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AN INDEPENDENCE RESULT FOR $(\Pi_1^1\text{-CA})+\text{BI}$

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Introduction

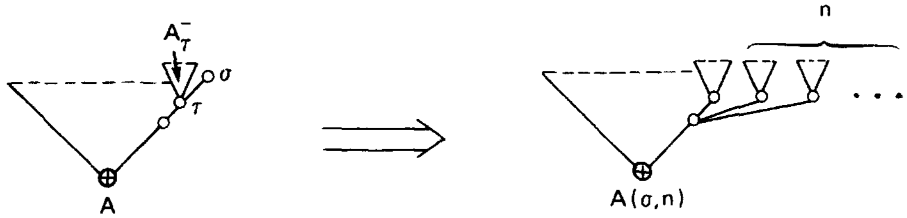
In Kirby and Paris [5] it was shown that a certain combinatorial statement (concerning finite trees) is independent of Peano Arithmetic. Here we present a not too complicated extension of this statement and prove its independence from the much stronger theory $(\Pi_1^1\text{-CA})+\text{BI}$. This is done by refining the methods which we have developed in [2, Ch. IV, §1–§4].

Using the terminology of Kirby and Paris our result can be described as follows. A *hydra* is a finite labeled tree A which has the following properties:

(i) the root of A has label $+$,
(ii) any other node of A is labeled by some ordinal $v \leq \omega$,
(iii) all nodes immediately above the root of A have label 0 (zero).

If Hercules chops off a head (i.e. top node) σ of a given hydra A, the hydra will choose an arbitrary number $n \in \mathbb{N}$ and transform itself into a new hydra $A(\sigma, n)$ as follows. Let τ denote that node of A which is immediately below σ, and let A^- denote that part of A which remains after σ has been chopped off. The definition of $A(\sigma, n)$ depends on the label of σ:

Case 1: label $(\sigma)=0$. If τ is the root of A, we set $A(\sigma, n) := A^-$. Otherwise $A(\sigma, n)$ results from A^- by sprouting n replicas of A_τ^- from the node immediately below τ. Here A_τ^- denotes the subtree of A^- determined by τ.



Case 2: label $(\sigma) = u+1$. Let ε be the first node below σ with label $v \leq u$. Let B be that tree which results from the subtree A_ε by changing the label of ε to u and the label of σ to 0. $A(\sigma, n)$ is obtained from A by replacing σ by B. In this case $A(\sigma, n)$ does not depend on n.

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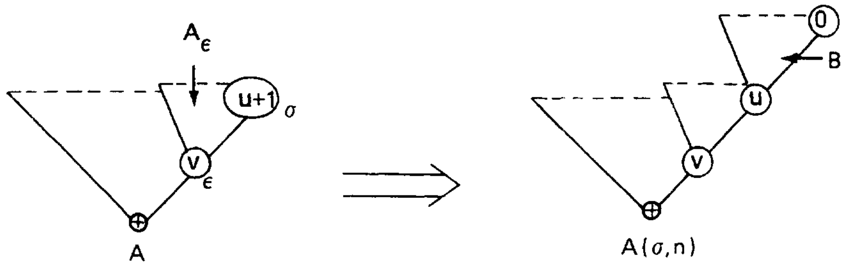
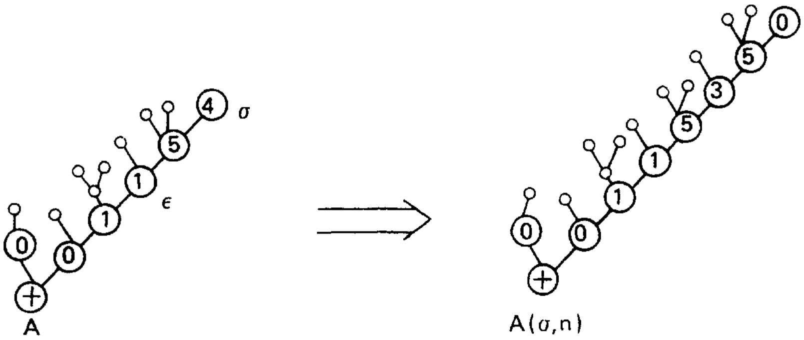


Example ($u = 3$, $v = 1$):



Case 3: label $(\sigma) = \omega$. $A(\sigma, n)$ is obtained from A simply by changing the label of σ: ω is replaced by $n + 1$.

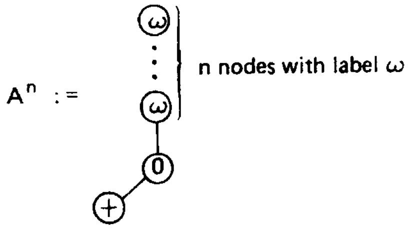
Notation. If σ is the rightmost head of A (as in the pictures above) we write $A(n)$ instead of $A(\sigma, n)$. In the following we consider only the operation $A \mapsto A(n)$. By $\oplus$ we denote the hydra which consists only of one node, namely its root.

The main results of the present paper are:

Theorem I. *By always chopping off the rightmost head, Hercules is able to kill every hydra in a finite number of steps, i.e., for each hydra A and any sequence* $(n_i)_{i\in\mathbb{N}}$ *of natural numbers there exists* $k \in \mathbb{N}$ *such that* $A(n_0)(n_1)\cdots(n_k) = \oplus$.

Theorem II. *For every fixed hydra A the statement* $\forall (n_i)_{i\in\mathbb{N}} \exists k\ A(n_0)(n_1)\cdots(n_k) = \oplus$ *is provable in* $(\Pi^1_1\text{-CA}) + \text{BI}$.

Theorem III. *Let*



Then the Π^0_2*-sentence* $\forall n\, \exists k\, A^n(1)(2)\cdots(k) = \oplus$ *is not provable in* $(\Pi^1_1\text{-CA}) +$ BI.

Contents

In Section 1 we prove Theorem I. In Section 2 we prove Theorem II. Section 3 contains some technical lemmata which will be used in Section 4 for the proof of Theorem III. In the appendix we characterize the proof-theoretic ordinals of the theories ID_ν $(\nu \leq \omega)$ for ν-times iterated inductive definitions by means of the term structure $(T, \cdot[\cdot])$.

1. Infinitary wellfounded trees and collapsing functions

In this section we introduce certain sets $\mathcal{T}_v$ $(v \leq \omega)$ of infinitary wellfounded trees together with a system of socalled collapsing functions $\mathcal{D}_v : \mathcal{T}_\omega \to \mathcal{T}_v$ $(v \leq \omega)$. These functions are then used to associate with every hydra A an element $\|A\|$ of $\mathcal{T}_0$ in such a way that, for each $n \in \mathbb{N}$, $\|A(n)\|$ is an immediate subtree of $\|A\|$. This yields Theorem I.

Definition of the tree classes $\mathcal{T}_v$ $(v \leq \omega)$

Suppose that $\mathcal{T}_u$ for $u < v$ is already defined. Then we define $\mathcal{T}_v$ to be the least set which contains 0 (the empty set) and is closed under the following rule:

$(\mathcal{T}_v)$ If $\alpha : I \to \mathcal{T}_v$ is a function with $I \in \{\{0\}, \mathbb{N}\} \cup \{\mathcal{T}_u : u < v\}$, then $\alpha \in \mathcal{T}_v$.

According to the inductive definition of $\mathcal{T}_v$ we have the following principle of *transfinite induction over $\mathcal{T}_v$*:

$$\forall \alpha \in \mathcal{T}_v\ (\forall x \in \mathrm{domain}(\alpha)\ \Psi(\alpha(x)) \to \Psi(\alpha)) \to \forall \alpha \in \mathcal{T}_v\ \Psi(\alpha).$$

Proposition. $u < v \Rightarrow \mathcal{T}_u \subsetneq \mathcal{T}_v$.

Notations. $(\alpha_x)_{x \in I} := \{\langle x, \alpha_x \rangle : x \in I\}$, i.e., $(\alpha_x)_{x \in I}$ denotes the function α with domain I and $\alpha(x) = \alpha_x$ for all $x \in I$.

$$\alpha^+ := (\alpha)_{x \in \{0\}} := \{\langle 0, \alpha \rangle\} \qquad \text{(the successor of } \alpha\text{)}.$$

In the following α, β, γ denote elements of $\mathcal{T}_\omega$.

Definition of $+ : \mathcal{T}_\omega \times \mathcal{T}_\omega \to \mathcal{T}_\omega$

We define $\alpha + \beta$ by transfinite induction on β:
(i) $\alpha + 0 := \alpha$,
(ii) $\alpha + (\beta_x)_{x \in I} := (\alpha + \beta_x)_{x \in I}$.

Proposition. (a) $\alpha + (\beta^+) = (\alpha + \beta)^+$.
(b) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
(c) $\alpha, \beta \in \mathcal{T}_v \Rightarrow \alpha + \beta \in \mathcal{T}_v$.

Definition. $\alpha \cdot 0 := 0$, $\alpha \cdot (n+1) := \alpha \cdot n + \alpha$.

Definition of $\mathcal{D}_v : \mathcal{T}_\omega \to \mathcal{T}_v$

$\mathcal{D}_v(\alpha)$ is defined by transfinite induction on $\alpha \in \mathcal{T}_\omega$ simultaneously for all $v \leqslant \omega$.

$$(\mathcal{D}1) \quad \mathcal{D}_0(0) := 0^+, \qquad \mathcal{D}_{n+1}(0) := (z)_{z \in \mathcal{T}_n}, \qquad \mathcal{D}_\omega(0) := (\mathcal{D}_{n+1}(0))_{n \in \mathbb{N}}.$$

$$(\mathcal{D}2) \quad \mathcal{D}_v((\alpha_x)_{x \in I}) := \begin{cases} (\mathcal{D}_v(\alpha_0) \cdot (n+1))_{n \in \mathbb{N}}, & \text{if } I = \{0\}, \\ (\mathcal{D}_v(\alpha_x))_{x \in I}, & \text{if } I \in \{\mathbb{N}\} \cup \{\mathcal{T}_u : u < v\}, \\ (\mathcal{D}_v(\alpha_z))_{n \in \mathbb{N}} \text{ with } z := \mathcal{D}_u(\alpha_{0^+}), & \text{if } I = \mathcal{T}_u \text{ with } v \leqslant u < \omega. \end{cases}$$

Remark. If domain$(\alpha) \in \{\mathcal{T}_u : v \leqslant u < \omega\}$, then $\mathcal{D}_v(\alpha)$ is a constant function with domain $\mathbb{N}$.

Definition of $\|A\|$

For every finite labeled tree A (with labels $\leqslant \omega$) we define $\|A\| \in \mathcal{T}_\omega$ by induction on the length (i.e. number of nodes) of A:

$$\|\textcircled{v}\| := \mathcal{D}_v(0),$$

$$\left\| \begin{array}{c} A_0 \cdots A_k \\ \diagdown | / \\ \textcircled{v} \end{array} \right\| := \mathcal{D}_v(\|A_0\| + \cdots + \|A_k\|).$$

If $A = \begin{array}{c} A_0 \cdots A_k \\ \diagdown | / \\ \oplus \end{array}$ is a hydra, we set $\|A\| := \|A_0\| + \cdots + \|A_k\|$. For $\alpha \in \mathcal{T}_0$ with domain$(\alpha) = \{0\}$ we set $\alpha(n) := \alpha(0)$.

1.1. Theorem. *For every hydra* $A \neq \oplus$ *and all* $n \in \mathbb{N}$ *the following holds*: $\|A\| \in \mathcal{T}_0$ *and* $\|A(n)\| = \|A\|(n)$.

Proof. Easy exercise.

From 1.1 we obtain Theorem I by transfinite induction over $\mathcal{T}_0$.

2. The term structure $(T, \cdot[\cdot])$

In this section we prove Theorem II. To this purpose we introduce the following set T of terms, where $D_0, \ldots, D_\omega$ is a sequence of formal symbols.

Inductive definition of the set T

(T1) $0 \in T$.
(T2) If $a \in T$ and $v \leqslant \omega$, then $D_v a \in T$; we call $D_v a$ a *principal term.*
(T3) If $a_0, \ldots, a_k \in T$ are principal terms and $k \geqslant 1$, then $(a_0, \ldots, a_k) \in T$.

For each term $a \in T$ we define its value $\bar{a} \in \mathcal{T}_\omega$ by

$$\bar{0} := 0, \qquad \overline{D_v a} := \mathcal{D}_v(\bar{a}), \qquad \overline{(a_0, \ldots, a_k)} := \bar{a}_0 + \cdots + \bar{a}_k.$$

This interpretation of terms as infinitary wellfounded trees will not be used in the proof of Theorem II. It serves only as a motivation for the following definitions of $a + b$, T_v, dom(a) and $a[z]$.

