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VARIATIONS ON THE THEOREMS OF JORDAN–HÖLDER AND SCHREIER

Rudolf FRITSCH
Universität Konstanz, Fachbereich Mathematik, Konstanz, B.R.D.

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Introduction

Much has been written about the theorems of Jordan–Hölder and Schreier after the publication of the original works of Jordan [9; 10, p. 42], Hölder [7] and Schreier [15]. The most famous papers on this subject are those of Zassenhaus [17] and Ore [13, 14]. More recent notes were written by Hilton and Ledermann [6], Calenko, Sul'geĭfer [2] and Wyler [16]. The aim of this paper is to give a common basis for these developments and to point out the difference between the Jordan–Hölder Theorem and the Schreier Theorem.

The material in this paper is organized in the following manner: § 1 a general description of situations, the so called "Jordan–Hölder–Schreier Situations", in which it is possible to ask whether the Jordan–Hölder and Schreier Theorems are valid. Some logical dependences between these two theorems are also mentioned. It seems hopeless to find a non-trivial necessary and sufficient condition for the validity of the theorems in general Jordan–Hölder–Schreier Situations. Sufficient conditions with respect to the Jordan–Hölder Theorem are given in § 2, and much stronger ones which imply the Schreier Theorem by means of a "Zassenhaus Lemma" are given in §4.

In this axiomatic approach the difference between the Jordan–Hölder Theorem and the Schreier Theorem lies in the fact that we need a notion of "union" (cf. section 0.4) in the latter but not in the former case.

As examples for the Jordan–Hölder case we obtain in §3 the Jordan–Dedekind Chain Condition for semimodular lattices and a generalization of the classical case which needs slightly fewer assumptions than are used in the literature, e.g. [6]. §5 contains examples for the Schreier Theorem, firstly the lattice theoretical case of Ore [14] again and some derivations from it, secondly a Schreier Theorem for semisimplicial sets as an example for a Schreier Theorem in functor categories and finally the classical case again under assumptions which are slightly weaker than those of Calenko and Sul'geĭfer [2] and Wyler [16].

§ 0. Notational preliminaries

0.0. $\mathbf{N}_0$ denotes the set of nonnegative integers and $[r]$ the set consisting of 0, r and all integers between 0 and r, for all $r \in \mathbf{N}_0$.

0.1. If $\mathcal{C}$ is a category we denote by

(0.1) $\mathcal{IC}, \mathcal{MC}, \mathcal{EC}$

the subcategories of all isomorphisms, monomorphisms and epimorphisms in $\mathcal{C}$.

0.2. For all morphisms f in a category $\mathcal{C}$ we denote by

(0.2) $\operatorname{dom} f$, $\operatorname{cod} f$

the domain and the codomain of f considered as identity morphisms. If N is a class of morphisms then "dom N" is the class of all objects in $\mathcal{C}$ which appear as domain of an element of N.

If $\mathcal{C}$ has a zero object we denote by

(0.3) $\ker f$, $\operatorname{coker} f$

the class of kernels and cokernels, respectively.

0.3. If g and h are monomorphisms (epimorphisms) of a category $\mathcal{C}$, then

(0.4) $f \cong g$

means that there is a $j \in \mathcal{IC}$ with

(0.5) $fj = g\ (jf = g)$.

0.4. *Unions.*

0.4.1. Definition. Let $\mathcal{C}$ be a category and g, h a coterminal pair in $\mathcal{MC}$. A *union of g and h*, denoted by $g \cup h$, is a morphism in $\mathcal{MC}$ with

(0.6) $g = (g \cup h)g^+ , h = (g \cup h)h^+$

for suitable g^+, $h^+ \in \mathcal{C}$ satisfying the following universal property:

(0.7) *If $f_0, g_0, h_0 \in M\,\mathcal{C}$ are given such that $f_0 g_0 = g$, $f_0 h_0 = h$ then there is a $j \in \mathcal{C}$ with $f_0 j = g \cup h$.*

We need some facts about these unions which are contained in the following lemmas.

0.4.2. Lemma. *Let $\mathcal{C}$ be a category with pullbacks and $f, g, h \in M\,\mathcal{C}$ be given such that*

(0.8) $\operatorname{dom} f = \operatorname{cod} g = \operatorname{cod} h.$

Then $g \cup h$ exists in $\mathcal{C}$ iff $(fg) \cup (fh)$ exists in $\mathcal{C}$. If this happens, the equation

(0.9) $f(g \cup h) = (fg) \cup (fh)$

holds.

0.4.3. Lemma. *Let $\mathcal{C}$ be a category and $g, h, k, m \in M\,\mathcal{C}$ be given such that*

(0.10) $h = km$

and

(0.11) $\operatorname{cod} g = \operatorname{cod} h.$

If $g \cup h \in \mathcal{I}\,\mathcal{C}$ then $g \cup k$ exists and belongs also to $\mathcal{I}\,\mathcal{C}$.

0.5. *Indecomposability.* Let $\mathcal{C}$ be a category and S a class of morphisms in $\mathcal{C}$.

