Range Restricted C^2 Interpolant to Scattered Data

A. Saaban[1], A. R. M. Piah[2], A. A. Majid[3]
School of Mathematical Sciences, Universiti Sains Malaysia
11800 USM Pulau Pinang, MALAYSIA
E-mail : [1]azizan.s@uum.edu.my, [2]arahni@cs.usm.my, [3]majid@cs.usm.my

Abstract

The construction of a range restricted bivariate C^2 interpolant to scattered data is considered in which the interpolant is positive everywhere if the original data are positive. Sufficient conditions are derived on Bézier points in order to ensure that surfaces comprising quintic Bézier triangular patches are always positive and satisfy C^2 continuity conditions. The first and second derivatives at the data sites are then calculated (and modified if necessary) to ensure that these conditions are satisfied. Its construction is local and easily extended to include as upper and lower constraints to the interpolating surfaces of the form z = C(x,y) where C is a polynomial of degree less or equal to 5. A number of examples are presented.

1. Introduction

Various methods concerning with visualization of surface data to preserve positivity using bivariate functions such as in [1], [2], [9], [11] and [12] concentrate only on generating the resulting C^1 smooth surfaces. As far as we know, very little effort or no attempt has been made to extend the various schemes to enable interested parties to visualize C^2 surfaces from scattered data. Motivated our previous work in [12] and the second author's work in [13], this paper will propose sufficient conditions be derived on Bézier points in order to ensure that surfaces comprising quintic Bézier triangular patches are always positive and satisfy C^2 continuity conditions. Each triangular patch of the interpolating surface is formed as a convex combination of three quintic Bézier triangular patches (as in [3]) and thus require inputs up to the second-order partial derivatives at vertices of the triangles. Initial values of inner Bézier ordinates will be estimated using cross boundary derivatives and C^2 continuity conditions across shared edge of adjacent triangles.

We begin by deriving sufficient conditions on Bézier ordinates which ensure the positivity of quintic Bézier patches in section 2. An outline of the surface construction is given in section 3, while section 4 presents the implementation of range-restricted interpolation. Examples and comparison in term of mean and maximum absolute errors with previous results in [13] are presented in section 5. Finally, the conclusion will be given in Section 6.

2. Sufficient Positivity Conditions for a Quintic Bézier Triangular Patch

Consider a triangle T (as in Figure 1), with vertices $V_1(x_1, y_1)$, $V_2(x_2\ y_2)$, $V_3(x_3, y_3)$, and barycentric coordinates u,v,w such that any point $V(x, y)$ on the triangle can be expressed as $V = uV_1 + vV_2 + wV_3$, where $u + v + w = 1$ and $u\ v,w \geq 0$.

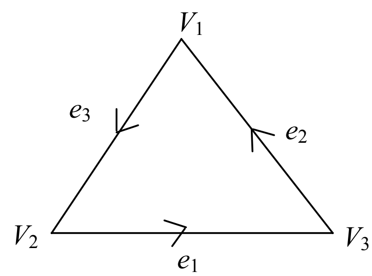

Figure 1 : Triangle T

A quintic Bézier triangular patch P on T is defined as

$$P(u,v,w) = \sum_{\substack{i+j+k=5 \\ i\geq 0,\, j\geq 0,\, k\geq 0}} b_{ijk} B^5_{ijk}(u,v,w) \qquad (1)$$

where $B^5_{ijk}(u,v,w) = \dfrac{5!}{i!\,j!\,k!} u^i v^j w^k$ and b_{ijk} are the Bézier ordinates or control points of P.

