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ON NETWORK FLOW PROBLEMS WITH CONVEX COST

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ABSTRACT

Minimum cost flow (MCF) problem is a typical example of network flow problems, for which an additional constraint of cost is added to each flow. Conventional MCF problems consider the cost constraints as linear functions of flow. In this paper, we extend the MCF problem to cover cost functions as strictly convex and differentiable, and refer to the problem as *convex cost flow problem*. To address this problem, we derive the optimality conditions for minimising convex and differentiable cost functions, and devise an algorithm based on the primal-dual algorithm commonly used in linear programming. The proposed algorithm minimises the total cost of flow by incrementing the network flow along augmenting paths of minimum cost. Simulation results are provided to demonstrate the efficacy of the proposed algorithm.

Key words: Minimum cost flow problem, convex cost functions, primal-dual algorithm, network flow.

1.0 INTRODUCTION

Network flows are of fundamental importance in computer science, communication networks, industrial engineering, operations research, and many other areas. One exhaustive reference on the subject is by Ahuja *et al.*, (1993). Similar to the shortest path problem and maximum flow problem, minimum cost flow (MCF) problem is a central problem in network flows. The MCF problem arises naturally in many contexts, including traffic flow routing in communication networks, VLSI layout (Klein *et al.*, 1994), production scheduling, and transportation (Kennington and Wang, 1991). Consequently, the MCF problem has been studied extensively in the literature, for example (Busacker and Gowen, 1961; Kapoor and Vaidya, 1986; Leighton and Rao, 1988).

In the conventional MCF problem, the cost associated to each arc is normally a linear function of the flow carried by the arc. A large number of methods have been proposed for solving this MCF problem. Among the most popular algorithms are the primal simplex method, primal-dual method, and the out-of-kilter method (Aashtiani and Magnanti, 1976; Ahuja *et al.*, 1993; Fulkerson, 1961; Klein 1967). Recently, Vygen (2002) proposed a new dual algorithm, which is a variant of the dual network simplex algorithm, to address the MCF problem. The algorithm can work directly with the capacitated network and thus it is applicable to more general problems (like those with sub-modular flow). Goldfarb and Lin (2002) designed combinatorial interior point methods for the generalised minimum cost flow (flow with losses and gains) based on the combinatorial interior point method for the MCF problem by Wallacher and Zimmermann (1992). Wayne (2002) developed fast combinatorial algorithms for the generalised minimum cost flow problem, which directly manipulate the underlying network.

We consider in this paper a directed flow network with two distinguished vertices–source s and sink t —and non-negative flow capacities on its edges. The MCF problem associates with each arc an additional parameter k_{ij}, where k_{ij} can be the cost of sending a unit of flow along arc (i, j). Considering the cost incurred by the network flows allows us to transport any given units of flow across a network from s to t such that the cost incurred is minimised. This has a practical usage in many network problems, and the unit cost incurred may vary from arc to arc, depending on the nature of the applications. In practice, the cost is often a nonlinear function of its flow, rather than a linear function like the conventional MFC problem. Hence, we investigate in this paper the MCF problem with strictly convex and differentiable cost functions.

The extension of the MCF problem to the convex cost flow problem can be used in a wide variety of useful applications, where the cost functions are not often linear. An application example can be found in computer network, where a group of computing devices are connected together by cables. The network can be modeled as a graph, in which each vertex represents a computing device and each network arc represents a cable (Bertsekas and Gallager, 1992; Jewell, 1958). A central problem of this computer network is on how to transfer data from one device to another while satisfying a given flow condition. For example, when a unit of data is transferred through the network, one may need to minimise the total loss probability, which can be formulated as the total cost of flow along the network arcs. Other arc attributes, such as transmission bandwidth (arc capacity) and cost can also be taken into account to satisfy different network constraints. This networking problem can be formulated as a convex cost flow problem.

A number of existing solutions can be applied to this convex cost flow problem. One possible way is to reduce the problem to a typical linear cost flow problem using piecewise linearisation of the arc cost functions (Meyer, 1979; Kamesam and Meyer, 1984). This approach assumes that each of the convex functions is linear between successive integers, and then introduces a separate arc for each linear segment. In this way, the convex cost flow problem is transformed into the conventional MCF problem, and solved by the existing MCF algorithms. The convex cost flow problem has also been addressed by Karzanov and McCormick (1997) with two approaches: (i) the minimum mean cycle cancelling method, and (ii) the cancel-and-tighten method for the MCF problem, based on work by Goldberg and Tarjan (1988), that proceeds by sending flows along negative cost cycles.

