Proceedings of the 2nd IMT-GT Regional Conference on Mathematics, Statistics and Applications
Universiti Sains Malaysia, Penang, June 13-15, 2006

Range Restricted Positivity-Preserving G^1 Scattered Data Interpolation

Abd. Rahni Mt. Piah[1], Azizan Saaban[2], Ahmad Abd. Majid[3]
School of Mathematical Sciences, Universiti Sains Malaysia
11800 USM Pulau Pinang, MALAYSIA
E-mail : [1]arahni@cs.usm.my, [2]azizan.s@uum.edu.my, [3]majid@cs.usm.my

Abstract : The construction of a range restricted bivariate G^1 interpolant to scattered data is considered in which the interpolant is positive everywhere if the original data are positive. This study is motivated by earlier work in which sufficient conditions are derived on Bézier points in order to ensure that surfaces comprising quartic Bézier triangular patches are always positive and satisfy G^1 continuity conditions. The gradients at the data sites are then calculated (and modified if necessary) to ensure that these conditions are satisfied. Its construction is local and easily extended to include as upper and lower constraints to the interpolating surfaces of the form $z = C(x,y)$ where C is a polynomial of degree less or equal to 4. Moreover, G^1 piecewise polynomial surfaces consisting of polynomial pieces of the form $z = C(x,y)$ on the triangulation of the data sites are also admissible constraints. A number of examples are presented.

Keywords: Scattered data, Interpolation, Positivity, G^1 continuity

1. Introduction

The properties that are most often used to quantify "shape" in shape preserving interpolation are positivity, convexity and monotonicity. The problem of positivity preserving interpolation, i.e. interpolation to positive data by a positive function, is often of interest. This problem could arise if one has data points on one side of a plane, and wishes to have an interpolating surface which is also on the same side of this plane. For instance, when the data from physical experiment are measured in the form of concentration or pressure or meteorology data such as amount of rainfall where negative values are meaningless, it is important for the interpolant to preserve positivity. This paper will propose sufficient conditions on the bivariate quartic function upon triangulation of the data in order to visualize the positive scattered data which may come from certain scientific phenomena. Various methods concerning visualization of positive data or range restricted interpolation using bivariate functions can be found in [1], [2], [8], [11], [12] and [13].

This study is motivated by previous works in [2], [13] and [15]. [2] describes the construction of range restricted bivariate C^1 interpolants to scattered data where sufficient non-negativity condition on the Bézier ordinates are derived to ensure the non-negativity of a cubic Bézier triangular patch.

In [2] and [13], a C^1 non-parametric surface is constructed comprising of cubic Bezier triangular patches. Each triangular patch of the interpolating surface is formed as a convex combination of three cubic Bézier triangular patches. An initial value of inner Bezier ordinates in each triangle are computed using the cubic precision method.

In this paper, we will construct bivariate G^1 interpolants to scattered data using similar approach adopted in [13] but each triangular patch of the interpolating surface is formed as a single quartic Bézier triangular patch. It has been established that quartic patch is the lowest possible degree suitable for the construction of a composite G^1 surface [14]. Initial values of

Bezier ordinates except for the values determined by gradients at vertices of each triangle are computed using the method of minimized sum of squares of principles curvartures ([7], [9]) with respect to the G^1 continuity conditions [5] on each non-boundary edge over the triangular mesh using the quadratic form of an objective function [15]. We also consider the construction of range restricted bivariate G^1 interpolants to the scattered data.

The sufficient conditions on Bézier ordinates which ensure the positivity of a quartic Bézier triangular patch using similar method as proposed in [13] will be described in section 2. An outline of the surface construction process is given in section 3, while section 4 presents the implementation of range-restricted interpolation. Examples are presented in section 5. Finally, the conclusions will be given in Section 6.

2. Sufficient positivity conditions for a quartic Bézier triangular patch

Consider a triangle T, with vertices V_1, V_2, V_3, and barycentric coordinates u, v, w such that any point V on the triangle can be expressed as

$V = uV_1 + vV_2 + wV_3$, where $u + v + w = 1$ and $u,v,w \geq 0$.

