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Contents

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The transcendence of π has been known for about a century – but who was the man who discovered it?

RUDOLF FRITSCH

Freiburg, April 12, 1882

A very well known German mathematical institution is the "Mathematische Forschungsinstitut Oberwolfach". Its director Professor Martin Barner of the University of Freiburg im Breisgau, built his private house on the Loretto hill, a part of the Black Forest belonging to the city of Freiburg. It stands on a place of extreme mathematical interest, because a young man had an important idea here. It was his 30th birthday, and he was alone on a stroll to Günterstal, a small village with a medieval monastery, today also part of Freiburg. Five years beforehand, in October 1877 as associate professor in Freiburg he had been invited to take similar walks in the company of the (full) professors Thomae[1] from Freiburg and du Bois-Reymond[2] from Tübingen, Thomae's friend and predecessor, who often came back for short visits. On the first of these walks he was unsuitably dressed, hiking through the creeks and brushwood of the Black Forest with top-hat and tails, whereas his colleagues looked like todays equivalent of "green-peacers". Besides enjoying the wonderful landscape, the group engaged in mathematical discussions. Thomae and du Bois-Reymond were specialists in (complex) analysis; our young man was brilliant in geometry, having learned a lot from Clebsch[3] and Felix Klein[4], especially the tools which he used so effectively a short time later. One topic touched on in their talks was the problem of the transcendence of π. Euler[5] and Lambert[6] had conjectured that π was transcendental. If this could be proved, then the very old question of squaring the circle would be settled[7]. Thomae and du Bois-Reymond proposed to attack this problem by means of continued fractions, a device which the French mathematician Liouville[8] had very successfully used in order to clarify the notion of transcendental numbers and to exhibit some of such numbers. But our young friend was not attracted to this approach. Some years previously he had spent a winter term (1876/77) in Paris, where Hermite[9] had shown him how to prove the transcendence of e using integration of real functions. He felt that this must be the right path leading to the goal, although for a long time he had had no idea how to begin. Meanwhile his personal circumstances had changed. On October 1, 1879 he had taken over the

chair of Thomae who had moved to Jena, and was too far away to come back just for walks. Reminiscing about the past, he frequently made such strolls alone now, just like the one on this pleasant day. Some weeks beforehand he had looked through his collection of reprints for Hermite's famous paper on the solution of the general equation of degree 5 by means of elliptic functions[10]. By chance, the proof of the transcendence of e caught his eye and he started to study it again. Hermite himself regarded this paper as his own most significant accomplishment. At the place where Barner's house stands today, the long hoped for idea flashed through his mind: $e^{\pi i} = -1$! He rushed home and wrote down the paper: "Über die Zahl π" (On the number π)[11]. When he went into his club, later for dinner his appearance must have been somewhat strange, for one of his friends, Lieutenant-Colonel von dem Busche, welcomed him by saying: "Sie sehen ja aus, als hätten Sie die Quadratur des Kreises gelöst"[12]. This well-meaning officer could not have made a more apt remark. The name of the new star in the mathematical heaven was *Ferdinand Lindemann.* In remembrance of this event the sculptor Rudolf Hofmann of Darmstadt modelled a bust in 1943 which originally stood in the University of Freiburg but now has its place in the Oberwolfach Institute.

Origin and youth

The Lindemanns were (and are) a neither wealthy nor poor middle class family. They originally were craftsmen, as for instance the brass-founder, Bartholomäus Lindemann, in Celle, who died in 1738. His son and his grandson became Lutheran pastors. One great-grandson, Ferdinand Johannes Heinrich, worked first as a teacher of modern languages in Hannover and later as a manager of his brother's gas works in Schwerin (Mecklenburg)[13]. This was the father of our Ferdinand. His wife Emilie[14] was the daughter of a famous teacher of classical languages, Gottlob Crusius[15]. It was from him that the grandson inherited an inclination to classical languages and ancient weight measures, which played an important role in his later life.

Our Carl Louis Ferdinand Lindemann was born in Hannover on April 12, 1852. He began elementary school in Schwerin. Since his father was not satisfied with the lessons there, he decided to teach his son himself and exposed him to a broad spectrum of general culture. One result was that his son, at the age of eight, was familiar with algebra involving letters and brackets, the rule of three and constructions with ruler and compass – the basis for his mathematical career. In 1861 Ferdinand entered the gymnasium in Schwerin where he experienced the good old classical education, which incidentally, has almost disappeared today. In the nine years up to his "Abitur" Ferdinand was quite a good student, often top

of his class, but without preference for a special subject. He liked Latin and classical Greek as well as science and mathematics; he even took a voluntary course in Hebrew! His first pocket money was used to buy Curtius' Greek history[16] as well as Humboldt's "Kosmos"[17]. It was the latter work which raised his enthusiasm for science and astronomy. Moreover, during his last years at the gymnasium he had an inspiring teacher of mathematics, Dr. Bastian[18].

At home his father argued strongly for a pure philological culture, but after a while became convinced that mathematics and science also bear ideal values. Thus, after the final examination in the summer of 1870 in which he was first in his class, Ferdinand decided to study astronomy and mathematics. But a war had just started between France and Germany – the war which decided the fate of Napoleon III and united the German local states in a new empire. He was lucky: because of his poor health, he was not called into the army. Until the university semester started, he studied some books which he got from his teacher: a school program of Liegnitz concerning conics[19], Stern's "Algebraische Analysis"[20] and Steiner's "Vorlesungen über Geometrie"[21]. But – what a surprise – in his father's library he found Schlömich's "Analytische Geometrie"[22].

University studies up to habilitation

For the winter semester 1870/1871 Ferdinand Lindemann enrolled at the University of Göttingen, which since the time of Gauß was the most famous place for mathematics, not only in Germany. His first academic teachers were Enneper (calculus and differential equations)[23], Weber (physics)[24], Wöhler (chemistry)[25] and Stern (algebraic analysis)[20]. Clearly the top-scientists among them were the aging Weber and Wöhler, who nevertheless gave impressive lectures. The summer term 1871 proved decisive for his further life. He attented Clebsch's[3] lecture on analytic plane geometry. Clebsch's method of lecturing was fascinating. He often stood near to the stove – at that time the professors had to pay for the heating from their salaries – and spoke freely about the equations at the blackboard. As far as style and rhetoric are concerned, his discourse is considered to have been most accomplished. One of his students said: "Ich kann bei Clebsch nichts lernen; man wird durch die Schönheit seiner Sprache so gefesselt, daß man auf den Inhalt gar nicht achtgeben kann"[26]. Lindemann's own lectures in later life had a different character. Perron[27] used to say that he never learned more in a lecture than when Lindemann started to extemporize, following a sudden inspiration or when a formula was not correct, and the lecture turned into a bright colloquium. According to one student: "Lindemann denkt ja in einer Minute mehr Geometrie, als ich im ganzen Semester begreifen kann"[28].

Besides the subjects already mentioned, Lindemann took some courses on mineralogy, given by Sartorius von Waltershausen, the friend and first biographer of Gauß[29]. Soon he cancelled his original purpose of studying astronomy too. One reason was that only the unfortunate Klinkerfues[30] offered lectures on astronomy, for which he very seldom got an audience; the other was that pure mathematics had totally engrossed him.