In this paper, we present a new algorithm based on a primal-dual algorithm used in linear programming to address this convex cost flow problem. We modify the optimality condition in the primal-dual algorithm so that it can be applied to convex and differentiable cost functions. In particular, we show that by using the new optimality condition, we can minimise the total cost of flow by incrementing the network flow along the augmenting paths of minimum cost.

The rest of the paper is organised as follows. Section 2 formulates the problem of minimum cost flow problem in which the cost functions are strictly convex and differentiable. In Sections 3 and 4, we present the optimality conditions and a primal-dual algorithm based on the optimality conditions for this minimum convex cost flow problem, respectively. Section 5 reports the simulation results, while Section 6 provides the concluding remarks and future work.

2.0 PROBLEM FORMULATION

We consider a directed graph $G(X,E)$ with the vertex set X and the arc set E. The vertex set X consists of N vertices denoted by $1,2, \quad , \quad$, while the arc set E consists of all the arcs (i,j) connecting vertices i and j, where $i,j \in X$. We denote the flow of arc (i,j) by f_{ij}. Given a directed graph associated with a flow network, the convex cost flow problem can then be formulated as finding the flow f that minimises the total cost function

$$z(f) = \sum_{(i,j)\in E} k_{ij}(f_{ij}), \tag{1}$$

subject to

$$0 \le f_{ij} \le u_{ij} \qquad \forall (i,j) \in E, \tag{2}$$

$$\sum_{j:(i,j)\in E} f_{ij} - \sum_{j:(j,i)\in E} f_{ji} = b_i \qquad \forall i \in X, \tag{3}$$

where $k_{ij}(f_{ij})$ is convex and differentiable cost functions of flow f_{ij}, u_{ij} is the capacity associated with arc (i,j), and b_i is the source strength of vertex i defined as the difference between the flows out and in of vertex i. Depending on whether it is greater or less than zero, the source strength b_i indicates whether vertex i is a source or a sink node.

We assume that there exists a feasible solution for the above convex cost flow problem. In fact, the existence of a feasible solution can be determined by solving a maximum flow problem (Ford and Fulkerson, 1956) as follows: (i) Introduce a source s and a sink t; for each vertex i with $b_i > 0$, add a source arc (s,i) with capacity b_i, and for each vertex i with $b_i < 0$, add a sink arc (i,t) with capacity $-b_i$. (ii) Solve the maximum flow problem from s to t; if the maximum flow saturates all the source arcs, there exists a feasible solution for the convex cost flow problem of concern. Such a solution also indicates the conservation of the flow; that is

$$\sum_{b_i>0} b_i + \sum_{b_i<0} b_i = 0 \quad \text{or} \quad \sum_{i\in X} b_i = 0. \tag{4}$$

For simplicity and without lost of generality, we assume that there is at most one arc associated with each ordered pair of vertices (i, j), and that all arc costs are non-negative. In addition, we consider only the case where the cost functions $k_{ij}(f_{ij})$ are convex and differentiable. Mathematically,

$$k_{ij}(\lambda x+(1-\lambda)y) \leq \lambda k_{ij}(x)+(1-\lambda)k_{ij}(y), \quad \forall \lambda \in [0,1]. \tag{5}$$

Denoting the right-hand and the left-hand derivatives of $k_{ij}(x)$ by $k_{ij}^{+}(x)$ and $k_{ij}^{-}(x)$, respectively, two useful properties of the convex cost functions of interest are stated as follows:

Lemma 1. If the cost function $k_{ij}(x)$ is differentiable at every point in which it is defined, then $k_{ij}^{-}(x) = k_{ij}^{+}(x)$.

Lemma 2. Since condition (2) needs to be satisfied, the cost function $k_{ij}(x)$ is defined only with positive x. The point x_o minimises $k_{ij}(x)$ if and only if $k_{ij}^{+}(x_o) \geq 0$ and $k_{ij}^{-}(x_o) \leq 0$ for $x_o > 0$, and $k_{ij}^{+}(x_o) \geq 0$ for $x_o = 0$.

3.0 OPTIMALITY CONDITIONS

In this section, we describe the optimality conditions for the convex cost flow problem. The conditions are extended according to the primal-dual method of linear programming and modified to suit the case of convex and differentiable cost functions. In the primal-dual method, a variable known as the *dual variable* is associated with each constraint, and optimality conditions for these dual variables are established to make the flow optimum (Dantzig, 1963; Trustrum, 1971). In our approach, we assign a price P_i to each vertex i. We shall show that an optimal solution for the convex cost flow problem can be found based on the optimality conditions as shown in the following theorem.

