Threshold copulas and positive dependence

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Abstract

Starting with a notion of positive dependence $\mathcal{P}$ and with the family of the lower threshold copulas C_t associated with a bivariate distribution having copula C, we define different notions of positive dependence for C, reflecting the dependence properties of the copulas C_t for some t.

Then, we analyze some structural aspects of lower threshold copulas and of the given definitions. Furthermore we consider several specific cases arising from relevant special choices of $\mathcal{P}$ (e.g., PQD, LTD, TP_2, PLR). Our analysis, in particular, allows us to present a number of relevant examples and counter-examples, which can be useful in the study of the tail dependence for a bivariate distribution.

Key words: Copulas, tail dependence, evolution of dependence, hyper-dependence.

1 Introduction

Let (X, Y) be a pair of two continuous random variables whose joint distribution function is given by $F(x, y) = \mathbb{P}(X \le x, Y \le y)$. For every real t such that $\mathbb{P}(X \le t, Y \le t) > 0$, we consider the new distribution function

$$F_t(x, y) = P\left(X \le x, Y \le y | X \le t, Y \le t\right). \tag{1}$$

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In different applied fields, in fact, interest arises in the study of the conditional distribution of X, Y, given the event that X, Y fall under a short threshold. Below, we will also occasionally use the notation $F_t^{(F)}$ in order to stress that F is the joint distribution function of the pair (X, Y) in (1).

Recently several authors studied the limit behaviour of F_t for $t \to 0$, mainly motivated by applications to risk management: see, for example, the papers by Juri and Wüthrich (2002, 2003) and Charpentier and Juri (2006). In particular, the attention is focused on the properties of stochastic dependence for F_t, in the limit, also called *tail-dependence* properties.

In our paper, we are rather interested in the *evolution* of dependence properties of F_t, for t spanning the interval $(0, 1]$. An analysis in the same direction has been already approached by some authors (Bassan and Spizzichino, 2003; Charpentier, 2006; Foschi and Spizzichino, 2008), and, restricted to "Archimedean" models, by Pellerey (2007). Here we aim at formalizing and extending some of these results by considering, for example, different notions of positive dependence and weaker assumptions on the copula of F (which, e.g., is not necessarily exchangeable), in the sense that we are going to describe.

First, we recall that a *copula* C is the restriction to $[0, 1] \times [0, 1]$ of a bivariate distribution function with uniform margins. A dependence property $\mathcal{P}$ holds for a continuous joint distribution F if and only if it holds for the *copula* associated with F. In other words, the copula is exactly the concept that captures the dependence of a random vector apart from its marginal behaviour (Joe, 1997; Nelsen, 2006). For this reason (see also the next Section 2), we are practically allowed, without any loss of generality, to consider, in place of (X, Y), a pair of random variables U, V uniformly distributed over $[0, 1]$ and with joint distribution determined by C. We then replace F with C, C being a copula. Furthermore we consider, for $0 < t \leq 1$, the copula C_t defined as the copula associated with $F_t^{(C)}$. The copulas C_t $(0 < t \leq 1)$ are called *lower threshold copulas* associated with C. In Section 2 we introduce some structural aspects, relevant to our analysis, concerning C_t and the relations between C_t and C.

We can now explain the purposes of our analysis.
Let $\mathcal{P}$ be a positive dependence property. The condition "C_t satisfies $\mathcal{P}$" (for some t) can actually be interpreted as a condition on C. Now, let Λ be an interval of $(0, 1]$.

Definition 1 *We say that C is $\langle \mathcal{P}; \Lambda \rangle$ if C_t is $\mathcal{P}$ for every $t \in \Lambda$. In particular, we say that C is hyper-$\mathcal{P}$ if C is $\langle \mathcal{P}; (0, 1] \rangle$.*

An *hyper-$\mathcal{P}$* property may be considered a property of positive dependence. As a main purpose of this paper we are interested in comparing the properties $\mathcal{P}$, $\langle \mathcal{P}; \Lambda \rangle$ and *hyper-$\mathcal{P}$*.

Our study will be carried out in Section 3, where we analyze basic aspects of the Definition 1 and derive some conclusions that can be of interest in the investigation of tail dependencies. In particular we are interested in considering whether *hyper-*$\mathcal{P}$ coincides with some other known properties, stronger than $\mathcal{P}$. For particular choices of $\mathcal{P}$, namely PQD, LTD, TP_2, PLR, we specifically analyze the properties $\langle \mathcal{P}; \Lambda \rangle$ and *hyper-*$\mathcal{P}$, providing a complete comprehension of Definition 1 by means of some useful examples and counter-examples.