The letters a, b, c, z now always denote elements of T.

For principal terms $a_0, \ldots, a_k$ and $k \in \{-1, 0\}$ we set

$$(a_0, \ldots, a_k) := \begin{cases} 0, & \text{if } k = -1, \\ a_0, & k = 0. \end{cases}$$

Definition of $a + b$ and $a \cdot n \in T$

$$a + 0 := 0 + a := a,$$
$$(a_0, \ldots, a_k) + (b_0, \ldots, b_m) := (a_0, \ldots, a_k, b_0, \ldots, b_m) \qquad (k, m \geqslant 0),$$
$$a \cdot 0 := 0, \qquad a \cdot (n+1) := a \cdot n + a.$$

Proposition. $(a + b) + c = a + (b + c)$.

Definition of T_v for $v \leqslant \omega$

$$T_v := \{0\} \cup \{(D_{u_0}a_0, \ldots, D_{u_k}a_k) : k \geqslant 0, a_0, \ldots, a_k \in T, u_0, \ldots, u_k \leqslant v\}.$$

Remark. $T_0 \subsetneqq T_1 \subsetneqq \cdots \subsetneqq T_\omega = T$.

Abbreviation. $1 := D_0 0$.

Convention. We identify $\mathbb{N}$ with the subset $\{0, 1, 1+1, 1+1+1, \ldots\}$ of T_0.

Now we define, for every $a \in T$, a subset dom(a) of T and a function $z \mapsto a[z]$ from dom(a) into T. This will be done in such a way that $\bar{z} \in \text{domain}(\bar{a})$ and $\overline{a[z]} = \bar{a}(\bar{z})$, for all $z \in \text{dom}(a)$.

Definition of dom(a) *and $a[z]$ for $a \in T$, $z \in$* dom(a)

([ ].0) $\text{dom}(0) := \emptyset$.
([ ].1) $\text{dom}(1) := \{0\}$; $1[0] := 0$.

([ ].2) $\mathrm{dom}(D_{u+1}0) := T_u$; $(D_{u+1}0)[z] := z$.
([ ].3) $\mathrm{dom}(D_\omega 0) := \mathbb{N}$; $(D_\omega 0)[n] := D_{n+1}0$.
([ ].4) Let $a = D_v b$ with $b \neq 0$:
(i) $\mathrm{dom}(b) = \{0\} : \mathrm{dom}(a) = \mathbb{N}$, $a[n] := (D_v b[0]) \cdot (n+1)$.
(ii) $\mathrm{dom}(b) = T_u$ with $v \leqslant u < \omega : \mathrm{dom}(a) := \mathbb{N}$, $a[n] := D_v b[D_u b[1]]$.
(iii) $\mathrm{dom}(b) \in \{\mathbb{N}\} \cup \{T_u : u < v\} : \mathrm{dom}(a) := \mathrm{dom}(b)$, $a[z] := D_v b[z]$.
([ ].5) $a = (a_0, \ldots, a_k)(k \geqslant 1) : \mathrm{dom}(a) := \mathrm{dom}(a_k)$,
$a[z] := (a_0, \ldots, a_{k-1}) + a_k[z]$.

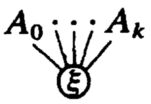
Definition. $0[n] := 0$, $a[n] := a[0]$ for $a \in T$ with $\mathrm{dom}(a) = \{0\}$.

Proposition. (a) $a \neq 0 \Leftrightarrow \mathrm{dom}(a) \neq \emptyset$.
(b) $\mathrm{dom}(a) = \{0\} \Leftrightarrow a = a[0] + 1$.
(c) $0 \neq a \in T_v \Rightarrow \mathrm{dom}(a) \in \{\{0\}, \mathbb{N}\} \cup \{T_u : u < v\}$, *and* $a[z] \in T_v$ *for all* $z \in \mathrm{dom}(a)$.

Now we are going to compare terms and hydras. It will turn out that the term structure $(T_0, \cdot[\cdot])$ is isomorphic to the structure $(\mathscr{H}, \cdot(\cdot))$, where $\mathscr{H}$ denotes the set of all hydras.

In fact $(\mathscr{H}, \cdot(\cdot))$ is nothing else than a geometric representation of $(T_0, \cdot[\cdot])$. $(\mathscr{H}, \cdot(\cdot))$ has been defined just in such a way that it becomes isomorphic to $(T_0, \cdot[\cdot])$.

Definition of $|A|$

If $A =$ 

$(k \geqslant -1)$ is a hydra or any finite labeled tree with labels $\leqslant \omega$ we define $|A|$ to be that term $a \in T$ which implicitely is given by the definition of $\|A\|$ in Section 1, namely:

$$|A| := \begin{cases} D_\xi(|A_0|, \ldots, |A_k|), & \text{if } \xi \leqslant \omega, \\ (|A_0|, \ldots, |A_k|), & \text{if } \xi = +. \end{cases}$$

2.1. Theorem. (a) *The operation* $A \mapsto |A|$ *yields a* 1–1 *correspondence between the set of all hydras and the set* T_0.
(b) $|A(n)| = |A|\,[n]$, *for each hydra* A *and all* $n \in \mathbb{N}$.

Proof. (a) Obvious.
(b) Definition (for $c, z \in T$, $c \neq 0$)

$$c[*/z] := \begin{cases} z, & \text{if } c = D_v 0, \\ D_v b[*/z], & \text{if } c = D_v b \text{ with } b \neq 0, \\ (c_0, \ldots, c_{k-1}) + c_k[*/z], & \text{if } c = (c_0, \ldots, c_k), \quad k \geqslant 1. \end{cases}$$

Now the reader can easily verify the following propositions and then also part (b) of the theorem.

Proposition 1. *If z is a principal term, then $c[*/z]$ results from c by replacing the rightmost subterm D_v0 of c by z.*

Proposition 2. *If $z \in T_u = \mathrm{dom}(a)$, then $a[z] = a[*/z]$.*

Proposition 3. *If $\mathrm{dom}(a) \in \{\{0\}, \mathbb{N}\}$, then one of the following cases holds:*

(i) $a = (a_0, \ldots, a_{k-1}, 1)$ *and* $a[n] = (a_0, \ldots, a_{k-1})$.

(ii) $a = c[*/D_v(a_0, \ldots, a_{k-1}, 1)]$ *and* $a[n] = c[*/D_v(a_0, \ldots, a_{k-1}) \cdot (n+1)]$.

(iii) $a = c[*/D_\omega 0]$ *and* $a[n] = c[*/D_{n+1}0]$.

(iv) $a = c[*/D_v b]$, $\mathrm{dom}(b) = T_u$, $v \leq u$ *and* $a[n] = c[*/D_v b[D_u b[1]]]$.

Let W_0 denote the least subset of T_0 which contains 0 and is closed under the following rule:

$$a \in T_0 \quad \text{and} \quad \forall n \in \mathbb{N}(a[n] \in W_0) \quad \Rightarrow \quad a \in W_0.$$

Since every $a \in T_0$ corresponds to an infinitary wellfounded tree $\bar{a} \in \mathcal{T}_0$ with $\bar{a}(n) = \overline{a[n]}$ (for all $n \in \mathbb{N}$), it follows that $W_0 = T_0$ and consequently $\forall a \in T_0 \forall (n_i)_{i\in\mathbb{N}} \exists k\, a[n_0][n_1] \cdots [n_k] = 0$.

Now we want to give a proof of "$a \in W_0$" which, for every fixed term $a \in T_0$, can be formalized in ID_ω, the formal theory of ω-times iterated inductive definitions. There we have to use methods which do not depend on the nonconstructive tree classes $\mathcal{T}_v$. In fact, we will establish a more general result:

2.2. Theorem. *Let $0 < \nu \leq \omega$. If $a \in T_0$ contains no symbol D_v with $\nu < v$, then "$a \in W_0$" is provable in ID_ν.*

Since ID_ω is contained in $(\Pi^1_1\text{-CA}) + \mathrm{BI}$ and since $(\Pi^1_1\text{-CA}) + \mathrm{BI}$ proves "$a \in W_0 \rightarrow \forall (n_i)_{i\in\mathbb{N}} \exists k\, a[n_0] \cdots [n_k] = 0$", we obtain from 2.2:

2.3. Theorem. $(\Pi^1_1\text{-CA}) + \mathrm{BI} \vdash \forall (n_i)_{i\in\mathbb{N}} \exists k\, a[n_0] \cdots [a_k] = 0$, *for each $a \in T_0$.*

This theorem together with 2.1 yields Theorem II.

In the following let $\nu \leq \omega$ be fixed. We use u, v to denote numbers $\leq \nu$.

Iterated inductive definition of sets W_v ($v < \nu$)

(W1) $0 \in W_v$.

(W2) $a \in T_v$, $\mathrm{dom}(a) \in \{\{0\}, \mathbb{N}\}$, $\forall n\, (a[n] \in W_v) \Rightarrow a \in W_v$.

(W3) $a \in T_v$, $\mathrm{dom}(a) = T_u$ with $u < v$, $\forall z \in W_u\, (a[z] \in W_v) \Rightarrow a \in W_v$.

Proposition. $u \leq v < \nu \Rightarrow W_u \subseteq W_v \subseteq T_v$

Abbreviations. Let X range over subsets of T which are definable in the language of ID_ν.

1. By $A_v(X, a)$ we denote the following statement:

$$a = 0 \vee [\mathrm{dom}(a) \in \{\{0\}, \mathbb{N}\} \wedge \forall n\, (a[n] \in X)]$$
$$\vee\ \exists u < v\, [\mathrm{dom}(a) = T_u \wedge \forall z \in W_u(a[z] \in X).$$

2. $A_v(X) := \{x \in T : A_v(X, x)\}$.
3. $X^{(a)} := \{y \in T : a + y \in X\}$.
4. $\bar{X} := \{y \in T : \forall x(x \in X \rightarrow x + D_\nu y \in X)\}$.
5. $W^* := \{x \in T : \forall u < \nu(D_u x \in W_u)\}$.

By the definition of W_v, for all $v < \nu$ we have:
(A1) $A_v(W_v) = W_v$,
(A2) $A_v(X) \subseteq X \Rightarrow W_v \subseteq X$.

2.4. Lemma. (a) *$A_v(X) \subseteq X$ and $a \in X \Rightarrow A_v(X^{(a)}) \subseteq X^{(a)}$ $(v \leq \nu)$.*
(b) *$a, b \in W_v \Rightarrow a + b \in W_v$ $(v < \nu)$.*

Proof. (a) Suppose $A_v(X) \subseteq X$, $a \in X$, $A_v(X^{(a)}, b)$. We have to prove $a + b \in X$:
1. $b = 0$: Then $a + b = a \in X$.
2. $\mathrm{dom}(b) \in \{\{0\}, \mathbb{N}\}$ and $\forall n\, (b[n] \in X^{(a)})$: Then we have $\mathrm{dom}(a + b) = \mathrm{dom}(b)$ and $(a + b)[n] = a + b[n] \in X$, for all $n \in \mathbb{N}$. It follows that $a + b \in A_v(X) \subseteq X$.
3. $\mathrm{dom}(b) = T_u$ with $u < v$: similar to 2.
(b) From (a) together with (A1), (A2) we obtain, for $v < \nu$, $a \in W_v \rightarrow W_v \subseteq W_v^{(a)}$, i.e., $a \in W_v \rightarrow (b \in W_v \rightarrow a + b \in W_v)$.

2.5. Lemma. *$A_\nu(X) \subseteq X \Rightarrow A_\nu(\bar{X}) \subseteq \bar{X}$*

Proof. Assumptions: $A_\nu(X) \subseteq X$, $A_\nu(\bar{X}, b)$, $a \in X$.
We have to prove $a + D_\nu b \in X$. First we prove: (1) $\forall u < \nu\, (a + D_{u+1}0 \in X)$.
We have $\mathrm{dom}(a + D_{u+1}0) = T_u$ and $(a + D_{u+1}0)[z] = a + z$. By 2.4 we obtain $A_\nu(X^{(a)}) \subseteq X^{(a)}$. Since $A_u(X^{(a)}) \subseteq A_\nu(X^{(a)})$, it follows by (A2) that $W_u \subseteq X^{(a)}$, i.e., $\forall z \in W_u(a + z \in X)$. Hence $A_\nu(X, a + D_{u+1}0)$ and therefore $a + D_{u+1}0 \in X$, since $A_\nu(X) \subseteq X$.

