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***Abstract*—We provide two interpretations, over nondeterministic and probabilistic processes, of PML, the probabilistic version of Hennessy-Milner logic used by Larsen and Skou to characterize bisimilarity of probabilistic processes without internal nondeterminism. Then, we exhibit two new bisimulation-based equivalences for nondeterministic and probabilistic processes, which are in full agreement with the two different interpretations of PML. The new equivalences are coarser than the bisimilarity for nondeterministic and probabilistic processes proposed by Segala and Lynch, which instead is in agreement with a version of Hennessy-Milner logic extended with an additional probabilistic operator interpreted over state distributions rather than over individual states. The modal logic characterizations provided for the new equivalences thus offer a uniform framework for reasoning on purely nondeterministic processes, reactive probabilistic processes, and alternating and non-alternating nondeterministic and probabilistic processes.**

I. Introduction

Modal logics and behavioral equivalences play a key rôle in the specification and verification of concurrent systems. The former are useful for model checking, in that they can be employed for specifying the properties to be verified. The latter are ancillary to the former, in the sense that they enable the transformation/minimization of models to be checked while guaranteeing that specific classes of properties are preserved.

Because of this, whenever a new behavioral relation is proposed, the quest starts for the associated modal logic, i.e., for a logic such that two systems are behaviorally equivalent if and only if they satisfy the same logical formulae. The first result along this line is due to Hennessy and Milner [15]. They showed that *bisimilarity* over *fully nondeterministic processes*, each modeled as a labeled transition system (LTS) [18], is in full agreement with a very simple modal logic, now known as HML. This logic has only the four operators $\texttt{true}$, $\cdot \wedge \cdot$, $\neg\cdot$, and $\langle a \rangle \cdot$, where the last one is called *diamond* and is used to describe the existence of a-labeled transitions. After this result, whenever any of the many quantitative variants of process description languages and process models has been introduced, other behavioral equivalences and modal logics have been defined and analogous results have been established to handle features such as probability and time.

Most of the works along the lines outlined above take as starting point a behavioral equivalence and then look for the logic in agreement with it. Obviously, it is also interesting, once one has fixed a model and a logic to reason about it, to find out the "right" behavioral relation.

A first work in the latter direction was [4]; starting from CTL interpreted over Kripke structures (state-labeled transition systems) [6], it showed that bisimilarity and stuttering bisimilarity are respectively in full agreement with the logical equivalences induced by CTL* and CTL* without the next-time operator. In the subsequent work [1], it was shown that the equivalence induced by PCTL* interpreted over probabilistic Kripke structures coincides with probabilistic bisimilarity. A more recent work is [28]; starting from PCTL interpreted over nondeterministic and probabilistic Kripke structures [3], it defined new probabilistic bisimilarities that fully characterize the logical equivalences induced by PCTL, PCTL*, and their variants without the next-time operator.

In this paper, we concentrate on the results obtained for extended LTS models that have been developed to deal with *probabilistic systems*. We look for bisimilarities that are in agreement with a probabilistic variant of HML known as PML [19], [20]. This is obtained by simply decorating the diamond operator with a probability bound. Formula $\langle a \rangle_p \phi$ is satisfied by state s if an a-labeled transition is possible from s after which a set of states satisfying ϕ is reached with probability at least p.

Modal logic characterizations for probabilistic bisimilarities have been studied for the first time by Larsen and Skou [19], [20]. They introduced probabilistic bisimilarity for *reactive probabilistic processes* [30] and showed that it is in full agreement with PML. Subsequently, Desharnais et al [11] showed that PML without negation is sufficient to characterize probabilistic bisimilarity for the same class of processes. Reactive probabilistic processes are LTS-based models where (i) every action-labeled transition reaches a probability distribution over states and (ii) the actions labeling transitions departing from the same state are all different from each other.

Segala and Lynch [25] defined, instead, a probabilistic bisimilarity for a more expressive model that also admits *internal nondeterminism*, i.e., the possibility for a state to have several outgoing transitions labeled with the same action. For this probabilistic bisimilarity over *nondeterministic and probabilistic processes*, Segala and collaborators [21], [16] exhibited a logical characterization in terms of an extension of HML, in which formulae satisfaction is defined over *probability distributions on states* rather than over single states. The logic is obtained from HML by giving the diamond operator a universal interpretation (all states in the support

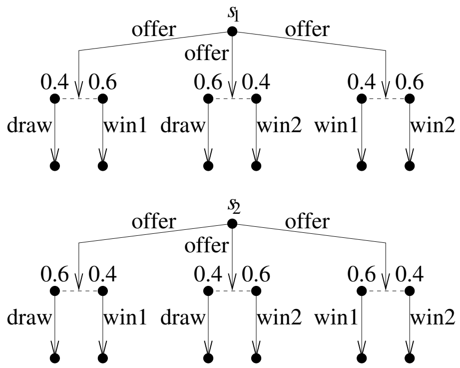

Fig. 1. Two games with the same set of winning probabilities ($\sim_{\mathrm{PB,gbg,=}}$)

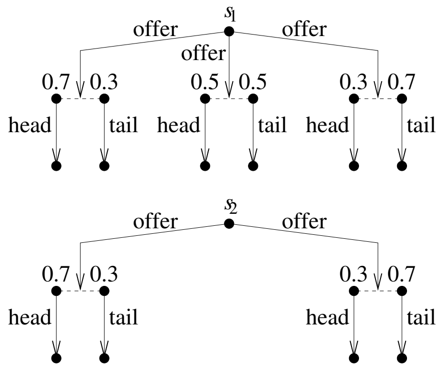

Fig. 2. Two games with the same extremal probabilities ($\sim_{\mathrm{PB,gbg,\leq}}$)

of a distribution must satisfy the formula) and by adding a unary operator $[\cdot]_p$ such that $[\phi]_p$ is true on a state distribution whenever the probability of the set of states that satisfy formula ϕ is at least p.

The above-mentioned variant of HML has been reconsidered in a number of subsequent works. In [8], D'Argenio et al revised the logic by distinguishing between state formulae including the diamond operator (interpreted as in HML over states) and measure formulae including the new unary operator (interpreted as before over state distributions). More recently, Crafa and Ranzato [7] showed an equivalent formulation of the logic that retrieves the HML interpretation of the diamond operator by lifting the transition relation to state distributions. Following a similar lifting, Hennessy [14] proposed an alternative logical characterization based on what he calls pHML, where a binary operator $\cdot \oplus_p \cdot$ is added to HML (instead of the unary operator $[\cdot]_p$) such that $\phi_1 \oplus_p \phi_2$ asserts decomposability of a state distribution to satisfy the two subformulae.

Now, the difference between PML and the two probabilistic extensions of HML in [21] and [14] is quite striking. It is thus interesting to understand whether such a difference is due to the different expressive power of the models in [19] and [25] – i.e., the absence or the presence of internal nondeterminism – or to the way probabilistic bisimilarity was defined on those two models. Since in [21] it was shown that PML characterizes probabilistic bisimilarity over processes *alternating* nondeterminism and probability like those in [13] (strictly alternating) and [32], [22] (non-strictly alternating), we feel it is worth exploring alternative definitions of probabilistic bisimilarity rather than alternative models.

The aim of this paper is to show that it is possible to define new probabilistic bisimilarities for *non-alternating* nondeterministic and probabilistic processes [24] that are characterized by PML. Our result is somehow similar to the one established in [28], where new probabilistic bisimilarities over nondeterministic and probabilistic Kripke structures were exhibited that are characterized by variants of PCTL. In both cases, the starting point for defining the new probabilistic bisimilarities is the consideration (see also [9]) that sometimes the definition of Segala and Lynch [25] might be overdiscriminating and thus differentiate processes that, according to intuition, should be identified.

Indeed, to compare systems where both nondeterminism and probabilistic choices coexist, in [24], [25] the notion of *scheduler* (or *adversary*) is used to resolve internal nondeterminism. A scheduler can be viewed as an external entity that selects the next action to perform according to the current state and the past history. When a scheduler is applied to a system, a fully probabilistic model called a *resolution* is obtained. The basic idea is deeming equivalent two systems if and only if for each resolution of one system (the *challenger*) there exists a resolution of the other (the *defender*) such that the two resolutions are probabilistic bisimilar in the sense of [19].

Let us consider the two scenarios in Fig. 1 modeling the offer to Player1 and Player2 of three differently biased dice. The game is conceived in such a way that if the outcome of a throw gives 1 or 2 then Player1 wins, while if the outcome is 5 or 6 then Player2 wins. In case of 3 or 4, the result is a draw. For instance, with the biased die associated with the leftmost branch of the first scenario, it happens that 3 or 4 (draw) will appear with probability 0.4, while 1 or 2 (Player1 wins) will appear with probability 0.6. Numbers 5 and 6 will never appear (no chance for Player2 to win).

The probabilistic bisimilarity proposed in [25] distinguishes the two models in Fig. 1 even if the *set of probabilities* of winning and of drawing for each player are the same. To identify these models, from a bisimulation perspective the impact of schedulers needs to be weakened. While in [25] the challenger and the defender must stepwise behave the same along any two matching resolutions, here the defender should be allowed to choose different resolutions in response to different directions taken by the challenger. In other words, instead of requiring as in [25] that for each resolution of the challenger there is a *fully matching resolution* of the defender, it might be admissible to consider bisimulation games with *partially matching resolutions* in the same vein as [29].

Other two systems differentiated (under deterministic schedulers) by the probabilistic bisimilarity in [25] are those in Fig. 2. In the first scenario, the two players are offered a choice among a fair coin and two biased ones. In the second scenario, the players can simply choose between the two biased coins of the former scenario. In both scenarios, Player1 wins with head while Player2 wins with tail. In our view, the two scenarios should be identified if what matters is that in each of them the two players have exactly the *same extremal (i.e., minimal and maximal) probabilities* of winning.

The first probabilistic bisimilarity we shall introduce – denoted by $\sim_{\rm PB,gbg,=}$ – identifies the two systems in Fig. 1, but distinguishes those in Fig. 2. Our second probabilistic bisimilarity – denoted by $\sim_{\rm PB,gbg,\leq}$ – instead identifies both the two systems in Fig. 1 and the two systems in Fig. 2. Notably, the same identifications are induced by one of the probabilistic bisimilarities in [28]. Indeed, once the appropriate transformations (eliminating actions from transitions and labeling each state with the set of possible next-actions) are applied to get nondeterministic and probabilistic Kripke structures from the four systems in Figs. 1 and 2, we have that no PCTL* formula distinguishes the two systems in Fig. 1 and the two systems in Fig. 2. We shall, however, see that neither $\sim_{\rm PB,gbg,=}$ nor $\sim_{\rm PB,gbg,\leq}$ coincides with the probabilistic bisimilarities in [28].

We shall show that $\sim_{\rm PB,gbg,\leq}$ is precisely characterized by the original PML as defined by Larsen and Skou [19], [20], with the original interpretation of the diamond operator: state s satisfies $\langle a \rangle_p \phi$ if s has an a-transition that reaches with probability at least p a set of states satisfying ϕ. In contrast, $\sim_{\rm PB,gbg,=}$ is characterized by a variant of PML having an interval-based operator $\langle a \rangle_{[p_1,p_2]}\cdot$ instead of $\langle a \rangle_p\cdot$: state s satisfies $\langle a \rangle_{[p_1,p_2]}\phi$ if s has an a-transition that reaches with probability between p_1 and p_2 a set of states satisfying ϕ. We shall refer to the interpretation of these two diamond operators as *existential* because it simply requires that there exists a way to resolve internal nondeterminism that guarantees satisfaction of formula ϕ within a certain probability range.

For both logics, we shall also provide an alternative interpretation of the diamond operator, which is inspired by the actual interpretation of PCTL* in [3]. We shall call *universal* this interpretation that might appear more appropriate in a nondeterministic and probabilistic setting. With this interpretation, state s satisfies $\langle a \rangle_p \phi$ (resp. $\langle a \rangle_{[p_1,p_2]}\phi$) if it has an a-transition that enjoys the same property as before and each a-transition departing from s enjoys that property, meaning that the formula is satisfied by s *no matter how internal nondeterminism is resolved*. We shall see that both universally interpreted variants of the logic lead to the same equivalence as the one characterized by the original interpretation of the original PML. Indeed, $\sim_{\rm PB,gbg,\leq}$ has also many other characterizations, and this leads us to the convincement that it is an interesting behavioral relation for nondeterministic and probabilistic processes.

The rest of the paper is organized as follows. In Sect. II, we provide the necessary background about the non-alternating model of nondeterministic and probabilistic processes, the bisimilarities in [15], [19], [25], and their modal logic characterizations. The two interpretations of PML over the non-alternating model are introduced in Sect. III and the new probabilistic bisimilarities that they characterize are presented in Sect. IV. In Sect. V, we provide further motivations, variants and results for the new probabilistic bisimilarities. Finally, Sect. VI draws some conclusions and hints at possible future work. All proofs of our results are collected in Appendix A.

II. Background

In this section, we present a model for nondeterministic and probabilistic processes. Then, we recast in this general model the bisimilarity in [15] and the probabilistic bimilarity in [19], together with their modal logic characterizations respectively based on HML and PML. Finally, we recall the probabilistic bisimilarity in [25] and its modal logic characterization for the non-alternating case and for the alternating case.

A. The NPLTS Model

Processes combining nondeterminism and probability are typically described by means of extensions of the LTS model, in which every action-labeled transition goes from a source state to a *probability distribution over target states* rather than to a single target state. The resulting processes are essentially Markov decision processes [10] and are representative of a number of slightly different probabilistic computational models including internal nondeterminism such as, e.g., concurrent Markov chains [31], strictly alternating models [13], probabilistic automata in the sense of [24], and the denotational probabilistic models in [17] (see [27] for an overview). We formalize them as a variant of simple probabilistic automata [24].

Definition 2.1: A *nondeterministic and probabilistic labeled transition system*, NPLTS for short, is a triple $(S, A, \longrightarrow)$ where:

- S is an at most countable set of states.
- A is a countable set of transition-labeling actions.
- $\longrightarrow \subseteq S \times A \times Distr(S)$ is a transition relation, where $Distr(S)$ is the set of probability distributions over S. ■

A transition $(s, a, \mathcal{D})$ is written $s \xrightarrow{a} \mathcal{D}$. We say that $s' \in S$ is not reachable from s via that a-transition if $\mathcal{D}(s') = 0$, otherwise we say that it is reachable with probability $p = \mathcal{D}(s')$. The reachable states form the support of $\mathcal{D}$, i.e., $supp(\mathcal{D}) = \{s' \in S \mid \mathcal{D}(s') > 0\}$. We write $s \xrightarrow{a}$ to indicate that s has an a-transition. The choice among all the transitions departing from s is external and nondeterministic, while the choice of the target state for a specific transition is internal and probabilistic.

The notion of NPLTS yields a non-alternating model [24] and embeds the following restricted models:

- *Fully nondeterministic processes*: every transition is Dirac, i.e., it leads to a distribution that concentrates all the probability mass into a single target state.
- *Fully probabilistic processes*: every state has at most one outgoing transition.
- *Reactive probabilistic processes*: no state has two or more outgoing transitions labeled with the same action [30]. These processes include the probabilistic automata in the sense of [23].
- *Alternating processes*: every state that enables a non-Dirac transition enables only that transition. Similar to [32] and [22], these processes consist of a non-strict

alternation of fully nondeterministic states and fully probabilistic states, with the addition that transitions departing from fully probabilistic states are labeled with actions.

An NPLTS can be depicted as a directed graph-like structure in which vertices represent states and action-labeled edges represent action-labeled transitions. Given a transition $s \xrightarrow{a} \mathcal{D}$, the corresponding a-labeled edge goes from the vertex representing state s to a set of vertices linked by a dashed line, each of which represents a state $s' \in \mathit{supp}(\mathcal{D})$ and is labeled with $\mathcal{D}(s')$ – label omitted if $\mathcal{D}(s') = 1$. Four NPLTS models are shown in Figs. 1 and 2.

We say that an NPLTS $(S, A, \longrightarrow)$ is *image finite* iff for all $s \in S$ and $a \in A$ the set $\{\mathcal{D} \in \mathit{Distr}(S) \mid s \xrightarrow{a} \mathcal{D}\}$ is finite. Following [19], we say that it satisfies the *minimal probability assumption* iff there exists $\epsilon \in \mathbb{R}_{>0}$ such that, whenever $s \xrightarrow{a} \mathcal{D}$, then for all $s' \in S$ either $\mathcal{D}(s') = 0$ or $\mathcal{D}(s') \geq \epsilon$; this implies that $\mathit{supp}(\mathcal{D})$ is finite because it can have at most $\lceil 1/\epsilon \rceil$ elements. If $\mathcal{D}(s')$ is a multiple of ϵ for all $s' \in S$, then the *minimal deviation assumption* is also satisfied.

Sometimes, instead of ordinary transitions, we will consider *combined transitions* [25], each being a convex combination of equally labeled transitions. Given an NPLTS $(S, A, \longrightarrow)$, $s \in S$, $a \in A$, and $\mathcal{D} \in \mathit{Distr}(S)$, in the following we write $s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}$ iff there exist $n \in \mathbb{N}_{>0}$, $\{p_i \in \mathbb{R}_{]0,1]} \mid 1 \leq i \leq n\}$, and $\{s \xrightarrow{a} \mathcal{D}_i \mid 1 \leq i \leq n\}$ such that $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} p_i \cdot \mathcal{D}_i = \mathcal{D}$.

B. Bisimilarity for Fully Nondeterministic Processes

We recast in the NPLTS model the definition of bisimilarity for fully nondeterministic processes in [15]. In this case, the target of each transition is a Dirac distribution δ_s for $s \in S$, i.e., $\delta_s(s) = 1$ and $\delta_s(s') = 0$ for all $s' \in S \setminus \{s\}$.

Definition 2.2: Let $(S, A, \longrightarrow)$ be an NPLTS in which the target of each transition is a Dirac distribution. A relation $\mathcal{B}$ over S is a *bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$:

- If $s_1 \xrightarrow{a} \delta_{s'_1}$, then $s_2 \xrightarrow{a} \delta_{s'_2}$ such that $(s'_1, s'_2) \in \mathcal{B}$.
- If $s_2 \xrightarrow{a} \delta_{s'_2}$, then $s_1 \xrightarrow{a} \delta_{s'_1}$ such that $(s'_1, s'_2) \in \mathcal{B}$.

We denote by $\sim_{\mathrm{B}}$ the largest bisimulation. ■

Given an image-finite NPLTS $(S, A, \longrightarrow)$ in which the target of each transition is a Dirac distribution, the relation $\sim_{\mathrm{B}}$ is characterized by the so-called Hennessy-Milner logic (HML) [15]. The set $\mathbb{F}_{\mathrm{HML}}$ of its formulae is generated by the following grammar ($a \in A$):

$$\phi \ ::= \ \mathtt{true} \mid \neg\phi \mid \phi \wedge \phi \mid \langle a \rangle \phi$$

The semantics of HML can be defined through an interpretation function $\mathcal{M}_{\mathrm{HML}}$ that associates with any formula in $\mathbb{F}_{\mathrm{HML}}$ the set of states satisfying the formula:

$$\begin{aligned} \mathcal{M}_{\mathrm{HML}}[\![\mathtt{true}]\!] &= S \\ \mathcal{M}_{\mathrm{HML}}[\![\neg\phi]\!] &= S \setminus \mathcal{M}_{\mathrm{HML}}[\![\phi]\!] \\ \mathcal{M}_{\mathrm{HML}}[\![\phi_1 \wedge \phi_2]\!] &= \mathcal{M}_{\mathrm{HML}}[\![\phi_1]\!] \cap \mathcal{M}_{\mathrm{HML}}[\![\phi_2]\!] \\ \mathcal{M}_{\mathrm{HML}}[\![\langle a \rangle \phi]\!] &= \{s \in S \mid \exists s' \in \mathcal{M}_{\mathrm{HML}}[\![\phi]\!].\, s \xrightarrow{a} \delta_{s'}\} \end{aligned}$$

C. Bisimilarity for Reactive Probabilistic Processes

We recast in the NPLTS model also the definition of probabilistic bisimilarity for reactive probabilistic processes in [19]. In the following, we let $\mathcal{D}(S') = \sum_{s' \in S'} \mathcal{D}(s')$ for $\mathcal{D} \in \mathit{Distr}(S)$ and $S' \subseteq S$.