Let the Bézier ordinates at vertices be strictly positive, i.e. b_{500}, b_{050}, $b_{005} > 0$. Sufficient conditions on the remaining Bézier ordinates shall be derived to ensure the entire Bézier patch to be positive. For simplicity in writing the Bézier ordinates at vertices, let $A = b_{500}$, $B = b_{050}$, and $C = b_{005}$. Our approach is to find the lowest bound on the remaining Bézier ordinates, such that if all the Bézier ordinates apart from A, B, C, have this value, then $P(u,v,w) = 0$. We thus assume that, the remaining Bézier ordinates have the same value $-r$ (where $r > 0$). So, (1) can now be written as,

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$$P(u,v,w) = Au^5 + Bv^5 + Cw^5 - \sum_{\substack{i+j+k=5 \\ i\neq 5,\, j\neq 5,\, k\neq 5}} rB^5_{ijk}(u,v,w). \quad (2)$$

Since $\sum_{\substack{i+j+k=5 \\ i\geq 0,\, j\geq 0,\, k\geq 0}} B^5_{ijk}(u,v,w) = 1$, then

$$\sum_{\substack{i+j+k=5 \\ i\neq 5,\, j\neq 5,\, k\neq 5}} B^5_{ijk}(u,v,w) + u^5 + v^5 + w^5 = 1.$$

Thus,

$$P(u,v,w) = Au^5 + Bv^5 + Cw^5 - r(1 - u^5 - v^5 - w^5) \quad (3)$$

Clearly $P(u, v, w) > 0$, when $r = 0$. We are interested to find the value of $r = r_0$ when the minimum value of $P(u, v, w) = 0$. The partial derivatives of P in (3) with respect to u, v and w are given by,

$$\left.\begin{aligned} &\frac{\partial P}{\partial u} = 5(A+r)u^4,\ \frac{\partial P}{\partial v} = 5(B+r)v^4, \\ &\frac{\partial P}{\partial w} = 5(C+r)w^4. \end{aligned}\right\} \quad (4)$$

At the minimum value of P, $\frac{\partial P}{\partial u} - \frac{\partial P}{\partial v} = 0$ and $\frac{\partial P}{\partial u} - \frac{\partial P}{\partial w} = 0$. Thus,

$$\frac{\partial P}{\partial u} = \frac{\partial P}{\partial v} = \frac{\partial P}{\partial w}. \quad (5)$$

Using (4) and (5), we have

$$\frac{u^4}{v^4} = \frac{B+r}{A+r} \text{ and } \frac{u^4}{w^4} = \frac{C+r}{A+r}.$$

Hence,

$$u^4 : v^4 : w^4 = \frac{1}{A+r} : \frac{1}{B+r} : \frac{1}{C+r}$$

or $u : v : w = \frac{1}{(A+r)^{\frac{1}{4}}} : \frac{1}{(B+r)^{\frac{1}{4}}} : \frac{1}{(C+r)^{\frac{1}{4}}}$.

Since $u + v + w = 1$, we obtain

$$u = \frac{\frac{1}{(A+r)^{\frac{1}{4}}}}{\frac{1}{(A+r)^{\frac{1}{4}}} + \frac{1}{(B+r)^{\frac{1}{4}}} + \frac{1}{(C+r)^{\frac{1}{4}}}},$$

$$v = \frac{\frac{1}{(B+r)^{\frac{1}{4}}}}{\frac{1}{(A+r)^{\frac{1}{4}}} + \frac{1}{(B+r)^{\frac{1}{4}}} + \frac{1}{(C+r)^{\frac{1}{4}}}} \text{ and}$$

$$w = \frac{\frac{1}{(C+r)^{\frac{1}{4}}}}{\frac{1}{(A+r)^{\frac{1}{4}}} + \frac{1}{(B+r)^{\frac{1}{4}}} + \frac{1}{(C+r)^{\frac{1}{4}}}}.$$

From the above and (3), the minimum value of $P(u,v,w)$ is

$$P_{min}(u,v,w) = \frac{r}{\left(\frac{1}{(\frac{A}{r}+r)^{\frac{1}{4}}} + \frac{1}{(\frac{B}{r}+r)^{\frac{1}{4}}} + \frac{1}{(\frac{C}{r}+r)^{\frac{1}{4}}}\right)^4} - r. \quad (6)$$

