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A Continuous Time GARCH Process of Higher Order

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Abstract

A continuous time GARCH model of order (p, q) is introduced, which is driven by a single Lévy process. It extends many of the features of discrete time GARCH(p,q) processes to a continuous time setting. When $p = q = 1$, the process thus defined reduces to the COGARCH(1,1) process of Klüppelberg, Lindner and Maller (2004). We give sufficient conditions for the existence of stationary solutions and show that the volatility process has the same autocorrelation structure as a continuous time ARMA process. The autocorrelation of the squared increments of the process is also investigated, and conditions ensuring a positive volatility are discussed.

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1 Introduction

In financial econometrics, GARCH (generalised autoregressive conditionally heteroscedastic) processes are commonly used to model daily returns on stocks, currency investments and other assets. These models capture many of the so called *stylised features* of such data which include tail heaviness, volatility clustering and dependence without significant correlation. For GARCH processes with finite fourth moments, the autocorrelation functions (ACFs) of both the squared process and of the associated volatility process are those of ARMA processes. For the GARCH(1,1) process they both decay exponentially, but for higher-order GARCH processes they can exhibit damped oscillatory behaviour.

Various attempts have been made to capture these features with continuous-time models. One approach is to use the GARCH diffusion approximation of Nelson [24]. (See also Duan [14] and Drost and Werker [13].) These diffusion limits have the property of being driven by two independent Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$. For example, the GARCH(1,1) diffusion limit satisfies

$$dG_t = \sigma_t \, dW_t^{(1)}, \quad d\sigma_t^2 = \theta(\gamma - \sigma_t^2) + \rho \sigma_t^2 \, dW_t^{(2)}, \quad t \geq 0. \tag{1.1}$$

Another approach is via the stochastic volatility model of Barndorff-Nielsen and Shephard [3, 4]. In this model the volatility process $(\sigma_t^2)_{t\geq 0}$ is an Ornstein–Uhlenbeck (O-U) process driven by a subordinator (or a superposition of such processes), and thus exhibits jumps. The logarithm of the asset price G_t at time t is assumed to satisfy an equation of the form $dG_t = \mu dt + \sigma_t \, dW_t$, with W_t being a Brownian motion independent of the Lévy process. The autocorrelation function of the Lévy-driven O-U volatility process is restricted to functions of the form $\rho(h) = \exp(-c|h|)$ for some $c > 0$. By taking either superpositions of O-U type processes as in Barndorff–Nielsen [2], or by specifying the volatility to be a Lévy driven CARMA (continuous-time ARMA) process as in Brockwell [8], it is possible to obtain a much larger class of autocorrelation functions for the volatility process, some of which exhibit damped oscillatory behaviour. As in Nelson's diffusion limit, the price process $(G_t)_{t\geq 0}$ is again driven by two independent noise processes.

In Klüppelberg et al. [20], a different approach was taken, leading to the introduction of a continuous time GARCH(1,1) model, called COGARCH(1,1), which is driven by a single noise process only. To summarise their approach, recall that the discrete time GARCH(1,1) process $(\xi_n)_{n\in\mathbb{N}_0}$ is defined by the equations,

$$\xi_n = \varepsilon_n \sigma_n, \quad \sigma_n^2 = \alpha_0 + \alpha_1 \xi_{n-1}^2 + \beta_1 \sigma_{n-1}^2, \quad n \in \mathbb{N}, \tag{1.2}$$

where $\alpha_0, \alpha_1, \beta_1 > 0$ are parameters and $(\varepsilon_n)_{n\in\mathbb{N}_0}$ is a sequence of iid (independent and identically distributed) random variables. This recursion can be solved to give

$$\sigma_n^2 = \left(\sigma_0^2 + \alpha_0 \int_0^n \exp\left(-\sum_{j=0}^{\lfloor s \rfloor} \log(\beta_1 + \alpha_1 \varepsilon_j^2)\right) ds\right) \exp\left(\sum_{j=0}^{n-1} \log(\beta_1 + \alpha_1 \varepsilon_j^2)\right),$$

where $\lfloor s \rfloor$ denotes the integer part of $s \in \mathbb{R}$. The COGARCH(1,1) model is then motivated by replacing ε_n by the jumps $\Delta L_t = L_t - L_{t-}$ of a Lévy process. More precisely, observing that $\sum_{j=0}^{n-1} \log(\beta_1 + \alpha_1 \varepsilon_j^2) = n \log \beta_1 + \sum_{j=0}^{n-1} \log(1 + (\alpha_1/\beta_1)\varepsilon_j^2)$ for $\beta_1 > 0$, and writing η for $-\log \beta_1$, ω_0 for α_0 and ω_1 for α_1, the COGARCH(1,1) process $G = (G_t)_{t\geq 0}$ with left-continuous volatility process $(\sigma_t^2)_{t\geq 0}$, driven by the Lévy process $(L_t)_{t\geq 0}$, is defined as

$$dG_t = \sigma_t \, dL_t, \quad t > 0, \quad G_0 = 0, \tag{1.3}$$

$$\sigma_t^2 = \left(\sigma_0^2 + \omega_0 \int_0^t e^{X_s} \, ds\right) e^{-X_{t-}}, \quad t \geq 0, \tag{1.4}$$

where

$$X_t := \eta t - \sum_{0<s\leq t} \log\left(1 + \omega_1 e^{\eta} (\Delta L_s)^2\right). \tag{1.5}$$

Here, $\omega_0 > 0, \omega_1 \geq 0, \eta > 0$ and σ_0^2 is independent of $(L_t)_{t\geq 0}$. The COGARCH(1,1) process displays many of the features of the discrete time GARCH(1,1) process. As shown in Klüppelberg et al. [20, 21], the COGARCH(1,1) process has uncorrelated increments, but the autocorrelation function of the squared volatility $(\sigma_t^2)_{t\geq 0}$ as well as of the squared increments of G decay exponentially. Further, the COGARCH(1,1) process has heavy tails and its volatility clusters on high levels, see [21] and Fasen et al. [16]. While cluster behaviour can be also achieved in the aforementioned volatility model of Barndorff-Nielsen and Shephard if the driving Lévy process has regularly varying tails (see [16] or Fasen [15]), this is impossible for the GARCH diffusion (1.1). For an overview of extremes of stochastic volatility models, see [16]. Also, observe that many of the features of the COGARCH(1,1) process can be obtained in a more general setting, as in Lindner and Maller [22].

It is not clear how the approach outlined above leading to the COGARCH(1,1) process can be generalised to higher order GARCH processes in continuous time. In particular, the recursion corresponding to (1.2) cannot be solved easily and generalised to a continuous time setting. In this paper we adopt a different but related approach which allows us to define a continuous time GARCH process of order (p, q) with $1 \leq p \leq q$. The process is driven by a single Lévy process and, when $p = q = 1$, it reduces to the COGARCH(1,1) process. It will therefore be referred to as a COGARCH(p, q) process. While the COGARCH(1,1) process is restricted to have decreasing ACF, for higher orders this is not necessarily the case and we can obtain damped oscillatory behaviour. The paper is organised as follows:

In the next section, the COGARCH(p, q) process is defined, in such a way that the volatility process satisfies a CARMA type stochastic differential equation, which is driven by the quadratic covariation of the COGARCH process itself. This is directly motivated by the corresponding structure of the discrete GARCH(p, q) process. We then show that the process defined in this way reduces to the COGARCH(1,1) process if $p = q = 1$. Further definitions and notations used throughout the paper are given at the end of Section 2.

In Section 3, we give sufficient conditions for the existence of a strictly stationary COGARCH(p, q) volatility process. In the COGARCH(1,1) case, these are exactly the necessary and sufficient conditions obtained in [20]. The case where the driving Lévy process is compound Poisson is considered in detail. The proofs will rely on the fact that the state vector of the COGARCH(p,q) process satisfies a multivariate random recurrence equation when sampled at discrete times.

In Section 4 we focus on the autocorrelation structure of the volatility process. It turns out that the stationary COGARCH(p,q) process has the ACF of a CARMA process. This is completely analogous with the discrete-time GARCH, which has the ACF of an ARMA process.

Section 5 deals with conditions which ensure positivity of the volatility, while the autocorrelation structure of the squared increments of the COGARCH process itself is obtained in Section 6. Finally, an example with simulations is given in Section 7. In order not to disturb the flow of arguments, proofs are postponed to Sections 8 – 11.

2 Definition of the COGARCH(p,q) process

Let $(\varepsilon_n)_{n\in\mathbb{N}_0}$ be an iid sequence of random variables. Let $p, q \geq 0$. Then the GARCH(p,q) process $(Y_n)_{n\in\mathbb{N}_0}$ is defined by the equations,

$$\begin{aligned} \xi_n &= \sigma_n \varepsilon_n, \\ \sigma_n^2 &= \alpha_0 + \alpha_1 \xi_{n-1}^2 + \ldots + \alpha_p \xi_{n-p}^2 + \beta_1 \sigma_{n-1}^2 + \ldots + \beta_q \sigma_{n-q}^2, \quad n \geq s, \end{aligned} \tag{2.1}$$

where $s := \max(p, q)$, $\sigma_0^2, \ldots, \sigma_{s-1}^2$ are iid and independent of the iid sequence $(\varepsilon_n)_{n\geq s}$, and $\xi_n = G_{n+1} - G_n$ represents the increment at time n of the log asset price process $(G_n)_{n\in\mathbb{N}_0}$.

The squared volatility process $(\sigma_n^2)_{n\in\mathbb{N}_0}$ can thus be viewed as a "self-exciting" ARMA$(q, p-1)$ process driven by the noise sequence $(\sigma_{n-1}^2\varepsilon_{n-1}^2)_{n\in\mathbb{N}}$. Motivated by this observation, we will define a continuous time GARCH model for the log asset price process $(G_t)_{t\geq 0}$ of order (p, q) by

$$dG_t = \sigma_t \, dL_t, \quad t > 0, \quad G_0 = 0,$$

where $(\sigma_t^2)_{t\geq 0}$ is a CARMA$(q, p-1)$ process driven by a suitable replacement for the discrete time driving noise sequence $(\sigma_{n-1}^2\varepsilon_{n-1}^2)_{n\in\mathbb{N}}$. The state-space representation of a Lévy-driven CARMA$(q, p-1)$ process $(\psi_t)_{t\geq 0}$ with driving Lévy process L, location parameter c, moving average coefficients $\alpha_1, \ldots, \alpha_p$, autoregressive coefficients $\beta_1, \ldots, \beta_q$ and $q \geq p$ is (see Brockwell [9]),

$$\begin{aligned}
\psi_t &= c + \mathbf{a}'\zeta_t, \\
d\zeta_t &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -\beta_q & -\beta_{q-1} & -\beta_{q-2} & \cdots & -\beta_1 \end{bmatrix} \zeta_t dt + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} dL_t,
\end{aligned}$$

where $\mathbf{a}' = [\alpha_1, \ldots, \alpha_q]$ and $\alpha_j := 0$ for $j > q$. In order to obtain a continuous-time analog of the equation (2.1) we suppose that the volatility process $(\sigma_t^2)_{t\geq 0}$ has the state-space representation of a CARMA$(q, p-1)$ process in which the driving Lévy process (L_t) is replaced by a continuous-time analog of the driving process $(\sigma_{n-1}^2\varepsilon_{n-1}^2)_{n\in\mathbb{N}}$ in (2.1).

The increments of the driving process in continuous time should correspond to the increments of the discrete-time process,

$$R_n^{(d)} := \sum_{i=0}^{n-1} \xi_i^2 = \sum_{i=0}^{n-1} \sigma_i^2\varepsilon_i^2.$$

We therefore replace the innovations ε_n by the jumps ΔL_t of a Lévy process $(L_t)_{t\geq 0}$ to obtain the continuous-time analogue,

$$R_t := \sum_{0<s\leq t} \sigma_s^2 (\Delta L_s)^2, \quad t > 0.$$

If L has no Gaussian part (i.e. $\tau_L^2 = 0$ in (2.2) below), we recognise R as the quadratic covariation of G, i.e.

$$R_t = \sum_{0<s\leq t} \sigma_s^2 (\Delta L_s)^2 = \int_0^t \sigma_s^2 d[L, L]_s = [G, G]_t.$$

If L has a Gaussian part, then still $\sum_{0<s\leq t} (\Delta L_s)^2 = [L, L]^{(d)}$, the *discrete part of the quadratic covariation*, and we have in general

$$R_t = \int_0^t \sigma_s^2 d[L, L]_s^{(d)}, \quad \text{i.e.} \quad dR_t = \sigma_t^2 \, d[L, L]_t^{(d)}.$$

Recall that for a Lévy process $L = (L_t)_{t\geq 0}$ the characteristic function of L_t can be written in the form

$$E\left(e^{i\theta L_t}\right) = \exp\left(t\left(i\gamma_L\theta - \tau_L^2\frac{\theta^2}{2} + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \mathbf{1}_{|x|\leq 1}\right) d\nu_L(x)\right)\right), \quad \theta \in \mathbb{R}. \quad (2.2)$$

The constants $\gamma_L \in \mathbb{R}$, $\tau_L^2 \geq 0$ and the measure ν_L on $\mathbb{R}$ form the *characteristic triplet* of L. As usual, the Lévy measure ν_L is required to satisfy $\int_{\mathbb{R}} \min(1, x^2)\, d\nu_L(x) < \infty$. For more information on Lévy processes we refer to the books by Applebaum [1], Bertoin [5] or Sato [26].

We now define the COGARCH(p,q) process by specifying the volatility process $(V_t)_{t\geq 0}$, the analogue of the discrete-time process $(\sigma_n^2)_{n\in\mathbb{N}_0}$, to be a self-exciting *continuous-time* ARMA process driven by the process $(R_t)_{t\geq 0}$ defined above. As we shall see, when $p = q = 1$ the resulting process coincides with the COGARCH(1,1) process of Klüppelberg et al. [20]. (The parameters $\beta_1, \dots, \beta_q$ and $\alpha_1, \dots, \alpha_p$ in the following definition should not be confused with the parameters denoted by the same symbols in the defining equation (2.1) of the discrete-time GARCH process.)

Definition 2.1 Let $1 \leq p \leq q$, $\alpha_0 > 0$, $\alpha_1, \dots, \alpha_p \in \mathbb{R}$, $\beta_1, \dots, \beta_q \in \mathbb{R}$, $\alpha_p \neq 0$, $\beta_q \neq 0$, and $\alpha_{p+1} = \dots = \alpha_q = 0$. Define the $(q \times q)$–matrix B and the vectors $\mathbf{a}$ and $\mathbf{e}$ by

$$B = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\beta_q & -\beta_{q-1} & -\beta_{q-2} & \dots & -\beta_1 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{q-1} \\ \alpha_q \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(if $q = 1$, then $B = -\beta_1$). Let $L = (L_t)_{t\geq 0}$ be a Lévy process with non-trivial Lévy measure and define the *volatility process* $V = (V_t)_{t\geq 0}$ of a COGARCH(p,q) process (with matrix B, vector $\mathbf{a}$, scaling parameter α_0 and driving $(L_t)_{t\geq 0}$) by

$$V_t = \alpha_0 + \mathbf{a}'\mathbf{Y}_{t-}, \quad t > 0, \quad V_0 = \alpha_0 + \mathbf{a}'\mathbf{Y}_0,$$

where $(\mathbf{Y}_t)_{t\geq 0}$ is the unique càdlàg solution of the stochastic differential equation

$$d\mathbf{Y}_t = B\mathbf{Y}_t\, dt + \mathbf{e}V_t\, d[L, L]_t^{(d)}, \quad t > 0, \tag{2.3}$$

with initial value $\mathbf{Y}_0$, independent of the driving Lévy process $(L_t)_{t\geq 0}$. If the process $(V_t)_{t\geq 0}$ is non-negative almost surely, the *COGARCH*(p,q) *process* $G = (G_t)_{t\geq 0}$ is given by

$$dG_t = \sqrt{V_t}\, dL_t, \quad t > 0, \quad G_0 = 0.$$

That there is in fact a unique solution of (2.3) for any starting random variable $\mathbf{Y}_0$ follows from standard theorems on stochastic differential equations (e.g. Protter [25], Chapter V, Theorem 7). The stochastic integrals are interpreted with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, which is defined to be the smallest filtration satisfying the "usual hypotheses" such that $(L_t)_{t\geq 0}$ is adapted and $\mathbf{Y}_0$ is $\mathcal{F}_0$ measurable.