Proof of $a + D_\nu b \in X$:
1. $b = 0$ and $\nu = 0$: Then $a + D_\nu b = a + 1$; and $a + 1 \in X$ follows from $A_\nu(X) \subseteq X \wedge a \in X$.
2. $b = 0$ and $\nu = u + 1$: In this case we are done by (1).
3. $b = 0$ and $\nu = \omega$: Then $\mathrm{dom}(a + D_\nu b) = \mathbb{N}$ and $(a + D_\nu b)[n] = a + D_{n+1}0$. By (1) we obtain $A_\nu(X, a + D_\nu b)$. Hence $a + D_\nu b \in X$.
4. $b = b_0 + 1$ with $b_0 \in \bar{X}$: Then we have $\forall x \in X\, (x + D_\nu b_0 \in X)$. Using this and the assumption $a \in X$ we obtain $\forall n \in \mathbb{N}\, (a + (D_\nu b_0) \cdot (n + 1) \in X)$ by complete

induction. Since $\mathrm{dom}(a+D_\nu b)=\mathbb{N}$ and $(a+D_\nu b)[n]=a+(D_\nu b_0)\cdot(n+1)$ it follows that $a+D_\nu b\in A_\nu(X)\subseteq X$.

5. $\mathrm{dom}(b)=\mathbb{N}$ and $\forall n\,(b[n]\in\bar{X})$: Then we have $\mathrm{dom}(a+D_\nu b)=\mathbb{N}$ and $(a+D_\nu b)[n]=a+D_\nu b[n]\in X$, for all $n\in\mathbb{N}$. Hence $a+D_\nu b\in A_\nu(X)$.

6. $\mathrm{dom}(b)=T_u$, $u<\nu$ and $\forall z\in W_u(b[z]\in\bar{X})$: similar to 5.

2.6. Lemma. $A_\nu(W^*)\subseteq W^*$.

Proof. Suppose $b\in A_\nu(W^*)$ and $v<\nu$. We have to show $D_v b\in W_v$.

1. $b=0$ and $v=0$: From $0\in W_v$ we get $D_0 0=1\in W_v$ by (W2).

2. $b=0$ and $v=u+1$: Then $\mathrm{dom}(D_v b)=T_u$, $(D_v b)[z]=z$ and $W_u\subseteq W_v$. Hence $D_v b\in W_v$ by (W3).

3. $b=b_0+1$ and $b_0\in W^*$: Then we have $\mathrm{dom}(D_v b)=\mathbb{N}$, $(D_v b)[n]=(D_v b_0)\cdot(n+1)$ and $D_v b_0\in W_v$. Using 2.4(b) we obtain $\forall n\,(D_v b)[n]\in W_v$ by induction on n. Hence $D_v b\in W_v$.

4. $\mathrm{dom}(b)=T_u$, $u<\nu$ and $b[z]\in W^*$ for all $z\in W_u$:

4.1. $u<v$: Then we have $\mathrm{dom}(D_v b)=T_u$ and $(D_v b)[z]=D_v b[z]\in W_v$ for all $z\in W_u$, i.e., $D_v b\in W_v$.

4.2. $v\leqslant u<\nu$: Then we have $\mathrm{dom}(D_v b)=\mathbb{N}$ and $(D_v b)[n]=D_v b[z]$ with $z:=D_u b[1]$. Obviously $1\in W_u$ and therefore $b[1]\in W^*$. It follows that $z\in W_u$. From this we obtain $b[z]\in W^*$ and then $D_v b[z]\in W_v$, i.e., $\forall n\,(D_v b)[n]\in W_v$. Hence$D_v b\in W_v$.

5. $\mathrm{dom}(b)=\mathbb{N}$ and $b[n]\in W^*$ for all $n\in\mathbb{N}$: similar to 4.1.

2.7. Lemma. *If $a\in T$ contains no symbol D_v with $v>\nu$, then $A_\nu(X)\subseteq X\rightarrow a\in X$.*

Proof. By induction on the length of a: suppose $A_\nu(X)\subseteq X$.

1. $a=0$: In this case $a\in A_\nu(X)\subseteq X$.

2. $a=(a_0,\ldots,a_k)(k\geqslant 1)$: Let $c:=(a_0,\ldots,a_{k-1})$. Then we have:

(1) $c\in X\rightarrow A_\nu(X^{(c)})\subseteq X^{(c)}$ (by 2.4(a)).

(2) $c\in X$ (by induction hypothesis).

(3) $A_\nu(X^{(c)})\subseteq X^{(c)}\rightarrow a_k\in X^{(c)}$ (by induction hypothesis).

From this we get $a=c+a_k\in X$.

3. $a=D_\nu b$: From $A_\nu(X)\subseteq X$ we get $0\in X$ and $A_\nu(\bar{X})\subseteq\bar{X}$ by 2.5. By I.H. (induction hypothesis) we have $A_\nu(\bar{X})\subseteq\bar{X}\rightarrow b\in\bar{X}$. By definition of $\bar{X}$ we have $b\in\bar{X}\rightarrow(0\in X\rightarrow D_\nu b\in X)$. Hence $D_\nu b\in X$.

4. $a=D_v b$ with $v<\nu$: By I.H. we have $A_\nu(W^*)\subseteq W^*\rightarrow b\in W^*$. Using 2.6 we obtain $b\in W^*$. Hence $a=D_v b\in W_v$. From $A_\nu(X)\subseteq X$ we get $A_v(X)\subseteq X$ and then $W_v\subseteq X$.

2.8. Lemma. *If $a\in T_0$ contains no symbol D_v with $v>\nu$, then $a\in W_0$.*

Proof. Let $a \neq 0$. Then $a = D_0a_0 + \cdots + D_0a_k$ with $a_0, \ldots, a_k \in T$, and by Lemmata 2.6, 2.7 we have $a_0, \ldots, a_k \in W^*$. Hence $D_0a_0, \ldots, D_0a_k \in W_0$. From this we obtain $a \in W_0$ by 2.4(b).

By formalizing in ID_ν the definition of W_υ $(\upsilon < \nu)$ and the proofs of 2.4–2.8 we obtain Theorem 2.2.

3. The relations $\ll_k$ and the functions $H_a : \mathbb{N} \to \mathbb{N}$

In Section 4 we will use terms $a \in T$ instead of ordinals to measure the lengths of infinitary derivations. In this context we need certain relations $\ll_k$ on T which we introduce now. We also introduce a hierarchy $(H_a)_{a \in T_0}$ of number-theoretic functions which is closely related to the so called Hardy hierarchy. The relation $\ll_0$ restricted to T_0 is just the step-down relation of Schmidt [6]; cf. also Ketonen and Solovay [4] where similar relations are studied.

As before the letters a, b, c, d, e, z will always denote elements of T. As mentioned in Section 2 every $a \in T$ can be considered as a notation for a wellfounded tree $\bar{a} \in \mathcal{T}_\omega$ in such a way that $\bar{z} \in \mathrm{domain}(\bar{a})$ and $\bar{a}(\bar{z}) = \overline{a[z]}$ holds for all $z \in \mathrm{dom}(a)$. Consequently we have the following principle of *transfinite induction over T*:

$$\forall a \in T\, [\forall z \in \mathrm{dom}(a)\ \Psi(a[z]) \to \Psi(a)] \to \forall a \in T\ \Psi(a).$$

Definition of $c \ll_k a$ by transfinite induction on $a \in T$

$$c \ll_k a \quad :\Leftrightarrow \quad a \neq 0 \quad \text{and} \quad \forall z \in d_k(a)\, (c \leqq\!\!\!\!\ll_k a[z])$$

where

$$d_k(a) := \begin{cases} \{k\}, & \text{if } \mathrm{dom}(a) \in \{\{0\}, \mathbb{N}\} \\ \{D_u e : 0 \neq e \in T\}, & \text{if } \mathrm{dom}(a) = T_u \end{cases}$$

and

$$c \leqq\!\!\!\!\ll_k a \quad :\Leftrightarrow \quad c \ll_k a \quad \text{or} \quad c = a.$$

3.1. Lemma. (a) *$c \ll_k a$ and $a \ll_k b \Rightarrow c \ll_k b$.*

(b) *$c \ll_k b \Rightarrow a + c \ll_k a + b$.*

(c) *$b \neq 0 \Rightarrow a \ll_k a + b$.*

Proof by transfinite induction on b.

3.2. Lemma. (a) *$n \leqslant k + 1 \Rightarrow (D_\upsilon a) \cdot n \ll_k D_\upsilon(a + 1)$.*

(b) *$c \ll_k a \Rightarrow D_\upsilon c \ll_k D_\upsilon a$.*

Proof. (a) By 3.1(c) we have $(D_\upsilon a) \cdot n \leqq\!\!\!\!\ll_k (D_\upsilon a) \cdot (k + 1) = D_\upsilon(a + 1)[k]$. Hence $(D_\upsilon a) \cdot n \ll_k D_\upsilon(a + 1)$, since $d_k(D_\upsilon(a + 1)) = \{k\}$.

(b) Transfinite induction on a: Suppose $a \neq 0$ and $\forall z \in d_k(a)(c \lesseqq_k a[z])$.

1. $a = a_0 + 1$: By I.H. and 3.2(a) we have $D_v c \lesseqq_k D_v a_0 \ll_k D_v a$.

2. $\mathrm{dom}(a) \in \{\mathbb{N}\} \cup \{T_u : u < v\}$: Then $d_k(D_v a) = d_k(a)$ and $\forall z \in d_k(a)$ $((D_v a)[z] = D_v a[z])$. By I.H. we have $\forall z \in d_k(a)(D_v c \lesseqq_k D_v a[z])$. Hence $D_v c \ll_k D_v a$.

3. $\mathrm{dom}(a) = T_u$ with $v \leqslant u$: Then $d_k(D_v a) = \{k\}$ and $(D_v a)[k] = D_v a[z]$ with $z := D_u a[1] \in d_k(a)$. By I.H. we have $D_v c \lesseqq_k D_v a[z]$. Hence $D_v c \ll_k D_v a$.

3.3. Lemma. $\mathrm{dom}(a) = \mathbb{N} \Rightarrow a[n] \lesseqq_k a[n+1]$.

Proof. By induction on the length of a:

1. $a = D_\omega 0$: Then we have $a[n+1] = D_{n+2}0$ and therefore $d_k(a[n+1]) = \{D_{n+1}e : 0 \neq e \in T\}$, $a[n+1][z] = z$. Using 3.1(c) and 3.2(b) we obtain $\forall z \in d_k(a[n+1])(D_{n+1}0 \ll_k z)$. Hence $a[n] \ll_k a[n+1]$.

2. For $a = b + c$ or $a = D_v b$ with $\mathrm{dom}(b) = \mathbb{N}$ the assertion follows immediately from I.H. and 3.1(b), 3.2(b).

3. For $a = D_v b$ with $\mathrm{dom}(b) \in \{T_u : v \leqslant u\}$ we have $a[n] = a[n+1]$.

4. For $a = D_v(b_0 + 1)$ we have $a[n] = (D_v b_0)(n+1) \ll_k (D_v b_0)(n+2) = a[n+1]$ by 3.1(c).

3.4. Lemma. (a) *$a \ll_k b$ and $k \leqslant m \Rightarrow a \ll_m b$.*

(b) *$\mathrm{dom}(a) = \mathbb{N}$ and $n \leqslant k \Rightarrow a[n] \ll_k a$.*

Proof. (a) Transfinite induction on b: Suppose $b \neq 0$ and $\forall z \in d_k(b)(a \lesseqq_k b[z])$. For $\mathrm{dom}(b) = \{0\}$ or $\mathrm{dom}(b) = T_u$ the assertion follows immediately from I.H. Otherwise the I.H. and 3.3 yield $a \lesseqq_m b[k] \lesseqq_m b[m]$. Hence $a \ll_m b$.

(b) By 3.3 we get $a[n] \lesseqq_k a[k]$. Hence $a[n] \ll_k a$.

3.5. Lemma. (a) *$a \neq 0 \Rightarrow 1 \lesseqq_0 a$.*

(b) *$D_v a + 1 \ll_1 D_v(a+1)$.*

(c) *$D_u 1 \ll_0 D_{u+1}0$ and $D_0 1 \ll_0 D_\omega 0$.*

(d) *$a \neq 0$ or $v \neq 0 \Rightarrow k+1 \ll_k D_v a$ and for $k \neq 0$, $D_v a + k + 1 \ll_k D_v(a+1)$.*

Proof. (a) For $a \notin \{0, 1\}$ we have $\forall z \in d_0(a)(a[z] \neq 0)$. From this the assertion follows by transfinite induction on a.

(b) We have $D_v a + 1 \lesseqq_0 D_v a + D_v a = D_v(a+1)[1]$.

(c) By 3.5(a) and 3.2(b) we have $D_u 1 \lesseqq_0 z = (D_{u+1}0)[z]$ for all $z \in d_0(D_{u+1}0)$. Hence $D_u 1 \ll_0 D_{u+1}0$. Especially $D_0 1 \ll_0 D_1 0 = (D_\omega 0)[0]$ and thus $D_0 1 \ll_0 D_\omega 0$.