We now need to choose a value of $r = r_0$ so that this minimum value is zero. From (6), $P_{min}(u,v,w) = 0$ when

$$\frac{1}{\left(\frac{A}{r}+1\right)^{\frac{1}{4}}} + \frac{1}{\left(\frac{B}{r}+1\right)^{\frac{1}{4}}} + \frac{1}{\left(\frac{C}{r}+1\right)^{\frac{1}{4}}} = 1. \quad (7)$$

Let $s = \frac{1}{r}$ and

$$G(s) = \frac{1}{(As+1)^{\frac{1}{4}}} + \frac{1}{(Bs+1)^{\frac{1}{4}}} + \frac{1}{(Cs+1)^{\frac{1}{4}}},$$

then

$$G(s) = 1,\ s \geq 0. \quad (8)$$

Now, we describe the method to determine the value of $s_0 = \frac{1}{r_0}$ for each triangular patch. Recalling that $A, B, C > 0$, it is easy to show that for $s \geq 0$, $G'(s) < 0$ and $G''(s) > 0$. Let $M = \max(A,B,C)$ and $N = \min(A,B,C)$, then clearly $\frac{3}{(Ms+1)^{\frac{1}{4}}} \leq G(s) \leq \frac{3}{(Ns+1)^{\frac{1}{4}}}$. In particular, we get $G\left(\frac{80}{M}\right) \geq 1$ and $G\left(\frac{80}{N}\right) \leq 1$.

Figure 2 shows the form of $G(s)$, $s \geq 0$ with relative locations of $80/M$, $80/N$ and s_0.

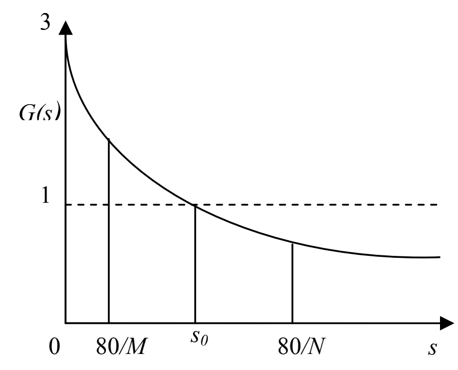

Figure 2. Function G(s) for s ≥ 0

To obtain the value of s_0 for given values of A, B and C, we need to calculate the root of (7) that will give a lower bound on the remaining Bézier ordinates, i.e. $r_0 = 1/s_0$. We can use a simple iterative scheme which must ensure one-sided convergence, i.e. that s_0 is approached from above. The convexity of *G(s)* means that this can be achieved using the method of false-position as in [13] (see [4] for further details). An initial estimate for the root will be the value of *s* for which the line joining 80/N and 80/M has the value 1 as shown in Figure 2. Thus, we obtain the following Proposition.

Proposition. *Consider the quintic Bézier triangular patch P(u, v, w) with* $b_{500} = A$, $b_{050} = B$, $b_{005} = C$, *where* $A, B, C > 0$. *If* $b_{ijk} \geq -r_0 = -1/s_0$, $(i,j,k) \neq (5,0,0), (0,5,0)$ *and* $(0,0,5)$, *where* s_0 *is the unique solution of*

$$\frac{1}{\sqrt[4]{As+1}} + \frac{1}{\sqrt[4]{Bs+1}} + \frac{1}{\sqrt[4]{Cs+1}} = 1$$

then $P(u,v,w) \geq 0,\ \forall\ u, v, w \geq 0,\ u + v + w = 1$.

Note that, in practice, if any of the values of *A, B*, or *C* are zero (i.e. the given data are not strictly positive), we will assign the value zero to r_0 for that triangle.

