Positivity-Preserving Scattered Data Interpolating Surface using C^1 Piecewise Cubic Triangular Patches

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Abstract

The construction of a bivariate C^1 interpolant to scattered data is considered in which the interpolant is positive everywhere if the original data are positive. This study is motivated by earlier work in which sufficient conditions are derived on Bézier points in order to ensure that surfaces comprising cubic Bézier triangular patches are always positive and satisfy C^1 continuity conditions. Initial gradients at the data sites are estimated and then modified if necessary to ensure that these conditions are satisfied. The construction is local and easy to implement. Graphical examples are presented using two test functions.

1. Introduction

In [2] and [10], the interpolating surface comprises cubic Bézier triangular patches with sufficient conditions imposed on the ordinates of the Bézier control points in each triangle to guarantee preservation of positivity. This paper will focus on relaxing further these sufficient conditions. Various methods concerning visualization of positive data which use either piecewise bicubic interpolant or piecewise cubic triangular interpolant can be found in the literature, see for examples, [1], [2], [9] and [10]. A piecewise bicubic function u(x, y) from the data on a rectangular mesh is constructed in [1], where u(x, y) is nonnegative (positive) everywhere. Sufficient conditions for positivity are derived in terms of the first and mixed partial derivatives at the grid points. These conditions form a basis of positive interpolation algorithm. In [2], sufficient non-negative conditions on the Bézier ordinates are derived to ensure the non-negativity of piecewise cubic Bézier triangular mesh. This is done by imposing the same lower bound value for the edge (other than the vertices) and inner Bézier ordinates. [10] follows a similar approach as in [2] but offers more relaxed sufficient conditions which are easier to compute on the ordinates of the Bézier control points. In both [2] and [10], C^1 non-parametric surfaces are constructed. Each triangular patch of the interpolating surface is formed as a convex combination of three cubic Bézier triangular patches. An initial value of inner Bézier ordinate in each triangle is computed using the cubic precision method. However, the disadvantage of both schemes in [2] and [10], is that the inner and edge Bézier ordinates are assigned the same lower bound.

Thus, in this paper, we shall derive sufficient conditions for the lower bounds of the edge and inner Bézier ordinates to be adjusted independently while still ensuring positivity of the triangular patches. We use a similar approach of construction as in [2] and [10] but the initial value of inner Bézier ordinate in each triangle is computed using the cross boundary derivative along the respective edges as in [6]. This paper is organized as follows. Sufficient conditions on Bézier ordinates which will ensure the positivity of cubic Bézier triangular patches will be described in Section 2. An outline of the surface construction is given in Section 3, while Section 4 presents graphical examples of our proposed scheme. Finally, the conclusions are given in Section 5.

2. Sufficient conditions to preserve positivity

Consider a triangle T with vertices V_1, V_2, V_3 and barycentric coordinates u, v, w such that a point V on the triangle can be expressed as $V=uV_1 + vV_2 + wV_3$ where $u,v,w \geq 0$ and $u+v+w = 1$. A cubic Bézier triangular patch P on T is defined as

$$P(u,v,w) = \sum_{\substack{i+j+k=3 \\ i\geq 0,\, j\geq 0,\, k\geq 0}} b_{ijk} B^3_{ijk}(u,v,w) \qquad (1)$$

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where $B_{ijk}^3(u,v,w) = \frac{3!}{i!\,j!\,k!} u^i v^j w^k$ and b_{ijk} are the Bézier ordinates or control points of P (see Figure 1).

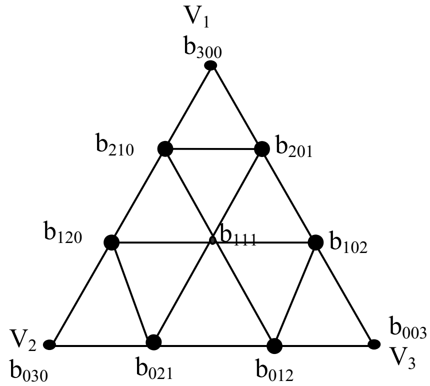

Figure 1. Control points of cubic triangular patch

We assume that the Bézier ordinates at the vertices are strictly positive, i.e. b_{300}, b_{030}, $b_{003} > 0$. We shall derive sufficient conditions on the remaining Bézier ordinates for the entire Bézier patch to be positive. Note that, in [2] and [10], apart from the Bézier points at the vertices, all the other Bézier ordinates are assumed to have the same value. Let us assume $b_{300} = A$, $b_{030} = B$, $b_{003} = C$ where A, B, C > 0. Assume also $b_{021} = b_{012} = -y_1$, $b_{201} = b_{102} = -y_2$, $b_{210} = b_{120} = -y_3$ and $b_{111} = -x$. Thus, (1) becomes

$$P(u, v, w) = Au^3 + Bv^3 + Cw^3 - 3y_1(v^2w + vw^2) - 3y_2(u^2w + uw^2) - 3y_3(u^2v + uv^2) - 6xuvw. \quad (2)$$

Clearly when $x = y_1 = y_2 = y_3 = 0$, $P(u, v, w) > 0$ [10].