(d) We have $k+1 = (D_0 1)[k]$ and therefore $k+1 \ll_k D_0 1$. By (c) it follows that $k+1 \ll_k D_v 0$ for all $v \neq 0$. If $a \neq 0$, then we have $k+1 \ll_k D_v 1 \lesseqq_0 D_v a$ by (a) and 3.2(b). Using $k+1 \ll_k D_v a$ we get $D_v a + k + 1 \ll_k (D_v a) \cdot 2 \ll_1 D_v(a+1)$.

Definition of $H_a:\mathbb{N}\to\mathbb{N}$ *for* $a\in T_0$

$$H_0(n):=n,$$
$$H_a(n):=H_{a[n]}(n+1),\quad \text{if } a\neq 0.$$

3.6. Lemma. Let $a, b, c\in T_0$.
(a) $H_a(n)=\min\{k>n: a[n][n+1]\cdots[k-1]=0\}$, *if* $a\neq 0$.
(b) $H_{a+b}=H_a\circ H_b$
(c) $H_a(n)<H_a(n+1)$.
(d) $c\ll_k a\Rightarrow H_c(n)<H_a(n)$, *for all* $n\geqslant k$.

Proof. (a) Let $m:=\min\{k>n: a[n][n+1]\cdots[k-1]=0\}$. Then we have

$$H_a(n)=H_{a[n]}(n+1)=\cdots=H_{a[n]\cdots[m-1]}(m)=H_0(m)=m.$$

(b) Let $b\neq 0$ and $m:=H_b(n)$. Then $(a+b)[n]\cdots[m-1]=a+(b[n]\cdots[m-1])=a+0=a$ and thus $H_{a+b}(n)=H_a(m)=H_a(H_b(n))$.

(c) and (d) are proved simultaneously by transfinite induction on a: Let $a\neq 0$.

(c) By 3.3 we have $a[n]\leqq_0 a[n+1]$, and therefore by I.H.

$$H_a(n)=H_{a[n]}(n+1)\leqslant H_{a[n+1]}(n+1)<H_{a[n+1]}(n+2)=H_a(n+1).$$

(d) Suppose $c\leqq_k a[k]$ and $n\geqslant k$: By 3.3 we get $c\leqq_k a[n]$ and then by I.H. $H_c(n)\leqslant H_{a[n]}(n)<H_{a[n]}(n+1)=H_a(n)$.

Definition.

$$D_v^0 a:=D_v a,\qquad D_v^{m+1}a:=D_vD_v^m a,\qquad c_v^m:=D_0D_v^m 0.$$

3.7. Lemma. (a) $(D_v^m a)\cdot n\ll_k D_v^m(a+1)$, *for* $n\leqslant k+1$.
(b) $(D_v^m 0)\cdot n\ll_k D_v^{m+1}0$, *for* $n\leqslant k+1$.

Proof. (a) From 3.1(c) and 3.2(b) we obtain $D_v^m a\ll_0 D_v^m(a+1)$. For $k\neq 0$ we proceed by induction on m:
1. $m=0$: $(D_v^m a)\cdot n=(D_v a)\cdot n\ll_k D_v(a+1)=D_v^m(a+1)$ by 3.2.
2. $m\neq 0$: Using 3.2(a), 3.5(a) and the I.H. we obtain

$$(D_v^m a)\cdot n=D_v(D_v^{m-1}a)\cdot n\ll_k D_v(D_v^{m-1}a+1)$$

and

$$D_v^{m-1}a+1\leqq_0(D_v^{m-1}a)\cdot 2\ll_1 D_v^{m-1}(a+1).$$

From this the assertion follows by 3.2(b).
(b) $(D_v^m 0)\cdot n\ll_k D_v^m 1\leqq_0 D_v^m D_v 0=D_v^{m+1}0$ by 3.7(a), 3.5(a), 3.2(b).

3.8. Lemma. (a) $m\geqslant 1$ *and* $n\geqslant 1\Rightarrow H_{c_v^m}(4n+6)<H_{c_v^{m+1}}(n)$.
(b) $n\geqslant m+1\Rightarrow H_{c_v^m}(n)<H_{c_v^n}(1)$.

Proof. (a) Let $a := D_v^m 0$. Obviously $H_i(n) = i + n$ and therefore $H_{D_0 1}(n) = H_{n+1}(n+1) = 2n+2$. By 3.6(b) we obtain $H_{D_0 a}(4n+6) = H_{D_0 a + D_0 1 + D_0 1}(n)$. By 3.5(d) we have $2 \ll_1 a$ (since $m \neq 0$) and thus

$$D_0 a + (D_0 1) \cdot 2 \ll_1 D_0 a + D_0 2 \ll_1 D_0 a + D_0 a \ll_1 D_0(a+1)$$

and $a + 1 \leqq_0 a + a = (D_v^m 0) \cdot 2 \ll_1 D_v^{m+1} 0$. From this together with 3.2(b) we get $D_0 a + (D_0 1) \cdot 2 \ll_1 D_0 D_v^{m+1} 0 = c_v^{m+1}$. Hence $H_{D_0 a}(4n+6) < H_{c_v^{m+1}}(n)$ for $n \geq 1$.

(b) By 3.7(b) and 3.2(b) we have $c_v^n \ll_0 c_v^{n+1}$. Hence $n \leq H_{c_v^n}(0)$ and $n + 1 \leq H_{c_v^n}(1)$ by 3.6(c, d). For $n \geq m+1$ we have

$$\begin{aligned} c_v^{n-1} + c_v^{n-1} &= (D_0 D_v^{n-1} 0) \cdot 2 \ll_1 D_0(D_v^{n-1} 0 + 1) \\ &\leqq_0 D_0(D_v^{n-1} 0 + D_v^{n-1} 0) \ll_1 D_0 D_v^n 0 = c_v^n \end{aligned}$$

and thus

$$H_{c_v^m}(n) \leq H_{c_v^{n-1}}(n) \leq H_{c_v^{n-1}}(H_{c_v^{n-1}}(1)) = H_{c_v^{n-1} + c_v^{n-1}}(1) < H_{c_v^n}(1).$$

4. The infinitary system $\mathrm{ID}_\omega^\infty$

In this section we prove the following theorem:

4.0. Theorem. *If a Π_2^0-sentence $\forall x \exists y \, \varphi(x, y)$ ($\varphi \in \Sigma_1^0$) is provable in* ID_ν *($\nu \leq \omega$), then there exists $p \in \mathbb{N}$ such that $\forall n \geq p \; \exists k < H_{D_0 D_\nu^n 0}(1) \; \varphi(n, k)$.*

Corollary. $\mathrm{ID}_\nu \nvdash \forall n \, \exists k \, (D_0 D_\nu^n 0)[1][2] \cdots [k] = 0$.

Proof. Suppose $\mathrm{ID}_\nu \vdash \forall n \, \exists k \, (D_0 D_\nu^n 0)[1] \cdots [k] = 0$. Then also $\mathrm{ID}_\nu \vdash \forall n \, \exists k \, (D_0 D_\nu^n 0)[1] \cdots [k-1] = 0$ and therefore by 4.0 there exists $p \in \mathbb{N}$ such that $\forall n \geq p \; \exists k < H_{D_0 D_\nu^n 0}(1) \, (D_0 D_\nu^n 0)[1] \cdots [k-1] = 0$. Hence $\min\{k \in \mathbb{N} : (D_0 D_\nu^p 0)[1] \cdots [k-1] = 0\} < H_{D_0 D_\nu^p 0}(1)$, which is a contradiction to 3.6(a).

From this corollary together with 2.1 and the fact that ID_ω proves the same arithmetic sentences as (Π_1^1-CA) + BI we obtain Theorem III, i.e.,

$$(\Pi_1^1\text{-CA}) + \mathrm{BI} \nvdash \forall n \, \exists k \, A^n(1) \cdots (k) = \oplus.$$

Theorem 4.0 is obtained by embedding ID_ν into an infinitary proof system $\mathrm{ID}_\omega^\infty$ which allows cut elimination.

Preliminaries. Let L denote the first-order language consisting of the following symbols:

(i) the logical constants $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$,

(ii) number variables (indicated by x, y),

(iii) a constant 0 (zero) and a unary function symbol $'$ (successor),

(iv) constants for primitive recursive predicates (among them the symbol $<$ for the arithmetic 'less' relation).

By s, t, $t_0, \ldots$ we denote arbitrary L-terms. The constant terms $0, 0', 0'', \ldots$ are called numerals; we identify numerals and natural numbers and denote them by i, j, k, m, n, u, v, w. A formula of the shape $Rt_1 \cdots t_n$ or $\neg Rt_1 \cdots t_n$, where R is a n-ary predicate symbol of L, is called an *arithmetic prime formula* (abbreviated by a.p.f.).

Let X be a unary and Y a binary predicate variable. A *positive operator form* is a formula $\mathfrak{A}(X, Y, y, x)$ of $L(X, Y)$ in which only X, Y, y, x occur free and all occurrences of X are positive. The *language* L_{ID} is obtained from L by adding a binary predicate constant $P^{\mathfrak{A}}$ and a 3-ary predicate constant $P^{\mathfrak{A}}_{<}$ for each positive operator form $\mathfrak{A}$.

Abbreviations

$$t \in P^{\mathfrak{A}}_s :\equiv P^{\mathfrak{A}}_s t :\equiv P^{\mathfrak{A}} st, \qquad t \notin P^{\mathfrak{A}}_s :\equiv \neg(t \in P^{\mathfrak{A}}_s),$$

$$P^{\mathfrak{A}}_{<s} t_0 t_1 :\equiv P^{\mathfrak{A}}_{<} s t_0 t_1, \qquad \mathfrak{A}_s(X, x) :\equiv \mathfrak{A}(X, P^{\mathfrak{A}}_{<s}, s, x).$$

The formal theory ID_ω is an extension of Peano Arithmetic, formulated in the language L_{ID}, by the following axioms:

$(P^{\mathfrak{A}}.1)$ $\forall y \forall x (\mathfrak{A}_y(P^{\mathfrak{A}}_y, x) \to x \in P^{\mathfrak{A}}_y)$.

$(P^{\mathfrak{A}}.2)$ $\forall y (\forall x (\mathfrak{A}_y(F, x) \to F(x)) \to \forall x (x \in P^{\mathfrak{A}}_y \to F(x)))$,
for every L_{ID}-formula $F(x)$.

$(P^{\mathfrak{A}}_{<})$ $\forall y \forall x_0 \forall x_1 (P^{\mathfrak{A}}_{<y} x_0 x_1 \leftrightarrow x_0 < y \wedge x_1 \in P^{\mathfrak{A}}_{x_0})$.

The *infinitary system* $\mathrm{ID}^\infty_\omega$ will be formulated in the language $L_{\mathrm{ID}}(N)$ which arises from L_{ID} by adding a new unary predicate symbol N. This is a technical tool which shall help us to keep control over the numerials n occurring in $\exists$-inferences $A(n) \vdash \exists x A(x)$ of $\mathrm{ID}^\infty_\omega$-derivations. Following Tait [8] we assume all formulas to be in *negation normal form,* i.e., the formulas are built up from atomic and negated atomic formulas by means of $\wedge$, $\vee$, $\forall$ $\exists$ If A is a complex formula we consider $\neg A$ as a notation for the corresponding negation normal form.

Definition of the length $|A|$ of a $L_{\mathrm{ID}}(N)$-formula A

1. $|Nt| := |\neg Nt| := 0$.
2. $|A| := 1$, if A is an a.p.f. or a formula $(\neg)P^{\mathfrak{A}}_s t$.
3. $|P^{\mathfrak{A}}_{<s} t_0 t_1| := |\neg P^{\mathfrak{A}}_{<s} t_0 t_1| := 2$.
4. $|A \wedge B| := |A \vee B| := \max\{|A|, |B|\} + 1$.
5. $|\forall x A| := |\exists x A| := |A| + 1$.

Proposition. $|\neg A| = |A|$, *for each* $L_{\mathrm{ID}}(N)$-*formula* A.

As before we use the letters u, v to denote numbers $\leqslant \omega$.

Inductive definition of formula sets Pos_v $(v < \omega)$

1. All $L(N)$-formulas belong to Pos_v.
2. All formulas $P_u^{\mathfrak{A}}t$, $(\neg)P_{<u}^{\mathfrak{A}}t_0t_1$ with $u \leqslant v$ belong to Pos_v.
3. All formulas $\neg P_u^{\mathfrak{A}}t$ with $u < v$ belong to Pos_v.
4. If A and B belong to Pos_v, then the formulas $A \wedge B$, $A \vee B$, $\forall x\, A$, $\exists x\, A$ also belong to Pos_v.

