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Main notation

d	:	the space dimension, $d = 2$ or 3
$\lvert \cdot \rvert_{\mathbb{R}^d},\ \lvert \cdot \rvert$	:	the classical Euclidean norm in $\mathbb{R}^d$
$\lVert \cdot \rVert_B$	:	norm in some Banach space B
$\langle \cdot ; \cdot \rangle$	:	scalar product in some Hilbert space, or duality product
$\mathcal{S}$	:	solid domain
$\mathcal{F}$	:	fluid domain
$\mathcal{O}$	:	domain of $\mathbb{R}^d$ which does not depend on time
Γ	:	boundary (a curve or a surface) separating $\mathcal{O}$ into two connected components
x	:	space variable, in time-dependent domains
y	:	space variable, in non-depending time domains
$t,\ \bar{t}$	:	time variables
s	:	curvilinear abscissa
$u,\ v,\ w,\ u_{\mathcal{S}}$	:	Eulerian velocities in time-dependent domains
$U,\ V,\ \tilde{u},\ \tilde{v},\ \hat{u},\ \hat{v}, \mathrm{w}$	:	fluid's velocities, in non-depending time domains
$\rho,\ \rho_{\mathcal{S}}$	:	densities
$h,\ H$	:	position of the solid's center of mass
$\omega,\ \Omega$	:	angular velocity of the solid
f	:	surface force (chosen as control), or some arbitrary additional volume force
$X,\ \tilde{X},\ X_{\mathcal{S}}$	:	Lagrangian mappings
$Y,\ \tilde{Y},\ Y_{\mathcal{S}}$	:	inverses - in space - of the mappings $X,\ \tilde{X},\ X_{\mathcal{S}}$ respectively
$X^*,\ Z^*$	:	Lagrangian mappings representing the solid's shape, and chosen as controls [1]
ν	:	kinematic viscosity of the fluid
M	:	mass of the solid
I	:	inertia matrix of the solid
Id_{Ω}	:	identity mapping $y \mapsto y$ of a subset $\Omega \subset \mathbb{R}^d$
$\mathrm{I}_{\mathbb{R}^d}$	:	identity matrix in $\mathbb{R}^{d \times d}$
$\mathcal{A},\ \mathbb{A},\ \mathcal{B},\ \mathbb{B},\ \mathcal{K}$	:	functional operators
$\mathbb{M},\ \mathbb{M}_0,\ \mathbb{M}_{add}$	:	functional operators in relation with a mass effect

Introduction - version Française

0.1 Motivation

Dans le cadre des problèmes d'interactions fluide-structure, nous nous intéressons à l'étude des phénomènes en jeu lorqu'un organisme animal se déplace dans un fluide. La problématique sous-jacente est la nage d'une structure déformable dans un liquide : comment cette structure peut-elle se déformer pour nager dans le fluide environnant ? On peut s'intéresser par exemple à la façon dont un poisson opère pour nager dans l'eau. Énoncons quelques questions en relation avec ce problème. Comment le fluide agit-il sur la structure ? Comment le solide perçoit-il les forces du fluide ? Comment peut-il réagir pour se servir de ces forces ? Comment une vitesse frontière imposée par le solide influence-t-elle le comportement du fluide ?
Ces questions relèvent de problèmes physiques, que l'étude de modèles mathématiques peut aider à comprendre.
À bas nombre de Reynolds, l'influence des phénomènes d'inertie est négligeable, et donc le nageur ne peut *a priori* qu'utiliser la viscosité du fluide et les variations de forme pour se déplacer. Pour cette situation, nous pouvons citer en exemple des micro-organismes tels que les spermatozoïdes.
À l'opposé, la nage d'un requin correspond à un nombre de Reynolds élevé ; dans cette situation les phénomènes liés à l'inertie sont prépondérants, et l'échange de moments mécaniques entre le requin et le fluide semblent *a priori* constituer l'essentiel du procédé qui permet au requin de nager. À une échelle intermédiaire, la nage d'un têtard par exemple peut s'effectuer à l'aide de deux phénomènes principaux, à savoir l'inertie du têtard, et l'influence que peut avoir la viscosité du fluide sur sa peau.
L'étude de tels problèmes s'avère complexe, autant d'un point de vue physique que d'un point de vue mathématique. C'est pourquoi dans cette thèse nous choisissons d'étudier principalement un seul modèle qui a pour but d'apporter un regard mathématique sur la nage à nombre de Reynolds intermédiaire. Le problème physique sous-jacent constitue seulement une source d'inspiration, et notre contribution se veut modeste. En effet, nous ne considérons que des petits déplacements, c'est-à-dire que seules des déformations de la structure arbitrairement proche de ce qui correspond à une structure rigide seront considérées.

0.2 Le modèle

Dans un domaine borné $\mathcal{O}$ de $\mathbb{R}^2$ ou $\mathbb{R}^3$, nous considérons un solide déformable dans un fluide. Le solide occupe à un instant t un domaine noté $\mathcal{S}(t)$, et le fluide occupe le domaine $\mathcal{F}(t) = \mathcal{O} \setminus \overline{\mathcal{S}(t)}$. Le cadre est celui de la mécanique des milieux continus.

$\mathcal{O} = \mathcal{F}(t) \cup \overline{S(t)} \subset \mathbb{R}^2$ ou $\mathbb{R}^3$.

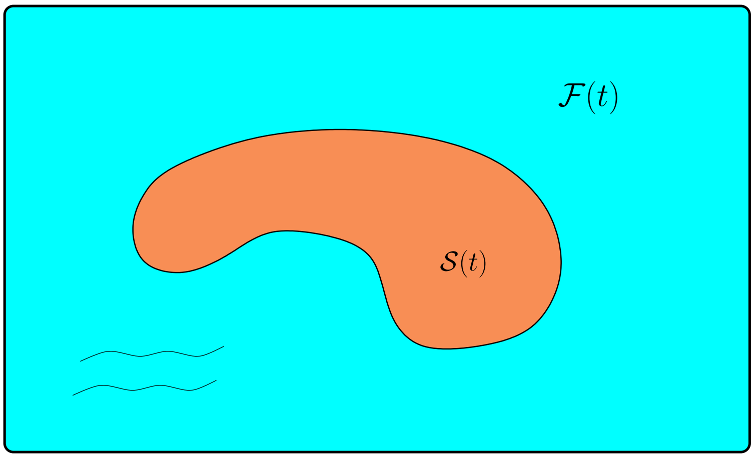

FIGURE 1 – Un solide déformable plongé dans un fluide.

La masse volumique

En mécanique des milieux continus, la masse volumique (pour le fluide ou pour le solide) vérifie le principe de conservation de la masse, que nous écrivons dans sa forme locale, en représentation Eulérienne, comme suit :

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) = 0.$$

Dans cette équation, ρ désigne la masse volumique du milieu et u le champ de vitesse Eulérien. À cette vitesse nous pouvons associer le flot Lagrangien X, solution du problème de Cauchy suivant :

$$\begin{cases} \dfrac{\partial X}{\partial t}(y,t) = u(X(y,t),t), \\ X(y,0) = y. \end{cases}$$

À un instant t, l'application $y \mapsto X(y,t)$ doit être inversible. Le principe de conservation de la masse réécrit en coordonnées Lagrangiennes n'est rien d'autre que

$$\frac{\partial}{\partial t}\Big(\rho(X(y,t),t)\mathrm{det}\nabla X(y,t)\Big) = 0,$$

ce qui conduit à la formule suivante :

$$\rho(X(y,t),t) = \frac{\rho(y,0)}{\mathrm{det}\nabla X(y,t)}.$$

Le rôle de la masse volumique dans cette thèse est tout à fait mineur. En effet, elle est considérée constante dans le fluide, et le rôle qu'elle joue dans le solide est occulté par le fait que nous considérons seulement des petits déplacements, si bien que la masse volumique du solide n'est perçue de manière significative qu'au travers de sa valeur initiale. De plus, cette valeur initiale sera supposée constante, dans un souci de simplicité.

Le fluide

Le fluide est supposé incompressible, c'est-à-dire que sa masse volumique est constante en temps et en espace. Cela revient à supposer que la vitesse du fluide u satisfait la condition de divergence nulle :

$$\operatorname{div} u = 0.$$

Les fonctions d'état pour le fluide sont sa vitesse u et sa pression p. À un nombre de Reynolds intermédiaire, le couple (u, p) est supposé vérifier l'équation de Navier-Stokes

$$\frac{\partial \rho u}{\partial t} + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\sigma(u,p) \quad = \quad 0.$$

Le tenseur des contraintes de Cauchy $\sigma(u, p)$ a pour expression

$$\sigma(u,p) \quad = \quad 2\nu D(u) - p\mathrm{I}_{\mathbb{R}^d} = \nu\left(\nabla u + \nabla u^T\right) - p\mathrm{I}_{\mathbb{R}^d},$$

où ν désigne la viscosité cinématique du fluide. En ajoutant la condition de divergence homogène écrite plus haut, nous fixons la densité ρ constante égale à 1, et le système qui en résulte est appelé *système des équations de Navier-Stokes incompressibles* :

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = 0 \qquad & \text{dans} \quad \mathcal{F}(t), \\ \operatorname{div} u = 0 \qquad & \text{dans} \quad \mathcal{F}(t). \end{aligned}$$

La pression p, telle qu'elle apparaît dans ce système, peut ête considérée - d'un point de vue mathématique - comme un multiplicateur de Lagrange associé à la contrainte d'incompressibilité. Cette fonction est alors déterminée à une constante près.

Le solide, sa déformation choisie comme fonction de contrôle, et ses contraintes

Pour le modèle considéré dans cette thèse, nous verrons, qu'une partie importante de l'interaction entre le fluide et le solide est localisée à l'interface. Toutefois, une autre partie importante du couplage est due aux échanges de moments mécaniques. Décrivons les quantités liées à l'inertie du solide. D'abord, la masse du solide est supposée constante :

$$M \quad = \quad \int_{\mathcal{S}(t)} 1 \mathrm{d}x.$$

Son moment d'inertie s'exprime comme suit :

$$\begin{aligned} I(t) = \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)|x|^2 \mathrm{d}x \qquad & \text{en dimension 2,} \\ I(t) = \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)\left(|x|^2\mathrm{I}_{\mathbb{R}^3} - x \otimes x\right) \mathrm{d}x \qquad & \text{en dimension 3.} \end{aligned}$$

Il s'agit d'un scalaire en dimension 2, et d'une matrice en dimension 3. Lorsque le solide se déforme, le moment d'inertie dépend du temps de façon non triviale (alors que dans le cas rigide la dépendance s'exprime simplement à l'aide de rotations). Mais après linéarisation obtenue pour des petits déplacements, seul le moment d'inertie à $t = 0$ a un rôle significatif.

À partir de ces deux quantités nous pouvons définir $h(t)$, le vecteur coordonnées du centre de gravité du solide, et $\omega(t)$, sa vitesse angulaire, comme suit

$$\begin{aligned} Mh(t) &= \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t) x \mathrm{d}x, \\ I(t)\omega(t) &= \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)(x-h(t)) \wedge (u_{\mathcal{S}}(x,t) - h'(t)) \mathrm{d}x. \end{aligned}$$

La vitesse angulaire $\omega(t)$ induit une rotation $\mathbf{R}(t)$, qui peut être déterminée en dimension 3 en résolvant :

$$\left\{ \begin{array}{rcl} \dfrac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} &=& \mathbb{S}(\omega)\,\mathbf{R} \\ \mathbf{R}(0) &=& \mathrm{I}_{\mathbb{R}^3}, \end{array} \right. \quad \text{avec } \mathbb{S}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Le mouvement du solide, dans un référentiel Galiléen, est représenté par une application Lagrangienne $X_{\mathcal{S}}$, qui définit alors

$$\mathcal{S}(t) = X_{\mathcal{S}}(\mathcal{S}(0), t)$$

à un instant t donné. Nous pouvons décomposer cette application de la manière suivante :

$$X_{\mathcal{S}}(\cdot, t) = h(t) + \mathbf{R}(t) X^*(\cdot, t).$$

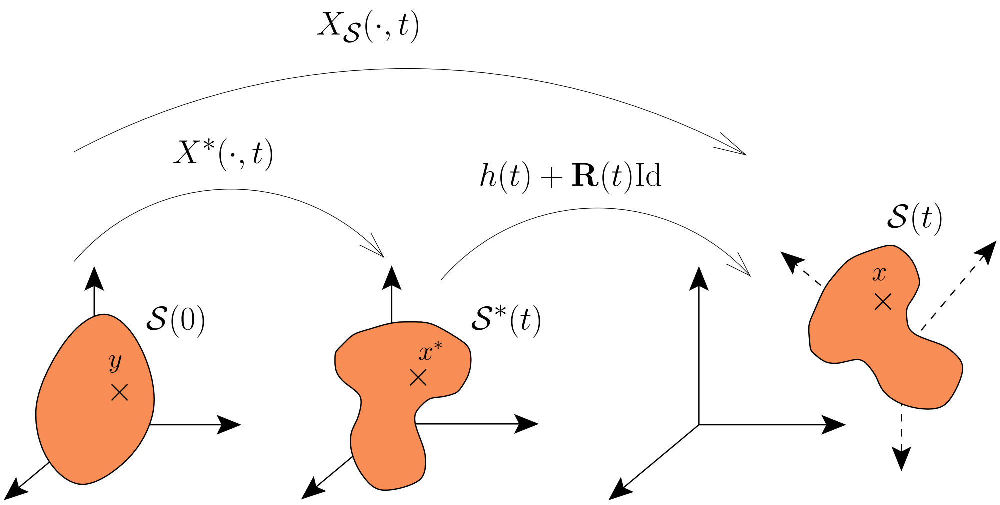

FIGURE 2 – Décomposition du movement du solide.

La transformation rigide $y \mapsto h(t) + \mathbf{R}(t)y$ détermine la position et l'orientation du solide. Il s'agit d'une inconnue du problème. L'application $X^*(\cdot, t)$ peut être vue comme la déformation du solide dans son propre référentiel, c'est-à-dire une quantité qui représente sa forme. Nous la supposons donnée. Cette application doit satisfaire un ensemble de contraintes que nous donnons ci-dessous :

H1 Pour tout temps $t \geq 0$, nous supposons que l'application $X^*(\cdot,t)$ est un C^1-difféomorphisme de $\overline{\mathcal{S}(0)}$ sur $\overline{\mathcal{S}^*(t)}$. Cela permet en particulier de conserver dans le temps la régularité et l'injectivité de la frontière initiale du solide $\partial\mathcal{S}(0)$.

H2 Puisque le fluide est supposé incompressible, et puisque le domaine $\mathcal{O} = \mathcal{F}(t) \cup \overline{\mathcal{S}(t)}$ occupé par le fluide et le solide est borné et immobile, le volume du solide doit être constant au cours du temps. Cela revient à supposer que

$$\int_{\partial\mathcal{S}(0)} \frac{\partial X^*}{\partial t} \cdot (\mathrm{cof}\nabla X^*)\, n\mathrm{d}\Gamma \quad = \quad 0.$$

H3 La déformation ne modifie pas la quantité de mouvement du solide :

$$\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t)\mathrm{d}y \quad = \quad 0.$$

H4 La déformation ne modifie pas son moment cinétique :

$$\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t) \wedge \frac{\partial X^*}{\partial t}(y,t)\mathrm{d}y \quad = \quad 0.$$

Les hypothèses **H1** et **H2** sont invariantes par rotations et translations. Les hypothèses **H3** et **H4** garantient le caractère *autopropulsé* du solide, c'est-à-dire que le solide ne bénéficie d'aucune aide « extérieure »(comme la traction d'une corde, ou la puissance d'une quelconque propulsion chimique par exemple) afin de se propulser dans le fluide. Le seul moyen dont il dispose est la réaction du fluide environnant, réaction qu'il provoque en se déformant.
La déformation du solide, imposée au travers de l'application X^*, est choisie comme fonction de contrôle dans le chapitre 3. Cela implique que le solide est assez puissant pour imposer sa propre forme, et ainsi contrebalancer les forces du fluide susceptibles de pousser la frontière du solide et de modifier sa forme. De tels effets dûs au fluide peuvent toutefois être négligés, voire même être inexistants, si les forces de frottement interne (friction) sont assez fortes à l'intérieur du solide.