Definition 2.3: Let $(S, A, \longrightarrow)$ be an NPLTS in which the transitions of each state have different labels. An equivalence relation $\mathcal{B}$ over S is a *probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and equivalence classes $C \in S/\mathcal{B}$ it holds that $s_1 \xrightarrow{a} \mathcal{D}_1$ implies $s_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(C) = \mathcal{D}_2(C)$. We denote by $\sim_{\mathrm{PB}}$ the largest probabilistic bisimulation. ■

Given an NPLTS $(S, A, \longrightarrow)$ satisfying the minimal deviation assumption in which the transitions of each state have different labels, the relation $\sim_{\mathrm{PB}}$ is characterized by PML [19], [20]. The set $\mathbb{F}_{\mathrm{PML}}$ of its formulae is generated by the following grammar ($a \in A$, $p \in \mathbb{R}_{[0,1]}$):

$$\phi \ ::= \ \mathtt{true} \mid \neg\phi \mid \phi \wedge \phi \mid \langle a \rangle_p \phi$$

The semantics of PML can be defined through an interpretation function $\mathcal{M}_{\mathrm{PML}}$ that differs from $\mathcal{M}_{\mathrm{HML}}$ only for the last clause, which becomes as follows:

$$\mathcal{M}_{\mathrm{PML}}[\![\langle a \rangle_p \phi]\!] \ = \ \{s \in S \mid \exists \mathcal{D} \in \mathit{Distr}(S).\ s \xrightarrow{a} \mathcal{D} \wedge \mathcal{D}(\mathcal{M}_{\mathrm{PML}}[\![\phi]\!]) \geq p\}$$

Note that, in this reactive setting, if an a-labeled transition exists that goes from s to $\mathcal{D}$, then it is the only a-labeled transition departing from s, and hence $\mathcal{D}$ is unique.

In [11], it was subsequently shown that probabilistic bisimilarity for reactive probabilistic processes can be characterized by PML without negation and that the existence of neither a minimal deviation nor a minimal probability needs to be assumed to achieve the characterization result.

D. Bisimilarity for Non-Alternating and Alternating Processes

For NPLTS models in their full generality, we now recall two probabilistic bisimulation equivalences defined in [25]. Both of them check whether the probabilities of *all* classes of equivalent states – i.e., the *class distributions* – reached by the two transitions considered in the bisimulation game are equal.

The first equivalence relies on deterministic schedulers for resolving nondeterminism. This means that, when responding to an a-transition of the challenger, the defender can only select a single a-transition (if any).

Definition 2.4: Let $(S, A, \longrightarrow)$ be an NPLTS. An equivalence relation $\mathcal{B}$ over S is a *class-distribution probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ it holds that $s_1 \xrightarrow{a} \mathcal{D}_1$ implies $s_2 \xrightarrow{a} \mathcal{D}_2$ such that, for all equivalence classes $C \in S/\mathcal{B}$, $\mathcal{D}_1(C) = \mathcal{D}_2(C)$. We denote by $\sim_{\mathrm{PB,dis}}$ the largest class-distribution probabilistic bisimulation. ■

While in Def. 2.3 the quantification over $C \in S/\mathcal{B}$ can be placed before or after the implication because s_1 and s_2 can have at most one outgoing a-transition each, in Def. 2.4 it is important for the quantification to be after the implication.

The second equivalence relies instead on randomized schedulers. This means that, when responding to an a-transition of the challenger, the defender can select a convex combination of a-transitions (if any). In the following, the acronym ct stands for "based on combined transitions".

Definition 2.5: Let $(S, A, \longrightarrow)$ be an NPLTS. An equivalence relation $\mathcal{B}$ over S is a *class-distribution ct-probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ it holds that $s_1 \xrightarrow{a} \mathcal{D}_1$ implies $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}_2$ such that, for all equivalence classes $C \in S/\mathcal{B}$, $\mathcal{D}_1(C) = \mathcal{D}_2(C)$. We denote by $\sim_{\rm PB,dis}^{\rm ct}$ the largest class-distribution ct-probabilistic bisimulation. ■

In order to obtain a modal logic characterization for $\sim_{\rm PB,dis}$ and $\sim_{\rm PB,dis}^{\rm ct}$, in [21] and [16] an extension of HML much richer than PML was defined. The main differences are that (i) formulae are interpreted over probability distribution on states rather than over single states and (ii) the modal operator $\langle a \rangle_p \cdot$ is split into the original modal operator $\langle a \rangle \cdot$ of HML and an additional unary operator $[\cdot]_p$ such that state distribution $\mathcal{D}$ satisfies $[\phi]_p$ if $\mathcal{D}$ associates with the set of states satisfying ϕ a probability that is at least p.

In [14], the same equivalences (lifted to state distributions) were differently characterized by adding to HML a binary operator $\cdot \oplus_p \cdot$, where $\phi_1 \oplus_p \phi_2$ asserts decomposability of a state distribution to satisfy the two subformulae.

For alternating processes, i.e., NPLTS models in which every state that enables a non-Dirac transition enables only that transition, the following holds:

- $\sim_{\rm PB,dis}$ and $\sim_{\rm PB,dis}^{\rm ct}$ collapse into a single equivalence that coincides with those defined in [13] and [22] for alternating processes, as shown in [26].
- $\sim_{\rm PB,dis}$ is again characterized by the original PML, as shown in [21].

III. Interpreting PML over NPLTS Models

PML was originally interpreted in [19], [20] on reactive probabilistic processes and then in [21] on alternating processes. The same interpretation can be applied to general NPLTS models by establishing that state s satisfies formula $\langle a \rangle_p \phi$ iff *there exists* a resolution of internal nondeterminism such that s can perform an a-transition and afterwards reaches with probability at least p a set of states that satisfy ϕ. This *existential interpretation* only provides a *weak guarantee* of fulfilling properties, as it depends on how internal nondeterminism is resolved.

A different interpretation can be adopted by following [3]: s satisfies $\langle a \rangle_p \phi$ iff, *for each* resolution of internal nondeterminism, s can perform an a-transition and afterwards reaches with probability at least p a set of states that satisfy ϕ. The resulting *universal interpretation* provides a *strong guarantee* of fulfilling properties because, no matter how internal nondeterminism is resolved, a certain behavior is ensured.

We denote by $\mathrm{PML}_{\exists,\geq}$ and $\mathrm{PML}_{\forall,\geq}$ the logics resulting from the two different interpretations of the diamond operator, which we formalize as follows:

$$\mathcal{M}_{\mathrm{PML}_{\exists,\geq}}[\![\langle a \rangle_p \phi]\!] = \{s \in S \mid \exists \mathcal{D}.\, s \xrightarrow{a} \mathcal{D} \wedge \mathcal{D}(\mathcal{M}_{\mathrm{PML}_{\exists,\geq}}[\![\phi]\!]) \geq p\}$$

$$\mathcal{M}_{\mathrm{PML}_{\forall,\geq}}[\![\langle a \rangle_p \phi]\!] = \{s \in S \mid s \xrightarrow{a} \wedge \forall \mathcal{D}.\, s \xrightarrow{a} \mathcal{D} \Longrightarrow \mathcal{D}(\mathcal{M}_{\mathrm{PML}_{\forall,\geq}}[\![\phi]\!]) \geq p\}$$

We also introduce the two variants $\mathrm{PML}_{\exists,\leq}$ and $\mathrm{PML}_{\forall,\leq}$ in which the probability value p decorating the diamond operator is intended as an upper bound rather than a lower bound.

Finally, we denote by $\mathrm{PML}_{\exists,\mathrm{I}}$ and $\mathrm{PML}_{\forall,\mathrm{I}}$ two further variants generalizing the previous four logics, in which the probability value p is replaced by a probability interval $[p_1, p_2]$ – where $p_1, p_2 \in \mathbb{R}_{[0,1]}$ are such that $p_1 \leq p_2$ – and the resulting diamond operator is interpreted as follows:

$$\mathcal{M}_{\mathrm{PML}_{\exists,\mathrm{I}}}[\![\langle a \rangle_{[p_1,p_2]} \phi]\!] = \{s \in S \mid \exists \mathcal{D}.\, s \xrightarrow{a} \mathcal{D} \wedge p_1 \leq \mathcal{D}(\mathcal{M}_{\mathrm{PML}_{\exists,\mathrm{I}}}[\![\phi]\!]) \leq p_2\}$$

$$\mathcal{M}_{\mathrm{PML}_{\forall,\mathrm{I}}}[\![\langle a \rangle_{[p_1,p_2]} \phi]\!] = \{s \in S \mid s \xrightarrow{a} \wedge \forall \mathcal{D}.\, s \xrightarrow{a} \mathcal{D} \Longrightarrow p_1 \leq \mathcal{D}(\mathcal{M}_{\mathrm{PML}_{\forall,\mathrm{I}}}[\![\phi]\!]) \leq p_2\}$$

Note that $\langle a \rangle_p \phi$ can be encoded as $\langle a \rangle_{[p,1]} \phi$ when p is a lower bound and as $\langle a \rangle_{[0,p]} \phi$ when p is an upper bound.

In the following, if L is one of the above variants of PML, then we denote by $\mathcal{F}_{\rm L}(s)$ the set of formulae in $\mathbb{F}_{\rm L}$ that are satisfied by state s and we let $s_1 \sim_{\rm L} s_2$ iff $\mathcal{F}_{\rm L}(s_1) = \mathcal{F}_{\rm L}(s_2)$. Interestingly enough, in Sect. V-C we shall see that the equivalences induced by the universally interpreted variants are the same and coincide with the equivalences induced by the existentially interpreted variants with probabilistic bound. In contrast, the equivalence induced by $\mathrm{PML}_{\exists,\mathrm{I}}$ is finer.

IV. Bisimilarities Characterized by PML

In this section, we introduce the probabilistic bisimilarities for NPLTS models that are characterized by PML as interpreted in the previous section. Before presenting their definition, we highlight the differences with respect to $\sim_{\rm PB,dis}$.

Firstly, the new equivalences focus on *a single equivalence class at a time* instead of comparing the probability distributions over all classes of equivalent states reached by the transitions considered in the bisimulation game. Therefore, given an action a, the probability distribution over all classes of equivalent states reached by an a-transition of the challenger can now be matched by means of *several* (not just by one) a-transitions of the defender, each taking care of a different class. This is similar to the approach followed in [29] to prove a logical characterization in the setting of approximate probabilistic relations.

Secondly, the new equivalences take into account the probability of reaching *groups of equivalence classes* rather than only individual classes. This is similar to the approach followed in [12] and in [5], [8] to ensure transitivity of probabilistic bisimilarity over probabilistic processes without and with internal nondeterminism, respectively, when the state space is continuous. Considering also groups would make no difference in the case of $\sim_{\rm PB,dis}$, while here it significantly changes the discriminating power as will be illustrated in Sect. V-A. Due to the previous and the current difference with respect to $\sim_{\rm PB,dis}$, we call our equivalences *group-by-group probabilistic bisimilarities*.

Thirdly, the new equivalences come in several variants depending on whether, in the bisimulation game, the probabilities of reaching a certain group of classes of equivalent states are compared based on $=$, $\leq$, or $\geq$. Again, this would make no difference in the case of $\sim_{\rm PB,dis}$.

In the following, we let $\bigcup \mathcal{G} = \bigcup_{C \in \mathcal{G}} C$ when $\mathcal{G} \in 2^{S/\mathcal{B}}$ is a group of equivalence classes with respect to an equivalence relation $\mathcal{B}$ over S.

Definition 4.1: Let $(S, A, \longrightarrow)$ be an NPLTS and the relational operator $\bowtie \in \{=, \leq, \geq\}$. An equivalence relation $\mathcal{B}$ over S is a $\bowtie$-*group-by-group probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \xrightarrow{a} \mathcal{D}_1$ implies $s_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) \bowtie \mathcal{D}_2(\bigcup \mathcal{G})$. We denote by $\sim_{\rm PB,gbg,\bowtie}$ the largest $\bowtie$-group-by-group probabilistic bisimulation. ■

The definition of $\sim_{\rm PB,gbg,\bowtie}$ assumes the use of deterministic schedulers, but it can be easily extended to the case of randomized schedulers by analogy with $\sim^{\rm ct}_{\rm PB,dis}$, thus yielding $\sim^{\rm ct}_{\rm PB,gbg,\bowtie}$. Note that, while in Def. 2.4 the quantification over $C \in S/\mathcal{B}$ is after the implication, in Def. 4.1 the quantification over $\mathcal{G} \in 2^{S/\mathcal{B}}$ is before the implication thus allowing a transition of the challenger to be matched by several transitions of the defender depending on the target groups.

The relation $\sim_{\rm PB,gbg,=}$ identifies the two systems in Fig. 1, whilst the relations $\sim_{\rm PB,gbg,\leq}$ and $\sim_{\rm PB,gbg,\geq}$ also identify the two systems in Fig. 2. In Sect. V-C, we shall see that $\sim_{\rm PB,dis}$ is finer than $\sim_{\rm PB,gbg,=}$ and that the latter is finer than $\sim_{\rm PB,gbg,\leq}$, which in turn coincides with $\sim_{\rm PB,gbg,\geq}$.

Before moving to the modal logic characterization results, as a sanity check we show that the three group-by-group probabilistic bisimilarities and their ct-variants (i) are backward compatible with the bisimilarity in [15] for fully nondeterministic processes (see Def. 2.2) and with the probabilistic bisimilarity in [19] for reactive probabilistic processes (see Def. 2.3) and (ii) coincide with $\sim_{\rm PB,dis}$ and $\sim^{\rm ct}_{\rm PB,dis}$ when restricting attention to alternating processes.

Theorem 4.2: Let $(S, A, \longrightarrow)$ be an NPLTS in which the target of each transition is a Dirac distribution. Let $s_1, s_2 \in S$ and $\bowtie \in \{=, \leq, \geq\}$. Then:

$$s_1 \sim_{\rm PB,gbg,\bowtie} s_2 \iff s_1 \sim^{\rm ct}_{\rm PB,gbg,\bowtie} s_2 \iff s_1 \sim_{\rm B} s_2 \quad ■$$

Theorem 4.3: Let $(S, A, \longrightarrow)$ be an NPLTS in which the transitions of each state have different labels. Let $s_1, s_2 \in S$ and $\bowtie \in \{=, \leq, \geq\}$. Then:

$$s_1 \sim_{\rm PB,gbg,\bowtie} s_2 \iff s_1 \sim^{\rm ct}_{\rm PB,gbg,\bowtie} s_2 \iff s_1 \sim_{\rm PB} s_2 \quad ■$$

Theorem 4.4: Let $(S, A, \longrightarrow)$ be an NPLTS in which every state that enables a non-Dirac transition enables only that transition. Let $s_1, s_2 \in S$ and $\bowtie \in \{=, \leq, \geq\}$. Then:

$$s_1 \sim_{\rm PB,gbg,\bowtie} s_2 \iff s_1 \sim_{\rm PB,dis} s_2$$
$$s_1 \sim^{\rm ct}_{\rm PB,gbg,\bowtie} s_2 \iff s_1 \sim^{\rm ct}_{\rm PB,dis} s_2 \quad ■$$

The relation $\sim_{\rm PB,gbg,=}$ turns out to be characterized by $\rm PML_{\exists,I}$ under assumptions of image finiteness and minimal probability.

Theorem 4.5: Let $(S, A, \longrightarrow)$ be an image-finite NPLTS satisfying the minimal probability assumption. Let $s_1, s_2 \in S$. Then:

$$s_1 \sim_{\rm PB,gbg,=} s_2 \iff s_1 \sim_{\rm PML_{\exists,I}} s_2 \quad ■$$

The proof of this result follows the same pattern as in [15]. First, an alternative characterization of $\sim_{\rm PB,gbg,=}$ as the limit of a sequence of equivalence relations $\sim^i_{\rm PB,gbg,=}$ is provided. For an NPLTS $(S, A, \longrightarrow)$, the family $\{\sim^i_{\rm PB,gbg,=} \mid i \in \mathbb{N}\}$ of equivalence relations over S is inductively defined as follows:

- $\sim^0_{\rm PB,gbg,=} \; = \; S \times S$.
- $\sim^{i+1}_{\rm PB,gbg,=}$ is the set of all pairs $(s_1, s_2) \in \sim^i_{\rm PB,gbg,=}$ such that for all actions $a \in A$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\sim^i_{\rm PB,gbg,=}}$ it holds that $s_1 \xrightarrow{a} \mathcal{D}_1$ implies $s_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$.

Each equivalence relation $\sim^i_{\rm PB,gbg,=}$ identifies those states that cannot be distinguished within i steps of computation. The following lemma guarantees that two states of an image-finite NPLTS are equivalent according to $\sim_{\rm PB,gbg,=}$ iff they are equivalent according to all the relations $\sim^i_{\rm PB,gbg,=}$.

Lemma 4.6: Let $(S, A, \longrightarrow)$ be an image-finite NPLTS. Then:

$$\sim_{\rm PB,gbg,=} \; = \; \bigcap_{i \in \mathbb{N}} \sim^i_{\rm PB,gbg,=} \quad ■$$

The second step of the proof is to show that two states are equated by $\sim^i_{\rm PB,gbg,=}$ iff they satisfy the same formulae in $\mathbb{F}^i_{\rm PML_{\exists,I}}$, which is the set of formulae in $\mathbb{F}_{\rm PML_{\exists,I}}$ whose maximum number of nested diamond operators is at most i.

Lemma 4.7: Let $(S, A, \longrightarrow)$ be an image-finite NPLTS satisfying the minimal probability assumption. Let $s_1, s_2 \in S$. Then for all $i \in \mathbb{N}$:

$$s_1 \sim^i_{\rm PB,gbg,=} s_2 \iff \mathcal{F}^i_{\rm PML_{\exists,I}}(s_1) = \mathcal{F}^i_{\rm PML_{\exists,I}}(s_2) \quad ■$$

Now Thm. 4.5 directly follows from Lemma 4.6 and Lemma 4.7. The same result would not hold if $\rm PML_{\exists,\geq}$ or $\rm PML_{\exists,\leq}$ were used. For instance, the two states s_1 and s_2 in Fig. 2, which are not related by $\sim_{\rm PB,gbg,=}$ as can be seen by considering the $\rm PML_{\exists,I}$ formula $\langle \text{offer} \rangle_{[0.5,0.5]} \langle \text{head} \rangle_{[1,1]} \texttt{true}$, cannot be distinguished by any $\rm PML_{\exists,\geq}$ or $\rm PML_{\exists,\leq}$ formula.

Following the same strategy, we can prove that the relations $\sim_{\rm PB,gbg,\leq}$ and $\sim_{\rm PB,gbg,\geq}$ are respectively characterized by $\rm PML_{\exists,\geq}$ and $\rm PML_{\exists,\leq}$.

Theorem 4.8: Let $(S, A, \longrightarrow)$ be an image-finite NPLTS satisfying the minimal probability assumption. Let $s_1, s_2 \in S$. Then:

$$s_1 \sim_{\rm PB,gbg,\leq} s_2 \iff s_1 \sim_{\rm PML_{\exists,\geq}} s_2$$
$$s_1 \sim_{\rm PB,gbg,\geq} s_2 \iff s_1 \sim_{\rm PML_{\exists,\leq}} s_2 \quad ■$$

It is easy to see that $\sim^{\rm ct}_{\rm PB,gbg,=}$, $\sim^{\rm ct}_{\rm PB,gbg,\leq}$, and $\sim^{\rm ct}_{\rm PB,gbg,\geq}$ are respectively characterized by $\rm PML^{ct}_{\exists,I}$, $\rm PML^{ct}_{\exists,\geq}$, and $\rm PML^{ct}_{\exists,\leq}$, in which the interpretation of the diamond operator relies on combined transitions instead of ordinary ones.

V. Variants of Group-by-Group Bisimilarities

In this section, we present further motivations, alternative characterizations based on extremal probabilities and universal interpretations, relationships determined by the distinguishing power, and multistep variants for the group-by-group probabilistic bisimilarities of the previous section.

A. Class-by-Class Probabilistic Bisimilarities

In order to motivate the use of groups of equivalence classes in Def. 4.1, we now introduce class-by-class variants of $\sim_{\rm PB,dis}$ by simply anticipating the quantification over equivalence classes of target states in Def. 2.4.