A quartic Bézier triangular patch P on T is defined as,

$$P(u,v,w) = u^4b_{400} + v^4b_{040} + w^4b_{004} + 4u^3vb_{310} + 4u^3wb_{301} + 4v^3ub_{130} + 4v^3wb_{031} + 4w^3ub_{103} + 4w^3vb_{013} + 6u^2v^2b_{220} + 6u^2w^2b_{202} + 6v^2w^2b_{022} + 12u^2vwb_{211} + 12v^2uwb_{121} + 12w^2uvb_{112} \quad (1)$$

where b_{ijk} are the Bézier ordinates of P as shown in figure 1.

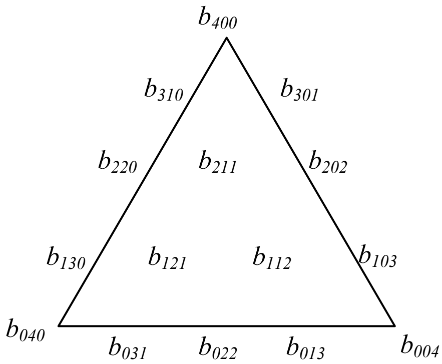


Figure 1. Relative locations of Bezier ordinates for $P(u, v, w)$

We assume that the Bézier ordinates at vertices are strictly positive, i.e. $b_{400}, b_{040}, b_{004} > 0$. The sufficient conditions on the remaining Bézier ordinates will be derived to ensure the positivity of the entire patch. Let $A = b_{400}$, $B = b_{040}$, and $C = b_{004}$ where A, B, C > 0. Our approach is to find the lower bounds of the remaining Bézier ordinates, so that $P(u,v,w) = 0$. We also assume that, apart from the vertices, the remaining Bézier ordinates have the same value $-r$ (where $r > 0$). Thus, (1) can be written as,

$$\begin{aligned} P(u,v,w) &= Au^4 + Bv^4 + Cw^4 - r(4u^3v + 4u^3w + 4v^3u + 4v^3w + 4w^3u + 4w^3v + 6u^2v + 6u^2w^2 + 6v^2w^2 + 12u^2vw + 12v^2uwb_{121} + 12w^2uvb_{112} \\ &= Au^4 + Bv^4 + Cw^4 - r(1 - u^4 - v^4 - w^4) \\ &= (A + r)u^4 + (B + r)v^4 + (C + r)w^4 - r \end{aligned} \quad (2)$$

From (2), clearly when $r = 0$, $P(u, v, w) > 0$. We are interested to find the value $r = r_0$ when the minimum value of $P(u, v, w)$ is equal to zero. The derivatives of P in (2) with respect to u, v and w are given by,

$$\frac{\partial P}{\partial u} = 4(A+r)u^3,\ \frac{\partial P}{\partial v} = 4(B+r)v^3,\ \frac{\partial P}{\partial w} = 4(C+r)w^3. \qquad (3)$$

We know that at the minimum value of *P(u, v, w),* $\frac{\partial P}{\partial u} - \frac{\partial P}{\partial v} = 0$ and $\frac{\partial P}{\partial u} - \frac{\partial P}{\partial w} = 0$.

Thus, we have $\frac{\partial P}{\partial u} = \frac{\partial P}{\partial v} = \frac{\partial P}{\partial w}$. (4)

By substituting (3) into (4), we get $\frac{u^3}{v^3} = \frac{B+r}{A+r}$ *and* $\frac{u^3}{w^3} = \frac{C+r}{A+r}$.

Hence, $u^3 : v^3 : w^3 = \frac{1}{A+r} : \frac{1}{B+r} : \frac{1}{C+r}$ or $u : v : w = \frac{1}{\sqrt[3]{A+r}} : \frac{1}{\sqrt[3]{B+r}} : \frac{1}{\sqrt[3]{C+r}}$.