Remark. If $P_u^{\mathfrak{A}}t \in \mathrm{Pos}_v$, then also $\mathfrak{A}_u(P_u^{\mathfrak{A}}, t) \in \mathrm{Pos}_v$.

Notations

- In the following A, B, C always denote closed $L_{\mathrm{ID}}(N)$-formulas.
- Γ, Γ', Δ denote finite sets of closed $L_{\mathrm{ID}}(N)$-formulas; we write, e.g., Γ, Δ, A for $\Gamma \cup \Delta \cup \{A\}$.
- A^N denotes the result of restricting all quantifiers in A to N.
- $t \in N :\equiv Nt$, $t \notin N :\equiv \neg Nt$.
- As before we use the letters a, b, c, d, z to denote elements of T.

Definition

$$c \ll_\Gamma a \quad :\Leftrightarrow \quad c \ll_k a, \quad \text{where } k := \max(\{2\} \cup \{3n : \neg Nn \in \Gamma\}).$$

4.1. Proposition. (a) *$c \ll_\Gamma a$ and $\Gamma \subseteq \Delta \Rightarrow c \ll_\Delta a$ (cf.* 3.4(a)).
(b) $c \ll_{\Gamma \cup \{0 \notin N\}} a \Rightarrow c \ll_\Gamma a$.

Basic inference rules

$(\wedge)$	$A_0, A_1 \vdash A_0 \wedge A_1$.
$(\vee)$	$A \vdash A \vee B$; $\quad B \vdash A \vee B$.
$(\forall^\infty)$	$(A(n))_{n \in \mathbb{N}} \vdash \forall x\, A(x)$.
$(\exists)$	$A(n) \vdash \exists x\, A(x)$.
(N)	$n \in N \vdash n' \in N$.
$(P_{<u}^{\mathfrak{A}})$	$P_j^{\mathfrak{A}}n \vdash P_{<u}^{\mathfrak{A}}jn$, if $j < u < \omega$.
$(\neg P_u^{\mathfrak{A}})$	$\neg P_j^{\mathfrak{A}}n \vdash \neg P_{<u}^{\mathfrak{A}}jn$, if $j < u < \omega$.

Every instance $(A_i)_{i \in I} \vdash A$ of these rules is called a *basic inference*. If $(A_i)_{i \in I} \vdash A$ is a basic inference with $A \in \mathrm{Pos}_v$, then $A_i \in \mathrm{Pos}_v$ for all $i \in I$. This property will be used in the proof of 4.6.

The *system* $\mathrm{ID}_\omega^\infty$ consists of the *language* $L_{\mathrm{ID}}(N)$ and a certain *derivability relation* $\vdash_m^a \Gamma$ ("Γ is derivable with order $a \in T$ and cutdegree $m \in \mathbb{N}$") which we introduce below by an iterated inductive definition similar to that of the tree classes $\mathcal{T}_v$ in Section 1. The main feature in the definition of $\vdash_m^a \Gamma$ is the Ω_{u+1}-rule which we have developed in Buchholz [1], [2]. We try to give a short explanation of this inference rule. To this purpose let us consider "$\vdash_1^a A$" as a notion of realizability similar to modified realizability. So we read "$\vdash_1^a A$" as "a realizes A". Now suppose that $\vdash_1^z \Gamma$ is already defined for all $z \in T_u$. Then, according to the fact that

$$f^{\sigma\to\tau}\,\mathrm{mr}\,A \to B \quad \text{iff} \quad \forall g^\sigma\,(g^\sigma\,\mathrm{mr}\,A \Rightarrow f^{\sigma\to\tau}(g^\sigma)\,\mathrm{mr}\,B),$$

it seems reasonable to define:

$$a \text{ realizes } (P_u^{\mathfrak{A}} n \to B) \quad :\Leftrightarrow \quad \begin{cases} \mathrm{dom}(a) = T_u \quad \text{and} \\ \forall z \in T_u\ (z \text{ realizes } P_u^{\mathfrak{A}} n \Rightarrow a[z] \text{ realizes } B). \end{cases}$$

This motivates the following inference rule:

$$(\Omega_{u+1})' \quad \left.\begin{array}{l} \mathrm{dom}(a) = T_u \quad \text{and} \\ \forall z \in T_u (\vdash_1^z P_u^{\mathfrak{A}} n \Rightarrow \vdash_m^{a[z]} B) \end{array}\right\} \Rightarrow \vdash_m^a P_u^{\mathfrak{A}} n \to B.$$

The next step is a straightforward modification of this rule:

$$(\Omega_{u+1})'' \quad \left.\begin{array}{l} \mathrm{dom}(a) = T_u \quad \text{and} \\ \forall z \in T_u\ \forall A \in \mathrm{Pos}_u\ (\vdash_1^z A \vee P_u^{\mathfrak{A}} n \Rightarrow \vdash_m^{a[z]} A \vee B) \end{array}\right\} \Rightarrow \vdash_m^a P_u^{\mathfrak{A}} n \to B.$$

For technical reasons we combine every application of $(\Omega_{u+1})''$ with a cut $B \vee P_u^{\mathfrak{A}} n,\ P_u^{\mathfrak{A}} n \to B \vdash B$. This gives the final version of the Ω_{u+1}-rule.

Inductive definition of $\vdash_m^a \Gamma$ $(a \in T,\ m \in \mathbb{N})$

(Ax1) $\vdash_m^a \Gamma, A$, if A is a true a.p.f. or $A \equiv 0 \in N$ or $A \equiv \neg P_{<u}^{\mathfrak{A}} jn$ with $u \leq j$.

(Ax2) $\vdash_m^a \Gamma, \neg A, A$, if $A \equiv n \in N$ or $A \equiv P_u^{\mathfrak{A}} n$.

(Bas) If $(A_i)_{i\in I} \vdash A$ is a basic inference with $A \in \Gamma$ and $\forall i \in I\ (\vdash_m^a \Gamma, A_i)$, then $\vdash_m^{a+1} \Gamma$.

($P_u^{\mathfrak{A}}$) $\vdash_m^a \Gamma,\ n \in N \wedge \mathfrak{A}_u^N(P_u^{\mathfrak{A}}, n)$ and $P_u^{\mathfrak{A}} n \in \Gamma \Rightarrow \vdash_m^{a+3} \Gamma$.

(Cut) $\vdash_m^a \Gamma, \neg C$ and $\vdash_m^a \Gamma, C$ and $|C| < m \Rightarrow \vdash_m^{a+1} \Gamma$.

$$(\Omega_{u+1}) \quad \left.\begin{array}{l} \mathrm{dom}(a) = T_u \text{ and } \vdash_m^{a[1]} \Gamma, P_u^{\mathfrak{A}} n \text{ and} \\ \forall z \in T_u\ \forall \Delta \subseteq \mathrm{Pos}_u\ (\vdash_1^z \Delta, P_u^{\mathfrak{A}} n \Rightarrow \vdash_m^{a[z]} \Delta, \Gamma) \end{array}\right\} \Rightarrow \vdash_m^a \Gamma.$$

($\ll$) $\vdash_m^b \Gamma$ and $b \ll_\Gamma a \Rightarrow \vdash_m^a \Gamma$.

4.2. Lemma. (a) $\vdash_m^a \Gamma$ *and* $m \leq k,\ \Gamma \subseteq \Delta \Rightarrow \vdash_k^a \Delta$.

(b) $\vdash_m^a \Gamma \Rightarrow \vdash_m^{c+a} \Gamma$.

(c) $\vdash_m^a \Gamma,\ 0 \notin N \Rightarrow \vdash_m^a \Gamma$.

Proof. By transfinite induction on a using 3.1(b) and 4.1 and the fact that $(c + a)[z] = c + a[z]$ for all $z \in \mathrm{dom}(a)$.

4.3. Lemma (Inversion). *Let $(A_i)_{i\in I} \vdash A$ be a basic inference* $(\wedge)$, $(\forall^\infty)$, $(P^{\mathfrak{A}}_{<u})$, $(\neg P^{\mathfrak{A}}_{<u})$. *Then $\vdash^a_m \Gamma, A$ implies $\forall i \in I\, (\vdash^a_m \Gamma, A_i)$.*

Proof. By transfinite induction on a.

4.4. Lemma (Reduction). *Suppose $\vdash^a_m \Gamma_0, \neg C$ and $|C| \leq m$, where C is a formula of the shape $A \vee B$ or $\exists x\, A(x)$ or $P^{\mathfrak{A}}_{<u} jn$ or $\neg P^{\mathfrak{A}}_u n$ or a false a.p.f. Then $\vdash^b_m \Gamma, C$ implies $\vdash^{a+b}_m \Gamma_0, \Gamma$.*

Proof. By transfinite induction on b:

(Ax1) If $\vdash^b_m \Gamma, C$ holds by (Ax1), then also $\vdash^{a+b}_m \Gamma$ by (Ax1).

(Ax2) If $\vdash^b_m \Gamma, C$ holds by (Ax2), then either $\vdash^{a+b}_m \Gamma$ by (Ax2) or $\neg C \in \Gamma$. In the latter case $\vdash^{a+b}_m \Gamma_0, \Gamma$ follows from $\vdash^a_m \Gamma_0, \neg C$.

(Bas) Suppose $b = b_0 + 1$, $A \in \Gamma \cup \{C\}$ and $\forall i \in I\, (\vdash^{b_0}_m \Gamma, C, A_i)$ where $(A_i)_{i\in I} \vdash A$ is a basic inference $(\mathscr{I})$. Then by I.H. we have (1) $\forall i \in I\, (\vdash^{a+b_0}_m \Gamma_0, \Gamma, A_i)$.

Case 1: $A \in \Gamma$. Then the assertion follows immediately from (1).

Case 2: $A \equiv C$. Then, according to the assumption we have made on C, $(\mathscr{I})$ is an inference $(\vee)$, $(\exists)$, $(P^{\mathfrak{A}}_{<u})$ with $I = \{0\}$. By 4.3, 4.2(a) and $(\ll)$ from $\vdash^a_m \Gamma_0, \neg C$ we get (2) $\vdash^{a+b_0}_m \Gamma_0, \Gamma, \neg A_0$. From (1), (2) and $|A_0| < |C| \leq m$ we obtain $\vdash^{a+b}_m \Gamma_0, \Gamma$ by a cut with cutformula A_0.

$(\ll)$ Suppose $\vdash^{b_0}_m \Gamma, C$ with $b_0 \ll_{\Gamma, C} b$. Since C is not a formula $n \notin N$, it follows that $a + b_0 \ll_{\Gamma_0, \Gamma} a + b$. By I.H. we have $\vdash^{a+b_0}_m \Gamma_0, \Gamma$. Hence $\vdash^{a+b}_m \Gamma_0, \Gamma$ by $(\ll)$.

In all other cases the assertion follows immediately from I.H.

4.5. Theorem (Cutelimination). *$\vdash^a_{m+1} \Gamma$ and $a \in T_\rho$, $\rho \leq \omega$, $m > 0 \Rightarrow \vdash^{D_\rho a}_m \Gamma$.*

Proof. By transfinite induction on a:

1. If $\vdash^a_{m+1} \Gamma$ holds by (Ax1) or (Ax2), then the assertion is trivial.

2. Suppose $a = a_0 + 1$, $A \in \Gamma$ and $\forall i \in I\, (\vdash^{a_0}_{m+1} \Gamma, A_i)$, where $(A_i)_{i\in I} \vdash A$ is a basic inference $(\mathscr{I})$. Then by I.H. we have $\forall i \in I\, (\vdash^{D_\rho a_0}_m \Gamma, A_i)$. By $(\mathscr{I})$ we obtain $\vdash^{D_\rho a_0 + 1}_m \Gamma$ and then $\vdash^{D_\rho a}_m \Gamma$ by $(\ll)$ and 3.5(a).

3. Suppose $a = a_0 + 3$, $P^{\mathfrak{A}}_u n \in \Gamma$ and $\vdash^{a_0}_{m+1} \Gamma, B$ with $B \equiv n \in N \wedge \mathfrak{A}^N_u(P^{\mathfrak{A}}_u, n)$. Then by I.H. and $(\ll)$ we have $\vdash^{D_\rho(a_0+2)}_m \Gamma, B$. By $(P^{\mathfrak{A}}_u)$ we get $\vdash^{D_\rho(a_0+2)+3}_m \Gamma$ and then $\vdash^{D_\rho a}_m \Gamma$ by $(\ll)$ and 3.5(d).