3. Construction of C^2 positivity-preserving interpolating surface

We want to construct a C^2 positivity-preserving surface *F(x,y)* which interpolates given scattered data, (x_i, y_i, z_i), $i=1,2,\ldots,N$, where $z_i > 0$. The surface comprises quintic Bézier triangular patches, each of which is guaranteed to remain positive. We use Delaunay triangulation to triangulate the convex hull of the data points (x_i, y_i). Estimation of first order partial derivatives of *F* will be obtained by using the method proposed in [7]. For second order partial derivatives estimation, we will use the quadratic approximation of least squares method i.e.

$$\min \sum_i \left(a x_i^2 + b x_i y_i + c y_i^2 + d x_i + e y_i + f - z_i\right)^2$$

where a, b, c, d, e and f are the coefficients to be determined. The second order partial derivatives at vertex V_i can then be estimated by $F_{xx} = 2a$, $F_{xy} = b$, $F_{yy} = 2c$. We will refer to b_{500}, b_{050}, b_{005} as Bézier ordinates at vertices, b_{410}, b_{401}, b_{140}, b_{041}, b_{014}, b_{104}, b_{320}, b_{302}, b_{230}, b_{032}, b_{023}, b_{203} as boundary Bézier ordinates and b_{311}, b_{131}, b_{113}, b_{122}, b_{212}, b_{221} as inner Bézier ordinates respectively (see Figure 3). From given data, $F(V_i)$ and estimated partial derivatives F_x, F_y, F_{xx}, F_{yy}, F_{xy} at vertex V_i, we can determine all the control points b_{ijk} except for the three inner control points b_{122}, b_{221}, b_{212}. For instance (refer to Figures 1 and 3), at vertex V_1, we shall obtain the following six control points:

$b_{500} = F(V_1)$, $b_{410} = b_{500} + D_{e3}(V_1)/5$,

$b_{401} = b_{500} - D_{e2}(V_1)/5$,

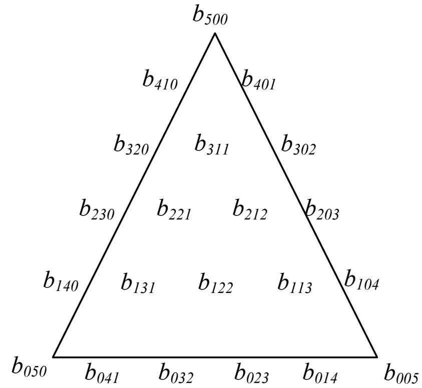

Figure 3. Control points of quintic triangular patch

$$b_{320} = b_{500} + \frac{1}{5}[2D_{e3}(V_1) + D^2_{e3e3}(V_1)/4],$$

$$b_{302} = b_{500} - \frac{1}{5}[2D_{e2}(V_1) - D^2_{e2e2}(V_1)/4] \text{ and}$$

$$b_{311} = b_{500} + \frac{1}{5}[D_{e3}(V_1) - D_{e2}(V_1) - D^2_{e2e3}(V_1)/4]$$

where

$$D_{e3}(V_1) = (x_2 - x_1)F_x(V_1) + (y_2 - y_1)F_y(V_1)$$

$$D_{e2}(V_1) = (x_1 - x_3)F_x(V_1) + (y_1 - y_3)F_y(V_1)$$

$$D^2_{e3e3}(V_1) = (x_2 - x_1)^2 F_{xx}(V_1) + 2(x_2 - x_1)(y_2 - y_1)F_{xy}(V_1) + (y_2 - y_1)^2 F_{yy}(V_1)$$

$$D^2_{e2e2}(V_1) = (x_1 - x_3)^2 F_{xx}(V_1) + 2(x_1 - x_3)(y_1 - y_3)F_{xy}(V_1) + (y_1 - y_3)^2 F_{yy}(V_1)$$

$$D^2_{e3e2}(V_1) = (x_1 - x_3)(x_2 - x_1)F_{xx}(V_1) + [(x_1 - x_3)(y_2 - y_1) + (x_2 - x_1)(y_1 - y_3)]F_{xy} + (y_1 - y_3)(y_2 - y_1)F_{yy}(V_1).$$