In the following semesters Lindemann attended Clebsch's lectures on analytic spatial geometry, algebraic curves, elliptic functions and the theory of algebraic forms. Suddenly, on November 7, 1872, Clebsch died of diphtheria, at the age of 39 years. Lindemann had only once had an opportunity[31] for an intensive discussion with Clebsch but he had written down the notes of Clebsch's lectures very carefully. The lecturer Neesen[32], who was asked to continue the lectures on spatial geometry borrowed these notes and proceeded along their lines. During the preparations for Clebsch's funeral, Lindemann demonstrated his ability in administrative matters for the first time. This later led him to act very effectively as dean, rector in Königsberg and in München, and as director of the "Verwaltungsausschuß" (administrative committee) of the university in München for about 25 years. There were two student fraternities in Göttingen – the "Burschenschaft" and the "Corps" – which struggled for precedence at the funeral procession, both claiming that Clebsch had been one of their members. At that time, the student fraternities played an important role in German academic life. Lindemann did not belong to a real fraternity, but he was the chairman of the "Mathematische Verein" (mathematical union) in Göttingen which functioned in a similar manner. Clebsch had studied in the town of his birth, Königsberg, which was quite a distance away but nobody knew anything about his activities there. The janitor decided that Clebsch had not been a colour-wearing student, and Lindemann won. The guard of honour at the rector's coffin consisted of members of the mathematical union.

It was at one of the meetings of the mathematical union that the young lecturer Felix Klein[33] first realized the presence of a very bright student named Lindemann. By chance during a meeting of the union Klein listened to Lindemann reporting on his (= Klein's) new papers about non-euclidean geometry[34]. Klein was immediately inspired to propose to Lindemann that he writes his PhD thesis on non-euclidean mechanics. Thus Lindemann became the second PhD student[35] of Klein; less than three years after starting his university studies, at the end of the summer semester 1873, he was awarded the degree from the University of Erlangen[37], for the thesis: "Über unendlich kleine Bewegungen und Kraftsysteme bei allgemeiner projektivischer Maßbestimmung"[36].

It was under the strong pressure of Klein, that Lindemann was persuaded to finish his doctoral examinations as quickly as possible. This meant taking his PhD

examination with little preparation, resulting in a bad mark in the oral physics examination[38]. But Klein had another project he wanted Lindemann to tackle.

Together with Clebsch, Klein had prepared some of Plücker's[39] work posthumously for publication. Recalling the precise notes Lindemann had written in Clebsch's lectures, Klein–now acting as executor of Clebsch's scientific heritage[40]–thought Lindemann could prepare Clebsch's lectures as a text book. He had asked Lindemann not to go home early for the Christmas holidays in 1872, because he wanted to introduce Lindemann to former friends and students of Clebsch. Thus, Lindemann met Gordan[41] and Noether[42] for the first time. Having made a good impression, he received the formal offer from Klein to write such a text book. That represented a great honour, and a good deal of work, but without real monetary reward; the honorarium was assigned to Clebsch's family. The 21 year old student could barely live on the 75 marks which his father sent monthly, plus the care packages from his girl cousin Ida v. Witzendorff, but he accepted nevertheless. Since Klein wanted to write the preface himself as well as to supervise the work, Lindemann had to be reachable. Klein and Gordan urged Lindemann to include new results which were not known to Clebsch. The end product really deserved the name "Clebsch–Lindemann" under which the book[43] was known and used for quite a few decades. At the occasion of Lindemann's 70th birthday, his Munich colleague Voss[44] described this work as follows: "Mit jugendlichem Wagemut haben Sie, noch vor Vollendung Ihres Universitätsstudiums, diese große und schwierige Aufgabe übernommen und so ein Werk geschaffen, das zugleich auch durch und durch Ihr eigenes geworden ist. Ich staune so oft ich in dasselbe hineinsehe, noch jedesmal über die Tiefe und Weite des Blickes, mit dem Sie alles zu einem harmonischen Ganzen zu verschmelzen wußten, was Clebsch in den letzten Jahren seines Lebens in Vorlesungen zum Teil ausgeführt oder auch, wie z.B. die Invariantentheorie der Konnexe, unvollendet hinterlassen hatte. Seit fast 50 Jahren ist Ihr Werk noch von der gleichen Bedeutung für jeden Geometer geblieben, dem keine Nation eines von ähnlicher umfassender Bedeutung an die Seite stellen kann"[45]. However it took a long time to complete the book. Klein's intention to have it published in 1874 was not realised. The first edition appeared in 1876 and served as a basis for Lindemann's habilitation at the university of Würzburg 1877.

During his short stay in Erlangen, Lindemann had an interesting living situation. His landlady, Mrs. Brater, widow of a Bavarian politician[46], was a sister of Klein's predecessor Hans Pfaff[37]. Lindemann's living room was filled with Pfaff's furniture. For the discussions on Clebsch's lectures, Klein preferred to visit Lindemann. He had a good reason: namely, the girls who served coffee, the landlady's daughter Agnes[47] and her friend Miss Hegel,[48] who later became Mrs. Klein.

For the summer semester 1875, Klein accepted a position at the Technical School of München, and Lindemann had to follow again. The status of the Technical School at that time was not that of a university. This raised problems, not for Klein but for Lindemann. Klein had been a full professor at a university and was consequently established. Because of the larger number of students at the Technical School, he got a higher income[49] there. But the Technical School could neither award the PhD degree nor a habilitation. The professors of mathematics at the University of München did not even like to be referred to as "colleagues" by mathematicians at the Technical School. At that time there were two full professors of mathematics at the University of München, Seidel[50] and Bauer[44]. Lindemann asked them to sponsor his habilitation, but Seidel refused. He did not like the methods of Riemann[51] and Klein, and he did not like the idea of having a student of Klein as a lecturer at the University of München. Did he foresee that this student would become his immediate successor? Lindemann seemed not to have any academic future in München, but at the moment he was too busy with Clebsch's lectures anyway. He got an opportunity nevertheless.

The Technical School was fighting to gain an equal status to the universities. For instance the money which the royal court got from people asking for titles[52] was given to graduate students of the Bavarian universities for scientific excursions. The Technical School also wanted to take advantage of these funds, and Klein proposed that Lindemann takes a trip to London to see an exhibition of scientific gadgets. Lindemann received the funds and was asked to thank the government. He visited the nearly almighty Minister of Education, Johann von Lutz[53], who took an interest in Lindemann's personal difficulties. Lutz knew that the University of Würzburg had requested a lecturer for mathematics. Prym[54] was full professor at Würzburg and Lutz had absolute confidence in Prym's judgement. Prym accepted Lindemann for a later habilitation, and thus Lindemann could start the trip which led him to London and Paris, without a worry in the world.

