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INDICE

Transcendental Numbers in Secondary Education?

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What I am going to talk about is based on the system of German high schools and the curricula for the teaching of mathematics in the upper grades. Therefore I think I should first explain in a few words how this is organised. After four years of common elementary school for all children aged between six and ten years there is a splitting to three types of secondary education. The highest level is offered at schools which we call "Gymnasium" and it takes nine further years. We have such schools with different orientations: classical languages, living languages, mathematics and science, economics, domestic science and so on, but with a a common structure. In the first seven years the students are taught in classes. That means all students of the same grade are collected in classes of a reasonable size where they learn the same subjects, depending on the orientation of the school. In the last two years they choose courses, independent from former classes. They have to choose two major courses and some minor ones according to certain rules which are not worth explaining in detail since they will change from time to time and are different in the local states of the Federal Republic of Germany. I only mention that every student has the possibility of choosing mathematics as a major subject and then he has to study two years of detailed calculus, linear algebra with analytic geometry and stochastics. An introduction to the differential calculus is already taught in grade eleven (= grade seven of the high school) to all students, before the courses start. So our students with major subject mathematics have a lot of material in calculus which they can use for theoretical and applied problems. Therefore, I can guarantee that all the techniques of integration which I will use later on in this talk are available to those students. This statement has been confirmed in several lectures which I presented to school teachers in Germany and also in an experiment in which I personally taught a major course in mathematics for students in grade 13 in a high school near München.

The basic idea of this has been the following: Although the extensive mathematical education in the schools is mostly justified by the many applications of mathematics in daily life one should not forget that mathematics is also determined to teach thinking and that it is an essential part of the cultural development of mankind. From this point of view one should also integrate problems in the teaching of mathematics which - although of purely theoretical interest - have given work to great thinkers over the centuries. In this connection the structure of our system of numbers is a main theme and what I am going to discuss is the notion of transcendental numbers. Now I shall present to you an outline of what I was teaching. I started with the definition:

Definition - A real number x is an *algebraic* number, if there are a natural number n (> 0) and integers $a_0, a_1, \dots, a_n$ with $a_n \neq 0$ such that

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 = 0, \tag{1}$$

ie. if x is a zero of a polynomial function of positive degree with integer coefficients.

This notion has been explained by simple

Examples - (1) Rational numbers are algebraic numbers: Any rational number x can be written in the form $x = p/q$ with $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ which leads to the equation

$$qx - p = 0 \quad .$$

(2) The number $x = \sqrt{2}$ is an algebraic number which follows from the equation

$$x^2 - 2 = 0 \quad .$$

(3) The number $x = \sqrt{2} + \sqrt{3}$ is an algebraic number. To see this compute first $x^2 = 5 + 2\sqrt{6}$ which is equivalent to $x^2 - 5 = 2\sqrt{6}$. By repeated squaring one obtains

$$x^4 - 10x^2 + 1 = 0.$$

The teacher knows from his course in algebra that all numbers are algebraic numbers which can be obtained from rational numbers by means of extracting roots, forming of linear combinations with rational coefficients and iteration of these processes. Sometimes it is very subtle to find the equation for a given number of this form but the process always works! So one has a lot of examples and exercises for the classroom. Somewhere in this discussion the following fact should be observed.

Remark - If x is an algebraic number different from zero then one can find an equation of the form (1) with the additional property $a_0 \neq 0$. Indeed, if in the first instance some $k > 0$ is the smallest index with $a_k \neq 0$ ie,

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_k x^k = 0,$$

then one can divide by x^k and obtains

$$a_n x^{n-k} + a_{n-1} x^{n-1-k} + \dots + a_k = 0,$$

which is an equation of the desired kind. (Under the assumption $a_n \neq 0 \neq x$ one has obviously $k < n$ ie, $n - k > 0$.)