Let the boundary curve along the edge V_2V_3 be

$$\begin{aligned} B_1(w) &= P(0, 1-w, w) \\ &= Bv^3 + Cw^3 - 3y_1(v^2w + vw^2) \\ &= Bv^3 + Cw^3 - y_1(1 - v^3 - w^3) \\ &= (B+y_1)(1-w)^3 + (C+y_1)w^3 - y_1. \end{aligned}$$

$B'_1(w) = -3(B+y_1)(1-w)^2 + 3(C+y_1)w^2 = 0$ gives us $\frac{1-w}{w} = \sqrt{\frac{C+y_1}{B+y_1}}$ or

$$w = \frac{\sqrt{B+y_1}}{\sqrt{B+y_1} + \sqrt{C+y_1}}.$$

It follows that $0 < w \le 1$, implies $B_1''(w) > 0$, thus, the minimum value of B_1 is

$$\frac{(B+y_1)(C+y_1)}{\left(\sqrt{B+y_1} + \sqrt{C+y_1}\right)^2} - y_1.$$

The lower bound of y_1 occurs when the minimum value of B_1 equals 0, i.e.

$$\frac{(B+y_1)(C+y_1)}{\left(\sqrt{B+y_1} + \sqrt{C+y_1}\right)^2} = y_1$$

or

$$3y_1^4 + 4(B+C)y_1^3 + 6BCy_1^2 - B^2C^2 = 0. \quad (3)$$

We can show that the real root of (3) for $y_1 > 0$ exists. Let $f(y) = 3y^4 + 4(B+C)y^3 + 6BCy^2 - B^2C^2$. Since $f'(y) = 12y^3 + 12(B+C)y^2 + 12BCy$, f is an increasing function for y_1, B and C > 0. Since $f(0) = -B^2C^2 < 0$ and $f(B+C) = 7(B+C)^4 + BC(6B^2+6C^2+11BC) > 0$, there exists $0 < y_1^0 < B + C$, such that $f(y_1^0) = 0$ where y_1^0 is a unique solution of (3) in $(0, \infty)$ as shown in Figure 2.

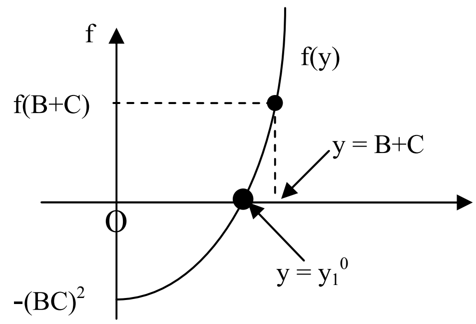

Figure 2. The unique solution, y_1^0 of f(y) = 0

By following a similar approach for the other two respective boundary curves, we can find the lower bounds, y_2 and y_3 as the positive real roots of the following equations respectively:

$$3y_2^4 + 4(A+C)y_2^3 + 6ACy_2^2 - A^2C^2 = 0, \quad (4)$$

$$3y_3^4 + 4(A+B)y_3^3 + 6Aby_3^2 - A^2B^2 = 0. \quad (5)$$

The lower bound of inner Bézier ordinates, x, can be obtained as follows:

Let $b_{210} = b_{120} = b_{201} = b_{021} = b_{012} = -y_0$, where $y_0 = \max(y_1, y_2, y_3)$. Equation (2) can then be written as

$$P(u, v, w) = Au^3 + Bv^3 + Cw^3 - 3y_0(u^2v + u^2w + v^2u + v^2w + w^2u + w^2v) - 6xuvw = Q(u, v, w) - 6xuvw.$$

If $P(u, v, w) \ge 0$ then $Q(u, v, w) - 6xuvw \ge 0$

or $$0 < x \le \frac{Q(u,v,w)}{6uvw}. \quad (6)$$

The RHS of (6) is a homogenous function. We can then use a Lagrange multiplier in order to solve this problem, i.e. to find $x_0 = \min Q(u,v,w)$, the lower bound of x, subject to $uvw = 1$.