Si on note $Y^*(\cdot,t)$ l'inverse de l'application $X^*(\cdot,t)$, on définit la vitesse

$$w^*(x^*,t) \quad = \quad \frac{\partial X^*}{\partial t}(Y^*(x^*,t),t), \quad x^* \in \mathcal{S}^*(t).$$

Si on note $Y_{\mathcal{S}}(\cdot,t)$ l'inverse de l'application $X_{\mathcal{S}}(\cdot,t)$, la vitesse Eulérienne qui est associée à cette déformation du solide dans le référentiel Galiléen est définie par :

$$u_{\mathcal{S}}(x,t) \quad = \quad \frac{\partial X_{\mathcal{S}}}{\partial t}(Y_{\mathcal{S}}(x,t),t), \quad x \in \mathcal{S}(t),$$

et peut être décomposée comme suit :

$$u_{\mathcal{S}}(x,t) \quad = \quad h'(t) + \omega(t) \wedge (x - h(t)) + w(x,t), \quad x \in \mathcal{S}(t),$$

où la vitesse w est définie à partir de w^* via le changement de référentiel

$$w(x,t) \quad = \quad \mathbf{R}(t) w^*(\mathbf{R}(t)^T(x-h(t)),t).$$

Rappelons que u désigne la vitesse du fluide. L'égalité des vitesses à l'interface fluide-solide s'écrit

$$u(x,t) \quad = \quad u_{\mathcal{S}}(x,t), \quad x \in \partial\mathcal{S}(t).$$

Cela constitue une condition de Dirichlet non homogène pour le fluide, à laquelle nous ajoutons une condition de Dirichlet homogène sur $\partial\mathcal{O}$. La réponse du fluide est la trace normale sur $\partial\mathcal{S}(t)$ du tenseur des contraintes de Cauchy $\sigma(u,p)$, à savoir $\sigma(u,p)n$. Cela correspond aux forces que le fluide applique sur la frontière du solide. La partie rigide inconnue de la déformation du solide, donnée par $h(t)$ et $\omega(t)$, est alors reliée à ces forces grâce aux lois de Newton :

$$\begin{aligned} Mh''(t) &= -\int_{\partial\mathcal{S}(t)} \sigma(u,p)n\mathrm{d}\Gamma, \\ (I\omega)'(t) &= -\int_{\partial\mathcal{S}(t)} (x-h(t))\wedge\sigma(u,p)n\mathrm{d}\Gamma. \end{aligned}$$

Le système complet

Le système que nous étudions principalement dans cette thèse est le suivant :

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, \qquad x\in\mathcal{F}(t), \quad t\in(0,T), \tag{0.1}$$

$$\mathrm{div}\ u = 0, \qquad x\in\mathcal{F}(t), \quad t\in(0,T), \tag{0.2}$$

$$u = 0, \qquad x\in\partial\mathcal{O}, \quad t\in(0,T), \tag{0.3}$$

$$u = h'(t) + \omega(t)\wedge(x-h(t)) + w(x,t), \qquad x\in\partial\mathcal{S}(t), \quad t\in(0,T), \tag{0.4}$$

$$Mh''(t) = -\int_{\partial\mathcal{S}(t)} \sigma(u,p)n\mathrm{d}\Gamma, \qquad t\in(0,T), \tag{0.5}$$

$$(I\omega)'(t) = -\int_{\partial\mathcal{S}(t)} (x-h(t))\wedge\sigma(u,p)n\mathrm{d}\Gamma, \qquad t\in(0,T), \tag{0.6}$$

$$u(y,0) = u_0(y),\ y\in\mathcal{F}(0), \quad h(0) = h_0\in\mathbb{R}^d, \quad h'(0) = h_1\in\mathbb{R}^d, \quad \omega(0) = \omega_0\in\mathbb{R}^3. \tag{0.7}$$

L'inconnue du système est le quadruplet (u,p,h,ω). La vitesse w sur $\partial\mathcal{S}(t)$ est exprimée en fonction de (h,ω) et de la déformation X^* (qui est une donnée), au travers du changement de référentiel suivant :

$$w(x,t) = \mathbf{R}(t)w^*(\mathbf{R}(t)^T(x-h(t)),t), \quad x\in\partial\mathcal{S}(t),$$

où la rotation $\mathbf{R}(t)$ est associée à la vitesse angulaire $\omega(t)$, et où

$$\frac{\partial X^*}{\partial t}(y,t) = w^*(X^*(y,t),t), \quad X^*(y,t) = y - h_0, \quad y\in\mathcal{S}(0).$$

Le moment d'inertie $I(t)$ peut s'exprimer en dimension 2 de la manière suivante :

$$I(t) = \int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0)\left|X^*(y,t)\right|^2\mathrm{d}y,$$

et en dimension 3 de la manière suivante :

$$I(t) = \mathbf{R}(t)\left(\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0)\left(\left|X^*(y,t)\right|^2\mathrm{I}_{\mathbb{R}^3} - X^*(y,t)\otimes X^*(y,t)\right)\mathrm{d}y\right)\mathbf{R}(t)^T.$$

Les domaines $\mathcal{S}(t)$ et $\mathcal{F}(t)$ sont définis par :

$$\mathcal{S}(t) = h(t) + \mathbf{R}(t)X^*(\mathcal{S}(0), t), \qquad \mathcal{F}(t) = \mathcal{O} \setminus \overline{\mathcal{S}(t)}.$$

Ce système couplé constitue un problème avec données initiales et conditions limites. Remarquons que ce système est écrit à l'aide de domaines qui dépendent du temps. Dans un souci de simplicité, nous supposerons souvent que $h_0 = 0$, sans perte de généralité. Les autres données initiales, à savoir u_0, h_1 et ω_0, sont uniquement des vitesses.

0.3 Résultats principaux

Résultat principal du chapitre 1

Dans le premier chapitre nous reconsidérons un résultat établi dans [SMSTT08], à savoir l'existence en dimension 2 de solutions fortes globales en temps pour le système (0.1)–(0.7). Nous adaptons et complètons ce résultat dans le cas de la dimension 3. En particulier, le changement d'inconnues suggéré dans [SMSTT08] (afin de réécrire le système principal en domaines cylindriques) est classique, mais conduirait à des calculs compliqués. Ainsi nous utilisons un changement de variables plus approprié, et nous étudions le système linéarisé qui est associé au système en domaines cylindriques qui résulte de ce changement de variables. Le prix à payer est l'étude d'un système linéaire avec une condition de divergence non homogène. La méthode que nous utilisons afin de prouver l'existence locale de solutions fortes est la même que celle qui est détaillée dans [Tak03]. L'existence globale est obtenue en dimension 3 pour des données petites, comme dans [CT08], à la différence que dans notre cas nous devons quantifier la classe de fonctions à laquelle la vitesse de déformation w^* doit appartenir. Le résultat principal de ce chapitre est le théorème 1.18, que nous énoncons comme suit :

Théorème. *Soit w^* une vitesse Eulérienne assez régulière. Supposons que la solution X^* du problème suivant*

$$\left\{ \begin{array}{rcl} \dfrac{\partial X^*}{\partial t}(y,t) & = & w^*(X^*(y,t),t), \quad y \in \mathcal{S}(0) \\ X^*(y,0) & = & y - h_0, \quad y \in \mathcal{S}(0) \end{array} \right.$$

vérifie les hypothèses **H1**–**H4** *données plus haut. Supposons que* $\mathrm{dist}(\mathcal{S}(0), \partial\mathcal{O}) > 0$*, et que* $u_0 \in \mathbf{H}^1(\mathcal{F})$ *vérifie*

$$\mathrm{div}\ u_0 = 0 \ \text{in } \mathcal{F}, \quad u_0 = 0 \ \text{on } \partial\mathcal{O}, \quad u_0(y) = h_1 + \omega_0 \wedge (y - h_0) \ \text{on } \partial\mathcal{S}.$$

Supposons que $\|w^*\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S}^*(t)))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S}^*(t)))}$, $\|u_0\|_{\mathbf{H}^1(\mathcal{F})}$, $|h_1|_{\mathbb{R}^3}$ *et* $|\omega_0|_{\mathbb{R}^3}$ *sont assez petits. Alors le système* (0.1)–(0.7) *admet une unique solution forte* (u, p, h', ω) *dans*

$$\mathcal{U}(0,\infty;\mathcal{F}(t)) \times \mathrm{L}^2(0,\infty;\mathbf{H}^1(\mathcal{F}(t))) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3).$$

Les espaces fonctionnels mentionnés dans cet énoncé sont définis dans des domaines non cylindriques. Par exemple, on peut définir $\mathcal{U}(0,T;\mathcal{F}(t))$ à l'aide de la norme donnée par :

$$\|u\|^2_{\mathcal{U}(0,T;\mathcal{F}(t))} = \int_0^T \|u(\cdot,t)\|^2_{\mathbf{H}^2(\mathcal{F}(t))}\mathrm{d}t + \int_0^T \left\| \frac{\partial u}{\partial t}(\cdot,t) \right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} \mathrm{d}t + \sup_{t\in(0,T)} \|u(\cdot,t)\|^2_{\mathbf{H}^1(\mathcal{F}(t))}.$$

Dans ce chapitre nous considérons des déformations du solide sans restriction sur leurs régularités. Cela permet facilement de considérer la vitesse Eulérienne w^* comme donnée principale, avec une régularité maximale, et ainsi de définir le changement de variables comme dans [Tak03] ou [SMSTT08].

Résultat principal du chapitre 2

Dans un cadre où la régularité de la déformation du solide est limitée, considérer la vitesse Eulérienne w^* comme la donnée principale n'est plus si évident, puisqu'une telle application est définie sur $\mathcal{S}^*(t)$, c'est-à-dire un domaine qui est directement défini par X^*. Ainsi nous considérons désormais l'application Lagrangienne X^* comme la donnée principale en relation avec la déformation du solide. L'application $X^*(\cdot,t)$ représente la déformation du solide dans son propre référentiel, et est définie sur le domaine fixe $\mathcal{S}(0)$. L'objectif de ce deuxième chapitre est alors de reconsidérer l'étude du chapitre précédent lorsque l'application X^* vérifie

$$\frac{\partial X^*}{\partial t} \in \mathrm{L}^2(0,T;\mathbf{H}^m(\mathcal{S})) \cap \mathrm{H}^1(0,T;\mathbf{H}^1(\mathcal{S})),$$
$$X^*(\cdot,0) = \mathrm{Id}_{\mathcal{S}}, \quad \frac{\partial X^*}{\partial t}(\cdot,0) = 0,$$

où $m \geq 3$ est un entier. Pour le cas limite $m = 3$, l'espace $\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S}))$ peut être obtenu par injection continue à partir de $\mathrm{H}^{3,3/2}(\mathcal{S}\times(0,\infty))$. On définit alors plus généralement l'espace $\mathcal{W}_0^m(0,T;\mathcal{S})$ comme suit :

$$X^* \in \mathcal{W}_0^m(0,T;\mathcal{S}) \Leftrightarrow \begin{cases} \dfrac{\partial X^*}{\partial t} \in \mathrm{H}^{m,m/2}(\mathcal{S}\times(0,T)), \\ \\ X^*(y,0) = y, \ \dfrac{\partial X^*}{\partial t}(y,0) = 0 \quad \forall y \in \mathcal{S}. \end{cases}$$

Le résultat principal de ce deuxième chapitre est le théorème 2.16, qui peut être énoncé de la manière suivante :

Théorème. *Supposons que X^* est assez proche de $\mathrm{Id}_{\mathcal{S}}$ dans $\mathcal{W}_0^m(0,\infty;\mathcal{S})$ - avec $m \geq 3$ - et satisfait les hypothèses* **H1** − **H4** *données plus haut. Supposons que* $\mathrm{dist}(\mathcal{S}(0),\partial\mathcal{O}) > 0$*, et que $u_0 \in \mathbf{H}^1(\mathcal{F})$ vérifie*

$$\mathrm{div}\ u_0 = 0 \ \text{in}\ \mathcal{F}, \quad u_0 = 0 \ \text{on}\ \partial\mathcal{O}, \quad u_0(y) = h_1 + \omega_0 \wedge y \ \text{on}\ \partial\mathcal{S}.$$

Supposons aussi que $\|u_0\|_{\mathbf{H}^1(\mathcal{F})}$, $|h_1|_{\mathbb{R}^3}$ and $|\omega_0|_{\mathbb{R}^3}$ sont assez petites. Alors le système (0.1)–(0.7) *admet une unique solution forte (u,p,h',ω) dans*

$$\mathcal{U}(0,\infty;\mathcal{F}(t)) \times \mathrm{L}^2(0,\infty;\mathbf{H}^1(\mathcal{F}(t))) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3).$$

Pour établir ce résultat, nous définissons d'une nouvelle façon le changement de variables qui permet de réécrire le système principal en des domaines qui ne dépendent pas du temps, et nous vérifions que nous pouvons faire le lien avec le cadre du chapitre 1. L'approche adoptée dans ce chapitre nous permet de faire une transition avec le chapitre suivant, où l'application X^* est choisie comme fonction de contrôle.

Résultat principal du chapitre 3

La partie théorique la plus importante de cette thèse est le chapitre 3, où nous étudions en dimension 2 ou 3 la stabilisation à zéro du système (0.1)–(0.7), en choisissant comme contrôle l'application X^* pour laquelle nous devons imposer des contraintes. D'abord, nous considérons les déplacements $Z^* = X^* - \mathrm{Id}_{\mathcal{S}}$ dans un espace fonctionnel approprié ; pour $\lambda > 0$, nous définissons :

$$Z^* \in \mathcal{W}_\lambda(S^0_\infty) \Leftrightarrow \begin{cases} e^{\lambda t}\dfrac{\partial Z^*}{\partial t} \in \mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S})), \\ \\ Z^*(y,0) = 0, \ \dfrac{\partial Z^*}{\partial t}(y,0) = 0 \quad \forall y \in \mathcal{S}. \end{cases}$$

Le résultat principal de ce chapitre est le théorème 3.24 :

Théorème. *Pour* (u_0, h_1, ω_0) *assez petit dans* $\mathbf{H}^1_{cc}$*, le système* (0.1)–(0.7) *est stabilisable avec un taux de décroissance exponentiel* $\lambda > 0$ *arbitraire, c'est-à-dire qu'il existe une constante* C_0 *strictement positive telle que pour tout* $t \geq 0$ *on a :*

$$\|(u(\cdot,t), h'(t), \omega(t))\|_{\mathrm{L}^2(\mathcal{F}(t))\times\mathbb{R}^d\times\mathbb{R}^3} \quad \leq \quad C_0 e^{-\lambda t}.$$

La constante C_0 *dépend uniquement de* (u_0, h_1, ω_0).