As already mentioned, Lindemann spent the winter term of 1876/77 in Paris. It was the beginning of his long friendship with Hermite. During a visit to Hermite's apartment he had to sit in the same chair where Jacobi[55] and Riemann[51] had sat before. Saying goodbye, Hermite asked Lindemann to support scientific cooperation between the nations. Remembering this Lindemann later was very active in founding the "Association Internationale des Sciences" in Paris in the summer of 1900[56]. Besides the contact with Hermite, Lindemann participated in the lectures of Bertrand[57] and Jordan[58], which were given for an audience of exactly three persons and sometimes four, when a worker who appreciated the heating in the room joined them. The first edition of "Clebsch-Lindemann" had just appeared and because of this he was able to establish

contact with other famous mathematicians living in Paris at that time, for instance Darboux[59], Halphen[60], Bourget[61], Broch[62], Haton de la Goupillière[63], Chasles[64], and Fouret[65], the secretary of the French mathematical society, who introduced him to these mathematicians.

Habilitation and initiation of his academic career

From Paris, Lindemann sent his request for habilitation to the philosophical faculty of the University of Würzburg. In view of "Clebsch-Lindemann", the faculty only asked a public lecture be given on probation, which Lindemann held on May 2, 1877. The subject was the notion of a function, and one of the theses he had to defend was concerned with modern axiomatics. Here the psychologist Stumpf[66] asked if Lindemann believed that even among the angels $3+4=7$ would hold. Lindemann answered that mathematics was an empirical science, and he had never seen an angel.... The faculty wanted Lindemann to start his lectures immediately, but King Ludwig II had disappeared[67] and could not sign the letter of appointment. The rector, Lexer[68], assumed the responsibility, and on May 14, 1877 Lindemann gave his first lecture. The subject was analytical mechanics. It was absolutely necessary that the course was held since three (3!) students were waiting for it. The signature of the king came on May 23. This was the official permission to lecture but it was not combined with a salary unfortunately. Since it was not clear when he could get the promised position in Würzburg, Lindemann was forced to accept the unexpected offer of an associate professorship at the University of Freiburg.

In Freiburg Lindemann stayed for six years, from the winter semester of 1877/78 to the summer semester of 1883. His teaching duties were concerned with calculus, geometry and geodesy. Only later he could also give advanced courses. Among the audience for these we name Mangoldt[69], who already taught at the Protestantic Gymnasium in nearby Straßburg (Elsaß) and who got his habilitation in Freiburg in 1880. In addition to the mathematicians, Lindemann had good contact with the physicist Warburg[70], who gave a weekly colloquium to colleagues and high school teachers introducing new physical apparatus. Thus the invention of the microphone inspired Lindemann's paper on the vibration of strings[71]. From his contact with Thomae, a paper on special functions[72] developed, and a small paper on Fourier series[73] was inspired by du Bois-Reymond. At the same time, Lindemann was working on a continuation of Clebsch's lectures, namely on the part concerning spatial geometry, which did not appear until 1891[74].

Who dares to judge, to confirm the correctness of the result?

On April 12, 1882, as already mentioned, he suddenly had great success. From Hermite's theorem it follows immediately that e^r is transcendental for every nonzero rational number r; now Lindemann could prove the irrationality of e^r for irrational *algebraic* numbers r which – because of $e^{\pi i} = -1$ – implies the transcendence of π. To this end he first observed that certain algebraic relations between integrals of the form

$$\int_{z_0}^{z} \frac{(e^{-z}p^m)}{z - z_i}\, dz$$

where p is a polynomial in z with z_0 and z_i among its roots, do not only hold in Hermite's case where the coefficients of p are integers but also are true for Gaußian integers as coefficients. Then a similar estimation as that carried out by Hermite led him to the fact that whenever $z_1, \ldots, z_n$ are the pairwise different roots of an irreducible polynomial with Gaußian integer coefficients, the number $\sum_{i=1}^{n} e^{z_i}$ is irrational. Next simple manipulations gave that also all the symmetric functions s_ρ of these numbers $e^{z_1}, \ldots, e^{z_n}$ are linearly independent over the field of rational numbers ($1 \leq \rho = \deg s_\rho \leq n$) and that $s_0 = 1$ is either linearly independent of $(s_1, \ldots, s_n)$ or a rational multiple of s_n. But now, if e^{z_i} would be rational (for one i), then the polynomial $\sum_{j=0}^{n} s_j z^j$ would have the rational solution e^{z_i} and one would get the relation $0 = \sum_{j=0}^{n} (e^{z_i})^j s_j$ contradicting the described linear independence property.

Lindemann sent the paper to Klein in Leipzig[75] for publication in the Mathematische Annalen. But Klein was suspicious[76] and showed the paper to Gordan who often came for visits from Erlangen. Gordan checked the paper and could not find a mistake, but he also did not trust the proof. They sent the manuscript to Georg Cantor[1] in Halle who reacted in a similar manner. But he knew whom to ask: Weierstraß[69] in Berlin. Weierstraß understood at once what he had in his hands and became very excited. He asked Lindemann for permission to present the result to the Berlin academy of sciences. Lindemann agreed, but felt obliged to inform Hermite who submitted Lindemann's letter to the Paris academy session of July 10[77]. On June 22 Weierstraß spoke in Berlin, and immediately after the session he hurried personally to the printer; at the end of June he was able to send the reprints to Lindemann[78]! The response to the three publications[11,77,78] was overwhelming. Clearly, Weierstraß and Hermite were impressed, but also Dedekind[79], Kronecker[80], Zeuthen[81], Stephanos[82] and Cremona[83]. Sylvester wrote a few years later: "Lindemann, whom I am wont to call the Vanquisher of π, a prouder title in my eyes than if he had been the conqueror at Solferino or Sadowa"[84].

Even today it is still rumored that Lindemann's proof contained a gap which was filled by Weierstraß. This seems not to be true but could have the following background. Reading the proof sheets of his major paper[11], Lindemann added a generalization of his main theorem, announcing a detailed proof for later. He did not do this immediately, but since the subject was of general interest, Weierstraß continued to work on it. Weierstraß got a proof of the generalization and could simplify Lindemann's method, but he published this improvement only after asking Lindemann for permission. The preprints[85] were distributed by Weierstraß himself on the celebration of his 70th birthday (October 31, 1885) with respectful acknowledgement to the 33-year-old Lindemann, who was also present.

Lindemann in Königsberg

An immediate consequence of Lindemann's fame was a call to a chair of mathematics at the Albertus-Universität of Königsberg ("Albertina"). That was also a famous place of mathematics since Gauß's friend Bessel[86], Jacobi[55] and Neumann[87] worked there. Lindemann used his reputation and, as a stipulation for his acceptance, he requested an associate professorship for Hurwitz[88], who had some difficulties in finding a position because of his Jewish origin. During the ten years he stayed at Königsberg, Lindemann had a lot of PhD students – among them, Minkowski[56], Hilbert[56] and Sommerfeld[38] – wrote many papers and became rector.

But he also married, in 1887 Lisbeth (= Elisabeth) Küssner. She was born in Königsberg on July 22, 1861, where her father Albert Küssner directed a school and where she had finished all the exams for becoming a schoolteacher herself before she became an actress, a successful actress[89]. Lindemann met her in Königsberg where she stayed at the home of her parents for vacations during an engagement in his home town Schwerin. As Mrs. Lindemann and later wife of a Privy Councillor, she gave up the stage and showed her literary skills by writing stories[90]. Moreover, she also had mathematical merits – she helped Lindemann to translate certain essential works from French into German, above all some books of Poincaré[91]. She died in München three years before her husband, on February 28, 1936. The Lindemanns had two children, both born in Königsberg: the son Reinhart (23.5.1889–9.7.1911) and the daughter Irmgard (4.11.1891–26.2.1971). Reinhart, a promising student of mining, died in a mountain accident during a private excursion to the "Wilder Kaiser" (part of the Alps). Irmgard married the physician Dr. Balser and bore seven children; six of them currently living in different parts of Europe.