Cardinality considerations in the sense of set theory imply immediately that non-algebraic real numbers must exist; but in the usual teaching you will not talk about cardinalities. There I said that it would not have been worth to have the notion of "algebraic (real) number" if all real numbers were algebraic, and then I introduced the new notion.

Definition - A real number is a *transcendental* number, if it is not an algebraic number.

In this context one should also tell the students something about the historical development of the idea of number. For details, I refer you to the relevant literature. Now, I want only to mention that the final settling of transcendental numbers was done about 1840 by the French mathematician JOSEPH LIOUVILLE who also performed the first proofs of transcendence [5]. The numbers which he recognized as transcendental numbers are nowadays - in his honour - called "Liouville numbers". But in my opinion they are not suited for the classroom because they are constructed quite artificially and only for that purpose far away from the daily mathematical life. An interesting question is the transcendence of numbers like the circle number π - which is connected to the age-long problem of squaring the circle - and, of Euler's number e, the base of the exponential function and the natural logarithms. An other French mathematician, CHARLES HERMITE, was able to prove in 1873 [2]:

Euler's number e is a transcendental number.

Extending this result by means of complex integration the German FERDINAND LINDEMANN proved in 1882 [4]:

The circle number π is a transcendental number.

Undoubtedly, it will be clear for every student that proving the transcendence of a given number will, in general, be much harder than the search for coefficients for a given algebraic number because - for such a proof - one has to look at all polynomial functions of positive degree with integer coefficients. Therefore one might ask if genuine proofs of transcendence can be dealt within secondary education. My claim is that the answer to this question is: yes, it can be proved in our classrooms that Euler's number is a transcendental number. According to my experience the best way to this result is to follow the ideas of HILBERT who - in 1893 - essentially simplified [3] the considerations of HERMITE and LINDEMANN. Other "more elementary" proofs, for instance by reducing to the use of power series, do not give real simplifications or more insight. Unfortunately, the same idea does not work for proving the transcendence of π; HILBERT's proof, too, needs complex integration which we do not teach in our German schools. I assume that the same is true here in Italy.

Before sketching details of the proof, I want to be honest and to tell you that I did not check the success of my teaching by tests. Nevertheless, the ordinary teacher of the experimental class and the director of the school - both attended the experiment - confirmed that they could see that the students had a certain understanding of what was going on.

The proof does not start from a possible definition of e but uses a derived property of the exponential function. It is based on the following fact.

Foundation: *For all $k \in \mathbf{N}_0$ one has*

$$\int_0^\infty x^k \cdot e^{-x} dx = k! \quad . \tag{2}$$

That can be easily proved by means of partial integration and induction. Also, if these tools are not available it is not hard to get the result directly in the following manner (where the above mentioned procedures are hidden): First verify that

$$\lim_{x \to \infty} x^k \cdot e^{-x} = 0 \qquad \text{for all } k \in \mathbf{N}_0 \quad . \tag{3}$$

Then define for all $k \in \mathbf{N}_0$ the function f_k by

$$x \longmapsto x^k \cdot e^{-x} \quad ; \quad x \in \mathbf{R} \quad ,$$

the function F_k by

$$\bar{x} \longmapsto \int_0^{\bar{x}} f_k(x)dx \quad ; \quad \bar{x} \in \mathbf{R}$$

and, for $k > 0$ the function $\bar{F}_k$ by

$$\bar{x} \longmapsto k \cdot F_{k-1}(\bar{x}) \quad - \quad \bar{x}^k \cdot e^{-\bar{x}} \quad ; \quad \bar{x} \in \mathbf{R} \quad .$$