Pour prouver ce résultat, les méthodes utilisées sont fortement inspirées du travail de [Ray10]. Nous distinguons deux grandes parties dans ce chapitre 3. La première consiste à stabiliser le système linéarisé suivant :

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} - \lambda\hat{u} - \mathrm{div}\ \sigma(\hat{u},\hat{p}) &= 0, \quad \text{dans } (0,\infty)\times\mathcal{F}, \\ \mathrm{div}\ \hat{u} &= 0, \quad \text{dans } (0,\infty)\times\mathcal{F}, \end{aligned}$$

$$\begin{aligned} \hat{u} &= 0, && \text{dans } \partial\mathcal{O}\times(0,\infty), \\ \hat{u} &= \hat{h}'(t) + \hat{\omega}(t)\wedge y + \zeta(y,t), && y \in \partial\mathcal{S}, \quad t\in(0,\infty), \end{aligned}$$

$$\begin{aligned} M\hat{h}''(t) &= -\int_{\partial\mathcal{S}} \sigma(\hat{u},\hat{p})n\mathrm{d}\Gamma, && t \in (0,\infty), \\ I_0\hat{\omega}'(t) &= -\int_{\partial\mathcal{S}} y\wedge\sigma(\hat{u},\hat{p})n\mathrm{d}\Gamma, && t\in(0,\infty), \end{aligned}$$

$$\hat{u}(y,0) = u_0(y), \ y\in\mathcal{F}, \quad \hat{h}'(0) = h_1 \in \mathbb{R}^d, \quad \hat{\omega}(0) = \omega_0 \in \mathbb{R}^3,$$

par des fonctions $\zeta = e^{\lambda t}\dfrac{\partial X^*}{\partial t} \in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))$ satisfaisant les contraintes linéarisées qui correspondent aux hypothèses **H1** − **H4**. Pour cela, on prouve d'abord la contrôlabilité approchée de ce système. On introduit le système adjoint associé à ce système linéaire, dont les inconnues sont (ϕ, ψ, k', r), et nous sommes amenés à considérer un problème de continuation unique avec l'égalité suivante :

$$\int_0^T \int_{\partial\mathcal{S}} \zeta\cdot\sigma(\phi,\psi)n\mathrm{d}\Gamma \quad = \quad 0, \tag{0.8}$$

pour tout ζ dans $\in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))$, tel que

$$\int_{\partial\mathcal{S}} \zeta\cdot n\mathrm{d}\Gamma \quad = \quad 0.$$

Dans cette égalité le couple (ϕ, ψ) joue le rôle de la vitesse et de la pression du fluide. À partir du résultat de controllabilité approchée nous obtenons la stabilisation par feedback d'un système linéaire non homogène. Ensuite dans un second temps nous voulons stabiliser le système non linéaire réécrit en domaines cylindriques. Pour cela nous devons d'abord définir des déformations qui satisfont les contraintes non linéaires, à partir de l'opérateur de feedback obtenu précédemment, qui ne vit lui que sur la frontière $\partial\mathcal{S}$. Pour cela nous montrons que ce contrôle frontière peut induire à l'intérieur du solide une déformation X_l^* qui satisfait les contraintes linéarisées. Ensuite nous projetons la déformation X_l^* sur un ensemble correspondant aux contraintes non linéaires. Cette méthode de projection nous permet de considérer un contrôle décomposé en une partie qui stabilise par feedback la composante linéaire du système, et une partie qui a de bonnes propriétés Lipschitz. On conclut la preuve du résultat principal par une méthode de point fixe.

Résultats principaux du chapitre 5

Le contenu de ce chapitre a fait l'objet d'un article qui a été soumis en collaboration avec Michel Fournié et Alexei Lozinski.

L'étude du système principal d'un point de vue contrôlabilité (voir l'équation (0.8)) a souligné l'importance du rôle joué par les forces du fluide sur le bord du solide, à savoir la trace normale du tenseur des contraintes de Cauchy $\sigma(u,p)n$. De plus, cette quantité détermine les expressions données par les lois de Newton. Dans la perspective de simulations numériques, il est crucial de pouvoir obtenir une bonne approximation de cette quantité. Plus précisémment, nous considérons le problème de Stokes suivant avec des conditions de Dirichlet non homogènes :

$$\begin{aligned} -\nu\Delta u + \nabla p &= f \quad \text{dans } \mathcal{F}, \\ \operatorname{div} u &= 0 \quad \text{dans } \mathcal{F}, \\ u &= 0 \quad \text{sur } \partial\mathcal{O}, \\ u &= g \quad \text{sur } \partial\mathcal{S}. \end{aligned}$$

Ce problème est relativement simple, comparé au système fluide-solide complet. Cela nous permet de mettre en lumière les principaux aspects du problème, à savoir : prendre en compte des conditions aux bords non homogènes (correspondant à l'égalité des vitesses à l'interface fluide-structure), considérer des frontières à géométrie quelconque (car dans notre cas la géométrie du solide est inconnue), et obtenir une bonne approximation de la trace normale du tenseur des contraintes de Cauchy.
Pour cela, nous adaptons au problème de Stokes la méthode de domaines fictifs initialement introduite pour le problème de Poisson dans [HR09], et qui est basée sur les idées de la méthode Xfem. Cette approche nous permet d'effectuer des simulations pour des domaines dont la frontière ne dépend pas du maillage. Une technique de stabilisation numérique est réalisée dans le but de recouvrer la convergence pour la quantité $\sigma(u,p)n$, et une condition inf-sup est prouvée théoriquement pour le problème stabilisé discret. Des calculs de taux de convergence sont réalisés, et soulignent les intérêts de la méthode.

0.4 Quelques mots sur les changements de variables

Remarquons que la formulation la plus appropriée pour décrire l'état du fluide est la formulation Eulérienne, alors que celle qui est la plus appropriée pour l'évolution d'un solide est la formulation Lagrangienne. Le point de vue que nous adoptons est celui qui correspond au contrôle, c'est-à-dire que nous préférons manipuler des applications Lagrangiennes pour le solide comme pour le fluide. Un premier travail consiste alors à effectuer un changement d'inconnues pour les grandeurs écrites en formalisme Eulérien, à savoir la vitesse et la pression du fluide :

$$u(\cdot,t) \ : \ \mathcal{F}(t) \longrightarrow \mathbb{R}^d, \qquad p(\cdot,t) \ : \ \mathcal{F}(t) \longrightarrow \mathbb{R}.$$

Pour réécrire ces inconnues dans des domaines qui ne dépendent pas du temps, nous choisissons d'abord une configuration de référence, la configuration qui correspond à la géométrie à l'état initial, c'est-à-dire celle donnée par $\mathcal{F}(0)$ et $\mathcal{S}(0)$. Nous cherchons alors à définir un changement de variables :

$$X(\cdot,t) \ : \ \mathcal{F}(0) \longrightarrow \mathcal{F}(t).$$

Pour la partie solide, le changement de variables choisi est bien sûr celui induit par la déformation du solide, à savoir l'application

$$X_{\mathcal{S}}(\cdot,t) \ : \ \mathcal{S}(0) \longrightarrow \mathcal{S}(t).$$

Pour la partie fluide, nous construisons des changements de variables $X(\cdot,t)$ qui prolongent $X_{\mathcal{S}}(\cdot,t)$ au reste du domaine $\mathcal{F}(0) = \mathcal{O} \setminus \overline{\mathcal{S}(0)}$, et qui ne dépendent pas de certaines inconnues, comme la vitesse du fluide par exemple. En effet, un changement de variables qui pourrait être induit par la vitesse du fluide u ne serait *a priori* pas pratique : par exemple la régularité d'un tel changement de variables peut être directement limitée par celle de u, et u est une fonction qui n'a un sens que sur le domaine $\mathcal{F}(t)$ qui dépend du temps.
Les changements de variables que nous construisons dans cette thèse pour la partie fluide n'ont pas de sens physique, mais nous permettent de définir des changements d'inconnues qui ont de bonnes propriétés. Expliquons comment nous définissons une application $X(\cdot,t)$, qui est un C^1-difféomorphisme de $\mathcal{F}(0)$ sur $\mathcal{F}(t)$, et qui vérifie

$$\begin{cases} \det\nabla X = 1, & \text{dans } \mathcal{F}\times(0,\infty),\\ X = X_{\mathcal{S}}, & \text{sur } \partial\mathcal{S}\times(0,\infty),\\ X = \mathrm{Id}_{\partial\mathcal{O}}, & \text{sur } \partial\mathcal{O}\times(0,\infty). \end{cases}$$

Rappelons que l'application $X_{\mathcal{S}}$ représente la déformation du solide dans le référentiel Galiléen, et est décomposée comme suit :

$$X_{\mathcal{S}}(y,t) \quad = \quad h(t) + \mathbf{R}(t)X^*(y,t), \quad y \in \mathcal{S}(0),$$

où X^* représente la déformation du solide dans propre référentiel.

Le changement de variables dans le chapitre 1

Lorsque nous ne supposons pas de restriction sur la régularité de l'application X^* - comme dans le chapitre 1 - nous pouvons associer à cette dernière une vitesse Eulérienne w^*, pouvant être définie par

$$w^*(x^*,t) \quad = \quad \frac{\partial X^*}{\partial t}(Y^*(x^*,t),t), \quad x^* \in \mathcal{S}^*(t) = X^*(\mathcal{S}(0),t),$$

et qui a un maximum de régularité en espace. Il est alors facile de prolonger cette vitesse Eulérienne en une vitesse $\overline{w}^*$ à tout le domaine $\mathcal{O}$. Pour cela, nous considérons le problème classique de Dirichlet suivant :

$$\begin{cases} \operatorname{div} \overline{w}^* = 0 & \text{dans } \mathbb{R}^3 \setminus \overline{\mathcal{S}^*(t)},\ t \in (0,T), \\ \overline{w}^*(x^*,t) = 0 & \text{si } \operatorname{dist}(x^*, \mathcal{S}^*(t)) \geq \eta > 0,\ t \in (0,T), \\ \overline{w}^*(x^*,t) = w^*(x^*,t) & \text{si } x^* \in \mathcal{S}^*(t),\ t \in (0,T). \end{cases}$$

Alors un prolongement de $X^*(\cdot,t)$ à $\mathbb{R}^3 \setminus \mathcal{S}(0)$, noté $\overline{X}^*(\cdot,t)$, peut être obtenu comme la solution du problème suivant :

$$\frac{\partial \overline{X}^*}{\partial t}(y,t) = \overline{w}^*(\overline{X}^*(y,t),t), \quad \overline{X}^*(y,0) = y - h_0, \quad y \in \mathcal{F}(0).$$

Ensuite on définit grâce à un changement de référentiel la fonction suivante :

$$\overline{w}(x,t) \quad = \quad \mathbf{R}(t)\overline{w}^*(\mathbf{R}(t)^T(x-h(t)),t), \quad x \in \mathcal{F}(t).$$

Enfin, l'application X recherchée peut être obtenue comme solution du problème

$$\frac{\partial X}{\partial t}(y,t) = \Lambda(X(y,t),t), \quad X(y,0) = y, \quad y \in \mathcal{F}(0),$$

où Λ est définie à l'aide d'une fonction de troncature ξ (qui vaut 1 dans un voisinage de $\partial\mathcal{S}(0)$, et 0 dans un voisinage de $\partial\mathcal{O}$) et d'un vecteur potentiel $\mathfrak{F}_R$ tel que $\operatorname{curl}(\mathcal{F}_R)(x,t) = h'(t) + \omega(t) \wedge (x - h(t))$, comme suit :

$$\Lambda \quad = \quad \operatorname{curl}(\xi \mathfrak{F}_R) + \overline{w},$$

Notons que Λ vérifie

$$\begin{cases} \Lambda = 0 \text{ sur } \mathbb{R}^3 \setminus \mathcal{O}, \\ \operatorname{div} \Lambda = 0 \text{ dans } \mathbb{R}^3 \setminus \overline{\mathcal{S}(t)}, \\ \Lambda(x,t) = h'(t) + \omega \wedge (x - h(t)) + w(x,t) \text{ si } x \in \mathcal{S}(t). \end{cases}$$

Le changement de variables dans le chapitre 2

Dans le chapitre 2, l'application Lagrangienne X^* est considérée comme la principale donnée en relation avec la déformation du solide. Elle est limitée en régularité. Au lieu d'obtenir les prolongements des applications Lagrangiennes $X^*(\cdot,t)$ et $X_{\mathcal{S}}(\cdot,t)$ en prolongeant d'abord des vitesses Eulériennes $w^*(\cdot,t)$ et $w(\cdot,t)$ (qui deviendraient bien plus délicates à définir), nous prolongeons directement $X^*(\cdot,t)$ à tout le domaine $\mathcal{O}$. Pour cela, nous résolvons le problème suivant :

$$\begin{cases} \det \nabla \tilde{X} = 1 & \text{dans } \mathcal{F} \times (0,T), \\ \tilde{X} = X^* & \text{sur } \partial\mathcal{S} \times (0,T), \\ \tilde{X}(y,t) = \mathbf{R}(t)^T(y - h(t)) & (y,t) \in \partial\mathcal{O} \times (0,T). \end{cases}$$

Ce problème tient compte de toutes les propriétés recherchées, puisque nous pouvons ensuite directement poser

$$X(y,t) \quad = \quad h(t) + \mathbf{R}(t)\tilde{X}(y,t).$$

Un tel prolongement pour l'application X^* est obtenu seulement pour T assez petit. Par rapport à la méthode choisie dans le chapitre 1 pour définir l'application $X(\cdot,t)$ comme prolongement de

$X_{\mathcal{S}}(\cdot,t)$, cette méthode est plus directe. De plus elle évite de manipuler des vitesses telles que $w^*(\cdot,t)$ qui sont définies sur des domaines qui dépendent du temps, et donc pour lesquels le cadre fonctionnel n'est pas standard. Enfin, la dépendance du prolongement $X(\cdot,t)$ ainsi obtenue vis-à-vis des données X^*, h et $\mathbf{R}$ est plus facile à obtenir. Quantifier cette dépendance est essentielle dans le chapitre 3, où la déformation X^* est choisie à l'aide d'un opérateur feedback, et donc dépend entièrement des inconnues du problème.

Le changement de variables dans le chapitre 3

Le but de ce chapitre est d'obtenir la stabilisation exponentielle des vitesses du fluide et du solide, en horizon de temps infini. Ainsi, l'approche du chapitre 2, pour prolonger l'application Lagrangienne X^*, ne peut être reconduite dans cette optique, puisqu'elle nous conduirait en particulier à supposer les vitesses inconnues h' et ω (utilisée comme données pour définir le prolongement de X^*) arbitrairement petites dans $\mathrm{H}^1(0,\infty;\mathbb{R}^d)$ et $\mathrm{H}^1(0,\infty;\mathbb{R}^3)$ respectivement. Ainsi, pour définir le changement de variables X, nous procédons en utilisant les idées des chapitres 1 et 2. Plus précisémment, nous prolongeons d'abord l'application Lagrangienne $X^*(\cdot,t)$ en résolvant plutôt le système

$$\left\{\begin{array}{ll} \det\nabla\overline{X}^* = 1 & \text{dans } \mathcal{F}\times(0,\infty), \\ \overline{X}^* = X^* & \text{sur } \partial\mathcal{S}\times(0,\infty), \\ \overline{X}^*(y,t) = y - h_0 & (y,t)\in\partial\mathcal{O}\times(0,\infty). \end{array}\right.$$

Notons qu'un tel prolongement $\overline{X}^*(\cdot,t)$ a des propriétés semblables à celles du prolongement obtenu dans le chapitre 1. Après cela, nous définissons un prolongement $\overline{X}^R$ de l'application $y\mapsto h(t)+\mathbf{R}(t)y$, obtenue comme solution du problème suivant :

$$\frac{\partial\overline{X}^R}{\partial t}(\overline{x}^*,t) = \mathrm{curl}(\xi\mathfrak{F}_R)(\overline{X}^R(\overline{x}^*,t),t), \quad \overline{X}^R(\overline{x}^*,0) = \overline{x}^*, \quad \overline{x}^R \in \mathcal{O}\setminus\overline{X}^*(\mathcal{S}(0),t).$$

Le changement de variables finalement utilisé pour réécrire le système principal en domaines cylindriques est donné en posant

$$X(y,t) \quad = \quad \overline{X}^R(\overline{X}^*(y,t),t), \quad y\in\mathcal{F}(0).$$

0.5 Plan de la thèse

Dans un premier chapitre, nous adaptons et complétons en dimension 3 l'étude effectuée en dimension 2 dans l'article de [SMSTT08]. En particulier, nous choisissons un changement d'inconnues approprié, et nous étudions le système linéarisé associé au système vérifié par les nouvelles inconnues. Cela nous conduit à un résultat d'existence locale de solutions fortes (pour des déformations du solide régulières), et l'existence globale est obtenue pour des données petites, en particulier pour des vitesses de déformation du solide assez petites, dans une classe de fonctions que nous précisons.
Dans le chapitre 2, nous étendons le travail effectué dans le chapitre 1 au cas de déformations solides limitées en régularité. Nous proposons une nouvelle façon de définir le changement de variables, qui s'avère plus appropriée lorsque la déformation du solide est vue principalement au travers d'une application Lagrangienne. Nous vérifions alors que les résultats obtenus dans le premier chapitre sont encore vrais dans ce cadre. Ce chapitre 2 peut être vu comme une transition vers le chapitre suivant où la déformation du solide - limitée en régularité - est choisie comme fonction de contrôle.
Ainsi dans le chapitre 3 nous établissons un résultat de stabilisation pour le système fluide-solide. Ce chapitre peut être divisé en deux grandes parties. Dans une première partie nous prouvons que le système linéarisé est stabilisable à zéro par des vitesses définies sur $\partial\mathcal{S}$. Ces vitesses peuvent être obtenues à partir de déformations du solide vérifiant un ensemble de contraintes physiques linéarisées. La seconde partie de ce chapitre est consacrée à la preuve de la stabilisation du système non linéaire complet, par une méthode de point fixe. Afin de prendre en considération les contraintes non linéaires imposées, nous considérons une décomposition du contrôle ; la première partie de cette décomposition vérifie les contraintes linéarisées, et est choisie à l'aide d'un opérateur de feedback qui permet de stabiliser la composante linéaire du système, alors que le terme résiduel de cette décomposition a de bonnes propriétés Lipschitz.
Dans le chapitre 4 nous proposons des moyens pratiques de décrire des déformations d'un solide, en agissant seulement sur quelques paramètres restreints, au lieu de considérer des déformations générales comme dans le chapitre 3. La façon dont nous obtenons ces déformations particulières est censée modéliser la locomotion d'animaux munis d'une colonne vertébrale sur laquelle ils peuvent agir pour se déformer.
Enfin dans le chapitre 5 nous développons une méthode numérique qui nous permet d'obtenir une bonne approximation de la trace normale du tenseur des contraintes de Cauchy, pour des frontières qui ne dépendent pas du maillage. Cette méthode combine une approche de type *domaines fictifs* basée sur les idées de Xfem, et une méthode de Lagrangien augmenté. Du point de vue des interactions fluide-structure, l'intérêt de cette méthode réside dans l'importance du rôle joué par les forces du fluide à l'interface fluide-solide.

