KYBERNETIKA — VOLUME ?? (????), NUMBER ?, PAGES 1–19

HYPER-DEPENDENCE, HYPER-AGEING PROPERTIES AND ANALOGIES BETWEEN THEM: A SEMIGROUP-BASED APPROACH

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In previous papers, evolution of dependence and ageing, for vectors of non-negative random variables, have been separately considered. Some analogies between the two evolutions emerge however in those studies. In the present paper, we propose a unified approach, based on semigroup arguments, explaining the origin of such analogies and relations among properties of stochastic dependence and ageing.

Keywords: copulas, semi-copulas, evolution of ageing and dependence, action of a semi-group.

Classification: 62N05, 60E15, 37L05

1. INTRODUCTION

Dependence among random variables and its evolution, conditionally on a given sequence of observations on the variables, are topics of interest in many fields.

In the case when one considers non-negative random variables, which may have the meaning of lifetimes, such an evolution can be represented by means of the family of copulas of the residual lifetimes, given the observation of survival data. Namely, if $X_1, \ldots, X_n$ are the afore-mentioned lifetimes, one can be interested in studying the conditional distribution $\overline{F}_t$ of $(X_1 - t, \ldots, X_n - t) \mid X_1 > t, \ldots, X_n > t$, its conditional survival copula $\hat{C}_t$ and the evolution of its dependence properties at increase of the survival time t. We point out straight away that the notation $(X_1 - t, \ldots, X_n - t) \mid X_1 > t, \ldots, X_n > t$ just denotes the conditional distribution function of $(X_1 - t, \ldots, X_n - t)$ given $\{X_1 > t, \ldots, X_n > t\}$ and not a proper random vector.

The concept of *hyper-dependence* was introduced at first in [5], in the context of conditioning on $X_1, \ldots, X_n$ falling below a threshold t, by focusing on the case determined by $n = 2$. In this paper as well, we will consider the bivariate case. A property of hyper-dependence is a special property of dependence. We remark that stochastic dependence properties can be represented as subsets of the class of all the copulas of a fixed dimension. A property of hyper-dependence can be seen as a suitable subset of the class of copulas as well; its special feature consists in the fact that such a set is defined in terms of dependence properties of the whole family $\{\hat{C}_t\}_{t \geq 0}$.

Similarly, in this paper, a concept of *hyper-ageing* is formalized. In [2, 3], some bivariate ageing properties have been introduced, that can be represented by a suitable semi-copula B. In parallel with dependence, studying evolution of ageing is also of interest. Such an interest has provided the motivation for the definition of a family of semi-copulas $\{B_t\}_{t\geq 0}$ in order to specifically describe the evolution of ageing (see [12]).

The two topics of evolution of dependence and of ageing have been so far separately considered. However, from previous papers, some analogies between the two evolutions emerge. A parallelism between the treatment of the evolution of dependence and of the evolution of ageing has been sketched out in [12].

In this paper we propose a unified approach for their analysis, that may explain the origin of such analogies and relations among properties of stochastic dependence and ageing. In order to analyze hyper-properties from a general point of view, we introduce an abstract setting, based on the language of semigroups and of their actions on a set. This approach permits a clearer understanding of the concept of *hyper-property* and of its theoretical and applied interest. Such an abstract approach is also used to obtain some new results about evolution of ageing. It is shown, furthermore, how the analogies between evolution of dependence and of ageing can be explained in terms of different applications of the same general results.

The concept of hyper-property can be also applied to the study of the tail behaviour of dependence (the so-called *tail dependence*) or of ageing. In fact a hyper-property is in particular a property that is automatically preserved by some transformation (or under some conditioning, in the specific case of tail dependence) and therefore it is also asymptotically satisfied.

The paper is organized as follows. In Section 2, we recall basic notation and facts about survival models and families of copulas or semi-copulas we use for describing dependence and ageing, namely, the family of survival copulas $\{\hat{C}_t\}_{t\geq 0}$ and the family of ageing functions $\{B_t\}_{t\geq 0}$. Section 3 is devoted to general results on semigroup actions and their orbits. In this algebraic setting, a hyper-property is represented by a class of copulas, that is closed under the action of a given semigroup. In Section 4, we apply the results of Section 3 to dependence and ageing. We study in detail some relevant dependence and ageing properties and the corresponding hyper-dependence and hyper-ageing properties. We also provide some examples of the evolution of such properties on the families $\{\hat{C}_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$. Furthermore, we use the common algebraic structure of the two families to explain the similarities between properties of dependence and of ageing, that already emerged in [5, 12]. Section 5 contains conclusions.

2. SURVIVAL MODELS AND RELATED FAMILIES OF SEMI-COPULAS

This section is devoted to recalling basic notions and results about survival models and related dependence and ageing structures.

Let X, Y be non-negative random variables, that we can interpret as the lifetimes of two units, and let $\overline{F}(x,y) = P(X > x, Y > y)$ be their joint survival function, with univariate margins $\overline{G}_X$, $\overline{G}_Y$. We will assume from now on $\overline{F}$ to be continuous and strictly decreasing on $\mathbb{R}_+$ in each variable.

The dependence structure of $\overline{F}$ is described by suitable analytical properties of the survival copula (see e.g. [14, 15]) defined by $\hat{C}(u,v) = \overline{F}\left\{\overline{G}_X^{-1}(u), \overline{G}_Y^{-1}(y)\right\}$.

An item of general interest is the conditional model $\overline{F}_t$:

$$\overline{F}_t(x,y) = P(X > t+x, Y > t+y | X > t, Y > t) = \frac{\overline{F}(x+t, y+t)}{\overline{F}(t,t)},$$

with margins

$$\overline{G}_t^X(x) = P(X > t+x | X > t, Y > t), \quad \overline{G}_t^Y(y) = P(Y > t+y | X > t, Y > t).$$

Concerning the family $\{\overline{F}_t\}_{t\geq 0}$, it is interesting studying the evolution in time of the family of copulas $\hat{C}_t(u,v) = \overline{F}_t\left\{(\overline{G}_t^X)^{-1}(u), (\overline{G}_t^Y)^{-1}(v)\right\}$, for which we have

$$\hat{C}_t(u,v) = \frac{\hat{C}\left[\overline{G}_X((\overline{G}_t^X)^{-1}(u)+t), \overline{G}_Y((\overline{G}_t^Y)^{-1}(v)+t)\right]}{\hat{C}(\overline{G}_X(t), \overline{G}_Y(t))}. \tag{1}$$

Let $\mathcal{C}$ denote the set of all bivariate copulas, such that $C(u,v) \neq 0$ for any $(u,v) \neq (0,0)$. For any fixed $C \in \mathcal{C}$, Eq. (1) defines a transformation mapping the survival copula of the model $\overline{F}$, $\hat{C}$, into the survival copula of the conditional model $\overline{F}_t$, $\hat{C}_t$. We denote this transformation by

$$\hat{C}_t = \Phi_{dep}(\hat{C}, t). \tag{2}$$

Remark 2.1. Actually, the transformation defined in (2) also depends on $\overline{G}_X, \overline{G}_Y$. However, once we fixed the model $\overline{F}$, the functions $\overline{G}_X, \overline{G}_Y$ are fixed as well; therefore we can drop them from the notation.
Furthermore, when X, Y are exchangeable, and therefore $\overline{G}_X = \overline{G}_Y = \overline{G}$, we will show in the following (see Example 3.3) that Φ_{dep} may be made independent of $\overline{G}$ by means of a suitable change of parameter.

Eq. (2) defines the family $\{\hat{C}_t\}_{t\geq 0}$. Since the family is obtained by applying $\Phi_{dep}(\cdot, t)$, for $t \in [0, +\infty)$, to $\hat{C}$, we say that $\hat{C}$ is the *generator* of the family. We can also interpret $\hat{C}$ as the *starting element* of the family, since it can be reobtained as $\Phi_{dep}(\hat{C}, 0)$.