Both functions F_k and $\bar{F}_k$ have the derivative f_k and agree at zero; thus, they are equal:

$$F_k = \bar{F}_k \quad . \tag{4}$$

This implies

$$\begin{aligned}
\int_0^\infty x^k e^{-x} dx &= \lim_{\bar{x}\to\infty} \int_0^{\bar{x}} x^k e^{-x} dx = \\
&= \lim_{\bar{x}\to\infty} (F_k(\bar{x}) - F_k(0)) = \lim_{\bar{x}\to\infty} F_k(\bar{x}) = \\
&\qquad \text{since } F_k(0) = 0 \\
&= \lim_{\bar{x}\to\infty} \bar{F}_k(\bar{x}) = \lim_{\bar{x}\to\infty} (k \cdot F_{k-1}(\bar{x}) - \bar{x}^k \cdot e^{-\bar{x}}) = \\
&\qquad \text{by (4) and the definition of } \bar{F}_k \\
&= \lim_{\bar{x}\to\infty} k \cdot F_{k-1}(\bar{x}) = k \cdot \lim_{\bar{x}\to\infty} F_{k-1}(\bar{x}) \\
&\qquad \text{by (3),}
\end{aligned}$$

if $\lim_{\bar{x}\to\infty} F_{k-1}(\bar{x})$ exists. Now observe that $F_0(\bar{x}) = 1 - e^{-\bar{x}}$ for all $\bar{x} \in \mathbf{R}$; thus $\lim_{\bar{x}\to\infty} F_0(\bar{x})$ exists and takes the value 1. The displayed chain of equation shows that this gives stepwise:

$$\begin{aligned}
\int_0^\infty x \cdot e^{-x} dx &= \lim_{\bar{x}\to\infty} F_1(\bar{x}) = 1 \\
\int_0^\infty x^2 \cdot e^{-x} dx &= \lim_{\bar{x}\to\infty} F_2(\bar{x}) = 2 \\
\int_0^\infty x^3 \cdot e^{-x} dx &= \lim_{\bar{x}\to\infty} F_3(\bar{x}) = 6 \\
&\vdots \\
\int_0^\infty x^k \cdot e^{-x} dx &= \lim_{\bar{x}\to\infty} F_k(\bar{x}) = k! \quad ,
\end{aligned}$$

which is the desired result. Incidentally note that the equation (4) allows an explicit presentation of the functions F_k by means of recursive computation and that the equation (2) is fundamental for the definition of the Γ-function.

As tools I had to use the limit

$$\lim_{k\to\infty} \frac{x^k}{k!} = 0 \tag{5}$$

which holds for all $x \in \mathbf{R}$ and the generalized triangle inequality:

For arbitrary real numbers $x_0, x_1, \ldots, x_n$ one has

$$|x_0 + x_1 + \ldots + x_n| \le |x_0| + |x_1| + \ldots + |x_n|. \tag{6}$$

In order to prove (5) one notes that the $(k+1)$st member of the sequence is obtained from the kth member by multiplying with the factor $x/(k+1)$. Given x one has for all $k > 2|x|$ that the factor in question is smaller than $1/2$; that means that from a certain point on the next member of the sequence is always - with respect to its absolute value - smaller than half of the preceding. Therefore one has a null sequence. - The generalized triangle inequality (6) holds by evidence if all appearing x_i have the same sign; then one even has equality. In the other case on the lefthand side something is cancelled while on the righthand side all x_i again submit their full value.

Now let us turn to the actual proof. Usually one uses of a proof by contradiction. Experience shows that one often comes to the question of where the contradiction is really hidden. Mathematicians are accustomed to the bad habit of proving nearly anything by contradiction although a direct path to a desired result might be much simpler. I recommend that we get rid of this habit. In particular in our case the idea of the proof can be much more easily understood if the claim is formulated in a positive manner. One shows:

For each choice of $n \in \mathbf{N}$, $a_0, a_1, \ldots, a_n \in \mathbf{Z}$ with $a_0 \neq 0$ and $a_n \neq 0$ one obtains

$$a_n e^n + a_{n-1}e^{n-1} + \ldots + a_2 e^2 + a_1 e + a_0 \neq 0. \tag{7}$$

Idea of Proof: Let such numbers $n, a_0, \ldots, a_n$ be given. We construct numbers $r, s \in \mathbf{R}$, $p \in \mathbf{Z}$ such that

$$r \cdot (a_n e^n + a_{n-1}e^{n-1} + \ldots + a_2 e^2 + a_1 e + a_0) = s + p \tag{8}$$

with

$$|s| < 1 \tag{9}$$

and

$$p \neq 0. \tag{10}$$

Since p shall be an integer one gets from (9) and (10) that the righthand side of the equation (8) is different from zero; then also none of the factors on the lefthand side can vanish which implies (7). (I should remark that some other proofs of transcendence and irrationality use similar ideas.)