Without restrictions on α_0, $\mathbf{a}$ and B, the process $(V_t)_{t\geq 0}$ is not necessarily non-negative, as required for the definition of $(G_t)_{t\geq 0}$ to make sense. Conditions which ensure that V_t is non-negative will be discussed in Section 5. In particular, it will be shown that if $\mathbf{a}' e^{Bt}\mathbf{e} \geq 0$ for all $t \geq 0$ and $\mathbf{Y}_0$ is such that $\alpha_0 + \mathbf{a}' e^{Bt}\mathbf{Y}_0 \geq 0$ for all $t \geq 0$, then $V_t \geq 0$ as well. But even if V_t has negative values, it is still of some interest in its own right and many of the results we obtain for the process $(V_t)_{t\geq 0}$ will be valid without the non-negativity restriction.

We next show that if $p = q = 1$, then the COGARCH process just defined is indeed the COGARCH(1,1) process as defined by Klüppelberg et al. [20].

Theorem 2.2 *Suppose that $p = q = 1$, and that $\alpha_0, \alpha_1, \beta_1 > 0$. Then the processes $(G_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ of Definition 2.1, driven by the Lévy process $(L_t)_{t\geq 0}$, are exactly the COGARCH(1,1) process and its volatility process $(\sigma_t^2)_{t\geq 0}$ driven by $(L_t)_{t\geq 0}$ as given by* (1.3) – (1.5)*, with parameters $\omega_0 = \alpha_0\beta_1$, $\omega_1 = \alpha_1 e^{-\beta_1}$ and $\eta = \beta_1$.*

Proof. From $d\mathbf{Y}_t = -\beta_1\mathbf{Y}_t\,dt + V_t\,d[L,L]_t^{(d)}$ and $V_{t+} = \alpha_0 + \alpha_1\mathbf{Y}_t$ follows that

$$dV_{t+} = \alpha_1\,d\mathbf{Y}_t = -\alpha_1\beta_1\,\frac{V_{t+} - \alpha_0}{\alpha_1}\,dt + \alpha_1 V_t\,d[L,L]_t^{(d)},$$

and hence that

$$V_{t+} = \alpha_0\beta_1 t - \beta_1\int_0^t V_s\,ds + \alpha_1\sum_{0<s\leq t} V_s(\Delta L_s)^2 + V_0.$$

But this equation is also satisfied by the volatility process $(\sigma_t^2)_{t\geq 0}$ of (1.4) when $\omega_0 = \alpha_0\beta_1$, $\eta = \beta_1$ and $\omega_1 = \alpha_1 e^{-\beta_1}$, as shown in Proposition 3.2 of [20], and uniqueness of the solution gives the claim. □

We conclude this section with a few definitions and some notation which will be used throughout the paper.

Definition 2.3 Let $\mathbf{a}$ and B be as in Definition 2.1. Then the *characteristic polynomials* associated with $\mathbf{a}$ and B are given by

$$\begin{aligned} a(z) &:= \alpha_1 + \alpha_2 z + \ldots + \alpha_p z^{p-1}, \quad z \in \mathbb{C}, \\ b(z) &:= z^q + \beta_1 z^{q-1} + \ldots + \beta_q, \quad z \in \mathbb{C}. \end{aligned}$$

The eigenvalues of the matrix B (which are exactly the zeroes of b) will be denoted by $\lambda_1, \ldots, \lambda_q$ and assumed to be ordered in such a way that

$$\Re\lambda_q \leq \Re\lambda_{q-1} \leq \ldots \leq \Re\lambda_1$$

(where $\Re\lambda_i$ denotes the real part of λ_i). Further, define

$$\lambda := \lambda(B) := \Re\lambda_1.$$

For the rest of the paper, convergence in probability will be denoted by "$\xrightarrow{P}$", uniform convergence on compacts in probability by "ucp", or in symbols "$\xrightarrow{ucp}$", and equality in distribution by "$\stackrel{d}{=}$". For $x \in \mathbb{R}$ we denote $\log^+(x) = \log(\max\{1, x\})$. The transpose of a coloumn vector $\mathbf{c} \in \mathbb{C}^q$ will be denoted by $\mathbf{c}'$. If $\|\cdot\|$ is a vector norm in $\mathbb{C}^q$, then the natural matrix norm for a $(q \times q)$-matrix C will also be denoted by $\|C\|$ and is given by $\|C\| = \sup_{\mathbf{c} \in \mathbb{C}^q \setminus \{0\}} \frac{\|C\mathbf{c}\|}{\|\mathbf{c}\|}$. Correspondingly, for $r \in [1, \infty]$ we denote by $\|\cdot\|_r$ both the vector L^r-norm and the associated natural matrix norm. Recall that the natural matrix norms of the L^1, L^2 and L^∞ vector norms are the coloumn-sum norm, the spectral norm and the row-sum norm, respectively.

The $(q \times q)$–identity matrix will be denoted by I_q or simply I, and the i^{th} canonical vector $(0, \ldots, 0, 1, 0, \ldots, 0)'$ in $\mathbb{C}^q$, where the 1 is on the i^{th} position, by $\mathbf{e}_i$. For $\mathbf{e}_q$ we simply write $\mathbf{e}$. By $\operatorname{diag}(\lambda_1, \ldots, \lambda_q)$ we mean a diagonal $(q \times q)$-matrix with these entries on the diagonal. The Kronecker product of two $(q \times q)$–matrices A and B will be denoted by $A \otimes B$, and by $\operatorname{vec}(A)$ we denote the coloumn vector in $\mathbb{C}^{q^2}$ which arises from A by stacking the coloumns of A in a vector (starting with the first coloumn). For the properties of the Kronecker product we refer to Lütkepohl [23].

3 Stationarity conditions

Throughout this section, we fix the parameters B, $\mathbf{a}$ and α_0, and $(V_t)_{t \geq 0}$ and $(\mathbf{Y}_t)_{t \geq 0}$ will be the processes as defined in Definition 2.1. In the next theorem we shall give sufficient conditions implying the existence of a strictly stationary solution of the COGARCH volatility. Rather than trying to obtain the most general conditions possible, we will concentrate on conditions which are easy to check and still give room to many interesting examples. We restrict ourselves to the case where B can be diagonalised. Since any eigenvector to the eigenvalue λ_i must necessarily be a multiple constant of $[1, \lambda_i, \lambda_i^2, \ldots, \lambda_i^{q-1}]'$, we see that B can be diagonalised if and only if all the eigenvalues of B are distinct. Let S be a matrix such that $S^{-1}BS$ is a diagonal matrix. For example, a possible choice for S is given by

$$S = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_q \\ \vdots & \cdots & \vdots \\ \lambda_1^{q-1} & \cdots & \lambda_q^{q-1} \end{bmatrix}, \tag{3.1}$$

so that

$$S^{-1}BS = \operatorname{diag}(\lambda_1, \ldots, \lambda_q).$$

Theorem 3.1 *Let $(\mathbf{Y}_t)_{t \geq 0}$ be the state vector process of the COGARCH(p, q) process with parameters B, $\mathbf{a}$ and α_0. Suppose that all the eigenvalues of B are distinct. Let L be a*

Lévy process with non-trivial Lévy measure ν_L, and suppose there is some $r \in [1, \infty]$ such that

$$\int_{\mathbb{R}} \log(1 + \|S^{-1}\mathbf{e}\mathbf{a}'S\|_r \, y^2) \, d\nu_L(y) < -\lambda = -\lambda(B), \tag{3.2}$$

where S is such that $S^{-1}BS$ is diagonal. Then $\mathbf{Y}_t$ converges in distribution to a finite random variable $\mathbf{Y}_\infty$, as $t \to \infty$. In particular, if $\mathbf{Y}_0 \stackrel{d}{=} \mathbf{Y}_\infty$, then $(\mathbf{Y}_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ are strictly stationary.

Remark 3.2 *(a)* If $(V_t)_{t\geq 0}$ is the volatility of a COGARCH(1,1) process with parameters $B = -\beta_1 < 0$, $\alpha_0, \alpha_1 > 0$, then $\|S^{-1}\mathbf{e}\mathbf{a}'S\|_r = \alpha_1$ and (3.2) is exactly the necessary and sufficient condition for the COGARCH(1,1) volatility process to be strictly stationary, as shown in [20], Theorem 3.1.
(b) For general $q \geq 2$, the quantity $\|S^{-1}\mathbf{e}\mathbf{a}S\|_r$ depends on the specific choice of S and on r. Observe that it is sufficient to find some S and some r such that (3.2) holds. □

The proof of Theorem 3.1 will make heavy use of the general theory of multivariate random recurrence equations, as discussed for example in Bougerol and Picard [7] or Kesten [19], or Brandt [6] in the one–dimensional case. The COGARCH state vector satisfies such a multivariate random recurrence equation, which is the contents of the next theorem:

Theorem 3.3 *Let $(\mathbf{Y}_t)_{t\geq 0}$ be the state vector process of the COGARCH(p, q) process with parameters B, $\mathbf{a}$ and α_0, and driving Lévy process L. Then there exists a family $(J_{s,t}, \mathbf{K}_{s,t})_{0\leq s\leq t}$ of random $(q \times q)$–matrices $J_{s,t}$ and random vectors $\mathbf{K}_{s,t}$ in $\mathbb{R}^q$ such that*

$$\mathbf{Y}_t = J_{s,t}\mathbf{Y}_s + \mathbf{K}_{s,t}, \quad 0 \leq s \leq t. \tag{3.3}$$

Further, the distribution of $(J_{s,t}, \mathbf{K}_{s,t})$ depends only on $t - s$, and $(J_{s_1,t_1}, \mathbf{K}_{s_1,t_1})$ and $(J_{s_2,t_2}, \mathbf{K}_{s_2.t_2)}$ are independent for $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$. For $0 \leq s \leq u \leq t$ it holds

$$J_{s,t} = J_{u,t} \, J_{s,u}. \tag{3.4}$$

If additionally the conditions of Theorem 3.1 hold, then the distribution of the vector $\mathbf{Y}_\infty$ is for any $h > 0$ the unique solution of the random fixed point equation

$$\mathbf{Y}_\infty \stackrel{d}{=} J_{0,h}\mathbf{Y}_\infty + \mathbf{K}_{0,h}, \tag{3.5}$$

with $\mathbf{Y}_\infty$ independent of $(J_{0,h}, \mathbf{K}_{0,h})$ on the right hand side of (3.5).

Remark 3.4 *(a)* We have concentrated on giving the feasible stationarity condition (3.2) which is easy to check. Actually, as the proof of Theorems 3.1 and 3.3 shows, it would be sufficient to a find a vector norm $\|\cdot\|$ and $t_0 > 0$ such that for J_{0,t_0} and $\mathbf{K}_{0,t_0}$ it holds

$$E \log \|J_{0,t_0}\| < 0 \quad \text{and} \quad E \log^+ \|\mathbf{K}_{0,t_0}\| < \infty. \tag{3.6}$$

By (3.4), $E\log\|J_{0,t_0}\| < 0$ is equivalent to the requirement that there is $t_1 > 0$ such that the Lyapunov exponent of the iid sequence $(J_{t_1 n, t_1(n+1)})_{n\in\mathbb{N}_0}$, which is given by

$$\lim_{n\to\infty}\frac{1}{n}E\left(\log\|J_{t_1(n-1),t_1 n}\cdots J_{0,t_1}\|\right) = \inf_{n\in\mathbb{N}}\left(\frac{1}{n}E\left(\log\|J_{t_1(n-1),t_1 n}\cdots J_{0,t_1}\|\right)\right)$$

and independent of the specific norm, is strictly negative. As shown by Bougerol and Picard [7], provided that $E\log^+\|J_{0,t_1}\| < \infty$, $E\log^+\|\mathbf{K}_{0,t_1}\| < \infty$ and a suitable irreducibility condition holds, then strict negativity of the Lyapunov exponent is not only sufficient but also necessary for the existence of stationary solutions of such random recurrence equations. We shall not go into further details.
(b) The conditions of Theorem 3.1 are sufficient conditions which imply (3.6). The specific vector norm used is given by

$$\|\mathbf{c}\|_{B,r} := \|S^{-1}\,\mathbf{c}\|_r, \quad \mathbf{c}\in\mathbb{C}^q, \tag{3.7}$$

so that the associated natural matrix norm of any $(q\times q)$-matrix A is $\|A\|_{B,r} = \|S^{-1}AS\|_r$. Observe, however, that the conditions of Theorem 3.1 are in general not necessary, in particular one can also find stationary solutions if there are multiple eigenvalues. For example, using similar methods as in the proof of Theorems 3.1 and 3.3, it can be shown that for any vector norm $\|\cdot\|$ it holds

$$\|J_{0,t}\| \le \|e^{Bt}\| + e^{\|B\|t}\exp\left(\sum_{0<s\le t}\log\left(1+(\Delta L_s)^2\|\mathbf{e}\mathbf{a}'\|\right)\right)\|\mathbf{e}\mathbf{a}'\|\sum_{0<s\le t}(\Delta L_s)^2, \quad t\ge 0.$$

Now if $\lambda(B) < 0$, then $\|e^{Bt}\|\to 0$ as $t\to\infty$, and (3.6) can be fulfilled without assuming that all the eigenvalues of B are distinct, but choosing $\|\mathbf{a}\|$ sufficiently small and imposing certain integrability conditions on L. We shall not pursue this further here but concentrate on the feasible conditions of Theorem 3.1. □

The matrices $J_{s,t}$ and the vector $\mathbf{K}_{s,t}$ of Theorem 3.3 will be constructed explicitly when L is compound Poisson, and in the general case will be obtained as the limit of the corresponding quantities for compound Poisson driven processes. We shall pay special attention to the compound Poisson process. In particular, we show that in that case the stationary state vector satisfies a distributional fixed point equation which is much easier to handle than (3.5). Also, we compare the stationary distribution of $\mathbf{Y}_\infty$ with the stationary distribution of the state vector when sampled at the jump times of the Lévy process. This is the contents of the next theorem:

Theorem 3.5 (a) *Let $(\mathbf{Y}_t)_{t\ge 0}$ be the state vector process of a COGARCH(p,q) process with parameters B, $\mathbf{a}$ and α_0. Suppose that the Lévy measure ν_L of the driving Lévy process*

L is finite. Let the compound Poisson process $[L,L]^{(d)}$ have representation

$$[L,L]_t^{(d)} = \sum_{0<s\le t} (\Delta L_s)^2 = \sum_{i=1}^{N(t)} Z_i,$$

where the sojourn times $(T_i)_{i\in\mathbb{N}}$ are iid exponentially distributed with parameter $\nu_L(\mathbb{R})$, and the iid sequence $(Z_i)_{i\in\mathbb{N}}$ describes the jump sizes. Further, let (T_0, Z_0) be independent of $(T_i, Z_i)_{i\in\mathbb{N}}$ and have the same distribution as (T_1, Z_1). For $i \in \mathbb{N}_0$, let

$$\begin{aligned} C_i &= (I + Z_i \mathbf{e}\mathbf{a}')e^{BT_i}, \\ \mathbf{D}_i &= \alpha_0 Z_i \mathbf{e}, \end{aligned}$$

and let $\Gamma_n := \sum_{i=1}^n T_i$, $n \in \mathbb{N}_0$, be the time of the n^{th} jump. Then the discrete time process $(\mathbf{Y}_{\Gamma_n})_{n\in\mathbb{N}_0}$ satisfies the random recurrence equation

$$\mathbf{Y}_{\Gamma_{n+1}} = C_{n+1}\mathbf{Y}_{\Gamma_n} + \mathbf{D}_{n+1}, \quad n \in \mathbb{N}_0. \tag{3.8}$$