Since $u + v + w = 1$, we obtain $u = \frac{\frac{1}{\sqrt[3]{A+r}}}{denom}$, $v = \frac{\frac{1}{\sqrt[3]{B+r}}}{denom}$ and

$$w = \frac{\frac{1}{\sqrt[3]{C+r}}}{denom} \text{ where } denom = \frac{1}{\sqrt[3]{A+r}} + \frac{1}{\sqrt[3]{B+r}} + \frac{1}{\sqrt[3]{C+r}}.$$

Substituting these *u, v* and *w* into (2), we obtain the minimum value of *P*(*u, v, w*),

$$P(u,v,w) = \frac{r}{\left(\frac{1}{\sqrt[3]{\frac{A}{r}+1}} + \frac{1}{\sqrt[3]{\frac{B}{r}+1}} + \frac{1}{\sqrt[3]{\frac{C}{r}+1}}\right)^3} - r. \qquad (5)$$

We need to choose a value $r = r_0$ so that this minimum value of *P* is zero. From (5) and $r > 0$, we know that, *P(u, v, w)* = 0 when

$$\frac{1}{\sqrt[3]{\frac{A}{r}+1}} + \frac{1}{\sqrt[3]{\frac{B}{r}+1}} + \frac{1}{\sqrt[3]{\frac{C}{r}+1}} = 1. \qquad (6)$$

Let $s = \frac{1}{r}$ and $G(s) = \frac{1}{\sqrt[3]{As+1}} + \frac{1}{\sqrt[3]{Bs+1}} + \frac{1}{\sqrt[3]{Cs+1}}$ (7)

If A, B and C are strictly positive, then $s_0 = 1/r_0$ is the solution of G(s) = 1.

Now, let us describe the method to determine the value of s_0 for each triangular patch. Since *A, B, C* > 0, it is easy to show that for $s \geq 0$, $G'(s) < 0$ and $G''(s) > 0$. Let $M = \max(A, B, C)$ and $N = \min(A, B, C)$ and it can be verified that $\frac{3}{\sqrt[3]{Ms+1}} \leq G(s) \leq \frac{3}{\sqrt[3]{Ns+1}}$ or in particular we get $G(\frac{26}{M}) \geq 1$ *and* $G(\frac{26}{N}) \leq 1$.

Figure 2 shows the form of *G(s)*, $s \geq 0$ with the relative location of *26/M, 26/N* and s_0.

To obtain the value of s_0 for given values of *A*, *B* and *C*, we need to find the root of (7) and G(s) = 1 that will give us a lower bound on the remaining Bézier ordinates, i.e. $r_0 = 1/s_0$ using simple iterative scheme. In choosing these scheme, we must ensure of one sided convergence of the root. This can be achieved by the method of false-position [3] with an initial estimate for the root will be the value of *s* for which the line joining *26/N* and 26/M has the value 1. The following proposition prescribes the lower bounds for the Bézier ordinates to ensure the positivity of Bézier patch.

PROPOSITION. *Consider the quartic Bézier triangular patch P(u, v, w) with* $b_{400} = A$, $b_{040} = B$, $b_{004} = C$, *where A, B, C > 0. If* $b_{rst} \geq -r_0 = -1/s_0$, $(r, s, t) \neq (4,0,0), (0,4,0)$ *and* $(0,0,4)$, *where* s_0 *is the unique solution of (7) and G(s)=1 then* $P(u, v, w) \geq 0, \forall\, u, v, w \geq 0$, $u+v+w = 1$.

Note that, if any of the values of *A, B* or *C* are zero (i.e. the given data are not strictly positive), we will assign the value of r_0 to be zero for that triangle.