Similarly, the other 12 Bezier ordinates can be obtained using input data at vertices V_2 and V_3, respectively. However, initial estimates of the above ordinates may not satisfy the positivity condition for P. With regard to our Proposition, we need these Bézier ordinates to be greater or equal to $-r_0$. If they are not, then the magnitudes of F_x, F_y, F_{xx}, F_{yy} and F_{xy} at the vertices need to be reduced so that the condition is satisfied. The modification of these partial derivatives at vertex V_i, is achieved by multiplying each derivative at that vertex, by a scaling factor $0 < \alpha_i < 1$, $i = 1, 2, 3$. The smallest value of α_i is obtained by considering all triangles T_t, $t = 1, \ldots, k$ that meet at vertex *O*, which satisfy the positivity condition of all these triangles (see Figure 4). Determination of scaling factor α_i for each vertex is done as follows: Consider say, a triangle T_1 of a triangulation domain with vertices

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O, A and B. Let O $=V_1$, B $=V_2$, C $=V_3$. At vertex V_1, scalar β_j , $j=1,2,\ldots 5$ are defined as follows:

1. If $F(V_0) + D_{e1}/5 < -r_0$, then $\beta_1 = -5((r_0)_i + F(V_0))/D_{e1}$, otherwise $\beta_1 = 1$,
2. if $F(V_0) - D_{e2}/5 < -r_0$, then $\beta_2 = 5(r_0 + F(V_0))/D_{e2}$, otherwise $\beta_2 = 1$,
3. if $F(V_0) + C_1/5 < -r_0$ then $\beta_3 = 5(r_0 + F(V_0))/C_1$, otherwise $\beta_3 = 1$,
4. if $F(V_0) - C_2/5 < -r_0$ then $\beta_4 = 5((r_0)_i + F(V_0))/C_2$, otherwise $\beta_4 = 1$ and
5. if $F(V_0) + C_3/5 < -r_0$ then $\beta_5 = -5((r_0)_i + F(V_0))/C_3$, otherwise $\beta_5 = 1$

where $C_1 = [2D_{e1}(V_0) + D^2_{e1e1}(V_0)/4]$, $C_2 = [2D_{e2}(V_0) - D^2_{e2e2}(V_0)/4]$ and $C_3 = [D_{e1}(V_0) + D_{e2}(V_0) - D^2_{e2e2}(V_0)/4]$.

Then we define $\gamma_1 = \min(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ at vertex V_0 of T_1. We can obtain all γ_t, $t = 2, 3, \ldots, k$ using similar steps as above. Finally, in order for all five Bezier ordinates adjacent to V_0 to fulfill the positivity condition in the Proposition, we will choose $\alpha_1 = \min(\gamma_1, \gamma_2, \ldots, \gamma_k)$. The above process is then repeated at all V_i of the triangulation in order to obtain all the scaling factors α_i, i = 1,2,3. The adjusted Bézier ordinates except for b_{122}, b_{221}, b_{212} of the whole triangular patches that satisfy the Proposition can then be calculated.