For a better comparison with the literature about tail dependence, in this paper we preferred to express our results in terms of distribution functions, copulas, and lower threshold copulas. In other applied fields, such as, e.g., reliability, survival analysis, interacting defaults, X, Y are typically non-negative variables and, for given $\overline{F}(x, y) = \mathbb{P}(X > x, Y > y)$, one rather considers

$$\overline{F}_t(x, y) = P\left(X > t + x, Y > t + y | X > t, Y > t\right),$$

in place of (1). For every $t \geq 0$ such that $\mathbb{P}(X > t, Y > t) > 0$, $\overline{F}_t$ is then the survival function of $(X - t, Y - t)$ conditional on the fact that $(X > t, Y > t)$. Denoting by $(X_t,\ Y_t)$ the random pair whose survival function coincides with $\overline{F}_t$, X_t and Y_t are interpreted as *residual lifetimes.* The evolution of the dependence among X_t and Y_t can be studied in terms of the *upper threshold copulas*, that are the *survival copulas* of the $\overline{F}_t$'s. Our results about lower threshold copulas can be equivalently reformulated for upper threshold copulas by means of a simple transformation.

2 Threshold copulas

Let U, V be two random variables (=r.v.'s) defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and uniformly distributed on $[0, 1]$, whose joint distribution function (=d.f.) is given by the copula C. For every $t \in (0, 1]$, suppose that $C(t, t) > 0$. We are interested in the conditional distribution F_t of (U, V) given that $U \leq t$ and $V \leq t$. For every $x, y \in [0, t]$, we have:

$$F_t(x, y) = \mathbb{P}(U \leq x, V \leq y \mid U \leq t, V \leq t) = \frac{C(x, y)}{C(t, t)}. \tag{2}$$

The univariate marginal d.f.'s G_t and H_t are given, for every $x \in [0, t]$, by

$$G_t(x) = \mathbb{P}(U \leq x, V \leq t \mid U \leq t, V \leq t) = \frac{C(x, t)}{C(t, t)}.$$

and

$$H_t(x) = \mathbb{P}(U \leq t, V \leq x \mid U \leq t, V \leq t) = \frac{C(t, x)}{C(t, t)}.$$

For any fixed $t \in (0,1]$, let $h_t : [0,1] \longrightarrow [0,t]$, $h_t(u) = C(u,t)$, be the *horizontal section* of C at the level t, and, analogously let $k_t : [0,1] \longrightarrow [0,t]$, $k_t(u) = C(t,u)$, be the *vertical section* of C at the level t. Then, we can rewrite

$$G_t(x) = \frac{h_t(x)}{h_t(t)} \quad \text{and} \quad H_t(x) = \frac{k_t(x)}{k_t(t)}. \tag{3}$$

Notice that, for any fixed $t \in (0,1]$, $h_t(t) = k_t(t) = C(t,t)$.

We now need to recall that generally, for a bivariate distribution $F(x,y)$ with marginal distributions $G(x)$ and $H(y)$, the copula associated with F is the copula defined by

$$D(u,v) = F\left(G^{[-1]}(u), H^{[-1]}(v)\right), \tag{4}$$

where $G^{[-1]}$ and $H^{[-1]}$ are the *pseudo–inverse* functions of G and H, respectively, also called *quantile inverse* (Nelsen, 2006). Then, it is obvious that

$$F(x,y) = D(G(x), H(y)). \tag{5}$$

When $G(x)$ and $H(y)$ are continuous, D is the unique copula for which Eq. (5) holds.

Now, we come back to consider the distribution F_t defined above (by (2)) and we notice that the univariate margins G_t and H_t are not uniform.