0.5.1. Definition. $f \in S$ is *S-indecomposable,* if any equation

(0.12) $f = g_0 g_1 \ldots g_r$

with $g_i \in S$ for all $i \in [r]$ implies the existence of an $j \in [r]$ and morphisms $f', f'' \in \mathcal{C}$ such that

(0.13) $g_j = f'ff''$,

(0.14) $\operatorname{cod} f = \begin{cases} g_0 g_1 \dots g_{j-1} f', & \text{if } j \neq 0, \\ f' & \text{otherwise,} \end{cases}$

(0.15) $\operatorname{dom} f = \begin{cases} f'' g_{j+1} \dots g_r, & \text{if } j \neq r, \\ f'' & \text{otherwise.} \end{cases}$

0.5.2. Lemma. *Let S be contained either in $\mathsf{M}\,\mathcal{C}$ or in $\mathsf{E}\,\mathcal{C}$. Then $f \in S$ is S-indecomposable iff any equation*

(0.16) $f = g_0 g_1 \dots g_r$

with $g_i \in S$ for all $i \in [r]$ implies that all except exactly one of the g_i's are isomorphisms.

0.5.3. Notation. If S is a class of morphisms in a category $\mathcal{C}$ then we denote by SI the class of S-indecomposable morphisms in $\mathcal{C}$.

§ 1. Jordan–Hölder–Schreier Situations

1.1. Definition. A *Jordan–Hölder–Schreier Situation* (abbreviated JHSS) is a triple $(\mathcal{C}, S, \sim)$ such that

(1.1) *$\mathcal{C}$ is a category,*

(1.2) *S is a subclass of $\mathcal{C}$* consisting of the socalled "*subinvariant*" morphisms,

(1.3) *$\sim$ is an equivalence relation on S.*

In a JHSS one may ask if the theorems of Jordan–Hölder and Schreier are valid or not. To make this more precise we need some further definitions.

Let a fixed JHSS $(\mathcal{C}, S, \sim)$ be given.

1.2. Definition. A *subinvariant series* is a finite sequence

(1.4) $\mathfrak{g} = (g_0, g_1, \dots, g_{l_\mathfrak{g}})$

of subinvariant morphisms such that

(1.5) $\operatorname{dom} g_l = \operatorname{cod} g_{l+1}$

for all $l \in [l_\mathfrak{g} - 1]$. $l_\mathfrak{g}$ is called the *length* of $\mathfrak{g}$.

Later we will also need infinite sequences of this kind: A *subinvariant sequence* is an infinite sequence

(1.6) $\mathfrak{G} = (g_0, g_1, \ldots, g_l, \ldots)$

of subinvariant morphisms such that (1.6) holds for all $l \in \mathbf{N}_0$.

1.3. Notation. Subinvariant series are denoted by small and sequences by capital gothic letters, their elements by the corresponding latin letters plus a lower index. Further we set

(1.7) $|\mathfrak{g}| := g_0 g_1 \ldots g_{l_\mathfrak{g}}$

and

(1.8) $|\mathfrak{G}_l| := g_0 g_1 \ldots g_l$

for all $l \in \mathbf{N}_0$.

1.4. Definition. *The Jordan–Hölder Theorem is valid in the JHSS* (C, S, $\sim$) if for any pair ($\mathfrak{g}$, $\mathfrak{h}$) of subinvariant series with

(1.9) $|\mathfrak{g}| = |\mathfrak{h}|$

a bijection $\pi : [l_\mathfrak{g}] \to [l_\mathfrak{h}]$ exists such that

(1.10) $g_l \sim h_{\pi l}$

for all $l \in [l_\mathfrak{g}]$.

In order to formulate the analogous definition for the Schreier Theorem we need further the notion of refinement.

1.5. Definition. The subinvariant series $\tilde{\mathfrak{g}}$ is a *refinement* of the subinvariant series $\mathfrak{g}$ if there is a strongly increasing map $\alpha : [l_\mathfrak{g}] \to [l_{\tilde{\mathfrak{g}}}]$ such that

(1.11) $g_0 = \tilde{g}_0 \tilde{g}_1 \ldots \tilde{g}_{\alpha 0}$,

(1.12) $g_{l+1} = \tilde{g}_{(\alpha l)+1} \tilde{g}_{(\alpha l)+2} \ldots \tilde{g}_{\alpha(l+1)}$ for all $l \in [l_\mathfrak{g} - 1]$,

and

(1.13) $\alpha(l_\mathfrak{g}) = l_{\tilde{\mathfrak{g}}}$.

1.6. Definition. *The Schreier Theorem is valid in the JHSS* (C, S, $\sim$), if for any pair ($\mathfrak{g}$, $\mathfrak{h}$) of subinvariant series with (1.9) refinements $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and a bijection $\pi : [l_{\tilde{\mathfrak{g}}}] \to [l_{\tilde{\mathfrak{h}}}]$ exist such that

(1.10~) $\widetilde{g}_l \sim \widetilde{h}_{\pi l}$

for all $l \in [l_{\tilde{\mathfrak{g}}}]$.

The two following theorems are some trivial consequences of these definitions.

1.7. Theorem. *If the Jordan–Hölder Theorem is valid in the JHSS* $(\mathcal{C}, S, \sim)$, *then*

(*i*) *the Jordan–Hölder Theorem is valid in all JHSS* $(\mathcal{C}, S, \sim_a)$ *with* $\sim_a$ *coarser than* $\sim$, *i.e.* $\sim \subset \sim_a$,

(*ii*) *the Schreier Theorem is valid in the JHSS* $(\mathcal{C}, S, \sim)$,

(*iii*) *no finite composition of subinvariant morphisms is subinvariant. This implies the subinvariant morphisms which are not isomorphisms are S-indecomposable.*

1.8. Theorem. *If the Schreier Theorem is valid in the JHSS* $(\mathcal{C}, S, \sim)$, *then*

(*i*) (*analogous to* (*i*) *in* Theorem 1.7.),

(*ii*) *the Jordan–Hölder Theorem is valid in the JHSS* $(\mathcal{C}, SI, \sim)$ *provided that no S-indecomposable morphism is equivalent to an isomorphism and*

(1.14) $ifj \sim f$

for all $f \in SI$ *and suitable* $i, j \in \mathcal{I}\,\mathcal{C}$.