THEOREM 1. *A flow* $\{f_{ij}\}$ *in* $G(X,E)$ *satisfying the condition*

$$\sum_{j:(i,j)\in E} f_{ij} - \sum_{j:(j,i)\in E} f_{ji} = b_i \qquad \forall i \in X \tag{6}$$

minimises the total cost

$$z(f) = \sum_{(i,j)\in E} k_{ij}(f_{ij}), \tag{7}$$

with convex and differentiable cost function $k_{ij}(f_{ij}) > 0$ *if and only if there exists for each vertex* i *a price* P_i *for which the following conditions hold:*

$$P_j - P_i \le k_{ij}^{+}(f_{ij}) \qquad \text{if } f_{ij} = 0 \tag{8}$$

$$k_{ij}^{-}(f_{ij}) \le P_j - P_i \le k_{ij}^{+}(f_{ij}) \quad \text{if } f_{ij} > 0. \tag{9}$$

Proof:

1. Necessary condition: Let conditions (8) and (9) be satisfied for a flow $\{\overline{f}_{ij}\}$. We will show that $\{\overline{f}_{ij}\}$ is optimum; that is, it minimises the total cost of flow. A flow is not optimum if there exists an arc (i,j) such that $f_{ij} > 0$ and $f_{ji} > 0$; this is because we can reduce the cost by setting $f_{ij} = f_{ij} - f_{ji}$ and $f_{ij} = 0$, an adjustment that still preserves the conservation of flow requirement at the vertex. We also impose $f_{ij} = 0$ if arc (i,j) does not lie in G at all. Hence, equation (6) can be rewritten as follows.

$$\sum_{j=1}^{N} f_{ij} - \sum_{j=1}^{N} f_{ji} = b_i \qquad \forall i \in X. \tag{10}$$

Substituting (10) into the total cost function (7), we have

$$\begin{aligned}\sum_{i,j=1}^{N} k_{ij}(f_{ij}) &= \sum_{i,j=1}^{N} k_{ij}(f_{ij}) + \sum_{i=1}^{N} P_i\left(\sum_{j=1}^{N} f_{ij} - \sum_{j=1}^{N} f_{ji} - b_i\right) \\ &= \sum_{i,j=1}^{N} k_{ij}(f_{ij}) + \sum_{i=1}^{N} P_i \sum_{j=1}^{N} f_{ij} - \sum_{i=1}^{N} P_i \sum_{j=1}^{N} f_{ji} - \sum_{i=1}^{N} P_i b_i \end{aligned} \tag{11}$$

since indexes i and j have the same role in the summation $\sum_{i=1}^{N} P_i \sum_{j=1}^{N} f_{ji}$. Therefore,

$$\sum_{i=1}^{N} P_i \sum_{j=1}^{N} f_{ji} = \sum_{j=1}^{N} P_j \sum_{i=1}^{N} f_{ij} = \sum_{i,j=1}^{N} P_j f_{ij}. \tag{12}$$

Substituting (12) into (11), we have

$$\sum_{i,j=1}^{N} k_{ij}(f_{ij}) = \sum_{i,j=1}^{N} (k_{ij}(f_{ij}) + (P_i - P_j)f_{ij}) - \sum_{i=1}^{N} P_i b_i. \tag{13}$$

Since the flow $\overline{f}_{ij}$ satisfies conditions (8) and (9), we have

$$k_{ij}^{-}(\overline{f}_{ij}) \leq P_j - P_i \leq k_{ij}^{+}(\overline{f}_{ij}) \qquad \text{if } \overline{f}_{ij} > 0, \tag{14}$$

$$P_j - P_i \leq k_{ij}^{+}(\overline{f}_{ij}) \qquad \text{if } \overline{f}_{ij} = 0. \tag{15}$$

Moreover, letting $g_{ij}(f_{ij}) = k_{ij}(f_{ij}) + (P_i - P_j)f_{ij}$, we have

$$g_{ij}^{+}(f_{ij}) = k_{ij}^{+}(f_{ij}) + (P_i - P_j), \tag{16}$$

$$g_{ij}^{-}(f_{ij}) = k_{ij}^{-}(f_{ij}) + (P_i - P_j). \tag{17}$$

Hence, we can see that $g^{+}(\overline{f}_{ij}) \geq 0$ and $g^{-}(\overline{f}_{ij}) \leq 0$ for $\overline{f}_{ij} > 0$, and $g^{+}(\overline{f}_{ij}) \geq 0$ for $\overline{f}_{ij} = 0$. Obviously, because cost function $k_{ij}(f_{ij})$ is convex, $g_{ij}(f_{ij}) = k_{ij}(f_{ij}) + (P_i - P_j)f_{ij}$ is also convex. We can then apply **Lemma 2** and conclude that the flow $\overline{f}_{ij}$ makes the cost function $g(f_{ij})$ minimum for every arc (i, j). From (13), since $\sum_{i=1}^{N} P_i b_i$ is a constant, this indicates that the total cost is minimized. Hence, the flow $\overline{f}_{ij}$, which satisfies conditions (8) and (9), is optimum.