A similar argument can be applied to the study of bivariate ageing of the survival model defined by $\overline{F}$ (see [2, 3, 12]). In order to exploit the tools defined therein, we will consider from now on X, Y to be exchangeable. We point out that, exchangeability of X, Y is preserved under conditioning on the observation $\{X > t, Y > t\}$ for any $t \geq 0$. In other words, since the conditioning event is symmetric in the two variables, the distribution of the *residual lifetimes* at time t, $(X-t, Y-t)|X > t, Y > t$, is exchangeable as well (for a detailed proof, see e.g. [11, Prop. 2.2.1]). Therefore, for all $t \geq 0$, $\overline{F}_t$ admits a unique margin $\overline{G}_t(x) = \dfrac{\overline{F}\ (x+t, t)}{\overline{F}\ (t,t)}$.

In order to describe some properties of bivariate ageing, the function

$$B(u,v) = \exp\left\{-\overline{G}^{-1}\left(\overline{F}(-\log u, -\log v)\right)\right\} \tag{3}$$

has been introduced (see [1, 2]). The function B is called *ageing function*. Studying time evolution of ageing properties corresponds to studying the evolution in time of

$$B_t(u,v) = \exp\left[-\overline{G}_t^{-1}\{\overline{F}_t\left(-\log u, -\log v\right)\}\right], \tag{4}$$

obtained by replacing $\overline{F}$ and $\overline{G}$ in Eq. (3) with the survival functions $\overline{F}_t$ and $\overline{G}_t$.

Along the same line of what has been said about the family $\{\hat{C}_t\}_{t\geq 0}$, we can write the relation

$$B_t = \Phi_{ag}(B,t) \tag{5}$$

between B_t and the generator of the family $\{B_t\}_{t\geq 0}$, $B_0 = B$. The explicit expression for Φ_{ag} is provided by the following corollary of [2, Lemma 12].

Corollary 2.2. Let $b_z : [0,1] \to [0,z]$, $b_z(u) = B(u,z)$, be the section of B at level z. We have

$$B_t(u,v) = e^t b_{e^{-t}}^{-1}\left(B\left(ue^{-t}, ve^{-t}\right)\right). \tag{6}$$

We point out that the ageing function B (and consequently the B_t's) is not necessarily a copula, but it is a *semi-copula* (see e.g. [9]), in agreement with the following definition (see e.g. [8, 10]):

Definition 2.3. A semi-copula S is a function $S : [0,1]^2 \to [0,1]$ increasing in any variable and such that, for any $u \in [0,1]$, $S(u,1) = S(1,u) = u$.

As a consequence, we also have $S(u,0) = S(0,u) = 0$ for any $u \in [0,1]$.

We denote with $\mathcal{S}$ the set of all the bivariate semi-copulas.

Taking the cue from [5, 12], we want to consider and analyze here in a more general framework the properties in Definition 2.4. The following definition is given for symmetric semi-copulas. In fact we are considering X, Y exchangeable and this implies $\hat{C}_t, B_t$ being symmetric for any $t \geq 0$.

Definition 2.4. Let $S \in \mathcal{S}$ be symmetric in the two variables.

1. S is PQD (Positive Quadrant Dependent) if, for any $u, v \in [0,1]$,
$$S(u,v) \geq uv;$$
2. S is LTD (Left Tail Decreasing) if, for any $v \in [0,1]$, $\dfrac{S(u,v)}{u}$ is decreasing in u on $(0,1]$;
3. S is TP_2 (Totally Positive of order 2) if, for any $u', u'', v', v'' \in [0,1]$, $u' \leq u''$, $v' \leq v''$,
$$S(u'',v'')S(u',v') \geq S(u',v'')S(u'',v');$$
4. S is SI (Stochastically Increasing) if, for any $v \in [0,1]$, $S(u,v)$ is concave in u on $[0,1]$;

5. S is SM (Supermigrative) if, for any $0 < s < 1$, $0 \leq v \leq u \leq 1$,

$$S(us, v) \geq S(u, sv). \tag{7}$$

Notice that properties 1.-4., when restricted to the set $\mathcal{C} \subset \mathcal{S}$, have the meaning of dependence properties (see e.g. [15]). Property 5. is not a dependence property (see e.g. [7]), but it emerges specifically in the study of the ageing function B; in this context, it has been shown (see [3]) to be equivalent to a bivariate notion of *Increasing Failure Rate*.

3. HYPER-PROPERTIES AND SEMIGROUPS

In this section, we study, in an abstract frame, properties of semi-copulas that we call *hyper-properties.*

Hyper-properties have been introduced at first in [5], in a context exclusively concerning dependence properties of copulas. We aim here to set and study hyper-properties in a more general and theoretical framework. In the next sections, we will analyze the hyper-properties corresponding to the properties in Definition 2.4, so that such hyper-properties will be not necessarily related to dependence and to families of copulas.

In order to formalize the concept of hyper-property, it will be convenient to recall some basic notation about semigroups, actions and orbits (see e.g. [13]).

In the following, $\mathcal{U}$ will denote an arbitrary non-empty set and $\oplus$ a binary operation on $\mathcal{U}$. $(\mathcal{U}, \oplus)$ is a *semigroup* if $\mathcal{U}$ is closed with respect to $\oplus$ and $\oplus$ is associative. If, furthermore, $\mathcal{U}$ contains a neutral element $1_\oplus$ for the operation $\oplus$, $(\mathcal{U}, \oplus, 1_\oplus)$ is said to be a *monoid* or a *unitary semigroup.* Since, along the whole paper, we will consider unitary semigroups, from now on we will refer to them simply as *semigroups.*

Let $\mathcal{T}$ be an arbitrary non-empty set.

Definition 3.1. An *action* of $(\mathcal{U}, \oplus, 1_\oplus)$ on $\mathcal{T}$ is a transformation

$$\Phi : \mathcal{T} \times \mathcal{U} \to \mathcal{T}$$

such that

(i) for any $\zeta \in \mathcal{T}$, $\Phi(\zeta, 1_\oplus) = \zeta$;

(ii) for any $t,\ s \in \mathcal{U}$, $\Phi(\Phi(\zeta, t), s) = \Phi(\zeta, t \oplus s)$.

The set

$$\mathcal{O}_\Phi(\zeta) = \Phi(\zeta, \mathcal{U}) = \{\zeta' \in \mathcal{T} :\ \exists s \in \mathcal{U} :\ \Phi(\zeta, s) = \zeta'\}$$

is the *orbit* of ζ under the action Φ.

In what follows, we will consider the semigroup $(\mathcal{U}, \oplus, 1_\oplus)$ coinciding with $(\mathbb{R}_+, +, 0)$ and $\mathcal{T}$ coinciding with $\mathcal{S}$ or with $\mathcal{C} \subset \mathcal{S}$.

Sometimes however, it may be convenient considering a different semigroup acting on $\mathcal{S}$ (see e.g. Example 3.3).

It can be proven that a monotonic transformation of the semigroup (in a sense that we will specify in short), leaves unchanged the orbits of $\mathcal{S}$ under Φ, as the following proposition states.

Let Φ be an action of $(\mathbb{R}_+, +, 0)$ on $\mathcal{S}$ and define

$$\Phi^{(\psi)} : \mathcal{S} \times \psi(\mathbb{R}_+) \to \mathcal{S},$$

$$\Phi^{(\psi)}(S, z) = \Phi(S, \psi^{-1}(z)).$$

Proposition 3.2. For any strictly monotonic ψ, $\Phi^{(\psi)}$ is an action of the semigroup $(\psi(\mathbb{R}_+), \oplus, \psi(0))$ on $\mathcal{S}$.

Furthermore, for any fixed $S \in \mathcal{S}$,

$$\mathcal{O}_{\Phi^{(\psi)}}(S) = \mathcal{O}_{\Phi}(S).$$

Proof. By letting

$$w \oplus z = \psi(\psi^{-1}(w) + \psi^{-1}(z)), \tag{8}$$

$\Phi^{(\psi)}$ satisfies conditions (i) and (ii) in Definition 3.1.