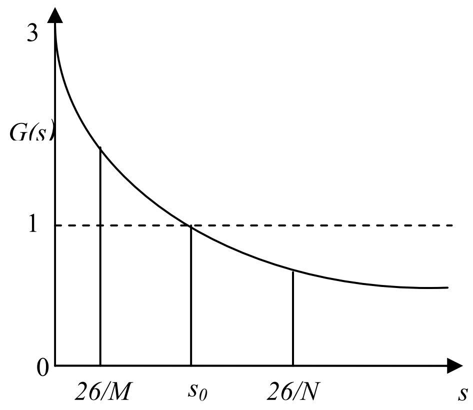


Figure 2. Function G(s) for s ≥ 0

3. Construction of positivity preserving interpolating surface

Having established sufficient conditions on the Bézier points as described in section 2, we are now able to construct the interpolating surface. Given a positive scattered data (x_i, y_i, z_i), $z_i \geq 0$, $i=1,2,\ldots,N$, we want to construct a G^1 positivity-preserving surface $z = F(x,y)$ that interpolate the given data. The surface comprises of quartic Bézier triangular patches each of which is guaranteed to remain positive. We use Delaunay triangulation ([4]) to triangulate the convex hull of the data points and the estimation of the first order partial derivative of *F* with respect to x and y is obtained using the method proposed in [6]. Let V_i, $i=1, 2, 3$ be the vertices of a triangle, such that $F(V_i) = z_i$, and the first partial derivatives $F_x(V_i)$ and $F_y(V_i)$. Then for each triangular patch *P* as given by (1), the derivative along the triangle edge e_{jk} joining (x_j, y_j) to (x_k, y_k) is given by $De_{jk} P(V_j) = (x_k - x_j) F_x(V_j) + (y_k - y_j) F_y(V_j)$.
From the given data, together with estimated derivatives at all (x_i, y_i) we can now determine all b_{rst} except of b_{220}, b_{202}, b_{022}, b_{211}, b_{121} and b_{112}. For example, we have: $b_{400} = F(V_1)$, $b_{310} = F(V_1) + \dfrac{D_{e_{12}}(V_1)}{3}$ and $b_{301} = F(V_1) - \dfrac{D_{e_{13}}(V_1)}{3}$.
By symmetry, we can obtain the remaining 6 control points. However, the initial estimate of the above edge ordinates may not satisfy the positivity conditions for P. In view of Proposition, we shall impose upon these Bézier ordinates the conditions $b_{310}, b_{301}, b_{130}, b_{103}, b_{031}, b_{013} \geq -r_0$. If it does not, the magnitudes of F_x, F_y at the vertices need to be reduced so that the conditions are satisfied. The modification of these partial derivatives is achieved by multiplying each derivative at that vertex by a scaling factor $0 < \alpha < 1$. The smallest value of α is obtained by considering all triangles that meet at vertex *V* that will guarantee satisfaction of the positivity condition for all these triangles. For example $(b_{310})_j = F(V_1) + \alpha \dfrac{D_{(e_{12})j}(V_1)}{3} \geq -(r_0)_j$ where subscript *j* represents quantities corresponding to triangle *j*. Having adjusted these derivatives, if necessary, the Bézier ordinates are recalculated using the above formulae.

Next, we shall calculate the remaining edge ordinates b_{220}, b_{202}, b_{022} and inner Bézier ordinates b_{211}, b_{121} b_{121} which need to be done in order to guarantee preservation of positivity and to ensure G^1 continuity across patch boundaries. We adopt a similar approach presented in [13] which guarantee G^1 continuity and has minimized sum of squares of principle curvartures. Two patches with a common boundary curve satisfy G^1 continuity if both have continuously varying tangent plane along the common curve. Figure 3 shows an example of Bézier control points of two adjacent quartic Bézier triangular patches (denoted by R and S respectively), where h_0 and h_4 are Bézier points of the vertices, g_0, f_0, h_1, g_3 , h_3 and f_3 are obtained from the patch gradients while g_1, f_1, h_2, g_2 and f_2 are points to be determined. We only have to consider $\{h_i,\ i = 0, 1,\ldots 4\}$ as the common boundary curve and $\{g_i, f_i,\ i = 0, 1\ldots 3\}$ which consist of the control points in each patch. Details of derivation with regard to the G^1 conditions can be found in [5]. Then, the conditions satisfying G^1 continuity between the two adjacent patches can be written as

$$\alpha g_0 + (1-\alpha)f_0 = \beta h_0 + (1-\beta)h_1 \quad (8)$$

$$\alpha g_1 + (1-\alpha)f_1 = \beta h_1 + (1-\beta)h_2 \quad (9)$$

$$\alpha g_2 + (1-\alpha)f_2 = \beta h_2 + (1-\beta)h_3 \quad (10)$$

$$\alpha g_3 + (1-\alpha)f_3 = \beta h_3 + (1-\beta)h_4 \quad (11)$$

where α and β are constants.