For each triangle, the inner Bézier ordinates b_{122}, b_{221}, b_{212} remain to be calculated, in such a way to guarantee preservation of positivity and to ensure C^2 continuity across patch boundaries. We shall use similar method as in [3] to determine initial estimates of these ordinates. A local scheme P_i, i = 1,2, 3 is defined by replacing b_{122}, b_{221}, b_{212} with b^i_{122}, b^i_{212}, b^i_{221} respectively which will satisfy C^2 conditions across boundary e_i. Ordinates b^1_{122}, b^2_{212}, b^3_{221} are obtained using cross boundary derivatives on edges e_1, e_2, e_3 respectively. These ordinates will then be used to estimate the remaining local ordinates i.e. b^1_{212}, b^1_{221}, b^2_{122}, b^2_{221}, b^3_{122}, b^3_{212} (see [3] for further details). Initial estimates of these Bézier ordinates in each triangle may not satisfy the positivity condition of *P(u,v,w)* as stated in the Proposition. Now, we shall describe a method to adjust b^i_{122}, b^i_{212}, b^i_{221} for each local scheme in order to satisfy the Proposition. Let T_A and T_B be two adjacent quintic Bézier triangular patches with a common boundary curve (see Figure 4). In order to achieve C^2 continuity along shared edge, the conditions involving ordinates to be adjusted are given as follows [6]:

$$c^1_{122} = ub_{023} + vb_{032} + wb^1_{122} \qquad (9)$$

$$c^1_{221} = uc_{131} + vc^1_{122} + w(ub_{113} + vb^1_{122} + wb^1_{212}) \qquad (10)$$

$$c^1_{212} = uc^1_{122} + vc_{113} + w(ub^1_{122} + vb_{131} + wb^1_{221}) \qquad (11)$$

where (u,v,w) is barycentric coordinates of W_1 with respect to triangle T_A.

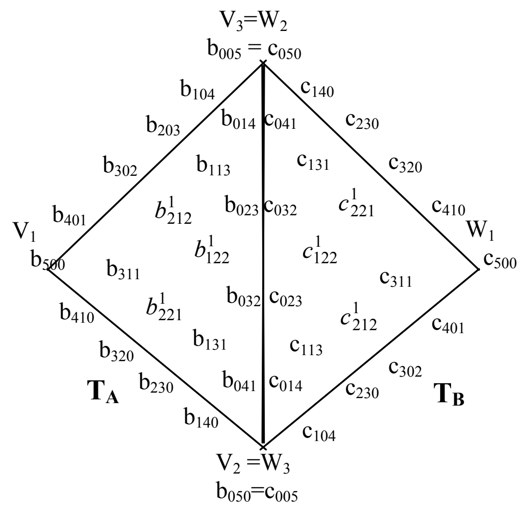

Figure 4. Adjacent quintic triangular patches

The modification of three local inner Bézier ordinates to satisfy the Proposition and C^2 continuity conditions is done as follows: It suffices to show adjustments to b^1_{122}, b^1_{212}, b^1_{221}. b^i_{122}, b^i_{212}, b^i_{221}, i=2,3 are adjusted accordingly using similar approach. Let e_1 be a common edge to two triangles as in Figure 4. Let lower bounds of ordinates for T_A, T_B be denoted by $-(r_0)_1$ and $-(r_0)_2$, respectively. First, consider adjustments to b^1_{122} and c^1_{122}. If $b^1_{122} \ge -(r_0)_1$ and $c^1_{122} < -(r_0)_2$, then c^1_{122} is set to be $-(r_0)_2$ and b^1_{122} is then adjusted according to (9). Similarly, if $c^1_{122} \ge -(r_0)_2$ and $b^1_{122} < -(r_0)_2$, then b^1_{122} is set to be $-(r_0)_1$ and c^1_{122} will be adjusted according to (9). If $b^1_{122} < -(r_0)_1$, $c^1_{122} < -(r_0)_2$ and $b^1_{122} < c^1_{122}$, then b^1_{122} will be reset as $-(r_0)_1$ and c^1_{122} is then adjusted according to (9). Similar approach will be done for the case of $b^1_{122} > c^1_{122}$. When e_1 is on the boundary of the domain, and $b^1_{122} < -r_0$ then b^1_{122} is reset to equal to $-r_0$. Using adjusted value of b^1_{122} and c^1_{122}, we can now adjust b^1_{212}, c^1_{221}, b^1_{221} and c^1_{212} to satisfy the Proposition and C^2 conditions in (10) and (11) respectively. For the boundary case, if b^1_{212} and $c^1_{221} < -r_0$, we reset both ordinate values to be $-r_0$.