The copula associated with F_t is defined, for every $u, v \in [0,1]$, by

$$C_t(u,v) = F_t(G_t^{[-1]}(u), H_t^{[-1]}(v)).$$

By recalling (3), we can obtain

$$G_t^{[-1]}(u) = h_t^{[-1]}(uh_t(t)) \quad \text{and} \quad H_t^{[-1]}(v) = k_t^{[-1]}(vk_t(t)), \tag{6}$$

where, for any fixed $t \in (0,1]$,

$$h_t^{[-1]}(u) = \sup\{z \in [0,1] \mid C(z,t) \leq u\}$$

is the pseudo–inverse of h_t, and, analogously for $k_t^{[-1]}$.
In view of (2) we can conclude

$$C_t(u,v) = \frac{C(h_t^{[-1]}(uh_t(t)), k_t^{[-1]}(vk_t(t)))}{C(t,t)}, \tag{7}$$

In the rest of the paper, we will assume that, for any fixed $t \in (0,1]$, h_t and k_t are strictly increasing on $[0,t]$ and, therefore, $h_t^{[-1]}$ and $k_t^{[-1]}$ are their respective standard inverse functions on $[0, C(t,t)]$.

Remark 2 *Eq.* (7) *shows that, for any t, C_t only depends on the restriction of C on $[0,t]^2$. More generally, it is also interesting to notice that, if C is the copula associated with a distribution F, then the copula associated with $F_t^{(F)}$ only depends on the restriction of C on $[0,t]^2$.*

The copulas C_t defined by (7) are called *lower threshold copula* associated with C. In the sequel, we will denote by $\mathcal{C}$ the class of all copulas C satisfying our assumptions, i.e. $C(t,t) > 0$ for every $t \in (0,1]$, and C has horizontal and vertical sections (at a fixed $t \in (0,1]$) strictly increasing on $[0,t]$. In particular, every $C \in \mathcal{C}$ generates a family of copulas $\{C_t\}_{t\in(0,1]}$, where we set $C_1 := C$.

Remark 3 *Given a copula C, the left–residuum of C is the function $R_C^l : [0,1]^2 \to [0,1]$ defined by $R_C^l(x,y) = \sup\{z \in [0,1] \mid C(z,x) \leq y\}$ and the right–residuum of C is the function $R_C^r : [0,1]^2 \to [0,1]$ defined by $R_C^r(x,y) = \sup\{z \in [0,1] \mid C(x,z) \leq y\}$. These two functions have been proved to be useful in multivalued logic. Here, it is important to note that, by using Theorem 3.3 by Durante et al. (2007), both R_C^l and R_C^r are continuous in each argument with $R_C^l(t,u) = h_t^{[-1]}(u)$ and $R_C^r(t,u) = k_t^{[-1]}(u)$.*

Remark 4 *In previous papers, the lower and upper threshold copulas are called lower and upper tail dependence copulas. Here, we prefer to adopt a different terminology following McNeil et al. (2005, section 7.6.3), also in order to avoid confusion with the (different) notion of tail copula recently presented by Einmahl et al. (2006).*

Before investigating the evolution of the dependence along the family $\{C_t\}_{t\in(0,1]}$, it is important to note that, with t spanning $(0,1]$, the family has "no jump", in the sense that the copulas C_{t_0} and C_{t_1} are close each other with respect to the L^∞–norm for sufficiently close t_0 and t_1, as stated in the following result.

Proposition 5 *Let $C \in \mathcal{C}$. The mapping $\Psi : (0,1] \to \mathcal{C}$, $t \mapsto C_t$, is continuous, in the sense that, for every $(u,v) \in [0,1]^2$, $C_t(u,v)$ converges to $C_{t_0}(u,v)$ when t tends to t_0.*

PROOF. We have to prove that, for all $u,v \in [0,1]$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that, if $|t - t_0| < \delta$, then $|C_t(u,v) - C_{t_0}(u,v)| < \varepsilon$.

First, given $(u,v) \in [0,1]^2$, consider that

$$\begin{aligned}|C_t(u,v) - C_{t_0}(u,v)| &= \left|\frac{C(x,y)}{C(t,t)} - \frac{C(x_0,y_0)}{C(t_0,t_0)}\right| \\ &= \frac{|C(x,y)C(t_0,t_0) - C(x_0,y_0)C(t,t)|}{C(t,t)C(t_0,t_0)},\end{aligned}$$

where

$$x = h_t^{-1}(uh_t(t)),\ y = k_t^{-1}(vk_t(t)),\ x_0 = h_{t_0}^{-1}(uh_{t_0}(t_0)),\ y_0 = k_{t_0}^{-1}(vk_{t_0}(t_0)),$$

for $t, t_0 \in (0,1]$, $|t-t_0| < \delta$ for a suitable $\delta > 0$. Set $\alpha := \frac{1}{C(t,t)C(t_0,t_0)}$. We have:

$$\begin{aligned}
&|C(x,y)C(t_0,t_0) - C(x_0,y_0)C(t,t)| \\
\leq &|C(x,y)C(t_0,t_0) - C(x,y)C(t,t)| + |C(x,y)C(t,t) - C(x_0,y_0)C(t,t)| \\
= &C(x,y)|C(t_0,t_0) - C(t,t)| + C(t,t)|C(x,y) - C(x_0,y_0)|.
\end{aligned}$$