Let $F(u, v, w, \lambda) = Q(u, v, w) + \lambda(uvw-1)$, where λ is a constant. Thus,

$$\frac{\partial F}{\partial u} = \frac{\partial Q}{\partial u} + \lambda vw, \quad \frac{\partial F}{\partial v} = \frac{\partial Q}{\partial v} + \lambda uw,$$

$$\frac{\partial F}{\partial w} = \frac{\partial Q}{\partial w} + \lambda uv \text{ and } \frac{\partial F}{\partial \lambda} = uvw - 1.$$

For minimization of Q, we get

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial v} = \frac{\partial F}{\partial w} = \frac{\partial F}{\partial \lambda} = 0, \text{ or}$$

$$u\frac{\partial Q}{\partial u} = v\frac{\partial Q}{\partial v} = w\frac{\partial Q}{\partial w} = -\lambda. \quad (7)$$

$Q(u, v, w) = (A+y_0)u^3 + (B+y_0)v^3 + (C+y_0)w^3 + 5y_0$, implies that

$$u\frac{\partial Q}{\partial u} = 3(A+y_0)u^3 \;, \; v\frac{\partial Q}{\partial v} = 3(B+y_0)v^3 \text{ and}$$

$$w\frac{\partial Q}{\partial u} = 3(C+y_0)w^3 \;.$$

From (7), we have

$3(A+y_0)u^3 = 3(B+y_0)v^3$, $3(A+y_0)u^3 = 3(C+y_0)w^3$ and $3(B+y_0)v^3 = 3(C+y_0)w^3$ or

$$\frac{u^3}{v^3} = \frac{B+y_0}{A+y_0} \quad (8)$$

$$\frac{u^3}{w^3} = \frac{C+y_0}{A+y_0} \quad (9)$$

$$\frac{v^3}{w^3} = \frac{C+y_0}{B+y_0}. \quad (10)$$

From (8), $u^6 w^3 = \frac{B+y_0}{A+y_0}$. (11)

From (9), $w^3 = \frac{(A+y_0)}{(C+y_0)}u^3$ and (11) becomes

$u^9 = \frac{(B+y_0)(C+y_0)}{(A+y_0)^2}$. Hence,

$$u = \frac{(B+y_0)^{\frac{1}{9}}(C+y_0)^{\frac{1}{9}}}{(A+y_0)^{\frac{2}{9}}},$$

$$v = \frac{(A+y_0)^{\frac{1}{9}}(C+y_0)^{\frac{1}{9}}}{(B+y_0)^{\frac{2}{9}}} \text{ and}$$

$$w = \frac{(B+y_0)^{\frac{1}{9}}(A+y_0)^{\frac{1}{9}}}{(C+y_0)^{\frac{2}{9}}}.$$

Thus,

$$x_0 = \frac{Q(u,v,w)}{6}$$

$$= \frac{(A+y_0)^{\frac{1}{3}}(B+y_0)^{\frac{1}{3}}(C+y_0)^{\frac{1}{3}}}{2} + \frac{5}{6}y_0. \quad (12)$$

Proposition

Consider the cubic Bézier triangular patch P(u, v, w) with $b_{300} = A$, $b_{030} = B$, $b_{003} = C$, $A, B, C > 0$. *If* b_{021}, $b_{012} \geq -y_1$, b_{201}, $b_{102} \geq -y_2$, b_{210}, $b_{120} \geq -y_3$ *and* $b_{111} \geq -x_0$ *(where* y_i, $x_0 > 0$*) such that* y_1 *is the unique solution of (3),* y_2 *is the unique solution of (4),* y_3 *is the unique solution of (5) and* x_0 *is given by (12) with* $y_0 = \max(y_1, y_2, y_3)$, *then* $P(u, v, w) \geq 0$ *for all* $u, v, w \geq 0$, $u + v + w = 1$.

We note that in practice if any of the values of A, B or C are zero then x_0 and y_0 are both assigned the value zero for that triangle.

3. Construction of positivity preserving interpolating surface

We want to construct a C^1 positivity-preserving functional surface $F(x,y)$ which interpolates a given scattered data, (x_i, y_i, z_i), $i=1,2,\ldots,N$, where $z_i > 0$. The surface comprises cubic Bézier triangular patches, each of which is guaranteed to remain positive. The construction process follows the steps in [2] and [10] which consists of the following:

1. Triangulate the domain data (x_i, y_i), $i=1,2,\ldots,N$.
2. Initialise partial derivatives at each data point, then modify them (if necessary) to be consistent with the positivity constraints imposed on the Bezier ordinates.
3. Generate the final surface.