Now, for any fixed $S \in \mathcal{S}$,

$$\mathcal{O}_{\Phi^{(\psi)}}(S) = \{S' \in \mathcal{S} : \exists z \in \psi(\mathbb{R}_+) : \Phi^{(\psi)}(S, z) = S'\}.$$

Since ψ is strictly monotonic and it is acting on the totally ordered set $\mathbb{R}_+$,

$$\mathcal{O}_{\Phi^{(\psi)}}(S) = \{S' \in \mathcal{S} : \exists t \in \mathbb{R}_+ : \Phi^{(\psi)}(S, \psi(t)) = S'\}.$$

Now, by definition, $\Phi^{(\psi)}(S, \psi(t)) = \Phi(S, t)$. Thus, for any $S \in \mathcal{S}$,

$$\mathcal{O}_{\Phi^{(\psi)}}(S) = \mathcal{O}_{\Phi}(S).$$

$\square$

Example 3.3. In [5] the family $\{C_z\}_{z \in (0,1]}$ is considered. The circumstance that the parameter z spans the interval $(0, 1]$ presents two different advantages. As first, it allows us to represent the conditioning on events of the kind $\{U < z, V < z\}$, where U, V, are random variables with support $[0, 1]$. Secondly, it makes the family of copulas independent of the margins of the considered variables. Since $z \in (0, 1]$, the semigroup acting on $\mathcal{C}$ is no more $(\mathbb{R}_+, +, 0)$, but $((0, 1], \oplus, 1_{\oplus})$, obtained by applying to $(\mathbb{R}_+, +, 0)$ the strictly monotonic function $\psi = \overline{G}$, $\overline{G} : \mathbb{R}_+ \to (0, 1]$. Hence, $1_{\oplus} = 1$ and $\oplus$ is defined by Eq. (8). The family $\{C_z\}_{z \in (0,1]}$ can hence be obtained from $\{\hat{C}_t\}_{t \geq 0}$ by defining $C_z = \hat{C}_z^{(\overline{G})} = \hat{C}_{\overline{G}^{-1}(z)}$.

For our purposes, it is more convenient, to state our results in terms of classes of semi-copulas characterized by the properties in Definition 2.4. Namely, let $\mathcal{P}_{PQD}$, $\mathcal{P}_{LTD}$, $\mathcal{P}_{TP_2}$, $\mathcal{P}_{SI}$, $\mathcal{P}_{SM}$ respectively denote the classes of PQD, LTD, TP_2, SI, SM semi-copulas.

In general, let $\mathcal{P}$ be the class of all the semi-copulas satisfying a property **P**. By using the algebraic notation introduced above, we are now in a position to formally define hyper-properties as follows:

Definition 3.4. $S \in \mathcal{S}$ is *hyper*$_\Phi$-**P** if $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}$.

We denote by $hyper_\Phi$-$\mathcal{P}$ the class of all the $hyper_\Phi$-**P** semi-copulas.

Remark 3.5. As an obvious consequence of Definition 3.4, the class $hyper_\Phi$-$\mathcal{P}$ is contained in $\mathcal{P}$.

When $\Phi = \Phi_{dep}$ or $\Phi = \Phi_{ag}$, we can define respectively *hyper-dependence* and *hyper-ageing* properties.

Definition 3.6. $C \in \mathcal{C}$ is $hyper_{dep}$-**P** if $\mathcal{O}_{\Phi_{dep}}(C) \subseteq \mathcal{P}$.

$Hyper_{dep}$-$\mathcal{P}$ is the class of all $hyper_{dep}$-**P** copulas.
Analogously

Definition 3.7. $S \in \mathcal{S}$ is $hyper_{ag}$-**P** if $\mathcal{O}_{\Phi_{ag}}(S) \subseteq \mathcal{P}$.

$Hyper_{ag}$-$\mathcal{P}$ is the class of all $hyper_{ag}$-**P** semi-copulas.

Remark 3.8. Both Φ_{dep} and Φ_{ag}, defined in Eq.s (2),(5), are actions of the semigroup $(\mathbb{R}_+, +, 0)$ on $\mathcal{C}$ and $\mathcal{S}$ respectively. For the proof of these statements, we refer to [12].

Remark 3.9. As it emerges from the above Definitions 3.4, 3.6, 3.7, it is important to mention in the notation the action we are referring to each time in defining hyper-properties. We want to clarify the role the action Φ has in defining hyper-properties. In fact, for a given $\mathcal{P}$, $hyper_\Phi$-$\mathcal{P}$ obviously depends on the orbit of the action Φ and, therefore, on the particular choice of the transformation.

The following example shows that, since in general $\Phi_{dep}(S) \neq \Phi_{ag}(S)$, for a given class $\mathcal{P}$, $hyper_{dep}$-$\mathcal{P}$, $hyper_{ag}$-$\mathcal{P} \subseteq \mathcal{P}$ are two different classes of $\mathcal{S}$.

Example 3.10. Let $S(u,v) = uv[1 + (1-u)(1-v)]$.
$S \in \mathcal{P}_{TP_2}$ and therefore, as proven in [12], $\hat{C}_t = \Phi_{dep}(S,t) \in \mathcal{P}_{TP_2}$ for any $t \geq 0$.
On the contrary, for some $t \geq 0$, $B_t = \Phi_{ag}(S,t) \notin \mathcal{P}_{TP_2}$. In fact, for $t = 1$ and $u' = \frac{1}{5}$, $u'' = v' = \frac{1}{2}$, $v'' = \frac{3}{5}$,

$$B_t(u'', v'')B_t(u', v') - B_t(u', v'')B_t(u'', v') = -0.2099 < 0.$$

Hence $hyper_{dep}$-$\mathcal{P}_{TP_2} \neq hyper_{ag}$-$\mathcal{P}_{TP_2}$.

Let now the two classes $\mathcal{P}, \mathcal{P}' \subset \mathcal{S}$ be characterized by two properties $\mathbf{P}, \mathbf{P}'$ in Definition 2.4. Some relationships between such classes are well known in the literature; for example,

$$\mathcal{P}_{PQD} \supseteq \mathcal{P}_{LTD} \supseteq \mathcal{P}_{TP_2};\ \mathcal{P}_{TP_2} \nsubseteq \mathcal{P}_{SI},\ \mathcal{P}_{TP_2} \nsupseteq \mathcal{P}_{SI}.$$

Such relations are included in the following case record of comparisons:

- $\mathcal{P} \subset \mathcal{P}'$ (or $\mathcal{P}' \subset \mathcal{P}$);
- $\mathcal{P} \nsubseteq \mathcal{P}'$ and $\mathcal{P}' \nsubseteq \mathcal{P}$; (9)
- $\mathcal{P} = \mathcal{P}'$.

We notice that (9) is an exhaustive survey of the possible relations between two classes $\mathcal{P}, \mathcal{P}'$.

When, besides a pair $\mathcal{P}, \mathcal{P}'$, we also consider the classes $hyper_\Phi$-$\mathcal{P}$, $hyper_\Phi$-$\mathcal{P}'$, obviously a more heterogeneous landscape emerges. In particular, we are interested in the following situations:

i) $\mathcal{P} = hyper_\Phi$-$\mathcal{P}$;

ii) $\mathcal{P} \subset \mathcal{P}'$, $hyper_\Phi$-$\mathcal{P} = hyper_\Phi$-$\mathcal{P}'$;

iii) $\mathcal{P} \nsubseteq \mathcal{P}'$, $\mathcal{P}' \nsubseteq \mathcal{P}$, $hyper_\Phi$-$\mathcal{P}' \subset \mathcal{P}$;

iv) $\mathcal{P} \nsubseteq \mathcal{P}'$, $\mathcal{P}' \nsubseteq \mathcal{P}$, $hyper_\Phi$-$\mathcal{P} = hyper_\Phi$-$\mathcal{P}'$.

The relations appearing in the items i)-iv) are suggested by some examples of dependence and hyper-dependence properties.

Example 3.11. 1. $\mathcal{P}_{TP_2} = hyper_{dep}$-$\mathcal{P}_{TP_2}$ (see [5, 12]);

2. $hyper_{dep}$-$\mathcal{P}_{TP_2} = hyper_{dep}$-$\mathcal{P}_{LTD}$, with $\mathcal{P}_{TP_2} \subsetneq \mathcal{P}_{LTD}$ (see [5, 12]);

3. $hyper_{dep}$-$\mathcal{P}_{SI} \subsetneq \mathcal{P}_{TP_2}$, with $\mathcal{P}_{TP_2}$ and $\mathcal{P}_{SI}$ not comparable (see Proposition 4.8 below).