Since a copula is a Lipschitz function (with constant 1),

$$|C(t_0,t_0) - C(t,t)| \leq 2|t-t_0| < 2\delta.$$

Analogously,

$$|C(x,y) - C(x_0,y_0)| \leq |x-x_0| + |y-y_0|.$$

In order to estimate $|x-x_0|$ and $|y-y_0|$, we notice that $h_t^{-1}(w)$ and $k_t^{-1}(w)$ are alternative notations for the left- and right- residua of C. Then

$$\begin{aligned}
&|x-x_0| = |R_C^l(t, uC(t,t)) - R_C^l(t_0, uC(t_0,t_0))| \\
\leq &|R_C^l(t, uC(t,t)) - R_C^l(t_0, uC(t,t))| + |R_C^l(t_0, uC(t,t)) - R_C^l(t_0, uC(t_0,t_0))|.
\end{aligned}$$

Since R_C^l is continuous in each argument and

$$u|C(t,t) - C(t_0,t_0)| \leq 2u|t-t_0| \leq 2u\delta \leq 2\delta,$$

it follows that we can consider a suitable η such that

$$|x-x_0| \leq \frac{\eta}{2} + \frac{\eta}{2} < \eta.$$

By the same arguments applied for R_C^r, we have also that $|y-y_0| < \eta$. Thus

$$|C_t(u,v) - C_{t_0}(u,v)| \leq \alpha(2\delta C(x,y) + 2\eta C(t,t)) \leq 2\alpha(\delta+\eta).$$

Choosing a suitable η such that $0 < \delta < \dfrac{\varepsilon}{2\alpha} - \eta$, the proof is concluded. □

3 Lower threshold copulas and generated dependence properties

Consider $C \in \mathcal{C}$ and let $\{C_t\}_{t\in(0,1]}$ be the family of corresponding lower threshold copulas.

As mentioned in the Introduction, in this Section we aim at comparing, for a given dependence property $\mathcal{P}$, the properties $\mathcal{P}$, *hyper-*$\mathcal{P}$, and $\langle \mathcal{P}; \Lambda \rangle$. As a first step, we point out some basic aspects of Definition 1. Then, for a number of relevant notions of dependence $\mathcal{P}$, we analyze the properties $\langle \mathcal{P}; \Lambda \rangle$, *hyper*-$\mathcal{P}$, and relations among them.

Remark 6 *Let Λ be an arbitrary proper subset of $(0,1]$, $\bar{\Lambda}$ being the closure of Λ, and consider the two conditions $\langle \mathcal{P}; \Lambda \rangle$, $\langle \mathcal{P}; \bar{\Lambda} \rangle$. As an immediate consequence of the continuity property in Proposition 5, these two conditions coincide. We may then argue that, in considering the property $\langle \mathcal{P}; \Lambda \rangle$, we can limit attention to the cases when Λ is a closed subset of $(0,1]$, i.e., an closed interval or a union of disjoint closed intervals.*

Actually we are only interested in subsets Λ of the form $(0, \lambda]$, for some constant $\lambda \in (0, 1]$.

We notice that the condition $\langle \mathcal{P}; (0, \lambda] \rangle$ for a copula C has the following immediate meaning: it means that C may not satisfy $\mathcal{P}$ and that the property $\mathcal{P}$ possibly holds for all C_t, with t below a given λ. This notion can be of interest in the field of tail dependence. In fact we are typically interested in proving that C_t satisfies a dependence property $\mathcal{P}$ in the limit for $t \to 0$. Thus, proving that C is $\langle \mathcal{P}; (0, \lambda] \rangle$ guarantees the above condition without explicit computation of $\lim_{t \to 0} C_t$.

As a consequence of Eq. (7) (see Remark 2), we can state that $\langle \mathcal{P}; \Lambda \rangle$ only depends on the behaviour of C on Λ^2 (see, in this respect, Propositions 9, 13, 14, 18 below).

Generally, the property $\langle \mathcal{P}; \Lambda \rangle$ does not imply $\mathcal{P}$ (see Example 12) nor does $\mathcal{P}$ imply $\langle \mathcal{P}; \Lambda \rangle$ (see Example 10).