Since the values of g_0, f_0, h_0, h_1, g_3, h_3 , h_4 and f_3 are already known, α and β for each triangle can thus be determined from (8) and (11). We can also obtain expressions (9) and (10) on all non-boundary edges over the whole triangular mesh and be represented as

$$A\mathbf{x} = \boldsymbol{b} \quad (12)$$

where A is a l x n $(l< n)$ coefficient matrix , $\mathbf{x}$ is a n x 1 unknown vector consisting of all remaining Bézier ordinates to be determined for the entire triangular mesh, and $\boldsymbol{b}$ is a l x 1 constant vector. We shall follow a similar approach as in [15] using minimized sum of squares of principle curvatures with respect to the constraint in (12) to obtain unknown vector $\mathbf{x}$. Our aim is to find the function $F(u,v)$ which will minimize the functional $I(F(u, v)) = \sum_{t=1}^{m} I(P^t(u,v,w))$, where m is a number of triangles in a mesh with $u + v + w = 1$ or in the form of matrix-vector representation $I(F(u, v)) = \mathbf{x}^T M\mathbf{x} + \boldsymbol{e}\mathbf{x} + c$, where M is a real $(n\ x\ n)$ symmetric matrix, $\mathbf{e}$ is a $(1x\ n)$ row vector, $\mathbf{x}$ is a $(n\ x1)$ column vector representing the unknown Bézier points for the entire triangular mesh and c as a real constant.

In order to find $F(u, v)$ which will minimize $I(F(u, v))$ lead us to an optimisation problem, $\mathbf{x}^T M\mathbf{x} + \boldsymbol{e}\mathbf{x} + c$ subjects to the G^1 continuity constraints $A\mathbf{x} = \mathbf{b}$ (see [15] for further details of the method). An initial estimate of the edge ordinates b_{220}, b_{202}, b_{022} and inner Bézier ordinates b_{211}, b_{121} b_{121} in each triangle obtained by the above method may not satisfy the positivity conditions for $P(u, v, w)$ as stated in proposition. We may need to adjust the above coefficients in order to fulfill the proposition and the G^1 continuity conditions in (9) and (10). Thus, for every common edge of the two patches R and S as in Figure 3, we may need to adjust the coefficients of g_1, f_1, g_2, f_2 and h_2. We have adopted similar approach previously done for C^1 triangular cubic patches in [2] but with slight modification for the G^1 triangular quartic patches because more coefficients are involved. We also note that, this modification is local. For each patch P, when all the Bézier ordinates have been assigned, the final positive interpolating surface can be generated using (1) for the barycentric coordinates u, v and w.

4. Range-restricted interpolation

In section 3, we have described the construction of G^1 interpolating surface which is constrained to lie above the plane z = 0. We shall extend our scheme to include a larger set of constraint surfaces that are of the form z = C(x, y) where C(x, y) is a constant, linear, quadratic, cubic or quartic polynomial, i.e. $C(x, y) = ax^4 + bx^3y + cx^2y^2 + dy^4 + exy^3 + fx^3 +$