4. Suppose $\mathrm{dom}(a) = T_u$, $\vdash^{a[1]}_{m+1} \Gamma, P^{\mathfrak{A}}_u n$ and $\vdash^{a[z]}_{m+1} \Delta, \Gamma$ for all $z \in T_u$, $\Delta \subseteq \mathrm{Pos}_u$ with $\vdash^z_1 \Delta, P^{\mathfrak{A}}_u n$. Since $a \in T_\rho$, we have $u < \rho$ and thus $\mathrm{dom}(D_\rho a) = T_u$ and $(D_\rho a)[z] = D_\rho a[z]$. By I.H. we have $\vdash^{D_\rho a[1]}_m \Gamma, P^{\mathfrak{A}}_u n$ and $\vdash^{D_\rho a[z]}_m \Delta, \Gamma$ for all $z \in T_u$, $\Delta \leq \mathrm{Pos}_u$ with $\vdash^z_1 \Delta, P^{\mathfrak{A}}_u n$. From this we obtain $\vdash^{D_\rho a}_m \Gamma$ by an application of (Ω_{u+1}).

5. Suppose $\vdash^{a_0}_{m+1} \Gamma$ and $a_0 \ll_\Gamma a$. Then by I.H. and 3.2(b) we have $\vdash^{D_\rho a_0}_m \Gamma$ and $D_\rho a_0 \ll_\Gamma D_\rho a$. Hence $\vdash^{D_\rho a}_m \Gamma$.

6. Suppose $a = a_0 + 1$, $\vdash^{a_0}_{m+1} \Gamma, \neg C$, $\vdash^{a_0}_{m+1} \Gamma, C$ and $|C| < m + 1$. Then by I.H. we have $\vdash^{D_\rho a_0}_m \Gamma, \neg C$ and $\vdash^{D_\rho a_0}_m \Gamma, C$.

6.1. $|C| < m$: In this case we obtain $\vdash_m^{D_\rho a_0 + 1} \Gamma$ by a cut with cutformula C. The assertion follows by ($\ll$) and 3.5(b).

6.2. $|C| = m$: Since $m > 0$, we may assume that C fulfills the condition of 4.4. Then by 4.4 we obtain $\vdash_m^{D_\rho a_0 + D_\rho a_0} \Gamma$, and from this $\vdash_m^{D_\rho a} \Gamma$ by ($\ll$) and 3.2(a).

The following theorem shows that if $\Gamma \subseteq \mathrm{Pos}_v$ is derivable with cutdegree 1, then one can eliminate all Ω_{u+1}-inferences with $u \geqslant v$ from the derivation of Γ.

4.6 Theorem (Collapsing). *$\vdash_1^a \Gamma$ and $\Gamma \subseteq \mathrm{Pos}_v \Rightarrow \vdash_1^{D_v a} \Gamma$.*

Proof. By transfinite induction on a:

1. Suppose $\mathrm{dom}(a) = T_u$, $\vdash_1^{a[1]} \Gamma, P_u^{\mathfrak{A}} n$ and $\vdash_1^{a[z]} \Delta, \Gamma$ for all $z \in T_u$, $\Delta \subseteq \mathrm{Pos}_u$ with $\vdash_1^z \Delta, P_u^{\mathfrak{A}} n$.

Case 1: $u < v$. Then by I.H. we have $\vdash_1^{D_v a[1]} \Gamma, P_u^{\mathfrak{A}} n$ and $\vdash_1^{D_v a[z]} \Delta, \Gamma$ for all $z \in T_u$, $\Delta \subseteq \mathrm{Pos}_u$ with $\vdash_1^z \Delta, P_u^{\mathfrak{A}} n$. Moreover, $\mathrm{dom}(D_v a) = T_u$ and $(D_v a)[z] = D_v a[z]$. The assertion follows by (Ω_{u+1}).

Case 2: $u \geqslant v$. Then $\Gamma \cup \{P_u^{\mathfrak{A}} n\} \subseteq \mathrm{Pos}_u$ and therefore by I.H. $\vdash_1^{D_u a[1]} \Gamma, P_u^{\mathfrak{A}} n$. Since $z := D_u a[1] \in T_u$, we get $\vdash_1^{a[z]} \Gamma$. Now we apply the I.H. again and obtain $\vdash_1^{D_v a[z]} \Gamma$. But $D_v a[z] = (D_v a)[0] \ll_\Gamma D_v a$, and therefore $\vdash_1^{D_v a} \Gamma$.

2. In all other cases the assertion follows immediately from the I.H. by 3.5(b, d), 3.4(a), ($\ll$).

Definition

$$L(N)_+ := \{A : A \text{ is a sentence of } L(N) \text{ in which } N \text{ occurs only positively}\}.$$

For $\Gamma = \{A_1, \ldots, A_n\} \subseteq L(N)_+$ we define:

$$\vDash \Gamma(k) \quad :\Leftrightarrow \quad \begin{cases} A_1 \vee \cdots \vee A_n \text{ is true in the standard model} \\ \text{when } N \text{ is interpreted as } \{i \in \mathbb{N} : 3i < k\}. \end{cases}$$

4.7. Lemma.

$$\left.\begin{array}{l} \vdash_1^a i_1 \notin N, \ldots, i_m \notin N, \Gamma \quad \mathit{and} \\ \Gamma \subseteq L(N)_+,\ n \geqslant \max\{2, 3i_1, \ldots, 3i_m\} \end{array}\right\} \Rightarrow \vDash \Gamma(H_{D_0 a}(n)).$$

Proof. By transfinite induction on a: Let

$$\Gamma_0 := \{i_1 \notin N, \ldots, i_m \notin N\}, \qquad k := \max\{2, 3i_1, \ldots, 3i_m\} \leqslant n.$$

1. If $\vdash_1^a \Gamma_0, \Gamma$ holds by (Ax1), then the assertion is trivial.
2. If $\vdash_1^a \Gamma_0, \Gamma$ holds by (Ax2), then the assertion follows from $n < H_{D_0 a}(n)$.
3. If $\vdash_1^a \Gamma_0, \Gamma$ is the conclusion of a basic inference $\neq(N)$, then the assertion follows immediately from the I.H. and the relation $H_{D_0 b}(n) < H_{D_0(b+1)}(n)$.
4. Suppose $a = b + 1$, $N(j+1) \in \Gamma$, $\vdash_1^b \Gamma_0, \Gamma, Nj$. By I.H. we obtain $\vDash \Gamma \cup$

$\{Nj\}(H_{D_0b}(n))$. By 3.1(c), 3.2(a), 3.6(d) we have $H_{D_0b}(n) < H_{(D_0b)\cdot 2}(n) < H_{(D_0b)\cdot 3}(n) < H_{D_0a}(n)$ and therefore $H_{D_0b}(n) + 3 \leq H_{D_0a}(n)$. Hence $\models \Gamma(H_{D_0a}(n))$.

5. Suppose $\vdash_1^b \Gamma_0, \Gamma$ with $b \ll_{\Gamma_0 \cup \Gamma} a$. Then we have $D_0 b \ll_k D_0 a$ and therefore $H_{D_0b}(n) < H_{D_0a}(n)$, since $n \geq k$. Now the assertion follows immediately from the I.H.

6. Suppose $a = b + 1$, $\vdash_1^b \Gamma_0, \Gamma, i_0 \in N$ and $\vdash_1^b i_0 \notin N, \Gamma_0, \Gamma$. Let $\bar{n} := H_{D_0b}(n)$. Then we have

$$n < \bar{n} < H_{D_0b}(\bar{n}) = H_{(D_0b)\cdot 2}(n) < H_{D_0a}(n).$$

6.1. $\bar{n} < 3i_0$: From $\vdash_1^b \Gamma_0, \Gamma, i_0 \in N$ we obtain by the I.H. $\models \Gamma \cup \{i_0 \in N\}(\bar{n})$ and then $\models \Gamma(\bar{n})$, since $3i_0 \not< \bar{n}$. Using $\bar{n} < H_{D_0a}(n)$ we get the assertion.

6.2. $3i_0 \leq \bar{n}$: From $\vdash_1^b i_0 \notin N, \Gamma_0, \Gamma$ and $\max\{k, 3i_0\} \leq \bar{n}$ we obtain by the I.H. $\models \Gamma(H_{D_0b}(\bar{n}))$ and thus $\models \Gamma(H_{D_0a}(n))$.

7. Suppose $\mathrm{dom}(a) = T_u$, $\vdash_1^{a[1]} \Gamma_0, \Gamma, P_{uj}^{\mathfrak{A}}j$ and $\vdash_1^{a[z]} \Delta, \Gamma_0, \Gamma$ for all $z \in T_u$, $\Delta \subseteq \mathrm{Pos}_u$ with $\vdash_1^z \Delta, P_{uj}^{\mathfrak{A}}j$. By 4.6 we obtain $\vdash_1^z \Gamma_0, \Gamma, P_{uj}^{\mathfrak{A}}j$ with $z := D_u a[1] \in T_u$. From this we get $\vdash_1^{a[z]} \Gamma_0, \Gamma$. Now we apply the I.H. and obtain $\models \Gamma(H_{D_0a[z]}(n))$. Hence $\models \Gamma(H_{D_0a}(n))$, since $D_0 a[z] = (D_0 a)[0]$.

4.8. Theorem. *If* $\vdash_1^{D_v^m 0} \forall x \in N\, \exists y \in N\, \varphi^N(x, y)$, *where* $v \leq \omega$, $m \neq 0$ *and* $\varphi(x, y)$ *a* Σ_1^0*-formula of the language L, then there exists* $p \in \mathbb{N}$ *such that* $\forall n \geq p\, \exists k < H_{D_0 D_v^m 0}(1)\, \varphi(n, k)$.

Proof. Let $a := D_v^m 0$. From the premise we obtain $\vdash_1^a n \notin N, \exists y \in N\, \varphi^N(n, y)$ for all $n \in \mathbb{N}$. Then by 4.7 we get $\models \exists y \in N\, \varphi^N(n, y)\, (H_{D_0a}(\bar{n}))$ for all $n \in \mathbb{N}$ and all $\bar{n} \geq \max\{2, 3n\}$. Hence $\forall n\, \exists k < H_{D_0a}(3n+2)\, \varphi(n, k)$. By 3.8 we have $H_{D_0a}(3n+2) < H_{D_0 D_v^m 0}(1)$ for all $n \geq m + 2$.

In the remaining part of this section we show that ID_v $(v \leq \omega)$ can be embedded into $\mathrm{ID}_\omega^\infty$ and finally we prove Theorem 4.0. Let $v \leq \omega$ be fixed.

Abbreviations

$$\bar{k} := D_v^{k+2} 0,$$

$$a \longrightarrow_n b \quad :\Leftrightarrow \quad \exists a_0, \ldots, a_n (a_0 = a \wedge a_n = b \wedge \forall i < n\, (a_i + 1 \lneqq_2 a_{i+1})).$$

4.9. Lemma. (a) $\bar{k} \ll_1 \widetilde{k+1}$, (b) $\bar{k} \longrightarrow_6 \widetilde{k+1}$.

Proof. (a) follows from 3.7(b).

(b) By 3.5(d) and 3.7(b) we have $3 \ll_2 \bar{k}$ and $\bar{k} \cdot 3 \ll_2 \widetilde{k+1}$. Hence $\bar{k} + 3 \ll_2 \bar{k} \cdot 2$, $\bar{k} \cdot 2 + 3 \ll_2 \bar{k} \cdot 3 \ll_2 \widetilde{k+1}$ and consequently $\bar{k} \longrightarrow_3 \bar{k} \cdot 2 \rightarrow_3 \widetilde{k+1}$.

4.10. Lemma. $\vdash_0^{\bar{k}} \neg A, A$ *where* $k := |A|$

Proof. By induction on $|A|$:

1. If A is atomic, then $\vdash_0^{\tilde{k}} \neg A, A$ by (Ax1) or (Ax2).

2. $A = A_0 \wedge A_1$: Then $k = m+1$ with $m := \max\{|A_0|, |A_1|\}$. By I.H., 4.9(a) and ($\ll$) we get $\vdash_0^{\tilde{m}} \neg A_i, A_i$ for $i = 0, 1$, and then $\vdash_0^{\widetilde{m+1}} \neg A_0 \vee \neg A_1, A_0 \wedge A_1$ by ($\vee$), ($\wedge$), 4.9(b).

3. $A = \forall x\, B(x)$: This case is treated as 2.

4.11. Lemma. *$\vdash_0^{\tilde{k}+D_0 1} \neg F(0), \neg \forall x \in N\, (F(x) \to F(x')),\ n \notin N,\ F(n)$, where $k := |F|$.*

Proof. Let $G := \forall x \in N\, (F(x) \to F(x'))$. By induction on n we show:

(1) $\quad \vdash_0^{\tilde{k}+3n} \neg F(0), \neg G, F(n).$

From (1) we obtain $\vdash_0^{\tilde{k}+D_0 1} \neg F(0), \neg G, F(n), n \notin N$, since

$$\tilde{k} + 3n \ll_{3n} \tilde{k} + D_0 1.$$

Proof of (1). For $n = 0$ the assertion holds by 4.10.