2. Sufficient condition: Let $\overline{f}_{ij}$ be an optimum flow. We will show that there exists a price set $\{P_i\}$ that satisfies conditions (8) and (9). First, we assign an appropriate price set as follows: Consider the sub-graph $\overline{G}(X, \overline{E})$ of $G(X, E)$ which has the same set of vertices as G and the arc set $\overline{E} = \{(i, j) : k_{ij}^{+}(f_{ij}) = k_{ij}^{-}(f_{ij}) = k'_{ij}(\overline{f}_{ij})\}$. Obviously, on every arc (i, j) of $\overline{G}$ the flow $f_{ij} > 0$, since $f_{ij} = 0$ the left derivative $k_{ij}^{-}(f_{ij})$ is not defined. Assume the graph $\overline{G}$ consists of a number of connected sub-graphs. We choose an arbitrary vertex from each connected sub-graph, and assign the price 0. We then assign the prices to all other vertices in the following manner: Let P_i be the price already assigned to vertex i. We assign

$$P_j = P_i + k'_{ij}(\overline{f}_{ij}) \qquad \text{if } \overline{f}_{ij} > 0, \tag{18}$$

$$P_j = P_i - k'_{ij}(\overline{f}_{ij}) \qquad \text{if } \overline{f}_{ji} > 0. \tag{19}$$

The price P_j is said to be coordinated to j along the arc (i, j). Now, each vertex has been provided with a price. We now show that in this way, the price set assigned will satisfy conditions (8) and (9).

Assume that in $\overline{G}$ there exists an arc (u,v) with $\overline{f}_{uv} > 0$ and $P_v - P_u \neq k'_{uv}(\overline{f}_{uv})$; this violates condition (9). According to the way we assign the prices, the price P_u has been coordinated to u along a chain $(i_o, i_1, \ldots, i_m = u)$ and the price P_v to v along a chain $(i_o = i_p, i_{p-1}, \ldots, i_{m+1} = v)$. Together with the arc (u,v) we obtain a coordinated cycle (Figure 1):

$$\mu = (i_o, i_1, \ldots, i_m = u, i_{m+1} = v, \ldots, i_p = i_o). \tag{20}$$

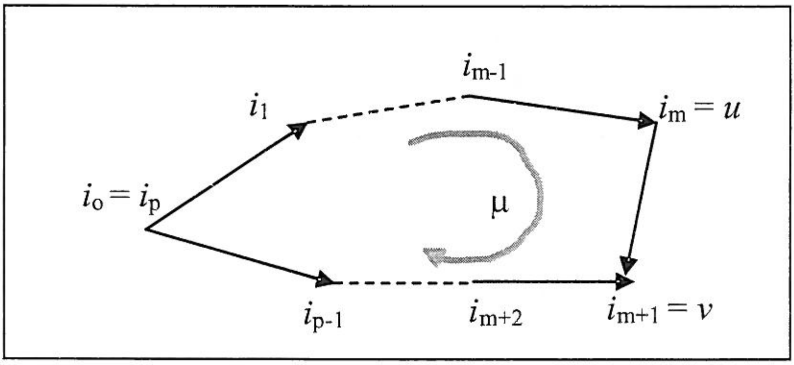

Fig. 1: Illustration of the μ Coordinated Cycle

The flow going on each arc of μ is always positive. Now, we modify the flows on the arcs of μ slightly in the following way: On the arcs of μ^+ (arcs on which the flow goes in the same direction as the orientation of μ), we increase the flows by a value h; on the arcs of μ^-, we decrease the flows by the value h. The flows on all the other arcs remain unchanged. Obviously, the vertex condition (6) remains unaffected.

Consider the cost difference resulting between the old and the new flows

$$\Delta = \sum_{(i,j)\in\mu^+} \left(k_{ij}(\overline{f}_{ij}) - k_{ij}(\overline{f}_{ij} + h)\right) + \sum_{(i,j)\in\mu^-} \left(k_{ij}(\overline{f}_{ij}) - k_{ij}(\overline{f}_{ij} - h)\right). \quad (21)$$

Using the Taylor series, we obtain

$$\Delta = h\left(- \sum_{(i,j)\in\mu^+} k'_{ij}(\overline{f}_{ij}) + \sum_{(i,j)\in\mu^-} k'_{ij}(\overline{f}_{ij}) \right) + o(h). \quad (22)$$