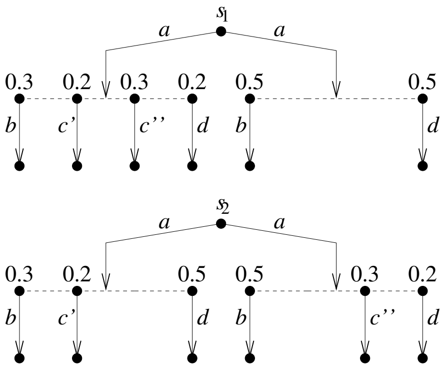

Fig. 3. Models related by $\sim_{\mathrm{PB,cbc,=}}$ and distinguished by all PML variants

Definition 5.1: Let $(S, A, \longrightarrow)$ be an NPLTS and the relational operator $\bowtie \in \{=, \leq, \geq\}$. An equivalence relation $\mathcal{B}$ over S is a $\bowtie$-*class-by-class probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and equivalence classes $C \in S/\mathcal{B}$ it holds that $s_1 \xrightarrow{a} \mathcal{D}_1$ implies $s_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(C) \bowtie \mathcal{D}_2(C)$. We denote by $\sim_{\mathrm{PB,cbc},\bowtie}$ the largest $\bowtie$-class-by-class probabilistic bisimulation. ■

The relations $\sim_{\mathrm{PB,cbc},\bowtie}$ are too coarse. For example, in Fig. 3 it holds that $s_1 \sim_{\mathrm{PB,cbc,=}} s_2$, as witnessed by the equivalence relation that pairs states with identically labeled transitions. However, after performing a, from s_2 it is always possible to reach a state in which c' or c'' is enabled, whereas this is not the case from s_1.

From a modal logic perspective, none of the relations $\sim_{\mathrm{PB,cbc},\bowtie}$ is characterized by the PML variants of Sect. III. For instance, in Fig. 3 it holds that only s_1 satisfies the following existentially interpreted formulae:

$$\mathrm{PML}_{\exists,\geq}: \quad \langle a \rangle_{0.5}(\langle c' \rangle_1 \mathsf{true} \vee \langle c'' \rangle_1 \mathsf{true})$$

$$\mathrm{PML}_{\exists,\leq}: \quad \langle a \rangle_{0}(\langle c' \rangle_1 \mathsf{true} \vee \langle c'' \rangle_1 \mathsf{true})$$

$$\mathrm{PML}_{\exists,\mathrm{I}}: \quad \neg \langle a \rangle_{[0.2,0.3]}(\langle c' \rangle_{[1,1]} \mathsf{true} \vee \langle c'' \rangle_{[1,1]} \mathsf{true})$$

while only s_2 satisfies the following universally interpreted formulae:

$$\mathrm{PML}_{\forall,\geq}: \quad \langle a \rangle_{0.7}(\langle b \rangle_1 \mathsf{true} \vee \langle d \rangle_1 \mathsf{true})$$

$$\mathrm{PML}_{\forall,\leq}: \quad \langle a \rangle_{0.8}(\langle b \rangle_1 \mathsf{true} \vee \langle d \rangle_1 \mathsf{true})$$

$$\mathrm{PML}_{\forall,\mathrm{I}}: \quad \langle a \rangle_{[0.7,0.8]}(\langle b \rangle_{[1,1]} \mathsf{true} \vee \langle d \rangle_{[1,1]} \mathsf{true})$$

where as usual $\phi_1 \vee \phi_2$ stands for $\neg(\neg\phi_1 \wedge \neg\phi_2)$. The presence of the logical disjunction in the distinguishing formulae above clearly indicates that – having anticipated the quantification over the target states – it is necessary to group equivalence classes together if one wants to obtain the same identifications as the equivalences induced by the variants of PML.

B. Group-by-Group Bisimilarities and Extremal Probabilities

The group-by-group probabilistic bisimilarities of Def. 4.1 are directly characterized by the *existentially* interpreted variants of PML. We consider below variants of the group-by-group approach in which only the supremum ($\sqcup$) and/or the infimum ($\sqcap$) of the probabilities of reaching a certain group after a certain action are considered. It turns out that the resulting probabilistic bisimilarities are directly characterized by the *universally* interpreted variants of PML.

Definition 5.2: Let $(S, A, \longrightarrow)$ be an NPLTS. An equivalence relation $\mathcal{B}$ over S is a $\sqcup\sqcap$-*group-by-group probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \xrightarrow{a}$ implies $s_2 \xrightarrow{a}$ with:

$$\bigsqcup_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$

$$\bigsqcap_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$

We denote by $\sim_{\mathrm{PB,gbg},\sqcup\sqcap}$ the largest $\sqcup\sqcap$-group-by-group probabilistic bisimulation. ■

Theorem 5.3: Let $(S, A, \longrightarrow)$ be an image-finite NPLTS satisfying the minimal probability assumption. Let $s_1, s_2 \in S$. Then:

$$s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2 \iff s_1 \sim_{\mathrm{PML}_{\forall,\mathrm{I}}} s_2$$ ■

Definition 5.4: Let $(S, A, \longrightarrow)$ be an NPLTS and symbol $\# \in \{\sqcup, \sqcap\}$. An equivalence relation $\mathcal{B}$ over S is a $\#$-*group-by-group probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \xrightarrow{a}$ implies $s_2 \xrightarrow{a}$ with:

$$\#_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \#_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$

We denote by $\sim_{\mathrm{PB,gbg},\#}$ the largest $\#$-group-by-group probabilistic bisimulation. ■

Theorem 5.5: Let $(S, A, \longrightarrow)$ be an image-finite NPLTS satisfying the minimal probability assumption. Let $s_1, s_2 \in S$. Then:

$$s_1 \sim_{\mathrm{PB,gbg},\sqcup} s_2 \iff s_1 \sim_{\mathrm{PML}_{\forall,\leq}} s_2$$

$$s_1 \sim_{\mathrm{PB,gbg},\sqcap} s_2 \iff s_1 \sim_{\mathrm{PML}_{\forall,\geq}} s_2$$ ■

C. Relating the Various Probabilistic Bisimilarities

If we investigate the spectrum of relations considered so far, we discover that five of the six group-by-group probabilistic bisimilarities boil down to the same equivalence, and this extends to the corresponding PML-based equivalences.

Theorem 5.6: Let $\mathcal{U} = (S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$. Then:

1) $s_1 \sim_{\mathrm{PB,dis}} s_2 \implies s_1 \sim_{\mathrm{PB,gbg,=}} s_2 \implies s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$.
2) $s_1 \sim_{\mathrm{PB,gbg},\leq} s_2 \iff s_1 \sim_{\mathrm{PB,gbg},\sqcup} s_2$ when $\mathcal{U}$ is image finite.
3) $s_1 \sim_{\mathrm{PB,gbg},\geq} s_2 \iff s_1 \sim_{\mathrm{PB,gbg},\sqcap} s_2$ when $\mathcal{U}$ is image finite.
4) $s_1 \sim_{\mathrm{PB,gbg},\sqcup} s_2 \iff s_1 \sim_{\mathrm{PB,gbg},\sqcap} s_2 \iff s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$. ■

The two implications above cannot be reversed: Fig. 1 shows that $\sim_{\mathrm{PB,dis}}$ is strictly finer than $\sim_{\mathrm{PB,gbg,=}}$ and Fig. 2 shows that $\sim_{\mathrm{PB,gbg,=}}$ is strictly finer than $\sim_{\mathrm{PB,gbg},\sqcup\sqcap}$. Note that the result relating $\sim_{\mathrm{PB,gbg},\sqcup}$, $\sim_{\mathrm{PB,gbg},\sqcap}$, and $\sim_{\mathrm{PB,gbg},\sqcup\sqcap}$ holds because *groups* of equivalence classes are considered. Analogous bisimilarities defined in a class-by-class fashion would not coincide.

Another interesting property is that the five coinciding group-by-group probabilistic bisimilarities are the same as their ct-variants, and hence are insensitive to whether deterministic or randomized schedulers are employed to re-

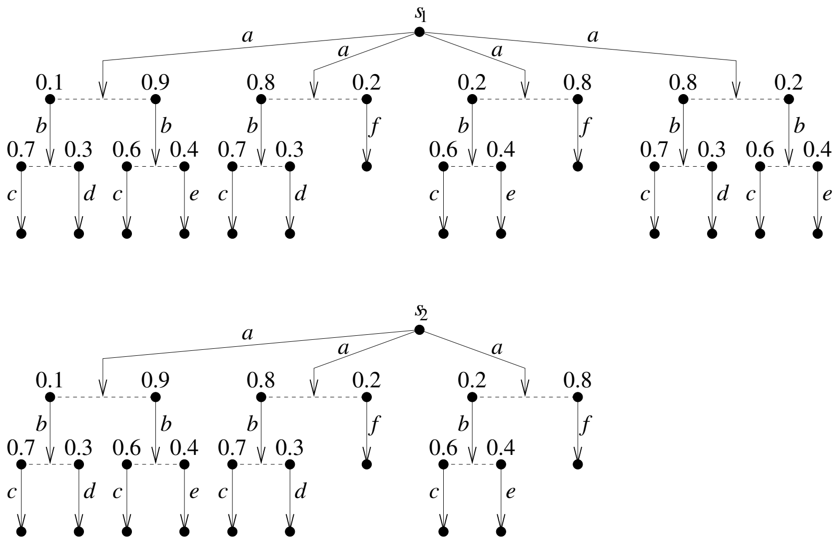

Fig. 4. Two models identified by $\sim_{\mathrm{PB,gbg,=}}$ and $\sim_{\mathrm{PB,gbg,\leq}}$ that are distinguished by PCTL*

solve nondeterminism. This is not the case with $\sim_{\mathrm{PB,dis}}$ and $\sim_{\mathrm{PB,gbg,=}}$. Moreover, the ct-variants of all the six group-by-group probabilistic bisimilarities boil down to the same equivalence ($\sim_{\mathrm{PB,gbg,\leq}}$), meaning that, in the bisimulation game, randomized schedulers reduce the discriminating power of the $=$-comparison of probabilities to that of the $\leq$-comparison.

Theorem 5.7: Let $\mathcal{U} = (S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$. Then:

1) $s_1 \sim_{\mathrm{PB},\triangleleft} s_2 \implies s_1 \sim^{\mathrm{ct}}_{\mathrm{PB},\triangleleft} s_2$ for $\triangleleft \in \{\text{"dis"}, \text{"gbg,="}\}$.
2) $s_1 \sim_{\mathrm{PB,gbg},\triangleright} s_2 \iff s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg},\triangleright} s_2$ for $\triangleright \in \{\leq, \geq, \sqcup\sqcap, \sqcup, \sqcap\}$ when $\mathcal{U}$ is image finite.
3) $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,dis}} s_2 \Longrightarrow s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg,=}} s_2$.
4) $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg,=}} s_2 \iff s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg,\sqcup\sqcap}} s_2$ when $\mathcal{U}$ is image finite. ■

The inclusions of $\sim_{\mathrm{PB,dis}}$ and $\sim_{\mathrm{PB,gbg,=}}$ in $\sim^{\mathrm{ct}}_{\mathrm{PB,dis}}$ and $\sim^{\mathrm{ct}}_{\mathrm{PB,gbg,=}}$, respectively, are strict, as shown by Fig. 2; the central offer-transition of s_1 can be matched by a convex combination of the two offer-transitions of s_2 both weighted by 0.5. Moreover, Fig. 1 shows that the inclusion of $\sim^{\mathrm{ct}}_{\mathrm{PB,dis}}$ in $\sim^{\mathrm{ct}}_{\mathrm{PB,gbg,=}}$ is strict. Finally, Figs. 1 and 2 show that $\sim^{\mathrm{ct}}_{\mathrm{PB,dis}}$ and $\sim_{\mathrm{PB,gbg,=}}$ are incomparable with each other.

D. Multistep Variants of Probabilistic Bisimilarities

Further relations can be defined by considering entire computations instead of individual transitions in the bisimulation game. Given an NPLTS $\mathcal{U} = (S, A, \longrightarrow)$, we say that $c \equiv s_0 \stackrel{a_1}{\longmapsto} s_1 \stackrel{a_2}{\longmapsto} s_2 \ldots s_{n-1} \stackrel{a_n}{\longmapsto} s_n$ is a *computation* of $\mathcal{U}$ of length n going from s_0 to s_n iff for all $i = 1, \ldots, n$ there exists a transition $s_{i-1} \xrightarrow{a_i} \mathcal{D}_i$ such that $s_i \in \mathit{supp}(\mathcal{D}_i)$, with $\mathcal{D}_i(s_i)$ being the execution probability of step $s_{i-1} \stackrel{a_i}{\longmapsto} s_i$ of c conditioned on the selection of transition $s_{i-1} \xrightarrow{a_i} \mathcal{D}_i$ of $\mathcal{U}$ at state s_{i-1}. We call combined computation a computation in which every step arises from a combined transition.

The *multistep* variants of probabilistic bisimilarities for NPLTS models can be defined in different ways. The first option, inspired by bisimilarity for fully nondeterministic processes, consists of changing the one-step definitions by considering traces $\alpha \in A^*$ in place of actions $a \in A$ and $\stackrel{\alpha}{\Longrightarrow}$ in place of $\stackrel{a}{\longrightarrow}$ (resp. $\stackrel{\alpha}{\Longrightarrow}_{\mathrm{c}}$ in place of $\stackrel{a}{\longrightarrow}_{\mathrm{c}}$), where $s \stackrel{\alpha}{\Longrightarrow} \mathcal{D}$ means that there exists a computation from s labeled with α whose last step is originated by a transition reaching distribution $\mathcal{D}$. When α is the empty sequence ε, we let $s \stackrel{\varepsilon}{\Longrightarrow} \delta_s$. It was shown in [15] that the discriminating power of bisimilarity for fully nondeterministic processes does not change if the multistep transition relation $\stackrel{\alpha}{\Longrightarrow}$ is used instead of the one-step relation $\stackrel{a}{\longrightarrow}$. As expected, this result carries over class-distribution and group-by-group probabilistic bisimilarities for nondeterministic and probabilistic processes (see Appendix B).

The second option, inspired by probabilistic bisimilarity for reactive probabilistic processes, does not only compare the probability values arising from the last step of the computations, but additionally considers the probability of performing the entire computations. While it can be shown that the discriminating power of the probabilistic bisimilarity for reactive probabilistic processes in [19] and of class-distribution probabilistic bisimilarities for nondeterministic and probabilistic processes does not change if multistep probability values are compared instead of one-step values, this is not the case with the group-by-group probabilistic bisimilarities (see Appendix C).

Finally, the third option, which is orthogonal to the previous two, consists of imposing some constraints along the computations, such as passing through specific sets of states at each step. This is the idea exploited in [28] in order to define probabilistic bisimilarities – following the second option above – over nondeterministic and probabilistic Kripke structures that are precisely characterized by PCTL, PCTL*, and their variants without the next-time operator, as interpreted in [3]. We note that the strong 1-depth bisimulation in [28] and our $\sim_{\text{PB,gbg},\leq}$ are strongly related. In contrast, the probabilistic bisimilarities built in [28] as the limit of a chain of n-depth bisimulations are provably finer than our group-by-group probabilistic bisimilarities. Consider for instance the two NPLTS models in Fig. 4. We have that $s_1 \sim_{\text{PB,gbg},=} s_2$ – and hence $s_1 \sim_{\text{PB,gbg},\leq} s_2$ – as witnessed by the equivalence relation that pairs states with identically labeled transitions and, in the case of b-transitions, identical target distributions. However, s_1 and s_2 are distinguished by the probabilistic bisimilarity in [28] that is characterized by PCTL*. In fact, let us view the two NPLTS models as two nondeterministic and probabilistic Kripke structures by eliminating actions from transitions and labeling each state with the set of its next-actions. Then the PCTL* formula $\text{Pr}_{\leq 0.61}(\mathbf{X}\,\mathbf{X}\,c)$ is satisfied by s_2 but it is not satisfied by s_1, because the probability of reaching in two steps a state that enables c in the maximal resolution of s_1 starting with the rightmost a-transition is $0.8 \cdot 0.7 + 0.2 \cdot 0.6 = 0.68$ and hence it is greater than 0.61.

VI. Conclusion

We have addressed the problem of defining behavioral relations for nondeterministic and probabilistic processes that are characterized by modal logics as close as possible to PML, the natural probabilistic version of the by now standard HML for fully nondeterministic processes. We have introduced two new probabilistic bisimilarities ($\sim_{\text{PB,gbg},=}$ and $\sim_{\text{PB,gbg},\leq}$) following a group-by-group approach and proved their relationships with an existential interpretation and a universal interpretation of two variants of PML, in which the diamond is respectively decorated with a probability bound and a probability interval. All the resulting logical equivalences, except the one based on existential interpretation and probability intervals (which corresponds to $\sim_{\text{PB,gbg},=}$), have turned out to coincide. This suggests that adopting a universal interpretation rather than an existential one does not matter for comparison purposes when using probability bounds.

For the new probabilistic bisimilarities, we have provided alternative definitions obtained by varying the requirements on the comparison between sets of probabilities ($=$, $\leq$, $\geq$) or by comparing only extremal probabilities ($\sqcup$ and/or $\sqcap$). Quite surprisingly, all relations but the one based on $=$ do collapse. We have also considered variants relying on combined transitions and have proved that all such variants, except the one based on $=$, coincide with the relations relying on ordinary transitions. This suggests that, in the group-by-group approach, resolving nondeterminism with deterministic schedulers or randomized ones leads to the same identifications except when checking for equality of probabilities.

The above-mentioned results for image-finite NPLTS models satisfying the minimal probability assumption are summarized in Fig. 5, where each arrow means more-discriminating-than and equivalences collected in the same dashed box coincide. Please notice that the top part of each dashed box contains behavioral equivalences while the bottom part contains logical ones.

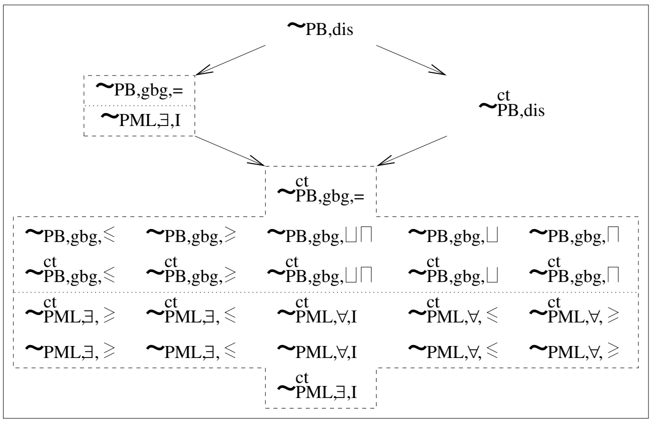

Fig. 5. Relating bisimulation-based and PML-based equivalences

Our modal logic characterization and backward compatibility results for $\sim_{\text{PB,gbg},\leq}$, together with the modal logic characterization in [19], [20] for probabilistic bisimilarity over reactive probabilistic processes and with the modal logic characterization in [21] for class-distribution probabilistic bisimilarity over alternating processes, show that PML provides a uniform framework for reasoning on different classes of processes including probability and various degrees of nondeterminism.

With regard to $\sim_{\text{PB,gbg},=}$, which is finer than $\sim_{\text{PB,gbg},\leq}$ and is characterized by an interval-based variant of PML rather than the original PML, we would like to mention that it has emerged quite naturally in a framework we have recently developed to provide a uniform model for different classes of widely used processes together with uniform definitions of the major behavioral equivalences [2].

Our work has some interesting points in common with [28], where new probabilistic bisimilarities over nondeterministic and probabilistic Kripke structures have been defined that are in full agreement with PCTL, PCTL*, and their variants without the next-time operator. Indeed, both [28] and our work witness that, in order to characterize the equivalences induced by PCTL/PCTL*/PML in a nondeterministic and probabilistic setting, it is necessary to:

- Anticipate the quantification over the sets of equivalent states to be reached in the bisimulation game, as done in [29] in the setting of approximate probabilistic relations. Placing this quantification after the comparison of probability values, like in [25], results in a much finer probabilistic bisimilarity that needs a modal logic much more expressive than PML as shown in [21], [16], [14].
- Consider groups of classes of equivalent states rather than only individual classes, as done in [12], [5], [8] in the setting of continuous-state probabilistic processes. Otherwise, due to the anticipation of the previously mentioned quantification, a much coarser probabilistic bisimilarity is obtained.

- Compare for equality only the extremal probabilities of reaching certain sets of states rather than all the probabilities, so that the presence of nondeterminism and the probabilistic bounds in logical formulae fit well together.

Our results and those in [28] also show that, in the case of nondeterministic and probabilistic processes, it is not possible to define a single probabilistic bisimilarity that is characterized by both PML – as interpreted in this paper – and PCTL* – as interpreted in [3]. Thus, for nondeterministic and probabilistic processes the situation is quite different from the case of fully nondeterministic processes, where probabilistic bisimilarity is characterized by both HML [15] and CTL* [4], and from the case of reactive probabilistic processes, where probabilistic bisimilarity is characterized by both PML [19], [20] and PCTL* [1].