Induction step: Suppose $\vdash_0^{\tilde{k}+3n} \neg F(0), \neg G, F(n)$. By 4.10 we have $\vdash_0^{\tilde{k}+3n} \neg F(n'), F(n')$. Hence $\vdash_0^{\tilde{k}+3n+1} \neg F(0), \neg G, F(n) \wedge \neg F(n'), F(n')$. By (Ax1) and n applications of (N) we get $\vdash^{\tilde{k}+3n+1} n \in N$, and then by ($\wedge$) $\vdash^{\tilde{k}+3n+2} \neg F(0), \neg G, n \in N \wedge (F(n) \wedge \neg F(n')), F(n')$. Now we apply ($\exists$) and obtain $\vdash_0^{\tilde{k}+3\cdot n'} \neg F(0), \neg G, F(n')$, since $\neg G \equiv \exists x\, (x \in N \wedge (F(x) \wedge \neg F(x')))$.

The following lemma will be used to show that the induction scheme $\forall x \in N\, (\mathfrak{A}_u^N(F, x) \to F(x)) \to \forall x \in N\, (P_u^{\mathfrak{A}} x \to F(x))$ is derivable in $\mathrm{ID}_\omega^\infty$.

4.12. Lemma.

$$\left.\begin{array}{l} a \in T_u,\ \Delta \subseteq \mathrm{Pos}_u,\ \vdash_1^a \Delta, P_u^{\mathfrak{A}} n \\ k = |F|,\ G \equiv \forall x \in N\, (\mathfrak{A}_u^N(F, x) \to F(x)) \end{array}\right\} \Rightarrow \vdash_1^{\tilde{k}+a} \Delta, \neg G, F(n).$$

Proof. Informal description: Let Π be a derivation of $\Delta, P_u^{\mathfrak{A}} n$. In Π we replace every occurrence of $P_u^{\mathfrak{A}}$, which is linked to the endformula $P_u^{\mathfrak{A}} n$, by $F(\cdot)$, Let Π' denote the result of this transformation. Π' may contain certain inferences of the kind $j \in N \wedge \mathfrak{A}_u^N(F, j) \vdash F(j)$, and therefore Π' may fail to be an $\mathrm{ID}_\omega^\infty$-derivation. From Π' we obtain an $\mathrm{ID}_\omega^\infty$-derivation of $\Delta, \neg G, F(n)$ as follows: First we adjoin $\neg G$ to each Γ in Π', and then we replace every inference $\neg G, \Gamma, j \in N \wedge \mathfrak{A}_u^N(F, j) \vdash \neg G, \Gamma, F(j)$ by the following inferences

$$\cfrac{\cfrac{\neg G, \Gamma, j \in N \wedge \mathfrak{A}_u^N(F, j) \qquad \neg F(j), F(j)}{\neg G, \Gamma, j \in N \wedge \mathfrak{A}_u^N(F, j) \wedge \neg F(j), F(j)}\ (\wedge)}{\neg G, \Gamma, F(j)}\ (\exists).$$

In order to get a rigorous proof of the lemma we have to prove a more general proposition.

Definition. For $A \in \mathrm{Pos}_u$ let A^* denote the result of replacing all occurrences of $P_u^{\mathfrak{A}}$ in A by $F(\cdot)$. $\{A_1, \ldots, A_m\}^* := \{A_1^*, \ldots, A_m^*\}$.

Proposition. $\Gamma_0 \cup \Gamma \subseteq \mathrm{Pos}_u$, $a \in T_u$, $k = |F|$, $\vdash_1^a \Gamma_0, \Gamma \Rightarrow \vdash_1^{\bar{k}+a} \Gamma_0, \neg G, \Gamma^*$.

Proof. By transfinite induction on a:

1. If $\vdash_1^a \Gamma_0, \Gamma$ holds by (Ax1) or (Ax2), then also $\vdash_1^{\bar{k}+a} \Gamma_0, \neg G, \Gamma^*$ by (Ax1), (Ax2), since $\neg P_u^{\mathfrak{A}}$ does not occur in $\Gamma_0 \cup \Gamma$.

2. Suppose that $a = a_0 + 1$ and $(A_i)_{i \in I} \vdash A$ is a basic inference with $A \in \Gamma_0 \cup \Gamma$ and $\forall i \in I\, (\vdash_1^{a_0} \Gamma_0, \Gamma, A_i)$. Then $\forall i \in I\, (A_i \in \mathrm{Pos}_u)$ and therefore we can apply the I.H. to Γ_0, Γ, A_i.

2.1. $A \in \Gamma_0$: By I.H. we get $\forall i \in I\, (\vdash_1^{\bar{k}+a_0} \Gamma_0, \neg G, \Gamma^*, A_i)$ and from this $\vdash_1^{\bar{k}+a} \Gamma_0, \neg G, \Gamma^*$ by the respective basic inference.

2.2. $A \in \Gamma$: Then $A^* \in \Gamma^*$ and $(A_i^*)_{i \in I} \vdash A^*$ is a basic inference. By I.H. we have $\forall i \in I\, (\vdash_1^{\bar{k}+a_0} \Gamma_0, \neg G, \Gamma^*, A_i^*)$. Hence $\vdash_1^{\bar{k}+a} \Gamma_0, \neg G, \Gamma^*$.

3. Suppose that $\mathrm{dom}(a) = T_w$, $\vdash_1^{a[1]} \Gamma_0, \Gamma, P_w^{\mathfrak{B}} j$ and $\vdash_1^{a[z]} \Delta, \Gamma_0, \Gamma$ for all $z \in T_w$, $\Delta \subseteq \mathrm{Pos}_w$ with $\vdash_1^z \Delta, P_w^{\mathfrak{B}} j$. Since $a \in T_u$, we have $w < u$ and therefore by I.H. $\vdash_1^{\bar{k}+a[1]} \Gamma_0, \neg G, \Gamma^*, P_w^{\mathfrak{B}} j$ and $\vdash_1^{\bar{k}+a[z]} \Delta, \Gamma_0, \neg G, \Gamma^*$ for all $z \in T_w$, $\Delta \subseteq \mathrm{Pos}_w$ with $\vdash_1^z \Delta, P_w^{\mathfrak{B}} j$. Now by an application of (Ω_{w+1}) we get the assertion.

4. Suppose $a = a_0 + 3$, $P_u^{\mathfrak{A}} j \in \Gamma$ and $\vdash_1^{a_0} \Gamma_0, \Gamma, j \in N \wedge \mathfrak{A}_u^N(P_u^{\mathfrak{A}}, j)$. Then $F(j) \in \Gamma^*$ and therefore $\vdash_1^{\bar{k}} \Gamma^*, \neg F(j)$ by 4.10. By I.H. and 4.3 we have $\vdash_1^{\bar{k}+a_0} \Gamma_0, \Gamma^*, \neg G, j \in N$ and $\vdash_1^{\bar{k}+a_0} \Gamma_0, \Gamma^*, \neg G, \mathfrak{A}_u^N(F, j)$. Now we obtain $\vdash_1^{k+a_0+2} \Gamma_0, \Gamma^*, \neg G, j \in N \wedge (\mathfrak{A}_u^N(F, j) \wedge \neg F(j))$ and then by $(\exists)$ $\vdash_1^{\bar{k}+a} \Gamma_0, \Gamma^*, \neg G$.

5. In all other cases the assertion follows immediately from I.H.

4.13. Lemma. $\vdash_1^{\bar{k}+D_{u+1}0} \neg \forall x \in N\, (\mathfrak{A}_u^N(F, x) \to F(x)), \neg P_u^{\mathfrak{A}} n, F(n)$, *with* $k := |F|$.

Proof. Let $b := \bar{k} + D_{u+1}0$ and $G := \forall x \in N\, (\mathfrak{A}_u^N(F, x) \to F(x))$. Then $\mathrm{dom}(b) = T_u$ and $b[z] = \bar{k} + z$. Therefore by 4.12 we have $\vdash_1^{b[z]} \Delta, \neg G, \neg P_u^{\mathfrak{A}} n, F(n)$ for all $z \in T_u$, $\Delta \subseteq \mathrm{Pos}_u$ with $\vdash_1^z \Delta, P_u^{\mathfrak{A}} n$. By (Ax2) we also have $\vdash_1^{b[1]} \neg G, \neg P_u^{\mathfrak{A}} n, F(n), P_u^{\mathfrak{A}} n$. Now we apply the Ω_{u+1}-rule and obtain $\vdash_1^b \neg G, \neg P_n^{\mathfrak{A}} n, F(n)$.

Remark. The theory ID_ν with $\nu < \omega$ is the same as ID_ω except that the axioms $(P^{\mathfrak{A}}.2)$ are replaced by

$(P^{\mathfrak{A}}.2)_{<\nu}$ $\quad \forall x\, (\mathfrak{A}_u(F, x) \to F(x)) \to \forall x\, (P_u^{\mathfrak{A}} x \to F(x))$,
for each L_{ID}-formula $F(x)$ and each $u < \nu$.

4.14. Theorem. *If the sentence* A *is provable in* ID_ν $(\nu \leq \omega)$, *then there exists* $k \in \mathbb{N}$ *such that* $\vdash_k^{D_\nu^k 0} A^N$.

Proposition 1. *For every mathematical axiom $A(v_1, \ldots, v_m)$ of* ID_ν *there exists* $k \in \mathbb{N}$ *such that* $\vdash_1^{\tilde{k}} A(i_1, \ldots, i_m)^N$ *for all* $i_1, \ldots, i_m \in \mathbb{N}$. ($v_1, v_2, \ldots$ denote variables of the language L.)

Proof. We assume $m = 1$.

1. $A(v) \equiv B(0, v) \wedge \forall x\,(B(x, v) \rightarrow B(x', v)) \rightarrow \forall x\, B(x, v)$.

Let $F(x) :\equiv B(x, i)^N$, $G :\equiv \forall x \in N\,(F(x) \rightarrow F(x'))$ and $k := |F(x)|$. By 4.11, 3.5(c), 4.9(a) we have $\vdash_1^{\tilde{k}\cdot 2} \neg F(0), \neg G, n \notin N, F(n)$ for all $n \in \mathbb{N}$. Since $\tilde{k}\cdot 2 \dashrightarrow_9 \widetilde{k+2}$, we obtain $\vdash_1^{\widetilde{k+2}} A(i)^N$.

2. For any other axiom of PA the assertion is trivial.

3. $A(v) \equiv \forall x\,(\mathfrak{A}_u(B(\cdot, v), x) \rightarrow B(x, v)) \rightarrow \forall x\,(P_u^{\mathfrak{A}} x \rightarrow B(x, v))$, $u < v < \omega$.

Let $F(x) :\equiv B(x, i)^N$, $G :\equiv \forall x \in N\,(\mathfrak{A}_u^N(F, x) \rightarrow F(x))$, $k := |F(x)|$. Then $A(i)^N \equiv \neg G \vee \forall x\,(x \notin N \vee (\neg P_u^{\mathfrak{A}} x \vee F(x)))$ and by 4.13 $\vdash_1^{\tilde{k}+D_{u+1}0} \neg G, \neg P_u^{\mathfrak{A}} n, F(n)$, for all $n \in \mathbb{N}$. Since $D_{u+1}0 \lneqq_0 D_\nu 0 \ll_0 \tilde{k}$ and $\tilde{k}\cdot 2 \dashrightarrow_9 \widetilde{k+2}$, we get $\vdash_1^{\widetilde{k+2}} A(i)^N$.

4. $A(v) \equiv \forall y\,(\forall x\,(\mathfrak{A}_y(B(\cdot, y, v), x) \rightarrow B(x, y, v)) \rightarrow \forall x\,(P_y^{\mathfrak{A}} x \rightarrow B(x, y, v))$ and $\nu = \omega$.

Let $F_u(x) :\equiv B(x, u, i)^N$, $G_u :\equiv \forall x \in N\,(\mathfrak{A}_u^N(F_u, x) \rightarrow F_u(x))$, $k := |F_u(x)|$. Then $A(i)^N \equiv \forall y\,(y \notin N \vee (\neg G_y \vee \forall x\,(x \notin N \vee (\neg P_y^{\mathfrak{A}} x \vee F_y(x)))))$ and by 4.13 $\vdash_1^{\tilde{k}+D_{u+1}0} \neg G_u, \neg P_u^{\mathfrak{A}} n, F_u(n)$, for all $u, n \in \mathbb{N}$. Since $\tilde{k} + D_{u+1}0 = (\tilde{k} + D_\omega 0)[u] \ll_u \tilde{k} + D_\omega 0$, we obtain by ($\ll$) $\vdash_1^{\tilde{k}+D_\omega 0} u \notin N, \neg G_u, \neg P_u^{\mathfrak{A}} n, F_u(n)$. From this we get by ($\vee$), ($\forall^\infty$), ($\ll$) $\vdash_1^{\widetilde{k+2}} A(i)^N$, since $\tilde{k} + D_\omega 0 \ll_0 \tilde{k}\cdot 2 \dashrightarrow_9 \widetilde{k+2}$.