According to the formation rules (18) and (19) for P_i, we can say that if there is a flow from i to j, then $k'_{ij}(f_{ij}) = P_j - P_i$. Hence, equation (22) assumes the following form

$$\begin{aligned}\Delta &= h(P_{io} - P_{i1} + P_{i1} - \cdots + P_{m-1} - P_u - k'_{uv}(\overline{f}_{ij}) + P_v - P_{m+2} + \cdots - P_{io}) + o(h) \\ &= h(P_v - P_u - k'_{uv}(\overline{f}_{uv})) + o(h). \end{aligned} \quad (23)$$

For sufficiently small $h > 0$, however, this expression is certainly greater than zero if $P_v - P_u > k'_{uv}(\overline{f}_{uv})$. But if $P_v - P_u < k'_{uv}(\overline{f}_{uv})$, we modify the flow in the opposite way with the above; that is, on the arcs μ^+, we decrease the flow by h; on that of μ^-, we increase it by h. The resulting difference can be obtained in an analogous way. It is easy to see that by modifying the flow in this way, the total cost of flow can be lower. However, this contradicts with the fact that $\overline{f}_{ij}$ is an optimum flow. Hence, we can conclude our assumption that there exists an arc (u,v) with $\overline{f}_{uv} > 0$ and $P_v - P_u \neq k'_{uv}$ is not true. In other words, the price set we defined satisfies conditions (8) and (9). Thus, the theorem follows.

Taking the capacity u_{ij} of arc (i,j) into account when $f_{ij} = 0$ or $f_{ij} = u_{ij}$, only either the right-hand or left-hand derivative of $k_{ij}(f_{ij})$ exists; hence the cost function $k_{ij}(f_{ij})$ is only differentiable for $0 < f_{ij} < u_{ij}$. From **Lemma 1**, we have

$k_{ij}^{+}(f_{ij}) = k_{ij}^{-}(f_{ij})$ for $0 < f_{ij} < u_{ij}$. Consequently, we can state the optimal conditions as follows.

THEOREM 2. *Let the cost functions be convex and differentiable, then a flow $\{f_{ij}\}$ is optimum if and only if there exists a price set $\{P_i\}$ such that*

$$P_j - P_i \le k_{ij}^{+}(f_{ij}) \quad \text{for} \quad f_{ij} = 0, \tag{24}$$

$$P_j - P_i = k_{ij}^{+}(f_{ij}) \quad \text{for} \quad 0 < f_{ij} < u_{ij}, \tag{25}$$

$$P_j - P_i \ge k_{ij}^{-}(f_{ij}) \quad \text{for} \quad f_{ij} = u_{ij}. \tag{26}$$

4.0 THE ALGORITHM

According to the optimal conditions obtained in the previous section, we can now present the proposed primal dual algorithm for solving the convex cost flow problem as follows.

Begin
  Find a feasible flow;
  While P_i not satisfying optimality conditions ***Do***
  Begin
    Build $\overline{G}(X, \overline{E})$ where $\overline{E} = \{(i,j) : k_{ij}^{+}(f_{ij}) = k_{ij}^{-}(f_{ij}) = k'_{ij}(f_{ij})\}$;
    Select a vertex u in every connected sub-graph in $\overline{G}$ and set $P_u = 0$;
    For every vertex j of arc (i, j) that P_i is assigned ***Do***
    Begin
      If $f_{ij} > 0$ ***then*** $P_j = P_i + k'_{ij}(f_{ij})$;
      If $f_{ji} > 0$ ***then*** $P_j = P_i - k'_{ij}(f_{ij})$;
    End
    If there exist P_u and P_v not satisfying optimality conditions ***then***
    Begin
      Find the coordinated cycle μ that contains u and v;
      Modify the flow along μ;
    End
  End
End

In the algorithm, we assign the price set according to criteria (18) and (19); that is, the price of vertex j is assigned based on the price P_i of vertex i which has previously been assigned. Price P_j is then said to be coordinated with the vertex j along the arc (i, j). If there exist vertices u and v such that P_u and P_v do not satisfy the optimality conditions, we will find the coordinated cycle $\mu = ({}_o, {}_1, \quad, {}_m = \quad, {}_{m+1} = \quad, \quad, {}_p = {}_o)$, where P_{i_t} is coordinated to i_t along the arc (i_{t-1}, i_t). The flow of μ is modified to lower the total cost as follows. On the arcs where the flow goes in the same (or opposite) direction as the orientation of μ, the flow is increased (or reduced) by a value h. The value of h is found to minimise the cost incurred in the cycle μ.

5.0 SIMULATION RESULTS

Two examples of simulation results are reported in this section to show the efficacy of the proposed algorithm.

Example 1:

This example is to show that our algorithm can be applied to simple minimum cost flow problems, in which the cost functions are linear functions of the flows. Consider sending 4 units of data from vertex s to vertex t in the network shown in Figure 2, in which the flow at each vertex is conservative except at s and t, that is $b_i = 0$ for $i \neq s, t$. On each arc (i, j) in the graph, we indicate in a bracket its cost function of k_{ij}, followed by its capacity u_{ij} and the current flow f_{ij}.