Further, for any $t > 0$,

$$\begin{aligned} \mathbf{Y}_t &= e^{B(t-\Gamma_{N(t)})}\left(\operatorname{sgn}(N(t))\mathbf{D}_{N(t)} + \sum_{i=0}^{N(t)-2} C_{N(t)}\cdots C_{N(t)-i}\mathbf{D}_{N(t)-i-1} + C_{N(t)}\cdots C_1\mathbf{Y}_0\right) \\ &\stackrel{d}{=} e^{B(t-\Gamma_{N(t)})}\left(\operatorname{sgn}(N(t))\,\mathbf{D}_1 + \sum_{i=1}^{N(t)-1} C_1\cdots C_i\mathbf{D}_{i+1} + C_1\cdots C_{N(t)}\mathbf{Y}_0\right). \end{aligned} \tag{3.9}$$

(b) *Assume additionally that the assumptions of Theorem 3.1 are satisfied. Then the infinite sum $\sum_{i=0}^\infty C_1\cdots C_i\mathbf{D}_{i+1}$ converges almost surely absolutely to a random vector $\widehat{\mathbf{Y}}$, which has the stationary distribution of the sequence $(\mathbf{Y}_{\Gamma_n})_{n\in\mathbb{N}_0}$. The stationary state vector $\mathbf{Y}_\infty$ satisfies*

$$\mathbf{Y}_\infty \stackrel{d}{=} e^{BT}\widehat{\mathbf{Y}}, \tag{3.10}$$

where T is independent of $(T_i, Z_i)_{i\in\mathbb{N}_0}$ and has distribution T_1. Further, $\mathbf{Y}_\infty$ is the unique solution in distribution of the distributional fixed point equation

$$\mathbf{Y}_\infty \stackrel{d}{=} Q\mathbf{Y}_\infty + \mathbf{R}, \tag{3.11}$$

where $\mathbf{Y}_\infty$ is independent of $(Q, \mathbf{R})$ on the right hand side of (3.11) *and Q and $\mathbf{R}$ are defined by*

$$\begin{aligned} Q &:= e^{BT_0}(I + Z_0\mathbf{e}\mathbf{a}'), \\ \mathbf{R} &:= \alpha_0\, Z_0\, e^{BT_0}\mathbf{e}. \end{aligned}$$

The fixed point equation (3.11) will play a crucial role for the determination of the covariance matrix of the stationary $\mathbf{Y}_\infty$ as studied in the next section.

4 Second order behaviour of the volatility

In the whole section, $(\mathbf{Y}_t)_{t\geq 0}$ will denote the state vector process of a COGARCH(p,q) volatility V_t with parameters B, $\mathbf{a}$ and α_0 and driving Lévy process L with Lévy measure ν_L. The aim of this section is to study the autocorrelation function of $(V_t)_{t\geq 0}$. Throughout, we will write

$$\mu := \int_{\mathbb{R}} y^2 \, d\nu_L(y) \quad \text{and} \quad \rho := \int_{\mathbb{R}} y^4 \, d\nu_L(y),$$

provided the quantities are finite. Further, provided $\mu < \infty$, we shall always denote

$$\widetilde{B} := B + \mu \mathbf{e}\mathbf{a}'. \tag{4.1}$$

Observe that $\widetilde{B}$ has the same form as B, but with last row given by $(-\beta_q + \mu\alpha_1, \ldots, -\beta_1 + \mu\alpha_q)$. We first give sufficient conditions for the moments of $\mathbf{Y}_t$ to exist:

Proposition 4.1 *Suppose that all eigenvalues of B are distinct and that $\lambda = \lambda(B) < 0$. Let $\|\cdot\|$ denote any vector norm in $\mathbb{C}^q$ and let $k \in \mathbb{N}$. Then holds:*
(a) If $E|L_1|^{2k} < \infty$ and $E\|\mathbf{Y}_0\|^k < \infty$, then

$$E\|\mathbf{Y}_t\|^k < \infty \quad \forall\, t \geq 0.$$

(b) If $E|L_1|^{2k} < \infty$ and there are a matrix S such that $S^{-1}BS$ is diagonal and some $r \in [1,\infty]$ such that

$$\int_{\mathbb{R}} \left((1 + \|S^{-1}\mathbf{e}\mathbf{a}'S\|_r \, y^2)^k - 1 \right) d\nu_L(y) < -\lambda k,$$

then (3.2) holds with the same S and r, and $E\|\mathbf{Y}_\infty\|^k < \infty$. In particular, $E(\mathbf{Y}_\infty)$ exists if

$$EL_1^2 < \infty \quad \text{and} \quad \|S^{-1}\mathbf{e}\mathbf{a}'S\|_r \, \mu < -\lambda, \tag{4.2}$$

and the covariance matrix $\operatorname{cov}(\mathbf{Y}_\infty)$ exists if

$$EL_1^4 < \infty \quad \text{and} \quad \|S^{-1}\mathbf{e}\mathbf{a}'S\|_r^2 \, \rho < 2(-\lambda - \|S^{-1}\mathbf{e}\mathbf{a}'S\|_r \mu). \tag{4.3}$$

Further, (4.3) implies (4.2), and (4.2) implies that all eigenvalues of $\widetilde{B}$ have negative real parts, in particular $\widetilde{B}$ is invertible, i.e. $\beta_q \neq \alpha_1 \mu$.

Next, we obtain the ACF of the (non-stationary) volatility process.

Theorem 4.2 *Let $(V_t)_{t\geq 0}$ be the COGARCH(p,q) volatility process with state vector process $(\mathbf{Y}_t)_{t\geq 0}$ and parameters B, $\mathbf{a}$ and α_0. Assume that $EL_1^4 < \infty$ and that $E\|\mathbf{Y}_t\|^2 < \infty$ $\forall\, t \geq 0$ (this holds for example if the conditions of Proposition 4.1 are satisfied). Then, with $\widetilde{B}$ as defined in (4.1), it holds*

$$\operatorname{cov}(V_{t+h}, V_t) = \mathbf{a}' e^{\widetilde{B}h} \operatorname{cov}(\mathbf{Y}_t)\, \mathbf{a}, \quad t, h \geq 0. \tag{4.4}$$

Equation (4.4) shows the qualitative behaviour of the ACF of V as a function in h. However, since we are mainly interested in the case when $(\mathbf{Y}_t)_{t\geq 0}$ is stationary, it seems desirable to have a formula for $\mathrm{cov}\,(\mathbf{Y}_\infty)$, too. This will be the topic of Theorem 4.4, but first we give a formula for $E(\mathbf{Y}_\infty)$.

Lemma 4.3 *Suppose that all the eigenvalues of B are distinct and that (4.2) holds. Then*

$$E(\mathbf{Y}_\infty) \quad = \quad -\alpha_0\mu\widetilde{B}^{-1}\mathbf{e} = \frac{\alpha_0\mu}{\beta_q-\alpha_1\mu}\mathbf{e}_1. \tag{4.5}$$

Now we are able to present the main theorem of this section. In particular, it shows that the ACF of the stationary COGARCH volatility is identical to the ACF of a CARMA process, however, with a different matrix. This is very much the same situation as for the volatility of the discrete GARCH, which is that of an ARMA process.

Theorem 4.4 *Suppose that all eigenvalues of the matrix B are distinct, that $\lambda(B)<0$ and that (4.3) holds. Then the matrix $(I\otimes\widetilde{B})+(\widetilde{B}\otimes I)+\rho((\mathbf{e}\mathbf{a}')\otimes(\mathbf{e}\mathbf{a}'))$ is invertible, and the covariance matrix of $\mathbf{Y}_\infty$ is the unique solution of*

$$\Big[(I\otimes\widetilde{B})+(\widetilde{B}\otimes I)+\rho((\mathbf{e}\mathbf{a}')\otimes(\mathbf{e}\mathbf{a}'))\Big]\ \mathrm{vec}\,(\mathrm{cov}(\mathbf{Y}_\infty)) = \frac{-\alpha_0^2\beta_q^2\rho}{(\beta_q-\mu\alpha_1)^2}\ \mathrm{vec}\,(\mathbf{e}\mathbf{e}'). \tag{4.6}$$

Denote by $(\zeta_t)_{t\geq 0}$ the state vector process of a CARMA$(q,p-1)$ process $(\psi_t)_{t\geq 0}$ with location parameter 0, moving average coefficients $\alpha_1,\dots,\alpha_p$, autoregressive coefficients $\beta_1-\mu\alpha_q,\beta_2-\mu\alpha_{q-1},\dots,\beta_q-\alpha_1\mu$ (corresponding to the matrix $\widetilde{B}$), and suppose it is driven by a Lévy process $\widetilde{L}$ such that $E(\widetilde{L}_1)=\mu$ and $\mathrm{var}\,(\widetilde{L}_1)=\rho$. Further, define

$$m := \rho\int_0^\infty \mathbf{a}'e^{\widetilde{B}t}\mathbf{e}\mathbf{e}'e^{\widetilde{B}'t}\mathbf{a}\,dt = \mathrm{var}\,(\psi_\infty).$$

Then $0\leq m<1$, and

$$\begin{aligned}
\mathrm{cov}\,(\mathbf{Y}_\infty) &= \frac{\alpha_0^2\beta_q^2}{(\beta_q-\mu\alpha_1)^2(1-m)}\,\mathrm{cov}\,(\zeta_\infty)\\
&= \frac{\alpha_0^2\beta_q^2\rho}{(\beta_q-\mu\alpha_1)^2(1-m)}\int_0^\infty e^{\widetilde{B}t}\mathbf{e}\mathbf{e}'e^{\widetilde{B}'t}\,dt, \qquad (4.7)\\
\mathrm{var}\,(V_\infty) &= \frac{\alpha_0^2\beta_q^2}{(\beta_q-\mu\alpha_1)^2}\,\frac{m}{1-m}, \qquad (4.8)\\
E(V_\infty) &= \frac{\alpha_0\beta_q}{\beta_q-\mu\alpha_1}, \qquad (4.9)\\
E(\psi_\infty) &= \frac{\alpha_1\mu}{\beta_q-\mu\alpha_1}. \qquad (4.10)
\end{aligned}$$

If $(V_t)_{t\ge0}$ *and* $(\psi_t)_{t\ge0}$ *are the stationary versions of the COGARCH volatility and CARMA process, respectively, then*

$$\operatorname{cov}(V_{t+h},V_t)=\frac{\alpha_0^2\beta_q^2}{(\beta_q-\mu\alpha_1)^2(1-m)}\operatorname{cov}(\psi_{t+h},\psi_t),\quad t,h\ge0. \tag{4.11}$$

In particular, the autocorrelation functions of the stationary V *and the stationary* ψ *are the same. If furthermore the eigenvalues* $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_q$ *of* $\widetilde{B}$ *are distinct, and* $a(z)$ *and* $\widetilde{b}(z)$ *denote the characteristic polynomials associated with* $\mathbf{a}$ *and* $\widetilde{B}$*, then*

$$\operatorname{cov}(V_{t+h},V_t)=\frac{\alpha_0^2\beta_q^2\rho}{(\beta_q-\mu\alpha_1)^2(1-m)}\sum_{j=1}^{q}\frac{a(\widetilde{\lambda}_j)\,a(-\widetilde{\lambda}_j)}{\widetilde{b}'(\widetilde{\lambda}_j)\,\widetilde{b}(-\widetilde{\lambda}_j)}\,e^{\widetilde{\lambda}_j h},\quad t,h\ge0, \tag{4.12}$$

where $\widetilde{b}'$ *denotes the derivative of* $\widetilde{b}$*.*

5 Positivity conditions for the volatility

In order for the definition of the COGARCH price process $dG_t=\sqrt{V_t}\,dt$ to make sense it is necessary that V_t is non-negative for all $t\ge0$. The following Theorem now gives necessary and sufficient conditions when this happens:

Theorem 5.1 (a) *Let* $(\mathbf{Y}_t)_{t\ge0}$ *be the state vector of a COGARCH(p,q) volatility process* $(V_t)_{t\ge0}$ *with parameters* B, $\mathbf{a}$ *and* $\alpha_0>0$*. Let* $\gamma\ge-\alpha_0$ *be a real constant. Suppose that the following two conditions hold:*

$$\mathbf{a}'e^{Bt}\mathbf{e}\ \ge\ 0\quad\forall\,t\ge0, \tag{5.1}$$

$$\mathbf{a}'e^{Bt}\mathbf{Y}_0\ \ge\ \gamma\quad a.s.\quad\forall\,t\ge0. \tag{5.2}$$

Then, whatever the driving Lévy process is, it holds almost surely

$$V_t\ge\alpha_0+\gamma\ge0\quad\forall\,t\ge0. \tag{5.3}$$

Conversely, if either (5.2) *fails, or* (5.2) *holds with* $\gamma>-\alpha_0$ *and* (5.1) *fails, then there exists a driving compound Poisson process* L *and* $t_0\ge0$ *such that* $P(V_{t_0}<0)>0$*.*
(b) *Suppose that all the eigenvalues of* B *are distinct and that* (3.2) *and* (5.1) *both hold. Then the stationary version* $(V_t)_{t\ge0}$ *of the COGARCH(p,q) volatility process satisfies almost surely*

$$V_t\ge\alpha_0>0\quad\forall\,t\ge0.$$

Since we are mainly interested in stationary solutions or in processes started with $\mathbf{Y}_0=\mathbf{0}$, the condition to check is (5.1). The question when (5.1) holds arises similarly for

CARMA processes, and some results in this direction have been recently obtained by Tsai and Chan [29]. We state their result in the next theorem in the language of COGARCH rather than CARMA processes. Statement (e) below has also been obtained by Todorov and Tauchen [28]. Recall that a function ϕ on $(0,\infty)$ is called *completely monotone* if it possesses derivatives of all orders and satisfies $(-1)^n \frac{d^n\phi}{dt^n}(t) \geq 0$ for all $t > 0$ and all $n \in \mathbb{N}_0$.

Theorem 5.2 *Let B and $\mathbf{a}$ be the parameters of a COGARCH(p, q) process. Suppose that $\lambda(B) < 0$ and $\alpha_1 > 0$. Then the following holds:*
(a) For the COGARCH(p, q) process, equation (5.1) holds if and only the ratio of the characteristic polynomials $a(\cdot)/b(\cdot)$ is completely monotone on $(0,\infty)$.
(b) A sufficient condition for (5.1) to hold for the COGARCH(1, q) process is that either (i) all eigenvalues of B are real and negative, or
(ii) if $(\lambda_{i_1}, \lambda_{i_1+1}), \ldots, (\lambda_{i_r}, \lambda_{i_r+1})$ is a partition of the set of all pairs of complex conjugate eigenvalues of B (counted with multiplicity), then there exists an injective mapping $u : \{1, \ldots, r\} \to \{1, \ldots, n\}$ such that $\lambda_{u(j)}$ is a real eigenvalue of B satisfying $\lambda_{u(j)} \geq \Re(\lambda_{i_j})$.
(c) A necessary condition for (5.1) to hold for the COGARCH(1, q) process is that there exists a real eigenvalue of B not smaller than the real part of all other eigenvalues of B.
(d) Suppose $2 \leq p \leq q$, that all eigenvalues of B are negative and ordered as in Definition 2.3, and that the roots γ_j of $a(z) = 0$ are negative and ordered such that $\gamma_{p-1} \leq \ldots \leq \gamma_1 < 0$. Then a sufficient condition for (5.1) to hold for the COGARCH(p, q) process is that

$$\sum_{i=1}^{k} \gamma_i \leq \sum_{i=1}^{k} \lambda_i \quad \forall\, k \in \{1, \ldots, p-1\}.$$

(e) A necessary and sufficient condition for (5.1) to hold for the COGARCH(2, 2) process is that both eigenvalues of B are real, that $\alpha_2 \geq 0$ and that $\alpha_1 \geq -\alpha_2 \lambda(B)$.