Carrying out this idea, define two auxiliary functions:

$$g : x \longmapsto x(x-1)(x-2)\ldots(x-n); \quad x \in \mathbf{R},$$

$$h : x \longmapsto (x-1)(x-2)\ldots(x-n)e^{-x}; \quad x \in \mathbf{R}.$$

Furthermore, fix some $k \in \mathbf{N}$, first arbitrarily, later in such a way that certain conditions are satisfied and define - depending on this k - another function

$$f : x \longmapsto g(x)^k \cdot h(x); \quad x \in \mathbf{R},$$

that means

$$f(x) = x^k \cdot [(x-1) \cdot \ldots \cdot (x-n)]^{k+1} \cdot e^{-x} \quad \text{for all } x \in \mathbf{R}.$$

Multiplying out the parantheses one obtains for $f(x)$ a presentation of the form

$$f(x) = e^{-x} \cdot \sum_{j=k}^{k+n(k+1)} b_j \cdot x^j \tag{11}$$

with $b_j \in \mathbf{Z}$ for all j which appear. In particular one has $b_k = \pm(n!)^{k+1}$; the exact sign of b_k and the exact values of the other b_j $(j > k)$ do not play any role in the following.

Here I should mention a surprise in my teaching. German school teachers are cautious with the use of the sum symbol "$\sum$" in the classroom; it is introduced in our textbooks but seldom used and not trained. Thus, I tried to avoid it and I said to the students something like: "Multiplying out one obtains a sum of terms of the form $b \cdot x^j$ with $k \leq j \leq k + n(k+1)$ and $b \in \mathbf{Z}$. If $j = k$ then b has the particular value $\pm(n!)^{k+1}$." In the sequel I wrote only one summand at the blackboard and I said that they always should have the complete sums in mind. The students did not understand that; they could not imagine the interplay between these summands. Luckily at this stage in the game the teaching was interrupted by a fifteen minute break during which the students explained to me that they would get a better idea of the subject if I used the sum symbol.

Back to the main stream: Consider the integral

$$w_0 = \int_0^\infty f(x)dx \quad = \sum_{j=k}^{k+n(k+1)} b_j \cdot \int_0^\infty x^j \cdot e^{-x} dx \quad .$$

In view of (2) the single integrals under the sum are integers, divisible by $k!$, for $j > k$ even divisible by $(k+1)!$. Since also all coefficients b_j are integers one obtains

$$w_0 = \pm(n!)^{k+1} \cdot k! + c_0 \cdot (k+1)! \tag{12}$$

with $c_0 \in \mathbf{Z}$. Now take

$$r = \frac{w_0}{k!} = \pm(n!)^{k+1} + c_0 \cdot (k+1) \quad ; \tag{13}$$

later on conditions for the still arbitrary number k will be formulated which guarantee that this number r really can be used for the construction of an equation of the form (8) with the desired additional properties.

Next for $i = 1, \ldots, n$ split w_0 in the form

$$w_0 = v_i + w_i$$

with

$$v_i = \int_0^i f(x)dx \quad , \quad w_i = \int_i^\infty f(x)dx \quad .$$

Thus one obtains for $r \cdot (a_n e^n + a_{n-1}e^{n-1} + \ldots + a_2e^2 + a_1e + a_0)$ a decomposition of the form

$$\frac{v_n a_n e^n + \ldots + v_1 a_1 e}{k!} \quad + \quad \frac{w_n a_n e^n + \ldots + w_1 a_1 e + w_0 a_0}{k!} \quad ;$$

it remains to show that for a suitable choice of k the number

$$s = \frac{v_n a_n e^n + \ldots + v_1 a_1 e}{k!}$$

and

$$p = \frac{w_n a_n e^n + \ldots + w_1 a_1 e + w_0 a_0}{k!}$$

have the desired properties.