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Introduction - English version

0.6 Motivation

In fluid-structure interactions, we are interested in studying the phenomena which are at stake when an animal organism moves itself in a fluid. The underlying problems are about the swim of a deformable structure in a liquid: How this structure can deform itself in order to swim in the surrounding fluid? We can take an interest, for instance, to the way a fish proceeds in order to swim in water. Let us enunciate some questions related to these problems: How does the fluid act on the structure? How does the solid feel the fluid's forces? How can it react in order to play with these forces? How does a boundary velocity influence the fluid's behavior?
These questions deal with physical problems that the study of mathematical models can help to understand.
At low Reynolds number, the influence of the inertia phenomena is negligible so that the swimmer can only use the fluid's viscosity and the variations of shapes in order to move itself. For this situation, we can take as an example some microorganisms like the spermatozoon. At the opposite, the swim of a shark corresponds to a high Reynolds number; In this situation the inertia phenomena are paramount, and the exchanges of momenta between the shark and the fluid seem *a priori* to constitute the main part of the process which enables the shark to swim. At an intermediate level, the swim of a tadpole for instance can be made with the help of two main phenomena, that is to say the inertia of the tadpole, and the influence that can have the fluid's viscosity on its skin.
The study of such problems is quite complicated, from a physical point of view as well as from a mathematical point of view. That is why we choose in this thesis to study mainly a single model which aims to bring a mathematical look on the swim at an intermediate Reynolds number. The underlying physical problems constitute only a source of inspiration, and our contribution claims to be modest. Indeed, we only consider small displacements, that is to say that only structure's deformations which are arbitrarily close to what corresponds to a rigid structure will be considered.

0.7 The model

In a bounded domain $\mathcal{O}$ of $\mathbb{R}^2$ or $\mathbb{R}^3$, let us consider a deformable solid immersed in a fluid. The solid fulfills at time t a domain denoted by $\mathcal{S}(t)$, and the fluid fulfills the domain $\mathcal{F}(t) = \mathcal{O} \setminus \overline{\mathcal{S}(t)}$. The framework is the one of continuum mechanics.

$\mathcal{O} = \mathcal{F}(t) \cup \overline{\mathcal{S}(t)} \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

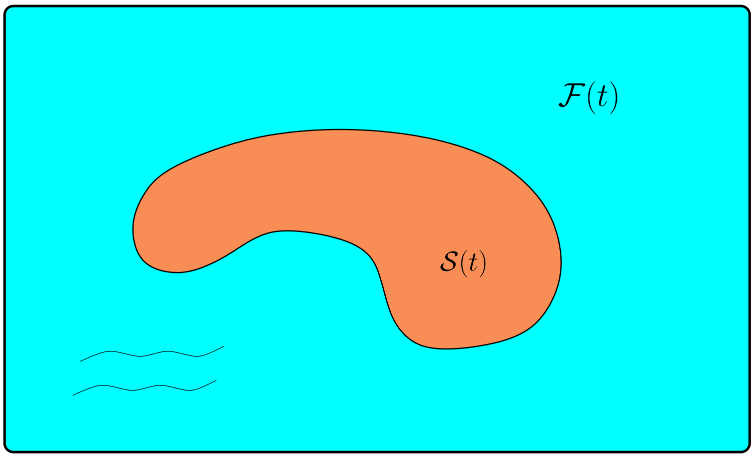

Figure 3: A deformable solid immersed in a fluid

The density

In continuum mechanics, the density (for the fluid or the solid) satisfies the conservation of mass principle, that we write in the local form, in Eulerian representation, as follows

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) = 0.$$

In this equation, ρ denotes the density and u the Eulerian velocity field. We can associate to this velocity the Lagrangian flow X, solution of the following Cauchy problem

$$\left\{ \begin{array}{rcl} \dfrac{\partial X}{\partial t}(y,t) & = & u(X(y,t),t), \\ X(y,0) & = & y. \end{array} \right.$$

At some time t, the mapping $y \mapsto X(y,t)$ has to be invertible. The conservation of mass principle rewritten in Lagrangian coordinates is nothing else than

$$\frac{\partial}{\partial t}\Big(\rho(X(y,t),t)\mathrm{det}\nabla X(y,t)\Big) = 0,$$

which leads us to the following formula

$$\rho(X(y,t),t) = \frac{\rho(y,0)}{\mathrm{det}\nabla X(y,t)}.$$

The role of the density in this thesis is absolutely minor. Indeed, it is considered constant in the fluid, and the role that it plays in the solid is occulted by the fact that we only consider small displacements, so that the solid's density is seen significantly only through its initial value. Moreover, this initial value will be assumed to be constant, for a sake of simplicity.

The fluid

The fluid is assumed to be incompressible, that is to say that its density is constant in time and space variables. It is equivalent to assume that the fluid's velocity u satisfies the homogeneous divergence condition

$$\text{div } u = 0.$$

The functions of state for the fluid are its velocity field u and its pressure p. At an intermediate Reynolds number, the couple (u, p) is assumed to satisfy the Navier-Stokes equation

$$\frac{\partial \rho u}{\partial t} + \text{div}(\rho u \otimes u) - \text{div}\sigma(u,p) \quad = \quad 0.$$

The expression of the Cauchy stress tensor $\sigma(u,p)$ is the following

$$\sigma(u,p) \quad = \quad 2\nu D(u) - p\mathrm{I}_{\mathbb{R}^d} = \nu\left(\nabla u + \nabla u^T\right) - p\mathrm{I}_{\mathbb{R}^d},$$

where ν denotes the kinematic viscosity of the fluid. In adding the homogeneous divergence condition written above, we fix the density ρ constant equal to 1, and the system so obtained is called *the incompressible Navier-Stokes equations*:

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = 0, \qquad & \text{in} \quad \mathcal{F}(t) \\ \text{div } u = 0, \qquad & \text{in} \quad \mathcal{F}(t). \end{aligned}$$

The pressure p, as it appears in this system, can be considered - from a mathematical point of view - as a Lagrange multiplier associated to the incompressibility constraint. This function is then determined up to a constant.

The solid, its deformation chosen as a control function, and its constraints

For the model considered in this thesis, we will see that an important part of the interaction between the fluid and the solid is located at the interface. However, the other important part of the coupling is due to the exchanges of momenta. Let us describe the quantities related to the inertia of the solid. First, the mass of the whole solid is assumed to be constant:

$$M \quad = \quad \int_{\mathcal{S}(t)} 1 \mathrm{d}x.$$

Its moment of inertia is expressed as follows

$$\begin{aligned} I(t) = \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)|x|^2 \mathrm{d}x \qquad & \text{in dimension 2,} \\ I(t) = \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)\left(|x|^2\mathrm{I}_{\mathbb{R}^d} - x\otimes x\right)\mathrm{d}x \qquad & \text{in dimension 3.} \end{aligned}$$

It is a scalar function in dimension 2, and a matrix function in dimension 3. When the solid is deforming itself, the moment of inertia depends on time in a nontrivial way (whereas the dependence is merely expressed in terms of rotations in the rigid case). But after linearization for small displacements, only the moment of inertia at time $t = 0$ has a significant role.
From these two quantities we can define $h(t)$, the coordinates of the solid's center of mass, and $\omega(t)$, its angular velocity, as follows

$$\begin{aligned} Mh(t) \quad &= \quad \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t) x \mathrm{d}x, \\ I(t)\omega(t) \quad &= \quad \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)(x - h(t)) \wedge (u_{\mathcal{S}}(x,t) - h'(t))\mathrm{d}x. \end{aligned}$$

The angular velocity $\omega(t)$ induces a rotation $\mathbf{R}(t)$, which can be determined in dimension 3 in solving

$$\left\{\begin{array}{rcl}\dfrac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} & = & \mathbb{S}\left(\omega\right)\mathbf{R} \\ \mathbf{R}(0) & = & \mathrm{I}_{\mathbb{R}^3},\end{array}\right. \quad \text{with } \mathbb{S}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

The movement of the solid, in an inertial frame of reference, is represented by a Lagrangian mapping $X_{\mathcal{S}}$, which then defines

$$\mathcal{S}(t) = X_{\mathcal{S}}(\mathcal{S}(0), t)$$

at some time t. We can decompose this mapping as follows

$$X_{\mathcal{S}}(\cdot, t) \quad = \quad h(t) + \mathbf{R}(t)X^*(\cdot, t).$$

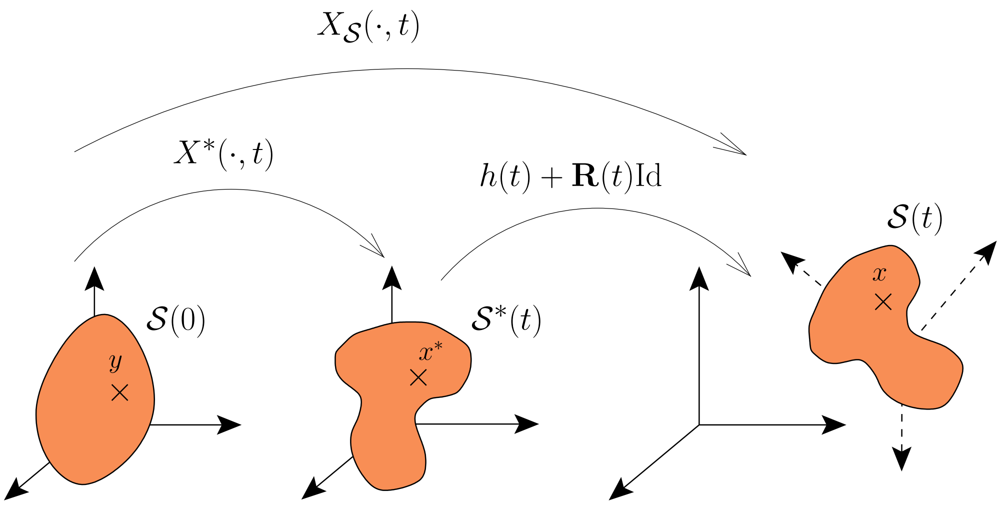

Figure 4: Decomposition of the solid's movement

The rigid transformation $y \mapsto h(t) + \mathbf{R}(t)y$ determines the position and the orientation of the whole solid. It is an unknown of the problem. The mapping $X^*(\cdot, t)$ can be seen as the solid's deformation in its own frame of reference, that is to say a quantity which represents its shape. We assume it as given. This mapping has to satisfy a set of constraints that we sum up below:

H1 For all $t \geq 0$, we assume that the mapping $X^*(\cdot, t)$ is a C^1-diffeomorphism from $\overline{\mathcal{S}(0)}$ onto $\overline{\mathcal{S}^*(t)}$. This particularly enables us to conserve the regularity and the injectivity of the initial solid's boundary $\partial\mathcal{S}(0)$ through the time.

H2 Since the fluid is assumed to be incompressible, and since the whole domain $\mathcal{O}$ occupied by the fluid and the solid is bounded and immobile, the solid's volume has to be constant through the time. It leads us to assume that

$$\int_{\partial\mathcal{S}(0)} \frac{\partial X^*}{\partial t} \cdot \left(\mathrm{cof}\nabla X^*\right) n \mathrm{d}\Gamma \quad = \quad 0.$$

H3 The deformation does not modify the linear momentum of the solid:

$$\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t) \mathrm{d}y \quad = \quad 0.$$

H4 The deformation does not modify its angular momentum:

$$\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t) \wedge \frac{\partial X^*}{\partial t}(y,t) \mathrm{d}y \quad = \quad 0.$$

The hypotheses **H1** and **H2** are invariant by rotations and translations. The hypotheses **H3** and **H4** guarantee the *self-propelled* nature of the solid, that is to say that the solid does not receive any “exterior”help (like the traction of a rope, or the power of some chemical propulsions for instance) in order to propel itself in the fluid. The only means it can use is the reaction of the environing fluid, a reaction that it causes by deforming itself.
The deformation of the solid, imposed through the mapping X^*, is chosen as a control function in Chapter 3. It implies that the solid is strong enough to impose its own shape, and thus to counterbalance the fluid's forces which could push on the solid's boundary and so modify its shape. Such effects due to the fluid can however be neglected, or even be non-existent, if the internal friction forces are strong enough inside the solid.

If we denote $Y^*(\cdot,t)$ the inverse mapping of $X^*(\cdot,t)$, we define the velocity

$$w^*(x^*,t) \quad = \quad \frac{\partial X^*}{\partial t}(Y^*(x^*,t),t), \quad x^* \in \mathcal{S}^*(t).$$

If we denote $Y_{\mathcal{S}}(\cdot,t)$ the inverse mapping of $X_{\mathcal{S}}(\cdot,t)$, the Eulerian velocity which is associated with this solid's deformation in the inertial frame of reference is defined by

$$u_{\mathcal{S}}(x,t) \quad = \quad \frac{\partial X_{\mathcal{S}}}{\partial t}(Y_{\mathcal{S}}(x,t),t), \quad x \in \mathcal{S}^*(t),$$

and can be decomposed as follows

$$u_{\mathcal{S}}(x,t) \quad = \quad h'(t) + \omega(t) \wedge (x - h(t)) + w(x,t), \quad x \in \mathcal{S}(t), \quad x \in \mathcal{S}(t),$$

where the velocity w is defined from w^* via the change of frame

$$w(x,t) \quad = \quad \mathbf{R}(t) w^*(\mathbf{R}(t)^T (x - h(t)), t).$$

Let us keep in mind that we denote u the fluid's velocity. The equality of the velocities at the fluid-solid interface is written as

$$u(x,t) \quad = \quad u_{\mathcal{S}}(x,t), \quad x \in \partial\mathcal{S}(t).$$

This constitutes a nonhomogeneous Dirichlet boundary condition for the fluid, on which we add an homogeneous one on $\partial\mathcal{O}$. The response of the fluid is the normal trace on $\partial\mathcal{S}(t)$ of the Cauchy stress tensor $\sigma(u,p)$, that is to say $\sigma(u,p)n$. It corresponds to the forces that the fluid applies on the solid's boundary. The unknown rigid part of the solid's deformation, given by $h(t)$ and $\omega(t)$, is then related to these forces through the Newton's laws:

$$\begin{aligned} M h''(t) \quad &= \quad -\int_{\partial\mathcal{S}(t)} \sigma(u,p) n \mathrm{d}\Gamma, \\ (I\omega)'(t) \quad &= \quad -\int_{\partial\mathcal{S}(t)} (x - h(t)) \wedge \sigma(u,p) n \mathrm{d}\Gamma. \end{aligned}$$

The complete system

The system which is mainly studied in this thesis is the following:

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T), \tag{0.9}$$

$$\mathrm{div}\ u = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T), \tag{0.10}$$

$$u = 0, \qquad x \in \partial\mathcal{O}, \quad t \in (0,T), \tag{0.11}$$

$$u = h'(t) + \omega(t)\wedge(x - h(t)) + w(x,t), \qquad x \in \partial\mathcal{S}(t), \quad t \in (0,T), \tag{0.12}$$

$$Mh''(t) = -\int_{\partial\mathcal{S}(t)} \sigma(u,p)n\mathrm{d}\Gamma, \qquad t \in (0,T), \tag{0.13}$$

$$(I\omega)'(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t))\wedge\sigma(u,p)n\mathrm{d}\Gamma, \qquad t \in (0,T), \tag{0.14}$$

$$u(y,0) = u_0(y),\ y \in \mathcal{F}(0), \quad h(0) = h_0 \in \mathbb{R}^3, \quad h'(0) = h_1 \in \mathbb{R}^3, \quad \omega(0) = \omega_0 \in \mathbb{R}^3. \tag{0.15}$$

The unknown of the system is the quadruplet (u,p,h,ω). The velocity w on $\partial\mathcal{S}(t)$ is expressed in terms of (h,ω) and the deformation X^* (which is given), through the following change of frame

$$w(x,t) \quad = \quad \mathbf{R}(t)w^*(\mathbf{R}(t)^T(x-h(t)),t), \quad x \in \partial\mathcal{S}(t),$$

where the rotation $\mathbf{R}(t)$ is associated with the angular velocity $\omega(t)$, and where

$$\frac{\partial X^*}{\partial t}(y,t) = w^*(X^*(y,t),t), \quad X^*(y,t) = y - h_0, \quad y \in \mathcal{S}(0).$$

The inertia moment $I(t)$ can be expressed in dimension 2 as follows

$$I(t) = \int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0)\left|X^*(y,t)\right|^2 \mathrm{d}y,$$

and in dimension 3 as follows

$$I(t) = \mathbf{R}(t)\left(\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0)\left(\left|X^*(y,t)\right|^2 \mathrm{I}_{\mathbb{R}^3} - X^*(y,t)\otimes X^*(y,t)\right)\mathrm{d}y\right)\mathbf{R}(t)^T.$$

The domains $\mathcal{S}(t)$ and $\mathcal{F}(t)$ are defined by

$$\mathcal{S}(t) = h(t) + \mathbf{R}(t)X^*(\mathcal{S}(0),t), \qquad \mathcal{F}(t) = \mathcal{O}\setminus\overline{\mathcal{S}(t)}.$$

This coupled system constitutes an initial and boundary value problem. Notice that this system is written in time-depending domains. For a sake of simplicity, we will often assume that $h_0 = 0$, without loss of generality. The other initial data, that is to say u_0, h_1 and ω_0, are only velocities.

0.8 Main results

Main result of Chapter 1

In the first chapter we reconsider a result investigated in [SMSTT08], that is to say the existence in dimension 2 of global strong solutions for system (0.9)–(0.15). We adapt and complete this result in the case of dimension 3. In particular, the change of unknowns suggested in [SMSTT08] (in order to rewrite the main system in cylindrical domains) is classical, but would lead to unappropriate complicated calculations. Thus we use a more suitable change of unknowns, and we study the linearized system associated with the system which results from this change of variables. The price to pay is the study of a linear system with a nonhomogeneous divergence condition. The method we utilize in order to prove the local existence of strong solutions is the same as the one detailed in [Tak03]. The global existence is obtained in dimension 3 for small initial data, as in [CT08], with the difference that in our case we have to quantify the class of functions that the velocity of deformation w^* has to lie in. The main result of this chapter is Theorem 1.18, that we state as follows:

Theorem. *Let w^* be a regular deformation. Assume that the solution X^* of the following problem*

$$\begin{cases} \dfrac{\partial X^*}{\partial t}(y,t) & = & w^*(X^*(y,t),t), \quad y \in \mathcal{S}(0) \\ X^*(y,0) & = & y - h_0, \quad y \in \mathcal{S}(0) \end{cases}$$

satisfies the hypotheses **H1**–**H4** *given above. Assume that* $\mathrm{dist}(\mathcal{S}(0), \partial\mathcal{O}) > 0$*, and that* $u_0 \in \mathbf{H}^1(\mathcal{F})$ *satisfies*

$$\mathrm{div}\ u_0 = 0 \text{ in } \mathcal{F}, \quad u_0 = 0 \text{ on } \partial\mathcal{O}, \quad u_0(y) = h_1 + \omega_0 \wedge (y - h_0) \text{ on } \partial\mathcal{S}.$$

Assume that $\|w^*\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S}^*(t)))\cap \mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S}^*(t)))}$, $\|u_0\|_{\mathbf{H}^1(\mathcal{F})}$, $|h_1|_{\mathbb{R}^3}$ *and* $|\omega_0|_{\mathbb{R}^3}$ *are small enough. Then system* (0.9)–(0.15) *admits a unique strong solution* (u, p, h', ω) *in*

$$\mathcal{U}(0,\infty;\mathcal{F}(t)) \times \mathrm{L}^2(0,\infty;\mathbf{H}^1(\mathcal{F}(t))) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3).$$

The functional spaces mentioned in this statement are defined in non-cylindrical domains. For instance, we can define $\mathcal{U}(0,T;\mathcal{F}(t))$ with the norm given by

$$\|u\|^2_{\mathcal{U}(0,T;\mathcal{F}(t))} = \int_0^T \|u(\cdot,t)\|^2_{\mathbf{H}^2(\mathcal{F}(t))} \mathrm{d}t + \int_0^T \left\| \frac{\partial u}{\partial t}(\cdot,t) \right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} \mathrm{d}t + \sup_{t\in(0,T)} \|u(\cdot,t)\|^2_{\mathbf{H}^1(\mathcal{F}(t))}.$$

In this chapter we consider solid's deformations without any restriction on their regularities. This enables us easily to consider the Eulerian velocity w^* as the main datum, with the maximum desired regularity, and thus to define the change of variables as in [Tak03] or [SMSTT08].

Main result of Chapter 2

In a framework where the regularity of the solid's deformation is limited, considering the Eulerian velocity w^* as the main datum is no more so obvious, since such a mapping is defined on $\mathcal{S}^*(t)$, that is to say a domain which is directly defined by X^*. Thus we now consider the Lagrangian mapping X^* as the main datum in relation with the deformation of the solid. This mapping represents the solid's deformation in its own frame of reference, and is defined on the fixed domain $\mathcal{S}(0)$. The goal of the second chapter is then to reconsider the study of the previous chapter when the mapping X^* satisfies

$$\frac{\partial X^*}{\partial t} \in \mathrm{L}^2(0,T;\mathbf{H}^m(\mathcal{S})) \cap \mathrm{H}^1(0,T;\mathbf{H}^1(\mathcal{S})),$$
$$X^*(\cdot,0) = \mathrm{Id}_{\mathcal{S}}, \quad \frac{\partial X^*}{\partial t}(\cdot,0) = 0,$$

where $m \geq 3$ is an integer. For the limit case $m = 3$, the space $\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S}))$ can be obtained by interpolation from $\mathrm{H}^{3,3/2}(\mathcal{S}\times(0,\infty))$. We then define more generally the space $\mathcal{W}_0^m(0,T;\mathcal{S})$ as follows

$$X^* \in \mathcal{W}_0^m(0,T;\mathcal{S}) \Leftrightarrow \begin{cases} \dfrac{\partial X^*}{\partial t} \in \mathrm{H}^{m,m/2}(\mathcal{S}\times(0,T)), \\ X^*(y,0) = y, \ \dfrac{\partial X^*}{\partial t}(y,0) = 0 \quad \forall y \in \mathcal{S}. \end{cases}$$

The main result of this second chapter is Theorem 2.16, which can be stated as follows:

Theorem. *Assume that X^* is close enough to $\mathrm{Id}_{\mathcal{S}}$ in $\mathcal{W}_0^m(0,\infty;\mathcal{S})$ - with $m \geq 3$ - and satisfies the hypotheses* **H1** $-$ **H4** *given above. Assume that $\mathrm{dist}(\mathcal{S}(0),\partial\mathcal{O}) > 0$, and that $u_0 \in \mathbf{H}^1(\mathcal{F})$ satisfies*

$$\mathrm{div}\ u_0 = 0 \ \text{in } \mathcal{F}, \quad u_0 = 0 \ \text{on } \partial\mathcal{O}, \quad u_0(y) = h_1 + \omega_0 \wedge y \ \text{on } \partial\mathcal{S}.$$

Assume also that $\|u_0\|_{\mathbf{H}^1(\mathcal{F})}$, $|h_1|_{\mathbb{R}^3}$ and $|\omega_0|_{\mathbb{R}^3}$ are small enough. Then system (0.9)–(0.15) *admits a unique strong solution (u,p,h',ω) in*

$$\mathcal{U}(0,\infty;\mathcal{F}(t)) \times \mathrm{L}^2(0,\infty;\mathbf{H}^1(\mathcal{F}(t))) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3).$$

For setting this result, we introduce a new means for defining the change of variables which enables us to rewrite the main system in non-depending time domains, and we verify that we can make the connection with the framework of Chapter 1. The approach adopted in this chapter enables us to make a transition to the next chapter, where the mapping X^* is chosen as a control function.

Main result of Chapter 3

The most important theoretical part of this thesis is Chapter 3, where we study in dimension 2 or 3 the stabilization to zero of system (0.9)–(0.15), in choosing as a control function the mapping X^* on which we have to assume some constraints. First, we consider displacements $Z^* = X^* - \mathrm{Id}_{\mathcal{S}}$ in a suitable functional set; For some $\lambda > 0$, we define

$$Z^* \in \mathcal{W}_\lambda(S_\infty^0) \Leftrightarrow \begin{cases} e^{\lambda t}\dfrac{\partial Z^*}{\partial t} \in \mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S})), \\ Z^*(y,0) = 0, \ \dfrac{\partial Z^*}{\partial t}(y,0) = 0 \quad \forall y \in \mathcal{S}. \end{cases}$$

The main result of this chapter is Theorem 3.24:

Theorem. *For* (u_0, h_1, ω_0) *small enough in* $\mathbf{H}^1_{cc}$*, system* (0.9)–(0.15) *is stabilizable with an arbitrary exponential decay rate* $\lambda > 0$*, that is to say that there exists a positive constant* C_0 *such that for all* $t \geq 0$ *we have*

$$\|(u(\cdot,t), h'(t), \omega(t))\|_{\mathrm{L}^2(\mathcal{F}(t))\times\mathbb{R}^d\times\mathbb{R}^3} \quad \leq \quad C_0 e^{-\lambda t}.$$

The constant C_0 *depends only on* (u_0, h_1, ω_0)*.*

In order to prove this result, the methods used are strongly inspired from the work of [Ray10]. We distinguish two big parts in Chapter 3. The first one consists in stabilizing the following linearized system

$$\begin{aligned}
\frac{\partial u}{\partial t} - \mathrm{div}\ \sigma(u,p) &= 0, \quad \text{in } (0,\infty)\times\mathcal{F},\\
\mathrm{div}\ u &= 0, \quad \text{in } (0,\infty)\times\mathcal{F},
\end{aligned}$$

$$\begin{aligned}
u &= 0, && \text{in } \partial\mathcal{O}\times(0,\infty),\\
u &= h'(t) + \omega(t)\wedge y + \zeta(y,t), && y\in\partial\mathcal{S}, \quad t\in(0,\infty),
\end{aligned}$$

$$\begin{aligned}
Mh''(t) &= -\int_{\partial\mathcal{S}} \sigma(u,p)n\mathrm{d}\Gamma, && t\in(0,\infty),\\
I_0\omega'(t) &= -\int_{\partial\mathcal{S}} y\wedge\sigma(u,p)n\mathrm{d}\Gamma, && t\in(0,\infty),
\end{aligned}$$

$$u(y,0) = u_0(y),\ y\in\mathcal{F}, \quad h'(0) = h_1\in\mathbb{R}^d, \quad \omega(0) = \omega_0\in\mathbb{R}^3,$$

by mappings $\zeta = e^{\lambda t}\dfrac{\partial X^*}{\partial t} \in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))$ satisfying the linearized constraints which correspond to **H1**−**H4**. For that, we first prove the approximate controllability of this system. We introduce the adjoint system associated with this linear system, whose unknowns are (ϕ,ψ,k',r), and we are lead to consider a unique continuation problem with the following equality

$$\int_0^T \int_{\partial\mathcal{S}} \zeta\cdot\sigma(\phi,\psi)n\mathrm{d}\Gamma \quad = \quad 0, \tag{0.16}$$

for all ζ in $\in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))$, such that

$$\int_{\partial\mathcal{S}} \zeta\cdot n\mathrm{d}\Gamma \quad = \quad 0.$$

In this equality the couple (ϕ,ψ) stands for the velocity and the pressure of the fluid. From this approximate controllability result we deduce the stabilization by feedback of a nonhomogeneous linear system. Then in a second time we aim at proving the stabilization of the nonlinear system rewritten in fixed domains. For that we have to define in a first time deformations satisfying the nonlinear constraints, from the feedback operator obtained previously, which exists only on the boundary $\partial\mathcal{S}$. For that we show that this boundary control can induce inside the solid a deformation X_l^* which satisfies the linearized constraints. Then we project the deformation X_l^* on a set corresponding to the nonlinear constraints. This projection method enables us to consider a control decomposed into a part which stabilizes the linear component of the system, and a part which has suitable Lipschitz properties. We conclude the proof of the main result by a fixed point method.

Main results of Chapter 5

This chapter is the transcription of a submitted article, in collaboration with Michel Fournié and Alexei Lozinski.

The study of the main system from a controllability point of view (see equation (0.16)) has underlined the importance of the fluid's forces on the solid's boundary, that is to say the normal trace of the Cauchy stress tensor $\sigma(u,p)n$. Moreover, this quantity determines the expressions given by the Newton's laws. In the perspective of numerical simulations, it is crucial to be able to get a good approximation of this quantity. More precisely, we consider the following Stokes problem with nonhomogeneous Dirichlet conditions

$$\begin{array}{rcll} -\nu\Delta u + \nabla p & = & f & \text{in } \mathcal{F}, \\ \operatorname{div} u & = & 0 & \text{in } \mathcal{F}, \\ u & = & 0 & \text{on } \partial\mathcal{O}, \\ u & = & g & \text{on } \partial\mathcal{S}. \end{array}$$

This problem is quite simple, in comparison with the full fluid-solid system. This enables us to highlight the main aspects of the problem, that is to say: Taking into account nonhomogeneous boundary conditions (corresponding to the equality of velocities at the fluid-solid interface), handling boundaries with arbitrary geometries (because in our case the geometry of the solid is unknown), and obtaining a good approximation for the normal trace of the Cauchy stress tensor. For that, we adapt to the Stokes problem a fictitious domain method which has been initially introduced for the Poisson problem in [HR09], and which is based on the ideas of Xfem. This approach enables us to perform computations in domains whose boundaries do not depend on the mesh. A numerical stabilization technique is carried out in order to recover the convergence for the quantity $\sigma(u,p)n$, and an inf-sup condition is theoretically proven for the stabilized discrete problem. Computations of the rates of convergence are performed, and underline the interests of the method.