The cases listed above can be alternatively formulated as

i) $S \in \mathcal{P} \Rightarrow \mathcal{O}_\Phi(S) \subseteq \mathcal{P}$;

ii) $\mathcal{P} \subset \mathcal{P}'$ and $\mathcal{O}_\Phi(S) \subseteq \mathcal{P} \Leftrightarrow \mathcal{O}_\Phi(S) \subseteq \mathcal{P}'$;

iii) $\mathcal{P} \nsubseteq \mathcal{P}'$, $\mathcal{P}' \nsubseteq \mathcal{P}$ and $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}' \Rightarrow S \in \mathcal{P}$;

iv) $\mathcal{P} \nsubseteq \mathcal{P}'$, $\mathcal{P}' \nsubseteq \mathcal{P}$ and $\mathcal{O}_\Phi(S) \subseteq \mathcal{P} \Leftrightarrow \mathcal{O}_\Phi(S) \subseteq \mathcal{P}'$.

The four conditions above are alternative, but not exhaustive. The following propositions will show how such conditions can be combined to obtain further relations involving hyper-properties.

As above, let $\mathcal{P}, \mathcal{P}' \subset \mathcal{S}$ and Φ denote the action of a semigroup.

Proposition 3.12. Let $\mathcal{P} \subset \mathcal{P}'$. Then

$$\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}' \tag{10}$$

implies

$$\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}. \tag{11}$$

Proof. By (10), $S \in \mathcal{P}$ if and only if $S \in hyper_\Phi$-$\mathcal{P}'$, i.e. if and only if $\mathcal{O}_\Phi(S) \subset \mathcal{P}'$. We want to prove that in this case $\mathcal{O}_\Phi(S) \subset \mathcal{P}$ also holds.

To this aim, let us consider $\tilde{S} \in \mathcal{S}$ such that $\tilde{S} = \Phi(S, t_0) = S_{t_0}$, for some $t_0 > 0$, and therefore $\tilde{S} \in \mathcal{O}_\Phi(S)$. By definition of action of a semigroup, $\mathcal{O}_\Phi(S_{t_0}) \subseteq \mathcal{O}_\Phi(S)$. In fact,

$$\mathcal{O}_\Phi(S_{t_0}) = \{S' \in \mathcal{S} : \exists s \in \mathbb{R}_+ : \Phi(S_{t_0}, s) = S'\}.$$

But $\Phi(S_{t_0}, s) = \Phi(S, t_0 + s)$ and, therefore, $\mathcal{O}_\Phi(S_{t_0})$ coincides with the sub-orbit of S,

$$\Phi(S, t_0 + \mathbb{R}_+) = \{S' \in \mathcal{S} : \exists t \in t_0 + \mathbb{R}_+ : \Phi(S, t) = S'\},$$

where

$$t_0 + \mathbb{R}_+ = \{t : t = t_0 + s, s \in \mathbb{R}_+\} = \mathbb{R}_+ \cap \{t \geq t_0\}.$$

Thus, $\mathcal{O}_\Phi(S_{t_0}) \subseteq \mathcal{O}_\Phi(S) \subseteq \mathcal{P}'$ for any $t_0 \geq 0$.

This means that $S \in hyper_\Phi\text{-}\mathcal{P}'$ implies $S_{t_0} \in hyper_\Phi\text{-}\mathcal{P}'$ for any $t_0 \geq 0$. By (10), we get $S_{t_0} \in \mathcal{P}$ for any $t_0 \geq 0$, that, in view of the arbitrariness of t_0, is equivalent to $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}$, and hence (11) follows. □

The previous proposition establishes an equivalence between two identities of classes. More precisely, if conditions for property $\mathbf{P}$ coincide with conditions for preservation of property $\mathbf{P}'$, then the property $\mathbf{P}$ is automatically preserved by the action Φ.

In the next proposition, we analyze a different relation that may exist between two classes $\mathcal{P}, \mathcal{P}'$, again such that $\mathcal{P} \subset \mathcal{P}'$; this time we require that conditions for preservation of the weakest property $\mathbf{P}'$ just imply the strongest one $\mathbf{P}$. We obtain that such conditions for preservation of $\mathbf{P}'$ coincide with conditions for preservation of $\mathbf{P}$. However, since the hypothesis (12) of Proposition 3.13 is weaker than the one of Proposition 3.12, in the thesis of Proposition 3.13, we miss the equivalence between $\mathcal{P}$ and $hyper_\Phi\text{-}\mathcal{P}$.

Proposition 3.13. Let $\mathcal{P} \subset \mathcal{P}'$. Then

$$hyper_\Phi\text{-}\mathcal{P}' \subset \mathcal{P} \tag{12}$$

if and only if

$$hyper_\Phi\text{-}\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}'. \tag{13}$$

P r o o f. Starting from (12), we have to prove that both the inclusions

$$hyper_\Phi\text{-}\mathcal{P} \subseteq hyper_\Phi\text{-}\mathcal{P}' \text{ and } hyper_\Phi\text{-}\mathcal{P} \supseteq hyper_\Phi\text{-}\mathcal{P}'$$

hold. Since $\mathcal{P} \subset \mathcal{P}'$, $hyper_\Phi\text{-}\mathcal{P} \subseteq hyper_\Phi\text{-}\mathcal{P}'$ obviously follows. To prove the converse inclusion, we use the fact that, for any $t_0 \geq 0$, $\mathcal{O}_\Phi(S_{t_0}) \subseteq \mathcal{O}_\Phi(S)$.

Let us consider $S \in hyper_\Phi\text{-}\mathcal{P}'$, i.e. $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}'$. By definition of semigroup action, $\mathcal{O}_\Phi(S_{t_0}) \subseteq \mathcal{P}'$ as well, for any $t_0 \geq 0$, and therefore, by (12), $S_{t_0} \in \mathcal{P}$ for any $t_0 \geq 0$, i.e. $\mathcal{O}_\Phi(S) \subseteq \mathcal{P}$. Hence $hyper_\Phi\text{-}\mathcal{P}' \subseteq hyper_\Phi\text{-}\mathcal{P}$.

The converse implication, (13)⇒(12), can be easily proven, by taking into account Remark 3.5. □

An analog of Proposition 3.13, when two properties ($\mathbf{P}'$ and $\mathbf{P}'' = hyper_\Phi\text{-}\mathbf{P}$) are not comparable is given by the following proposition. In this case, we also lose the equivalence between $hyper_\Phi\text{-}\mathbf{P}'$ and $hyper_\Phi\text{-}\mathbf{P}$.

Proposition 3.14. Let $\mathcal{P}$, $\mathcal{P}' \subset \mathcal{S}$, $hyper_\Phi\text{-}\mathcal{P} \nsubseteq \mathcal{P}'$, $\mathcal{P}' \nsubseteq hyper_\Phi\text{-}\mathcal{P}$. If

$$\mathcal{P}' \subset \mathcal{P}, \tag{14}$$

then

$$hyper_\Phi\text{-}\mathcal{P}' \subset hyper_\Phi\text{-}\mathcal{P}. \tag{15}$$

Proof. Let $S \in hyper_\Phi$-$\mathcal{P}'$, i.e. $S_{t_0} \in \mathcal{P}'$ for any $t_0 \geq 0$. By (14), it straightly implies that $S_{t_0} \in \mathcal{P}$ for any $t_0 \geq 0$, i.e. $S \in hyper_\Phi$-$\mathcal{P}$. □

Remark 3.15. Conditions stated in Proposition 3.14 implicitly require that $hyper_\Phi$-$\mathcal{P} \subsetneq \mathcal{P}$. In fact, if $hyper_\Phi$-$\mathcal{P} = \mathcal{P}$, in the hypothesis two incompatible conditions would appear.

Some corollaries of Propositions 3.12, 3.13, 3.14 will be obtained below. In order to illustrate their meaning and usefulness, we need to point out the following fact. Since the semigroup acting on $\mathcal{S}$, $(\mathbb{R}_+, +, 0)$, is totally ordered, it is possible to define an orientation on any orbit, corresponding to the natural orientation on $\mathbb{R}_+$. Thus an orbit can be seen as a trajectory in the space of semi-copulas, parametrized by a time.