As far as the property *hyper*-$\mathcal{P}$ is concerned, we notice that *hyper*-$\mathcal{P} \Rightarrow \mathcal{P}$, just by definition. We can also state a sufficient (not a necessary) condition on $\mathcal{P}$ under which the implication $\mathcal{P} \Rightarrow$ *hyper*-$\mathcal{P}$ even holds. Let $\mathcal{P}$ and $\mathcal{P}'$ be two different properties of positive dependence. We have the following result.

Proposition 7 *If $\mathcal{P}$ is equivalent to hyper-$\mathcal{P}'$, then $\mathcal{P}$ is equivalent to hyper-$\mathcal{P}$*

PROOF. For a dependence property $\mathcal{Q}$, it can be shown that C *hyper*-$\mathcal{Q}$ implies C_t *hyper*-$\mathcal{Q}$ for any t. We can then write the following chain of implications:

$$\begin{aligned} C \text{ satisfies } \mathcal{P} \quad &\Rightarrow \quad C \text{ is } hyper-\mathcal{P}' \quad \Rightarrow \quad C_t \text{ is } hyper-\mathcal{P}' \\ &\Leftrightarrow \quad \forall t \quad C_t \text{ is } \mathcal{P} \quad \Leftrightarrow \quad C \text{ is } hyper-\mathcal{P}, \end{aligned}$$

which proves the desired assertion. □

Two different examples of this last situation are provided by the properties of TP_2 and PLR (as we will see below).

Now, we study the properties $\langle \mathcal{P}; \Lambda \rangle$ and *hyper*-$\mathcal{P}$ for relevant notions of dependence recalled below (see also (Kimeldorf and Sampson, 1989; Nelsen, 2006) for a more complete overview).

Definition 8 *Let $C \in \mathcal{C}$ the d.f. of the random pair (U, V).*

- *C is* PQD *(i.e. positively quadrant dependent) if, and only if, $C(u,v) \geq uv$ for every $(u,v) \in [0,1]^2$.*
- *C is* $\mathrm{LTD}(V \mid U)$ *(i.e. V is left tail decreasing in U) if, and only if, $u \mapsto \frac{C(u,v)}{u}$ is decreasing on $[0,1]$. Analogously, C is* $\mathrm{LTD}(U \mid V)$ *(i.e. U is left tail decreasing in V) if, and only if, $v \mapsto \frac{C(u,v)}{v}$ is decreasing on $[0,1]$.*
- *C is* TP_2 *(i.e. totally positive of order 2) if, and only if, for all x, x', y, y' in $[0,1]$, $x \leq x'$ and $y \leq y'$*

$$C(x,y)C(x',y') \geq C(x,y')C(x',y). \tag{8}$$

- *C is* PLR *(i.e. positively likelihood ratio dependent) if, and only if, C is absolutely continuous and its density c satisfies* (8)*, with C replaced by c.*

The following chain of implications holds (and none of the converse implications is satisfied)

$$\mathrm{PLR} \quad \Rightarrow \quad \mathrm{TP}_2 \quad \Rightarrow \quad \mathrm{LTD}(U \mid V) \text{ (or } \mathrm{LTD}(V \mid U)) \quad \Rightarrow \quad \mathrm{PQD}.$$

Let $\Lambda = (0, \lambda]$ be an interval of $(0,1]$ and consider $C \in \mathcal{C}$.

Proposition 9 *C is $\langle \mathrm{PQD}; \Lambda \rangle$ if, and only if, C satisfies*

$$C(x,y)C(t,t) \geq C(x,t)C(t,y) \tag{9}$$

for every $x, y, t \in \Lambda$, $x \leq t$ and $y \leq t$.

PROOF. By definition, C is $\langle \mathrm{PQD}; \Lambda \rangle$ means that $C_t(u,v) \geq uv$ for all $t \leq \lambda$, that is $C\left(h_t^{-1}(uh_t(t)), k_t^{-1}(vk_t(t))\right) \geq uvC(t,t)$. Putting $x = h_t^{-1}(uh_t(t))$, $y = k_t^{-1}(vk_t(t))$, we get

$$C(x,y) \geq \frac{h_t(x)}{h_t(t)} \frac{k_t(y)}{k_t(t)} h_t(t),$$

that is inequality (9). □

According to the general Definition 1, the copulas $\Pi(u,v) = uv$ and $M(u,v) = \min(u,v)$ are *hyper*-PQD.