Although characterisation (a) may be difficult to check in general, it gives a method of constructing further pairs of $\mathbf{a}$'s and B's, since the product of two completely monotone functions is again completely monotone.

6 The autocorrelation of the squared increments

In Section 4 we have investigated the behaviour of the autocorrelation function of the volatility process. However, it is also important to know something about the second order properties of the increments of the COGARCH process itself. In order to do that, suppose that $V_t \geq 0$ almost surely for all $t \geq 0$, and define for $r > 0$,

$$G_t^{(r)} := G_{t+r} - G_t = \int_{(t,t+r]} \sqrt{V_s}\, dL_s, \quad t \geq 0.$$

We shall restrict ourselves to the case of stationary $(\mathbf{Y}_t)_{t\geq 0}$ such that (5.1) holds. It is easy to see that in that case, also $(G_t^{(r)})_{t\geq 0}$ is a stationary process. Let μ and $\widetilde{B}$ be defined as in Section 4. We then have:

Theorem 6.1 *Let B, $\mathbf{a}$ and α_0 be parameters of a COGARCH(p,q) process such that* (5.1) *holds and such that all the eigenvalues of B are distinct. Assume that the driving Lévy process $(L_t)_{t\geq 0}$ has no Gaussian part and that $EL_1 = 0$. Further assume* (4.2)*. Let $(V_t)_{t\geq 0}$ be the stationary volatility process. Then for any $t \geq 0$ and $h \geq r > 0$,*

$$E(G_t^{(r)}) = 0, \tag{6.1}$$

$$E((G_t^{(r)})^2) = \frac{\alpha_0 \beta_q r}{\beta_q - \mu\alpha_1} E(L_1^2), \tag{6.2}$$

$$\operatorname{cov}(G_t^{(r)}, G_{t+h}^{(r)}) = 0. \tag{6.3}$$

Assume further that (4.3) *holds. Then*

$$\operatorname{cov}((G_t^{(r)})^2, (G_{t+h}^{(r)})^2) = \mathbf{a}' e^{\widetilde{B}h} \mathbf{H}_r, \quad h \geq r, \tag{6.4}$$

where

$$\mathbf{H}_r := E(L_1^2)\, \widetilde{B}^{-1}(I - e^{-\widetilde{B}r}) \operatorname{cov}(\mathbf{Y}_r, G_r^2)$$

(here, $\operatorname{cov}(\mathbf{Y}_r, G_r^2) = E(\mathbf{Y}_r G_r^2) - E(\mathbf{Y}_r)E(G_r^2)$*).*

Equation (6.4) describes the behaviour of the autocovariance function of the squared increments as a function of h. It can be seen that it has a similar structure like the autocovariance function of a CARMA process with matrix $\widetilde{B}$. In particular, if $\widetilde{B}$ has a pair of complex conjugate eigenvalues, then the autocorrelation function of $(G_t^{(r)})^2$ should show damped oscillatory behaviour.

7 An Example

In this section we consider the COGARCH(1,3) process driven by a compound Poisson process with jump-rate 2 and normally distributed jumps with mean zero and variance 0.74. The COGARCH coefficients are $\alpha_0 = \alpha_1 = 1, \beta_1 = 1, \beta_2 = .48 + \pi^2$, and $\beta_3 = .64 + .4\pi^2$, from which we find that the eigenvalues of B are $-.4$, $-.4 + \pi i$ and $-.4 - \pi i$. With S defined as in (3.1), $||S^{-1}\mathbf{e}\mathbf{a}'S||_2 = 0.21493$ and it is easy to check from this that the conditions (4.2) and (4.3) are satisfied. Condition (c) of Theorem 5.2 also implies that the volatility process is non-negative.

The eigenvalues of the matrix $\widetilde{B} = B + \mu \mathbf{e}\mathbf{a}'$ are $-.25038$, $-.47481 + 3.14426i$ and $-.47481 - 3.14426i$. From (4.12) we conclude that the autocorrelation of the volatility in this case is a linear combination of $\exp(-.25038t)$ and a damped sinusoid with period close to 2 and damping factor $\exp(-.47481t)$.

The top graph in Figure 1 shows the values at integer times $1, \ldots, 8000$ of a simulated series $(G_t)_{0<t<1000000}$ with the parameters specified above. The second graph shows the differenced series $(G_{t+1} - G_t)_{t=0,\ldots,7999}$ and the last graph shows the volatility sequence $(\sigma_t^2)_{t=1,\ldots,8000}$.

As is the case for a discrete-time GARCH process, the increments $(G_{t+1} - G_t)$ exhibit no significant correlation, but the squared increments $((G_{t+1}-G_t)^2)$ have highly significant correlations as shown in the second graph of Figure 2. The first graph in Figure 2 shows the sample autocorrelation function of the volatility process at integer lags. This too is highly significant for large lags, reflecting the long-memory property frequently observed in financial time series. As expected from the remarks in the first paragraph above, it has the form of an exponentially decaying term plus a small damped sinusoidal term with period approximately equal to two.

8 Proofs for Section 3

We start by proving Theorem 3.5, since equation (3.9) will be needed in the proof of Theorems 3.1 and 3.3.

Proof of Theorem 3.5. (a) It follows from (2.3) that $\mathbf{Y}_t$ satisfies $d\mathbf{Y}_t = B\mathbf{Y}_t\,dt$ for $t \in [\Gamma_n, \Gamma_{n+1})$, so that

$$\mathbf{Y}_t = e^{B(t-\Gamma_n)}\mathbf{Y}_{\Gamma_n}, \quad t \in [\Gamma_n, \Gamma_{n+1}), \quad n \in \mathbb{N}_0. \tag{8.1}$$

At time Γ_{n+1} a jump of size $\mathbf{e}(\alpha_0 + \mathbf{a}'\mathbf{Y}_{\Gamma_{n+1}-})Z_{n+1}$ occurs, so that

$$\begin{aligned}
\mathbf{Y}_{\Gamma_{n+1}} &= \mathbf{Y}_{\Gamma_{n+1}-} + \mathbf{e}(\alpha_0 + \mathbf{a}'\mathbf{Y}_{\Gamma_{n+1}-})Z_{n+1} \\
&= (I + Z_{n+1}\mathbf{e}\mathbf{a}')\mathbf{Y}_{\Gamma_{n+1}-} + \alpha_0 Z_{n+1}\mathbf{e} \\
&= C_{n+1}\mathbf{Y}_{\Gamma_n} + \mathbf{D}_{n+1}, \quad n \in \mathbb{N}_0,
\end{aligned}$$

which is (3.8). Solving this recursion gives

$$\mathbf{Y}_{\Gamma_n} = \mathbf{D}_n + \sum_{i=0}^{n-2} C_n \cdots C_{n-i}\mathbf{D}_{n-i-1} + C_n \cdots C_1 \mathbf{Y}_0, \quad n \in \mathbb{N},$$

and the first equality in (3.9) follows from this and $\mathbf{Y}_t = e^{B(t-\Gamma_{N(t)})}\mathbf{Y}_{\Gamma_{N(t)}}$. The second equality in (3.9) is a consequence of the fact that the infinite random element $(N(t), \Gamma_{N(t)}, C_{N(t)}, \mathbf{D}_{N(t)}, \ldots, C_1, \mathbf{D}_1, 0, 0, \ldots)$ has the same distribution as $(N(t), \Gamma_{N(t)}, C_1, \mathbf{D}_1, \ldots, C_{N(t)}, \mathbf{D}_{N(t)}, 0, 0, \ldots)$; indeed, for any $n \in \mathbb{N}_0$ and $c \geq 0$ the random vectors $(C_1, \mathbf{D}_1), \ldots, (C_n, \mathbf{D}_n)$ are iid and depend on the restriction $\{N(t) = n,\ \Gamma_{N(t)} \geq c\}$ only in terms of $\sum_{i=1}^n T_i$ and T_{n+1}, but not on the T_i, $i = 1, \ldots, n$, individually.

(*b*) Let S be such that $S^{-1}BS =: \Lambda$ is diagonal and define the vector norm $\|\mathbf{c}\|_{B,r} = \|S^{-1}\mathbf{c}\|_r$ as in equation (3.7), so that the associated natural matrix norm is $\|A\|_{B,r} = \|S^{-1}AS\|_r$. Then we have for $t \geq 0$,

$$\left\|e^{Bt}\right\|_{B,r} = \left\|S\, e^{\Lambda t}\, S^{-1}\right\|_{B,r} = \left\|e^{\Lambda t}\right\|_r = e^{\lambda t}. \tag{8.2}$$

This gives $\|C_1\|_{B,r} \leq (1 + Z_1\|\mathbf{ea}'\|_{B,r})e^{\lambda T_1}$ and $\|\mathbf{D}_1\|_{B,r} = \alpha_0\|\mathbf{e}\|_{B,r}\, Z_1$, so that, using $\nu_{[L,L]}([x,\infty)) = \nu_L(\{y \in \mathbb{R} : |y| \geq \sqrt{x}\}$ for $x \geq 0$,

$$\begin{aligned} E\log\|C_1\|_{B,r} &\leq \lambda\, E(T_1) + E\log(1 + Z_1\|\mathbf{ea}'\|_{B,r}) \\ &= \frac{\lambda}{\nu_L(\mathbb{R})} + \frac{1}{\nu_L(\mathbb{R})}\int_{(0,\infty)} \log(1 + \|\mathbf{ea}'\|_{B,r}\, y^2)\, d\nu_L(y) < 0 \end{aligned}$$

by (3.2) and

$$E\log^+(Z_1) = \frac{1}{\nu_L(\mathbb{R})}\int_{\mathbb{R}} \log^+(y^2)\, d\nu_L(y) < \infty.$$

From the general theory of random recurrence equations this implies the almost sure absolute convergence of $\sum_{i=0}^\infty C_1\cdots C_i\mathbf{D}_{i+1}$ to $\widehat{\mathbf{Y}}$ which has the stationary distribution of $(\mathbf{Y}_{\Gamma_n})_{n\in\mathbb{N}}$, see e.g. Bougerol and Picard [7].

To prove (3.10), for $m \in \mathbb{N}$ let

$$\widehat{\mathbf{Y}}_m := \sum_{i=0}^{m-1} C_1\cdots C_i D_{i+1} + C_1\cdots C_m\mathbf{Y}_0,$$

and

$$\mathbf{Y}_{t,m} := e^{B(t-\Gamma_{N(t)})}\widehat{\mathbf{Y}}_m, \quad t \geq 0.$$

Since the random variable $(t - \Gamma_{N(t)})$ is asymptotically independent of $T_1, Z_1, \ldots, T_m, Z_m$ (for $t \to \infty$, m fixed), it follows that $e^{B(t-\Gamma_{N(t)})}$ is asymptotically independent of $\widehat{\mathbf{Y}}_m$, and hence $\mathbf{Y}_{t,m}$ converges in distribution to $e^{BT}\widehat{\mathbf{Y}}_m$, as $t \to \infty$, where T is exponentially distributed with parameter $\nu_L(\mathbb{R})$ (e.g. Taylor and Karlin [27], Section 7.4.4) and independent of $T_1, Z_1, \ldots, T_m, Z_m$ and hence can be chosen to be independent of $(T_i)_{i\in\mathbb{N}}$, $(Y_i)_{i\in\mathbb{N}}$ (as in the statement of the theorem). Moreover, $e^{BT}\widehat{\mathbf{Y}}_m$ converges almost surely, hence in distribution to $e^{BT}\widehat{\mathbf{Y}}$, as $m \to \infty$. Denote by $\widetilde{\mathbf{Y}}_t$ the expression in the lower line of (3.9). Then (3.10), and in particular the existence of the limit variable $\mathbf{Y}_\infty$ in the compound

Poisson case, follow from (3.9) and a variant of Slutsky's Theorem (e.g. Brockwell and Davis [11], Proposition 6.3.9), provided

$$\lim_{m\to\infty}\limsup_{t\to\infty} P(\|\widetilde{\mathbf{Y}}_t - \mathbf{Y}_{t,m}\|_{B,r} > \varepsilon) = 0, \quad \forall\, \varepsilon > 0. \tag{8.3}$$

Since $\|e^{B(t-\Gamma_{N(t)})}\|_{B,r} \le 1$, and $\mathrm{sgn}(N(t))\mathbf{D}_1 + \sum_{i=1}^{N(t)-1} C_1\cdots C_i\mathbf{D}_{i+1} + C_1\cdots C_{N(t)}\mathbf{Y}_0 - \widehat{\mathbf{Y}}_m$ converges almost surely, hence in probability as $t\to\infty$ to $\sum_{i=m}^{\infty} C_1\cdots C_i\mathbf{D}_{i+1} - C_1\cdots C_m\mathbf{Y}_0$, which itself converges almost surely to 0 as $m\to\infty$, (8.3) is true and (3.10) follows. That $\mathbf{Y}_\infty$ satisfies (3.11) is clear from (3.10), and that it is the unique solution follows from $E\log\|Q\|_{B,r} < 0$ and $E\log^+\|\mathbf{R}\|_{B,r} < \infty$. □

The proof of Theorem 3.5 (b) already showed the existence of the limit variable $\mathbf{Y}_\infty$ for the case of a driving compound Poisson process. Nevertheless, this existence will be reestablished in the proof of Theorems 3.1 and 3.3 for the general case, making use of Theorem 3.5 (a) only. We shall use an approximation argument and introduce the following notation:

Definition 8.1 Let L be a Lévy process. Then for any $\varepsilon > 0$, the *$\sqrt{\varepsilon}$-cut Lévy process* $(L_t^{(\varepsilon)})_{t\ge0}$ is defined by

$$L_t^{(\varepsilon)} := \sum_{0<s\le t, |\Delta L_s|\ge\sqrt{\varepsilon}} |\Delta L_s|, \quad t \ge 0.$$

If $(\mathbf{Y}_t)_{t\ge0}$ is a state vector process of a COGARCH(p,q) process driven by L, then the COGARCH(p,q) process with the same parameters and starting vector but driving Lévy process $(L_t^{(\varepsilon)})_{t\ge0}$ will be denoted by $(\mathbf{Y}_t^{(\varepsilon)})_{t\ge0}$.