Starting from any point on it, an orbit can be gone along in both directions: forwards and backwards. More precisely: let us consider the orbit generated by $S \in \mathcal{S}$; any element of the orbit, $D = S_{t_0} = \Phi(S, t_0)$, is identified by an element $t_0 \in \mathbb{R}_+$. Starting from S_{t_0}, it is possible to go along the orbit both forwards, by taking t increasing on $(t_0, +\infty)$, and backwards, by taking t decreasing on $[0, t_0)$. If, furthermore, we interpret $t \in \mathbb{R}_+$ as a time parameter (as we will do in the following), then “forwards” and “backwards” respectively mean “for future times” and “for past times”.

We are now in a position to state the afore-mentioned corollaries about the evolution of properties along an orbit. They are related to the preservation of a property on a sub-orbit, but not necessarily on an entire orbit.

The proofs of such corollaries derive from a basic consequence of the fact that Φ is an action of a semigroup on $\mathcal{S}$. Actually, in proving previous propositions, we have already exploited the total ordering of $\mathbb{R}_+$ and the consequent fact that for any $t_0 \in \mathbb{R}_+$, the orbit $\Phi(S_{t_0}, \mathbb{R}_+)$, generated by the element $S_{t_0} = \Phi(S, t_0) \in \mathcal{O}_\Phi(S)$ for some $S \in \mathcal{S}$, is a sub-orbit of the orbit generated by S. In the following corollaries, we are furthermore considering the occurrence of the situation

$$S_{t_0} \in \mathcal{P}, \text{ but } S_t \notin \mathcal{P} \quad \forall t \in [0, t_0).$$

This means that the property $\mathbf{P}$ is satisfied at time t_0, but not for $t < t_0$. In other words, we can say that the property $\mathbf{P}$ “manifests” or “arises” at a certain time t_0.

If the corresponding class $\mathcal{P}$ is closed under the action Φ, the property $\mathbf{P}$ can arise along the orbit, but, from that time on, it is necessarily preserved.

We also have that, if two properties $\mathbf{P}, \mathbf{P}'$ are preserved under the same conditions, the arising of the weakest one, at a time t_0, and its preservation, from that time on, imply the arising of the strongest property, at the same time t_0. Hence, it may not occur that $S \in hyper_\Phi$-$\mathcal{P}' \setminus \mathcal{P}$. This argument will be applied in Remarks 4.7, 4.10 and Corollaries 4.6, 4.14 below.

Corollary 3.16. Let $\mathcal{P}, \mathcal{P}' \subseteq \mathcal{S}$ such that $\mathcal{P} = hyper_\Phi$-$\mathcal{P} \subset \mathcal{P}'$.

1. Then
$$S_{t_0} \in \mathcal{P} \text{ for some } t_0 \in \mathbb{R}_+ \;\Rightarrow\; \Phi(S, t_0 + \mathbb{R}_+) \subseteq \mathcal{P}.$$

2. If, furthermore,
$$hyper_\Phi\text{-}\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}', \tag{16}$$

then $S_{t_0} \in \mathcal{P}$ for some $t_0 \in \mathbb{R}_+$ if and only if

$$\Phi(S, t_0 + \mathbb{R}_+) \subseteq \mathcal{P}'.$$

Proof.

1. The statement straightly follows by the fact that $\mathcal{O}_\Phi(S_{t_0}) = \Phi(S, t_0 + \mathbb{R}_+)$.
2. By hypothesis and condition (16), $\mathcal{P} = hyper_\Phi\text{-}\mathcal{P}'$ holds and therefore $S_{t_0} \in \mathcal{P}$ is equivalent to $\mathcal{O}_\Phi(S_{t_0}) \subseteq \mathcal{P}'$.

□

The following corollary can be easily proved starting from Proposition 3.14 and finds its application in Corollaries 4.9, 4.17.

Corollary 3.17. Let be $\mathcal{P} \nsubseteq \mathcal{P}'$ and $\mathcal{P}' \nsubseteq \mathcal{P}$. If

$$hyper_\Phi\text{-}\mathcal{P}' \subset \mathcal{P},$$

then

$$\Phi(S, t_0 + \mathbb{R}_+) \subseteq \mathcal{P}' \text{ for some } t_0 \in \mathbb{R}_+ \Rightarrow S_{t_0} \in \mathcal{P}.$$

Since we are considering actions of semigroups (and not of groups), nothing can be said instead about the backward behaviour of an orbit. This is a consequence of the fact that, under actions of semigroups, the orbits are not a partition of the set: this means that a given $\tilde{S} \in \mathcal{S}$ may belong to two different orbits, i.e.

$$\exists\ S', S'' \in \mathcal{S},\ S' \neq S'',\ \text{and } t', t'' \in \mathbb{R}_+$$

(t', t'' not necessarily different), such that

$$\tilde{S} = \Phi(S', t') = \Phi(S'', t''). \tag{17}$$

Thus, given an element $\tilde{S}$, it is not possible to univocally reconstruct backwards the orbit it belongs to. We will provide below some examples of different constructions.

Instead, as concerns the forward behaviour, condition (17) straightly implies, for any $t \geq 0$,

$$\Phi(\tilde{S}, t) = \Phi(S', t' + t) = \Phi(S'', t'' + t),$$

that is,

$$\mathcal{O}_\Phi(S'_{t'}) = \mathcal{O}_\Phi(S''_{t''}) = \mathcal{O}_\Phi(\tilde{S}).$$

Since the orbit of a given element $\tilde{S}$ is univocally and established, once that two orbits have a common element, from that time on they cannot be distinguished.

If we consider, instead, an action Ψ of a group $(\mathcal{G}, \oplus, 1_\oplus)$ on $\mathcal{S}$, the orbits of the elements of $\mathcal{S}$ constitute a partition of $\mathcal{S}$. In view of our comparison with the semigroup $(\mathbb{R}_+, +, 0)$, let us suppose also $\mathcal{G}$ to be a totally ordered set. Under the action Ψ, if two orbits have a common element, they necessarily entirely coincide and, starting from any element of $\mathcal{S}$, we can reconstruct its orbit, both forward and backward. This fact in particular implies that, if $\mathcal{P}$ is closed under Ψ, the situation

$$\Psi(S,t_0) \in \mathcal{P} \quad \Psi(S,t) \notin \mathcal{P} \;\; \forall\, t < t_0$$

cannot occur; but, if t_0 exists such that $\Psi(S,t_0) \in \mathcal{P}$, it must be $\Psi(S,t) \in \mathcal{P}\ \forall\, t \in \mathcal{G}$, that is $\mathcal{O}_\Psi(S) \subset \mathcal{P}$. In this case, an analog of Corollary 3.16 holds.

Corollary 3.18. Let $\mathcal{P}, \mathcal{P}' \subseteq \mathcal{S}$ be such that $S \in \mathcal{P} \Leftrightarrow \mathcal{O}_\Psi(S) \subseteq \mathcal{P}'$. Then $S_{t_0} \in \mathcal{P}$ for some $t_0 \in \mathcal{G}$ if and only if $\mathcal{O}_\Psi(S) \subseteq \mathcal{P}'$.

Proof. For any fixed $t_0 \in \mathcal{G}$, by the hypothesis, $S_{t_0} \in \mathcal{P} \Leftrightarrow \mathcal{O}_\Psi(S_{t_0}) \subseteq \mathcal{P}'$ follows. By definition of action, $\mathcal{O}_\Psi(S_{t_0}) = \Psi(S, t_0 \oplus \mathcal{G})$. Since $\mathcal{G}$ is a group, the coset $t_0 \oplus \mathcal{G}$ coincides with $\mathcal{G}$, therefore

$$\Psi(S, t_0 \oplus \mathcal{G}) = \Psi(S, \mathcal{G}) = \mathcal{O}_\Psi(S).$$

$\square$

4. HYPER-DEPENDENCE AND HYPER-AGEING PROPERTIES OF SEMI-COPULAS

We devote this section to the application of the results of Section 3 to dependence and bivariate ageing.

The theoretical frame developed in the previous section allows us both to explain some observed analogies between dependence and bivariate ageing and to obtain new results about preservation in time of dependence and of ageing properties.