München 1893–1939

For the winter semester 1893/94, Lindemann accepted a call to the Ludwig-Maximilians-Universität where he stayed until his death on March 6, 1939. The important dates of his later life are:

1894 extraordinary member of the Bavarian academy of sciences
1895 ordinary member
1904/5 rector of the Ludwig-Maximilians-Universität
1905 awarded the "Maximilians-Orden für Wissenschaft und Kunst"
1908–1932 director of the "Verwaltungsausschuß"
1907 "Geheimer Hofrat" } different stages
1916 "Geheimer Rat" } of privy councillor
1918 "Ritterkreuz des Verdienstordens der bayerischen Krone": this implied peerage and changing his name to "Ferdinand Ritter von Lindemann".
1923 Professor Emeritus[92]

During the 46 years of his München life, Lindemann wrote many mathematical papers. Most of them presented solid research, but none could reach the importance of "Uber die Zahl π"[11]. That was hardly surprising, since only very few men in the history of science achieved more than one such top result during their life. However grudging colleagues started to say that Lindemann just had had a stroke of luck[93]. One can understand that Lindemann was hurt by such gossip, but he reacted in the wrong manner. He tried to attack the next famous outstanding problem: Fermat's Last Theorem. He wrote a series of papers on this, each correcting a mistake in the preceding paper but he was unable to get the desired result[94]. Clearly, that confirmed the opinion of his ill-intentioned colleagues and – as the world is – the defamation survived better than all the positive criticism which was made about Lindemann's work.

His teaching was better appreciated. On the occasion of Lindemann's 70th birthday Perron[27] counted more than 60 German and foreign PhD students. Besides Perron himself, we mention some from the München era: Loewy[95], Faber[96], Volk[97] and Kutta[98]. Lindemann was also interested in the teaching of mathematics in high schools. During his inaugural lecture[99] as rector in 1904, he complained of the backwardness of this teaching. Some high school teachers felt offended and reacted with hard attacks against Lindemann; others found his proposals reasonable and looked for positive consequences[100]. Interesting even today is his idea of how to combine ancient Greek and modern mathematics!

In addition to mathematics, Lindemann had another scientific interest. It was the heritage of his grandfather, Crusius, which led him to undertake intensive prehistoric studies on weights found in Northern Italy. He developed a new

theory concerning the meaning of these weights but it seems that the professional archaeologists did not accept it[101].

The administration of a university was very difficult in those days too. Lindemann proved his administrative skills first by extending the mathematical seminar, then as dean of the philosophical faculty, as rector of the university and as director of the "Verwaltungsausschuß" for about 25 years. These successes led to a peerage – the king could not judge his mathematical merits – and to an honorary doctor's degree in the faculty of political economy, which was awarded to him after his 70th birthday.

Lindemann played a political role during the "Räterepublik"[102]. The rector Baeumker[103] was imprisoned to diminish the power of the university. Fortunately the revolutionary soldiers overlooked Lindemann in his small office[104]. According to old university law a rector who was unable to act should be substituted by the next available predecessor, in this case Lindemann. Thus Lindemann had all rights to direct the university which he did during this difficult period. He came and left through a small door at the back of the university facing Amalienstraße, and he always carried the cash of the university with him!

Several times we have mentioned the celebration of Lindemann's 70th birthday in 1922. He liked festivities[105] and this was a really big one[106]. The students praised him in the seminar at the beginning of the summer semester, and the colleagues congratulated him at a meeting on June 9. The speakers were Voß[44], as a senior colleague, and Perron[27] and Hilb[107] as established students of Lindemann, and Pringsheim[108] representing the German Mathematical Union. The next day, June 10, a bust[97] of Lindemann was unveiled, modelled by the sculptor Bleeker[109]. There the speakers were the present reactor of the university the geographer Drygalski[110], Voß again, and Döhlemann[111] substituting for the rector of the Technical High School, Dyck[112]. Finally there was a big dinner organized by students in the "Deutsches Museum" with a laudatio by Hartogs[113] and an amusing after-dinner speech by Pringsheim.

Deeming even all this not enough honour, Mrs. Lindemann paid for a copy of Bleeker's bust which was erected in the mathematical-physical seminar of the Albertina in Königsberg during another big ceremony. After that, Lindemann seemed to fade away. He still published mathematical papers and thought about problems up to the day before his death. But the mathematical community forgot him. For some of Lindemann's birthdays, O. Volk[97] wrote short laudations[114]. His successor in München, Caratheodory, gave a short obituary in the Bavarian academy of sciences[115]. Because of the German political situation at that time, no notice of Lindemann's death appeared in the "Jahresbericht der Deutschen Mathematiker – Vereinigung"[116]. Not a single mathematician visited his grave at the "Waldfriedhof" in München[117] on the occasion of his 100th birthday in 1952.

Later Mathematics

Next to Hermite's result Lindemann's discovery was the second step of a very fruitful development. Here we were only able to sketch some of the main stages. In view of Hermite's and Lindemann's work and subsequent papers of Hurwitz[88,118], Hilbert proposed as the 7th problem in his famous speech at the International Congress in Paris[56] to prove that *the expression* α^β, *for an algebraic base* α *and an irrational algebraic exponent* β, *e.g., the number* $2^{\sqrt{2}}$ *or* $e^\pi = i^{-2i}$, *always represents a transcendental or at least an irrational number.* This was a generalization of the conjecture of Euler[5] that ${}^a\log b$ $(= \log b/\log a)$ should be transcendental, whenever $a, b \in \mathbb{Q}$ with $a, b > 1$ and ${}^a\log b \notin \mathbb{Q}$[119]. In 1929 Gelfond[120] proved Hilbert's conjecture for the case when β is an imaginary quadratic irrational. Kuzmin[121] in 1930 and Siegel[119] (unpublished) extended Gelfond's method to real quadratic irrational β, which included $2^{\sqrt{2}}$. Siegel, moreover, refined the method of Hermite and Lindemann.

The complete affirmative solution of Hilbert's 7th problem was given independently by Gelfond[122] and Schneider[123] in 1934. This today is called the Gelfond-Schneider Theorem. It shows that many numbers are transcendental. But the transcendence question is still open for such "simple" numbers as $e + \pi$ or $e \cdot \pi$. Nowadays people are also working on the following conjecture, which would be a generalization of the Gelfond-Schneider Theorem: *Let* $\alpha_1, \ldots, \alpha_n$ *be non-zero algebraic numbers. If* $\log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over* $\mathbb{Q}$ *then they are linearly independent over the field of algebraic numbers.*

A technique which might help to solve this problem is the introduction of "approximation measures" and "transcendence measures". Some of the most fruitful research in this area is due to Alan Baker[124] who won the Fields Medal in 1970. The reader who is interested in becoming more familiar with the present stage of this art should first try to understand Baker's textbook[124] and then consult the proceedings of the conferences in Cambridge 1976[125] and Exeter 1980[126].