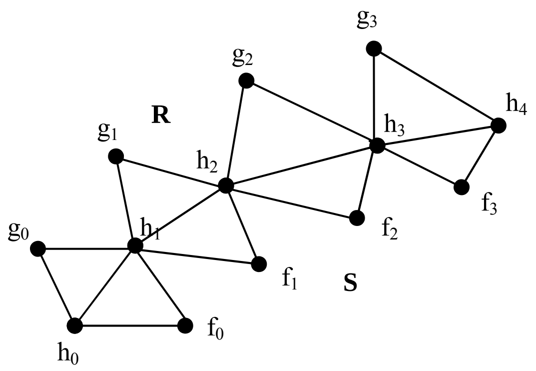


Figure 3. Control points of adjacent quartic Bézier triangular patches

$gx^2y + hy^3 + ixy^2 + jx^2 + kxy + ly^2 + mx + ny + o$ where a, b, c, d, e, f, g, h, j, k, l, m, n and o are real numbers. These surfaces are considered because they can be expressed as quartic Bézier triangular patches on each triangle of the triangular mesh. Thus, G^1 piecewise polynomial surfaces consisting of polynomial pieces of the form z = C(x , y) on the triangulation of data sites can also be treated as constraint surfaces.

We would like to generate a G^1 interpolating surface $z = F(x, y)$ through the data points (x_i, y_i, z_i) , $i = 1, 2, \ldots, N$ which lies either above or below the constraint surface or lie between both the constraint surfaces. This problem can easily be reduced to the positivity preserving interpolation case which we have considered earlier. Assume the data points lie above the constraint surface. The initial problem of constructing the interpolation surface F(x, y) with respect to the constraint surface C(x, y) is similar to the construction of a function $G(x, y) = F(x, y) - C(x, y)$ such that G is positive and G^1 with $G(x_i, y_i) = z_i^*$ where $z_i^* = z_i - C(x_i, y_i)$ is a new set of data points. The initial values of gradients of G at the data sites are obtained from $G_x(x_i, y_i) = F_x(x_i, y_i) - C_x(x_i, y_i)$ and $G_y(x_i, y_i) = F_y(x_i, y_i) - C_y(x_i, y_i)$. The gradients of G are modified if necessary using the method described in section 3. Thus, the positivity-preserving interpolating surface $F(x, y)$ is constructed piecewise as a single quartic triangular patch, where $G(x, y)$ is also a single piecewise quartic triangular patch. We can use a similar construction method if the data points lie below the constraint surface by writing $G(x, y)$ as $C(x, y) - F(x, y)$. The above construction method can also be extended to describe the interpolating surface that lie between both the upper and lower constraint surfaces (see [2] for further details).

5. Examples

In this section, we will illustrate our interpolating scheme using two test functions. The first example is the function g taken from [2] where

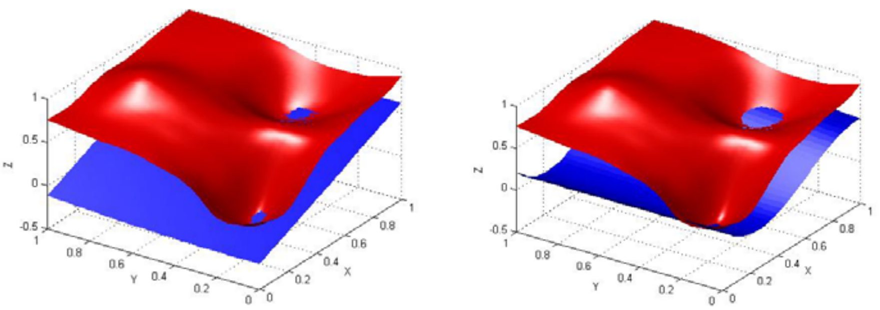
$$f(x,y) = 1.025 - 0.75\exp(-(6x-1)^2 - (6y-1)^2 - 0.75\exp(-(9x+1)^2/49 - (9y+1)/10)$$
$$- 0.50\exp(-((9x-7)^2 + (9y-3)^2)/4) + -0.50\exp(-(10x-4)^2 - (10y-7)^2 ,$$
$$(x,y) \in [0,1]x[0,1].$$

In this example, we use 33 data points which g interpolates. A linear function $C_1(x,y) = 0.1xy + 0.825x - 0.215$ and a cubic polynomial $C_2(x,y) = -3x^3 + 5.55x^2 + 0.2xy - 2.25x + 0.207$ are used as lower constraint surfaces respectively. Figures 4(a) and 4(b) show part of the unconstrained interpolating surfaces cross the two constraint surfaces at two different regions but for the range restricted interpolant with positivity conditions imposed, its stay above the two lower constrained surfaces as given in figures 5(a) and 5(b).