With regard to future work, we plan to investigate further properties of group-by-group probabilistic bisimilarities. Results along this direction would also be useful for a better understanding of the probabilistic bisimilarities in [28]. Another obvious direction of research would be to define the weak variants of the group-by-group probabilistic bisimilarities and find the corresponding modal logics. Finally, we would like to study the expressiveness of the variants of PML that we have introduced in this paper.

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APPENDIX

A: PROOFS OF RESULTS

Proof of Thm. 4.2. Since every transition of this specific NPLTS can reach with probability greater than 0 a single state and hence a single class of any equivalence relation – which are thus reached with probability 1 – the reflexive, symmetric, and transitive closure of a bisimulation is trivially a $=$-group-by-group (ct-)probabilistic bisimulation, a $\leq$-group-by-group (ct-)probabilistic bisimulation, and a $\geq$-group-by-group (ct-)probabilistic bisimulation. ■

Proof of Thm. 4.3. Since every state of this specific NPLTS has at most one transition labeled with a certain action, a probabilistic bisimulation is trivially a $=$-group-by-group (ct-)probabilistic bisimulation, a $\leq$-group-by-group (ct-)probabilistic bisimulation, and a $\geq$-group-by-group (ct-)probabilistic bisimulation. ■

Proof of Thm. 4.4. Since every state of this specific NPLTS has either zero or more Dirac transitions or a single non-Dirac transition, a class-distribution (ct-)probabilistic bisimulation is trivially a $=$-group-by-group (ct-)probabilistic bisimulation, a $\leq$-group-by-group (ct-)probabilistic bisimulation, and a $\geq$-group-by-group (ct-)probabilistic bisimulation. ■

Proof of Lemma 4.6. For each equivalence relation $\mathcal{B}$ over S, we define $E(\mathcal{B})$ as the following relation over S: $(s_1, s_2) \in E(\mathcal{B})$ iff for all actions $a \in A$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \xrightarrow{a} \mathcal{D}_1$ implies $s_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$. The proof then proceeds like in [15] by showing that:

- $E(\mathcal{B})$ is an equivalence relation.
- If $\mathcal{B}_1 \subseteq \mathcal{B}_2$, then $E(\mathcal{B}_1) \subseteq E(\mathcal{B}_2)$.
- For each $i \in \mathbb{N}$, $\sim^{i+1}_{\rm PB,gbg,=} = E(\sim^{i}_{\rm PB,gbg,=})$.
- $\bigcap_{i \in \mathbb{N}} \sim^{i}_{\rm PB,gbg,=}$ is the maximal solution of $\mathcal{B} = E(\mathcal{B})$ and is a $=$-group-by-group probabilistic bisimulation.
- $\sim_{\rm PB,gbg,=} = E(\sim_{\rm PB,gbg,=})$.

Directly from the last two items, it follows that $\sim_{\rm PB,gbg,=} = \bigcap_{i \in \mathbb{N}} \sim^{i}_{\rm PB,gbg,=}$. ■

Proof of Lemma 4.7. Given an image-finite NPLTS $(S, A, \longrightarrow)$ satisfying the minimal probability assumption, and given $s_1, s_2 \in S$, we proceed by induction on $i \in \mathbb{N}$.

Base of Induction ($i = 0$): Since $\sim^{0}_{\rm PB,gbg,=} = S \times S$ and $\mathcal{F}^{0}_{\rm PML_{\exists,I}}(s) = \{\phi \in \mathbb{F}^{0}_{\rm PML_{\exists,I}} \mid \phi \equiv \textsf{true}\}$ for all $s \in S$, it trivially holds that:

$$s_1 \sim^{0}_{\rm PB,gbg,=} s_2 \iff \mathcal{F}^{0}_{\rm PML_{\exists,I}}(s_1) = \mathcal{F}^{0}_{\rm PML_{\exists,I}}(s_2)$$

Induction Hypothesis: Given $i \in \mathbb{N}$, we assume that for all $j = 0, \dots, i$:

$$s_1 \sim^{j}_{\rm PB,gbg,=} s_2 \iff \mathcal{F}^{j}_{\rm PML_{\exists,I}}(s_1) = \mathcal{F}^{j}_{\rm PML_{\exists,I}}(s_2)$$

Induction Step: We prove both implications for $i + 1$ by reasoning on their corresponding contrapositive statements, i.e., we prove that:

$$\mathcal{F}^{i+1}_{\rm PML_{\exists,I}}(s_1) \neq \mathcal{F}^{i+1}_{\rm PML_{\exists,I}}(s_2) \iff s_1 \not\sim^{i+1}_{\rm PB,gbg,=} s_2$$

($\Longrightarrow$) If $\mathcal{F}^{i+1}_{\rm PML_{\exists,I}}(s_1) \neq \mathcal{F}^{i+1}_{\rm PML_{\exists,I}}(s_2)$, then there are two cases:

- If $\mathcal{F}^{i}_{\rm PML_{\exists,I}}(s_1) \neq \mathcal{F}^{i}_{\rm PML_{\exists,I}}(s_2)$, then by the induction hypothesis it holds that $s_1 \not\sim^{i}_{\rm PB,gbg,=} s_2$ and hence $s_1 \not\sim^{i+1}_{\rm PB,gbg,=} s_2$.
- If $\mathcal{F}^{i}_{\rm PML_{\exists,I}}(s_1) = \mathcal{F}^{i}_{\rm PML_{\exists,I}}(s_2)$, then from $\mathcal{F}^{i+1}_{\rm PML_{\exists,I}}(s_1) \neq \mathcal{F}^{i+1}_{\rm PML_{\exists,I}}(s_2)$ it follows that there exists $\phi \in \mathbb{F}^{i+1}_{\rm PML_{\exists,I}}$ such that $s_1 \in \mathcal{M}_{\rm PML_{\exists,I}}[\![\phi]\!]$ and $s_2 \notin \mathcal{M}_{\rm PML_{\exists,I}}[\![\phi]\!]$. We now proceed by induction on the syntactical structure of ϕ. Here we only consider the case $\phi = \langle a \rangle_{[p_1,p_2]} \phi'$ because the other cases are routine. From $s_1 \in \mathcal{M}_{\rm PML_{\exists,I}}[\![\langle a \rangle_{[p_1,p_2]} \phi']\!]$ and $s_2 \notin \mathcal{M}_{\rm PML_{\exists,I}}[\![\langle a \rangle_{[p_1,p_2]} \phi']\!]$, it follows that:
  - $p_1 \leq \mathcal{D}_1(\mathcal{M}_{\rm PML_{\exists,I}}[\![\phi']\!]) \leq p_2$ for some $\mathcal{D}_1$ such that $s_1 \xrightarrow{a} \mathcal{D}_1$.
  - $\mathcal{D}_2(\mathcal{M}_{\rm PML_{\exists,I}}[\![\phi']\!]) < p_1$ or $\mathcal{D}_2(\mathcal{M}_{\rm PML_{\exists,I}}[\![\phi']\!]) > p_2$ for all $\mathcal{D}_2$ such that $s_2 \xrightarrow{a} \mathcal{D}_2$.

  Since $\phi' \in \mathbb{F}^{i}_{\rm PML_{\exists,I}}$, by the induction hypothesis there exists $\mathcal{G} \in 2^{S/\sim^{i}_{\rm PB,gbg,=}}$ such that $\bigcup_{C \in \mathcal{G}} C = \mathcal{M}_{\rm PML_{\exists,I}}[\![\phi']\!]$. Then:
  - $\mathcal{D}_1(\bigcup \mathcal{G}) = q \in \mathbb{R}_{[p_1,p_2]}$.
  - $\mathcal{D}_2(\bigcup \mathcal{G}) \neq q$ for all $\mathcal{D}_2$ such that $s_2 \xrightarrow{a} \mathcal{D}_2$.

  Therefore $s_1 \not\sim^{i+1}_{\rm PB,gbg,=} s_2$.

($\Longleftarrow$) If $s_1 \not\sim^{i+1}_{\rm PB,gbg,=} s_2$, then there are two cases:

- If $s_1 \not\sim^{i}_{\rm PB,gbg,=} s_2$, then by the induction hypothesis it holds that $\mathcal{F}^{i}_{\rm PML_{\exists,I}}(s_1) \neq \mathcal{F}^{i}_{\rm PML_{\exists,I}}(s_2)$ and hence $\mathcal{F}^{i+1}_{\rm PML_{\exists,I}}(s_1) \neq \mathcal{F}^{i+1}_{\rm PML_{\exists,I}}(s_2)$.
- If $s_1 \sim^{i}_{\rm PB,gbg,=} s_2$, then from $s_1 \not\sim^{i+1}_{\rm PB,gbg,=} s_2$ it follows that there exist $p \in \mathbb{R}_{[0,1]}$ and $\mathcal{G} \in 2^{S/\sim^{i}_{\rm PB,gbg,=}}$ such that:
  - $\mathcal{D}_1(\bigcup \mathcal{G}) = p$ for some $\mathcal{D}_1$ such that $s_1 \xrightarrow{a} \mathcal{D}_1$.
  - $\mathcal{D}_2(\bigcup \mathcal{G}) \neq p$ for all $\mathcal{D}_2$ such that $s_2 \xrightarrow{a} \mathcal{D}_2$.

  Let $\mathcal{G}_1 = \{C \in S/\sim^{i}_{\rm PB,gbg,=} \mid \mathcal{D}_1(C) > 0\}$ and $\mathcal{G}_2 = \{C \in S/\sim^{i}_{\rm PB,gbg,=} \mid \exists \mathcal{D}_2 .\, s_2 \xrightarrow{a} \mathcal{D}_2 \wedge \mathcal{D}_2(C) > 0\}$. Thanks to the assumptions of image finiteness and minimal probability, both $\mathcal{G}_1$ and $\mathcal{G}_2$ are finite.

  By the induction hypothesis, there exists a distinguishing formula $\phi_{<C_1,C_2>} \in \mathbb{F}^{i}_{\rm PML_{\exists,I}}$ for all C_1 and C_2 in $S/\sim^{i}_{\rm PB,gbg,=}$ such that $C_1 \neq C_2$, i.e.:

  $$C_1 \subseteq \mathcal{M}_{\rm PML_{\exists,I}}[\![\phi_{<C_1,C_2>}]\!]$$
  $$C_2 \cap \mathcal{M}_{\rm PML_{\exists,I}}[\![\phi_{<C_1,C_2>}]\!] = \emptyset$$

  Then:

  $$\phi_{\mathcal{G}} = \bigvee_{C \in \mathcal{G}} \left(\bigwedge_{C_1 \in \mathcal{G}_1 \setminus \{C\}} \phi_{<C,C_1>} \wedge \bigwedge_{C_2 \in \mathcal{G}_2 \setminus \{C\}} \phi_{<C,C_2>} \right)$$

  where $\bigvee_{i \in I} \phi_i = \neg \bigwedge_{i \in I} \neg \phi_i$ for I finite and $\bigwedge_{i \in I} \phi_i = \textsf{true}$ for $I = \emptyset$, yields a distinguishing formula for s_1 and s_2 because:
  - $s_1 \in \mathcal{M}_{\rm PML_{\exists,I}}[\![\langle a \rangle_{[p,p]} \phi_{\mathcal{G}}]\!]$.
  - $s_2 \notin \mathcal{M}_{\rm PML_{\exists,I}}[\![\langle a \rangle_{[p,p]} \phi_{\mathcal{G}}]\!]$.

  Since $\langle a \rangle_{[p,p]} \phi_{\mathcal{G}} \in \mathbb{F}^{i+1}_{\rm PML_{\exists,I}}$, we derive that $\mathcal{F}^{i+1}_{\rm PML_{\exists,I}}(s_1) \neq \mathcal{F}^{i+1}_{\rm PML_{\exists,I}}(s_2)$. ■

Proof of Thm. 4.8. The proof of the first result is similar to the proof of Thm. 4.5 – based on Lemmata 4.6 and 4.7 – up to the use of $\leq$ in place of $=$ when comparing the probabilities of reaching a group of equivalence classes and the use of $\sim^i_{\rm PB,gbg,\le}$, $\mathcal{F}^i_{\rm PML_{\exists,\ge}}$, $\mathcal{M}_{\rm PML_{\exists,\ge}}$, and $\langle a \rangle_p$ in place of $\sim^i_{\rm PB,gbg,=}$, $\mathcal{F}^i_{\rm PML_{\exists,I}}$, $\mathcal{M}_{\rm PML_{\exists,I}}$, and $\langle a \rangle_{[p_1,p_2]}$ wherever necessary.

In particular, for the induction step of Lemma 4.7 we point out that:

- In the ($\Longrightarrow$) part, from $s_1 \in \mathcal{M}_{\rm PML_{\exists,\ge}}[\![\langle a \rangle_p \phi']\!]$ and $s_2 \notin \mathcal{M}_{\rm PML_{\exists,\ge}}[\![\langle a \rangle_p \phi']\!]$, it follows that:
  - $\mathcal{D}_1(\mathcal{M}_{\rm PML_{\exists,\ge}}[\![\phi']\!]) \ge p$ for some $\mathcal{D}_1$ such that $s_1 \xrightarrow{a} \mathcal{D}_1$.
  - $\mathcal{D}_2(\mathcal{M}_{\rm PML_{\exists,\ge}}[\![\phi']\!]) < p$ for all $\mathcal{D}_2$ such that $s_2 \xrightarrow{a} \mathcal{D}_2$.

  Since $\phi' \in \mathbb{F}^i_{\rm PML_{\exists,\ge}}$, by the induction hypothesis there exists $\mathcal{G} \in 2^{S/\sim^i_{\rm PB,gbg,\le}}$ such that $\bigcup_{C\in\mathcal{G}} C = \mathcal{M}_{\rm PML_{\exists,\ge}}[\![\phi']\!]$. Then:
  - $\mathcal{D}_1(\bigcup\mathcal{G}) \ge p$.
  - $\mathcal{D}_2(\bigcup\mathcal{G}) < p$ for all $\mathcal{D}_2$ such that $s_2 \xrightarrow{a} \mathcal{D}_2$.
- In the ($\Longleftarrow$) part, if $s_1 \sim^i_{\rm PB,gbg,\le} s_2$, then there exist $p \in \mathbb{R}_{[0,1]}$ and $\mathcal{G} \in 2^{S/\sim^i_{\rm PB,gbg,\le}}$ such that:
  - $\mathcal{D}_1(\bigcup\mathcal{G}) = p$ for some $\mathcal{D}_1$ such that $s_1 \xrightarrow{a} \mathcal{D}_1$.
  - $\mathcal{D}_2(\bigcup\mathcal{G}) < p$ for all $\mathcal{D}_2$ such that $s_2 \xrightarrow{a} \mathcal{D}_2$.

  The distinguishing formula in $\mathbb{F}^{i+1}_{\rm PML_{\exists,\ge}}$ for s_1 and s_2 is then $\langle a \rangle_p \phi_{\mathcal{G}}$.

The proof of the second result is similar to the proof of the first one up to the use of $\ge$ in place of $\le$ and $>$ in place of $<$ wherever necessary. ■

Proof of Thm. 5.3. Similar to the proof of Thm. 4.5 – based on Lemmata 4.6 and 4.7 – up to the use of $\bigsqcup$ and $\bigsqcap$ in place of individual values when comparing the probabilities of reaching a group of equivalence classes and the use of $\sim^i_{\rm PB,gbg,\sqcup\sqcap}$, $\mathcal{F}^i_{\rm PML_{\forall,I}}$, and $\mathcal{M}_{\rm PML_{\forall,I}}$ in place of $\sim^i_{\rm PB,gbg,=}$, $\mathcal{F}^i_{\rm PML_{\exists,I}}$, and $\mathcal{M}_{\rm PML_{\exists,I}}$ wherever necessary.

In particular, for the induction step of Lemma 4.7 we point out that:

- In the ($\Longrightarrow$) part, from $s_1 \in \mathcal{M}_{\rm PML_{\forall,I}}[\![\langle a \rangle_{[p_1,p_2]} \phi']\!]$ and $s_2 \notin \mathcal{M}_{\rm PML_{\forall,I}}[\![\langle a \rangle_{[p_1,p_2]} \phi']\!]$, it follows that:
  - $s_1 \xrightarrow{a}$ and $p_1 \le \mathcal{D}_1(\mathcal{M}_{\rm PML_{\forall,I}}[\![\phi']\!]) \le p_2$ for all $\mathcal{D}_1$ such that $s_1 \xrightarrow{a} \mathcal{D}_1$.
  - $s_2 \not\xrightarrow{a}$ or $\mathcal{D}_2(\mathcal{M}_{\rm PML_{\forall,I}}[\![\phi']\!]) < p_1$ or $\mathcal{D}_2(\mathcal{M}_{\rm PML_{\forall,I}}[\![\phi']\!]) > p_2$ for some $\mathcal{D}_2$ such that $s_2 \xrightarrow{a} \mathcal{D}_2$.

  Since $\phi' \in \mathbb{F}^i_{\rm PML_{\forall,I}}$, by the induction hypothesis there exists $\mathcal{G} \in 2^{S/\sim^i_{\rm PB,gbg,\sqcup\sqcap}}$ such that $\bigcup_{C\in\mathcal{G}} C = \mathcal{M}_{\rm PML_{\forall,I}}[\![\phi']\!]$. Then:
  - $s_1 \xrightarrow{a}$ and $\bigsqcup_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup\mathcal{G}) = q''$, $\bigsqcap_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup\mathcal{G}) = q'$ with $q', q'' \in \mathbb{R}_{[p_1,p_2]}$ such that $q' \le q''$.
  - $s_2 \not\xrightarrow{a}$ or $\bigsqcup_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup\mathcal{G}) > q''$ or $\bigsqcap_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup\mathcal{G}) < q'$.
- In the ($\Longleftarrow$) part, if $s_1 \sim^i_{\rm PB,gbg,\sqcup\sqcap} s_2$, then there exist $p'_1, p''_1, p'_2, p''_2 \in \mathbb{R}_{[0,1]}$ – with $p'_1 \le p''_1$ and $p'_2 \le p''_2$ – and $\mathcal{G} \in 2^{S/\sim^i_{\rm PB,gbg,\sqcup\sqcap}}$ such that:
  - $s_1 \xrightarrow{a}$ and $\bigsqcup_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup\mathcal{G}) = p''_1$, $\bigsqcap_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup\mathcal{G}) = p'_1$.
  - $s_2 \not\xrightarrow{a}$ or $\bigsqcup_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup\mathcal{G}) = p''_2 \neq p''_1$ or $\bigsqcap_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup\mathcal{G}) = p'_2 \neq p'_1$.

  Let $\mathcal{G}_1 = \{C \in S/\sim^i_{\rm PB,gbg,\sqcup\sqcap} \mid \exists \mathcal{D}_1 .\, s_1 \xrightarrow{a} \mathcal{D}_1 \wedge \mathcal{D}_1(C) > 0\}$ and $\mathcal{G}_2 = \{C \in S/\sim^i_{\rm PB,gbg,\sqcup\sqcap} \mid \exists \mathcal{D}_2 .\, s_2 \xrightarrow{a} \mathcal{D}_2 \wedge \mathcal{D}_2(C) > 0\}$. The distinguishing formula in $\mathbb{F}^{i+1}_{\rm PML_{\forall,I}}$ for s_1 and s_2 is then:
  - $\langle a \rangle_{[p'_1,p''_1]} \phi_{\mathcal{G}}$ if $s_2 \not\xrightarrow{a}$ or it is not the case that $p'_1 \le p'_2$ and $p''_2 \le p''_1$.
  - $\langle a \rangle_{[p'_2,p''_2]} \phi_{\mathcal{G}}$ if $s_2 \xrightarrow{a}$ and it is the case that $p'_1 \le p'_2$ and $p''_2 \le p''_1$. ■

Proof of Thm. 5.5. The proof of the first result is similar to the proof of Thm. 4.5 – based on Lemmata 4.6 and 4.7 – up to the use of $\bigsqcup$ in place of individual values when comparing the probabilities of reaching a group of equivalence classes and the use of $\sim^i_{\rm PB,gbg,\sqcup}$, $\mathcal{F}^i_{\rm PML_{\forall,\le}}$, and $\mathcal{M}_{\rm PML_{\forall,\le}}$ in place of $\sim^i_{\rm PB,gbg,=}$, $\mathcal{F}^i_{\rm PML_{\exists,I}}$, and $\mathcal{M}_{\rm PML_{\exists,I}}$ wherever necessary.