§ 2. The validity of the Jordan–Hölder Theorem

2.1. Many of the JHSS's in which the Jordan–Hölder Theorem is valid satisfy the following axioms, the existence axiom (JHE) and the quality axioms (JHQ.1) to (JHQ.3) or their duals.

Let $(\mathcal{C}, S, \sim)$ be a JHSS and let $S\mathcal{C}$ denote the subcategory of $\mathcal{C}$, which is generated by S.

2.2. Axioms.

(JHE) *A pullback exists in* $S\mathcal{C}$ *for any coterminal pair* $g, h \in S$.

(JHQ.0) $\mathcal{I}S\mathcal{C} \cdot S \cdot \mathcal{I}S\mathcal{C} \subset S \subset \mathcal{M}S\mathcal{C}$.

(JHQ.1) *If*

(2.1) $$\begin{array}{ccc} \cdot & \xleftarrow{\;g'\;} & \cdot \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ \cdot & \xleftarrow[\;g\;]{} & \cdot \end{array}$$

is a pullback in SC *with* $g, h \in S$ *and* $g \not\approx h$, *then* g' *and* h' *also belong to* S.* (If these axioms are valid, then SC has pullbacks).

(JHQ.2) *if* $fj \sim f$ *whenever* $f \in S$, $i, j \in ISC$ *and* ifj *is defined.*

Now we are able to state our first main theorem.

2.3. Theorem. *The Jordan–Hölder Theorem is valid in the JHSS* $(C, S, \sim)$ *satisfying* (JHE) *and* (JHQ.0) *to* (JHQ.2) *iff* (JHQ.4) *are valid.*

(JHQ.3) *Either* ($g \sim g'$ *and* $h \sim h'$) *or* ($g \sim h$ *and* $g' \sim h'$) *whenever* (2.1) *is a pullback over* S.

(JHQ.4) S *contains no isomorphism.*

2.4. Proof. The necessity of (JHQ.3) is trivial, that of (JHQ.4) follows from the equation

$$i^{-1}ii^{-1} = i^{-1} \tag{2.2}$$

for any isomorphism i, namely, $i \in S$ would imply $i^{-1} \in S$ by means of (JHQ.0), thus the equation (2.2) would contain only elements of S. But such an equation is impossible, if the Jordan–Hölder Theorem is valid.

The remainder of the proof of the theorem follows by means of the well known induction argument from

2.5. Proposition. *Given a JHSS* $(C, S, \sim)$ *satisfying* (JHE) *and* (JHQ.0) *to* (JHQ.4), *then any equation*

$$g_0 g_1 \ldots g_r = h_0 h \tag{2.3}$$

with $r > 0$, $g_l \in S$ *for all* $l \in [r]$, $h_0 \in S$ *and* $h \in SC$ *implies the existence of a decomposition of* h *in the form*

$$h = h_1 h_2 \ldots h_r \tag{2.4}$$

with $h_{l+1} \in S$ *for all* $l \in [r-1]$ *and of a bijection* $\pi : [r] \to [r]$ *with*

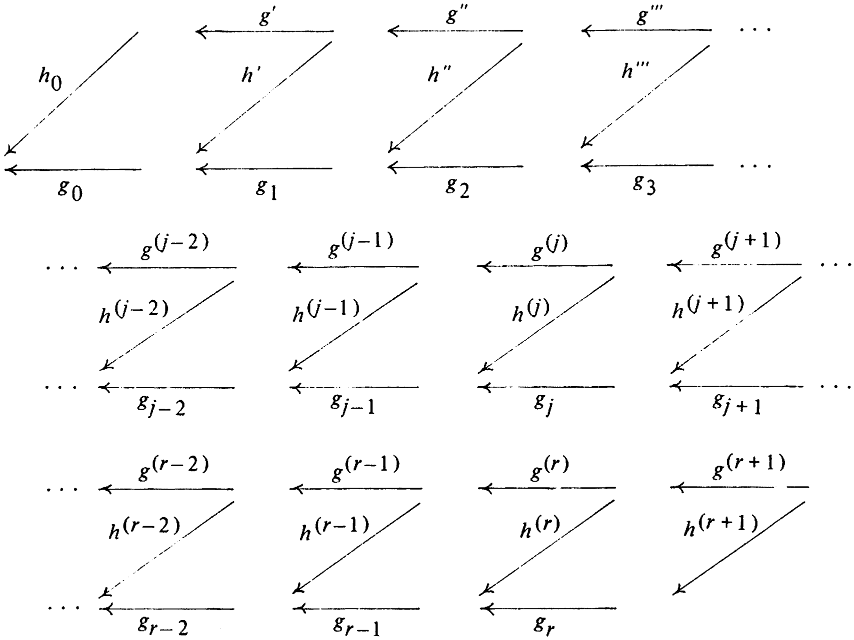
$$g_l \sim h_{\pi l} \tag{1.10}$$

for all $l \in [r]$.