5. $A \equiv \forall y\, \forall x\,(\mathfrak{A}_y(P_y^{\mathfrak{A}}, x) \rightarrow P_y^{\mathfrak{A}} x)$.

Let $k := |\mathfrak{A}_y^N(P_y^{\mathfrak{A}}, x)|$. By (Ax2) we have $\vdash_0^{\tilde{k}} n \notin N, n \in N$. By 4.10 we have $\vdash_0^{\tilde{k}} \neg\mathfrak{A}_u^N(P_u^{\mathfrak{A}}, n), \mathfrak{A}_u^N(P_u^{\mathfrak{A}}, n)$. Hence $\vdash_0^{\tilde{k}\cdot 2} n \notin N, \neg\mathfrak{A}_u^N(P_u^{\mathfrak{A}}, n), n \in N \wedge \mathfrak{A}_u^N(P_u^{\mathfrak{A}}, n)$. Now we apply ($P_u^{\mathfrak{A}}$) and get $\vdash_0^{\tilde{k}\cdot 2+3} n \notin N, \neg\mathfrak{A}_u^N(P_u^{\mathfrak{A}}, n), P_u^{\mathfrak{A}} n$. Some applications of ($\vee$), ($\forall^\infty$) and ($\ll$) yield $\vdash_0^{\widetilde{k+3}} A^N$, since $\tilde{k}\cdot 2 + 3 \ll_2 \tilde{k}\cdot 3 \ll_2 \widetilde{k+1} \dashrightarrow_{12} \widetilde{k+3}$.

6. $A \equiv \forall y\, \forall x_0\, \forall x_1\,(P_{<y}^{\mathfrak{A}} x_0 x_1 \leftrightarrow x_0 < y \wedge P_{x_0}^{\mathfrak{A}} x_1)$: Left to the reader.

Proposition 2. *By* PL1 *we denote Tait's calculus for first-order predicate logic in the language* L_{ID} *(cf.* [8]*). If $\Gamma(v_1, \ldots, v_m)$ is derivable in* PL1, *then there exists* $k \in \mathbb{N}$ *such that* $\vdash_0^{\tilde{k}} i_1 \notin N, \ldots, i_m \notin N, \Gamma(i_1, \ldots, i_m)^N$ *for all* $i_1, \ldots, i_m \in \mathbb{N}$.

Proof. By induction on the derivation of Γ: Let $m = 1$.

1. $\Gamma \equiv \Gamma_0 \cup \{\neg A, A\}$: cf. 4.10.

2. If Γ is the conclusion of a ($\wedge$)- or ($\vee$)-inference, then the assertion follows immediately from the I.H.

3. $\Gamma(v) \equiv \Gamma_0(v), \forall x\, A(v, x)$ and $\mathrm{PL1} \vdash \Gamma(v), A(v, x)$ with $x \not\equiv v$: By I.H. there exists k such that $\vdash_0^{\tilde{k}} i \notin N, n \notin N, \Gamma(i)^N, A(i, n)^N$ for all $i, n \in \mathbb{N}$. Then by ($\vee$) and ($\forall^\infty$) we get $\vdash_0^{\widetilde{k+1}} i \notin N, \Gamma(i)^N$.

4. $\Gamma(v) \equiv \Gamma_0(v), \exists x\, A(v, x)$ and $\mathrm{PL1} \vdash \Gamma(v), A(v, t)$:

4.1. $t \equiv y\underbrace{'\cdots'}_{k_0}$ ($y \not\equiv v$) or $t \equiv 0\underbrace{'\cdots'}_{k_0}$: By I.H. there exists $k \geq k_0$ such that $\vdash_0^{\tilde{k}} i \notin N, 0 \notin N, \Gamma(i)^N, A(i, k_0)^N$ for all $i \in \mathbb{N}$. From this we get by 4.2(c) $\vdash_0^{\tilde{k}} i \notin N, \Gamma(i)^N, A(i, k_0)^N$. Since $k \geq k_0$, we have $\vdash_0^{\tilde{k}} k_0 \in N$. Hence by ($\wedge$) $\vdash_0^{\tilde{k}+1} i \notin N, \Gamma(i)^N, k_0 \in N \wedge A(i, k_0)^N$. An application of ($\exists$) yields $\vdash_0^{\tilde{k}+1} i \notin N, \Gamma(i)^N$.

4.2. $t \equiv v\underbrace{'\cdots'}_{k_0}$: By I.H. there exists $k \geqslant k_0$ such that $\vdash_0^{\bar{k}} i \notin N, \; \Gamma(i)^N, A(i, i\underbrace{'\cdots'}_{k_0})^N$ for all $i \in \mathbb{N}$. Since $k \geqslant k_0$, we have $\vdash_0^{\bar{k}} i \notin N, \; i\underbrace{'\cdots'}_{k_0} \in N$ for all $i \in \mathbb{N}$. Hence $\vdash_0^{\bar{k}+1} i \notin N, \; \Gamma(i)^N, \; i'^{\cdots\prime} \in N \wedge A(i, i'^{\cdots\prime})^N$. Now we apply ($\exists$) and get $\vdash_0^{\widetilde{k+1}} i \notin N, \; \Gamma(i)^N$.

Proof of 4.14. Suppose $\mathrm{ID}_\nu \vdash A$ (A closed). Then $\mathrm{PL1} \vdash \neg(A_1 \wedge \cdots \wedge A_n), A$ where every A_i is the universal closure of an axiom of ID_ν. By Propositions 1 and 2 there exists m such that $\vdash_1^{\bar{m}} (A_1 \wedge \cdots \wedge A_n)^N$ and $\vdash_0^{\bar{m}} \neg(A_1 \wedge \cdots \wedge A_n)^N, A^N$. By a cut with cut formula $(A_1 \wedge \cdots \wedge A_n)^N$ we obtain now $\vdash_k^{\bar{k}} A^N$ with $k := \max\{|(A_1 \wedge \cdots \wedge A_n)^N|, m\} + 1$.

Conclusion. By combining the Theorems 4.14, 4.5, 4.8 we obtain Theorem 4.0 which was stated at the beginning of this section.

Appendix: The proof-theoretic ordinal of ID_ν

Definitions. 1. By transfinite induction on a we define an ordinal $\mathrm{rk}(a)$ for every $a \in T_0$:

$$\mathrm{rk}(a) := \sup\{\mathrm{rk}(a[n]) + 1 : n \in \mathrm{dom}(a)\}.$$

2. By transfinite induction on $\alpha \in \mathrm{On}$ we define the sets $I_{\mathfrak{A}}^{\alpha}$ and $I_{\mathfrak{A}}^{<\alpha}$ for every positive operator form $\mathfrak{A}$:

$$I_{\mathfrak{A}}^{\alpha} := \{n \in \mathbb{N} : \mathfrak{A}_0(I_{\mathfrak{A}}^{<\alpha}, n) \text{ is true in the standard model}\},$$

$$I_{\mathfrak{A}}^{<\alpha} := \bigcup_{\xi<\alpha} I_{\mathfrak{A}}^{\xi}.$$

3. For $n \in \bigcup_{\alpha \in \mathrm{On}} I_{\mathfrak{A}}^{\alpha}$ we set $|n|_{\mathfrak{A}} := \min\{\alpha : n \in I_{\mathfrak{A}}^{\alpha}\}$.

4. $|\mathrm{ID}_\nu| := \sup\{|n|_{\mathfrak{A}} : \mathrm{ID}_\nu \vdash P_0^{\mathfrak{A}} n\}$. $|\mathrm{ID}_\nu|$ is called the proof-theoretic ordinal of ID_ν.

We will prove the following *result*:

$$|\mathrm{ID}_\nu| = \sup\{\mathrm{rk}(D_0 D_\nu^k 0) : k \in \mathbb{N}\} \qquad (\nu \leqslant \omega).$$

Definition. Let $\Gamma = \{A_1, \ldots, A_n\} \subseteq \mathrm{Pos}_0$:

$$\vDash^{\alpha} \Gamma \quad :\Leftrightarrow \quad \begin{cases} A_1 \vee \cdots \vee A_n \text{ is true in the standard model when} \\ P_0^{\mathfrak{A}}, P_{<0}^{\mathfrak{A}}, N \text{ are interpreted by } I_{\mathfrak{A}}^{<\alpha}, \emptyset, \mathbb{N} \text{ resp.} \end{cases}$$

A.1. Lemma. $\vdash_1^a \Gamma, \; \Gamma \subseteq \mathrm{Pos}_0, \; a \in T_0, \; \mathrm{rk}(a) \leqslant \alpha \Rightarrow \vDash^{\alpha} \Gamma$.

Proof. By transfinite induction on a:

1. If $\vdash_1^a \Gamma$ holds by (Ax1), then $\vDash^{\alpha} \Gamma$ for every α.

2. Suppose that $\vdash_1^a \Gamma$ holds by (Ax2). Then, since $\Gamma \subseteq \mathrm{Pos}_0$, we have $\Gamma = \Gamma_0, n \notin N, n \in N$ and thus $\vDash^{\alpha} \Gamma$ for every α.

3. If $\vdash_1^a \Gamma$ is the conclusion of a basic inference $(\mathscr{I})$, then $(\mathscr{I})$ is an inference $(\wedge)$, $(\vee)$, $(\forall^\infty)$, $(\exists)$ or (N), and the assertion follows immediately from the I.H.

4. Suppose $\vdash_1^b \Gamma$, $n \in N \wedge \mathfrak{A}_0^N(P_0^{\mathfrak{A}}, n)$ with $a = b + 1$ and $\Gamma = \Delta, P_0^{\mathfrak{A}} n$. Then $\beta := \mathrm{rk}(b) < \alpha$. By I.H. we get "$\vDash^\beta \Delta$ or $n \in I_{\mathfrak{A}}^{<\beta}$ or $\mathfrak{A}_0(I_{\mathfrak{A}}^{<\beta}, n)$" and from this "$\vDash^\alpha \Delta$ or $n \in I_{\mathfrak{A}}^{<\alpha}$", i.e., $\vDash^\alpha \Gamma$.

5. If $\vdash_1^a \Gamma$ is the conclusion of a cut, then the cut formula is of the kind $n \in N$, and the assertion follows immediately from the I.H.

6. If $\vdash_1^b \Gamma$ with $b \ll_\Gamma a$, then $\mathrm{rk}(b) < \mathrm{rk}(a) \leq \alpha$ and thus $\vDash^\alpha \Gamma$ by I.H.

From $a \in T_0$ it follows that $\vdash_1^a \Gamma$ cannot be the conclusion of an application of the Ω_{u+1}-rule.

A.2. Lemma. $|\mathrm{ID}_\nu| \leq \sup\{\mathrm{rk}(D_0 D_\nu^k 0) : k \in \mathbb{N}\}$.

Proof. Suppose $\mathrm{ID}_\nu \vdash P_0^{\mathfrak{A}} n$. Then by 4.14, 4.5, 4.6 we obtain $\vdash_1^{D_0 D_\nu^k 0} P_0^{\mathfrak{A}} n$, for some $k \in \mathbb{N}$. By A.1 this yields $n \in I_{\mathfrak{A}}^{<\alpha}$ with $\alpha := \mathrm{rk}(D_0 D_\nu^k 0)$. Hence $|n|_{\mathfrak{A}} < \mathrm{rk}(D_0 D_\nu^k 0)$.

A.3. Lemma. $\mathrm{Sup}\{\mathrm{rk}(D_0 D_\nu^k 0) : k \in \mathbb{N}\} \leq |\mathrm{ID}_\nu|$.

Proof. Here we make use of Theorem 2.2 which claims that "$a \in W_0$" is provable in ID_ν, for every $a \in T_0$ which contains no symbol D_υ with $\upsilon > \nu$. From this we get, for all $k \in \mathbb{N}$,

(1) $\mathrm{ID}_\nu \vdash P_0^{\mathfrak{A}} \ulcorner D_0 D_\nu^k 0 \urcorner$

where $a \mapsto \ulcorner a \urcorner$ is any reasonable Gödel numbering of the terms in T, and $\mathfrak{A}$ is a positive operator form which on the basis of this Gödel numbering formalizes the inductive definition of the sets W_υ $(\upsilon < \nu)$ in Section 2. Then we also have

(2) $|\ulcorner a \urcorner|_{\mathfrak{A}} = \mathrm{rk}(a)$, for all $a \in T_0$.

The assertion follows immediately from (1) and (2).

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