The quadratic covariation of $L^{(\varepsilon)}$ is given by

$$[L^{(\varepsilon)}, L^{(\varepsilon)}]_t = [L^{(\varepsilon)}, L^{(\varepsilon)}]_t^{(d)} = \sum_{0<s\le t, |\Delta L_s|^2\ge\varepsilon} |\Delta L_s|^2.$$

In particular, the corresponding COGARCH volatility results in being driven by a compound Poisson process. With this notation, we have the following lemma:

Lemma 8.2 *Let $(\mathbf{Y}_t)_{t\ge0}$ be the state vector process of a COGARCH(p,q) process. Then $\mathbf{Y}_t^{(\varepsilon)}$ converges in ucp to $\mathbf{Y}_t$, as $\varepsilon\to0$.*

Proof of Lemma 8.2. This is an easy consequence of perturbation results in stochastic differential equations: recalling the definition of prelocal convergence in $\underline{\underline{H}}^p$, $1\le p\le\infty$, as in Protter [25], page 260, it is easy to see that $[L^{(\varepsilon)}, L^{(\varepsilon)}]$ converges prelocally to $[L, L]^{(d)}$ in $\underline{\underline{H}}^p$, $1\le p\le\infty$, as $\varepsilon\to0$ (for example, with stopping times as in the proof of [25], Theorem 4 of Chapter V, page 247). The claim then follows from Theorems 14 and 15 of Chapter V in [25]. □

Proof of Theorems 3.1 and 3.3. We shall first concentrate on (3.3) and (3.4) and then prove Theorem 3.1 and the rest of Theorem 3.3 simultaneously. Let $\varepsilon > 0$, and assume the representation

$$[L^{(\varepsilon)}, L^{(\varepsilon)}]_t = \sum_{i=1}^{\Gamma^{(\varepsilon)}_{N_\varepsilon(t)}} Z_i^{(\varepsilon)},$$

where $L^{(\varepsilon)}$ is the $\sqrt{\varepsilon}$–cut Lévy process of Definition 8.1. Define $C_i^{(\varepsilon)}$ and $\mathbf{D}_i^{(\varepsilon)}$ similarly as in Theorem 3.5. Further, let

$$\begin{aligned} J_{0,t}^{(\varepsilon)} &:= e^{B(t-\Gamma^{(\varepsilon)}_{N_\varepsilon(t)})} C^{(\varepsilon)}_{N_\varepsilon(t)} \cdots C_1^{(\varepsilon)}, \\ \mathbf{K}_{0,t}^{(\varepsilon)} &:= e^{B(t-\Gamma^{(\varepsilon)}_{N_\varepsilon(t)})} \left(\operatorname{sgn}(N_\varepsilon(t)) \mathbf{D}^{(\varepsilon)}_{N_\varepsilon(t)} + \sum_{i=0}^{N_\varepsilon(t)-2} C^{(\varepsilon)}_{N_\varepsilon(t)} \cdots C^{(\varepsilon)}_{N_\varepsilon(t)-i} \mathbf{D}^{(\varepsilon)}_{N_\varepsilon(t)-i-1} \right). \end{aligned}$$

Then, by Theorem 3.5 (a),

$$\mathbf{Y}_t^{(\varepsilon)} = J_{0,t}^{(\varepsilon)} \mathbf{Y}_0 + \mathbf{K}_{0,t}^{(\varepsilon)}. \tag{8.4}$$

From the previous Lemma we know that $\mathbf{Y}_t^{(\varepsilon)}$ converges in ucp to $\mathbf{Y}_t$ as $\varepsilon \to 0$. Since this is true for any starting value $\mathbf{Y}_0$, it holds in particular for $\mathbf{Y}_0 = 0$, and from (8.4) follows that $\mathbf{K}_{0,t}^{(\varepsilon)}$ converges in ucp to some $\mathbf{K}_{0,t}$, as $\varepsilon \to 0$. Hence, again from (8.4) follows that for arbitrary $\mathbf{Y}_0$,

$$J_{0,t}^{(\varepsilon)} \mathbf{Y}_0 = \mathbf{Y}_t^{(\varepsilon)} - \mathbf{K}_{0,t}^{(\varepsilon)} \stackrel{ucp}{\to} \mathbf{Y}_t - \mathbf{K}_{0,t}, \quad \text{as} \quad \varepsilon \to 0.$$

Since this holds for arbitrary $\mathbf{Y}_0$, we conclude that $J_{0,t}^{(\varepsilon)}$ converges in ucp to some $J_{0,t}$ as $\varepsilon \to 0$. From (8.4) then follows

$$\mathbf{Y}_t = J_{0,t} \mathbf{Y}_0 + \mathbf{K}_{0,t}.$$

By starting at an arbitrary time s instead of at time 0, we obtain (3.3). For example, $J_{s,t}^{(\varepsilon)}$ is given by

$$J_{s,t}^{(\varepsilon)} = e^{B(t-\Gamma^{(\varepsilon)}_{N_\varepsilon(t)})} C_{N_\varepsilon(t)} \dots C_{N_\varepsilon(s)+2} (I + Z_{N_\varepsilon(s)+1} \mathbf{e}\mathbf{a}') e^{B(\Gamma^{(\varepsilon)}_{N_\varepsilon(s)+1} - s)}, \quad 0 \le s \le t,$$

giving (3.4). The independence and stationarity assertions on $(J_{s,t}, \mathbf{K}_{s,t})$ are clear, since $J_{s,t}$ and $\mathbf{K}_{s,t}$ are constructed only from the segment $(L_u)_{s<u\le t}$ of the Lévy process L.

Now assume that all eigenvalues of B are distinct and that (3.2) holds. Applying (8.2) to $J_{0,t}^{(\varepsilon)}$ gives

$$\begin{aligned}
\|J_{0,t}^{(\varepsilon)}\|_{B,r} &\le \left\|e^{B(t-\Gamma^{(\varepsilon)}_{N_\varepsilon(t)})}\right\|_{B,r}\left\|C^{(\varepsilon)}_{N_\varepsilon(t)}\right\|_{B,r}\cdots\left\|C_1^{(\varepsilon)}\right\|_{B,r} \\
&\le e^{\lambda(t-\Gamma^{(\varepsilon)}_{N_\varepsilon(t)})}\prod_{i=1}^{N_\varepsilon(t)}\left((1+Z_i^{(\varepsilon)}\,\|\mathbf{ea}'\|_{B,r})e^{\lambda(\Gamma_i^{(\varepsilon)}-\Gamma_{i-1}^{(\varepsilon)})}\right) \\
&= e^{\lambda t}\exp\left(\sum_{i=1}^{N_\varepsilon(t)}\log(1+Z_i^{(\varepsilon)}\,\|S^{-1}\mathbf{ea}'S\|_r)\right) \qquad (8.5) \\
&\le e^{\lambda t}\exp\left(\sum_{0<s\le t}\log(1+(\Delta L_s)^2\|S^{-1}\mathbf{ea}'S\|_r)\right). \qquad (8.6)
\end{aligned}$$

Since $\|J_{0,t}\|_{B,r}\le\limsup_{\varepsilon\to 0}\|J_{0,t}^{(\varepsilon)}\|_{B,r}$, we conclude that

$$\log\|J_{0,t}\|_{B,r}\le\lambda t+\sum_{0<s\le t}\log(1+(\Delta L_s)^2\|S^{-1}\mathbf{ea}'S\|_r), \qquad (8.7)$$

giving

$$E\log\|J_{0,t}\|_{B,r}\le t\left(\lambda+\int_{\mathbb{R}}\log(1+\|S\mathbf{ea}'S^{-1}\|y^2)\,d\nu_L(y)\right)<0$$

by (3.2) (see e.g. Protter [25], Chapter I, Theorems 36 and 38). This is the left hand inequality of (3.6). To show that $E\log^+\|\mathbf{K}_{0,t}\|_{B,r}<\infty$, observe that

$$\begin{aligned}
&\|\mathbf{K}_{0,t}^{(\varepsilon)}\|_{B,r} \\
&\le e^{\lambda(t-\Gamma^{(\varepsilon)}_{N_\varepsilon(t)})}\operatorname{sgn}(N_\varepsilon(t))\,\alpha_0\|\mathbf{e}\|_{B,r}\,Z^{(\varepsilon)}_{N_\varepsilon(t)} \\
&\quad+\alpha_0\|\mathbf{e}\|_{B,r}\sum_{i=0}^{N_\varepsilon(t)-2}e^{\lambda(t-\Gamma^{(\varepsilon)}_{N_\varepsilon(t)-i-1})}\left(1+Z^{(\varepsilon)}_{N_\varepsilon(t)}\|\mathbf{ea}'\|_{B,r}\right)\cdots\left(1+Z^{(\varepsilon)}_{N_\varepsilon(t)-i}\|\mathbf{ea}'\|_{B,r}\right) \\
&\qquad\qquad\times Z^{(\varepsilon)}_{N_\varepsilon(t)-i-1} \qquad (8.8) \\
&\le \alpha_0\|\mathbf{e}\|_{B,r}\operatorname{sgn}(N_\varepsilon(t))Z^{(\varepsilon)}_{N_\varepsilon(t)} \\
&\quad+\alpha_0\|\mathbf{e}\|_{B,r}\sum_{i=0}^{N_\varepsilon(t)-2}\exp\left(\sum_{0<s\le t}\log(1+(\Delta L_s)^2\|\mathbf{ea}'\|_{B,r})\right)Z^{(\varepsilon)}_{N_\varepsilon(t)-i-1} \\
&\le \alpha_0\|S^{-1}\mathbf{e}\|_r\exp\left(\sum_{0<s\le t}\log(1+(\Delta L_s)^2\|S^{-1}\mathbf{ea}'S\|_r)\right)\sum_{0<s\le t}(\Delta L_s)^2. \qquad (8.9)
\end{aligned}$$

From this follows that

$$\log\|\mathbf{K}_{0,t}\|_{B,r}\le\log(\alpha_0\|S^{-1}\mathbf{e}\|_r)+\sum_{0<s\le t}\log(1+(\Delta L_s)^2\|S^{-1}\mathbf{ea}'S\|_r)+\log\,[L,L]_t^{(d)}.$$

The expectation of the second summand is finite as shown above, and $E(\log[L,L]_t^{(d)}) < \infty$ since $\int_{(1,\infty)} \log x \, d\nu_{[L,L]}(x) = \int_{\mathbb{R}\setminus[-1,1]} \log x^2 \, d\nu_L(x) < \infty$, showing the right hand inequality of (3.6).

Let $(J_n, \mathbf{K}_n)_{n\in\mathbb{N}}$ be an iid sequence with distribution $(J_{0,1}, \mathbf{K}_{0,1})$, independent of L and $\mathbf{Y}_0$. Let $\gamma \in [0,1)$ and $n \in \mathbb{N}$. Then it follows from (3.3) that

$$\begin{aligned} \mathbf{Y}_{n+\gamma} &= \mathbf{K}_{n+\gamma-1,n+\gamma} + \sum_{i=0}^{n-2} J_{n+\gamma-1,n+\gamma} \cdots J_{n+\gamma-i-1,n+\gamma-i} \mathbf{K}_{n+\gamma-i-2,n+\gamma-i-1} \\ &\quad + J_{n+\gamma-1,n+\gamma} \cdots J_{\gamma,\gamma+1} \mathbf{Y}_\gamma \\ &\stackrel{d}{=} \mathbf{K}_1 + \sum_{i=1}^{n-1} J_1 \cdots J_i \, \mathbf{K}_{i+1} + J_1 \cdots J_n \mathbf{Y}_\gamma \\ &=: \mathbf{G}_n + H_n \mathbf{Y}_\gamma, \quad \text{say.} \end{aligned}$$

Since $E \log \|J_1\|_{B,r} < 0$ and $E \log^+ \|\mathbf{K}_1\|_{B,r} < \infty$, it follows from the general theory of random recurrence equations (e.g. Bougerol and Picard [7]) that H_n converges almost surely to 0 as $n \to \infty$ and that $\mathbf{G}_n$ converges almost surely absolutely to some random vector $\mathbf{G}$, as $n \to \infty$. Since $\mathbf{Y}$ has càdlàg paths, it follows that $\sup_{\gamma\in[0,1)} \|\mathbf{Y}_\gamma\|_{B,r}$ is almost surely finite. Hence

$$\lim_{n\to\infty} \sup_{\gamma\in[0,1)} \|H_n \mathbf{Y}_\gamma\|_{B,r} = 0 \quad a.s.,$$

and it follows that $\mathbf{Y}_t$ converges in distribution to $\mathbf{Y}_\infty := \mathbf{G}$ as $t \to \infty$. That $\mathbf{Y}_\infty$ satisfies (3.5) and is the unique solution is clear by the theory of random recurrence equations. Equations (3.5) and (3.3) then imply that if $\mathbf{Y}_0 \stackrel{d}{=} \mathbf{Y}_\infty$, then $\mathbf{Y}_t \stackrel{d}{=} \mathbf{Y}_\infty$ for all $t > 0$, showing strict stationarity of $(\mathbf{Y}_t)_{t\geq 0}$ since it is a Markov process. □

9 Proofs for Section 4

In order to prove Proposition 4.1, we will show that the state vector process $(\mathbf{Y}_t)_{t\geq 0}$ can be majorised by the state vector process of a COGARCH(1,1) process, for which we can apply the moment conditions of [20]. We further show that under the conditions of Theorem 3.1, the stationary distribution $\mathbf{Y}_\infty$ can be approximated by stationary distributions of compound Poisson driven COGARCH processes, and that there is a majorant for this approximation. This will allow to restrict attention to compound Poisson driven processes when calculating autocorrelations, the general case following from Lebesgue's dominated convergence theorem. This is the contents of the next lemma:

Lemma 9.1 *Let $(\mathbf{Y}_t)_{t\geq 0}$ be the state vector process of a COGARCH(p,q) process with parameters B, $\mathbf{a}$ and $\alpha_0 > 0$ such that all eigenvalues of B are distinct and that $\lambda =$*

$\lambda(B) < 0$. Let $r \in [1, \infty]$, S such that $S^{-1}BS$ is diagonal, and denote by $\|\cdot\|_{B,r}$ the vector norm defined in (3.7). Further, denote by $(\overline{\mathbf{Y}}_t)_{t\geq 0}$ the state vector process of a COGARCH(1,1) process with (1×1)–matrix λ, vector $\|\mathbf{ea}'\|_{B,r} \in \mathbb{R}^1$, scaling parameter $\alpha_0\|\mathbf{e}\|_{B,r} > 0$ and initial state vector $\overline{\mathbf{Y}}_0 := \|\mathbf{Y}_0\|_{B,r}$. Then

$$\|\mathbf{Y}_t\|_{B,r} \leq \overline{\mathbf{Y}}_t, \quad t \geq 0. \tag{9.1}$$

If (3.2) is satisfied for this r, then there exist versions of $\mathbf{Y}_\infty$ and $\overline{\mathbf{Y}}_\infty$ such that

$$\|\mathbf{Y}_\infty\|_{B,r} \leq \overline{\mathbf{Y}}_\infty. \tag{9.2}$$

Further, if $(\mathbf{Y}_t^{(\varepsilon)})_{t\geq 0}$ is the process defined in Definition 8.1 for $\varepsilon > 0$, then versions of $\mathbf{Y}_\infty^{(\varepsilon)}$ can be chosen such that $\|\mathbf{Y}_\infty^{(\varepsilon)}\|_{B,r} \leq \overline{\mathbf{Y}}_\infty$ for all $\varepsilon > 0$ and $\mathbf{Y}_\infty^{(\varepsilon)} \xrightarrow{P} \mathbf{Y}_\infty$, as $\varepsilon \to 0$.