It is also easy to show that, if C is TP_2, then it is *hyper*-PQD. Moreover, as stated in general before, if C is *hyper*-PQD, then it is PQD. The converse implication is false, as the following example shows.

Example 10 *Let C be the copula given by $C(u,v) = \min\left(u, v, \frac{u^2+v^2}{2}\right)$. Then C is* PQD, *but not $\langle \mathrm{PQD}; \Lambda\rangle$ for $\lambda = \frac{3}{5}$. In fact, by considering $x = y = \frac{1}{2}$, we obtain that*

$$C(x,y)C(t,t) = \frac{36}{100}\frac{25}{100} < \left(\frac{61}{200}\right)^2 = C(x,t)C(t,y).$$

Therefore, $C(u,v) \geq uv$ on $[0,1]^2$, but $C_{\frac{3}{5}}(u,v) < uv$ for some $(u,v) \in [0,1]^2$. Actually, in view of Proposition 5, for every t belonging to a neighbourhood of $\frac{3}{5}$, C_t is not PQD. □

It is interesting to note that, for an Archimedean copula $C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$ Nelsen (2006), the notions of TP_2 and *hyper*-PQD coincide. Furthermore, these conditions are equivalent to $t \mapsto \varphi(e^{-t})$ being convex: the proof can be derived directly from Alsina et al. (2006, section 4.5).

We recall that the upper and lower bounds of a PQD copula are respectively $M(u,v) = \min(u,v)$ and $\Pi(u,v) = uv$. Since, in general, a PQD copula is not *hyper*-PQD, the lower bound for C is not preserved for C_t. But, also in this case, we can obtain for C a lower bound better than $W(u,v) = \max(u+v-1, 0)$.

Proposition 11 *If C is* PQD, *then, for every $t \in (0,1]$, we have that*

$$C_t(u,v) \geq \max(uvt^3, \max(u+v-1,0)). \tag{10}$$

PROOF. Since C is PQD, $uv \leq C(u,v) \leq \min(u,v)$. In particular, $ut \leq h_t(u) \leq \min(u,t)$ and, consequently,

$$u \leq h_t^{-1}(u) \leq \min\left(\frac{u}{t}, t\right) \quad \text{for all } u \in [0,t].$$

Since the fact that an analogous inequality can be proved for k_t, we obtain

$$C_t(u,v) \geq \frac{C(h_t^{-1}(ut^2), k_t^{-1}(vt^2))}{t} \geq \frac{h_t^{-1}(ut^2) \cdot k_t^{-1}(vt^2)}{t} \geq \ uvt^3,$$

and hence (10) holds. □

Now, note that if C satisfies $\langle \mathrm{PQD}, (0,\lambda]\rangle$ for a given $\lambda < 1$, then C needs not be PQD.

Example 12 *Consider, for example, the copula C given by*

$$C(u,v) = \begin{cases} \dfrac{uv}{\lambda}, & (u,v) \in [0,\lambda]^2, \\ \lambda + (1-\lambda)C'\left(\dfrac{u-\lambda}{1-\lambda}, \dfrac{v-\lambda}{1-\lambda}\right), & (u,v) \in [\lambda,1]^2, \\ \min(u,v), & \textit{otherwise}, \end{cases}$$

where $C'(u,v) = uv[1-(1-u)(1-v)]$ is a copula that is not PQD*, and hence nor C is* PQD*. Actually, C is an ordinal sum of the copulas Π and C' with respect to the partition $([0,\lambda],[\lambda,1])$. Now, compute $C_t(u,v)$ for any $t \in (0,\lambda]$. We obtain $C_t(u,v) = \frac{\lambda}{t^2} \cdot C(ut,vt) = uv$. It follows that C_t is* PQD *for any $t \leq \lambda$, even if C is not* PQD. □

Now, let us consider the other dependence properties, LTD, TP_2, PLR. Similarly to Proposition 9, the following result holds for the LTD property.

Proposition 13 *C is $\langle \mathrm{LTD}(V \mid U); \Lambda \rangle$ if, and only if,*

$$C(x,y)C(x',t) \geq C(x,t)C(x',y) \tag{11}$$

for all $t \in \Lambda$, $x, x', y \in [0,t]$ such that $x \leq x'$.

An analogous condition holds for $\langle \mathrm{LTD}(U \mid V); \Lambda \rangle$.