* Our considerations can be done analogously in the case where (JHE) and (JHQ.1) are weakened to the "quadrilateral condition" of Ore [13], but then (JHQ.3) has to be strengthened.

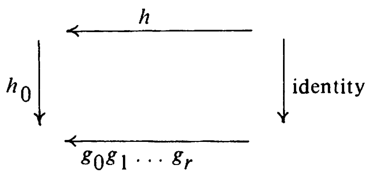
2.6. Proof. We form the diagram

(2.5)



choosing $g^{(i)}$ and $h^{(i)}$ step by step, beginning on the left, such that every small parallelogram is a pullback. Since $h_0 \in \mathcal{MSC}$ the square

(2.6)



is also a pullback. This implies $h^{(r+1)} \in \mathcal{ISC}$.

It may be that some other $h^{(l)}$'s are also isomorphisms. Let j be the smallest number such that $h^{(j)}$ is an isomorphism. Then we derive from (JHQ.1) and (JHQ.4)

(2.7) $\quad h_0 \cong g_0 \quad$ if $j = 1$,

(2.8) $\quad h^{(j-1)} \cong g_{j-1} \quad$ if $j-1 \in [r]$,

and further $g^{(j)} \in \mathcal{IC}$. Then

$$
(2.9) \qquad h_i := \begin{cases} g^{(i)} & \text{if } 1 \leqslant i < j \\ g^{(j)}(h^{(j)})^{-1} g_j & \text{if } i = j \\ g_i & \text{if } j < i \leqslant r \end{cases}
$$

gives the desired morphisms if $j < r + 1$. In case $j = r + 1$ (2.10) has to be slightly modified.

Using (JHQ.3) now it is easy to construct the required bijection π.

§ 3. Examples for the validity of the Jordan–Hölder Theorem

3.1. *Lattices.* Let L be a lattice, considered as a category, S the set of indecomposable morphisms in L* and

(3.1) $\sim := S \times S$.

In the JHSS $(L, S, \sim)$, the axioms (JHE), (JHQ.0), (JHQ.2) – (JHQ.4) and their duals automatically hold. (JHQ.1) is exactly the condition of lower semi-modularity and its dual corresponds to upper semi-modularity [5, p. 120]. The resulting Jordan–Hölder Theorem is known as "Jordan–Dedekind Chain Condition" (or "Dedekindscher Kettensatz" [3; 5]).

This and the more general case of preordered sets is treated excellently by Ore in [13] (cf. also [11]). Our approach in § 2 generalizes Ore's work in considering arbitrary equivalence relations instead of (3.1).

3.2. *The "classical" case*

3.2.1. The classical Jordan–Hölder Theorem works in a category C with zero object 0 where the subinvariant morphisms belong to the class K of all kernels having cokernels and any two such classes are equivalent if they have isomorphic cokernels.

In this section we do not take the K-indecomposable morphisms as subinvariant since the assumptions which we need in order to prove the corresponding Jordan–Hölder Theorem are strong enough to derive the Schreier Theorem. We therefore obtain this case by means of Theorem 1.8 from section 5.3.

3.2.2. Let E be a class of objects in C, S the class of all kernels having a cokernel whose codomain belongs to E and $\bar{S}$, the class of all cokernels having a kernel in S.

Then (JHQ.0) and (JHQ.2) hold automatically. (JHE) is valid iff a composition $\bar{g}h$ has a kernel whenever $\bar{g} \in \bar{S}$ and $h \in S$; (JHQ.4) is equivalent to

(3.2) $0 \notin E$.

Further we have, assuming (3.2),

* $f : A \to B$ is indecomposable iff B covers A (cf. [5, p. 115]).

3.2.3. Lemma. *If $\bar{g}h$ is either zero or a cokernel whenever $\bar{g} \in \bar{S}$ and $h \in S$, then* (JHQ.1) *and* (JHQ.3) *also hold.*

Proof. Let a pullback of the form (2.1) be given and let $\bar{g}$, $\bar{h}$ be cokernels of g and h, respectively, with codomain in E. The only nontrivial point to show is that $\bar{g}h = 0$ iff $\bar{h}g = 0$.

Let us assume $\bar{g}h = 0$. Then there is an epimorphism f such that

(3.3) $\quad \bar{g} = f\bar{h}$.

If $\bar{h}g \neq 0$, then by assumption $\bar{h}g$ would be a cokernel. Therefore

(3.4) $\quad f(\bar{h}g) = \bar{g}g$

would be also an epimorphism. But this gives

(3.5) $\quad \operatorname{cod} \bar{g} = 0$,

in contradiction to (3.2).

3.2.4. Corollary. *The Jordan–Hölder Theorem is valid in the JHSS* (C, S ~) *if* (3.2) *holds and $\bar{g}h$ either is zero or belongs to $\bar{S}$ whenever $\bar{g} \in \bar{S}$ and $h \in S$.*

More familiar conditions are obtained if E is the class of simple objects.

3.2.5. Definition. An object in C is *simple* if it is neither a zero nor a codomain of a nonisomorphic element of K.

3.2.6. Now let E be the class of all simple objects and $\bar{K}$ the class of all cokernels having kernels. Then the condition

(3.6) *Any composition $\bar{g}h$ with $\bar{g} \in \bar{K}$ and $h \in K$ can be decomposed into a composition $h'\bar{g}$, with $h'' \in K$ and $\bar{g}' \in \bar{K}$*

implies the hypotheses in Lemma 3.2.3 and Corollary 3.2.4.

