find other meaningful points and lines associated with these conics.

As an aside, we have derived a further result with respect to the point τ_3.

THEOREM 6. *Brocard's third point is collinear with the centre of the Spieker circle and the isotomic conjugate of the incentre.*

Proof. Since the determinant

$$\begin{vmatrix} \dfrac{b+c}{a} & \dfrac{c+a}{b} & \dfrac{a+b}{c} \\ \dfrac{1}{a^3} & \dfrac{1}{b^3} & \dfrac{1}{c^3} \\ \dfrac{1}{a^2} & \dfrac{1}{b^2} & \dfrac{1}{c^2} \end{vmatrix}$$

vanishes, the result follows.

Remark. The barycentric coordinates of Brocard's first and second points are $\tau_1 = (1/b^2, 1/c^2, 1/a^2)$ and $\tau_2 = (1/c^2, 1/a^2, 1/b^2)$, so that the barycentric coordinates of $\tau_3 = (1/a^2, 1/b^2, 1/c^2)$ complete the cyclic order. This may be the reason for the name *Brocard's third point*, which we only found in Casey [**5**, p. 66], in the coordinate form above, with no further information given. Kimberling [**18**] lists this point as just one of 91 *polynomial centers* of the reference triangle and mentions our THEOREM 6 in a slightly different form.

6. Conclusion

Even now there are other aspects of these conics that we have not touched upon as they seem to require a more thorough knowledge of the geometry of the triangle than that of the general reader. However, what is included should serve to convince the reader that Kiepert's hyperbola and Kiepert's parabola are not only interesting in their own right, but also, they constitute an important chapter of the geometry of the triangle. In FIGURE 11 we show them together for the first time. The reference triangle is deliberately chosen to be right-angled since the hyperbola is best illustrated with respect to an acute triangle while, in the case of the parabola, the obtuse case is more convenient.

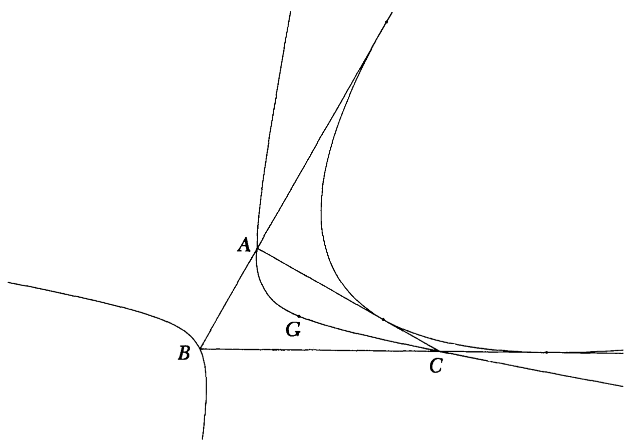

FIGURE 11

Acknowledgements The authors would like to thank the following:

- One of the referees for suggesting the present proof that the points N_a, N_b, N_c are collinear; this is much nicer than our original proof. Also, to the same referee, for a shortcut of our original argument concerning the projective derivation of the parabola.
- Mrs. B. Eddy for locating many of the references, some of which were not readily obtainable.

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"The eagle putt was long, and at that moment it seemed almost as long as the Iberian Peninsula that spawned the man who stood over it at the 15th hole of Augusta National today. The Masters is won and lost on such putts, and José Maria Olazabal added his name to the list of men who have accepted the challenge...

... He learned early the value of imagination around the greens, and it was that imagination that carried him to victory. On a day when the firm and fast putting surfaces were as difficult to solve as linear equations, Olazabal spent much of the day doing an impression of his more famous countryman, Seve Ballesteros, when it came to getting up and down..."

--Larry Dorman, *New York Times*, April 11, 1994, C1
(sent by Robert A. Russell, New York)