First consider p. The graph of the function f, restricted to the domain $x \geq i$, determines the same area as the graph of the function $\tilde{f} : \tilde{x} \mapsto f(\tilde{x} + i) \; ; \; \tilde{x} \geq 0$. Therefore one has for $i > 0$

$$w_i = \int_0^\infty (\tilde{x} + i)^k [(\tilde{x} + i - 1)(\tilde{x} + i - 2) \ldots \tilde{x} \ldots (\tilde{x} + i - n)]^{k+1} e^{-\tilde{x}-i} d\tilde{x} \quad .$$

Evidently this is an integration by substitution which was unknown to my students. Since we dealt only with a simple translation of the origin they did not have any difficulties.

Taking out the factor e^{-i} from the integral and renaming the variable of integration as x one obtains

$$w_i = e^{-i} \cdot \int_0^\infty (x + i)^k [(x + i - 1)(x + i - 2) \ldots x \ldots (x + i - n)]^{k+1} e^{-x} dx \quad .$$

Since x appears "pure" within the brackets the integrand has now the form

$$\sum_{j=k+1}^{k+n(k+1)} \tilde{b}_j \cdot x^j$$

with $\tilde{b}_j \in \mathbf{Z}$ for all j which appear; this means

$$w_i = e^{-i} \cdot \sum_{j=k+1}^{k+n(k+1)} \tilde{b}_j \int_0^\infty x^j e^{-x} dx \quad .$$

Here it follows from (2) that every single integral under the sum is an integer divisible by $(k+1)!$ that means that w_i has the form

$$w_i = e^{-i} \cdot c_i \cdot (k+1)!$$

with $c_i \in \mathbf{Z}$. Altogether one obtains

$$\begin{aligned} p &= (c_n a_n + \ldots + c_1 a_1 + c_0) \cdot (k+1) \quad \pm \quad (n!)^{k+1} \cdot a_0 \quad = \\ &= c \cdot (k+1) \quad \pm \quad (n!)^{k+1} \cdot a_0 \end{aligned}$$

with $c \in \mathbf{Z}$. Therefore p is an integer and certainly different from zero if k is chosen such that $k+1$ is a sufficiently large prime number, greater than n and a_0. (Since $(n!)^{k+1} \cdot a_0$ is surely different from zero the integer p could only vanish if $(n!)^{k+1} \cdot a_0$ would be divisible by $k+1$; this is impossible if k is chosen in the named manner.)

For the discussion of s consider the restrictions of the functions g, h and f to the interval $[0, n]$. Since these are again continuous functions they are bounded, which means that there are positive real numbers G, H such that for all $x \in [0, n]$,

$$|g(x)| \leq G \quad , \quad |h(x)| \leq H \quad ;$$

which implies for the same x,

$$|f(x)| \leq G^k \cdot H \quad .$$

If the theorem of the boundedness of continuous functions on finite closed intervals is not known to the students one can work with explicit bounds. Evidently, one has for all $x \in [0, n]$

$$|g(x)| \leq n^{n+1} \quad , \quad |h(x)| \leq n^n \quad ;$$

take $G = n^{n+1}$, $H = n^n$.