Condition (3.6) seems to be quite natural for generalizations of the notion of abelian categories which include the category of groups. It may be found for example in [2, p. 46–47; 6; 8; 16].

In [6] the following axioms are assumed.

(I). *Every epi has a kernel, every mono a cokernel.*

(II). *Every $f \in C$ may be expressed as $f = gh$ with g mono and h epi.*

(III). (Our condition (3.6)).

Our development shows that (II) is completely superfluous and that (I) is too strong if one deals with "simple" objects as the cokernels of subinvariants. Also a weaker condition than (I) (cf. (5.8)) is sufficient to imply the Schreier Theorem and the case of "indecomposables" mentioned above. The proof of the Jordan–Hölder Theorem in [6] uses "unions" of which the existence is established by means of (I), but which are not intrinsic to this theorem.

3.2.7. From 3.2.6 we can derive the Jordan–Hölder Theorem for the category of groups and its dual. We should remark that the notion "simple" and the condition (3.6) are self-dual. As MacLane [12] has already pointed out, the group theoretical Jordan–Hölder Theorem has nothing to do with the fact that the category of groups fails to be self-dual.

A further example is the category of pointed sets where the two-point sets are simple. For more examples see [16, §2].

§4. The validity of the Schreier Theorem

4.1. For the validity of Schreier's Refinement Theorem, we assume the following existence axioms (SE.1) to (SE.3) and the quality axioms (SQ.0) to (SQ.3) (or their duals). These axioms contain nothing about the equivalence relation, but we shall show that there is a canonical one such that the Schreier Theorem is true. Thus in view of Theorem 1.8 (i), we obtain sufficient conditions for the validity of the Schreier Theorem.

Let $\mathcal{C}$ be a category and S a class of morphisms in $\mathcal{C}$. Without loss of generality we may assume for the remainder of §4 that $\mathcal{C}$ is generated by S. For the applications later on we have to replace $\mathcal{C}$ by $S\mathcal{C}$ in the following axioms.

4.2. Axioms.

(SQ.0) $\operatorname{cod} S \subset S \subset \mathcal{M}\mathcal{C}$ $(\Rightarrow \mathcal{C} = \mathcal{M}\mathcal{C})$.

If g and h belong to S and are coterminal, then

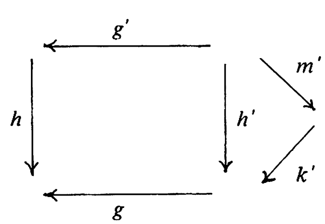
(SE.1) *a pullback for g and h exists in $\mathcal{C}$.*

(SQ.1) *If (2.1) is a pullback in $\mathcal{C}$, then g' and h' also belong to S.*

(SE.2) *$g \cup h$ exists in $\mathcal{C}$.*

(SQ.2) *$g \cup h$ belongs to S.*

If

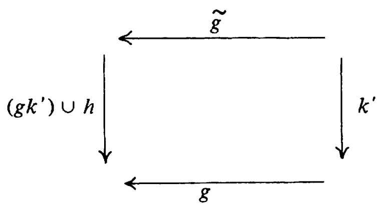
(4.1) 

is a commutative diagram over S such that the lefthand square is a pullback and $g \cup h$ an isomorphism, then

(SE.3) *$(gk') \cup h$ exists in $\mathcal{C}$,*

(SQ.3) *$(gk') \cup h$ belongs to S and the induced square*

(4.2)



is a pullback.

Now we give immediate consequences of these axioms.

4.3. Lemma. *(i). $I\mathcal{C} \subset S$, $I\mathcal{C} \cdot S \cdot I\mathcal{C} \subset S$.*

(ii). *$g \in S$ and $gh \in S$ implies $h \in S$.*

(iii). *$\mathcal{C}$ has pullbacks.*

(iv). *$f(g \cup h) \cong fg \cup fh$ for all $f, g, h \in \mathcal{C}$ with*

(4.3) $\quad \operatorname{dom} f = \operatorname{cod} g = \operatorname{cod} h,$

such that either $g \cup h$ or $fg \cup fh$ exists.

Next we define the canonical equivalence relation.

4.4. Definition. h is *(strongly) perspective to h^* along g,* if $g \in \mathcal{C}$ $(g \in S)$, $g \cup h$ exists and belongs to $I\mathcal{C}$ and h^* is a pullback of h along g. h is *(strongly) perspective to h^** if there is a g such that h is (strongly) perspective to h^* along g.

Thus perspectivity is a reflexive and transitive relation on $\mathcal{C}$ generated by the reflexive relation of strong perspectivity. These relations generate an equivalence relation of which the restriction to S is the canonical equivalence relation which we shall use in the following. It is denoted by $\sim_c$.

Using these notions we may express (SE.3) and (SQ.3) in another form.

4.5. Lemma. *If $h \in S$ is strongly perspective to $k'm'$ along g with $k' \in S$, then there are $k, m \in S$ with*

(4.4) $\quad h = km$

such that k is strongly perspective to k' along g and m is strongly perspective to m' along the pullback of g along k.