Acknowledgements

The author of this text is a German mathematician. For the historical background, he had to talk with – and received useful hints from – a lot of people whom he would like to take this opportunity to thank. It is impossible to name each and everyone. Special thanks go to one granddaughter of Lindemann, Mrs. Verholzer[127], Professor Volk[97] and Professor Keith Hardie from Cape Town, who corrected some of the English[128].

NOTES

[1] Johannes Thomae (1840–1921), full professor of mathematics in Freiburg from 1874 to 1879, then in Jena, is nearly forgotten today. In the "Mathematiker-Lexikon" by Herbert Meschkowski [Mannheim 1964], he is not even mentioned. Nevertheless, he left some traces. For instance, the German expression "Mächtigkeit" for "cardinality" was proposed by him to his friend Georg Cantor (1845–1918); more about him may be found in the obituary published in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* [Jber. DMV 30/1921, 133–144]. *Hint:* For biographical information on the mathematicians who are mentioned in this article without any further details see the above named encyclopedia.

[2] Paul du Bois-Reymond (1831–1889), full professor of mathematics in Freiburg from 1870 to 1874, then in Tübingen and from 1884 in Berlin.

[3] Rudolf Friedrich Alfred Clebsch (1833–1872).

[4] Felix Klein (1849–1925).

[5] Leonhard Euler (1707–1783): "Introductio in analysis infinitorum" [Lausanne 1748, Chapter IV; Opera omnia VIII, IX].

[6] Johann Heinrich Lambert (1728–1777): "Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques." [Hist. Acad. roy. sci. belles lettr. Berlin, 1761, 265–322; Opera Math. II, 112–159].

[7] The problem of squaring the circle had been posed by the Greek philosopher Anaxagoras (500?–428 B.C.) while he was imprisoned by the Athenians under a charge of impiety (±430 B.C.).

[8] Joseph Liouville (1809–1882): "Sur les classes très étundus de quantités dont la valeur n'est ni algébrique, ni même reductible á des irrationelles algébriques" [C.R. Acad. Sci. Paris 18/1844, 883–885, 900–911; J. Math. pures appl. (1) 16/1851, 133–142]. Today the transcendental numbers constructed by Liouville are called *Liouville Numbers.*

[9] Charles Hermite (1822–1901): "Sur la fonction exponentielle" [C.R. Acad. Sci. Paris 77/1873, 18–24, 74–79, 226–233, 285–293; see: (Œuvres II, 150–181].

[10] Charles Hermite: "Sur la résolution de l'équation du cinquiéme degré" [C.R. Acad. Sci. Paris 46/1858, 508–515; Œuvres II, 5–12].

[11] [Math. Ann. 20/1882, 213–225].

[12] You look as though you have just solved the squaring of the circle.

[13] Ferdinand Johannes Heinrich Lindemann, born in Hannoversch-Münden June 12, 1806, died in Schwerin April 14, 1880. He was teacher at a high school for girls in Hannover from 1843 to 1854; then he moved to Schwerin.

[14] Emilie Crusius, born in Hannover December 18, 1823, married to F. J. H. Lindemann in Hannover on October 3, 1847, died in Hannover May 31, 1907.

[15] Gottlob Christian Crusius (1785–1848). His dictionary of Homer: "Vollst. Griechisch-Deutsches Wörterbuch über die Gedichte des Homer und der Homeriden . . . " [1. ed. Hannover 1836] was used almost up to the present day.

[16] Ernst Curtius (1814–1896): "Griechische Geschichte bis zur Schlacht bei Chäronea" [3 vol., Berlin 1857–67].

[17] Alexander von Humboldt (1769–1859): "Kosmos, Entwurf einer physischen Weltbeschreibung" [5 vol., Stuttgart-Tübingen 1845–1862].

[18] This is Lindemann's own judgement. At the time this report was written, no further information on Dr. Bastian is available.

[19] In the 19th century the custom of editing the so-called "Schulprogramm" had a certain scientific signficance. In a gymnasium one teacher was asked every year to write a scientific exposition. In general this had nothing to do with a "program" but was often original research. It was combined with the invitation to attend some ceremonies of the school, for instance at the completion of a school year or on the occasion of a visit by monarch, and was intended to demonstrate the quality of the teachers and the teaching.

[20] Moritz Abraham Stern (1807–1894) spent all his academic life in Göttingen with the exception of one student year in Heidelberg: PhD 1829, associate professor 1849, full professor 1859, retired 1884. His oral exam for getting the PhD was the first one in which Gauß examined. Gauß later said that he was more afraid than Stern. His text book: "Lehrbuch der Algebraischen Analysis" [Leipzig-Heidelberg 1860] was the basis for most of his lectures. Bernhard Riemann (1826–1866) and Richard Dedekind (1831–1916) named Stern among their academic teachers. See his curriculum vitae in "Allgemeine Deutsche Biographie, vierundfünfzigster Band" [Leipzig 1908].

[21] Jakob Steiner (1796–1863): "Vorlesungen über synthetische Geometrie, 1. Teil", ed. by C. F. Geiser [Leipzig 1867].

[22] Oskar Schlömilch (1823–1901 [Biog. Jbuch. Deut. Nekr 6/1904, 119–122]): "Analytische Geometrie des Raumes" [Leipzig 1855]. Almost every textbook on calculus mentions his formula for the remainder in the Taylor expansion.

[23] Alfred Enneper (1830–1895).

[24] Wilhelm Eduard Weber (1804–1891) together with Gauß constructed the electro-magnetic telegraph.

[25] Friedrich Wöhler (1800–1882) produced urea from ammonia, thereby destroying the barrier between organic and inorganic chemistry.

[26] I cannot learn anything in Clebsch's lecture; one is so engrossed by the beauty of his language that one cannot pay heed to the content.

[27] Oskar Perron (1880–1975), we cite from his address delivered at the official celebration of Lindemann's 70th birthday [Jber. DMV 31/1922, 26–28]. Regarding Perron, see the obituary in the Jahrbuch der Bayerischen Akademie der Wissenschaften [Jbuch. Bay. Akad. Wiss. 1976, 217–227].

[28] Oh, this Lindemann thinks in one minute more geometry than I can learn in a complete semester.

[29] Wolfgang Sartorius von Waltershausen (1809–1876): Gauß zum Gedächtnis [Leipzig 1856]. His father Georg Sartorius (1765–1828) had made political economy a subject of academic research and teaching. On recognition of this, King Ludwig I of Bavaria raised him to a hereditary peerage (1827).

[30] Ernst Friedrich Wilhelm Klinkerfues (1827–1884) was the successor of Gauß as head of the Göttingen observatory. He was known as the inventor of an automatic gas lighter and a bifilarhygrometer. He ended his life by shooting himself in the observatory. For his personality see "Briefwechsel zwischen Carl Friedrich Gauss und Christian Ludwig Gerling" ed. by Clemens Schaefer [Berlin 1927].