Proof of Lemma 9.1. We use the notations and setup of the proof of Theorems 3.1 and 3.3. Let $\varepsilon > 0$ and define a COGARCH(1,1) state vector process $\overline{\mathbf{Y}}^{(\varepsilon)}$ similarly as above (with respect to $\mathbf{Y}^{(\varepsilon)}$). Let $\overline{J}_{0,t}^{(\varepsilon)}$ and $\overline{\mathbf{K}}_{0,t}^{(\varepsilon)}$ be defined similarly as $J_{0,t}^{(\varepsilon)}$ and $\mathbf{K}_{0,t}^{(\varepsilon)}$ (with respect to $\overline{\mathbf{Y}}^{(\varepsilon)}$). Then it is easy to see that $\overline{J}_{0,t}^{(\varepsilon)}$ and $\overline{\mathbf{K}}_{0,t}^{(\varepsilon)}$ are the right hand sides of (8.5) and (8.8), respectively. In particular, $\|J_{0,t}^{(\varepsilon)}\|_{B,r} \leq \overline{J}_{0,t}^{(\varepsilon)}$ and $\|\mathbf{K}_{0,t}^{(\varepsilon)}\|_{B,r} \leq \overline{\mathbf{K}}_{0,t}^{(\varepsilon)}$, and since $\overline{J}_{0,t}^{(\varepsilon)}$ and $\overline{\mathbf{K}}_{0,t}^{(\varepsilon)}$ converge in ucp as $\varepsilon \to 0$ to some $\overline{J}_{0,t}$ and $\overline{\mathbf{K}}_{0,t}$ such that

$$\overline{\mathbf{Y}}_t = \overline{J}_{0,t}\overline{\mathbf{Y}}_0 + \overline{\mathbf{K}}_{0,t},$$

it follows that $\|\mathbf{Y}_t\|_{B,r} \leq \overline{\mathbf{Y}}_t$ for fixed $t \geq 0$, giving (9.1).

Similar quantities such as $\overline{J}_{s,t}^{(\varepsilon)}$ and $\overline{J}_{s,t}$ can be defined when going from time s to time t, and similar results hold. Let $\overline{V}_t^{(\varepsilon)} := \alpha_0\|\mathbf{e}\|_{B,r} + \|\mathbf{ea}'\|_{B,r}\,\overline{\mathbf{Y}}_{t-}^{(\varepsilon)}$ be the COGARCH(1,1) volatility corresponding to $\overline{\mathbf{Y}}^{(\varepsilon)}$. Define

$$\begin{aligned} X_t &:= -\lambda t - \sum_{0<s\leq t} \log(1 + (\Delta L_s)^2 \|\mathbf{ea}'\|_{B,r}), \\ X_t^{(\varepsilon)} &:= -\lambda t - \sum_{0<s\leq t, (\Delta L_s)^2 \geq \varepsilon} \log(1 + (\Delta L_s)^2 \|\mathbf{ea}'\|_{B,r}). \end{aligned}$$

Then it follows from Theorem 2.2 and (1.4), that

$$\overline{V}_{t+}^{(\varepsilon)} = \left(\overline{V}_0 - \alpha_0\|\mathbf{e}\|_{B,r}\lambda \int_0^t e^{X_s^{(\varepsilon)}}\,ds\right) e^{-X_t^{(\varepsilon)}}.$$

Thus we have $\overline{J}_{0,t}^{(\varepsilon)} = e^{-X_t^{(\varepsilon)}}$ and obtain another formula for $\overline{\mathbf{K}}_{0,t}^{(\varepsilon)}$, namely

$$\overline{\mathbf{K}}_{0,t}^{(\varepsilon)} = \|\mathbf{ea}'\|_{B,r}^{-1}\left[\alpha_0\|\mathbf{e}\|_{B,r}\,e^{-X_t^{(\varepsilon)}} - \alpha_0\|\mathbf{e}\|_{B,r}\lambda \int_0^t e^{-(X_t^{(\varepsilon)} - X_s^{(\varepsilon)})}\,ds - \alpha_0\|\mathbf{e}\|_{B,r}\right].$$

From this it can be seen that $\overline{J}_{0,t}^{(\varepsilon)}$ and $\overline{\mathbf{K}}_{0,t}^{(\varepsilon)}$ are bounded by $\overline{J}_{0,t} = e^{-X_t}$ and

$$\overline{\mathbf{K}}_{0,t} = \|\mathbf{e}\mathbf{a}'\|_{B,r}^{-1} \alpha_0 \|\mathbf{e}\|_{B,r} \left[e^{-X_t} - \lambda \int_0^t e^{-(X_t - X_s)}\, ds - 1 \right],$$

respectively. Now define the versions

$$\begin{aligned}
\overline{\mathbf{Y}}_\infty &:= \sum_{i=0}^{\infty} \overline{J}_{0,1} \cdots \overline{J}_{i-1,i} \overline{\mathbf{K}}_{i,i+1}, \\
\mathbf{Y}_\infty^{(\varepsilon)} &:= \sum_{i=0}^{\infty} J_{0,1}^{(\varepsilon)} \cdots J_{i-1,i}^{(\varepsilon)} \mathbf{K}_{i,i+1}^{(\varepsilon)}, \\
\mathbf{Y}_\infty &:= \sum_{i=0}^{\infty} J_{0,1} \cdots J_{i-1,i} \mathbf{K}_{i,i+1}.
\end{aligned}$$

In the proof of Theorems 3.1 and 3.3 we have seen that (3.2) implies that the sum defining $\overline{\mathbf{Y}}_\infty$ converges almost surely. This then gives the claim, since

$$\|J_{i-1,i}\|_{B,r},\ \|J_{i-1,i}^{(\varepsilon)}\|_{B,r} \leq \overline{J}_{i-1,i}, \quad \|\mathbf{K}_{i,i+1}\|_{B,r},\ \|\mathbf{K}_{i,i+1}^{(\varepsilon)}\|_{B,r} \leq \overline{\mathbf{K}}_{i,i+1},$$

and $J_{i-1,i}^{(\varepsilon)}$ and $\mathbf{K}_{i,i+1}^{(\varepsilon)}$ converge in probability to $J_{i-1,i}$ and $\mathbf{K}_{i,i+1}$ as $\varepsilon \to 0$, respectively. □

Proof of Proposition 4.1. All assertions apart from the implication "(4.2) $\Longrightarrow \lambda(\widetilde{B}) < 0$" follow immediately from Lemma 9.1 (observing that the existence of $E\|Y_t\|^k$ is independent of the specific matrix norm) and the corresponding properties of the COGARCH(1,1) process, see Section 4 in [20]. That (4.2) implies $\lambda(\widetilde{B}) < 0$ is a consequence of the Bauer-Fike perturbation result on eigenvalues, stating that for every eigenvalue $\widetilde{\lambda}_j$ of $\widetilde{B}$ we have

$$\min_{i=1,\ldots,q} |\lambda_i - \widetilde{\lambda}_j| \leq \|S^{-1}(\widetilde{B} - B)S\|_r = \mu\, \|S^{-1}\mathbf{e}\mathbf{a}'S\|_r,$$

see e.g. Theorem 7.2.2 and its proof in Golub and van Loan [18]. □

Proof of Theorem 4.2. Since for fixed t, almost surely $V_t = V_{t+} = \alpha_0 + \mathbf{a}'\mathbf{Y}_t$, we obtain

$$\operatorname{cov}(V_{t+h}, V_t) = \mathbf{a}' \operatorname{cov}(\mathbf{Y}_{t+h}, \mathbf{Y}_t)\, \mathbf{a}. \tag{9.3}$$

For the ease of notation, we will assume that $t = 0$. Let $J_h := J_{0,h}$ and $\mathbf{K}_h := \mathbf{K}_{0,h}$ as constructed in the proof of Theorem 3.3. Then, using that $\|e^{Bt}\| \leq e^{\|B\|t}$ for any vector norm $\|\cdot\|$, it follows as in the proof of (8.6) that

$$E\|J_h\| \leq e^{\|B\|t}\, E\left\{ \exp\left(\sum_{0<s\leq h} \log(1 + (\Delta L_s)^2 \|\mathbf{e}\mathbf{a}'\|) \right) \right\} < \infty \tag{9.4}$$

by [20], Lemma 4.1 (a). Using that $\mathbf{Y}_h = J_h \mathbf{Y}_0 + \mathbf{K}_h$, we conclude that $E\|\mathbf{K}_h\| < \infty$ and that

$$\begin{aligned} E(\mathbf{Y}_h \mathbf{Y}_0') &= E(E(\mathbf{Y}_h \mathbf{Y}_0' | J_h, \mathbf{K}_h)) \\ &= E\left(J_h\, E(\mathbf{Y}_0 \mathbf{Y}_0') + \mathbf{K}_h\, E(\mathbf{Y}_0')\right) = E(J_h)\, E(\mathbf{Y}_0 \mathbf{Y}_0') + E(\mathbf{K}_h)\, E(\mathbf{Y}_0'). \end{aligned}$$

On the other hand,

$$E(\mathbf{Y}_h)\, E(\mathbf{Y}_0') = E(J_h)\, E(\mathbf{Y}_0)\, E(\mathbf{Y}_0') + E(\mathbf{K}_h)\, E(\mathbf{Y}_0'),$$

so that $\operatorname{cov}(\mathbf{Y}_h, \mathbf{Y}_0) = E(J_h) \operatorname{cov}(\mathbf{Y}_0)$, and (4.4) will follow from (9.3) once we have shown that

$$E(J_t) = e^{\widetilde{B}t}, \quad t \geq 0. \tag{9.5}$$

To do that, it suffices to assume that $[L, L]_t$ is a compound Poisson process. The general case then follows from the fact that $J_t^{(\varepsilon)}$ as defined in the proof of Theorem 3.1 converges to J_t in L^1 as $\varepsilon \to 0$, since it converges stochastically and since there is an integrable majorant by (9.4) and its proof. So suppose that $[L, L]_t = \sum_{i=1}^{N(t)} Z_i$ is compound Poisson with intensity $c > 0$ and let $C_i = (I + Z_i \mathbf{e}\mathbf{a}')\, e^{B(\Gamma_i - \Gamma_{i-1})}$. Then, for $0 \leq s, t$, it follows from (3.4) and the independence of $J_{0,s}$ and $J_{s,s+t}$ that

$$E(J_{s+t}) = E(J_s) E(J_t).$$

It is easy to see that $E(J_t)$ is a continuous function in $t \in [0, \infty)$. Further, $E(J_0) = I$, and we conclude that $(E(J_t))_{t \geq 0}$ is a semigroup. We shall show that its generator A_J satisfies

$$A_J := \lim_{t \to 0} \frac{1}{t}(E(J_t) - I) = B + \int_{\mathbb{R}} y^2 \, d\nu_L(y)\, \mathbf{e}\mathbf{a}' = \widetilde{B}. \tag{9.6}$$

This then implies (9.5), since $E(J_t) = e^{tA_J}$, see e.g. Goldstein [12], Proposition 2.5. To show (9.6), write

$$J_t = e^{Bt}\, \mathbf{1}_{\{N(t)=0\}} + e^{B(t-\Gamma_1)} C_1\, \mathbf{1}_{\{N(t)=1\}} + e^{B(t-\Gamma_{N(t)})} C_{N(t)} \cdots C_1\, \mathbf{1}_{\{N(t) \geq 2\}}. \tag{9.7}$$

Since $N(t)$ is Poisson distributed with parameter ct, we have $P(N(t) = k) = e^{-ct}(ct)^k/(k!)$. Then by (9.4),

$$\begin{aligned}
& E\left(e^{B(t-\Gamma_{N(t)})} C_{N(t)} \cdots C_1 \, \mathbf{1}_{\{N(t)\geq 2\}}\right) \\
&\leq e^{\|B\|t} \, E\left(\exp\left(\sum_{i=1}^{N(t)} \log(1 + Z_i\|\mathbf{ea}'\|)\right) \mathbf{1}_{\{N(t)\geq 2\}}\right) \\
&= e^{\|B\|t} \, E\left(\exp\left(\sum_{i=1}^{N(t)} \log(1 + Z_i\|\mathbf{ea}'\|)\right) \Big| N(t) \geq 2\right) P(N(t)\geq 2) \\
&\leq e^{\|B\|t} \, E\left(\exp\left(\sum_{i=1}^{N(t)+2} \log(1 + Z_i\|\mathbf{ea}'\|)\right)\right) P(N(t)\geq 2) \\
&= e^{\|B\|t} \, E\left((1 + Z_1\|\mathbf{ea}'\|)(1 + Z_2\|\mathbf{ea}'\|)\right) E\left(\exp\left(\sum_{0<s\leq t} \log(1 + (\Delta L_s)^2\|\mathbf{ea}'\|)\right)\right) \\
&\qquad \times P(N(t)\geq 2) \\
&= o(t) \quad \text{as} \quad t \to 0, \qquad (9.8)
\end{aligned}$$

since $P(N(t) \geq 2) = o(t)$ as $t \to 0$. Further, since Γ_1 is uniformly distributed on $(0, t)$, conditional that $N(t) = 1$, it follows that

$$\begin{aligned}
& E\left(e^{B(t-\Gamma_1)} C_1 \, \mathbf{1}_{\{N(t)=1\}}\right) \\
&= E\left(e^{B(t-\Gamma_1)} (I + Z_1\mathbf{ea}') e^{B\Gamma_1} \Big| N(t) = 1\right) P(N(t) = 1) \\
&= \int_0^t e^{B(t-s)} \, (I + E(Z_1)\mathbf{ea}') e^{Bs} \, \frac{ds}{t} \, e^{-ct} \, ct.
\end{aligned}$$

Since $\sup_{0\leq s\leq t} \|e^{Bs} - I\|$ converges to 0 as $t \to 0$, we conclude that

$$\lim_{t\to 0} \frac{1}{t} E\left(e^{B(t-\Gamma_1)} C_1 \, \mathbf{1}_{\{N(t)=1\}}\right) = (I + E(Z_1)\mathbf{ea}')c.$$

Now (9.7) and (9.8) give (9.6), since

$$\lim_{t\to 0} \frac{E(J_t) - I}{t} = \lim_{t\to 0} \frac{e^{Bt}e^{-ct} - I}{t} + c(I + E(Z_1)\mathbf{ea}') = -c\, I + B + c(I + E(Z_1)\mathbf{ea}') = \widetilde{B}.$$

□

We need the following lemma:

Lemma 9.2 *Let T be exponentially distributed with parameter c, and suppose that $\lambda(B) < 0$. Let*

$$M := E(e^{BT} \otimes e^{BT}).$$

Then

$$E(e^{BT}) = \left(I - c^{-1}B\right)^{-1}, \tag{9.9}$$

$$M^{-1} = I_{q^2} - \left(I \otimes (c^{-1}B)\right) - \left((c^{-1}B) \otimes I\right). \tag{9.10}$$

Further, $(I \otimes B) + (B \otimes I)$ is invertible, and for any real $(q \times q)$–matrix U the unique solution of $((I \otimes B) + (B \otimes I))\,\mathbf{x} = \mathrm{vec}\,(U)$ is given by

$$\mathbf{x} = \mathrm{vec}\left(-\int_0^\infty e^{Bt}\,U\,e^{B't}\,dt\right). \tag{9.11}$$

Here, we denote by I the $(q \times q)$–identity matrix, and by I_{q^2} the $(q^2 \times q^2)$–identity matrix.