Proof. By means of (SE.3) we define

(4.5) $k := (gk') \cup h.$

Let m and n be the inclusions of h and (gk'), respectively in k. The desired pullback properties then follow from (SQ.3). It remains to show that $g \cup k \in \mathcal{I}C$ and $n \cup m \in \mathcal{I}C$. From Lemma 4.3 we obtain the existence of $n \cup m$ and

(4.6) $k(n \cup m) \cong (gk') \cup h = k,$

which gives $n \cup m \in \mathcal{I}C$ since is monic. Since

(4.7) $g \cup h = (g \cup km)$

is an isomorphism we find by means of Lemma 0.4.3 that $g \cup k$ is also an isomorphism.

It is clear that we may replace (SE.3) and (SQ.3) by the statement of this lemma if the other axioms are valid. Further, a simple induction argument shows

4.6. Lemma. *The statement following from Lemma* 4.5 *by cancelling the word "strongly" is also true.*

The key for the validity of the Schreier Theorem in the JHSS $(C, S, \sim_c)$ is

4.7. Lemma. *Let* $\mathfrak{G}$ *be a subinvariant sequence and* $h \in S$ *with*

(4.8) $\operatorname{cod} h = \operatorname{cod} g_0 .$

Then there is a (necessarily unique up to isomorphisms) subinvariant sequence $\mathfrak{K}$ *with*

(4.9) $|\mathfrak{G}_l| \cup h \cong |\mathfrak{K}_l|$

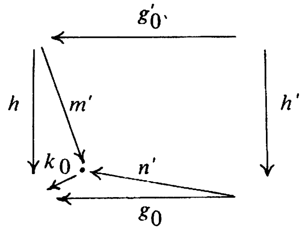
for all $l \in \mathbf{N}_0$. *If* $\mathfrak{K}$ *is such a subinvariant sequence then* k_l *is perspective to* $g_l \cup h^{(l)}$ *for all natural numbers* l *where* $h^{(l)}$ *denotes the pullback of* h *along* $|\mathfrak{G}_{l-1}|$.

Proof. We construct $\mathfrak{K}$ inductive by such that the inclusion $m^{(l)}$ of h in $|\mathfrak{K}_l|$ is perspective to $h^{(l+1)}$ along the inclusion $n^{(l)}$ of $|\mathfrak{G}_l|$ in $|\mathfrak{K}_l|$. Taking

(4.10) $k_0 = g_0 \cup h$

we obtain the diagram

(4.11)



and

(4.12) $\quad k_0(n' \cup m') \cong g_0 \cup h = k_0,$

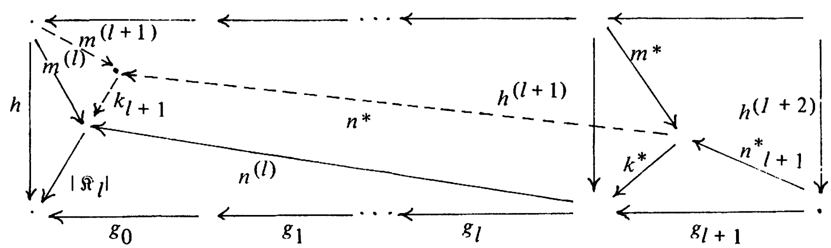
from which $n' \cup m' \in \mathcal{I}C$. Thus we can start the induction.

In order to perform the induction step from l to $l + 1$ we use the abbreviation

(4.13) $\quad k^* = g_{l+1} \cup h^{(l+1)}$

and denote by m^* and n^*_{l+1} the inclusions of $h^{(l+1)}$ and g_{l+1}, respectively in k^*. Then we have the following diagram (full lines):

(4.15)



Now Lemma 4.6 gives (dotted lines) k_{l+1} perspective to k^* and $m^{(l+1)}$ perspective to m^* along n^*. Since m^* is strongly perspective to $h^{(l+2)}$ along n^*_{l+1}, we get $m^{(l+1)}$ perspective to $h^{(l+2)}$ along

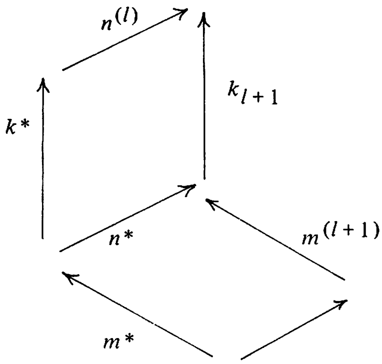
(4.16) $\quad n^{(l+1)} := n^* n^*_{l+1}\ .$

Finally we have

(4.17) $\quad |\mathfrak{K}_{l+1}| \cong |\mathfrak{K}_{l+1}|(n^{(l+1)} \cup m^{(l+1)}) \cong |\mathfrak{G}_{l+1}| \cup h.$

Rearranging the morphisms in this proof we obtain the essential part of the familiar butterfly diagram which occurs in the Zassenhaus lemma.

(4.18)



Thus we can perform the proof of the Schreier Theorem in the Zassenhaus way and obtain

4.8. Theorem. *The Schreier Theorem is valid in the JHSS* $(\mathcal{C}, S, \sim_c)$, *if* (SQ.0) *to* (SQ.3) *and* (SE.1) *to* (SE.3) *are satisfied.*

Since the canonical equivalence relation satisfies the conditions in Theorem 1.8 (ii), we obtain the validity of the Jordan–Hölder theorem in the JHSS $(\mathcal{C}, S, \sim_c)$. But this is nothing new since we may derive the axioms (JHE) and (JHQ.0) to (JHQ.4) in this situation.

In concrete situations it is an interesting problem to give other characterizations of the canonical equivalence relation or a coarser one, that is to say "invariants" of the equivalence classes generated by the perspectivity. We list some results of this kind in our examples.