We use Delaunay triangulation [3] to triangulate the convex hull of the data points. An estimation of the first order partial derivative of F with respect to x and y is obtained using the method proposed in [5]. Let $V_i=(x_i, y_i)$, $i=1, 2, 3$ be the vertices of a triangle, such that $F(V_i)= z_i$, and the first partial derivatives, $F_x(V_i)$ and $F_y(V_i)$. For each triangular patch P as in (1), the derivative along the edge e_{jk} joining (x_j, y_j) to (x_k, y_k) is given by

$$\frac{\partial P}{\partial e_{jk}} = (x_k - x_j)\frac{\partial F}{\partial x} + (y_k - y_j)\frac{\partial F}{\partial y}.$$

From the given data together with the estimated derivatives at all the (x_i,y_i), all the Bézier ordinates b_{rst} except b_{111} are determined. For example, we have:

$b_{300} = F(V_1)$, $\quad b_{210} = F(V_1) + \frac{1}{3}\frac{\partial F}{\partial e_{12}}(V_1)$ and

$b_{201} = F(V_1) - \frac{1}{3}\frac{\partial F}{\partial e_{31}}(V_1)$. However, the initial estimate for each edge ordinate may not satisfy the positivity conditions for P, in view of the Proposition, we need these Bézier ordinates to fulfill b_{021}, $b_{012} \geq -y_1$,

b_{201}, $b_{102} \geq -y_2$, b_{210}, $b_{120} \geq -y_3$. If it does not, the magnitudes of F_x, F_y at the vertices need to be reduced so that the conditions are satisfied. The modification of these partial derivatives at a vertex V_i, is achieved by multiplying each derivative at that vertex, by a scaling factor $0 < \gamma_i < 1$, $i = 1, 2, 3$. The smallest value of γ_i is obtained by considering all triangles that meet at vertex V_i, which satisfies the positivity conditions of all the triangles. For example, let T_i, $i = 1, \ldots, k$ be the triangles of the triangulation which have O as a vertex (see Figure 3 , where k = 6).

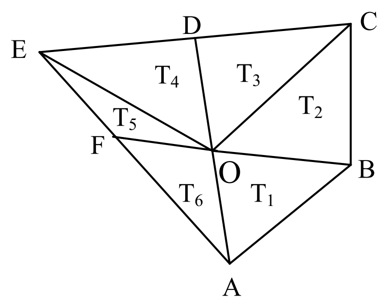

Figure 3. Triangles in the triangulation with the common vertex O

Consider the triangle T_1 and the lower bound of Bézier ordinates adjacent to O on the edge OA and OB as follows:

Case 1. If O is vertex V_1, the lower bound of Bézier ordinates adjacent to O on edge OA is $-y_3$ and on OB is $-y_2$.

Case 2. If O is vertex V_2, the lower bound of Bézier ordinates adjacent to O on edge OA is $-y_1$ and on OB is $-y_3$.

Case 3. If O is vertex V_3, the lower bound of Bézier ordinates adjacent to O on edge OA is $-y_1$ and on OB is $-y_2$.

It suffices to just consider Case 1 as the other two cases can be dealt with in a similar manner. Let the derivatives at O along OA and OB be denoted by $\frac{\partial F}{\partial e_{OA}}$ and $\frac{\partial F}{\partial e_{OB}}$, respectively. The scalar γ_{OA} and γ_{OB} are defined as follows:

If $F(O) + \frac{1}{3}\frac{\partial F}{\partial e_{OA}} \geq -y_3$, then $\gamma_{OA} = 1$, otherwise

$$\gamma_{OA} = -3\frac{(F(O)+y_3)}{\partial F/\partial e_{OA}}.$$

Similarly, if $F(O) - \frac{1}{3}\frac{\partial F}{\partial e_{OB}} \geq -y_2$, then $\gamma_{OB} = 1$,

otherwise $\gamma_{OB} = 3\frac{(F(O)+y_2)}{\partial F/\partial e_{OB}}$.