First of all, we point out that relationships among properties of copulas may be seen as relationships among classes of copulas and recall that a property $\mathbf{P}$ is automatically preserved in time when the class $\mathcal{P}$ is closed under the action of Φ, i.e. when $\mathcal{P}$ and $hyper_\Phi$-$\mathcal{P}$ coincide.

In this section, we are interested both in finding classes $\mathcal{P}$'s that are closed under the actions Φ_{dep} or Φ_{ag} and, when $\mathcal{P}$ is not closed, in finding conditions characterizing the classes $hyper_{dep}$-$\mathcal{P}$ and $hyper_{ag}$-$\mathcal{P}$. In this context, another situation of interest is

$$S_t \in \mathcal{P}\ \forall\, t \geq t_0, \quad \text{but } S_t \notin \mathcal{P}\ \forall\, t \in [0, t_0). \tag{18}$$

This means that only a part of the orbit of S is contained in $\mathcal{P}$. Therefore, when $\mathcal{P}$ is closed under Φ, S satisfies a weaker property than $\mathbf{P}$. In general, i.e. when $\mathcal{P}$ is not closed under Φ, (18) defines a further different property, that is obviously weaker than $hyper_\Phi$-$\mathbf{P}$, but that is not comparable to $\mathbf{P}$.

4.1. Dependence and hyper-dependence

We give the definition of the kind of properties, corresponding to the situation described in (18), for the case when we refer to the action Φ_{dep}. Let be $\Lambda \subset \mathbb{R}_+$.

Definition 4.1. We say that $C \in \mathcal{C}$ is $\langle \mathbf{P}; \Lambda \rangle_{dep}$ if $\Phi_{dep}(C,t) \in \mathcal{P}$ for any $t \in \Lambda$.

Remark 4.2. $\hat{C}$ being $\langle \mathbf{P}; [t_0, +\infty)\rangle_{dep}$ means that, for $t < t_0$, $\hat{C}_t$ does not necessarily satisfy $\mathbf{P}$, while the property $\mathbf{P}$ holds for all $\hat{C}_t$, with $t \geq t_0$. This notion can be of interest in the study of tail dependence. In fact we are typically interested in proving that $\hat{C}_t$ satisfies a dependence property $\mathbf{P}$ in the limit for $t \to +\infty$. Thus, proving that $\hat{C}$ is $\langle \mathbf{P}; [t_0, +\infty)\rangle_{dep}$ guarantees that $\mathbf{P}$ is asymptotically satisfied, without having recourse, when existing, to the computation of $\lim\limits_{t\to+\infty} C_t$.

A class closed under the action Φ_{dep}, and for which therefore $\langle \mathbf{P}; \Lambda\rangle_{dep}$ is weaker than $\mathbf{P}$, is $\mathcal{P}_{TP_2}$. It is known in fact (see [5, 12]) that, for any $C \in \mathcal{C}$,

$$C \text{ is } TP_2 \;\Leftrightarrow\; C \text{ is } hyper_{dep}\text{-}TP_2,$$

that reads as

$$\mathcal{P}_{TP_2} \cap \mathcal{C} = hyper_{dep} - \mathcal{P}_{TP_2} \cap \mathcal{C}. \tag{19}$$

Actually statement (19) can be extended to the set of semi-copulas, yielding the identity

$$\mathcal{P}_{TP_2} = hyper_{dep}\text{-}\mathcal{P}_{TP_2}, \tag{20}$$

as stated by the following proposition.

Proposition 4.3. Let be $S \in \mathcal{S}$. $S \in \mathcal{P}_{TP_2}$ if and only if $S \in hyper_{dep}\text{-}\mathcal{P}_{TP_2}$.

P r o o f. By [4, Lemma 3.1], $\mathcal{P}_{TP_2} \cap \mathcal{C} = \mathcal{P}_{TP_2} \cap \mathcal{S}$. Therefore, if $S \in \mathcal{P}_{TP_2} \cap \mathcal{S}$, then $S \in \mathcal{C}$ and, by (19), $S \in hyper_{dep}\text{-}\mathcal{P}_{TP_2}$. Conversely, if $S \in \mathcal{S} \setminus \mathcal{P}_{TP_2}$, it cannot be $S \in hyper_{dep}\text{-}\mathcal{P}_{TP_2}$, otherwise it would be also $S \in \mathcal{P}_{TP_2}$, against the hypothesis. Therefore the equivalence (20) holds. □

Since $\mathcal{P}_{TP_2}$ is closed under Φ_{dep}, by Corollary 3.16, it follows that

Corollary 4.4. If $t_0 \geq 0$ exists, such that $\hat{C}_{t_0}$ is TP_2, then $\hat{C}$ is $\langle TP_2; [t_0, +\infty)\rangle_{dep}$.

By interpreting the parameter $t \in \mathbb{R}_+$ as a time, we can say that the action Φ_{dep} preserves the TP_2 property "in the future", but not "in the past". In other words, an orbit can enter the class $\mathcal{P}_{TP_2}$, but it cannot go out. The following example shows that, if $\hat{C}$ is $\langle TP_2; [t_0, +\infty)\rangle_{dep}$, it has not necessarily to be $\langle TP_2; [0, t_0)\rangle_{dep}$.

Example 4.5. Let $z_0 \in (0, 1)$ and

$$\hat{C}(u,v) = \begin{cases} uv, & u, v \in [0, z_0], \\ z_0 + (1 - z_0) W\left(\frac{u-z_0}{1-z_0}, \frac{v-z_0}{1-z_0}\right), & u, v \in (z_0, 1], \\ \min(u, v), & \text{otherwise}, \end{cases}$$

where $W(u, v) = \max(u + v - 1, 0)$ is the lower Frechet bound.
We recall that, for any $t \geq 0$, $\hat{C}_t$ only depends on the behaviour of $\hat{C}$ on the square $[0, z]^2$, $z = \overline{G}(t)$. Thus $\hat{C}_{t_0}$ is TP_2, for $t_0 = \overline{G}^{-1}(z_0)$, and $\hat{C}_t$ continues to be TP_2 for $t > t_0$. For $t < t_0$ instead, $\hat{C}_t$ is not even PQD. In fact, if we consider $z_0 = \frac{1}{2}$ and $z_1 = \frac{3}{4}$: $C(z_1, z_1) = \frac{1}{2} < \frac{9}{16} = z_1^2$. Therefore, for $z_0 = \frac{1}{2}$, $\hat{C}_t$ is not PQD at least for $t \leq \overline{G}^{-1}(\frac{3}{4})$.

Heuristically speaking, Corollary 3.16 says that, if we observe at a certain time a strong dependence between the residual lifetimes, the structure of the dependence will not change. This preservation is not warranted instead by weaker dependence notions: the PQD property, for example, is not necessarily preserved in time (for a more detailed discussion about this topic, see [5]).

Since, for $t_0 > 0$, $\langle TP_2; [t_0, +\infty)\rangle_{dep}$ is weaker than TP_2, we also expect that a weaker property than $hyper_{dep}$-LTD is sufficient to guarantee $\langle TP_2; [t_0, +\infty)\rangle_{dep}$. In fact, by Corollary 3.16, it follows

Corollary 4.6. If $t_0 \geq 0$ exists, such that $\hat{C}$ is $\langle LTD; [t_0, +\infty)\rangle_{dep}$, then $\hat{C}$ is $\langle TP_2; [t_0, +\infty)\rangle_{dep}$.