[31] at a festivity in honour of the students who came back from the war.

[32] Friedrich Neesen (1849–1923) received his PhD from the University of Bonn in 1871 and became professor of physics in Berlin in 1877.

[33] Felix Klein – only three years older than Lindemann – received his habilitation in Göttingen just a year before.

[34] Felix Klein: "Uber die sogenannte Nicht-Euklidische Geometrie" [Nachr. kön. Ges. Wiss. 1871, 419–433; Math. Ann. 4/1871, 573–625].

[35] The first one was Franz Joseph Konrad Diekmann (1848–1905), later professor and director of the "Realgymnasium" in Viersen (Rheinland).
His thesis had the title: "Über die Modifikationen, welche die ebene Abbildung einer Fläche 3. Ordnung durch Auftreten von Singularitäten erhält" [Math. Ann. 4/1871, 442–475].

[36] [Math. Ann. 7/1874, 56–144].

[37] Klein had been promoted to full professor at the University of Erlangen for the wintersemester 1872/73. He succeeded Hans Pfaff (1824–1872], who was given the chair in 1869, after Hermann Hankel (1839–1873). Hankel had moved to Tübingen; in Erlangen he was (1867) the successor of Georg Karl Christian v. Staudt (1798–1867), the famous geometer (see: R. Fritsch: Ein Lehrer und zwei Schüler: Buzengeiger, v. Staudt und Feuerbach, in: Auf den Weg gebracht, ed. by H. Sund und M. Timmermann, [Konstanz 1979]). v. Staudt's predecessor was Hans Pfaff's father Wilhelm Pfaff (1774–1835) whose better known older brother, Johann Friedrich Pfaff, (1765–

1825) was the doctoral supervisor of Gauß. Klein's inaugural lecture in Erlangen was the famous "Erlanger Programm" which has greatly influenced the subsequent development of geometry, even up to the present time. The precise title was "Vergleichende Betrachtungen über neuere geometrische Forschungen" [Erlangen 1872; Math. Ann. 43/1893, 63–100; Ges. Math. Abh. I, 460–497].

[38] This did not bother him later. One of his PhD students in Königsberg was Arnold Sommerfeld (1868–1951), who became full professor of theoretical physics at the University of München in 1906. Sommerfeld's most famous PhD student was the Nobel-prize winner Werner Heisenberg (1901–1976).

[39] Julius Plücker (1801–1868) let the very young student of botany and physics Felix Klein help him to prepare his lectures on experimental physics. Under obligation to his late teacher, Klein turned to mathematics after Plücker's death.

[40] In particular, Klein took over the editorship of the famous "Mathematische Annalen", founded by Clebsch and his friend Carl Neumann (1832–1925).

[41] Paul Gordan (1837–1912); Klein succeeded in obtaining a second chair of mathematics at the university of Erlangen, which Gordan got in 1873.

[42] Max Noether (1844–1921), the father of Emmy Noether (1882–1935).

[43] Alfred Clebsch: "Vorlesungen über Geometrie", bearbeitet und herausgegeben von Dr. Ferdinand Lindemann [Leipzig 1876].

[44] Aurel Edmund Voß (1845–1931) held the second chair of mathematics at the University of München from 1903 to 1925, as successor of Gustav Bauer (1820–1906, see H. Gericke-H. Uebele: "Philipp Ludwig von Seidel und Gustav Bauer, zwei Erneuerer der Mathematik in München" in' Die Ludwig-Maximilians-Universität in ihren Fakultäten I [Berlin 1972]) and as predecessor of Heinrich Tietze (1880–1964, see [Jber. DMV 83/1981, 182–185]). The present chair holder is Bodo Pareigis (*1937).

[45] With youthful daring you have taken over this great and difficult task before finishing your university studies, and you have thus produced a work, which also has become your own, in every way. Whenever I look at it, I still admire the depth and the width of the viewpoint whereby you were able to solder together in a harmonic whole all that which Clebsch in the last years of his life had partially carried out in lectures or even had left unfinished, as e.g. the theory of invariants of connexes. For about 50 years your work has maintained the same importance for every geometer; no nation can offer anything of similar comprehensive importance [Jber. DMV 31/1922, 25–26]. (The notion of "Connex" as introduced by Clebsch [Abhandlungen der Königlichen Gesellschaft der Wissenschften zu Göttingen, mathematische Klasse 17/1872, 11–12] means a polynomial equation containing the coordinates of a variable point and a variable line, each in a homogenous manner.)

[46] Karl Brater (1819–1869) strove for the freedom of the press in Germany; see M. Spindler:" Bayerische Geschichte im 19. und 20. Jahrhundert" [München 1978] and A. Sapper[47]: "Frau Pauline Brater" [München 1908].

[47] Agnes Brater, married Sapper (1852–1929) and became very famous later as author of books for children and young people.

[48] Anna Hegel (1859–1927), a granddaughter of the philosopher Georg Wilhelm Friedrich Hegel (1770–1831), married Felix Klein in Erlangen 1875.

[49] Every student had to pay for the lectures he was attending. Klein was Professor at the "Polytechnische Schule", which was named in 1877: "Technische Hochschule". Today this is the "Technische Universität".

[50] Philipp Ludwig von Seidel (1821–1896) became full professor in 1855; before this he was recognized for applying statistics to the health sciences and thereby liberating München from regular epidemics of typhoid fever. His most important contribution to mathematics was the discovery of uniform convergence. He retired in 1893 and Lindemann became his successor.

[51] Bernhard Riemann (1826–1866).

[52] like "Hofbäcker" (court baker), "Hofbuchhändler" (court book seller) etc.

[53] Johann Freiherr von Lutz (1826–1890), a friend of the German chancellor Otto v. Bismarck (1815–1898), was Minister of Education in the Bavarian Kingdom from 1869 to 1890, Minister of Justice from 1867 to 1871, from 1880 to 1890 also Prime Minister. He played an essential role in the so-called "Kulturkampf" of the late 19th century.

[54] Friedrich Prym (1841–1915).

[55] Carl Gustav Jacobi (1804–1851).

[56] This, as well as the International Congress of Mathematicians, was held in conjunction with the famous world's fair in Paris 1900. During a reception by Prince Roland Bonaparte (1858–1924) Lindemann, Hilbert (1862–1943) and Minkowski (1864–1909) amused themselves by looking at the inner organs of Mrs. Minkowski shown by the newly developed X-ray apparatus; this is how light-headed people were about X-rays then. This was the international congress at which Hilbert posed his famous problems for the 20th century.

[57] Joseph Bertrand (1822–1900).

[58] Camille Jordan (1838–1922).

[59] Gaston Darboux (1842–1917).

[60] Georges Henry Halphen (1844–1889): "Œuvres" [4 vol., Paris 1916–1924].

[61] Justin Bourget (1822–1887, [J. Math. Elem. 11/1887] became rector of the academy of Aix-en-Provence in 1878.

[62] Ole Jacob Broch (1818–1889, [Acta Math. 12/1889 last page]) was professor of mathematics in Christiania (today Oslo, capital of Norway), he lived many years in Paris, he was member of the International Committee of Weights and Measures, in 1879 he became chief of the International Office of Weights and Measure in Sèvres.