PROOF. Since C is $\langle \mathrm{LTD}(V \mid U); \Lambda \rangle$, then, for any $u \leq u'$,

$$\frac{C(h_t^{-1}(uC(t,t)), k_t^{-1}(vC(t,t)))}{u} \geq \frac{C(h_t^{-1}(u'C(t,t)), k_t^{-1}(vC(t,t)))}{u'},$$

that is

$$\frac{C(t,t)}{C(x,t)} C(x,y) \geq C(x',y) \frac{C(t,t)}{C(x',t)}$$

for all $x, x', y \leq t$ such that $x \leq x'$. Thus, we obtain (11). Part (b) can be proved in the same way. □

Examples of *hyper*-LTD$(V \mid U)$ copulas are $\Pi(u,v) = uv$ and $M(u,v) = \min(u,v)$. Since they are symmetric with respect to u and v, they also are *hyper*-LTD$(U \mid V)$.

We now come to the TP_2. For such a property, we find some differences from the other dependence properties taken in account until now. As we can easily deduce from the following proposition, $\langle \mathrm{TP}_2; \Lambda \rangle$ is a weaker property than TP_2. We can build a copula satisfying $\langle \mathrm{TP}_2; \Lambda \rangle$, but not TP_2, following Example 12.

Proposition 14 *A copula C is $\langle \mathrm{TP}_2; \Lambda \rangle$ if, and only if, C satisfies* (8) *on* Λ^2.

PROOF. By definition, C is $\langle \mathrm{TP}_2; \Lambda \rangle$ if, and only if, for every $t \in \Lambda$ C_t is TP_2, i.e.

$$C_t(u, v)C_t(u', v') \geq C_t(u, v')C_t(u', v),$$

for all $u \leq u'$ and $v \leq v'$. Writing explicitly the multiplicands and making suitable substitutions, we obtain that, for every $x, x', y, y' \in [0, t]$, $x \leq x'$ and $y \leq y'$,

$$C(x, y)C(x', y') \geq C(x, y')C(x', y),$$

which is the desired assertion. □

In particular, C is *hyper*-TP_2 if, and only if, C is TP_2, as stated by the following corollary.

Corollary 15 *Let $C \in \mathcal{C}$ be the d.f. of the random pair (U, V). Then the following statements are equivalent:*

(a) *C is* TP_2*;*
(b) *C is hyper*-TP_2*;*
(c) *C is hyper*-$\mathrm{LTD}(V \mid U)$ *and hyper*- $\mathrm{LTD}(U \mid V)$.

It is easy to show that both *hyper*-$\mathrm{LTD}(V \mid U)$ and *hyper*-$\mathrm{LTD}(U \mid V))$ separately imply that C is PQD.

We note that, in order to obtain for C the stronger property TP_2, both *hyper*-$\mathrm{LTD}(V \mid U)$ and *hyper*-$\mathrm{LTD}(U \mid V)$ have to be satisfied. Statement (c) can be simplified if we have a condition that guarantees

$$LTD(V \mid U) \Leftrightarrow LTD(U \mid V) \tag{12}$$

(and therefore $hyper - \mathrm{LTD}(V \mid U) \Leftrightarrow hyper - \mathrm{LTD}(U \mid V)$). In this case we simply say that C is LTD (*hyper*-LTD). A sufficient condition for (12) is C being exchangeable.

Corollary 16 *Let $C \in \mathcal{C}$ be the d.f. of the exchangeable random pair (U, V). Then the following statements are equivalent:*

(a) *C is* $\langle \mathrm{LTD}(V \mid U); \Lambda \rangle$*;*
(b) *C is* $\langle \mathrm{LTD}(U \mid V); \Lambda \rangle$*;*
(c) *C satisfies* (8) *on* Λ^2.

In particular, C hyper-$\mathrm{LTD}(V \mid U)$ *(resp. hyper*-$\mathrm{LTD}(U \mid V)$*) is equivalent to C being* TP_2.

Finally, we consider the PLR property, which can be introduced only for a copula C that is *absolutely continuous* on $[0,1]^2$, viz.

$$C(u,v)=\int_0^u\int_0^v \partial^2_{12}C(\gamma,\theta)\,\mathrm{d}\gamma\mathrm{d}\theta,$$

where $\partial^2_{12}C$ denotes the second mixed derivative of C.

In order to state some properties about the family $\{C_t\}_{t\in(0,1]}$, it is hence important to consider whether the absolute continuity is preserved by every C_t.