Remark 4.7. By Proposition 3.13, an orbit cannot enter $\mathcal{P}_{TP_2}$ passing directly from $\mathcal{P}_{LTD}$ to $\mathcal{P}_{TP_2}$. In fact, let us suppose $\hat{C}_t \notin \mathcal{P}_{TP_2}$ for any $t \in [0, t_0)$, but $\hat{C}_{t_0} \in \mathcal{P}_{TP_2}$ (and therefore $\hat{C}_t \in \mathcal{P}_{TP_2}$ for any $t \geq t_0$) and $\hat{C}$ being $\langle \mathrm{LTD}; [t, t_0]\rangle_{dep}$ for some $t \in [0, t_0)$. Since $\mathcal{P}_{TP_2} \subset \mathcal{P}_{LTD}$, $\hat{C}$ would actually be $\langle \mathrm{LTD}; [t, +\infty)\rangle_{dep}$ and therefore, by Corollary 4.6, $\langle \mathrm{TP}_2; [t, +\infty)\rangle_{dep}$, against the hypothesis.
However, we cannot exclude instead that $\hat{C}$ is $\langle \mathrm{LTD}; \Lambda\rangle_{dep}$ for some set Λ such that $\overline{\Lambda} \cap [t_0, +\infty) = \emptyset$, where $\overline{\Lambda}$ denotes the closure of Λ; i.e., for any $\varepsilon \in (0, t_0)$, it is possible that $\hat{C}_t$ is LTD for some $t \in [0, t_0 - \varepsilon)$.

As the last dependence property to be discussed here, we consider SI, whose behaviour reflects the situation presented in Proposition 3.14. Without making explicit computations, the following inclusion can be proven.

Proposition 4.8. $Hyper_{dep}$-$\mathcal{P}_{SI} \subset \mathcal{P}_{TP_2}$.

Proof. Since $\mathcal{P}_{SI} \subset \mathcal{P}_{LTD}$, $hyper_{dep}$-$\mathcal{P}_{SI} \subset hyper_{dep}$-$\mathcal{P}_{LTD}$.
But $hyper_{dep}$-$\mathcal{P}_{LTD} = \mathcal{P}_{TP_2}$ and, therefore, $hyper_{dep}$-$\mathcal{P}_{SI} \subset \mathcal{P}_{TP_2}$. □

By Corollary 3.17, it follows

Corollary 4.9. If $t_0 \geq 0$ exists, such that $\hat{C}$ is $\langle SI; [t_0, +\infty)\rangle_{dep}$, then $\hat{C}$ is $\langle TP_2; [t_0, +\infty)\rangle_{dep}$.

We summarize the implications among the dependence properties considered here in the following table, similar to the one in [5]:

$$\begin{array}{ccc}
\langle \mathrm{TP}_2; \Lambda\rangle_{dep} & \Rightarrow & \\
\not\Uparrow\, \not\Downarrow & & \langle \mathrm{LTD}; \Lambda\rangle_{dep} \\
\langle \mathrm{SI}; \Lambda\rangle_{dep} & \Rightarrow & \\
\not\Uparrow\, \not\Downarrow & & \not\Uparrow\, \not\Downarrow \\
\mathrm{SI} & \Rightarrow & \\
\not\Uparrow\, \not\Downarrow & & \mathrm{LTD} \\
\mathrm{TP}_2 & \Rightarrow & \\
\Updownarrow & & \Uparrow \\
hyper_{dep}\text{-}\mathrm{TP}_2 & \Leftrightarrow & \\
\Uparrow\, \not\Downarrow & & hyper_{dep}\text{-}\mathrm{LTD} \\
hyper_{dep}\text{-}\mathrm{SI} & \Rightarrow &
\end{array} \tag{21}$$

For $\Lambda = [t_0, +\infty)$, $t_0 > 0$, the only different relationships are:

$$\begin{array}{ccc} \langle \mathrm{TP}_2; \Lambda\rangle_{dep} & \Leftrightarrow & \\ \Uparrow \not\Downarrow & & \langle \mathrm{LTD}; \Lambda\rangle_{dep} \\ \langle \mathrm{SI}; \Lambda\rangle_{dep} & \Rightarrow & \end{array} \tag{22}$$

Remark 4.10. In the Archimedean case, the relations among the classes

$$\mathcal{P}_{PQD} \supset \mathcal{P}_{LTD} \supset \mathcal{P}_{TP_2}, \quad hyper\text{-}\mathcal{P}_{PQD} \supset hyper\text{-}\mathcal{P}_{LTD} = hyper\text{-}\mathcal{P}_{TP_2}$$

change into

$$\mathcal{P}_{PQD} \supset \mathcal{P}_{LTD} = \mathcal{P}_{TP_2}, \quad hyper\text{-}\mathcal{P}_{PQD} = hyper\text{-}\mathcal{P}_{LTD} = hyper\text{-}\mathcal{P}_{TP_2}.$$

The following situation may be met: let us suppose $\hat{C}_t \notin \mathcal{P}_{TP_2}$ for any $t \in [0, t_0)$, but $\hat{C}_{t_0} \in \mathcal{P}_{TP_2}$ and therefore $\hat{C}_t \in \mathcal{P}_{TP_2}$ for any $t \geq t_0$. We can conclude that $\hat{C}$ is not $\langle \mathrm{PQD}; [t, t_0]\rangle_{dep}$ for any $t \in [0, t_0)$. However we cannot exclude that $\hat{C}$ is $\langle \mathrm{PQD}; \Lambda\rangle_{dep}$ for some set Λ such that $\overline{\Lambda} \cap [t_0, +\infty) = \emptyset$. In other words, the orbit of $\hat{C}$ cannot enter $\mathcal{P}_{TP_2}$ passing directly from $\mathcal{P}_{PQD}$ to $\mathcal{P}_{TP_2}$, but, for any $\varepsilon \in (0, t_0)$, $\hat{C}_t$ may be PQD for some $t \in [0, t_0 - \varepsilon)$. This conclusion derives by Corollary 3.16 and is analogous to the one in Remark 4.7.

4.2. Ageing and hyper-ageing

We apply now results of Section 3 to the evolution of ageing (see e.g. [3, 12]).

We limit the analysis of B's properties to SM and to its relations with PQD and TP_2. In fact, the scheme of the implications among PQD, SM and TP_2, investigated in [12], reflects the ones in (21) and (22).
Also in studying evolution of ageing, the following definition is needed:

Definition 4.11. $S \in \mathcal{S}$ is $\langle \mathbf{P}; \Lambda\rangle_{ag}$ if $\Phi_{ag}(S, t) \in \mathcal{P}$ for any $t \in \Lambda$.

Supermigrativity is a particularly interesting property for an ageing function, because of its meaning in terms of bivariate ageing; in fact B being SM is equivalent to $\overline{F}$ being Schur-concave (see [3, 6]). In [2] it was proven that $hyper_{ag}\text{-}\mathcal{P}_{PQD} = \mathcal{P}_{SM}$. It follows that $\mathcal{P}_{SM} = hyper_{ag}\text{-}\mathcal{P}_{SM}$. Thus, by Corollary 3.16, we have:

Corollary 4.12. If $t_0 \geq 0$ exists, such that B_{t_0} is SM, then B is $\langle SM; [t_0, +\infty)\rangle_{ag}$.

From this and other features of $\mathcal{P}_{SM}$, we notice an analogy between $\mathcal{P}_{SM}$ and $\mathcal{P}_{TP_2}$, due to the fact that both of them are closed under some semigroup actions, more precisely, $\mathcal{P}_{SM} = hyper_{ag}\text{-}\mathcal{P}_{SM}$ so like $\mathcal{P}_{TP_2} = hyper_{dep}\text{-}\mathcal{P}_{TP_2}$. As for TP_2 property for the survival copulas, we can say that the property SM for the ageing function can arise at some time t_0, but, once it has manifested, it is necessarily preserved for future times; i.e. the orbits of an ageing function under Φ_{ag} can enter $\mathcal{P}_{SM}$, but they cannot go out. On the other hand, it may happen that $B_t \in \mathcal{P}_{SM}$ for any $t \geq t_0$, but $B_t \notin \mathcal{P}_{SM}$ for any $t \in [0, t_0)$, as shown by Example 4.13 below. In this case, B_t cannot be $\langle PQD; [t, t_0]\rangle_{ag}$ for any $t < t_0$: the argument is analogous to the one discussed in Remark 4.7.

Example 4.13. We consider, for $z_0 \in (0,1)$, the ordinal sum

$$B(u,v) = \begin{cases} uv, & u,v \in [0,z_0], \\ z_0 + (1-z_0)W\left(\frac{u-z_0}{1-z_0}, \frac{v-z_0}{1-z_0}\right), & u,v \in (z_0,1], \\ \min(u,v), & \text{otherwise.} \end{cases}$$

B_t is SM for $t \geq -\log z_0$, but, for $t < -\log z_0$, B_t is not even PQD.