4.9. Remark. The original proof of Schreier [15] uses (SQ.0), (SE.1), (SE.2), (SQ.1), (SQ.2) and the following facts which are valid in an given JHSS $(\mathcal{C}, S, \sim)$.

(S.1) *If* (2.1) *is a pullback such that* $g \cup h \in \mathcal{I}\,\mathcal{C}$, *then* $g' \sim g$ *and* $h' \sim h$.

(S.2) *If* $h \sim h'$ *and* $h' = k'm'$ *with* $h, h', k', m' \in S$ *then there are* $k, m \in S$ *with* $k' \sim k$, $m' \sim m$ *and* $h = km$.

Clearly (S.1) and (S.2) follow for the canonical equivalence relation in the presence of (SE.3) and (SQ.3). But (S.2) can become wrong if we replace the canonical equivalence relation by a coarser one. This clarifies the difference between Schreier's and Zassenhaus' method for proving the Refinement Theorem.

§ 5. Examples for the validity of the Schreier Theorem

5.1. *Lattices*

5.1.1. Let L be a lattice, considered as a category, and

(5.1) $\quad S := L.$

Then all our axioms and their duals except the second part of (SQ.3) hold automatically. (SQ.3) is equivalent now to (using lattice theoretical notation)

(5.2) $\quad a \cap (b \cup c) = b$

whenever $(a \cap c) \subseteq b \subseteq a$. But this is exactly the condition of modularity which can be written in the self-dual form

(5.3) $\quad a \cap (b \cup c) = b \cup (a \cap c)$

whenever $b \subseteq a$.

An invariant of an equivalence class is the "quotient" of its morphisms (cf. [5, p. 118]).

5.1.2. If C is any category and

(5.4) $\quad S = MC,$

then the axioms in 4.2 are satisfied iff the subobjects of any object in C form a modular lattice. This is true for example for a topos, but not for arbitrary exact categories in the sense of Barr [1]. The latter follows from the fact that any preordered set can be considered as an exact category.

5.2. *Semisimplicial sets*

5.2.1. Let C be a category and S a subclass of C satisfying the axioms in 4.2; take for example, for C the category of sets and the class of injective maps. Let $\mathcal{D}$ be another category. We consider the functor category $C^{\mathcal{D}}$ and in $C^{\mathcal{D}}$ the class $S^{\mathcal{D}}$ of all natural transformations $F \to F'$ such that $FA \to FA'$ belongs to S for all objects A in $\mathcal{D}$. Then $S^{\mathcal{D}}$ satisfies also the axioms in 4.2.

5.2.2. In particular, we consider the category of semisimplicial sets, i.e., C the category of sets, $\mathcal{D}$ the category of finite ordinals. Let S be the class of injective maps, then $S^{\mathcal{D}}$ is the class of semisimplicial monomorphisms. If $h, h' \in S^{\mathcal{D}}$ are canonically equivalent, then they have isomorphic quotients.

A more interesting case is the canonical equivalence of $S^{\mathcal{D}}$-indecomposable morphisms. $h \in S^{\mathcal{D}}$ is indecomposable iff there is a pushout diagram (cf. [4] for the notations)

$$(5.5)\qquad \begin{array}{ccc} X & \xleftarrow{\quad h \quad} & Y \\ {\scriptstyle c_h}\uparrow & & \uparrow \\ \Delta[n] & \xleftarrow[\supset]{\qquad} & \dot{\Delta}[n] \end{array}$$

The epic part of the semisimplicial map c_h is an invariant of the equivalence class of an indecomposable semisimplicial monomorphism h.

5.2.3. Since the category of sets also satisfies the duals of the axioms in 4.2, we have a canonical equivalence relation and a Schreier Theorem as well if we take the semisimplicial epimorphisms as subinvariants. We restrict our attention again to the indecomposables. A semisimplicial epimorphism h is indecomposable, iff there is a diagram

$$(5.6)\qquad \dot{\Delta}[n] \xrightarrow{\subset} \Delta[n] \rightrightarrows X \xrightarrow{h} Y,$$

where the left-hand part is an equalizer and the right-hand part is a coequalizer. The nonnegative integer n then is an invariant of the equivalence class of h.

5.3. *The "classical" case* (notations as in 3.2)

5.3.1. If we take

$$(5.7)\qquad S := K,$$

then (SQ.0) automatically holds and (SE.1) as well as (SQ.1) are valid iff a composition $\bar{g}h$ has a kernel whenever $\bar{g} \in \bar{K}$ and $h \in K$. For example, this is satisfied if (3.6) is valid. But (3.6) is not enough to assure the existence of the desired unions.

5.3.2. **Lemma.** (SE.2) *and* (SQ.2) *are valid if* (3.6) *and*

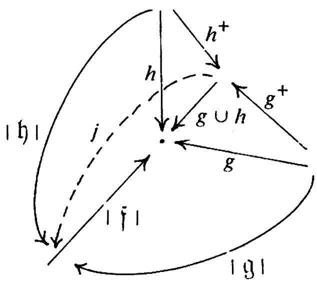
(5.8) *any composition* $\bar{h}''\bar{g}$ *has a kernel in* K *whenever* $\bar{h}'', \bar{g} \in \bar{K}$

hold.

Proof. We define $g \cup h$ to be the kernel of $\bar{h}''\bar{g}$ where $\bar{g}$ is a cokernel of g and $\bar{h}''$ a cokernel of the h'' which arises in the factorization of $\bar{g}h$ by means of (3.6).