[63] Napoléon Haton de la Goupillière (1833–1927 [C.R. Acad. Sci. Paris 184/1827, 50–52]).

[64] Michel Chasles (1793–1880).

[65] Georges Fouret (1845–1923) was also president of the "Société Mathématique de France" in 1887.

[66] Carl Stumpf (1848–1936) later founded the Institute of Psychology at the University of München and was the founder of the ethnology of music.

[67] Ludwig II (1845–1886) built the castles Herrenchiemsee, Linderhof and Neuschwanstein. He liked to be alone and often disappeared for weeks at a time in the mountains or the woods.

[68] Matthias (von) Lexer (1830–1892) was professor of German language and literature.

[69] Hans Carl Friedrich von Mangoldt (1854–1925) got his PhD in 1878 under the supervision of Weierstraß (1815–1897) in Berlin. In 1904, he became the first rector of the new Technical School of Danzig. Thousands of students learned mathematics from his "Einführung in die Mathematik" [3 vol. Leipzig 1911–1914], later revised by Konrad Knopp (1882–1957), familiar as "Mangoldt-Knopp".

[70] Emil Warburg (1846–1931) became full professor in Freiburg in 1876. He moved to Berlin in 1895 where he was president of the "Physikalisch-Technische Reichsanstalt" from 1905 to 1922. He was the founder of medicinal photochemistry.

[71] "Die Schwingungsformen gezupfter und gestrichener Seiten" [Freiburger Berichte 7/1879, 500–532; see also Jbuch. Fortschr. Math. 11/1879, 716–718].

[72] "Entwicklung der Funktionen einer komplexen Variablen nach Laméschen Funktionen und nach Zugeordneten der Kugelfunktionen" [Math. Ann. 19/1881, 323–386]. Together with his work on π, this paper was the reason that Lindemann was offered a chair in Königsberg.

[73] "Über das Verhalten der Fourier'schen Reihe an Sprungstellen" [Math. Ann. 19/1881, 517–523]. Here Lindemann improved a result of Seidel.[50]

[74] "Vorlesungen über Geometrie unter besonderer Benutzung der Vorträge von Alfred Clebsch. Die Flächen erster und zweiter Ordnung oder Klasse und der Lineare Complex" [Leipzig 1891].

[75] Klein moved to Leipzig in 1880 and returned to Göttingen in 1886, where he stayed until his death in 1925.

[76] He did not note the receipt of the paper in his private diary. Later on – as he was convinced – he made a remark misdating the event as Christmas 1881; see: "Felix Klein, Handschriftlicher

Nachlaß" edited by Konrad Jacobs [Erlangen 1977].

[77] "Sur le rapport de la circonférence au diamètre, et sur les logarithmes népériens des nombres commensurables ou des irrationelles algébriques". [C.R. Acad. Sci. Paris 115/1882, 72–74].

[78] "Über die Ludolph'sche Zahl" [Sber. Akad. Wiss. Berlin 1882, 679–686].

[79] Richard Dedekind (1831–1916).

[80] Leopold Kronecker (1821–1891).

[81] Hieronymus Georg Zeuthen (1839–1920); see [Math. Ann. 83/1921, 1–23].

[82] Cyparissos Stephanos (1857–1917) was professor of mathematics in Athens/Greece.

[83] Luigi Cremona (1830–1903).

[84] James Joseph Sylvester (1814–1897): "On the divisors of the sum of a geometrical series whose first term is unity and common ratio any positive or negative integer". [Nature 37/1888, 417–418]. The battle of Solferino between Napoleon III and the Austrian emperor Franz Joseph was an essential step toward the unification of Italy; the bloody battlefield inspired Henri Dunant to found the International Red Cross. The battle of Sadowa decided the Austrian-Prussian war of 1866; in Germany it is more familiar as the battle of "Königgrätz".

[85] "Zu Lindemann's Abhandlung: Über die Ludolphsche Zahl" [Sber. Akad. Wiss. Berlin 1885, 1067–1085; Math. Werke II, 341–362]. To make it crystal clear: Lindemann completely proved that e^z is transcendental for every non-zero algebraic number z; because of $e^{\pi i} = -1$ this implies the transcendence of π. Lindemann stated and Weierstraß published the proof that $e^{z_0}, \ldots, e^{z_n}$ are linearly independent over the field of algebraic numbers, whenever $z_0, \ldots, z_n$ are distinct algebraic numbers $n \geqq 1$. Weierstraß acknowledged Hermann Amadeus Schwarz (1843–1921) and Dedekind[79] for helpful comments. The next important simplification was obtained by Hilbert: "Über die Transzendenz der Zahlen e und π" [Math. Ann 43/1893, 216–219; Ges. Abh. I, 1–4].

[86] Friedrich Wilhelm Bessel (1784–1846).

[87] Franz Neumann (1798–1895) had the chair of physics and mineralogy; he is the father of Carl Neumann[40].

[88] Adolf Hurwitz (1859–1919) completed his PhD under the supervision of Felix Klein in Leipzig 1881 and his habilitation in Göttingen 1882. (He could not habilitate in Leipzig because he had been at a "Realgymnasium" instead of a "Classical Gymnasium").

[89] There is a picture of her as Marina in Shakespeare's "Pericles" from a performance in the royal München theatre which was given solely for King Ludwig II[68] in October 20, 1882 (nobody else was in the auditorium!).

[90] Her grandchildren enjoyed listening to her when she vividly narrated fairly tales.

[91] Henri Poincaré (1854–1912): "La Science et l'hypothèse" [Paris 1902] translated: "Wissenschaft und Hypothese" Autorisierte deutsche Ausgabe mit erläuternden Anmerkungen von F. und L. Lindemann [Leipzig 1904], and: "Science et méthode" [Paris 1908] translated: "Wissenschaft und Methode" [Leipzig und Berlin 1914]. The statesman Raymond Poincaré was a cousin of Henri Poincaré.

[92] His successors in the chair were: Constantin Caratheodory (1873–1950, appointed 1924), Eberhard Hopf (1902–1983, appointed 1944, moved 1949 to Indiana University, Bloomington [Notices AMS 28/1981, 508; 30/1983, 683–684]), Robert König (1885–1979, appointed 1950 [Jbuch. Bay. Akad. Wiss. 1981], Karl Stein (*1913, appointed 1954, he developed Stein spaces and Stein manifolds) and today Otto Forster (*1937, appointed 1982).

[93] "oh heilige Quadratur des Zirkels" (oh the holy squaring of the circle) Minkowski wrote in a letter to his friend Hilbert on July 20, 1898; see "Hermann Minkowski, Briefe an David Hilbert" ed. by L. Rüdenberg and H. Zassenhaus [Berlin-Heidelberg-New York 1973]. These letters contain some acid-tongued remarks about Lindemann. In contrast to Minkowski, Hilbert esteemed Lindemann throughout his whole life.

[94] "Über den sog. letzten Fermat'schen Satz" [Leipzig 1909], "Untersuchungen über den Fermatschen Satz" [München 1928, published by the author].

[95] Alfred Loewy (1873–1935); see M. Pinl "Kollegen in einer dunklen Zeit" [Jber. DMV 71/1969, 167–228].