Proposition 17 *Let C be a copula having non-zero first derivatives almost everywhere on $[0,1]^2$. If C is absolutely continuous, then C_t is absolutely continuous for every $t\in(0,1]$.*

PROOF. For every $t\in(0,1]$, the second mixed derivative of C_t is given by:

$$\partial^2_{12}C_t(u,v)=\frac{C(t,t)\partial^2_{12}C(h_t^{-1}(uh_t(t)),k_t^{-1}(vk_t(t)))}{\partial_1 C(h_t^{-1}(uh_t(t)),t)\cdot\partial_2 C(t,k_t^{-1}(vk_t(t)))}. \tag{13}$$

We have to check the equality

$$C_t(u,v)=\int_0^u\int_0^v \partial^2_{12}C_t(\gamma,\theta)\,\mathrm{d}\gamma\mathrm{d}\theta. \tag{14}$$

Applying (7) and (14), we obtain

$$\frac{C(x,y)}{C(t,t)}=\int_0^u\int_0^v \frac{C(t,t)\partial^2_{12}C(h_t^{-1}(\gamma h_t(t)),k_t^{-1}(\theta k_t(t)))}{\partial_1 C(h_t^{-1}(\gamma h_t(t)),t)\partial_2 C(t,k_t^{-1}(\theta k_t(t)))}\,\mathrm{d}\gamma\mathrm{d}\theta.$$

Changing variables by means of

$$\xi=h_t^{-1}(\gamma h_t(t)),\quad \eta=k_t^{-1}(\theta k_t(t)), \tag{15}$$

we obtain

$$C(x,y)=\int_0^x\int_0^y \partial^2_{12}C(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta,$$

which is the desired assertion. □

Proposition 18 *C is $\langle\mathrm{PLR};\Lambda\rangle$ if, and only if, $\partial^2_{12}C$ satisfies (8) on Λ^2.*

PROOF. By definition, C is $\langle\mathrm{PLR};\Lambda\rangle$ if, and only if, for every $t\in\Lambda$, C_t is PLR, i.e.

$$\frac{\partial^2 C_t(u,v)}{\partial u\partial v}\frac{\partial^2 C(u',v')}{\partial u\partial v}\geq\frac{\partial^2 C(u',v)}{\partial u\partial v}\frac{\partial^2 C(u,v')}{\partial u\partial v}$$

for every $u,u',v,v'\in[0,1]$, $u\leq u'$ and $v\leq v'$. Using Eq. (13), after some simplification, we obtain the desired assertion. □

Analogously to TP_2, C PLR is equivalent to C *hyper*-PLR. Moreover, there exists a copula C such that C is not PLR, even if C is $\langle \text{PLR}; (0,\lambda] \rangle$ for $\lambda < 1$.

The relationships among dependence properties and related *hyper*-dependence properties are summarized here.

$$
\begin{array}{ccccccc}
\langle \text{PLR}; \Lambda \rangle & \Rightarrow & \langle \text{TP}_2; \Lambda \rangle & \Rightarrow & \langle \text{LTD}(V \mid U); \Lambda \rangle \ (\text{or } \langle \text{LTD}(V \mid U); \Lambda \rangle) & \Rightarrow & \langle \text{PQD}; \Lambda \rangle \\
\Uparrow & & \Uparrow & & \not\Uparrow \ \not\Downarrow & & \not\Uparrow \ \not\Downarrow \\
\text{PLR} & \Rightarrow & \text{TP}_2 & \Rightarrow & \text{LTD}(U \mid V) \ (\text{or LTD}(V \mid U)) & \Rightarrow & \text{PQD} \\
\Updownarrow & & \Updownarrow & & \Uparrow & & \Uparrow \\
\textit{hyper}\text{-PLR} & \Rightarrow & \textit{hyper}\text{-TP}_2 & \Rightarrow & \textit{hyper}\text{-LTD}(U \mid V) \ (\text{or } \textit{hyper}\text{-LTD}(V \mid U)) & \Rightarrow & \textit{hyper}\text{-PQD} \\
 & & & & \Downarrow & & \Downarrow \\
 & & & & \langle \text{LTD}(V \mid U); \Lambda \rangle \ (\text{or } \langle \text{LTD}(V \mid U); \Lambda \rangle) & \Rightarrow & \langle \text{PQD}; \Lambda \rangle
\end{array}
$$

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