Proof. Equations (9.9) and (9.10) follow by simple calculations and a diagonalisation argument, while invertibility of $(I \otimes B) + (B \otimes I)$ and (9.11) are consequences of Lyapunov's theorem for the solution of Lyapunov equations, see e.g. Section 9.3 in Godunov [17]. □

Proof of Lemma 4.3. Suppose first that the Lévy measure of L is finite and let Q and $\mathbf{R}$ be as in Theorem 3.5 (b) (writing (T, Z) instead of (T_0, Z_0)). Then by Lemma 9.2,

$$E(Q) = \left(I - c^{-1}B\right)^{-1}(I + E(Z)\mathbf{e}\mathbf{a}'),$$

$$E(\mathbf{R}) = \alpha_0 E(Z)\left(I - c^{-1}B\right)^{-1}\mathbf{e},$$

so that (3.11) gives

$$(I - E(Q))E(\mathbf{Y}_\infty) = E(\mathbf{R}).$$

Further,

$$\left(I - c^{-1}B\right)(I - E(Q)) = \left[\left(I - c^{-1}B\right) - I - E(Z)\mathbf{e}\mathbf{a}'\right] = -\frac{1}{c}(B + \mu\mathbf{e}\mathbf{a}'),$$

giving

$$E(\mathbf{Y}_\infty) = -c(B + \mu\mathbf{e}\mathbf{a}')^{-1}\left(I - c^{-1}B\right)E(\mathbf{R}) = -\alpha_0\mu(B + \mu\mathbf{e}\mathbf{a}')^{-1}\mathbf{e}.$$

Denoting $\mathbf{u} = (u_1, \ldots, u_q)' := (B + \mu\mathbf{e}\mathbf{a}')^{-1}\mathbf{e}$, it is easy to see that $u_2 = \ldots = u_q = 0$ and $u_1 = 1/(\alpha_1\mu - \beta_q)$. In the case when ν_L is infinite the result follows from Lemma 9.1, using that $\overline{Y}_\infty$ is an integrable majorant by (4.2). □

Proof of Theorem 4.4. By Lemma 9.1 and the dominated convergence theorem, it is sufficient to assume that $[L, L]$ is a compound Poisson process. Hence, let Q and $\mathbf{R}$ be as in Theorem 3.5, writing (T, Z) instead of (T_0, Z_0), where T is exponentially distributed with parameter $c > 0$. Then

$$E(\mathbf{Y}_\infty\mathbf{Y}'_\infty) - E(Q\mathbf{Y}_\infty\mathbf{Y}'_\infty Q') = E(Q\mathbf{Y}_\infty\mathbf{R}') + E(\mathbf{R}\mathbf{Y}'_\infty Q') + E(\mathbf{R}\mathbf{R}') \tag{9.12}$$

by (3.11), and all these expectations exist by (4.3). Now

$$\begin{aligned} E(Q\mathbf{Y}_\infty\mathbf{Y}'_\infty Q') &= E\left(E[Q\mathbf{Y}_\infty\mathbf{Y}'_\infty Q'|Q]\right) \\ &= E\left(E[Q\,E(\mathbf{Y}_\infty\mathbf{Y}'_\infty)\,Q'|T]\right) \\ &= E\left(e^{BT}\,E\left[(I+Z\mathbf{e}\mathbf{a}')E(\mathbf{Y}_\infty\mathbf{Y}'_\infty)\,(I+Z\mathbf{a}\mathbf{e}')\right]e^{B'T}\right). \end{aligned}$$

Using that $\mathrm{vec}\,(A_1A_2A_3) = (A_3'\otimes A_1)\mathrm{vec}\,(A_2)$ for matrices A_1, A_2 and A_3 it follows with M as in Lemma 9.2 that

$$\begin{aligned} &\mathrm{vec}\,(E(Q\mathbf{Y}_\infty\mathbf{Y}'_\infty Q')) \\ &= M\,\mathrm{vec}\,(E\left((I+Z\mathbf{e}\mathbf{a}')E(\mathbf{Y}_\infty\mathbf{Y}'_\infty)(I+Z\mathbf{a}\mathbf{e}')\right)) \\ &= M\left(E\left((I+Z\mathbf{e}\mathbf{a}')\otimes(I+Z\mathbf{e}\mathbf{a}')\right)\right)\,\mathrm{vec}\,(E(\mathbf{Y}_\infty\mathbf{Y}'_\infty)) \\ &= M\big(I_{q^2}+E(Z)((\mathbf{e}\mathbf{a}')\otimes I)+E(Z)(I\otimes(\mathbf{e}\mathbf{a}'))+E(Z^2)((\mathbf{e}\mathbf{a}')\otimes(\mathbf{e}\mathbf{a}'))\big)\,\mathrm{vec}\,(E(\mathbf{Y}_\infty\mathbf{Y}'_\infty)). \end{aligned}$$

Similar expressions can be obtained for $\mathrm{vec}\,(E(Q\mathbf{Y}_\infty\mathbf{R}'))$, $\mathrm{vec}\,(E(\mathbf{R}\mathbf{Y}'_\infty Q'))$ and $\mathrm{vec}\,(E(\mathbf{R}\mathbf{R}'))$ and we obtain from (9.12) that

$$\begin{aligned} &\left[I_{q^2}-M\big(I_{q^2}+E(Z)((\mathbf{e}\mathbf{a}')\otimes I)+E(Z)(I\otimes(\mathbf{e}\mathbf{a}'))+E(Z^2)((\mathbf{e}\mathbf{a}')\otimes(\mathbf{e}\mathbf{a}'))\big)\right]\,\mathrm{vec}\,(E(\mathbf{Y}_\infty\mathbf{Y}'_\infty)) \\ &= M\,\mathrm{vec}\left[\alpha_0^2E(Z^2)\mathbf{e}\mathbf{e}'+\alpha_0(E(Z)I+E(Z^2)\mathbf{e}\mathbf{a}')E(\mathbf{Y}_\infty)\mathbf{e}'+\alpha_0\mathbf{e}E(\mathbf{Y}'_\infty)(E(Z)I+E(Z^2)\mathbf{a}\mathbf{e}')\right] \end{aligned}$$

Multiplying this equation by cM^{-1}, using (9.10), (4.5) as well as $\mu = c\,E(Z)$ and $\rho = c\,E(Z^2)$, we obtain

$$\begin{aligned} &-\Big[(I\otimes(B+\mu\mathbf{e}\mathbf{a}'))+((B+\mu\mathbf{e}\mathbf{a}')\otimes I)+\rho((\mathbf{e}\mathbf{a}')\otimes(\mathbf{e}\mathbf{a}'))\Big]\,\mathrm{vec}\,(E(\mathbf{Y}_\infty\mathbf{Y}'_\infty)) \\ &= \mathrm{vec}\Big[\alpha_0^2\rho\mathbf{e}\mathbf{e}'-\alpha_0^2(\mu I+\rho\mathbf{e}\mathbf{a}')\mu(B+\mu\mathbf{e}\mathbf{a}')^{-1}\mathbf{e}\mathbf{e}'-\alpha_0^2\mathbf{e}\mathbf{e}'(B'+\mu\mathbf{a}\mathbf{e}')^{-1}\mu(\mu I+\rho\mathbf{a}\mathbf{e}')\Big]. \end{aligned}$$

Adding to this

$$\begin{aligned} &\Big[(I\otimes\widetilde{B})+(\widetilde{B}\otimes I)+\rho((\mathbf{e}\mathbf{a}')\otimes(\mathbf{e}\mathbf{a}'))\Big]\,\mathrm{vec}\,(E(\mathbf{Y}_\infty)E(\mathbf{Y}'_\infty)) \\ &= \mathrm{vec}\Big[\widetilde{B}\,E(\mathbf{Y}_\infty)E(\mathbf{Y}'_\infty)+E(\mathbf{Y}_\infty)E(\mathbf{Y}'_\infty)\,\widetilde{B}'+\rho\mathbf{e}\mathbf{a}'E(\mathbf{Y}_\infty)E(\mathbf{Y}'_\infty)\mathbf{a}\mathbf{e}'\Big] \\ &= \alpha_0^2\,\mathrm{vec}\Big[\mu^2\mathbf{e}\mathbf{e}'(\widetilde{B}')^{-1}+\mu^2\widetilde{B}^{-1}\mathbf{e}\mathbf{e}'+\rho\mu^2\mathbf{e}\mathbf{a}'\widetilde{B}^{-1}\mathbf{e}\mathbf{e}'(\widetilde{B}')^{-1}\mathbf{a}\mathbf{e}'\Big] \end{aligned}$$

on both sides results in

$$\begin{aligned} &-\Big[(I\otimes\widetilde{B})+(\widetilde{B}\otimes I)+\rho((\mathbf{e}\mathbf{a}')\otimes(\mathbf{e}\mathbf{a}'))\Big]\,\mathrm{vec}\,(\mathrm{cov}(\mathbf{Y}_0)) \\ &= \alpha_0^2\rho\left[1-\mu\left(\mathbf{a}'\widetilde{B}^{-1}\mathbf{e}\right)\right]^2\,\mathrm{vec}\,(\mathbf{e}\mathbf{e}') = \frac{\alpha_0^2\beta_q^2\rho}{(\beta_q-\mu\alpha_1)^2}\,\mathrm{vec}\,(\mathbf{e}\mathbf{e}'), \end{aligned}$$

which is (4.6), where we used (4.5) in the last equation.

Now let $A := (I \otimes \widetilde{B}) + (\widetilde{B} \otimes I)$ and $\mathbf{x} := \operatorname{vec}(\operatorname{cov}(\mathbf{Y}_\infty))$. By Proposition 4.1 and Lemma 9.2, A is invertible. Observe that the matrix $\rho((\mathbf{ea}') \otimes (\mathbf{ea}'))$ has non-zero entries only in the last row. Denote this row by $\mathbf{c}'$. Further, set $\gamma := \rho\alpha_0^2\beta_q^2(\mu\alpha_1 - \beta_q)^{-2}$. Then (4.6) can be written as

$$A\mathbf{x} + (\mathbf{c}'\mathbf{x})\mathbf{e}_{q^2} = -\gamma\,\mathbf{e}_{q^2}.$$

We know already that a solution to this equation exists. Suppose there are two of them, call them $\mathbf{x}_1$ and $\mathbf{x}_2$. Then $A\mathbf{x}_1 = -(\gamma + \mathbf{c}'\mathbf{x}_1)\mathbf{e}_{q^2}$ and $A\mathbf{x}_2 = -(\gamma + \mathbf{c}'\mathbf{x}_2)\mathbf{e}_{q^2}$. Denoting the unique solution of $A\mathbf{y} = -n\,\mathbf{e}_{q^2}$ by $\mathbf{y}(n)$, $n \in \mathbb{R}$, it follows that $\mathbf{x}_1 = \mathbf{y}(\gamma + \mathbf{c}'\mathbf{x}_1)$ and $\mathbf{x}_2 = \mathbf{y}(\gamma + \mathbf{c}'\mathbf{x}_2)$. Since $\mathbf{x}_1 \neq \mathbf{0} \neq \mathbf{x}_2$, this implies $\gamma + \mathbf{c}'\mathbf{x}_1 \neq \mathbf{0} \neq \gamma + \mathbf{c}'\mathbf{x}_2$, and using the linearity of the solution $\mathbf{y}(n)$ in n it follows that there is $\kappa \neq 0$ such that $\mathbf{x}_2 = \kappa\mathbf{x}_1$. Thus we have $A\mathbf{x}_1 = -(\gamma + \mathbf{c}'\mathbf{x}_1)\mathbf{e}_{q^2}$ and $\kappa A\mathbf{x}_1 = -(\gamma + \kappa\mathbf{c}'\mathbf{x}_1)\mathbf{e}_{q^2}$, and this is only possible if $\kappa = 1$, so $\mathbf{x}_1 = \mathbf{x}_2$. So the solution of (4.6) is unique, implying that the matrix $A + \rho((\mathbf{ea}') \otimes (\mathbf{ea}'))$ is invertible.

By (9.11), the solution $y(n)$ of $Ay = -ne_{q^2}$ is given by

$$y(n) = \operatorname{vec}\left(n\int_0^\infty e^{\widetilde{B}t}\mathbf{ee}'e^{\widetilde{B}'t}\,dt\right). \tag{9.13}$$

This gives

$$\operatorname{cov}(\mathbf{Y}_\infty) = (\gamma + \mathbf{c}'\operatorname{vec}(\operatorname{cov}(\mathbf{Y}_\infty)))\int_0^\infty e^{\widetilde{B}t}\mathbf{ee}'e^{\widetilde{B}'t}\,dt.$$

Since both $\operatorname{cov}(\mathbf{Y}_\infty)$ and $\int_0^\infty e^{\widetilde{B}t}\mathbf{ee}'e^{\widetilde{B}'t}\,dt$ are positive semidefinite, it follows that $\gamma + \mathbf{c}'\operatorname{vec}(\operatorname{cov}(\mathbf{Y}_\infty)) > 0$. By Brockwell [9], the stationary CARMA state vector ζ_∞ has covariance matrix

$$\operatorname{cov}(\zeta_\infty) = \rho\int_0^\infty e^{\widetilde{B}t}\mathbf{ee}'e^{\widetilde{B}'t}\,dt,$$

so that there is $u > 0$ such that

$$\operatorname{cov}(\mathbf{Y}_\infty) = u\operatorname{cov}(\zeta_\infty). \tag{9.14}$$

Inserting (9.14) in (4.6) and using (9.13) shows

$$-u\rho\operatorname{vec}(\mathbf{ee}') + u\rho^2\operatorname{vec}\left(\mathbf{ea}'\int_0^\infty e^{\widetilde{B}t}\mathbf{ee}'e^{\widetilde{B}'t}\,dt\,\mathbf{ae}'\right) = \frac{-\alpha_0^2\beta_q^2\rho}{(\beta_q - \mu\alpha_1)^2}\operatorname{vec}(\mathbf{ee}'),$$

so that

$$-u(1-m)\operatorname{vec}(\mathbf{ee}') = \frac{-\alpha_0^2\beta_q^2}{(\beta_q - \mu\alpha_1)^2}\operatorname{vec}(\mathbf{ee}').$$

Since $u > 0$ and $\alpha_0, \beta_q \neq 0$, it follows that $0 \leq m < 1$ and that

$$u = \frac{\alpha_0^2\beta_q^2}{(\beta_q - \mu\alpha_1)^2(1-m)},$$

giving (4.7). This implies (4.8) using $V_\infty = \alpha_0 + \mathbf{a}'\mathbf{Y}_\infty$, and (4.9) follows from (4.5). Finally,

$$E(\psi_\infty) = \mathbf{a}'\, E \int_0^\infty e^{\widetilde{B}t}\mathbf{e}\, d\widetilde{L}_t = \mu \int_0^\infty \mathbf{a}' e^{\widetilde{B}t}\mathbf{e}\, dt = -\mu \mathbf{a}'\widetilde{B}^{-1}\mathbf{e},$$

giving (4.10), and (4.11) and (4.12) are direct consequences of (4.4), (4.7) and the autocovariance function of a CARMA process (see Brockwell [9]). □

10 Proofs for Section 5

Proof of Theorem 5.1. (a) Suppose that (5.1) and (5.2) both hold. By Lemma 8.2, it suffices to show (5.3) for the case that $[L,L] = \sum_{i=1}^{N(t)} Z_i$ is a compound Poisson process, with jump times $(\Gamma_n)_{n\in\mathbb{N}}$. Then it follows easily by induction from (2.3) and (8.1) that

$$\mathbf{Y}_t = e^{Bt}\mathbf{Y}_0 + \sum_{i=1}^{N(t)} e^{B(t-\Gamma_i)}\mathbf{e} V_{\Gamma_i} Z_i, \quad t \geq 0.$$

In view of the proof of (b) below, let $s \geq 0$. Then

$$\mathbf{a}' e^{Bs}\mathbf{Y}_t = \mathbf{a}' e^{B(s+t)}\mathbf{Y}_0 + \sum_{i=1}^{N(t)} \mathbf{a}' e^{B(s+t-\Gamma_i)}\,\mathbf{e}\, V_{\Gamma_i} Z_i \tag{10.1}$$

$$\geq \gamma + \sum_{i=1}^{N(t)} \mathbf{a}' e^{B(s+t-\Gamma_i)}\,\mathbf{e}\, V_{\Gamma_i} Z_i. \tag{10.2}$$

Setting $s = 0$, it follows that $V_t = \alpha_0 + \mathbf{a}'\mathbf{Y}_{t-} \geq \alpha_0 + \gamma$ for $t \in [0, \Gamma_1]$, hence also $V_{\Gamma_1+} \geq \alpha_0 + \gamma \geq 0$ by (5.1) and (10.2), and an induction argument shows that $V_t \geq \alpha_0 + \gamma$ for all $t \geq 0$, i.e. (5.3) holds.