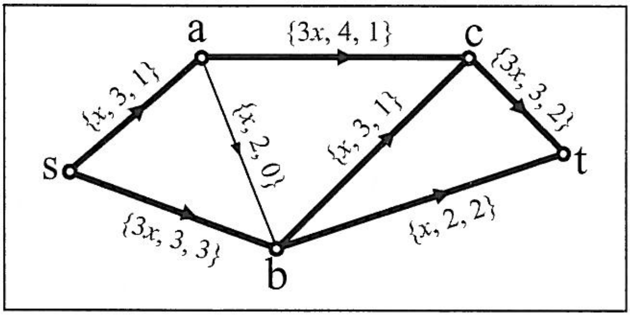

Fig. 2: Sending 4 Units of Data from Vertex s to Vertex t

Let us start with the feasible solution as indicated in Figure 2. We form the sub-graph $\overline{G}(X,\overline{E})$ from the graph G with the edge set $\overline{E}=\{(i,j):k_{ij}^{+}(f_{ij})=k_{ij}^{-}(f_{ij})=k'_{ij}(f_{ij})\}$. It can be seen that the sub-graph $\overline{G}$ consists of arcs (s,a), (a,c), (b,c) and (c,t). Arcs (s,b) and (b,t) are not included, since $k_{sb}^{+}(f_{sb})$ and $k_{bt}^{+}(f_{bt})$ are not defined. We define the price starting with vertex s with price $P_s=0$. The prices of other vertices are assigned according to the criteria (18) and (19) as shown in Figure 3.

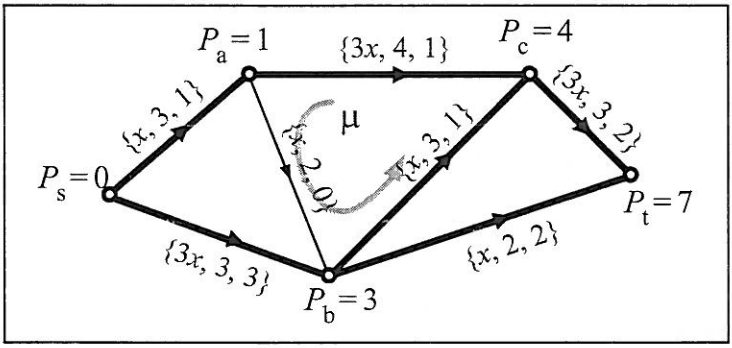

Fig. 3: Assigning According to the Criteria (18) and (19)

Apparently, the arc having the cost $k_{ab}=x$ and the flow $f_{ab}=0$ does not satisfy the optimality conditions. The price difference on this arc should be less than $k_{ab}^{+}(f_{ab})=1$, but in fact it is 2. It is possible to lower the total cost by reducing the cost along the cycle μ. On the arcs where the flow goes in the same (or opposite) direction as the orientation of μ, the flow is increased (or reduced) by a value h. The new cost for this cycle will be $3(1-h)+(1+h)+h$. To satisfy the capacity conditions on all the arcs along μ, the value h should be equal or smaller than 1. Hence, the cost achieves minimum at $h=1$. Figure 4 shows the new flow and modified price set.

As can be seen, the total cost has been decreased from 22 to 21. Again, we see that arc (s,b) does not satisfy the optimality conditions. The same steps described above can be repeated until the optimal conditions are satisfied in all sub-graphs. In the end, the total cost incurred by the optimum flow as shown in Figure 5 is 20, which is 2 units less than the total cost of the beginning feasible solution.