0.9 Some words on the changes of variables

Notice that the more suitable formulation for describing the fluid's state is the Eulerian formulation, whereas the more suitable one for the evolution of a solid is the Lagrangian formulation. The point of view we adopt is the one which corresponds to the control, that is to say we prefer to handle Lagrangian mappings for the solid as well as for the fluid. A first work then consists in making a change of unknowns for the mappings written in Eulerian formalism, that is to say the velocity and the pressure of the fluid:

$$u(\cdot,t) \ : \ \mathcal{F}(t) \longrightarrow \mathbb{R}^d, \qquad p(\cdot,t) \ : \ \mathcal{F}(t) \longrightarrow \mathbb{R}.$$

In order to rewrite these mappings in domains which do not depend on time, we first choose a reference configuration, the configuration which corresponds to the geometry at the initial state, that is to say the one which is given by $\mathcal{F}(0)$ and $\mathcal{S}(0)$. We want to define a change of variables

$$X(\cdot,t) \ : \ \mathcal{F}(0) \longrightarrow \mathcal{F}(t).$$

For the solid part, the change of variables chosen is of course the one induced by the deformation of the solid, that is to say the mapping

$$X_{\mathcal{S}}(\cdot,t) \ : \ \mathcal{S}(0) \longrightarrow \mathcal{S}(t).$$

For the fluid part, we construct changes of variables $X(\cdot,t)$ which extend $X_{\mathcal{S}}(\cdot,t)$ to the rest of the domain $\mathcal{F}(0) = \mathcal{O} \setminus \overline{\mathcal{S}(0)}$, and which do not depend on some unknowns, like the fluid's velocity for instance. Indeed, a change of variables which could be induced by the fluid's velocity u would be *a priori* not convenient: For instance the regularity of such a change of variables can be directly limited by the one of u, and u is a function which makes sense only on the domain $\mathcal{F}(t)$ which depends on time.
The changes of variables we construct in this thesis for the fluid part have no physical meanings, but enable us to define changes of unknowns which have good properties. Let us explain how we define a mapping $X(\cdot,t)$, which is a C^1-diffeomorphism from $\mathcal{F}(0)$ onto $\mathcal{F}(t)$, and which satisfies

$$\begin{cases} \det \nabla X = 1, & \text{in } \mathcal{F} \times (0,\infty), \\ X = X_{\mathcal{S}}, & \text{on } \partial\mathcal{S} \times (0,\infty), \\ X = \mathrm{Id}_{\partial\mathcal{O}}, & \text{on } \partial\mathcal{O} \times (0,\infty). \end{cases}$$

Let us keep in mind that the mapping $X_{\mathcal{S}}$ represents the solid's deformation in the inertial frame of reference, and is decomposed as follows

$$X_{\mathcal{S}}(y,t) \quad = \quad h(t) + \mathbf{R}(t)X^*(y,t), \quad y \in \mathcal{S}(0),$$

where X^* represents the solid's deformation in its own frame of reference.

The change of variables in Chapter 1

When we do not assume some restrictions on the regularity of the mapping X^* - like in Chapter 1 - we can associate with the latter an Eulerian velocity w^*, which can be defined by

$$w^*(x^*,t) \quad = \quad \frac{\partial X^*}{\partial t}(Y^*(x^*,t),t), \quad x^* \in \mathcal{S}^*(t) = X^*(\mathcal{S}(0),t),$$

and which has the maximum regularity in space. Thus it is easy to extend this Eulerian velocity in a velocity $\overline{w}^*$ to the whole domain $\mathcal{O}$. For that, we consider the classical Dirichlet problem

$$\begin{cases} \mathrm{div}\ \overline{w}^* = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{S}^*(t)},\ t \in (0,T), \\ \overline{w}^*(x^*,t) = 0 & \text{if } \mathrm{dist}(x^*, \mathcal{S}^*(t)) \geq \eta > 0,\ t \in (0,T), \\ \overline{w}^*(x^*,t) = w^*(x^*,t) & \text{if } x^* \in \mathcal{S}^*(t),\ t \in (0,T). \end{cases}$$

Then an extension of X^* to $\mathbb{R}^3 \setminus \mathcal{S}(0)$, denoted by $\overline{X}^*$, can be obtained as the solution of the following problem

$$\frac{\partial \overline{X}^*}{\partial t}(y,t) = \overline{w}^*(\overline{X}^*(y,t),t), \quad \overline{X}^*(y,0) = y - h_0, \quad y \in \mathcal{F}(0).$$

Then we define through a change of frame the following function

$$\overline{w}(x,t) \quad = \quad \mathbf{R}(t)\overline{w}^*(\mathbf{R}(t)^T(x - h(t)),t), \quad x \in \mathcal{F}(t).$$

Finally, the wanted mapping X can be obtained as the solution of the problem

$$\frac{\partial X}{\partial t}(y,t) = \Lambda(X(y,t),t), \quad X(y,0) = y, \quad y \in \mathcal{F}(0),$$

where Λ is defined with the help of a regular cut-off function ξ (which is equal to 1 in a vicinity of $\partial\mathcal{S}(0)$, and 0 in a vicinity of $\partial\mathcal{O}$) and a vector potential $\mathfrak{F}_R$ such that $\mathrm{curl}(\mathcal{F}_R)(x,t) = h'(t) + \omega(t) \wedge (x - h(t))$, as follows

$$\Lambda \quad = \quad \mathrm{curl}(\xi\mathfrak{F}_R) + \overline{w}.$$

Note that Λ satisfies

$$\begin{cases} \Lambda = 0 \text{ on } \mathbb{R}^3 \setminus \mathcal{O}, \\ \mathrm{div}\ \Lambda = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\mathcal{S}(t)}, \\ \Lambda(x,t) = h'(t) + \omega \wedge (x - h(t)) + w(x,t) \text{ if } x \in \mathcal{S}(t). \end{cases}$$

The change of variables in Chapter 2

In Chapter 2, the Lagrangian mapping X^* is considered as the main datum in relation with the deformation of the solid. It is limited in regularity. Instead of obtaining the extensions of the Lagrangian mappings $X^*(\cdot,t)$ and $X_{\mathcal{S}}(\cdot,t)$ by extending first the Eulerian velocities $w^*(\cdot,t)$ and $w(\cdot,t)$ (which would become much more delicate to define), we directly extend X^* to the whole domain $\mathcal{O}$. For that, we solve the following problem

$$\begin{cases} \det\nabla\tilde{X} = 1 & \text{in } \mathcal{F} \times (0,T), \\ \tilde{X} = X^* & \text{on } \partial\mathcal{S} \times (0,T), \\ \tilde{X}(y,t) = \mathbf{R}(t)^T(y - h(t)) & (y,t) \in \partial\mathcal{O} \times (0,T). \end{cases}$$

Note that this problem contains all the desired properties, since we can then directly set

$$X(y,t) \quad = \quad h(t) + \mathbf{R}(t)\tilde{X}(y,t).$$

Such an extension for the mapping X^* is obtained only for T small enough. With regards to the method chosen in Chapter 1 in order to define $X(\cdot,t)$ as an extension of $X_{\mathcal{S}}(\cdot,t)$, this method is more direct. Moreover it avoids handling velocities such as $w^*(\cdot,t)$ which are defined on domains which depend on time, and so for which the functional framework is not standard. Finally, the dependence of the extension $X(\cdot,t)$ then obtained on the data X^*, h and $\mathbf{R}$ is easier to obtain. Quantifying this dependence is essential in Chapter 3, where the deformation X^* is chosen with the help of a feedback operator, and thus depends entirely on the unknowns of the problem.

The change of variables in Chapter 3

The goal of this chapter is to get the exponential stabilization of the fluid and solid velocities, in infinite time horizon. Thus, the approach of Chapter 2, in order to extend the Lagrangian mapping X^*, cannot be used in this perspective, since it would lead in particular to assume that the unknown velocities h' and ω (utilized as data to define the extension of X^*) are arbitrary small enough in $\mathrm{H}^1(0,\infty;\mathbb{R}^d)$ and $\mathrm{H}^1(0,\infty;\mathbb{R}^3)$ respectively. Thus, in order to define the change of variables X, we proceed in mixing the ideas of Chapter 1 and Chapter 2. More precisely, we first extend the Lagrangian mapping X^* in solving rather the system

$$\begin{cases} \det\nabla\overline{X}^* = 1 & \text{in } \mathcal{F} \times (0,\infty), \\ \overline{X}^* = X^* & \text{on } \partial\mathcal{S} \times (0,\infty), \\ \overline{X}^*(y,t) = y - h_0 & (y,t) \in \partial\mathcal{O} \times (0,\infty). \end{cases}$$

Note that such an extension $\overline{X}^*$ has properties which are similar to the one of the extension obtained in Chapter 1. After that, we define an extension $\overline{X}^R$ of the mapping $y \mapsto h(t) + \mathbf{R}(t)y$, obtained as the solution of the following problem

$$\frac{\partial \overline{X}^R}{\partial t}(\overline{x}^*, t) = \mathrm{curl}(\xi \mathfrak{F}_R)(\overline{X}^R(\overline{x}^*, t), t), \quad \overline{X}^*(\overline{x}^*, 0) = \overline{x}^*, \quad \overline{x}^* \in \mathcal{O} \setminus \overline{X}^*(\mathcal{S}(0), t).$$

The change of variables finally used for rewriting the main system in cylindrical domains is given in setting

$$X(y,t) \quad = \quad \overline{X}^R(\overline{X}^*(y,t), t), \quad y \in \mathcal{F}(0).$$

0.10 Plan of the thesis

In a first chapter we adapt and complete in dimension 3 the study made in dimension 2 in the paper of [SMSTT08]. In particular, we choose a suitable change of unknowns, and we study the linearized system associated with the system satisfied by the new unknowns. It leads us to a local existence of strong solutions (for regular solid's deformations), and the global existence is obtained for small data, in particular for small deformation velocities, in a class of functions that we make precise.
In Chapter 2, we extend the work of Chapter 1 to the case where the deformation of the solid is limited in regularity. We propose a new means of defining the change of variables, which is more suitable when the deformation of the solid is mainly seen through a Lagrangian mapping. We then verify that the results obtained in the first chapter are still true in this framework. This chapter 2 can be seen as a transition to the next chapter where the deformation of the solid - limited in regularity - is chosen as a control function.
Thus in chapter 3 we state a stabilization result for the fluid-solid system. This chapter can be divided into two main parts. In a first part we prove that the linearized system is stabilizable to zero by boundary velocities defined on $\partial\mathcal{S}$. These boundary velocities can be obtained from solid's deformations satisfying a set of linearized physical constraints. The second part of this chapter is devoted to the proof of stabilization of the full nonlinear system, by a fixed point method. In order to take in consideration the imposed nonlinear constraints, we consider a decomposition of the control; The first part of this decomposition satisfies the linearized constraints, and is chosen with the help of a feedback operator which enables us to stabilize the linear component of the system, whereas the remaining corrective term of this decomposition has good Lipschitz properties.
In Chapter 4 we propose practical means of describing solid's deformations, in acting only on some restrained parameters, instead of considering general deformations like in Chapter 3. The way we obtain these particular deformations tends to modelize the locomotion of some animals endowed with a spine bone on which they can act to deform themselves.
Finally in Chapter 5 we develop a numerical method which enables us to get a good approximation of the normal trace of the Cauchy stress tensor, on boundaries which do not depend on the mesh. This method combines a fictitious domain approach based on the ideas of Xfem, and an augmented Lagrangian method. From a fluid-structure interactions point of view, the interest of such a method lies in the importance of the role played by the fluid's forces on the fluid-solid interface.

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Chapter 1

Wellposedness for a system modeling a deformable solid in a viscous incompressible fluid

This chapter is dedicated to the study of a system modeling a deformable solid in a fluid. For the solid we consider some deformation that has to obey several constraints. The motion of the fluid is modeled by the incompressible Navier-Stokes equations in a time-dependent bounded domain of $\mathbb{R}^3$, and the solid satisfies the Newton's laws. We rewrite the system in a domain which does not depend on time, by using a suitable change of unknowns. We study the corresponding linearized system before setting a local-in-time existence result. Global existence is obtained for small data.

1.1 Introduction

In this chapter we are interested in a deformable solid immersed in a viscous incompressible fluid in 3 dimensions. The domain occupied by the solid at time t is denoted by $\mathcal{S}(t)$. We assume that $\mathcal{S}(t) \subset \mathcal{O}$, where $\mathcal{O}$ is a bounded regular domain. The fluid surrounding the structure occupies the domain $\mathcal{O} \setminus \overline{\mathcal{S}(t)} = \mathcal{F}(t)$. The solid's position is unknown and is described by its center of mass, whose coordinates are given by the vector $h(t)$, and the rotation $\mathbf{R}(t)$ resulting from its angular velocity $\omega(t)$. The deformation of the solid is imposed in its own frame of reference, and can be viewed through an Eulerian velocity w^*. The latter induces a velocity w defined in the inertial frame of reference. The fluid flow is described by its velocity u and its pressure p. For w^* satisfying a set of hypotheses given further, we aim at proving the existence of strong solutions for the following coupled system

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \qquad x \in \mathcal{F}(t), \quad t \in (0,T), \tag{1.1}$$

$$\operatorname{div} u = 0, \qquad x \in \mathcal{F}(t), \quad t \in (0,T), \tag{1.2}$$

$$u = 0, \qquad x \in \partial\mathcal{O}, \quad t \in (0,T), \tag{1.3}$$

$$u = h'(t) + \omega(t) \wedge (x - h(t)) + w(x,t), \qquad x \in \partial\mathcal{S}(t), \quad t \in (0,T), \tag{1.4}$$

$$Mh''(t) = -\int_{\partial\mathcal{S}(t)} \sigma(u,p)n \mathrm{d}\Gamma, \qquad t \in (0,T), \tag{1.5}$$

$$(I\omega)'(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t)) \wedge \sigma(u,p)n \mathrm{d}\Gamma, \qquad t \in (0,T), \tag{1.6}$$

$$u(y,0) = u_0(y),\ y \in \mathcal{F}(0), \quad h(0) = h_0 \in \mathbb{R}^3, \quad h'(0) = h_1 \in \mathbb{R}^3, \quad \omega(0) = \omega_0 \in \mathbb{R}^3. \tag{1.7}$$

The symbol $\wedge$ denotes the cross product. Associated with the angular velocity $\omega(t)$, we introduce the rotation $\mathbf{R}(t)$ classically obtained as being the solution of the following Cauchy problem

$$\left\{ \begin{array}{rcl} \dfrac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} & = & \mathbb{S}(\omega)\,\mathbf{R} \\ \mathbf{R}(0) & = & \mathrm{I}_{\mathbb{R}^3} \end{array} \right. , \quad \text{with } \mathbb{S}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{1.8}$$

The linear map $\omega \wedge \cdot$ can be represented by the matrix $\mathbb{S}(\omega)$. The velocity field w is expressed in terms of w^*, h and ω, through the following change of frame

$$w(x,t) \quad = \quad \mathbf{R}(t)\ w^*\left(\mathbf{R}(t)^T(x-h(t)),t\right), \quad x \in \mathcal{S}(t). \tag{1.9}$$

The field w^* is an Eulerian velocity which can be considered as a given control function. The Lagrangian flow X^* associated with w^* is given through the following Cauchy problem

$$\frac{\partial X^*}{\partial t}(y,t) = w^*(X^*(y,t),t), \quad X^*(y,0) = y - h_0, \quad y \in \mathcal{S}(0). \tag{1.10}$$

The mapping X^* represents the deformation of the solid in its own frame of reference. Considering X^* - or w^* - as a datum is equivalent to assuming that the solid is strong enough to impose its own shape. Thus we assume that $X^*(\cdot,t)$ is given. The Lagrangian flow of the solid in the inertial frame of reference is decomposed as follows

$$X_{\mathcal{S}}(y,t) \quad = \quad h(t) + \mathbf{R}(t)X^*(y,t), \quad \text{for } y \in \mathcal{S}(0).$$

The map $X^R(y,t) = h(t) + \mathbf{R}(t)y$ denotes the unknown rigid part of the motion and $Y^R(x,t) = \mathbf{R}(t)^T(x-h(t))$ denotes its inverse. We define

$$\mathcal{S}^*(t) = X^*(\mathcal{S}(0),t), \qquad \tilde{\mathcal{F}}(t) = \mathcal{O} \setminus \overline{\mathcal{S}^*(t)}.$$

If $Y^*(\cdot,t)$ denotes the inverse of $X^*(\cdot,t)$, we have

$$w^*(x^*,t) \quad = \quad \frac{\partial X^*}{\partial t}(Y^*(x^*,t),t), \quad x^* \in \mathcal{S}^*(t). \tag{1.11}$$

In equations (1.5) and (1.6), the solid's mass M is constant, whereas the moment of inertia tensor depends on time, as

$$I(t) \quad = \quad \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)\left(|x-h(t)|^2\mathrm{I}_{\mathbb{R}^3} - (x-h(t))\otimes(x-h(t))\right)\mathrm{d}x.$$

The quantity $\rho_{\mathcal{S}}$ denotes the solid's density, and obeys the principle of mass conservation

$$\rho_{\mathcal{S}}(X_{\mathcal{S}}(y,t),t) \quad = \quad \frac{\rho_{\mathcal{S}}(y,0)}{\det\left(\nabla X_{\mathcal{S}}(y,t)\right)}, \quad y \in \mathcal{S}(0),$$

where $\nabla X_{\mathcal{S}}$ is the Jacobian matrix of mapping $X_{\mathcal{S}}$. We can define

$$\rho^*(x^*,t) \quad = \quad \frac{\rho_{\mathcal{S}}(Y^*(x^*,t),0)}{\det\left(\nabla X^*(Y^*(x^*,t),t)\right)}, \quad x^* \in \mathcal{S}^*(t).$$

In system (1.1)–(1.7), ν is the kinematic viscosity of the fluid, the normalized vector n is the normal at $\partial\mathcal{S}(t)$ exterior to $\mathcal{F}(t)$, and f represents some additional volume forces that can be applied to the fluid. It is a coupled system between the incompressible Navier-Stokes equations and the Newton's laws. The coupling is in particular made in the fluid-structure interface, through the equality of velocities (1.4) and through the Cauchy stress tensor

$$\sigma(u,p) = 2\nu D(u) - p \ \mathrm{Id} = \nu \left(\nabla u + (\nabla u)^T\right) - p \ \mathrm{Id}.$$

The velocity field w^* can be considered as the main data in relation with the imposed undulatory motion. We assume that the corresponding deformation X^* satisfies a set of hypotheses:

H1 For all $t \in [0,T]$, $X^*(\cdot,t)$ is a C^∞-diffeomorphism from $\overline{\mathcal{S}(0)}$ onto $\overline{\mathcal{S}^*(t)}$.