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The final interpolating surface P is then defined as a convex combination of all local schemes in order for sufficient conditions on all sides of the triangles to be satisfied, i.e.

$$P(u,v,w) = c_1P_1(u,v,w) + c_2P_2(u,v,w) + c_3P_3(u,v,w)$$

or

$$P(u,v,w) = \sum_{\substack{i+j+k=5,\\ i\neq 1, j\neq 2, k\neq 2,\\ i\neq 2, j\neq 1, k\neq 2,\\ i\neq 2, j\neq 2, k\neq 1}} b_{ijk} B^5_{ijk}(u,v,w) + 30uvw(c_1Q_1 + c_2Q_2 + c_3Q_3) \quad (12)$$

where

$$c_1 = \frac{vw}{vw + vu + uw}, \quad c_2 = \frac{uw}{vw + vu + uw}$$

$$c_3 = \frac{vu}{vw + vu + uw},$$

$Q_i = uvb^i_{122} + uwb^i_{212} + vwb^i_{221}$ i = 1,2,3 and *u, v, w* are barycentric coordinates.

4. Range-restricted interpolation

In the previous section, we have described the construction of C^2 interpolating surface which is constrained to lie above the plane z = 0. We shall extend our scheme to include a larger set of constraint surfaces that are of the form z = C(x, y) where C(x, y) is a constant, linear, quadratic, cubic quartic or quintic polynomial, i.e.

$$C(x,y) = \sum_{i=0}^{5} a_i x^{5-i} y^i + \sum_{j=0}^{4} b_j x^{4-j} y^j + \sum_{k=0}^{3} c_k x^{3-k} y^k + \sum_{l=0}^{2} d_l x^{2-l} y^l + \sum_{m=0}^{1} e_m x^{1-m} y^m + f$$

where a_i, $i=0,\dots,5$, b_j, $j=0,\dots,4$, c_k, $k=0,\dots,3$, d_l, $l=0,\dots,2$, e_m, $m=0,1$ and f are real numbers. These surfaces are considered because they can be expressed as quintic Bézier triangular patches on each triangle of the triangular mesh. Thus, C^2 piecewise polynomial surfaces consisting of polynomial pieces of the form z = C(x , y) on the triangulation of data sites can also be treated as constraint surfaces.

We would like to generate a C^2 interpolating surface $z = F(x, y)$ through data points (x_i, y_i, z_i) , i = 1, 2, . . ., N which lies either above or below the constraint surface or lie between both constraint surfaces. This problem can be easily reduced to the problem of positivity-preserving interpolation which we have considered earlier. Assume the data points lie above the constraint surface. The initial problem of constructing interpolation surface F(x,y) with respect to constraint surface C(x,y) is similar to the construction of a function $G(x,y) = F(x,y) - C(x,y)$ such that G is positive and C^2 with $G(x_i,y_i) = z_i^*$, where $z_i^* = z_i - C(x_i,y_i)$ is a new set of data points. As described earlier, the positivity-preserving interpolation surface $F(x,y)$ is constructed piecewise as a single quintic triangular patch, where $G(x,y)$ is also a single piecewise quintic triangular patch. We can use a similar construction method if the data points lie below the constraint surface by writing $G(x,y)$ as $C(x,y) - F(x,y)$. The above construction method can be extended to describe the interpolating surface that lie between both the upper and lower constraint surfaces.

5. Examples

Our first example taken from [9] consists of 63 data points of a positive function, f in domain $D = (-1.4)x(-1.4, 1.4)$, where

$f(x,y) = (x+1)^2(x-1)^2(y+1)^2(y-1)^2,\ (x,y) \in D.$

The interpolating surface that does not preserve positivity of the original data is given in Figure 5 where $\min_{(x,y)\in D} f(x,y) = -0.0907$. When our proposed positivity-preserving scheme is applied, the corresponding surface is shown in Figure 6 where $\min_{(x,y)\in D} f(x,y) = 0$.