[96] Georg Faber (1877–1966, [Jbuch. Bay. Ak. 1966, 207–210]).

[97] Otto Volk (*1892) was Lindemann's assistant from 1919 to 1923. He now lives in Würzburg, where he became professor of mathematics and astronomy in 1930. He collected the money for Lindemann's bust in the University of München from colleagues and institutions.

[98] Martin Wilhelm Kutta (1867–1944 Neue Deut. Biog. 13/1982, 348–350). Today every student of maths becomes familiar with the Runge-Kutta-Method for numerical integration of differential equations.

[99] "Lehren und Lernen in der Mathematik" [München 1904].

[100] Johann Waldvogel: "Die Gymnasialmathematik in der Beleuchtung des Herrn Prof. Dr. Lindemann" [Blätter für das Gymnasialschulwesen 41/1905, 50–59]; J. Lenauer: "Über neuere Vorschläge zur Reform des mathematischen Unterrichts". [ibid. 646–660].

[101] "Zur Geschichte der Polyeder und Zahlzeichen" [Sber. math. phys. Cl. Bay. Akad. Wiss. 26/1896, 625–758]; "Über einige prähistorische Gewichte aus deutschen und italienischen Museen I" [ibid. 29/1899, 71–136]; "Über einige Bleigewichte aus Pompeji" [Sber. math. nat. Abt. Bay. Akad. Wiss. 1935, 451–455].

[102] A communist movement proclaimed in Bavaria on April 7, 1919, the so called "Räterepublik", copying Soviet Russia. After a month of struggle it was over-powered.

[103] Clemens Baeumker (1853–1924), professor of philosophy, had the "Catholic" chair; he is known for his work on medieval philosophy.

[104] described by Werner Heisenberg[38] in "Der Teil und das Ganze" [München 1969]. However, the description which Heisenberg gives of his conversation with Lindemann is not correct. According to an eyewitness the dog did not bark!

[105] In one of his letters Minkowski wrote about the fairylike Italian nights which Lindemann organized as rector in Königsberg [l.c.[93]].

[106] It is described in [Jber. DMV, 31/1922, *24–30*].

[107] Emil Hilb (1882–1929, [Jber. DMV 42/1932, 183–199]) received his PhD under the supervision of Lindemann in 1903 and became professor of mathematics at the University of Würzburg in 1909.

[108] Alfred Pringsheim (1850–1941); his daughter Katja (1883–1980) loved Perron[27] but married the novelist and poet Thomas Mann (1875–1955). This fact is the reason for the description of the lecture notes on algebra and the appearance of Professor Lindemann (by name!) as a painter in Thomas Mann's novel "Königliche Hochheit" [Berlin 1909]; see also Peter de Mendelssohn: "Der Zauberer. Das Leben des deutschen Schriftstellers Thomas Mann" [Frankfurt am Main 1975].

[109] Bernhard Bleeker (1881–1968) was professor at the Academy of Fine Arts in München (since 1922). There is still a number of public monuments in München made by Bleeker, for instance, the statue of Prince Regent Luitpold in the entrance hall of the university (1908), the memorial of the unknown soldier (1924) and the fountain of Crown Prince Ruprecht (1961); see: Lothar Hennig: "Der Bildhauer Bernhard Bleeker" [Materialen-Dokumente zu Leben und Werk 5, Germanisches Nationalmuseum Nürnberg 1978].

[110] Erich v. Drygalski (1865–1949) oriented geographical research towards modern natural sciences.

[111] Karl Döhlemann (1864–1926, [Jber., DMV 37/1928, 209–212]) was a lecturer at the University of München when Lindemann arrived. According to a proposal of Lindemann he gave courses on descriptive geometry at the university, a subject which before was only taught at the Technical High School. He became such an expert on this subject that he later held the chair for geometry at the Technical High School (The former Technical School had been promoted meanwhile to a "High" School with status equal to a university).

[112] Walther Dyck (since 1901: W. von Dyck) (1856–1934, [Jber. DMV 45/1935, 89–98]) received his PhD under the supervision of Klein in 1879.

[113] Friedrich Hartogs (1874–1943); see M. Pinl: "Kollegen in einer dunklen Zeit III" [Jber. DMV 73/1971–72, 153–208].

[114] "Ferdinand Lindemann zu seinem 75. Geburtstage" [Forschungen und Fortschritte 3/1927, 88], "Ferdinand von Lindemann zum 80. Geburtstage" [ibid. 8/1932, 145]

[115] Constantin Carathéodory[96]: "Ferdinand von Lindemann" [Sber. math. nat. Abt. Bay. Akad. Wiss.

1940, 61–63].

[116] A list of the early publications: "Druckschriften-Verzeichnis von F. Lindemann' appeared in the "Almanach der Königlich Bayerischen Akademie der Wissenschaften zum 150. Stiftungsfest" [München 1909, p. 303–306].

[117] For tourists: The grave is situated in Section 43 of the Waldfriedhof, the nicest cemetery of München, near to the "Würmtalstraße", and ornamented with the figure π.

[118] "Über arithmetische Eigenschaften gewisser transzendenter Funktionen" [Math. Ann. 22/1883, 211–229; 32/1888, 583–588].

[119] For a more detailed presentation of the development of this problem see R. Tijdeman "On the Gel'fond-Baker method and its applications" in "Mathematical developments arising from Hilbert problems. I" [Providence, R. I. 1976]. There one can also find the nice story of the famous number theorist Carl Ludwig Siegel (1896–1981) concerning Hilbert's judgement of the difficulty of this problem.

[120] Alexander Osipovich Gelfond (1906–1968, [Dictionary of Scientific Biography], not to be confused with the famous functional analysist Izrail Moiseevich Gelfand, *1913) taught maths at the Moscow university from 1931 until his death: "Sur les nombres transcendants" [C.R. Acad. Sci. Paris 189/1929, 1224–1226].

[121] Rodion Osievich Kuzmin (1891–1949, [Izvestija Akad. Nauk SSSR, Ser. Mat. 13/1949, 385–388]): "Ob odnom novom klasse transzendentnych chisel" [Izvestija Akad. Nauk. SSSR, 7. Ser. Otd. Fyz.-Mat. Nauk 3/1930, 583–597].

[122] "Sur le septiéme probléme de Hilbert" [Izvestija Akad. Nauk SSSR, 7. Ser. Otd. Mat. Estest. Nauk, 7/1934, 623–630].

[123] Theodor Schneider (*1911), now Professor Emeritus of the University of Freiburg: "Transzendenzuntersuchungen periodischer Funktionen I: Transzendenz von Potenzen" [J. reine angew. Math. 72/1934, 65–69].

[124] Alan Baker (*1939), Professor of Pure Mathematics at the University of Cambridge since 1974.

[125] "Transcendence Theory: Advances and Applications" ed. by A. Baker and D. W. Masser [London-New York-San Francisco 1977].

[126] "Journées Arithmétiques 1980" ed. by J. V. Armitage [Cambridge 1982].

[127] Mrs. Verholzer showed me unpublished autobiographical notes by Lindemann, which are one main source for this article.

[128] All remaining language mistakes are the responsibility of the author.

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