In particular, for the induction step of Lemma 4.7 we point out that:

- In the ($\Longrightarrow$) part, from $s_1 \in \mathcal{M}_{\rm PML_{\forall,\le}}[\![\langle a \rangle_p \phi']\!]$ and $s_2 \notin \mathcal{M}_{\rm PML_{\forall,\le}}[\![\langle a \rangle_p \phi']\!]$, it follows that:
  - $s_1 \xrightarrow{a}$ and $\mathcal{D}_1(\mathcal{M}_{\rm PML_{\forall,\le}}[\![\phi']\!]) \le p$ for all $\mathcal{D}_1$ such that $s_1 \xrightarrow{a} \mathcal{D}_1$.
  - $s_2 \not\xrightarrow{a}$ or $\mathcal{D}_2(\mathcal{M}_{\rm PML_{\forall,\le}}[\![\phi']\!]) > p$ for some $\mathcal{D}_2$ such that $s_2 \xrightarrow{a} \mathcal{D}_2$.

  Since $\phi' \in \mathbb{F}^i_{\rm PML_{\forall,\le}}$, by the induction hypothesis there exists $\mathcal{G} \in 2^{S/\sim^i_{\rm PB,gbg,\sqcup}}$ such that $\bigcup_{C\in\mathcal{G}} C = \mathcal{M}_{\rm PML_{\forall,\le}}[\![\phi']\!]$. Then:
  - $s_1 \xrightarrow{a}$ and $\bigsqcup_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup\mathcal{G}) \le p$.
  - $s_2 \not\xrightarrow{a}$ or $\bigsqcup_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup\mathcal{G}) > p$.
- In the ($\Longleftarrow$) part, if $s_1 \sim^i_{\rm PB,gbg,\sqcup} s_2$, then there exist $p \in \mathbb{R}_{[0,1]}$ and $\mathcal{G} \in 2^{S/\sim^i_{\rm PB,gbg,\sqcup}}$ such that:
  - $s_1 \xrightarrow{a}$ and $\bigsqcup_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup\mathcal{G}) = p$.
  - $s_2 \not\xrightarrow{a}$ or $\bigsqcup_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup\mathcal{G}) = q \neq p$.

  Let $\mathcal{G}_1 = \{C \in S/\sim^i_{\rm PB,gbg,\sqcup} \mid \exists \mathcal{D}_1 .\, s_1 \xrightarrow{a} \mathcal{D}_1 \wedge \mathcal{D}_1(C) > 0\}$ and $\mathcal{G}_2 = \{C \in S/\sim^i_{\rm PB,gbg,\sqcup} \mid \exists \mathcal{D}_2 .\, s_2 \xrightarrow{a} \mathcal{D}_2 \wedge \mathcal{D}_2(C) > 0\}$. The distinguishing formula in $\mathbb{F}^{i+1}_{\rm PML_{\forall,\le}}$ for s_1 and s_2 is then:
  - $\langle a \rangle_p \phi_{\mathcal{G}}$ if $s_2 \not\xrightarrow{a}$ or $p < q$.
  - $\langle a \rangle_q \phi_{\mathcal{G}}$ if $s_2 \xrightarrow{a}$ and $q < p$.

The proof of the second result is similar to the proof of the first one up to the use of $\bigsqcap$ in place of $\bigsqcup$, $\geq$ in place of $\leq$, $<$ in place of $>$, and $>$ in place of $<$ wherever necessary. ■

Proof of Thm. 5.6. Let $\mathcal{U} = (S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$:

1) The fact that $s_1 \sim_{\mathrm{PB,dis}} s_2$ implies $s_1 \sim_{\mathrm{PB,gbg,=}} s_2$ is a straightforward consequence of the fact that a class-distribution probabilistic bisimulation is trivially a $=$-group-by-group probabilistic bisimulation.
   Suppose now that $s_1 \sim_{\mathrm{PB,gbg,=}} s_2$. This means that there exists a $=$-group-by-group probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. In other words, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:
   - If $s'_1 \xrightarrow{a} \mathcal{D}_1$, then $s'_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$.
   - If $s'_2 \xrightarrow{a} \mathcal{D}_2$, then $s'_1 \xrightarrow{a} \mathcal{D}_1$ such that $\mathcal{D}_2(\bigcup \mathcal{G}) = \mathcal{D}_1(\bigcup \mathcal{G})$.

   This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:
   - If $s'_1 \xrightarrow{a}$, then $s'_2 \xrightarrow{a}$ with $\bigcup\limits_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{\mathcal{D}_1(\bigcup \mathcal{G})\} \subseteq \bigcup\limits_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{\mathcal{D}_2(\bigcup \mathcal{G})\}$.
   - If $s'_2 \xrightarrow{a}$, then $s'_1 \xrightarrow{a}$ with $\bigcup\limits_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{\mathcal{D}_2(\bigcup \mathcal{G})\} \subseteq \bigcup\limits_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{\mathcal{D}_1(\bigcup \mathcal{G})\}$.

   Equivalently, if both s'_1 and s'_2 have at least one outgoing a-transition, then:
   $$\bigcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{\mathcal{D}_1(\textstyle\bigcup \mathcal{G})\} = \bigcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{\mathcal{D}_2(\textstyle\bigcup \mathcal{G})\}$$
   and hence:
   $$\bigsqcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\textstyle\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\textstyle\bigcup \mathcal{G})$$
   $$\bigsqcap_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\textstyle\bigcup \mathcal{G}) = \bigsqcap_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\textstyle\bigcup \mathcal{G})$$
   Therefore, $\mathcal{B}$ is also a $\sqcup\sqcap$-group-by-group probabilistic bisimulation, i.e., $s_1 \sim_{\mathrm{PB,gbg,}\sqcup\sqcap} s_2$.

2) Suppose that $s_1 \sim_{\mathrm{PB,gbg,}\leq} s_2$. This means that there exists a $\leq$-group-by-group probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. In other words, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:
   - If $s'_1 \xrightarrow{a} \mathcal{D}_1$, then $s'_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) \leq \mathcal{D}_2(\bigcup \mathcal{G})$.
   - If $s'_2 \xrightarrow{a} \mathcal{D}_2$, then $s'_1 \xrightarrow{a} \mathcal{D}_1$ such that $\mathcal{D}_2(\bigcup \mathcal{G}) \leq \mathcal{D}_1(\bigcup \mathcal{G})$.

   This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:
   - If $s'_1 \xrightarrow{a}$, then $s'_2 \xrightarrow{a}$ with $\bigsqcup\limits_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) \leq \bigsqcup\limits_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$.
   - If $s'_2 \xrightarrow{a}$, then $s'_1 \xrightarrow{a}$ with $\bigsqcup\limits_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G}) \leq \bigsqcup\limits_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G})$.

   Equivalently, if both s'_1 and s'_2 have at least one outgoing a-transition, then:
   $$\bigsqcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\textstyle\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\textstyle\bigcup \mathcal{G})$$
   Therefore, $\mathcal{B}$ is also a $\sqcup$-group-by-group probabilistic bisimulation, i.e., $s_1 \sim_{\mathrm{PB,gbg,}\sqcup} s_2$.
   The reverse implication holds too when the NPLTS $\mathcal{U}$ is image finite. In fact, this property guarantees that the following two sets:
   $$\bigcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{\mathcal{D}_1(\textstyle\bigcup \mathcal{G})\} \quad \text{and} \quad \bigcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{\mathcal{D}_2(\textstyle\bigcup \mathcal{G})\}$$
   are finite. In turn, the finiteness of those two sets ensures that their suprema respectively belong to the two sets themselves. As a consequence, starting from:
   $$\bigsqcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\textstyle\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\textstyle\bigcup \mathcal{G})$$
   or equivalently:
   $$\bigsqcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\textstyle\bigcup \mathcal{G}) \leq \bigsqcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\textstyle\bigcup \mathcal{G})$$
   $$\bigsqcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\textstyle\bigcup \mathcal{G}) \leq \bigsqcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\textstyle\bigcup \mathcal{G})$$
   when both s'_1 and s'_2 have at least one outgoing a-transition, the following holds:
   - If $s'_1 \xrightarrow{a} \mathcal{D}'_1$, then $s'_2 \xrightarrow{a} \mathcal{D}'_2$ with $\mathcal{D}'_1(\bigcup \mathcal{G}) \leq \mathcal{D}'_2(\bigcup \mathcal{G})$ because we can take $\mathcal{D}'_2$ such that $\mathcal{D}'_2(\bigcup \mathcal{G}) = \bigsqcup\limits_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$.
   - If $s'_2 \xrightarrow{a} \mathcal{D}'_2$, then $s'_1 \xrightarrow{a} \mathcal{D}'_1$ with $\mathcal{D}'_2(\bigcup \mathcal{G}) \leq \mathcal{D}'_1(\bigcup \mathcal{G})$ because we can take $\mathcal{D}'_1$ such that $\mathcal{D}'_1(\bigcup \mathcal{G}) = \bigsqcup\limits_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G})$.

3) Similar to the previous proof up to the use of $\geq$ in place of $\leq$ and $\sqcap$ in place of $\sqcup$ wherever necessary.

4) Suppose that $s_1 \sim_{\mathrm{PB,gbg,}\sqcup} s_2$. This means that there exists a $\sqcup$-group-by-group probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. In other words, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \xrightarrow{a}$ implies $s'_2 \xrightarrow{a}$ with:
   $$\bigsqcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\textstyle\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\textstyle\bigcup \mathcal{G})$$
   Then $\mathcal{B}$ must be a $\sqcap$-group-by-group probabilistic bisimulation as well and hence $s_1 \sim_{\mathrm{PB,gbg,}\sqcap} s_2$. In fact, if this were not the case, then there would exist $a' \in A$ and $\mathcal{G}' \in 2^{S/\mathcal{B}}$ such that $s'_1 \xrightarrow{a'}$, $s'_2 \xrightarrow{a}$, and:
   $$\bigsqcap_{s'_1 \xrightarrow{a'} \mathcal{D}_1} \mathcal{D}_1(\textstyle\bigcup \mathcal{G}') \neq \bigsqcap_{s'_2 \xrightarrow{a'} \mathcal{D}_2} \mathcal{D}_2(\textstyle\bigcup \mathcal{G}')$$
   As a consequence, denoting by $\mathcal{G}''$ the group of all the equivalence classes not in $\mathcal{G}'$, it would hold that $s'_1 \xrightarrow{a'}$, $s'_2 \xrightarrow{a}$, and:
   $$\begin{aligned} \bigsqcup_{s'_1 \xrightarrow{a'} \mathcal{D}_1} \mathcal{D}_1(\textstyle\bigcup \mathcal{G}'') &= 1 - \displaystyle\bigsqcap_{s'_1 \xrightarrow{a'} \mathcal{D}_1} \mathcal{D}_1(\textstyle\bigcup \mathcal{G}') \\ &\neq 1 - \displaystyle\bigsqcap_{s'_2 \xrightarrow{a'} \mathcal{D}_2} \mathcal{D}_2(\textstyle\bigcup \mathcal{G}') \\ &= \displaystyle\bigsqcup_{s'_2 \xrightarrow{a'} \mathcal{D}_2} \mathcal{D}_2(\textstyle\bigcup \mathcal{G}'') \end{aligned}$$
   thus contradicting the fact that $\mathcal{B}$ is a $\sqcup$-group-by-group

probabilistic bisimulation.

By proceeding in a similar way, we can prove that $s_1 \sim_{\mathrm{PB,gbg},\sqcap} s_2$ implies $s_1 \sim_{\mathrm{PB,gbg},\sqcup} s_2$. Therefore, $\sim_{\mathrm{PB,gbg},\sqcup}$ and $\sim_{\mathrm{PB,gbg},\sqcap}$ coincide.

Finally, we prove that $\sim_{\mathrm{PB,gbg},\sqcup\sqcap}$ and $\sim_{\mathrm{PB,gbg},\sqcup}$ coincide. If $s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$, then $s_1 \sim_{\mathrm{PB,gbg},\sqcup} s_2$ because a $\sqcup\sqcap$-group-by-group probabilistic bisimulation is trivially a $\sqcup$-group-by-group probabilistic bisimulation. Suppose now that $s_1 \sim_{\mathrm{PB,gbg},\sqcup} s_2$. This means that there exists a $\sqcup$-group-by-group probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. Since $\mathcal{B}$ must also be a $\sqcap$-group-by-group probabilistic bisimulation, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \xrightarrow{a}$ implies $s'_2 \xrightarrow{a}$ with:

$$\bigsqcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
$$\bigsqcap_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is also a $\sqcup\sqcap$-group-by-group probabilistic bisimulation, i.e., $s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$. ■

Proof of Thm. 5.7. Let $\mathcal{U} = (S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$:

1) Since an ordinary transition is a special case of combined transition in which a single transition is taken with weight 1, it trivially holds that (i) a class-distribution probabilistic bisimulation is a class-distribution ct-probabilistic bisimulation and (ii) a $=$-group-by-group probabilistic bisimulation is a $=$-group-by-group ct-probabilistic bisimulation.
2) The inclusion of $\sim_{\mathrm{PB,gbg},\rhd}$ in $\sim^{\mathrm{ct}}_{\mathrm{PB,gbg},\rhd}$ is a straightforward consequence of the fact that an ordinary transition is a special case of combined transition in which a single transition is taken with weight 1. We now prove the reverse inclusions:
   - Suppose that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg},\leq} s_2$. This means that there exists a $\leq$-group-by-group ct-probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. In other words, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \xrightarrow{a} \mathcal{D}_1$ implies $s'_2 \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) \leq \mathcal{D}_2(\bigcup \mathcal{G})$. On the side of s'_2, this means that there exist $n \in \mathbb{N}_{>0}$, $\{p_i \in \mathbb{R}_{]0,1]} \mid 1 \leq i \leq n\}$, and $\{s'_2 \xrightarrow{a} \mathcal{D}_{2,i} \mid 1 \leq i \leq n\}$ such that $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} p_i \cdot \mathcal{D}_{2,i} = \mathcal{D}_2$. As a consequence:
   $$\begin{aligned}\mathcal{D}_2(\bigcup \mathcal{G}) &\leq \sum_{i=1}^{n} p_i \cdot \max_{1 \leq i \leq n} \mathcal{D}_{2,i}(\bigcup \mathcal{G}) \\ &= \max_{1 \leq i \leq n} \mathcal{D}_{2,i}(\bigcup \mathcal{G}) \cdot \sum_{i=1}^{n} p_i \\ &= \max_{1 \leq i \leq n} \mathcal{D}_{2,i}(\bigcup \mathcal{G})\end{aligned}$$
   and hence there exists $\mathcal{D}'_2$ such that $s'_2 \xrightarrow{a} \mathcal{D}'_2$ with $\mathcal{D}_1(\bigcup \mathcal{G}) \leq \mathcal{D}'_2(\bigcup \mathcal{G})$. This means that $\mathcal{B}$ is also a $\leq$-group-by-group probabilistic bisimulation, i.e., $s_1 \sim_{\mathrm{PB,gbg},\leq} s_2$.
   - The proof that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg},\geq} s_2 \Longrightarrow s_1 \sim_{\mathrm{PB,gbg},\geq} s_2$ is similar to the previous proof up to the use of $\geq$ in place of $\leq$ and $\min$ in place of $\max$ wherever necessary.
   - Suppose that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$. This means that there exists a $\sqcup\sqcap$-group-by-group ct-probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. In other words, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \xrightarrow{a}$ implies $s'_2 \xrightarrow{a}$ with:
   $$\bigsqcup_{s'_1 \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
   $$\bigsqcap_{s'_1 \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s'_2 \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
   Given $a \in A$, $\mathcal{G} \in 2^{S/\mathcal{B}}$, and $s \in S$ having at least one outgoing a-transition, when $\mathcal{U}$ is image finite it holds that:
   $$\bigsqcup_{s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}} \mathcal{D}(\bigcup \mathcal{G}) = \bigsqcup_{s \xrightarrow{a} \mathcal{D}} \mathcal{D}(\bigcup \mathcal{G})$$
   $$\bigsqcap_{s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}} \mathcal{D}(\bigcup \mathcal{G}) = \bigsqcap_{s \xrightarrow{a} \mathcal{D}} \mathcal{D}(\bigcup \mathcal{G})$$
   because the supremum and the infimum on the left are respectively achieved by two ordinary a-transitions of s. In fact, let $\mathcal{D}_\sqcup$ (resp. $\mathcal{D}_\sqcap$) be the target of an a-transition of s assigning the maximum (resp. minimum) value to $\bigcup \mathcal{G}$ among all the a-transitions of s and consider an arbitrary convex combination of a subset $\{s \xrightarrow{a} \mathcal{D}_i \mid 1 \leq i \leq n\}$ of those transitions, with weights $p_1, \dots, p_n$ and $n \in \mathbb{N}_{>0}$. Then:
   $$\sum_{i=1}^{n} p_i \cdot \mathcal{D}_i(\bigcup \mathcal{G}) \leq \sum_{i=1}^{n} p_i \cdot \mathcal{D}_\sqcup(\bigcup \mathcal{G}) = \mathcal{D}_\sqcup(\bigcup \mathcal{G})$$
   $$\sum_{i=1}^{n} p_i \cdot \mathcal{D}_i(\bigcup \mathcal{G}) \geq \sum_{i=1}^{n} p_i \cdot \mathcal{D}_\sqcap(\bigcup \mathcal{G}) = \mathcal{D}_\sqcap(\bigcup \mathcal{G})$$
   As a consequence, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \xrightarrow{a}$ implies $s'_2 \xrightarrow{a}$ with:
   $$\bigsqcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
   $$\bigsqcap_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
   This means that $\mathcal{B}$ is also a $\sqcup\sqcap$-group-by-group probabilistic bisimulation, i.e., $s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$.
   - The proof that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg},\sqcup} s_2 \Longrightarrow s_1 \sim_{\mathrm{PB,gbg},\sqcup} s_2$ is similar to the proof that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg},\sqcup\sqcap} s_2 \Longrightarrow s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$.
   - The proof that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg},\sqcap} s_2 \Longrightarrow s_1 \sim_{\mathrm{PB,gbg},\sqcap} s_2$ is similar to the proof that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg},\sqcup\sqcap} s_2 \Longrightarrow s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$.
3) The fact that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,dis}} s_2$ implies $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg},=} s_2$ is a straightforward consequence of the fact that a class-distribution ct-probabilistic bisimulation is trivially a $=$-group-by-group ct-probabilistic bisimulation.
4) Suppose that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PB,gbg},=} s_2$. This means that there exists a $=$-group-by-group ct-probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. In other words, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:
   - If $s'_1 \xrightarrow{a} \mathcal{D}_1$, then $s'_2 \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$.

- If $s'_2 \xrightarrow{a} \mathcal{D}_2$, then $s'_1 \xrightarrow{a}_{\rm c} \mathcal{D}_1$ such that $\mathcal{D}_2(\bigcup \mathcal{G}) = \mathcal{D}_1(\bigcup \mathcal{G})$.

This implies that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

- If $s'_1 \xrightarrow{a}_{\rm c} \mathcal{D}_1$, then $s'_2 \xrightarrow{a}_{\rm c} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$.
- If $s'_2 \xrightarrow{a}_{\rm c} \mathcal{D}_2$, then $s'_1 \xrightarrow{a}_{\rm c} \mathcal{D}_1$ such that $\mathcal{D}_2(\bigcup \mathcal{G}) = \mathcal{D}_1(\bigcup \mathcal{G})$.

This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

- If $s'_1 \xrightarrow{a}$, then $s'_2 \xrightarrow{a}$ with $\bigcup_{s'_1 \xrightarrow{a}_{\rm c} \mathcal{D}_1} \{\mathcal{D}_1(\bigcup \mathcal{G})\} \subseteq \bigcup_{s'_2 \xrightarrow{a}_{\rm c} \mathcal{D}_2} \{\mathcal{D}_2(\bigcup \mathcal{G})\}$.
- If $s'_2 \xrightarrow{a}$, then $s'_1 \xrightarrow{a}$ with $\bigcup_{s'_2 \xrightarrow{a}_{\rm c} \mathcal{D}_2} \{\mathcal{D}_2(\bigcup \mathcal{G})\} \subseteq \bigcup_{s'_1 \xrightarrow{a}_{\rm c} \mathcal{D}_1} \{\mathcal{D}_1(\bigcup \mathcal{G})\}$.

Equivalently, if both s'_1 and s'_2 have at least one outgoing a-transition, then:

$$\bigcup_{s'_1 \xrightarrow{a}_{\rm c} \mathcal{D}_1} \{\mathcal{D}_1(\bigcup \mathcal{G})\} = \bigcup_{s'_2 \xrightarrow{a}_{\rm c} \mathcal{D}_2} \{\mathcal{D}_2(\bigcup \mathcal{G})\}$$

and hence:

$$\bigsqcup_{s'_1 \xrightarrow{a}_{\rm c} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \xrightarrow{a}_{\rm c} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
$$\bigsqcap_{s'_1 \xrightarrow{a}_{\rm c} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s'_2 \xrightarrow{a}_{\rm c} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$

Therefore, $\mathcal{B}$ is also a $\sqcup\sqcap$-group-by-group ct-probabilistic bisimulation, i.e., $s_1 \sim^{\rm ct}_{\rm PB,gbg,\sqcup\sqcap} s_2$.