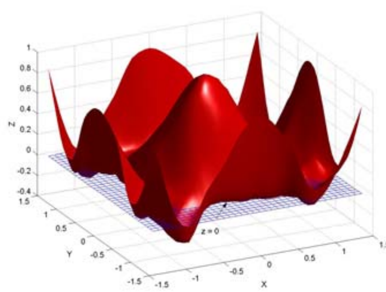

Figure 5. Surface does not preserve positivity

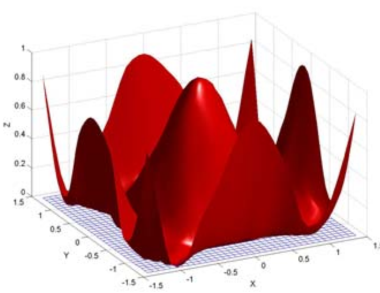

Figure 6. Positivity-preserving surface

The second example (from [10]) where 36 data points are obtained from a function g in domain $D = (0,2)x(0,1)$, where

$$g(x,y) = \begin{cases} 1.0 & \text{if } (y-x) \geq 0.5 \\ 2(y-x) & \text{if } 0.5 \geq (y-x) \geq 0.0 \\ 0.5(\cos(4\pi\sqrt{(x-1.5)^2+(y-0.5)^2}+1), & \\ \quad \text{if } (x-1.5)^2+(y-0.5)^2 \leq 1/16 & \\ 0, & \text{elsewhere on } D. \end{cases}$$

Note that $g \in [\mathbf{0},\mathbf{1}]$ in *D*. In this later example, we apply our range restricted method using lower and upper constraint planes, z = 0 and z = 1, respectively. The interpolating surface without imposing these constraints is displayed in Figure 7. Clearly, the surface crosses both constraint planes with $\min_{(x,y)\in D} g(x,y) = -0.1087$ and $\max_{(x,y)\in D} g(x,y)$ = 1.0866. After applying these constraints, the interpolating surface stays in between these two planes where $\min_{(x,y)\in D} g(x,y) = 0$ and $\max_{(x,y)\in D} g(x,y) = 1$ as shown in Figure 8.

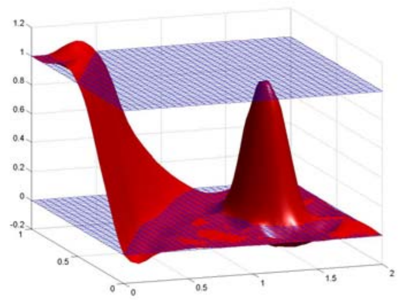

Figure 7. Unconstrained interpolating surface with constraint planes

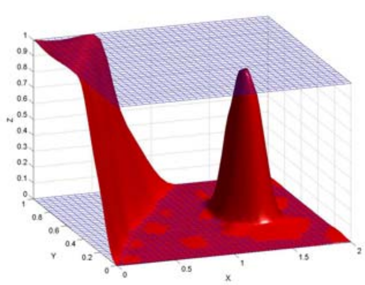

Figure 8. Constrained interpolating surface with constraint planes

6. Conclusion

In this study, we have considered the generation of positivity-preserving interpolation using quintic triangular Bézier patches by imposing relaxed and simpler conditions on Bézier ordinates which was similarly done previously for C^1 cubic triangular patches generated in [13]. We also extend the problem of C^2 positivity-preserving interpolants to the case of range restricted scattered data by imposing lower and upper constraints on the generated final surface.

7. Acknowledgements

The authors are very grateful to the Ministry of Higher Education of Malaysia and Universiti Sains Malaysia for providing us with the fundamental research grant scheme (FRGS) account number 203/PMATHS/671040 to enable us to pursue this research.

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