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REMARK ON THE SIMPLICIAL-COSIMPLICIAL TENSOR PRODUCT

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ABSTRACT. We show that the existence of canonical representatives for the elements of the tensor product (coend) of a simplicial and a cosimplicial set depends only on the Eilenberg-Zilber property of the given cosimplicial set. Thus the second condition which is used in [5] for achieving this result is superfluous.

Let $X\colon \Delta^{\mathrm{op}} \to S$ be a simplicial set and $Y\colon \Delta \to S$ a cosimplicial set. We consider X as a **N**-graded set with Δ acting on the right and correspondingly Y is a **N**-graded set with Δ acting on the left. The Eilenberg-Zilber Lemma states that every $x \in X$ has a unique decomposition

$$x = x^+ x^\circ \tag{1}$$

with x^+ nondegenerate and x° surjective. We assume that Y has the dual property, i.e.

$$\begin{gathered} \textit{every } y \in Y \textit{ has a unique decomposition} \\ y = y^+ y^\circ \\ \textit{with } y^+ \textit{ injective and } y^\circ \textit{ interior.} \end{gathered} \tag{2}$$

(That the proof of the Eilenberg-Zilber Lemma fails to be dualizable depends on the fact, that any surjective map in uniquely determined by the set of its sections; but different injective maps with the same one-element domain and the same codomain have the same set of retractions. Thus the Eilenberg-Zilber property for cosimplicial sets is a real restriction; see [**2**, 4.4 and **4**] for a further discussion of this phenomenon.)

Now take

$$T_n = \{(x, y) \mid x \in X,\ y \in Y,\ \text{degree}\, x = \text{degree}\, y = n\} \tag{3}$$

for every $n \in \mathbf{N}$ and

$$T = \coprod_{n \in \mathbf{N}} T_n; \tag{4}$$

thus T also is a **N**-graded set.

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A pair $(x, y) \in T$ is called *similar* to the pair $(x', y') \in T$ if there is an operator $\alpha \in \Delta$ such that

$$x = x'\alpha, \qquad \alpha y = y'. \tag{5}$$

Similarity generates an equivalence relation $\sim$ on T such that

$$(x'\alpha, y) \sim (x', \alpha y) \tag{6}$$

for all suitable α. The set of all equivalence classes is called the *tensor product* (*coend* [**3**]) of X and Y.

A pair $(x, y) \in T$ is called *minimal*, if x is nondegenerate and y is interior. Our aim is to prove

THEOREM. *Every equivalence class in T contains exactly one minimal element.*

To this end we follow the lines of the proof in [**1**, 2.1] formulated for "subdivision" functors; see also [**5**]. We define maps t_l, t_r, t: $T \to T$ (M_2, M_1, M in [**5**]) by taking

$$t_r(x, y) = (xy^+, y^\circ), \tag{7}$$

$$t_l(x, y) = (x^+, x^\circ y), \tag{8}$$

$$t = t_l \circ t_r. \tag{9}$$

Then clearly

$$(x, y) \sim t_l(x, y) \sim t_r(x, y) \sim t(x, y), \tag{10}$$

$$t(x, y) \neq (x, y) \Rightarrow \text{degree } t(x, y) < \text{degree } (x, y), \tag{11}$$

$$t_r(x, y) = (x, y) \Leftrightarrow y \text{ interior}, \tag{12}$$

$$t_l(x, y) = (x, y) \Leftrightarrow x \text{ nondegenerate}, \tag{13}$$

and finally

$$t(x, y) = (x, y) \Leftrightarrow (x, y) \text{ minimal}. \tag{14}$$

Since the set of degrees is bounded below, it follows from (11), that for any pair $(x, y) \in T$ the sequence

$$(t^n(x, y))_{n \in \mathbf{N}} \tag{15}$$

becomes stationary. Thus by (14) it contains a minimal pair, which by (10) is equivalent to the initial pair.

This proves the existence of a minimal pair in every class. The key to the uniqueness lies in the

LEMMA. *If* (x, y) *is similar to* (x', y') *then* $t_r(x', y')$ *is similar to* $t(x, y)$.

This comes out by taking the operator

$$\alpha' = \left(x'(\alpha y^+)^+\right)^\circ\left((\alpha y^+)^\circ y^\circ\right)^+ \tag{16}$$

where the exponent $^+$ on an operator denotes its injective part while the exponent $^\circ$ stands for the surjective component.

Now, for our goal it is enough to show that, if we start with two sequences (15) with similar pairs in T, we end up with the same minimal pair. So let the pair (x, y)

be similar to the pair (x', y'). The lemma provides us with the inductive argument for showing:

For all $n \in \mathbf{N}$ the pair $t_r t^n(x, y)$ is similar to the pair $t^n(x', y')$. But analogously as before, we have for sufficiently large n

$$t_r t^n(x, y) = t^n(x, y)$$

and both $t^n(x, y)$ and $t^n(x', y')$ are minimal pairs. In view of the equations (5) we see that in this case the corresponding α is injective and surjective, thus $\alpha = \mathrm{id}$, i.e.

$$t^n(x, y) = t^n(x', y')$$

which finishes the proof.

The reason for having written this note is that almost nobody seems to have read the German paper [**1**].

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