Suppose now that $s_1 \sim^{\rm ct}_{\rm PB,gbg,\sqcup\sqcap} s_2$. This means that there exists a $\sqcup\sqcap$-group-by-group ct-probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. Given $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$, assume that there exists $s_1 \xrightarrow{a} \mathcal{D}_1$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) = p$. Since $(s_1, s_2) \in \mathcal{B}$ and the NPLTS is image finite, there exist $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}'_2$ such that $\mathcal{D}'_2(\bigcup \mathcal{G}) = p' \leq p$ and $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}''_2$ such that $\mathcal{D}''_2(\bigcup \mathcal{G}) = p'' \geq p$. If $p' = p$ (resp. $p'' = p$), then $s_1 \xrightarrow{a} \mathcal{D}_1$ is trivially matched by $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}'_2$ (resp. $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}''_2$) with respect to $\sim^{\rm ct}_{\rm PB,gbg,=}$ when considering $\mathcal{G}$.

Assume that $p' < p < p''$ and note that $s_2 \xrightarrow{a}_{\rm c} (x \cdot \mathcal{D}'_2 + y \cdot \mathcal{D}''_2)$ for all $x, y \in \mathbb{R}_{]0,1]}$ such that $x + y = 1$. Indeed, directly from the definition of combined transition, we have that:

- Since $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}'_2$, there exist $n \in \mathbb{N}_{>0}$, $\{p'_i \in \mathbb{R}_{]0,1]} \mid 1 \leq i \leq n\}$, and $\{s_2 \xrightarrow{a} \hat{\mathcal{D}}'_i \mid 1 \leq i \leq n\}$ such that $\sum_{i=1}^{n} p'_i = 1$ and $\sum_{i=1}^{n} p'_i \cdot \hat{\mathcal{D}}'_i = \mathcal{D}'_2$.
- Since $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}''_2$, there exist $m \in \mathbb{N}_{>0}$, $\{p''_j \in \mathbb{R}_{]0,1]} \mid 1 \leq j \leq m\}$, and $\{s_2 \xrightarrow{a} \hat{\mathcal{D}}''_j \mid 1 \leq j \leq m\}$ such that $\sum_{j=1}^{m} p''_j = 1$ and $\sum_{j=1}^{m} p''_j \cdot \hat{\mathcal{D}}''_j = \mathcal{D}''_2$.

Hence, $(x \cdot \mathcal{D}'_2 + y \cdot \mathcal{D}''_2)$ can be obtained from the appropriate combination of:

$$\{s_2 \xrightarrow{a} \hat{\mathcal{D}}'_i \mid 1 \leq i \leq n\} \cup \{s_2 \xrightarrow{a} \hat{\mathcal{D}}''_j \mid 1 \leq j \leq m\}$$

with coefficients:

$$\{x \cdot p'_i \in \mathbb{R}_{]0,1]} \mid 1 \leq i \leq n\} \cup \{y \cdot p''_j \in \mathbb{R}_{]0,1]} \mid 1 \leq j \leq m\}$$

If we take $x = \frac{p''-p}{p''-p'}$ and $y = \frac{p-p'}{p''-p'}$, then $s_2 \xrightarrow{a}_{\rm c} \left(\frac{p''-p}{p''-p'} \cdot \mathcal{D}'_2 + \frac{p-p'}{p''-p'} \cdot \mathcal{D}''_2\right)$ with:

$$\begin{aligned}
&\left(\tfrac{p''-p}{p''-p'} \cdot \mathcal{D}'_2 + \tfrac{p-p'}{p''-p'} \cdot \mathcal{D}''_2\right)(\textstyle\bigcup \mathcal{G}) \\
&\quad = \tfrac{p''-p}{p''-p'} \cdot \mathcal{D}'_2(\textstyle\bigcup \mathcal{G}) + \tfrac{p-p'}{p''-p'} \cdot \mathcal{D}''_2(\textstyle\bigcup \mathcal{G}) \\
&\quad = \tfrac{p''-p}{p''-p'} \cdot p' + \tfrac{p-p'}{p''-p'} \cdot p'' \\
&\quad = \tfrac{p' \cdot p'' - p \cdot p' + p \cdot p'' - p' \cdot p''}{p''-p'} \\
&\quad = p \cdot \tfrac{p''-p'}{p''-p'} \\
&\quad = p = \mathcal{D}_1(\textstyle\bigcup \mathcal{G})
\end{aligned}$$

Due to the generality of $(s_1, s_2) \in \mathcal{B}$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$, it turns out that $\mathcal{B}$ is also a $=$-group-by-group ct-probabilistic bisimulation, i.e., $s_1 \sim^{\rm ct}_{\rm PB,gbg,=} s_2$. ■

B: Multistep Variants Inspired by $\sim_{\rm B}$

We start by introducing the multistep variant of $\sim_{\rm B}$ and proving that it coincides with $\sim_{\rm B}$ itself.

Definition A.1: Let $(S, A, \longrightarrow)$ be an NPLTS in which the target of each transition is a Dirac distribution. A relation $\mathcal{B}$ over S is a *multistep bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$:

- If $s_1 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_1}$, then $s_2 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_2}$ such that $(s'_1, s'_2) \in \mathcal{B}$.
- If $s_2 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_2}$, then $s_1 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_1}$ such that $(s'_1, s'_2) \in \mathcal{B}$.

We denote by $\sim_{\rm B,m}$ the largest multistep bisimulation. ■

Theorem A.2: Let $(S, A, \longrightarrow)$ be an NPLTS in which the target of each transition is a Dirac distribution. Let $s_1, s_2 \in S$. Then:

$$s_1 \sim_{\rm B,m} s_2 \iff s_1 \sim_{\rm B} s_2$$

Proof: Suppose that $s_1 \sim_{\rm B,m} s_2$. This means that there exists a multistep bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. As a consequence, it holds in particular that for all $(s'_1, s'_2) \in \mathcal{B}$ and $a \in A$:

- If $s'_1 \stackrel{a}{\Longrightarrow} \delta_{s''_1}$, then $s'_2 \stackrel{a}{\Longrightarrow} \delta_{s''_2}$ such that $(s''_1, s''_2) \in \mathcal{B}$.
- If $s'_2 \stackrel{a}{\Longrightarrow} \delta_{s''_2}$, then $s'_1 \stackrel{a}{\Longrightarrow} \delta_{s''_1}$ such that $(s''_1, s''_2) \in \mathcal{B}$.

Since $\stackrel{a}{\Longrightarrow}$ coincides with $\stackrel{a}{\longrightarrow}$, we have that $\mathcal{B}$ is also a bisimulation and hence $s_1 \sim_{\rm B} s_2$.

Suppose now that $s_1 \sim_{\rm B} s_2$. This means that there exists a bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. We prove that $\mathcal{B}$ is also a multistep bisimulation, so that $s_1 \sim_{\rm B,m} s_2$ will follow. Given $s'_1, s'_2 \in S$ such that $(s'_1, s'_2) \in \mathcal{B}$ and $\alpha \in A^*$, we proceed by induction on $|\alpha|$:

- If $|\alpha| = 0$, then $s'_1 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_1}$ and $s'_2 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_2}$ are the only possible computations from s'_1 and s'_2 labeled with α, hence the result trivially holds.
- Let $|\alpha| = n \in \mathbb{N}_{>0}$ and suppose that the result holds for all traces of length $n - 1$. Assume $\alpha = a\,\alpha'$. Since $(s'_1, s'_2) \in \mathcal{B}$ and $\mathcal{B}$ is a bisimulation, it holds that $s'_1 \stackrel{a}{\longrightarrow} \delta_{s'''_1}$ implies $s'_2 \stackrel{a}{\longrightarrow} \delta_{s'''_2}$ – and symmetrically $s'_2 \stackrel{a}{\longrightarrow} \delta_{s'''_2}$ implies $s'_1 \stackrel{a}{\longrightarrow} \delta_{s'''_1}$ – with $(s'''_1, s'''_2) \in \mathcal{B}$.

  Suppose that $s'_1 \stackrel{\alpha}{\Longrightarrow} \delta_{s''_1}$ with $s'_1 \stackrel{a}{\longrightarrow} \delta_{s'''_1}$ and $s'''_1 \stackrel{\alpha'}{\Longrightarrow} \delta_{s''_1}$. Then $s'_2 \stackrel{a}{\longrightarrow} \delta_{s'''_2}$ with $(s'''_1, s'''_2) \in \mathcal{B}$ and by the induction hypothesis we have that $s'''_2 \stackrel{\alpha'}{\Longrightarrow} \delta_{s''_2}$ with $(s''_1, s''_2) \in \mathcal{B}$. As a consequence, $s'_2 \stackrel{\alpha}{\Longrightarrow} \delta_{s''_2}$ with $(s''_1, s''_2) \in \mathcal{B}$. Symmetrically, with a similar argument we derive that $s'_2 \stackrel{\alpha}{\Longrightarrow} \delta_{s''_2}$ implies $s'_1 \stackrel{\alpha}{\Longrightarrow} \delta_{s''_1}$ with $(s''_1, s''_2) \in \mathcal{B}$.

■

We now provide the $\sim_{\rm B,m}$-inspired definition of each of the probabilistic bisimilarities considered in this paper and prove that it coincides with the original one-step equivalence. The ct-variants of the $\sim_{\rm B,m}$-inspired probabilistic bisimilarities can be defined similarly and satisfy an analogous coincidence property with respect to the original one-step ct-equivalences.

Definition A.3: Let $(S, A, \longrightarrow)$ be an NPLTS. An equivalence relation $\mathcal{B}$ over S is a *multistep class-distribution probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$ it holds that $s_1 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_1$ implies $s_2 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_2$ such that, for all equivalence classes $C \in S/\mathcal{B}$, $\mathcal{D}_1(C) = \mathcal{D}_2(C)$. We denote by $\sim_{\rm PB,dis,m}$ the largest multistep class-distribution probabilistic bisimulation. ■

Theorem A.4: Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$. Then:

$$s_1 \sim_{\rm PB,dis,m} s_2 \iff s_1 \sim_{\rm PB,dis} s_2$$

Proof: Suppose that $s_1 \sim_{\rm PB,dis,m} s_2$. This means that there exists a multistep class-distribution probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. As a consequence, it holds in particular that for all $(s'_1, s'_2) \in \mathcal{B}$ and $a \in A$, whenever $s'_1 \stackrel{a}{\Longrightarrow} \mathcal{D}_1$, then $s'_2 \stackrel{a}{\Longrightarrow} \mathcal{D}_2$ such that, for all $C \in S/\mathcal{B}$, $\mathcal{D}_1(C) = \mathcal{D}_2(C)$. Since $\stackrel{a}{\Longrightarrow}$ coincides with $\stackrel{a}{\longrightarrow}$, we have that $\mathcal{B}$ is also a class-distribution probabilistic bisimulation and hence $s_1 \sim_{\rm PB,dis} s_2$.

Suppose now that $s_1 \sim_{\rm PB,dis} s_2$. This means that there exists a class-distribution probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. We prove that $\mathcal{B}$ is also a multistep class-distribution probabilistic bisimulation, so that $s_1 \sim_{\rm PB,dis,m} s_2$ will follow. Given $s'_1, s'_2 \in S$ such that $(s'_1, s'_2) \in \mathcal{B}$ and $\alpha \in A^*$, we proceed by induction on $|\alpha|$:

- If $|\alpha| = 0$, then $s'_1 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_1}$ and $s'_2 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_2}$ are the only possible computations from s'_1 and s'_2 labeled with α and for all $C \in S/\mathcal{B}$ it holds that:
  $$\delta_{s'_1}(C) = \delta_{s'_2}(C) = \begin{cases} 1 & \text{if } \{s'_1, s'_2\} \subseteq C \\ 0 & \text{if } \{s'_1, s'_2\} \cap C = \emptyset \end{cases}$$
  because $(s'_1, s'_2) \in \mathcal{B}$ and C is an equivalence class with respect to $\mathcal{B}$.
- Let $|\alpha| = n \in \mathbb{N}_{>0}$ and suppose that the result holds for all traces of length $n - 1$. Assume $\alpha = a\,\alpha'$. Since $(s'_1, s'_2) \in \mathcal{B}$ and $\mathcal{B}$ is a class-distribution probabilistic bisimulation, it holds that $s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}'_1$ implies $s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}'_2$ such that, for all $C \in S/\mathcal{B}$, $\mathcal{D}'_1(C) = \mathcal{D}'_2(C)$.

  Suppose that $s'_1 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_1$ with $s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}'_1$, $s''_1 \stackrel{\alpha'}{\Longrightarrow} \mathcal{D}_1$, and $\mathcal{D}'_1(s''_1) > 0$. Then there exists $s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}'_2$ such that, for all $C \in S/\mathcal{B}$, $\mathcal{D}'_1(C) = \mathcal{D}'_2(C)$. If we take s''_2 such that $(s''_1, s''_2) \in \mathcal{B}$ and $\mathcal{D}'_2(s''_2) > 0$, by the induction hypothesis there exists $s''_2 \stackrel{\alpha'}{\Longrightarrow} \mathcal{D}_2$ such that, for all $C \in S/\mathcal{B}$, $\mathcal{D}_1(C) = \mathcal{D}_2(C)$. As a consequence, there exists $s'_2 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_2$ such that, for all $C \in S/\mathcal{B}$, $\mathcal{D}_1(C) = \mathcal{D}_2(C)$.

■

Definition A.5: Let $(S, A, \longrightarrow)$ be an NPLTS and the relational operator $\bowtie \in \{=, \leq, \geq\}$. An equivalence relation $\mathcal{B}$ over S is a *multistep $\bowtie$-group-by-group probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_1$ implies $s_2 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) \bowtie \mathcal{D}_2(\bigcup \mathcal{G})$. We denote by $\sim_{\rm PB,gbg,\bowtie,m}$ the largest multistep $\bowtie$-group-by-group probabilistic bisimulation. ■

Theorem A.6: Let $(S, A, \longrightarrow)$ be an NPLTS, $s_1, s_2 \in S$, and $\bowtie \in \{=, \leq, \geq\}$. Then:

$$s_1 \sim_{\rm PB,gbg,\bowtie,m} s_2 \iff s_1 \sim_{\rm PB,gbg,\bowtie} s_2$$

Proof: Suppose that $s_1 \sim_{\rm PB,gbg,\bowtie,m} s_2$. This means that there exists a multistep $\bowtie$-group-by-group probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. As a consequence, it holds in particular that for all $(s'_1, s'_2) \in \mathcal{B}$, $a \in A$, and

$\mathcal{G} \in 2^{S/\mathcal{B}}$, whenever $s'_1 \stackrel{a}{\Longrightarrow} \mathcal{D}_1$, then $s'_2 \stackrel{a}{\Longrightarrow} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) \bowtie \mathcal{D}_2(\bigcup \mathcal{G})$. Since $\stackrel{a}{\Longrightarrow}$ coincides with $\stackrel{a}{\longrightarrow}$, we have that $\mathcal{B}$ is also a $\bowtie$-group-by-group probabilistic bisimulation and hence $s_1 \sim_{\mathrm{PB,gbg},\bowtie} s_2$.

Suppose now that $s_1 \sim_{\mathrm{PB,gbg},\bowtie} s_2$. This means that there exists a $\bowtie$-group-by-group probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. We prove that $\mathcal{B}$ is also a multistep $\bowtie$-group-by-group probabilistic bisimulation, so that $s_1 \sim_{\mathrm{PB,gbg},\bowtie,\mathrm{m}} s_2$ will follow. Given $s'_1, s'_2 \in S$ such that $(s'_1, s'_2) \in \mathcal{B}$, $\alpha \in A^*$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$, we proceed by induction on $|\alpha|$:

- If $|\alpha| = 0$, then $s'_1 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_1}$ and $s'_2 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_2}$ are the only possible computations from s'_1 and s'_2 labeled with α and it holds that:
$$\delta_{s'_1}(\bigcup \mathcal{G}) = \delta_{s'_2}(\bigcup \mathcal{G}) = \begin{cases} 1 & \text{if } \{s'_1, s'_2\} \subseteq C \text{ for some } C \in \mathcal{G} \\ 0 & \text{if } \{s'_1, s'_2\} \cap C = \emptyset \text{ for all } C \in \mathcal{G} \end{cases}$$
because $(s'_1, s'_2) \in \mathcal{B}$ and $\mathcal{G}$ is a group of equivalence classes with respect to $\mathcal{B}$.
- Let $|\alpha| = n \in \mathbb{N}_{>0}$ and suppose that the result holds for all traces of length $n - 1$. Assume $\alpha = a\,\alpha'$. Since $(s'_1, s'_2) \in \mathcal{B}$ and $\mathcal{B}$ is a $\bowtie$-group-by-group probabilistic bisimulation, for all $\mathcal{G}' \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}'_1$ implies $s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}'_2$ such that $\mathcal{D}'_1(\bigcup \mathcal{G}') \bowtie \mathcal{D}'_2(\bigcup \mathcal{G}')$.
Suppose that $s'_1 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_1$ with $s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}'_1$, $s''_1 \stackrel{\alpha'}{\Longrightarrow} \mathcal{D}_1$, and $\mathcal{D}'_1(s''_1) > 0$. Let $\mathcal{G}' = \{C'\}$ with C' being the equivalence class containing s''_1. Then there exists $s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}'_2$ such that $\mathcal{D}'_1(\bigcup \mathcal{G}') \bowtie \mathcal{D}'_2(\bigcup \mathcal{G}')$. If we take s''_2 such that $(s''_1, s''_2) \in \mathcal{B}$ and $\mathcal{D}'_2(s''_2) > 0$ – it obviously exists in the case that $\bowtie \in \{=, \leq\}$ because $\mathcal{D}'_1(s''_1) > 0$, and it also exists in the case that $\bowtie$ is $\geq$ because, if s'_2 had no a-transition reaching $\mathcal{G}'$ with probability greater than 0, then all a-transitions of s'_2 would reach $\mathcal{G}'' = 2^{S/\mathcal{B}} \setminus \mathcal{G}'$ with probability 1 and hence for the transition $s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}'_1$ we would have $\mathcal{D}'_1(\bigcup \mathcal{G}'') = 1 - \mathcal{D}'_1(\bigcup \mathcal{G}') < 1 = \mathcal{D}'_2(\bigcup \mathcal{G}'')$ for all transitions $s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}'_2$, i.e., $\mathcal{B}$ would not be a $\geq$-group-by-group probabilistic bisimulation – by the induction hypothesis there exists $s''_2 \stackrel{\alpha'}{\Longrightarrow} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) \bowtie \mathcal{D}_2(\bigcup \mathcal{G})$. As a consequence, there exists $s'_2 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) \bowtie \mathcal{D}_2(\bigcup \mathcal{G})$.

■

Definition A.7: Let $(S, A, \longrightarrow)$ be an NPLTS. An equivalence relation $\mathcal{B}$ over S is a *multistep $\sqcup\sqcap$-group-by-group probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \stackrel{\alpha}{\Longrightarrow}$ implies $s_2 \stackrel{\alpha}{\Longrightarrow}$ with:
$$\bigsqcup_{s_1 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s_2 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
$$\bigsqcap_{s_1 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s_2 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
We denote by $\sim_{\mathrm{PB,gbg},\sqcup\sqcap,\mathrm{m}}$ the largest multistep $\sqcup\sqcap$-group-by-group probabilistic bisimulation. ■

Theorem A.8: Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$. Then:
$$s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap,\mathrm{m}} s_2 \iff s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$$

Proof: Suppose that $s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap,\mathrm{m}} s_2$. This means that there exists a multistep $\sqcup\sqcap$-group-by-group probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. As a consequence, it holds in particular that for all $(s'_1, s'_2) \in \mathcal{B}$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$, whenever $s'_1 \stackrel{a}{\Longrightarrow}$, then $s'_2 \stackrel{a}{\Longrightarrow}$ with:
$$\bigsqcup_{s'_1 \stackrel{a}{\Longrightarrow} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \stackrel{a}{\Longrightarrow} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
$$\bigsqcap_{s'_1 \stackrel{a}{\Longrightarrow} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s'_2 \stackrel{a}{\Longrightarrow} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
Since $\stackrel{a}{\Longrightarrow}$ coincides with $\stackrel{a}{\longrightarrow}$, we have that $\mathcal{B}$ is also a $\sqcup\sqcap$-group-by-group probabilistic bisimulation and hence $s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$.