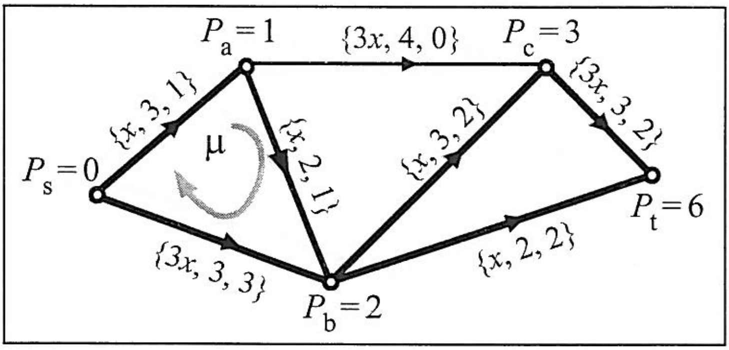

Fig. 4: New Flow and Modified Price Set

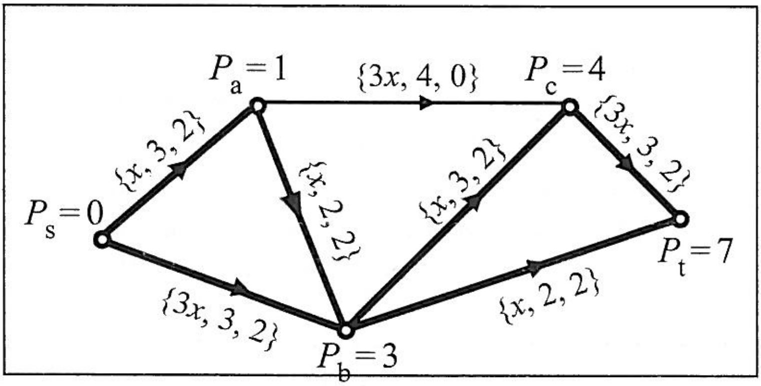

Fig. 5: Total Cost Incurred by the Optimum Flow

Example 2:

This example utilises the same network as in Example 1, but the cost function associated with each arc can be a convex and differentiable function. Consider sending 2 units of data from vertex s to vertex t. Similar to Example 1, we consider the case that the flow at each vertex is conservative except at s and t, that is $b_i = 0$ for $i \neq s,t$. On every arc of the graph shown in Figure 6, we indicate in a bracket the cost function k_{ij} followed by the current flow f_{ij}.

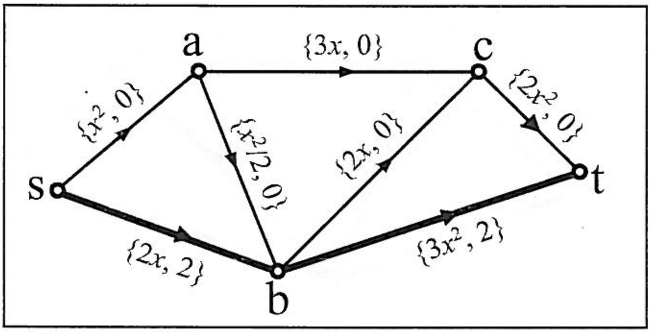

Fig. 6: Graph Comprising Cost Function and Current Flow

We start with a feasible solution by sending 2 units of data through the path $\{s,b,t\}$. A sub-graph $\overline{G}(X,\overline{E})$ is formed from the graph G with the edge set $\overline{E}=\{(i,j): k_{ij}^{+}(f_{ij})=k_{ij}^{-}(f_{ij})=k'_{ij}(f_{ij})\}$. It can be seen that the sub-graph $\overline{G}$ consists of only two arcs (s,b) and (b,t). Thus, there are 3 connected sub-graphs in $\overline{G}$: $\{s,b,t\}$, $\{a\}$, and $\{c\}$. We then assign a zero price to an arbitrary vertex in each sub-connected graph. In this case, we assign P_s, P_a, P_c to be 0. The prices of other vertices are assigned based on criteria (18) and (19); that is $P_b=2$ and $P_t=14$ (see Figure 7).

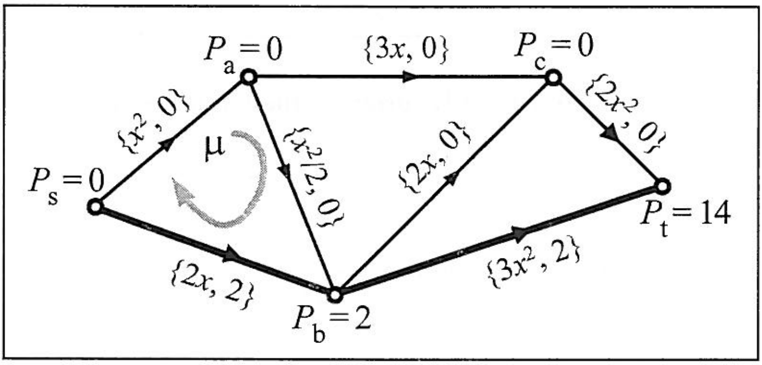

Fig. 7: Vertices Assigned Based on Criteria (18) and (19)

Obviously, the arc having the cost function $k_{ab}=x^2/2$ and the flow $f_{ab}=0$ does not satisfy the optimality conditions. The price difference on this arc should be less than $k_{ab}^{+}(f_{ab}) = 0$, but it turns out to be 2. Similar to Example 1, the total cost can be lowered by reducing the cost along the cycle μ. That is, on the arcs where the flow goes in the same (or opposite) direction as the orientation of μ, the flow is increased (or reduced) by a value h. The new cost for this cycle is now $h^2 + h^2/2 + 2(2-h)$. This cost achieves minimum at $h = 2/3$. Figure 8 shows the new flow and modified price set.