Then $\gamma_{T_1} = \min\{\gamma_{OA}, \gamma_{OB}\}$. We find γ_{T_i}, i =2,..,k, for the rest of triangles of the triangulation (as in Figure 3) by using the same argument above. Finally, in order that all the Bézier ordinates adjacent to O fulfill the positivity preserving conditions stated in the Proposition, let $\gamma_1 = \min\{\gamma_{T_1}, \gamma_{T_2}, \ldots, \gamma_{T_k}\}$. If $\gamma_1 < 1$, we redefine F_x and F_y at point O respectively as γ_1 times the corresponding initial values. By using the derivatives, $(\frac{\partial F}{\partial x})(O)$ and $(\frac{\partial F}{\partial y})(O)$, the Bézier ordinates adjacent to O in T_i are determined. For example, $(b_{210})_j = F(V_1) + \gamma_1 \frac{\partial F/\partial e_{12}}{3} \geq (y_3)_j$ and

$$(b_{201})_j = F(V_1) - \gamma_1 \frac{\partial F/\partial e_{31}}{3} \geq (y_2)_j,$$

where subscript j represents quantities corresponding to triangle j. Having adjusted these derivatives, if necessary, the Bézier ordinates are recalculated using the formulae above. The above process is repeated at all the nodes, V_i. For each triangle, the inner Bézier ordinate b_{111}, remains to be calculated, in such a way to guarantee preservation of positivity and to ensure C^1 continuity across patch boundaries. Let T_1 and T_2 be two adjacent cubic Bézier triangular patches with a common boundary curve (see Figure 4). C^1 continuity conditions between two adjacent patches are given by

$$c_{102} = \mu_1 b_{120} + \mu_2 b_{030} + \mu_3 b_{021} \quad (13)$$

$$c_{111} = \mu_1 b_{111} + \mu_2 b_{021} + \mu_3 b_{012} \quad (14)$$

$$c_{120} = \mu_1 b_{102} + \mu_2 b_{012} + \mu_3 b_{003} \quad (15)$$

where (μ_1, μ_2, μ_3) is the barycentric coordinates of W_3 with respect to triangle T_1. (13) and (15) are automatically satisfied since the Bézier ordinates b_{201}, b_{102}, b_{012}, b_{021}, b_{120}, and b_{210} have already been obtained.

We shall use similar method as in [6] to determine initial value of inner Bézier ordinate, b_{111}^i which is C^1 across the boundary e_i ,i = 1, 2, 3 where e_i is the edge of the triangle opposite vertex V_i. Local schemes P_i is defined by replacing b_{111} in (1) with b_{111}^i. The values of inner Bézier ordinates b_{111}^1, b_{111}^2 and b_{111}^3 for local schemes P_1, P_2, and P_3 respectively are given by

$$b^1_{111} = \frac{1}{2}(b_{120} + b_{102} + b_{012} + b_{021} - b_{003} - b_{030}) \quad (16)$$

$$b^2_{111} = \frac{1}{2}(b_{210} + b_{102} + b_{012} + b_{201} - b_{003} - b_{300}) \quad (17)$$

$$b^3_{111} = \frac{1}{2}(b_{021} + b_{201} + b_{120} + b_{210} - b_{300} - b_{030}). \quad (18)$$

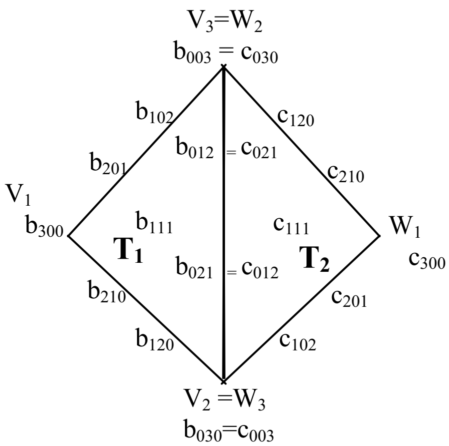

Figure 4. Adjacent cubic triangular patches

An initial estimate of the inner Bézier ordinates given by (16), (17) and (18) in each triangle may not satisfy the positivity conditions of *P(u, v, w)* as stated in the Proposition i.e. $b_{111} \geq -x_0$. For example, when e_1 is a common edge to two triangles as in Figure 3, and the initial value b^1_{111} or c^1_{111} is less than $-x_0$, then these ordinates are modified together to ensure that b^1_{111}, $c^1_{111} \geq -x_0$, and C^1 continuity condition along the common boundary curve between two adjacent patches given by (14) is satisfied. When e_1 is on the boundary of the domain, and b^1_{111} is less than $-x_0$ then b^1_{111} is reset to be equal to x_0. The ordinates b^2_{111} and b^3_{111} of P_2 and P_3 are adjusted accordingly by using a similar approach (see [2] for further detail).