Suppose now that $s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap} s_2$. This means that there exists a $\sqcup\sqcap$-group-by-group probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. We prove that $\mathcal{B}$ is also a multistep $\sqcup\sqcap$-group-by-group probabilistic bisimulation, so that $s_1 \sim_{\mathrm{PB,gbg},\sqcup\sqcap,\mathrm{m}} s_2$ will follow. Given $s'_1, s'_2 \in S$ such that $(s'_1, s'_2) \in \mathcal{B}$, $\alpha \in A^*$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$, we proceed by induction on $|\alpha|$:

- If $|\alpha| = 0$, then $s'_1 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_1}$ and $s'_2 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_2}$ are the only possible computations from s'_1 and s'_2 labeled with α and it holds that:
$$\delta_{s'_1}(\bigcup \mathcal{G}) = \delta_{s'_2}(\bigcup \mathcal{G}) = \begin{cases} 1 & \text{if } \{s'_1, s'_2\} \subseteq C \text{ for some } C \in \mathcal{G} \\ 0 & \text{if } \{s'_1, s'_2\} \cap C = \emptyset \text{ for all } C \in \mathcal{G} \end{cases}$$
because $(s'_1, s'_2) \in \mathcal{B}$ and $\mathcal{G}$ is a group of equivalence classes with respect to $\mathcal{B}$. Therefore:
$$\bigsqcup_{s'_1 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \delta_{s'_1}(\bigcup \mathcal{G}) = \delta_{s'_2}(\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
$$\bigsqcap_{s'_1 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \delta_{s'_1}(\bigcup \mathcal{G}) = \delta_{s'_2}(\bigcup \mathcal{G}) = \bigsqcap_{s'_2 \stackrel{\alpha}{\Longrightarrow} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
- Let $|\alpha| = n \in \mathbb{N}_{>0}$ and suppose that the result holds for all traces of length $n - 1$. Assume $\alpha = a\,\alpha'$. Since $(s'_1, s'_2) \in \mathcal{B}$ and $\mathcal{B}$ is a $\sqcup\sqcap$-group-by-group probabilistic bisimulation, for all $\mathcal{G}' \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \stackrel{a}{\longrightarrow}$ implies $s'_2 \stackrel{a}{\longrightarrow}$ with:
$$\bigsqcup_{s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}'_1} \mathcal{D}'_1(\bigcup \mathcal{G}') = \bigsqcup_{s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}'_2} \mathcal{D}'_2(\bigcup \mathcal{G}')$$
$$\bigsqcap_{s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}'_1} \mathcal{D}'_1(\bigcup \mathcal{G}') = \bigsqcap_{s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}'_2} \mathcal{D}'_2(\bigcup \mathcal{G}')$$
Suppose that $s'_1 \stackrel{\alpha}{\Longrightarrow}$ with $s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}'_1$, $s''_1 \stackrel{\alpha'}{\Longrightarrow}$, and $\mathcal{D}'_1(s''_1) > 0$. Let $\mathcal{G}' = \{C'\}$ with C' being the equivalence class containing s''_1. Then $s'_2 \stackrel{a}{\longrightarrow}$ with:
$$\bigsqcup_{s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}'_1} \mathcal{D}'_1(\bigcup \mathcal{G}') = \bigsqcup_{s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}'_2} \mathcal{D}'_2(\bigcup \mathcal{G}')$$
$$\bigsqcap_{s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}'_1} \mathcal{D}'_1(\bigcup \mathcal{G}') = \bigsqcap_{s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}'_2} \mathcal{D}'_2(\bigcup \mathcal{G}')$$
If we take s''_2 and $\mathcal{D}'_2$ such that $(s''_1, s''_2) \in \mathcal{B}$, $\mathcal{D}'_2(s''_2) > 0$, and $s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}'_2$, by the induction hypothesis we have that $s''_2 \stackrel{\alpha'}{\Longrightarrow}$ with:

$$\bigsqcup_{s_1'' \xRightarrow{\alpha'} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s_2'' \xRightarrow{\alpha'} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
$$\bigsqcap_{s_1'' \xRightarrow{\alpha'} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s_2'' \xRightarrow{\alpha'} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$

As a consequence, $s_2' \xRightarrow{\alpha}$ with:

$$\bigsqcup_{s_1' \xRightarrow{\alpha} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s_2' \xRightarrow{\alpha} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
$$\bigsqcap_{s_1' \xRightarrow{\alpha} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s_2' \xRightarrow{\alpha} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$

■

Definition A.9: Let $(S, A, \longrightarrow)$ be an NPLTS and symbol $\# \in \{\bigsqcup, \bigsqcap\}$. An equivalence relation $\mathcal{B}$ over S is a *multistep #-group-by-group probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \xRightarrow{\alpha}$ implies $s_2 \xRightarrow{\alpha}$ with:

$$\#_{s_1 \xRightarrow{\alpha} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \#_{s_2 \xRightarrow{\alpha} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$

We denote by $\sim_{\mathrm{PB,gbg},\#,\mathrm{m}}$ the largest multistep #-group-by-group probabilistic bisimulation. ■

Theorem A.10: Let $(S, A, \longrightarrow)$ be an NPLTS, $s_1, s_2 \in S$, and $\# \in \{\bigsqcup, \bigsqcap\}$. Then:

$$s_1 \sim_{\mathrm{PB,gbg},\#,\mathrm{m}} s_2 \iff s_1 \sim_{\mathrm{PB,gbg},\#} s_2$$

Proof: Similar to the proof of Thm. A.8. With regard to the induction step of the proof that $s_1 \sim_{\mathrm{PB,gbg},\#} s_2$ implies $s_1 \sim_{\mathrm{PB,gbg},\#,\mathrm{m}} s_2$, we observe that s_2'' and $\mathcal{D}_2'$ such that $(s_1'', s_2'') \in \mathcal{B}$, $\mathcal{D}_2'(s_2'') > 0$, and $s_2' \xrightarrow{a} \mathcal{D}_2'$ obviously exist in the case that # is $\sqcup$ because $\mathcal{D}_1'(s_1'') > 0$. They also exist in the case that # is $\sqcap$ because, if s_2' had no a-transition reaching $\mathcal{G}'$ (the group composed only of the equivalence class containing s_1'') with probability greater than 0, then all a-transitions of s_2' would reach $\mathcal{G}'' = 2^{S/\mathcal{B}} \setminus \mathcal{G}'$ with probability 1 and hence we would have:

$$\bigsqcap_{s_1' \xrightarrow{a} \mathcal{D}_1'} \mathcal{D}_1'(\bigcup \mathcal{G}'') < 1 = \bigsqcap_{s_2' \xrightarrow{a} \mathcal{D}_2'} \mathcal{D}_2'(\bigcup \mathcal{G}'')$$

i.e., the considered relation $\mathcal{B}$ would not be a $\sqcap$-group-by-group probabilistic bisimulation

■

We conclude by showing that all the considered $\sim_{\mathrm{B,m}}$-inspired probabilistic bisimilarities collapse into $\sim_{\mathrm{B,m}}$ when restricting attention to fully nondeterministic processes. An analogous result holds for their ct-variants.

Theorem A.11: Let $(S, A, \longrightarrow)$ be an NPLTS in which the target of each transition is a Dirac distribution. Let $s_1, s_2 \in S$ and $\circ \in \{=, \leq, \geq, \sqcup\sqcap, \sqcup, \sqcap\}$. Then:

$$s_1 \sim_{\mathrm{PB,dis,m}} s_2 \iff s_1 \sim_{\mathrm{PB,gbg},\circ,\mathrm{m}} s_2 \iff s_1 \sim_{\mathrm{B,m}} s_2$$

Proof: Since every multistep transition of this specific NPLTS can reach with probability greater than 0 a single state and hence a single class of any equivalence relation – which are thus reached with probability 1 – the reflexive, symmetric, and transitive closure of a multistep bisimulation is trivially a multistep class-distribution probabilistic bisimulation, a multistep =-group-by-group probabilistic bisimulation, a multistep $\leq$-group-by-group probabilistic bisimulation, a multistep $\geq$-group-by-group probabilistic bisimulation, a multistep $\sqcup\sqcap$-group-by-group probabilistic bisimulation, a multistep $\sqcup$-group-by-group probabilistic bisimulation, and a multistep $\sqcap$-group-by-group probabilistic bisimulation.

■

C: Multistep Variants Inspired by $\sim_{\rm PB}$

We start by introducing the multistep variant of $\sim_{\rm PB}$ and proving that it coincides with $\sim_{\rm PB}$ itself. Given an NPLTS $(S, A, \longrightarrow)$ in which the transitions of each state have different labels and given $s \in S$, $\alpha \in A^*$, and $S' \subseteq S$, we inductively define the multistep probability of reaching a state in S' from s via α as follows:

$$prob_{\rm m}(s, \alpha, S') \; = \; \begin{cases} \sum_{s' \in S} \mathcal{D}(s') \cdot prob_{\rm m}(s', \alpha', S') & \text{if } \alpha = a\,\alpha' \text{ and } s \xrightarrow{a} \mathcal{D} \\ 1 & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ 0 & \text{if } \alpha = a\,\alpha' \text{ and } s \not\xrightarrow{a} \\ & \text{or } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$$

Definition A.12: Let $(S, A, \longrightarrow)$ be an NPLTS in which the transitions of each state have different labels. An equivalence relation $\mathcal{B}$ over S is a *p-multistep probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$ and equivalence classes $C \in S/\mathcal{B}$ it holds that $s_1 \stackrel{\alpha}{\Longrightarrow}$ implies $s_2 \stackrel{\alpha}{\Longrightarrow}$ with:

$$prob_{\rm m}(s_1, \alpha, C) \; = \; prob_{\rm m}(s_2, \alpha, C)$$

We denote by $\sim_{\rm PB,pm}$ the largest p-multistep probabilistic bisimulation. ■

Theorem A.13: Let $(S, A, \longrightarrow)$ be an NPLTS in which the transitions of each state have different labels. Let $s_1, s_2 \in S$. Then:

$$s_1 \sim_{\rm PB,pm} s_2 \iff s_1 \sim_{\rm PB} s_2$$

Proof: Suppose that $s_1 \sim_{\rm PB,pm} s_2$. This means that there exists a p-multistep probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. As a consequence, it holds in particular that for all $(s'_1, s'_2) \in \mathcal{B}$, $a \in A$, and $C \in S/\mathcal{B}$, whenever $s'_1 \stackrel{a}{\Longrightarrow}$, then $s'_2 \stackrel{a}{\Longrightarrow}$ with:

$$prob_{\rm m}(s'_1, a, C) \; = \; prob_{\rm m}(s'_2, a, C)$$

Since $\stackrel{a}{\Longrightarrow}$ coincides with $\stackrel{a}{\longrightarrow}$ and for all $s \in S$ such that $s \xrightarrow{a} \mathcal{D}$ it holds that:

$$prob_{\rm m}(s, a, C) \; = \; \sum_{s' \in C} \mathcal{D}(s') \; = \; \mathcal{D}(C)$$

we have that $s'_1 \xrightarrow{a} \mathcal{D}_1$ implies $s'_2 \xrightarrow{a} \mathcal{D}_2$ with $\mathcal{D}_1(C) = \mathcal{D}_2(C)$. In other words, $\mathcal{B}$ is also a probabilistic bisimulation and hence $s_1 \sim_{\rm PB} s_2$.

Suppose now that $s_1 \sim_{\rm PB} s_2$. This means that there exists a probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. We prove that $\mathcal{B}$ is also a p-multistep probabilistic bisimulation, so that $s_1 \sim_{\rm PB,pm} s_2$ will follow. Given $s'_1, s'_2 \in S$ such that $(s'_1, s'_2) \in \mathcal{B}$, $\alpha \in A^*$, and $C \in S/\mathcal{B}$, we proceed by induction on $|\alpha|$:

- If $|\alpha| = 0$, then $s'_1 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_1}$ and $s'_2 \stackrel{\alpha}{\Longrightarrow} \delta_{s'_2}$ are the only possible computations from s'_1 and s'_2 labeled with α and it holds that:
$$prob_{\rm m}(s'_1, \alpha, C) \; = \; prob_{\rm m}(s'_2, \alpha, C) \; = \; \begin{cases} 1 & \text{if } \{s'_1, s'_2\} \subseteq C \\ 0 & \text{if } \{s'_1, s'_2\} \cap C = \emptyset \end{cases}$$
because $(s'_1, s'_2) \in \mathcal{B}$ and C is an equivalence class with respect to $\mathcal{B}$.
- Let $|\alpha| = n \in \mathbb{N}_{>0}$ and suppose that the result holds for all traces of length $n - 1$. Assume $\alpha = a\,\alpha'$. Since $(s'_1, s'_2) \in \mathcal{B}$ and $\mathcal{B}$ is a probabilistic bisimulation, for all $C' \in S/\mathcal{B}$ it holds that $s'_1 \xrightarrow{a} \mathcal{D}_1$ implies $s'_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(C') = \mathcal{D}_2(C')$.
Given $s \in S$ such that $s \stackrel{\alpha}{\Longrightarrow}$ with $s \xrightarrow{a} \mathcal{D}$, it holds that:
$$\begin{aligned} prob_{\rm m}(s, \alpha, C) \; &= \; \sum_{s' \in S} \mathcal{D}(s') \cdot prob_{\rm m}(s', \alpha', C) \\ &= \; \sum_{C' \in S/\mathcal{B}} \sum_{s' \in C'} \mathcal{D}(s') \cdot prob_{\rm m}(s', \alpha', C) \\ &= \; \sum_{C' \in S/\mathcal{B}} \sum_{s' \in C'} \mathcal{D}(s') \cdot prob_{\rm m}(s_{C'}, \alpha', C) \\ &= \; \sum_{C' \in S/\mathcal{B}} prob_{\rm m}(s_{C'}, \alpha', C) \cdot \sum_{s' \in C'} \mathcal{D}(s') \\ &= \; \sum_{C' \in S/\mathcal{B}} prob_{\rm m}(s_{C'}, \alpha', C) \cdot \mathcal{D}(C') \end{aligned}$$
where $s_{C'} \in C'$ and the factorization of $prob_{\rm m}(s_{C'}, \alpha', C)$ stems from the application of the induction hypothesis on α' to all states of each equivalence class C'. Since $s'_1 \xrightarrow{a} \mathcal{D}_1$ implies $s'_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(C') = \mathcal{D}_2(C')$ for all $C' \in S/\mathcal{B}$ – remember that the quantification over C' can be equivalently anticipated or postponed in the absence of internal nondeterminism – we derive that, whenever $s'_1 \stackrel{\alpha}{\Longrightarrow}$, then $s'_2 \stackrel{\alpha}{\Longrightarrow}$ with:
$$prob_{\rm m}(s'_1, \alpha, C) \; = \; prob_{\rm m}(s'_2, \alpha, C)$$

■

When considering an arbitrary NPLTS $(S, A, \longrightarrow)$, internal nondeterminism comes into play and hence there might be several computations labeled with the same trace belonging to *different* resolutions of nondeterminism. In that case, their multistep probabilities have to be kept separate, otherwise their multistep probabilities can be summed up like in the case of reactive probabilistic processes.

Since preserving the connection between each computation and the resolution of nondeterminism to which it belongs is important to define a $\sim_{\rm PB,m}$-inspired multistep variant of $\sim_{\rm PB,dis}$, we formalize below the notion of resolution. We call resolution of a state s of an NPLTS $\mathcal{U}$ any possible way of resolving nondeterminism starting from s. Each resolution is a tree-like structure whose branching points represent probabilistic choices. This is obtained by unfolding from s the graph structure underlying $\mathcal{U}$ and by selecting at each state a single transition of $\mathcal{U}$ – deterministic scheduler – or a convex combination of equally labeled transitions of $\mathcal{U}$ – randomized scheduler – among all the transitions possible from that state. A resolution of s can be formalized as an NPLTS $\mathcal{Z}$ rooted at a state z_s corresponding to s, in which every state has at most one outgoing transition and hence function $prob_{\rm m}$ can be safely applied.

Definition A.14: Let $\mathcal{U} = (S, A, \longrightarrow)$ be an NPLTS and $s \in S$. We say that an NPLTS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$ is a resolution of s obtained via a deterministic scheduler iff there exists a state correspondence function $corr : Z \to S$ such that $s = corr(z_s)$, for some $z_s \in Z$, and for all $z \in Z$:

- If $z \xrightarrow{a}_{\mathcal{Z}} \mathcal{D}$, then $corr(z) \xrightarrow{a} \mathcal{D}'$ with $\mathcal{D}(z') = \mathcal{D}'(corr(z'))$ for all $z' \in Z$.
- If $z \xrightarrow{a_1}_{\mathcal{Z}} \mathcal{D}_1$ and $z \xrightarrow{a_2}_{\mathcal{Z}} \mathcal{D}_2$, then $a_1 = a_2$ and $\mathcal{D}_1 = \mathcal{D}_2$.

We denote by $Res(s)$ the set of resolutions of s. ■

On the basis of the notion above, we provide a $\sim_{\rm PB,pm}$-inspired definition of $\sim_{\rm PB,dis}$ and show that it coincides with $\sim_{\rm PB,dis}$ itself. The ct-variant of the $\sim_{\rm PB,pm}$-inspired equivalence can be defined similarly and satisfies an analogous property with respect to the original one-step ct-equivalence.

Definition A.15: Let $(S, A, \longrightarrow)$ be an NPLTS. An equivalence relation $\mathcal{B}$ over S is a *p-multistep class-distribution probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$ it holds that $z_{s_1} \stackrel{\alpha}{\Longrightarrow}$ in a resolution $\mathcal{Z}_1 \in Res(s_1)$ implies $z_{s_2} \stackrel{\alpha}{\Longrightarrow}$ in a resolution $\mathcal{Z}_2 \in Res(s_2)$ such that for all equivalence classes $C \in S/\mathcal{B}$:

$$prob_{\rm m}(z_{s_1}, \alpha, corr_{\mathcal{Z}_1}^{-1}(C)) \ = \ prob_{\rm m}(z_{s_2}, \alpha, corr_{\mathcal{Z}_2}^{-1}(C))$$

We denote by $\sim_{\rm PB,dis,pm}$ the largest p-multistep class-distribution probabilistic bisimulation. ■

Theorem A.16: Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$. Then:

$$s_1 \sim_{\rm PB,dis,pm} s_2 \iff s_1 \sim_{\rm PB,dis} s_2$$

Proof: Suppose that $s_1 \sim_{\rm PB,dis,pm} s_2$. This means that there exists a p-multistep class-distribution probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. As a consequence, it holds in particular that for all $(s'_1, s'_2) \in \mathcal{B}$ and $a \in A$, whenever $z_{s_1} \stackrel{a}{\Longrightarrow}$ in a resolution $\mathcal{Z}_1 \in Res(s_1)$, then $z_{s_2} \stackrel{a}{\Longrightarrow}$ in a resolution $\mathcal{Z}_2 \in Res(s_2)$ such that for all $C \in S/\mathcal{B}$:

$$prob_{\rm m}(z_{s_1}, a, corr_{\mathcal{Z}_1}^{-1}(C)) \ = \ prob_{\rm m}(z_{s_2}, a, corr_{\mathcal{Z}_2}^{-1}(C))$$

Since $\stackrel{a}{\Longrightarrow}$ coincides with $\stackrel{a}{\longrightarrow}$ and for all $s \in S$ such that $z_s \stackrel{a}{\longrightarrow} \mathcal{D}$ in a resolution $\mathcal{Z} \in Res(s)$ it holds that:

$$prob_{\rm m}(z_s, a, corr_{\mathcal{Z}}^{-1}(C)) \ = \ = \sum_{z_{s'} \in corr_{\mathcal{Z}}^{-1}(C)} \mathcal{D}(z_{s'}) \ = \ \mathcal{D}(corr_{\mathcal{Z}}^{-1}(C))$$

we have that $s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}_1$ implies $s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}_2$ such that, for all $C \in S/\mathcal{B}$, $\mathcal{D}_1(C) = \mathcal{D}_2(C)$. In other words, $\mathcal{B}$ is also a class-distribution probabilistic bisimulation and hence $s_1 \sim_{\rm PB,dis} s_2$.