In order to prove the union property for this $g \cup h$ we have to consider a commutative diagram of the following form (full lines)

(5.9)



where $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ are subinvariant series and to construct a $j \in K$ (dotted line) such that

(5.10) $g \cup h = |\mathfrak{f}|\, j$.

Since Lemma 4.3 (ii) is trivial in this situation, we can assume

(5.11) $|\mathfrak{g}| = g_0$, $|\mathfrak{h}| = h_0$.

The remainder follows by induction on $l\,\mathfrak{f}$, showing moreover that the desired j is the kernel of $\bar{h}''_0\, \bar{g}_0$ (notation analogous to above), i.e.

(5.12) $j = g_0 \cup h_0$.

Thus we can assume

(5.13) $|\mathfrak{f}| = f_0$.

Then the proof of the existence of j is easy (see e.g. [6, Theorem 3.3]).

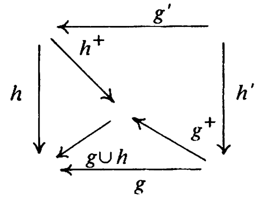
Thus it remains to show

(5.14) $\bar{h}_0\, \bar{g}_0\, j = 0$.

This will be done in 5.3.4. We need the following

5.3.3. Lemma. (First Noether Isomorphism Theorem). *If* (3.6) *and* (5.8) *hold and* *the outer square in the diagram*

(5.15)



is a pullback over K then any cokernel $\bar{g}'$ *of* g' *can be decomposed into* h^+ *followed by a cokernel* $\bar{g}^+$ *of* g^+.

Proof. We decompose $\bar{g}(g \cup h)$ according to (3.6) into a cokernel of g^+ and a kernel $\tilde{h} \in K$. $\bar{h}''$ turns out to be a cokernel of $\tilde{h}$. This proves the lemma.

5.3.4. In order to show (5.14) we observe first

(5.16) $\overline{g_0}\, jg^+ = 0.$

Therefore we get a j'' satisfying

(5.17) $j''\bar{g}^+ = \overline{g_0}\, j$.

Using Lemma 5.3.3 we find that (5.14) is equivalent to

(5.18) $\bar{h}_0''\, j''\bar{g}' = 0,$

which follows from the equation

(5.19) $\bar{h}_0''\, j''\bar{g}' = \bar{h}_0''\,\overline{g_0}\, h_0$.

This completes the proof of Lemma 5.3.2. For (SE.3) and (SQ.3) we need no further axioms.

5.3.5. Lemma. *If* (3.6) *and* (5.8) *hold, then* (SE.3) *and* (SQ.3) *are also valid.*

Proof. We consider a diagram of the form (4.1) and with the same properties. First we obtain

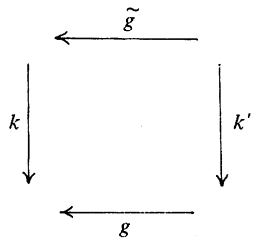
(5.20) $\bar{g}h \in \operatorname{coker} g'$, $\bar{h}g \in \operatorname{coker} h'$

for all cokernels $\bar{g}$ and $\bar{h}$ of g and h, respectively. Using (3.6) we decompose $(\bar{h}g)k'$

into a cokernel $\overline{m'}$ of m' followed by a kernel $k'' \in K$. Let $\overline{k''}$ be a cokernel of k'' and k a kernel of $\overline{k''}\ \overline{h}$, which we obtain by means of (5.8).

Since $\overline{k''}(\overline{h}g)$ is a cokernel of k', the induced square

(5.21)



turns out to be a pullback and $\tilde{g}$ is a kernel of $\overline{gk}$. Since $g \cup k$ is an isomorphism $\overline{gk}$ is also a cokernel of $\tilde{g}$. Now take $m \in K$ such that

(5.22) $\quad km = h.$

Then we have $\tilde{g} \cup m \in I\,C$ and therefore

(5.23) $\quad k \cong k(\tilde{g} \cup m) \cong (gk') \cup h.$

This gives the existence of $(gk') \cup h$ with the desired properties.

5.3.6. It is clear that the isomorphism classes of the codomains of the cokernels of elements of K are invariants with respect to the canonical equivalence.

Therefore we have the following

5.3.7. Theorem. *If* (3.6) *and* (5.8) *hold, then the Schreier Theorem holds in the JHSS* $(C, K, \sim)$, *where* C *is a category with zero and* K *the class of kernels having cokernels, any two of these classes being equivalent if they have cokernels with common codomain.*

5.3.8. Examples for the applicability of Theorem 5.3.7 are: The category of groups, the category of pointed sets, all categories satisfying the assumptions of [16] (where further examples are given).

Finally we mention that it is easy to construct a category with zero satisfying (3.6) but not (5.8). Then we have a Jordan–Hölder Theorem in the sense of 3.2 but not a Schreier Theorem as described here. But our construction gives a very artificial example and it may be void.

5.4. *Normal and principal series.* The theorems developed in 3.2 and 5.3 are generalizations of the classical theorems for subinvariant series (see e.g. [5]). One may ask if it is possible to generalize these theorems to include the case of principal and normal series in the sense of [5]. Without being precise we want to say that in both cases this needs a further axiom, viz. the dual of (5.8). This is not astonishing since in the category of groups one can consider principal series and composition series as being dual to each other [12].

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