H2 In order to respect the incompressibility condition given by (1.2), the volume of the whole solid is preserved through the time. That is equivalent to say that

$$\int_{\partial\mathcal{S}^*(t)} w^* \cdot n \mathrm{d}\Gamma = \int_{\partial\mathcal{S}(0)} \frac{\partial X^*}{\partial t} \cdot (\mathrm{cof}\nabla X^*)\, n \mathrm{d}\Gamma = 0, \tag{1.12}$$

where $\mathrm{cof}\nabla X^*$ denotes the cofactor matrix of ∇X^*.

H3 The linear momentum of the solid is preserved through the time, that means

$$\int_{\mathcal{S}^*(t)} \rho^*(x^*,t) w^*(x^*,t) \mathrm{d}x^* = \int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) \frac{\partial X^*}{\partial t}(y,t) \mathrm{d}y = 0. \tag{1.13}$$

H4 The angular momentum of the solid is preserved through the time, that means

$$\int_{\mathcal{S}^*(t)} \rho^*(x^*,t) x^* \wedge w^*(x^*,t) \mathrm{d}x^* = \int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t) \wedge \frac{\partial X^*}{\partial t}(y,t) \mathrm{d}y = 0. \tag{1.14}$$

Imposing constraints (1.13) and (1.14) enables us to get the two following constraints on the undulatory velocity w

$$\int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t) w(x,t) \mathrm{d}y \quad = \quad 0, \tag{1.15}$$

$$\int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)(x-h(t)) \wedge w(x,t) \mathrm{d}y \quad = \quad 0. \tag{1.16}$$

As equations (1.5) and (1.6) are written, equalities (1.15) and (1.16) are already assumed in system (1.1)–(1.7). Hypotheses **H3** and **H4** are made to guarantee the *self-propelled* nature of the solid's motion, that means no other help than its own deformation enables it to move in the fluid. By the undulatory motion induced by its own internal deformation, the solid imposes partially, through w, the nonhomogeneous Dirichlet condition (1.4). The latter induces the behavior of the environing fluid through (1.1)–(1.3), and thus the fluid's response - given by $\sigma(u,p)n$ on the interface - enables the whole solid to be carried, regarding to the ordinary differential equations (1.5) and (1.6). The other part of the interaction consists in the fact that domains occupied by the fluid and the solid change through the time, as follows

$$\mathcal{S}(t) = h(t) + \mathbf{R}(t)\mathcal{S}^*(t), \qquad \mathcal{F}(t) = \mathcal{O} \setminus \overline{\mathcal{S}(t)}.$$

This type of problem has been studied in [SMSTT08] in 2 dimensions. The case of weak solutions (in 3 dimensions) has been recently investigated in [NTT11].
The case of rigid bodies in a viscous incompressible fluid has been studied in several papers (see for instance [CSMHT00], [DE99], [GM00], [GLS00], [SMST02], [Fei03], [Tak03] in a bounded

domain, and [Gal99], [Gal02], [TT04], [CT08] when the system fills the whole space). The analysis of the interaction between an elastic body and a viscous incompressible fluid has been investigated in [DEGLT01], [Bou03], [CDEG05], [CS05] and [CS06], for instance, and in [BST12] more recently. Helpful calculations in continuum mechanics can be found in [Gur81], for instance. Our mathematical contribution is based on the approach introduced in [SMSTT08] and consists in investigating the whole 3D-case. In particular, compared to the 2D-case, the angular velocity is no more a scalar but a 3-dimensional vector. The first main result of the paper is stated in Theorem 1.17 in which we prove that under some regularity and compatibility assumptions for the data, system (1.1)–(1.7) admits a local unique strong solution. The second one is the existence of a unique solution global in time under smallness assumptions on the data (see Theorem 1.18). For the local existence result we assume that $f \in \mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))$, and for the global existence result we assume that $f \in \mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}(t))) \cap \mathrm{L}^{3/2}(0,\infty;\mathbf{L}^2(\mathcal{F}(t)))$. Theorem 1.17 is established with using a contracting fixed point method based on the study in section 1.4 of the linear system given by (1.46)–(1.52).
We first extend the Lagrangian flow $X_{\mathcal{S}}(\cdot,t)$ associated with the structure as a mapping $X(\cdot,t)$ defined on the whole domain, by using a method introduced in [Tak03] (see section 1.3.1). We denote by $Y(\cdot,t)$ the inverse of $X(\cdot,t)$. Then we rewrite system (1.1)–(1.7) in a cylindrical domain, in section 1.3.2. For that we use a well-chosen change of variables. There the novelty is that we introduce the following unknowns

$$\tilde{u}(y,t) = \mathbf{R}(t)^T u(X(y,t),t), \qquad \tilde{p}(y,t) = p(X(y,t),t),$$

rather than using the whole Jacobian matrix

$$\overline{u}(y,t) = \nabla Y(X(y,t),t)u(X(y,t),t), \qquad \overline{p}(y,t) = p(X(y,t),t), \tag{1.17}$$

which is used in [IW77] for instance, or in several papers which only consider a rigid solid (see [CT08], [Tak03], [TT04]), or simply suggested in [SMSTT08] to tackle with a local in time existence result for system (1.1)–(1.7). Let us notice that in our case the Jacobian matrix $\nabla Y(X(\cdot,t),t)$ actually depends on the space variable, and thus the calculations of the equations satisfied by $(\tilde{u},\tilde{p})$ would be quite complicated if we choose the change of unknowns (1.17). The corresponding nonlinear system, written in a cylindrical domain, is stated in (1.30)–(1.36). The change of variables we have chosen enables us to write this system in the simplest form we have found. In particular, the equation of velocities (1.4) on $\partial\mathcal{S}(t)$ becomes (1.33)

$$\tilde{u} = \tilde{h}' + \tilde{\omega} \wedge X^* + \frac{\partial X^*}{\partial t},$$

where the datum X^* and its time derivative appear in a simple way. The price to pay is that we have to study a system in a cylindrical domain in which the divergence of $\tilde{u}$ is not equal to 0. Properties of this nonhomogeneous condition are given in section 1.3.2. We introduce the linearized system in section 1.4.2. Another novelty is the study in section 1.4 of an existence result for the linearized system (1.46)–(1.52), in which we tackle the nonhomogeneous data in following the results of [Ray07] and [Ray10].

1.2 Definitions and notation

We denote by $\mathcal{F} = \mathcal{F}(0)$ and $\mathcal{S} = \mathcal{S}(0)$ the domains occupied at time $t = 0$ by the fluid and the solid respectively. We assume that $\mathcal{S}$ is regular enough. We denote by $\mathcal{S}^*(t) = X^*(\mathcal{S},t)$ and $\tilde{\mathcal{F}}(t) = \mathcal{O} \setminus \overline{\mathcal{S}^*(t)}$. We recall that the state of the system at $t = 0$ is chosen as the reference configuration. Note that the boundary of $\mathcal{F}$ is equal to $\partial\mathcal{O} \cup \partial\mathcal{S}$. In order to deal with some functional spaces, we use the notation

$$\begin{aligned} \mathbf{L}^2(\mathcal{O}) &= [\mathrm{L}^2(\mathcal{O})]^3, & \mathbf{H}^1(\mathcal{O}) &= [\mathrm{H}^1(\mathcal{O})]^3, \\ \mathbf{L}^2(\mathcal{F}) &= [\mathrm{L}^2(\mathcal{F})]^3, & \mathbf{L}^2(\mathcal{F}(t)) &= [\mathrm{L}^2(\mathcal{F}(t))]^3, \\ \mathbf{H}^s(\mathcal{F}) &= [\mathrm{H}^s(\mathcal{F})]^3, & \mathbf{H}^s(\mathcal{F}(t)) &= [\mathrm{H}^s(\mathcal{F}(t))]^3. \end{aligned}$$

Nevertheless this type of notation will be also used for spaces of type $[\mathrm{L}^2(\mathcal{F})]^{3\times 3}$, or more generally $[\mathrm{L}^2(\mathcal{F})]^{3^k}$ (where k is an integer). Let us now make precise the functional spaces that we will set in order to look for solutions to Problem (1.1)–(1.7).

Definition 1.1. *We denote by $\mathcal{U}(0,T;\mathcal{F}(t))$ the following functional space*

$$\mathcal{U}(0,T;\mathcal{F}(t)) \quad = \quad \mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}(t))) \cap \mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F}(t))) \cap C([0,T];\mathbf{H}^1(\mathcal{F}(t))),$$

that we endow and define with the norm given by

$$\|u\|^2_{\mathcal{U}(0,T;\mathcal{F}(t))} \quad = \quad \int_0^T \|u(\cdot,t)\|^2_{\mathbf{H}^2(\mathcal{F}(t))}\mathrm{d}t + \int_0^T \left\| \frac{\partial u}{\partial t}(\cdot,t)\right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} \mathrm{d}t + \sup_{t\in[0,T]} \|u(\cdot,t)\|^2_{\mathbf{H}^1(\mathcal{F}(t))}.$$

In the same way we can define $\mathcal{U}(0,T;\tilde{\mathcal{F}}(t))$. More classically, we set

$$\mathcal{U}(0,T;\mathcal{F}) \quad = \quad \mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F})) \cap \mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F})) \cap C([0,T];\mathbf{H}^1(\mathcal{F})).$$

Definition 1.2. *Let be $T > 0$. A quadruplet (u,p,h,ω) is called a strong solution of the system* (1.1)–(1.7) *when*

$$(u,p,h,\omega) \in \mathcal{U}(0,T;\mathcal{F}(t)) \times \mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{F}(t))) \times \mathrm{H}^2(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3),$$

$\mathrm{dist}(\mathcal{S}(t),\partial\mathcal{O}) > 0$ *for all $t \in [0,T)$, and when (u,p,h,ω) satisfies the system* (1.1)–(1.7) *almost everywhere in* $\displaystyle\bigcup_{t\in(0,T)} \mathcal{F}(t) \times \{t\}$*, or in the trace sense.*

Finally we set

$$\mathcal{H} = \left\{\phi \in \mathbf{L}^2(\mathcal{O}) \mid \mathrm{div}\ \phi = 0 \text{ in } \mathcal{O},\ D(\phi) = 0 \text{ in } \mathcal{S},\ \phi\cdot n = 0 \text{ on } \partial\mathcal{O}\right\},$$

$$\mathcal{V} = \left\{\phi \in \mathbf{H}^1(\mathcal{O}) \mid \mathrm{div}\ \phi = 0 \text{ in } \mathcal{O},\ D(\phi) = 0 \text{ in } \mathcal{S},\ \phi\cdot n = 0 \text{ on } \partial\mathcal{O}\right\}.$$

1.3 Change of variables and cylindrical domains

1.3.1 Extension of some mappings

Let (h,ω) be a couple in $\mathrm{H}^2(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3)$. Let w^* be a mapping which provides - through (1.10) - a flow X^* satisfying the hypotheses **H1-H2**. In this part we construct an extension $\overline{w}$ of w to $\mathbb{R}^3$. The goal is to construct a C^1-diffeomorphism X as being an extension of $X_{\mathcal{S}}$ to the whole domain $\mathcal{O}$.
Let us recall a result from [Lad69, page 27].

Proposition 1.3. *For all $\eta > 0$, there exists a function $(x^*,t) \mapsto \overline{w}^*(x^*,t)$ such that, for every $t \geq 0$ the map $x^* \mapsto \overline{w}^*(x^*,t)$ is C^∞ on $\mathbb{R}^3 \setminus \mathcal{S}^*(t)$, for every $x^* \in \mathbb{R}^3 \setminus \mathcal{S}^*(t)$ the function $t \mapsto \overline{w}^*(x^*,t)$ is of class C^∞, and $\overline{w}^*$ obeys*

$$\left\{\begin{array}{ll} \mathrm{div}\ \overline{w}^* = 0 & \textit{in } \mathbb{R}^3 \setminus \overline{\mathcal{S}^*(t)},\ t \in (0,T), \\ \overline{w}^*(x^*,t) = 0 & \textit{if } \mathrm{dist}(x^*, \mathcal{S}^*(t)) \geq \eta > 0,\ t \in (0,T), \\ \overline{w}^*(x^*,t) = w^*(x^*,t) & \textit{if } x^* \in \mathcal{S}^*(t),\ t \in (0,T). \end{array}\right. \tag{1.18}$$

The existence of $\overline{w}^*$ is guaranteed by Hypothesis **H2**:

$$\int_{\partial \mathcal{S}^*(t)} w^* \cdot n \mathrm{d}\Gamma = 0.$$

Then, in the same way we have defined w from w^*, we set:

$$\overline{w}(x,t) \quad = \quad \mathbf{R}(t)\overline{w}^* \left(\mathbf{R}(t)^T(x - h(t)), t\right),\ x \in \mathcal{O},\ t \in (0,T).$$

From now on $\eta > 0$ denotes a positive real number such that $0 < \eta \leq \mathrm{dist}(\mathcal{S}, \partial\mathcal{O})$. Thus by assuming that there exists $T > 0$ such that $\mathrm{dist}(\mathcal{S}(t), \partial\mathcal{O}) \geq \eta$ for all $t \in (0,T)$, the function $\overline{w}$ satisfies

$$\left\{\begin{array}{ll} \mathrm{div}\ \overline{w} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{S}(t)},\ t \in (0,T), \\ \overline{w} = 0 & \text{on } \partial\mathcal{O},\ t \in (0,T), \\ \overline{w} = w & \text{in } \mathcal{S}(t),\ t \in (0,T). \end{array}\right.$$

Let us now extend mapping $X_{\mathcal{S}}(\cdot,t)$ which is defined on $\overline{\mathcal{S}}$. Since $(h,\omega) \in \mathrm{H}^2(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3)$, we have in particular $h \in C^1([0,T];\mathbb{R}^3)$ and $\omega \in C([0,T];\mathbb{R}^3)$. We define

$$\begin{array}{rccc} \mathcal{V}_R : & \mathbb{R}^3 \times [0,T] & \longrightarrow & \mathbb{R}^3 \\ & (x,t) & \longmapsto & h'(t) + \omega(t) \wedge (x - h(t)). \end{array}$$

We can easily verify that, for all $t \in [0,T]$, the mapping $x \mapsto \mathcal{V}_R(x,t)$ lies in $C^\infty(\mathbb{R}^3,\mathbb{R}^3)$ and that, for all $x \in \mathbb{R}^3$, the mapping $t \mapsto \mathcal{V}(x,t)$ lies in $C([0,T];\mathbb{R}^3)$.
We set

$$\mathfrak{F}_R : \quad \mathbb{R}^3 \times [0,T] \quad \longrightarrow \quad \mathbb{R}^3,$$

$$\mathfrak{F}_R(x,t) = \frac{1}{2} h'(t) \wedge (x - h(t)) - \frac{1}{2}|x - h(t)|^2 \omega(t).$$