Rewriting this inequality as a double inequality:

$$-G^k \cdot H \leq f(x) \leq G^k \cdot H \quad ,$$

one obtains for $i = 1, \ldots, n$ the estimations

$$-G^k \cdot H \cdot i \leq v_i \leq G^k \cdot H \cdot i \quad ,$$

which means that

$$|v_i| \leq G^k \cdot H \cdot i \quad .$$

Taking these together the generalized triangle inequality (6) gives

$$\begin{aligned} |s| \cdot k! &= |v_n a_n e^n + \ldots + v_1 a_1 e| \leq \\ &\leq |v_n a_n e^n| + \ldots + |v_1 a_1 e| \leq \\ &\leq G^k \cdot H \cdot (n \cdot |a_n| \cdot e^n + \ldots + 1 \cdot |a_1| \cdot e) \end{aligned}$$

where the number

$$z = H \cdot (n \cdot |a_n| \cdot e^n + \ldots + 1 \cdot |a_1| \cdot e)$$

does not depend on k. From (5) now follows

$$\lim_{k \to \infty} \frac{G^k \cdot z}{k!} = (\lim_{k \to \infty} \frac{G^k}{k!}) \cdot z = 0 \quad ,$$

consequently, for sufficiently large k

$$|s| \leq \frac{G^k \cdot z}{k!} < 1 \quad .$$

Since there are infinitely many prime numbers one can certainly find a k such that $|s| < 1$ and $p \neq 0$. **qed**

The fact that the theorem of the existence of infinitely many prime numbers can be applied in this context has been the peak of the experiment for the students. - If you yourself would like to try something like that, you should note that there are some steps which can be done in the usual teaching as examples, exercises or homework. I also would recommend to treat some things more explicitly. For instance, one can compute completely the integral w_0 for small values of n und k. - Naturally, a representable proof of the transcendence of the circle number π would be more interesting for the students, but it does not exist. Let me close by challenging you to search for such a proof.

APPENDIX

In former discussions on this subject I have been asked for simple proofs of the irrationality of π . At this moment the simplest proof for use in the classroom is due to I. NIVEN [6] and proceeds as follows. Suppose $\pi = p/q$ with $p \in \mathbf{Z}$ and $q \in \mathbf{N}$. Dependent on an initially arbitrary $k \in \mathbf{N}$ one defines auxiliary functions

$$f : x \longmapsto \frac{x^k \cdot (p - qx)^k}{k!}; \quad x \in \mathbf{R},$$

$$F : x \longmapsto f(x) - f^{(2)}(x) + f^{(4)}(x) - \ldots + (-1)^k f^{(2k)}(x); \quad x \in \mathbf{R}.$$

One notes that the function f, all its derivatives, and therefore also the function F, take only integer values at the places 0 und π. Moreover one confirms by differentiation that the function

$$G : x \longmapsto F'(x) \cdot \sin x - F(x) \cdot \cos x; \quad x \in \mathbf{R},$$

has the function

$$g : x \longmapsto f(x) \cdot \sin x; \quad x \in \mathbf{R}$$

as derivative. This implies that the integral

$$\int_0^\pi g(x) \mathrm{d}x = G(x)|_0^\pi = F(\pi) + F(0)$$

is an integer for each choice of $k \in \mathbf{N}$. For all $x \in (0, \pi)$ one has

$$0 < g(x) < \frac{\pi^k \cdot p^k}{k!},$$

this means that

$$0 < \int_0^\pi g(x)\mathrm{d}x < \pi \cdot \frac{(\pi p)^k}{k!}.$$

In view of the limit (5) the value on the righthand side of this equation becomes smaller than 1 for sufficiently large k and therefore $\int_0^\pi g(x)\mathrm{d}x$ certainly fails to be an integer. Contradiction!

To prove the insolvability of the problem of squaring the circle it would be sufficient, as you know, to check that the number π does not belong to any algebraic extension of the field of rational numbers of degree 2^m with $m \in \mathbf{N}_0$. Although this claim seems to be much weaker than the transcendence of π no simple proof is known. One step in this direction, namely a classroom proof of the irrationality of π^2 was taken by the Japanese Y. IWAMOTO; his variation of NIVEN's sketched proof can be found in the very interesting but also more demanding book *Zahlen* [1].

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