If $\inf \Lambda > 0$, $\langle SM; \Lambda \rangle_{ag}$ is a weaker property than SM and we find a weaker property than *hyper*$_{ag}$-PQD implying it.

Corollary 4.14. If $t_0 \geq 0$ exists, such that B is $\langle PQD; [t_0, +\infty)\rangle_{ag}$, then B is $\langle SM; [t_0, +\infty)\rangle_{ag}$.

$\langle PQD; \Lambda\rangle_{ag}$ implies $\langle SM; \Lambda\rangle_{ag}$ only for intervals of the kind $\Lambda = [t_0, +\infty)$, for any $t_0 \geq 0$. This implication does not hold for a general interval $\Lambda = [t_0, t_1]$, $0 \leq t_0 \leq t_1$, as the following example shows.

Example 4.15. Let us consider, for $z_0 \in (0,1)$,

$$B(u,v) = \begin{cases} \min\left(u, v, \frac{u^2+v^2}{2}\right) & u,v \in [0,z_0], \\ z_0 + \frac{(u-z_0)(v-z_0)}{1-z_0}, & u,v \in (z_0,1], \\ \min(u,v), & \text{otherwise.} \end{cases}$$

B is at least $\langle PQD; [0, -\log z_0]\rangle_{ag}$, but B is not SM. In fact B does not satisfy Eq. (7), for $u = s = \frac{1}{2},\ v = \frac{1}{4},\ z_0 \in [\frac{1}{4}, \frac{1}{2}]$.

Another relation is provided by the following proposition, proven in [12].

Proposition 4.16. *hyper*$_{ag}$-$\mathcal{P}_{TP_2} \subset \mathcal{P}_{SM}$.

By Corollary 3.17, it follows:

Corollary 4.17. If $t_0 \geq 0$ exists, such that B is $\langle TP_2; [t_0, +\infty)\rangle_{ag}$, then B is $\langle SM; [t_0, +\infty)\rangle_{ag}$.

We summarize here the implications discussed in the present paragraph:

$$\begin{array}{ccccc} hyper_{ag}\text{-}TP_2 & \Rightarrow & TP_2 & \not\Rightarrow \not\Leftarrow & \langle TP_2; \Lambda\rangle_{ag} \\ \Downarrow & & \not\Uparrow \not\Downarrow & & \not\Downarrow \\ hyper_{ag}\text{-}SM & \Leftrightarrow & SM & \Rightarrow \not\Leftarrow & \langle SM; \Lambda\rangle_{ag} \\ \Updownarrow & & \Downarrow & & \Downarrow \not\Uparrow \\ hyper_{ag}\text{-}PQD & \Rightarrow & PQD & \not\Rightarrow \not\Leftarrow & \langle PQD; \Lambda\rangle_{ag} \end{array}$$

If $\Lambda = [t_0, +\infty)$, the only different implications in the table are:

$$\langle TP_2; \Lambda\rangle_{ag} \Rightarrow \langle SM; \Lambda\rangle_{ag} \Leftrightarrow \langle PQD; \Lambda\rangle_{ag}.$$

5. CONCLUSIONS

We consider the families $\{\hat{C}_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$, used to describe dependence and ageing of a model. Results in the same direction of the ones developed in the present paper are obtained in [12]. Therein the fact was considered that such results are based on the semigroup structure of the two families. In this paper, we developed this hint, and study more in detail the consequences of such a common semigroup structure. In this frame, we continued here the analysis of the concept of hyper-property, introduced in [5], and underpinned it from a theoretical point of view. We presented an algebraic approach to this investigation and found that the notion of hyper-property is something general, not only related to dependence properties. Thus, we applied the study of hyper-properties to ageing too. Results in Sections 4.1, 4.2 are obtained by means of general propositions, without having recourse to explicit computations for any particular case.

The common algebraic structure of $\{\hat{C}_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ allows us to explain some systematic analogies between dependence and ageing properties and between the structures of relations existing among them.

We notice in fact a parallelism between the properties TP_2 for $\hat{C}$ and SM for B, SI for $\hat{C}$ and TP_2 for B, LTD for $\hat{C}$ and PQD for B. The explanation of the behavioural similarities lies in the fact that each pair of classes of semi-copulas (corresponding to the afore-mentioned properties) has the same features with respect to the two different actions Φ_{dep} and Φ_{ag}. For example, we have $\mathcal{P}_{TP_2} \subset \mathcal{P}_{LTD}$, $\mathcal{P}_{SM} \subset \mathcal{P}_{PQD}$ and $\mathcal{P}_{TP_2} = hyper_{dep}\text{-}\mathcal{P}_{LTD}$, $\mathcal{P}_{SM} = hyper_{ag}\text{-}\mathcal{P}_{PQD}$. By Proposition 3.12 it follows that $\mathcal{P}_{TP_2}$ is closed under Φ_{dep} and $\mathcal{P}_{SM}$ is closed under Φ_{ag}. Again, we notice that $\mathcal{P}_{TP_2} \nsubseteq \mathcal{P}_{SI}$, $\mathcal{P}_{SI} \nsubseteq \mathcal{P}_{TP_2}$ and $\mathcal{P}_{TP_2} \nsubseteq \mathcal{P}_{SM}$, $\mathcal{P}_{SM} \nsubseteq \mathcal{P}_{TP_2}$. Since we know that $\mathcal{P}_{TP_2} = hyper_{dep}\text{-}\mathcal{P}_{TP_2}$ and $\mathcal{P}_{SM} = hyper_{ag}\text{-}\mathcal{P}_{SM}$, by Proposition 3.14, we get that $hyper_{dep}\text{-}\mathcal{P}_{SI} \subset \mathcal{P}_{TP_2}$ and $hyper_{ag}\text{-}\mathcal{P}_{TP_2} \subset \mathcal{P}_{SM}$.

All these analogies and the others discussed in Sections 4.1, 4.2, are a consequence of the fact that both Φ_{dep} and Φ_{ag} are actions of a semigroup on the set of semi-copulas.

Our results can be extended to the multivariate case. The fundamental difference between bivariate and n-variate case lies in the definitions of the specific dependence or ageing properties. In the n-variate case such definitions are more various and less immediate than the bivariate analogues. For example a bivariate dependence property, like PQD, can be extended in two different directions (see [14, Sect. 2.1]), giving rise to the positive lower orthant dependence (PLOD) and to the positive upper orthant dependence (PUOD). Between these two properties no implication relationship exists, while instead they turn out to be equivalent in the bivariate case. Another example of non-unique extension to the multivariate case is given in [6, Def. 2.1], where, as a generalization of PQD, both PLOD and a pairwise PLOD (PPLOD) are considered. Also PLOD and PPLOD are equivalent when stated for the bivariate case; otherwise, PPLOD strictly implies PLOD.
In the light of these differences with the bivariate case, conditions for preservation of multivariate dependence properties (along the line of [5]) have to be differently stated and proven.

The extension to the multivariate case of the results in Section 3 is straightforward, till we are considering symmetric conditioning events, i.e. the observation of survival data of the kind $\{X_1 > t, \ldots, X_n > t\}$. In this case the main difficulty lies in providing

a suitable extension of the properties in Definition 2.4 and in consequently adapting results in Sections 4.1, 4.2. However, the semigroup acting on the set $\mathcal{S}^{(n)}$ of all the n-variate semi-copulas, in order to represent the evolution of dependence and of ageing, is still $(\mathbb{R}_+, +, 0)$.
The situation is fundamentally different if we consider asymmetric conditioning events, i.e. the observation of survival data of the kind $\{X_1 > t_1, \ldots, X_n > t_n\}$. In this case, the semigroup acting on $\mathcal{S}^{(n)}$ is $(\mathbb{R}^n_+, +, \underline{0})$. The fact that $\mathbb{R}^n_+$ is only partially ordered brings on the loss of the interpretation of orbits as trajectories and a weakening of the results, that are based on the total order among the elements of $\mathbb{R}_+$, or the need for a different formulation of them.

ACKNOWLEDGEMENT

I would like to thank the two anonymous referees for criticism and comments that have led to an improvement of the previous version of this manuscript.

(Received November 16, 2011)

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