The interpolating surface P on the triangle T is then defined as a convex combination of all the local schemes so that the sufficient conditions on all sides of the triangles are satisfied, i.e.

$$P(u,v,w) = c_1P_1(u,v,w) + c_2P_2(u,v,w) + c_3P_3(u,v,w)$$

or

$$P(u,v,w) = \sum_{\substack{i+j+k=3 \\ i\neq 1, j\neq 1, k\neq 1}} b_{ijk} B^3_{ijk}(u,v,w) + 6uvw(c_1b^1_{111} + c_2b^2_{111} + c_3b^3_{111})$$

where

$$c_1 = \frac{vw}{vw + vu + uw}, \quad c_2 = \frac{uw}{vw + vu + uw} \quad \text{and}$$

$$c_3 = \frac{vu}{vw + vu + uw}.$$ u, v, w are the barycentric coordinates of T.

4. Graphical Examples

In this section, we will illustrate our interpolating scheme by using two positive test functions. The first example [8] comprises 36 data points obtained from the following function,

$$f(x,y) = \begin{cases} 1.0 & \text{if } (y-x) \geq 0.5 \\ 2(y-x) & \text{if } 0.5 \geq (y-x) \geq 0.0 \\ \dfrac{\cos(4\pi\sqrt{(x-1.5)^2 + (y-0.5)^2} + 1}{2}, & \\ & \text{if } (x-1.5)^2 + (y-0.5)^2 \leq \frac{1}{16} \\ 0, & \text{elsewhere on } [0,2] \times [0,1]. \end{cases}$$

Figure 5 shows the surface without imposing the positivity conditions and can be seen clearly that the surface actually falls below the xy-plane. When the positivity conditions are imposed, the interpolating surface remains positive as shown in Figure 6.

The second example is the function $g(x,y) = (x^2-1)^2(y^2-1)^2$, $(x,y) \in [-1.4,1.4] \times [-1.4,1.4]$ taken from [7]. We construct the interpolating suface using 63 data points from this function. The surface with no positivity conditions imposed is shown in Figure 7 and with positivity conditions imposed is given in Figure 8. Again, part of the surface in Figure 7 is below the xy-plane. After positivity conditions are imposed, the resulting surface remains positive.

5. Conclusions

In this study, we have considered the generation of non-parametric surfaces that interpolate positive scattered data. We imposed very general positivity conditions on Bézier ordinates which now could be adjusted independently while still ensuring positivity of the interpolating surface as compared to the results in [2] and [10] using cubic triangular patch. This gives more flexibility to the construction of the interpolating surfaces.

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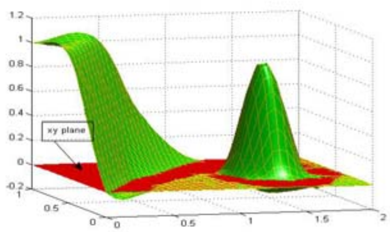

Figure 5. Interpolating surface with no positivity conditions imposed on data from f

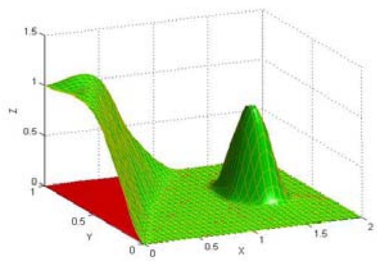

Figure 6. Interpolating surface with positivity conditions imposed on data from f

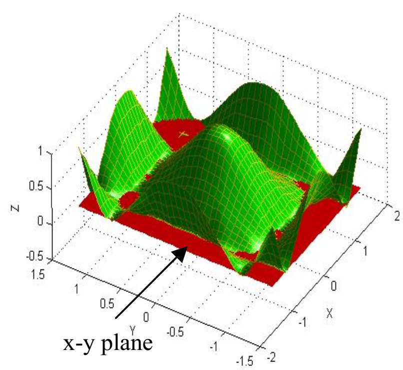

Figure 7. Interpolating surface with no positivity conditions imposed on data from g

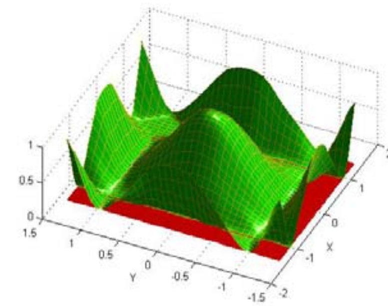

Figure 8. Interpolating surface with positivity conditions imposed on data from g

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