For the converse, suppose first that (5.2) fails. Then, using the continuity of the function $t \mapsto e^{Bt}$, it follows that there is $(t_1, t_2) \subset (0,\infty)$ such that $P(\alpha_0 + \mathbf{a}' e^{Bt}\mathbf{Y}_0 < 0 \;\forall\, t \in (t_1,t_2)) > 0$, and since $P(\Gamma_1 > t_2) > 0$ we get the claim from (10.1). So suppose that (5.2) holds with $\gamma > -\alpha_0$, but (5.1) fails. Suppose that the support of the Lévy measure of the compound Poisson process $[L,L]$ (and hence the support of the jump distribution Z_1) is unbounded. Let $(t_3,t_4) \subset (0,\infty)$ be an interval such that $\mathbf{a}' e^{Bt}\mathbf{e} \leq -c_1 < 0$ for all $t \in (t_3,t_4)$ for some $c_1 < 0$. Let $t_5 > t_4$. By (5.2) we have $P(V_{\Gamma_1} \geq \alpha_0 + \gamma) = 1$, so that it is easy to see that the set

$$A := \{\Gamma_1 < t_5 < \Gamma_2,\ t_5 - \Gamma_1 \in (t_3,t_4),\ V_{\Gamma_1} \geq \alpha_0 + \gamma\}$$

has positive probability. On A, we have by (10.1)

$$V_{t_5} = \alpha_0 + \mathbf{a}' e^{Bt_5}\mathbf{Y}_0 + \mathbf{a}' e^{B(t_5-\Gamma_1)}\,\mathbf{e}\, V_{\Gamma_1} Z_1.$$

Now $\mathbf{a}'e^{B(t_5-\Gamma_1)}\mathbf{e} \le -c_1$, and by choosing Z_1 (which is independent of Γ_1, Γ_2 and $\mathbf{Y}_0$) large enough we obtain $P(V_{t_5} < 0) > 0$.

(b) In view of (a) it remains to show that $\mathbf{Y}_\infty$ satisfies (5.2). For the proof of this, it suffices by Lemma 9.1 to assume that $[L,L]$ is compound Poisson. Let $(\widetilde{\mathbf{Y}}_t)_{t\ge 0}$ be a state vector process with $\widetilde{\mathbf{Y}}_0 = 0$. Then (5.2) holds for $\widetilde{\mathbf{Y}}_0$ with $\gamma = 0$, and it follows from (10.2), (5.1) and (5.3) that $\mathbf{a}'e^{Bs}\widetilde{\mathbf{Y}}_t \ge 0$ for all $s,t \ge 0$. Since $\widetilde{\mathbf{Y}}_t$ converges in distribution to $\mathbf{Y}_\infty$ as $t \to \infty$, (5.2) follows with $\gamma = 0$. □

11 Proofs for Section 6

Proof of Theorem 6.1. We mimic the proof of Proposition 5.1 of [20], i.e. in the COGARCH(1,1) case. Observe that (6.1) and (6.3) follow immediately, since $(L_t)_{t\ge0}$ is a zero-mean martingale. Further, $(G_t)_{t\ge0}$ is a square integrable martingale, and using the compensation formula (e.g. Bertoin [5], page 7), we have

$$EG_r^2 = E\int_0^r V_s\, d[L,L]_s = E\sum_{0<s\le r} V_s(\Delta L_s)^2 = E(L_1)^2 r E(V_\infty),$$

and (6.2) follows from (4.9). Before showing (6.4), we verify that $EG_t^4 < \infty$ if (4.3) is satisfied: it follows from the Burkholder-Davis-Gundy inequality (see e.g. Protter [25], page 222) that $EG_t^4 < \infty$ if $E[G,G]_t^2 < \infty$. Let $\overline{V}_t = \alpha_0\|\mathbf{e}\|_{B,r} + \|\mathbf{e}\mathbf{a}'\|_{B,r}\overline{\mathbf{Y}}_{t-}$ the volatility of the COGARCH(1,1) process constructed in Lemma 9.1, and let $\overline{G}_t = \int_0^t \sqrt{\overline{V}_t}\, dL_t$ the corresponding GOGARCH(1,1) price process. Then it follows from Lemma 9.1 that there is $C_1 > 0$ such that

$$0 \le V_s = \alpha_0 + \mathbf{a}'\mathbf{Y}_{s-} \le \alpha_0 + C_1\overline{\mathbf{Y}}_{s-} = \alpha_0 + C_1\frac{\overline{V}_s - \alpha_0\|\mathbf{e}\|_{B,r}}{\|\mathbf{e}\mathbf{a}'\|_{B,r}}.$$

Then

$$\begin{aligned}[G,G]_t &= \int_0^t V_s\, d[L,L]_s \le \frac{C_1}{\|\mathbf{e}\mathbf{a}'\|_{B,r}}\int_0^t \overline{V}_s\, d[L,L]_s + \left(\alpha_0 - \frac{C_1\alpha_0\|\mathbf{e}\|_{B,r}}{\|\mathbf{e}\mathbf{a}'\|_{B,r}}\right)[L,L]_t \\ &= \frac{C_1}{\|\mathbf{e}\mathbf{a}'\|_{B,r}}[\overline{G},\overline{G}]_t + \left(\alpha_0 - \frac{C_1\alpha_0\|\mathbf{e}\|_{B,r}}{\|\mathbf{e}\mathbf{a}'\|_{B,r}}\right)[L,L]_t,\end{aligned}$$

so that again by the Burkholder-Davis-Gundy inequality and Doob's maximal inequality finiteness of $E\overline{G}_t^4$ implies finiteness of $E[\overline{G},\overline{G}]_t^2$ and hence of EG_t^4. That $E\overline{G}_t^4 < \infty$ was already used in [20].

Denote by E_r the conditional expectation with respect to the σ-algebra $\mathcal{F}_r$. Using partial integration, we have

$$(G_h^{(r)})^2 = 2\int_{h+}^{h+r} G_{s-}\, dG_s + [G,G]_{h+}^{h+r} = 2\int_h^{h+r} G_{s-}\sqrt{V_s}\, dL_s + \sum_{h<s\le h+r} V_s(\Delta L_s)^2.$$

Since the increments of L on the interval $(h, h+r]$ are independent of $\mathcal{F}_r$ and since L has expectation 0, it follows that

$$E_r \int_{h+}^{h+r} G_{s-} \sqrt{V_s}\, dL_s = 0.$$

Recall that $\mathbf{Y}_s = J_{r,s}\mathbf{Y}_r + \mathbf{K}_{r,s}$ by (3.3). Hence we also have $\mathbf{Y}_{s-} = J_{r,s-}\mathbf{Y}_r + \mathbf{K}_{r,s-}$, so that by the compensation formula

$$\begin{aligned} E_r(G_h^{(r)})^2 &= E_r \sum_{h<s\le h+r} (\alpha_0 + \mathbf{a}'\mathbf{Y}_{s-})(\Delta L_s)^2 \\ &= E_r \sum_{h<s\le h+r} (\alpha_0 + \mathbf{a}' J_{r,s-}\mathbf{Y}_r + \mathbf{a}'\mathbf{K}_{r,s-})(\Delta L_s)^2 \\ &= E(L_1^2)\alpha_0 r + E(L_1^2)\mathbf{a}' \int_{h+}^{h+r} (EJ_{r,s-})\mathbf{Y}_r\, ds + E(L_1^2)\mathbf{a}' \int_{h+}^{h+r} (E\mathbf{K}_{r,s-})\, ds \\ &= E(L_1^2) \int_h^{h+r} E_r(V_s)\, ds. \end{aligned} \tag{11.1}$$

Since $\mathbf{Y}_\infty \stackrel{d}{=} J_{r,s}\mathbf{Y}_\infty + \mathbf{K}_{r,s}$ by (3.5), with $\mathbf{Y}_\infty$ independent on the right hand side, and $EJ_{r,s} = e^{\widetilde{B}(s-r)}$ by the proof of Theorem 4.2, it follows from (4.5) that

$$E\mathbf{K}_{r,s} = (I - e^{\widetilde{B}(s-r)}) \frac{\alpha_0 \mu}{\beta_q - \alpha_1 \mu} \mathbf{e}_1.$$

Hence

$$\begin{aligned} E_r(V_s) &= \alpha_0 + \mathbf{a}' e^{\widetilde{B}(s-r)} \mathbf{Y}_r + \mathbf{a}' \frac{\alpha_0 \mu}{\beta_q - \alpha_1 \mu} (I - e^{\widetilde{B}(r-s)}) \mathbf{e}_1 \\ &= \frac{\alpha_0 \beta_q}{\beta_q - \alpha_1 \mu} + \mathbf{a}' e^{\widetilde{B}(s-r)} \left(\mathbf{Y}_r - \frac{\alpha_0 \mu}{\beta_q - \alpha_1 \mu} \mathbf{e}_1 \right). \end{aligned} \tag{11.2}$$

Combining $\int_h^{h+r} e^{\widetilde{B}(s-r)}\, ds = e^{\widetilde{B}h} \widetilde{B}^{-1}(I - e^{-\widetilde{B}r})$ with (11.1), (11.2) and (4.5) gives

$$E_r(G_h^{(r)})^2 = E(L_1^2) \left(\frac{\alpha_0 r \beta_q}{\beta_q - \alpha_1 \mu} + \mathbf{a}' e^{\widetilde{B}h} \widetilde{B}^{-1}(I - e^{-\widetilde{B}r}) (\mathbf{Y}_r - E\mathbf{Y}_r) \right),$$

and we conclude with (6.2) that

$$\begin{aligned} E((G_0^{(r)})^2 (G_h^{(r)})^2) &= E(E_r((G_h^{(r)})^2 G_r^2)) \\ &= E(L_1^2)\, E\left(\frac{\alpha_0 r \beta_q}{\beta_q - \alpha_1 \mu} G_r^2 + \mathbf{a}' e^{\widetilde{B}h} \widetilde{B}^{-1}(I - e^{-\widetilde{B}r})(\mathbf{Y}_r - E\mathbf{Y}_r) G_r^2 \right) \\ &= (E(G_r^2))^2 + E(L_1^2)\mathbf{a}' e^{\widetilde{B}h} \widetilde{B}^{-1}(I - e^{-\widetilde{B}r}) \left[E(\mathbf{Y}_r G_r^2) - (E\mathbf{Y}_r)(EG_r^2) \right], \end{aligned}$$

showing (6.4). □

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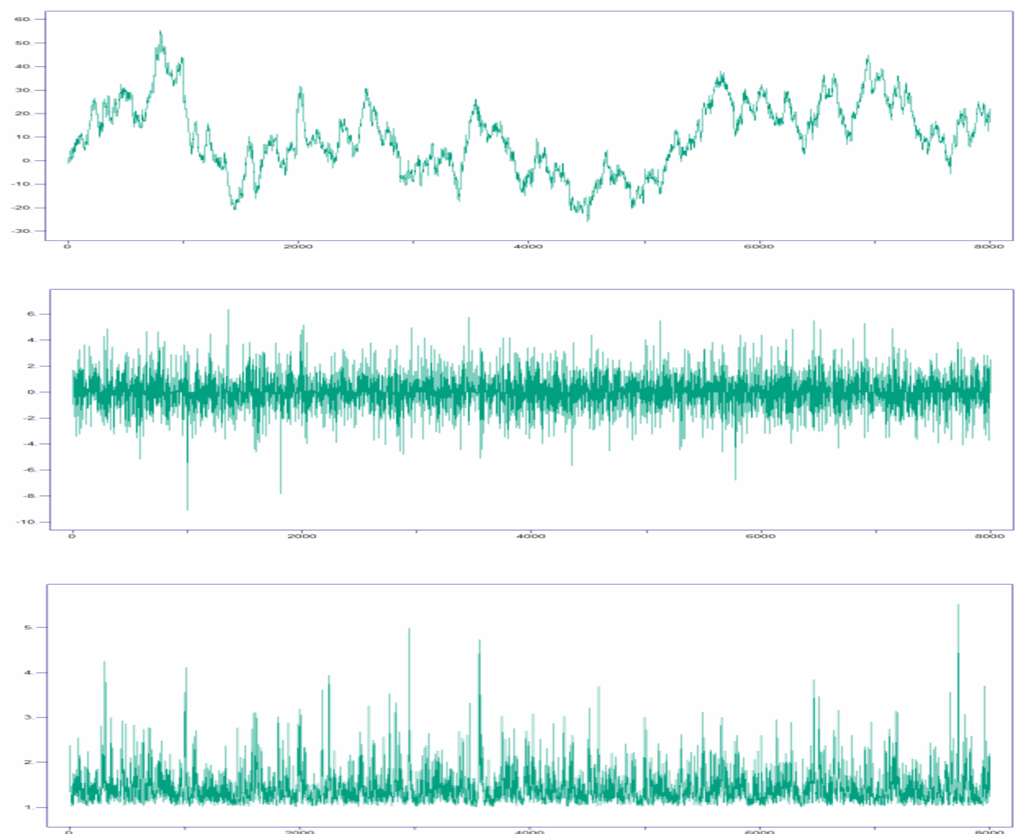

Figure 1: The simulated compound-Poisson driven COGARCH(1,3) process with jump-rate 2, normally distributed jumps with mean zero and variance 0.74 and coefficients $\alpha_0 = \alpha_1 = 1, \beta_1 = 1, \beta_2 = .48 + \pi^2$, and $\beta_3 = .64 + .4\pi^2$. The graphs show the process (G_t) sampled at integer times (top), the corresponding increments $((G_{t+1} - G_t))$ (centre), and the corresponding volatility sequence (σ_t^2) (bottom).

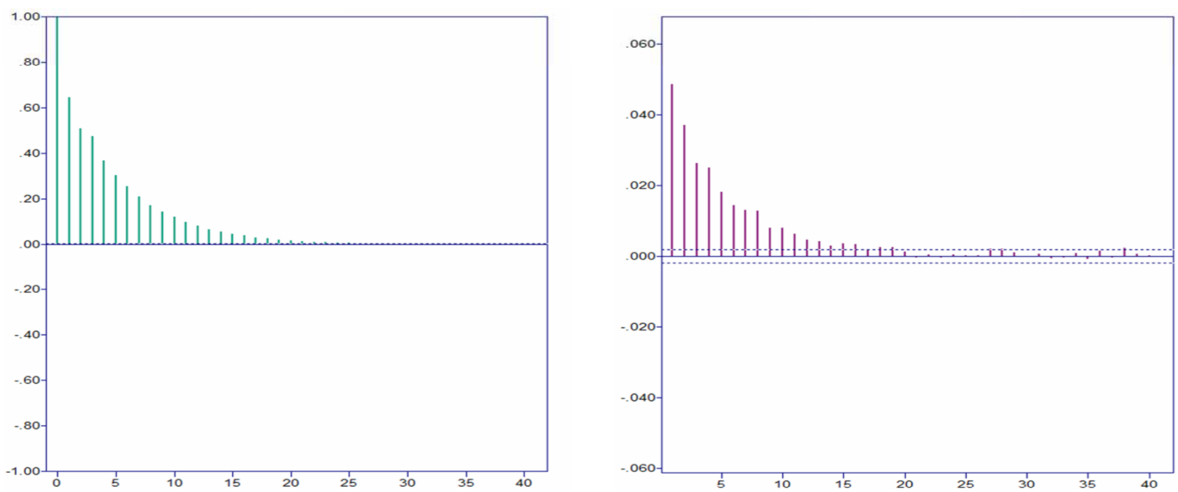

Figure 2: The sample autocorrelation functions of the volatilities $(\sigma_t)^2$ (left) and of the squared CO-GARCH increments $(G_{t+1} - G_t)^2$ (right) of a realisation of (σ_t^2, G_t) of length 1000000, the first 8000 values of which are shown in the graphics of Figure 1. The dashed lines show the 95 % confidence bounds.