The second example from the well-known data set taken from [10] comprises 36 data points of

$$g(x,y) = \begin{cases} 1.0 & \text{if } \ (y-x) \geq 0 \\ 2(y-x) & \text{if } \ 0.5 \geq (y-x) \geq 0.0 \\ \dfrac{\cos(4\pi\sqrt{(x-1.5)^2+(y-0.5)^2} + 1}{2}, & \text{if } \ (x-1.5)^2+(y-0.5)^2 \leq \frac{1}{16} \\ 0 & \text{otherwise} \end{cases} .$$

For this example, we have use z = 1.001 as the upper constraint and z = -0.001 as the lower constraint similar to [2]. Figure 6 show the unconstrained interpolating surfaces cross the upper and lower constraints. When we imposed positivity conditions together with both the upper and lower constraints, the interpolating surface does not oscillate unnecessarily and stays in-between the two constrained surfaces as shown in figure 7.



(a) (b)

Figure 4. (a) Unconstrained interpolating surface with constraint surface $C_1(x, y)$
(b) Unconstrained interpolating surface with constraint surface $C_2(x, y)$
(Data from f)

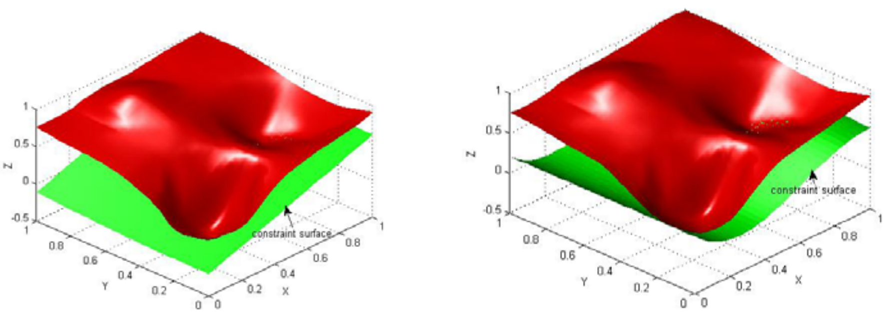


Figure 5. (a) Constrained interpolating surface with constraint surface $C_1(x, y)$
(b) Constrained interpolating surface with constraint surface $C_2(x, y)$
(Data from f)

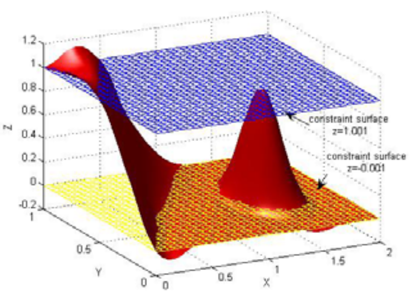


Figure 6. Unconstrained interpolating surface with constraint surfaces z = 1.001 and z=-0.001 (Data from g)

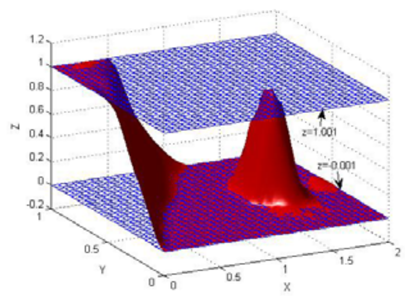


Figure 7. Constrained interpolating surface with constraint surfaces z = 1.001 and z = -0.001 (Data from g)

6. Conclusions

In this study, we have considered the generation of non-parametric surfaces that interpolate positive scattered data. We also imposed relaxed and simpler conditions on Bézier ordinates by modifying the previous work in [13] and [15]. We also extend the problem of G^1 positivity preserving interpolants to the range restricted scattered data cases similar to those considered for C^1 interpolants described in [2].

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