Suppose now that $s_1 \sim_{\rm PB,dis} s_2$. This means that there exists a class-distribution probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. We prove that $\mathcal{B}$ is also a p-multistep class-distribution probabilistic bisimulation, so that $s_1 \sim_{\rm PB,dis,pm} s_2$ will follow. Given $s'_1, s'_2 \in S$ such that $(s'_1, s'_2) \in \mathcal{B}$ and $\alpha \in A^*$, we proceed by induction on $|\alpha|$:

- If $|\alpha| = 0$, then $z_{s'_1} \stackrel{\alpha}{\Longrightarrow} \delta_{z_{s'_1}}$ and $z_{s'_2} \stackrel{\alpha}{\Longrightarrow} \delta_{z_{s'_2}}$ are the only possible computations labeled with α in any resolution $\mathcal{Z}_1 \in Res(s'_1)$ and any resolution $\mathcal{Z}_2 \in Res(s'_2)$, respectively, and for all $C \in S/\mathcal{B}$ it holds that:
$$prob_{\rm m}(z_{s'_1}, \alpha, corr_{\mathcal{Z}_1}^{-1}(C)) \ = \ prob_{\rm m}(z_{s'_2}, \alpha, corr_{\mathcal{Z}_2}^{-1}(C)) \ = \begin{cases} 1 & \text{if } \{s'_1, s'_2\} \subseteq C \\ 0 & \text{if } \{s'_1, s'_2\} \cap C = \emptyset \end{cases}$$
because $(s'_1, s'_2) \in \mathcal{B}$ and C is an equivalence class with respect to $\mathcal{B}$.
- Let $|\alpha| = n \in \mathbb{N}_{>0}$ and suppose that the result holds for all traces of length $n-1$. Assume $\alpha = a\,\alpha'$. Since $(s'_1, s'_2) \in \mathcal{B}$ and $\mathcal{B}$ is a class-distribution probabilistic bisimulation, it holds that $s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}_1$ implies $s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}_2$ such that, for all $C \in S/\mathcal{B}$, $\mathcal{D}_1(C) = \mathcal{D}_2(C)$.
Given $s \in S$ such that $z_s \stackrel{\alpha}{\Longrightarrow}$ with $z_s \stackrel{a}{\longrightarrow} \mathcal{D}$ in a resolution $\mathcal{Z} \in Res(s)$, for all $C \in S/\mathcal{B}$ it holds that:
$$\begin{aligned} & prob_{\rm m}(z_s, \alpha, corr_{\mathcal{Z}}^{-1}(C)) \ = \\ & = \sum_{z_{s'} \in Z} \mathcal{D}(z_{s'}) \cdot prob_{\rm m}(z_{s'}, \alpha', corr_{\mathcal{Z}}^{-1}(C)) \\ & = \sum_{C' \in S/\mathcal{B}} \ \sum_{z_{s'} \in corr_{\mathcal{Z}}^{-1}(C')} \mathcal{D}(z_{s'}) \cdot prob_{\rm m}(z_{s'}, \alpha', corr_{\mathcal{Z}}^{-1}(C)) \\ & = \sum_{C' \in S/\mathcal{B}} \ \sum_{z_{s'} \in corr_{\mathcal{Z}}^{-1}(C')} \mathcal{D}(z_{s'}) \cdot prob_{\rm m}(z_{s_{C'}}, \alpha', corr_{\mathcal{Z}}^{-1}(C)) \\ & = \sum_{C' \in S/\mathcal{B}} prob_{\rm m}(z_{s_{C'}}, \alpha', corr_{\mathcal{Z}}^{-1}(C)) \cdot \sum_{z_{s'} \in corr_{\mathcal{Z}}^{-1}(C')} \mathcal{D}(z_{s'}) \\ & = \sum_{C' \in S/\mathcal{B}} prob_{\rm m}(z_{s_{C'}}, \alpha', corr_{\mathcal{Z}}^{-1}(C)) \cdot \mathcal{D}(corr_{\mathcal{Z}}^{-1}(C')) \end{aligned}$$
where $s_{C'} \in C'$ and the factorization of $prob_{\rm m}(z_{s_{C'}}, \alpha', corr_{\mathcal{Z}}^{-1}(C))$ stems from the application of the induction hypothesis on α' to all states of each equivalence class C'. Since $s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}_1$ implies $s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}_2$ such that, for all $C' \in S/\mathcal{B}$, $\mathcal{D}_1(C') = \mathcal{D}_2(C')$, we derive that, whenever $z_{s'_1} \stackrel{\alpha}{\Longrightarrow}$ in a resolution $\mathcal{Z}_1 \in Res(s'_1)$, then $z_{s'_2} \stackrel{\alpha}{\Longrightarrow}$ in a resolution $\mathcal{Z}_2 \in Res(s'_2)$ such that for all $C \in S/\mathcal{B}$:
$$prob_{\rm m}(z_{s'_1}, \alpha, corr_{\mathcal{Z}}^{-1}(C)) \ = \ prob_{\rm m}(z_{s'_2}, \alpha, corr_{\mathcal{Z}}^{-1}(C))$$

■

Using the notion of resolution, we can also provide a $\sim_{\rm PB,pm}$-inspired definition of each of the six group-by-group probabilistic bisimilarities. The ct-variants of the six $\sim_{\rm PB,pm}$-inspired group-by-group probabilistic bisimilarities can be defined similarly.

Definition A.17: Let $(S, A, \longrightarrow)$ be an NPLTS and the relational operator $\bowtie \in \{=, \leq, \geq\}$. An equivalence relation $\mathcal{B}$ over S is a *p-multistep $\bowtie$-group-by-group probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $z_{s_1} \stackrel{\alpha}{\Longrightarrow}$ in a resolution $\mathcal{Z}_1 \in Res(s_1)$ implies $z_{s_2} \stackrel{\alpha}{\Longrightarrow}$ in a resolution $\mathcal{Z}_2 \in Res(s_2)$ with:

$$prob_{\rm m}(z_{s_1}, \alpha, corr_{\mathcal{Z}_1}^{-1}(\bigcup \mathcal{G})) \ \bowtie \ prob_{\rm m}(z_{s_2}, \alpha, corr_{\mathcal{Z}_2}^{-1}(\bigcup \mathcal{G}))$$

We denote by $\sim_{\rm PB,gbg,\bowtie,pm}$ the largest p-multistep $\bowtie$-group-by-group probabilistic bisimulation. ■

Definition A.18: Let $(S, A, \longrightarrow)$ be an NPLTS. An equivalence relation $\mathcal{B}$ over S is a *p-multistep $\sqcup\sqcap$-group-by-group probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \stackrel{\alpha}{\Longrightarrow}$ implies $s_2 \stackrel{\alpha}{\Longrightarrow}$ with:

$$\bigsqcup_{\mathcal{Z}_1 \in Res(s_1) \text{ s.t. } z_{s_1} \stackrel{\alpha}{\Longrightarrow}} prob_{\rm m}(z_{s_1}, \alpha, corr_{\mathcal{Z}_1}^{-1}(\bigcup \mathcal{G})) \ = \bigsqcup_{\mathcal{Z}_2 \in Res(s_2) \text{ s.t. } z_{s_2} \stackrel{\alpha}{\Longrightarrow}} prob_{\rm m}(z_{s_2}, \alpha, corr_{\mathcal{Z}_2}^{-1}(\bigcup \mathcal{G}))$$

$$\bigsqcap_{\mathcal{Z}_1 \in Res(s_1) \text{ s.t. } z_{s_1} \stackrel{\alpha}{\Longrightarrow}} prob_{\rm m}(z_{s_1}, \alpha, corr_{\mathcal{Z}_1}^{-1}(\bigcup \mathcal{G})) \ = \bigsqcap_{\mathcal{Z}_2 \in Res(s_2) \text{ s.t. } z_{s_2} \stackrel{\alpha}{\Longrightarrow}} prob_{\rm m}(z_{s_2}, \alpha, corr_{\mathcal{Z}_2}^{-1}(\bigcup \mathcal{G}))$$

We denote by $\sim_{\rm PB,gbg,\sqcup\sqcap,pm}$ the largest p-multistep $\sqcup\sqcap$-group-by-group probabilistic bisimulation. ■

Definition A.19: Let $(S, A, \longrightarrow)$ be an NPLTS and symbol $\# \in \{\bigsqcup, \bigsqcap\}$. An equivalence relation $\mathcal{B}$ over S is a *p-multistep $\#$-group-by-group probabilistic bisimulation* iff,

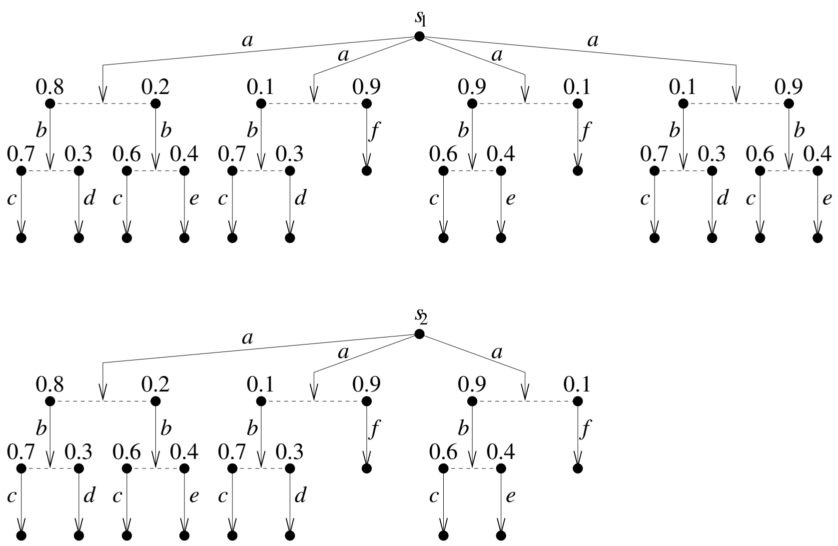

Fig. 6. Two models related by $\sim_{\rm PB,gbg,=}$ that are distinguished by $\sim_{\rm PB,gbg,=,pm}$

whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \stackrel{\alpha}{\Longrightarrow}$ implies $s_2 \stackrel{\alpha}{\Longrightarrow}$ with:

$$\underset{\mathcal{Z}_1 \in Res(s_1) \text{ s.t. } z_{s_1} \stackrel{\alpha}{\Longrightarrow}}{\#} prob_{\rm m}(z_{s_1}, \alpha, corr^{-1}_{\mathcal{Z}_1}(\textstyle\bigcup \mathcal{G})) \; = \; \underset{\mathcal{Z}_2 \in Res(s_2) \text{ s.t. } z_{s_2} \stackrel{\alpha}{\Longrightarrow}}{\#} prob_{\rm m}(z_{s_2}, \alpha, corr^{-1}_{\mathcal{Z}_2}(\textstyle\bigcup \mathcal{G}))$$

We denote by $\sim_{\rm PB,gbg,\#,pm}$ the largest p-multistep $\#$-group-by-group probabilistic bisimulation. ■

The six $\sim_{\rm PB,pm}$-inspired group-by-group probabilistic bisimilarities can be alternatively defined without making explicit use of the notion of resolution. Given $s \in S$, $\alpha \in A^*$, and $S' \subseteq S$, we inductively define the *set* of multistep probabilities of reaching a state in S' from s via α as follows:

$$probset_{\rm m}(s, \alpha, S') \; = \begin{cases} \bigcup\limits_{s \stackrel{a}{\longrightarrow} \mathcal{D}} \{ \sum\limits_{s' \in S} \mathcal{D}(s') \cdot p_{s'} \mid p_{s'} \in probset_{\rm m}(s', \alpha', S') \} & \text{if } \alpha = a\,\alpha' \text{ and } s \stackrel{a}{\longrightarrow} \\ \{1\} & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ \{0\} & \text{if } \alpha = a\,\alpha' \text{ and } s \stackrel{a}{\not\longrightarrow} \\ & \text{or } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$$

Since $probset_{\rm m}(s, \alpha, S') = \{prob_{\rm m}(z_s, \alpha, corr^{-1}_{\mathcal{Z}}(S')) \mid \mathcal{Z} \in Res(s)\}$, it is easy to see that in Defs. A.17 to A.19 we could have used $probset_{\rm m}(s_i, \alpha, \bigcup \mathcal{G})$ in place of $prob_{\rm m}(z_{s_i}, \alpha, corr^{-1}_{\mathcal{Z}_i}(\bigcup \mathcal{G}))$ for $i = 1, 2$. This is not possible in Def. A.15 because the use of $probset_{\rm m}$ causes the connection between each computation and the resolution to which it belongs to be broken.

Each of the six $\sim_{\rm PB,pm}$-inspired group-by-group probabilistic bisimilarities is contained in the corresponding original one-step equivalence. The ct-variants of the six $\sim_{\rm PB,pm}$-inspired group-by-group probabilistic bisimilarities satisfy an analogous inclusion property with respect to the original one-step ct-equivalences.

Theorem A.20: Let $(S, A, \longrightarrow)$ be an NPLTS, $s_1, s_2 \in S$, and $\circ \in \{=, \leq, \geq, \sqcup\sqcap, \sqcup, \sqcap\}$. Then:

$$s_1 \sim_{\rm PB,gbg,\circ,pm} s_2 \implies s_1 \sim_{\rm PB,gbg,\circ} s_2$$

Proof: Let $\circ = \bowtie \in \{=, \leq, \geq\}$ and suppose that $s_1 \sim_{\rm PB,gbg,\bowtie,pm} s_2$. This means that there exists a p-multistep $\bowtie$-group-by-group probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. As a consequence, it holds in particular that for all $(s'_1, s'_2) \in \mathcal{B}$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$, whenever $z_{s'_1} \stackrel{a}{\Longrightarrow}$ in a resolution $\mathcal{Z}_1 \in Res(s'_1)$, then $z_{s'_2} \stackrel{a}{\Longrightarrow}$ in a resolution $\mathcal{Z}_2 \in Res(s'_2)$ with:

$$prob_{\rm m}(z_{s'_1}, a, corr^{-1}_{\mathcal{Z}_1}(\textstyle\bigcup \mathcal{G})) \bowtie prob_{\rm m}(z_{s'_2}, a, corr^{-1}_{\mathcal{Z}_2}(\textstyle\bigcup \mathcal{G}))$$

Since $\stackrel{a}{\Longrightarrow}$ coincides with $\stackrel{a}{\longrightarrow}$ and for all $s \in S$ such that $z_s \stackrel{a}{\longrightarrow} \mathcal{D}$ in a resolution $\mathcal{Z} \in Res(s)$ it holds that:

$$prob_{\rm m}(z_s, a, corr^{-1}_{\mathcal{Z}}(\textstyle\bigcup \mathcal{G})) = \sum\limits_{z_{s'} \in corr^{-1}_{\mathcal{Z}}(\bigcup \mathcal{G})} \mathcal{D}(z_{s'}) = \mathcal{D}(corr^{-1}_{\mathcal{Z}}(\textstyle\bigcup \mathcal{G}))$$

we have that $s'_1 \stackrel{a}{\longrightarrow} \mathcal{D}_1$ implies $s'_2 \stackrel{a}{\longrightarrow} \mathcal{D}_2$ with $\mathcal{D}_1(\bigcup \mathcal{G}) \bowtie \mathcal{D}_2(\bigcup \mathcal{G})$. In other words, $\mathcal{B}$ is also a $\bowtie$-group-by-group probabilistic bisimulation and hence $s_1 \sim_{\rm PB,gbg,\bowtie} s_2$.

Suppose now that $s_1 \sim_{\rm PB,gbg,\sqcup\sqcap,pm} s_2$. This means that there exists a p-multistep $\sqcup\sqcap$-group-by-group probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. As a consequence, it holds in particular that for all $(s'_1, s'_2) \in \mathcal{B}$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$, whenever $s'_1 \stackrel{a}{\Longrightarrow}$, then $s'_2 \stackrel{a}{\Longrightarrow}$ with:

$$\bigsqcup_{\mathcal{Z}_1 \in Res(s'_1) \text{ s.t. } z_{s'_1} \stackrel{a}{\Longrightarrow}} prob_{\rm m}(z_{s'_1}, \alpha, corr^{-1}_{\mathcal{Z}_1}(\textstyle\bigcup \mathcal{G})) = \bigsqcup_{\mathcal{Z}_2 \in Res(s'_2) \text{ s.t. } z_{s'_2} \stackrel{a}{\Longrightarrow}} prob_{\rm m}(z_{s'_2}, \alpha, corr^{-1}_{\mathcal{Z}_2}(\textstyle\bigcup \mathcal{G}))$$

$$\bigsqcap_{\mathcal{Z}_1 \in Res(s'_1) \text{ s.t. } z_{s'_1} \stackrel{a}{\Longrightarrow}} prob_{\rm m}(z_{s'_1}, \alpha, corr^{-1}_{\mathcal{Z}_1}(\textstyle\bigcup \mathcal{G})) = \bigsqcap_{\mathcal{Z}_2 \in Res(s'_2) \text{ s.t. } z_{s'_2} \stackrel{a}{\Longrightarrow}} prob_{\rm m}(z_{s'_2}, \alpha, corr^{-1}_{\mathcal{Z}_2}(\textstyle\bigcup \mathcal{G}))$$

Since $\stackrel{a}{\Longrightarrow}$ coincides with $\stackrel{a}{\longrightarrow}$ and for all $s \in S$ such that $z_s \stackrel{a}{\longrightarrow} \mathcal{D}$ in a resolution $\mathcal{Z} \in Res(s)$ it holds that:

$$prob_{\rm m}(z_s, a, corr_{\mathcal{Z}}^{-1}(\bigcup \mathcal{G})) = \\ = \sum_{z_{s'} \in corr_{\mathcal{Z}}^{-1}(\bigcup \mathcal{G})} \mathcal{D}(z_{s'}) = \mathcal{D}(corr_{\mathcal{Z}}^{-1}(\bigcup \mathcal{G}))$$

we have that $s'_1 \xrightarrow{a}$ implies $s'_2 \xrightarrow{a}$ with:

$$\bigsqcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$

$$\bigsqcap_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$

In other words, $\mathcal{B}$ is also a $\sqcup\sqcap$-group-by-group probabilistic bisimulation and hence $s_1 \sim_{\rm PB,gbg,\sqcup\sqcap} s_2$.

Finally, the proof that $s_1 \sim_{\rm PB,gbg,\#,pm} s_2$ implies $s_1 \sim_{\rm PB,gbg,\#} s_2$ for $\# \in \{\sqcup, \sqcap\}$ is similar to the proof that $s_1 \sim_{\rm PB,gbg,\sqcup\sqcap,pm} s_2$ implies $s_1 \sim_{\rm PB,gbg,\sqcup\sqcap} s_2$.

■

Unlike Thm. A.16, the reverse implication of Thm. A.20 does not hold in general. For example, in Fig. 6 we have that $s_1 \sim_{\rm PB,gbg,=} s_2$ but $s_1 \not\sim_{\rm PB,gbg,=,pm} s_2$ because, for $\alpha = a\,b\,c$ and $\mathcal{G}$ containing all the states with no outgoing transitions, it turns out that the multistep probability of reaching $\mathcal{G}$ via α in the maximal resolution of s_1 starting with the rightmost a-transition – which is $0.1 \cdot 0.7 + 0.9 \cdot 0.6 = 0.61$ – is not matched by any of the multistep probabilities of reaching $\mathcal{G}$ via α in the three maximal resolutions of s_2 starting with the three a-transitions – which are $0.8 \cdot 0.7 + 0.2 \cdot 0.6 = 0.68$, $0.1 \cdot 0.7 = 0.07$, and $0.9 \cdot 0.6 = 0.54$.

We conclude by showing that all the considered $\sim_{\rm PB,pm}$-inspired probabilistic bisimilarities collapse into $\sim_{\rm PB,pm}$ when restricting attention to reactive probabilistic processes. An analogous result holds for their ct-variants.

Theorem A.21: Let $(S, A, \longrightarrow)$ be an NPLTS in which the transitions of each state have different labels. Let $s_1, s_2 \in S$ and $\circ \in \{=, \leq, \geq, \sqcup\sqcap, \sqcup, \sqcap\}$. Then:

$$s_1 \sim_{\rm PB,dis,pm} s_2 \iff s_1 \sim_{\rm PB,gbg,\circ,pm} s_2 \iff s_1 \sim_{\rm PB,pm} s_2$$

Proof: Since every state of this specific NPLTS has at most one transition labeled with a certain action, a p-multistep probabilistic bisimulation is trivially a p-multistep $=$-group-by-group probabilistic bisimulation, a p-multistep $\leq$-group-by-group probabilistic bisimulation, a p-multistep $\geq$-group-by-group probabilistic bisimulation, a p-multistep $\sqcup\sqcap$-group-by-group probabilistic bisimulation, a p-multistep $\sqcup$-group-by-group probabilistic bisimulation, and a p-multistep $\sqcap$-group-by-group probabilistic bisimulation.

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