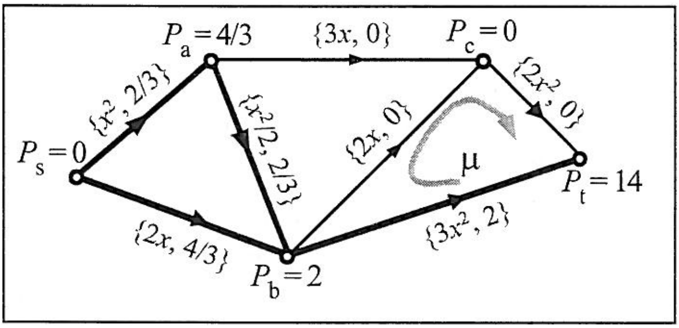

Fig. 8: New Flow and Price Set

At $h = 2/3$, the total cost has been decreased from 16 to $15\frac{1}{3}$. However, the arc (c,t) has not yet satisfied the optimality conditions. The same steps described above can be repeated until the optimal conditions are satisfied in all sub-graphs. In the end, the total cost incurred by the optimum flow as shown in Figure 9 is $10\frac{1}{3}$, which is a significant reduction from the initial total cost of 16.

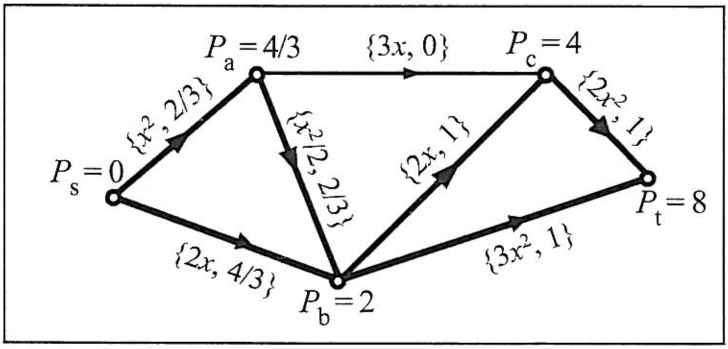

Fig. 9: Cost Incurred by the Optimum Flow

6.0 CONCLUDING REMARKS AND FUTURE WORK

We have presented an algorithm for the convex cost flow problem. This algorithm is based on the primal-dual method extended from the conventional MCF problem and optimality conditions modified for the case of convex and differentiable cost functions. Experimental results show that the proposed algorithm can find the minimum cost flows with both linear and convex cost functions.

The proposed algorithm can be applied to solve practical network flow problems of convex and differentiable cost functions. However, in practice, the cost functions are even more complex than just either linear or convex and differentiable to apply our approach directly. As an example, let us consider a data network whose network attributes, such as delay, loss probability, transmission cost, depend on the traffic load carried by each transmission link. The goal is to select the link capacities (or transmission bandwidths) in order to minimise a linear cost $\sum_{(i,j)} p_{ij}B_{ij}$, subject to the constraint that the average delay per packet should not exceed a given constant T, where B_{ij} is the capacity of each link and p_{ij} is the cost per unit capacity. According to the network model (see Bertsekas and Gallager, 1992), we can express the average delay constraint as

$$\frac{1}{\gamma}\sum_{(i,j)}\frac{f_{ij}}{B_{ij}-f_{ij}} \leq T, \tag{27}$$

where γ is the total data arrival rate of the network. Intuitively, the constraint can be satisfied as an equality to minimise the capacity cost. Let β be a Lagrange multiplier of the following Lagrangian function.

$$L = \sum_{(i,j)}\left(p_{ij}B_{ij} + \frac{\beta}{\gamma}\frac{f_{ij}}{B_{ij}-f_{ij}}\right). \tag{28}$$

In accordance with the Lagrange multiplier technique, solving $\partial L/\partial B_{ij} = 0$ gives

$$B_{ij} = f_{ij} + \sqrt{\frac{\beta f_{ij}}{\gamma p_{ij}}}. \tag{29}$$

From (27) and (29), we have

$$\sqrt{\beta} = \frac{1}{T} \sum_{(i,j)} \sqrt{\frac{p_{ij} f_{ij}}{\gamma}}. \quad (30)$$

Substitute into the cost function, the optimal cost can be expressed as

$$z(f) = \sum_{(i,j)} p_{ij} f_{ij} + \frac{1}{\gamma T} \left(\sum_{(i,j)} \sqrt{p_{ij} f_{ij}} \right)^2. \quad (31)$$

The cost function (31) has many local minima and is very difficult to minimize. It is a hard combinatorial problem. One possible future work is to extend the proposed algorithm, in conjunction with some heuristic methods suggested in (Dantzig, 1963; Trustrum, 1971), to solve this problem.

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