This enables us to express $\mathfrak{F}_R$ as a vector potential of the velocity field $\mathcal{V}_R$

$$\mathcal{V}_R \quad = \quad \mathrm{curl}\ \mathfrak{F}_R.$$

If $\eta > 0$ is such that $0 < \eta \leq \mathrm{dist}(\mathcal{S}, \partial\mathcal{O})$, we set

$$\mathcal{O}_\eta = \{x \in \mathcal{O}\ /\ \mathrm{dist}(x, \partial\mathcal{O}) > \eta\}.$$

We choose a function $\xi \in C^\infty(\mathbb{R}^3, \mathbb{R})$ equal to 1 in $\overline{\mathcal{O}_\eta}$ with a compact support included in $\mathcal{O}_{\eta/2}$.
We set

$$\Lambda \quad = \quad \mathrm{curl}\,(\xi \mathfrak{F}_R) + \overline{w}. \tag{1.19}$$

We can verify that Λ is continuous from $\mathbb{R}^3 \times [0,T]$ to $\mathbb{R}^3$, that for all $t \in [0,T]$ the map $x \mapsto \Lambda_{|\overline{\mathcal{F}(t)}}(x,t)$ lies in $C^\infty(\overline{\mathcal{F}(t)};\mathbb{R}^3)$, and that for all $x \in \mathbb{R}^3$ the function $t \mapsto \Lambda(x,t)$ lies in $C([0,T];\mathbb{R}^3)$. Moreover we have the following properties:

Lemma 1.4. *If* $\mathrm{dist}(\mathcal{S}(t), \partial\mathcal{O}) \geq \eta > 0$*, the function* Λ *defined by* (1.19) *satisfies*

$$\begin{array}{ll} (i) & \Lambda = 0 \text{ on } \mathbb{R}^3 \setminus \mathcal{O}, \\ (ii) & \mathrm{div}\ \Lambda = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\mathcal{S}(t)}, \\ (iii) & \Lambda(x,t) = h'(t) + \omega \wedge (x - h(t)) + w(x,t) \text{ if } x \in \mathcal{S}(t). \end{array}$$

Proof. The assertion (i) comes from the fact that the support of ξ is included in $\mathcal{O}_{\eta/2}$. In calculating

$$\mathrm{div}\ \Lambda = \mathrm{div}\,(\mathrm{curl}\,(\xi \mathfrak{F}_r)) + \mathrm{div}\ w = 0,$$

we obtain (ii). We notice that if $\mathcal{S}(t) \subset \mathcal{O}_\eta$, then $\xi \equiv 1$ in $\mathcal{S}(t)$. By coming back to the expression of Λ given by (1.19), we get (iii). □

Let us now consider the following Cauchy problem

$$\frac{\partial X}{\partial t}(y,t) = \Lambda(X(y,t),t), \quad X(y,0) = y, \quad y \in \mathbb{R}^3. \tag{1.20}$$

Lemma 1.5. *For all* $T > 0$ *and for all* $y \in \mathbb{R}^3$*, the Cauchy problem* (1.20) *admits a unique solution* $t \mapsto X(y,t)$ *in* $[0,T]$*. Moreover, the derivatives*

$$\frac{\partial^{i+\alpha_1+\alpha_2+\alpha_3} X}{\partial t^i \partial y_1^{\alpha_1} \partial y_2^{\alpha_2} \partial y_3^{\alpha_3}}, \quad i \in \{0,1\}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}$$

exist and are continuous on $\overline{\mathcal{F}} \times [0,T]$*. Besides, the mapping* $y \mapsto X(y,t)$ *is a diffeomorphism from* $\overline{\mathcal{F}}$ *onto* $\overline{\mathcal{F}(t)}$*. Its inverse, that we denote by* $Y(x,t)$*, is such that its derivatives*

$$\frac{\partial^{i+\alpha_1+\alpha_2+\alpha_3} Y}{\partial t^i \partial y_1^{\alpha_1} \partial y_2^{\alpha_2} \partial y_3^{\alpha_3}}, \quad i \in \{0,1\}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}$$

exist and are continuous on $\displaystyle\bigcup_{t \in [0,T]} \overline{\mathcal{F}(t)} \times \{t\}$.

Proof. From the Cauchy-Lipschitz theorem, since Λ is of class $\mathbf{W}^{1,\infty}$ in space and continuous in time, the problem (1.20) admits a unique maximal solution $t \mapsto X(y,t)$ on an interval included in $[0,T]$. Note that Λ is equal to 0 on the boundary of the connected domain $\mathcal{O}$. Therefore we have by connexity $X(y,t) \in \overline{\mathcal{O}}$ for all t . Since $\overline{\mathcal{O}}$ is compact, the maximal solution is actually global (that is to say defined on $[0,T]$). The global existence and the uniqueness of the solution to problem (1.20) imply that $X(\cdot,t)$ is bijective from $\mathbb{R}^3$ onto $\mathbb{R}^3$. On the other hand we notice that

$$X(\cdot,t)_{|\mathbb{R}^3 \setminus \mathcal{O}} \quad = \quad \mathrm{Id}_{|\mathbb{R}^3 \setminus \mathcal{O}}.$$

We deduce that $X(\cdot,t)$ is a bijection from $\mathcal{O}$ onto $\mathcal{O}$. With regards to the regularity of Λ, from a classical result of regularity (see for instance [Dem06, pages 301, 302]) the derivatives

$$\frac{\partial^{i+\alpha_1+\alpha_2+\alpha_3} X}{\partial t^i \partial y_1^{\alpha_1} \partial y_2^{\alpha_2} \partial y_3^{\alpha_3}}, \quad i \in \{0,1\}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}$$

exist and are continuous on $\overline{\mathcal{F}} \times [0,T]$. Moreover, by recalling the classical property $X(y,t) = X\left(X(y,t-\overline{t}),\overline{t}\right)$, we can see that

$$Y(x,\overline{t}) \quad = \quad X\left(Y(x,t), t - \overline{t}\right).$$

Then we can show that, for t given, the inverse $Y(\cdot,t)$ of $X(\cdot,t)$ satisfies a similar Cauchy problem

$$\frac{\partial Y}{\partial t}(x,\bar{t}) = -\Lambda(Y(x,\bar{t}),t-\bar{t}), \quad Y(x,0)=x. \tag{1.21}$$

Thus it follows that $X(\cdot,t)$ is a C^∞-diffeomorphism on $\overline{\mathcal{F}}$. Besides, if $\mathrm{dist}(\partial\mathcal{S}(t),\partial\mathcal{O}) > \eta$, we can easily verify that for all $y \in \mathcal{S}$ the function

$$t \mapsto h(t) + \mathbf{R}(t)X^*(y,t) \tag{1.22}$$

is a solution of problem (1.20). By uniqueness, this solution is the solution of (1.20), and thus we get $X(\cdot,t)(\mathcal{S}) \subset \mathcal{S}(t)$. In a similar way we have $Y(\cdot,t)(\mathcal{S}(t)) \subset \mathcal{S}$. Consequently, $X(\cdot,t)(\mathcal{S}) = \mathcal{S}(t)$, and $X(\cdot,t)$ is a diffeomorphism from $\overline{\mathcal{F}}$ onto $\overline{\mathcal{F}(t)}$. □

1.3.2 The equations in a cylindrical domain

Let us assume that the quadruplet (u,p,h,ω) is a strong solution of (1.1)–(1.7).
In order to deal with a domain which does not depend on time, we make the change of unknowns

$$\begin{array}{ll} \tilde{u}(y,t) = \mathbf{R}(t)^T u(X(y,t),t), & u(x,t) = \mathbf{R}(t)\tilde{u}(Y(x,t),t), \\ \tilde{p}(y,t) = p(X(y,t),t), & p(x,t) = \tilde{p}(Y(x,t),t). \end{array} \tag{1.23}$$

We also set

$$\begin{array}{ll} \tilde{X}(y,t) = \mathbf{R}(t)^T\left(X(y,t) - h(t)\right), & y \in \mathcal{F}, \\ \tilde{Y}(\tilde{x},t) = Y\left(h(t) + \mathbf{R}(t)\tilde{x},t\right), & \tilde{x} \in \mathbf{R}(t)^T\left(\mathcal{F}(t) - h(t)\right). \end{array}$$

Remark 1.6. *The mappings $\tilde{X}$ and $\tilde{Y}$ defined above have the same regularity properties as the mappings X and Y (see Lemma 1.5). This can be made obvious by noticing that the mapping $\tilde{X}$ satisfies the following Cauchy problem*

$$\frac{\partial \tilde{X}}{\partial t}(y,t) = \tilde{\Lambda}(\tilde{X}(y,t),t), \quad \tilde{X}(y,0) = y - h_0, \quad y \in \mathbb{R}^3,$$

with

$$\tilde{\Lambda}(\tilde{x},t) \quad = \quad \mathbf{R}(t)^T\left(\Lambda(h(t)+\mathbf{R}(t)\tilde{x},t) - h'(t) - \omega(t)\wedge\tilde{x}\right).$$

Some practical calculations

Let us begin with some technical calculations which enable us to express the equations (1.5)–(1.6) with integrals on the non-depending time domain $\partial\mathcal{S}$. In the following $\nabla\tilde{X}$ will denote a Jacobian matrix, and the same kind of notation will be used for other mappings. The term $\mathrm{cof}\left(\nabla\tilde{X}\right)$ will denote the cofactor matrix of $\nabla\nabla\tilde{X}$:

$$\mathrm{cof}\left(\nabla\tilde{X}\right)(y,t) \quad = \quad \det\left(\nabla\tilde{X}(y,t)\right)\nabla\tilde{Y}(\tilde{X}(y,t),t)^T.$$

Notice that since $\mathrm{div}\Lambda = 0$, we have

$$\det\left(\nabla\tilde{X}(y,t)\right) = \det\left(\nabla X(y,t)\right) = 1, \quad y \in \mathcal{F}.$$

Lemma 1.7. *Let us assume that* $h \in \mathrm{H}^2(0,T;\mathbb{R}^3)$ *et* $\omega \in \mathrm{H}^1(0,T;\mathbb{R}^3)$ *are such that* $\mathcal{S}(t) \subset \mathcal{O}_\eta$ *for all* $t \in [0,T)$*. Then we have*

$$\begin{aligned} \int_{\partial\mathcal{S}} \tilde{u}\cdot\left(\mathrm{cof}\left(\nabla\tilde{X}\right)n\right)\mathrm{d}\Gamma &= 0, & (1.24)\\ \int_{\partial\mathcal{S}(t)} \sigma(u,p)n\mathrm{d}\Gamma &= \mathbf{R}(t)\int_{\partial\mathcal{S}} \tilde{\Sigma}(\tilde{u},\tilde{p})\mathrm{cof}\left(\nabla\tilde{X}\right)n\mathrm{d}\Gamma, & (1.25)\\ \int_{\partial\mathcal{S}(t)} (x-h(t))\wedge\sigma(u,p)n\mathrm{d}\Gamma &= \mathbf{R}(t)\int_{\partial\mathcal{S}} \tilde{X}\wedge\left(\tilde{\Sigma}(\tilde{u},\tilde{p})\mathrm{cof}\left(\nabla\tilde{X}\right)n\right)\mathrm{d}\Gamma, & (1.26)\end{aligned}$$

where

$$\tilde{\Sigma}(\tilde{u},\tilde{p})(y,t) = \nu\left(\nabla\tilde{u}(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t) + \nabla\tilde{Y}(\tilde{X}(y,t),t)^T\nabla\tilde{u}(y,t)^T\right) - \tilde{p}\mathrm{I}_{\mathbb{R}^3}. \quad (1.27)$$

Proof. Let us first observe that at time t the mapping $X(\cdot,t)$ is C^1, so that we have the formula

$$X(y,t) = h(t) + \mathbf{R}(t)\tilde{X}(y,t), \quad \forall y \in \overline{\mathcal{F}}.$$

The corresponding Jacobian matrices are expressed as follows

$$\nabla X(y,t) = \mathbf{R}(t)\nabla X^*(y,t), \quad \nabla Y(X(y,t),t) = \nabla Y^*(X^*(y,t),t)\mathbf{R}(t)^T. \quad (1.28)$$

The equality (1.24) comes from the divergence formula combined with the incompressibility condition (1.2)

$$\int_{\partial\mathcal{F}(t)} u\cdot n = \int_{\mathcal{F}(t)} \mathrm{div}\ u = 0,$$

and, in using (1.3), from the formula of the change of variables

$$\begin{aligned}\int_{\partial\mathcal{F}(t)} u\cdot n &= \int_{\partial\mathcal{S}(t)} u\cdot n\\ &= \int_{\partial\mathcal{S}} (\mathbf{R}(t)\tilde{u}(y,t))\cdot(\mathrm{cof}X(y,t)n)\,\mathrm{d}\Gamma\\ &= \int_{\partial\mathcal{S}} (\mathbf{R}(t)\tilde{u}(y,t))\cdot\left(\mathbf{R}(t)\mathrm{cof}\tilde{X}(y,t)n\right)\mathrm{d}\Gamma\\ &= \int_{\partial\mathcal{S}} \tilde{u}(y,t)\cdot\left(\nabla\mathrm{cof}\tilde{X}(y,t)n\right)\mathrm{d}\Gamma.\end{aligned}$$

For the two other equalities, let us see that

$$\forall x \in \mathcal{F}(t), \quad u(x,t) = \mathbf{R}(t)\tilde{u}(Y(x,t),t),$$

implies

$$\forall y \in \mathcal{F}, \quad \nabla u(X(y,t),t) = \mathbf{R}(t)\nabla\tilde{u}(y,t)\nabla Y(X(y,t),t),$$

which means, from (1.28), that

$$\forall y \in \mathcal{F}, \quad \nabla u(X(y,t),t) = \mathbf{R}(t)\nabla\tilde{u}(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t)\mathbf{R}(t)^T.$$

This leads us to

$$\forall y \in \mathcal{F}, \quad D(u)(X(y,t),t) = \frac{\nu}{2}\mathbf{R}(t)\left(\nabla\tilde{u}(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t) + \nabla\tilde{Y}(\tilde{X}(y,t),t)^T\nabla\tilde{u}(y,t)^T\right)\mathbf{R}(t)^T,$$

and thus to the equality (1.25) obtained as follows

$$\begin{aligned}\int_{\partial\mathcal{S}(t)} \sigma(u,p)n\mathrm{d}\Gamma &= \int_{\partial\mathcal{S}} \mathbf{R}(t)\tilde{\Sigma}(\tilde{u},\tilde{p})\mathbf{R}(t)^T\nabla Y(X(y,t),t)^T n\mathrm{d}\Gamma \\ &= \mathbf{R}(t)\int_{\partial\mathcal{S}} \tilde{\Sigma}(\tilde{u},\tilde{p})\nabla\tilde{Y}(\tilde{X}(y,t),t)^T n\mathrm{d}\Gamma.\end{aligned}$$

From there, we prove the equality (1.26) in the same way as above:

$$\begin{aligned}&\int_{\partial\mathcal{S}(t)} (x-h(t))\wedge\sigma(u,p)(x,t)n\mathrm{d}\Gamma \\ &= \int_{\partial\mathcal{S}} \left(\mathbf{R}(t)\tilde{X}(y,t)\right)\wedge\left(\mathbf{R}(t)\tilde{\Sigma}(\tilde{u},\tilde{p})\mathbf{R}(t)^T\nabla Y(X(y,t),t)^T n\right)\mathrm{d}\Gamma \\ &= \mathbf{R}(t)\left(\int_{\partial\mathcal{S}} \tilde{X}(y,t)\wedge\left(\tilde{\Sigma}(\tilde{u},\tilde{p})\nabla\tilde{Y}(\tilde{X}(y,t),t)^T n\right)\mathrm{d}\Gamma\right).\end{aligned}$$

□

Statement

For $h\in\mathrm{H}^2(0,T;\mathbb{R}^3)$ et $\omega\in\mathrm{H}^1(0,T;\mathbb{R}^3)$, let us recall and complete the change of unknowns introduced at the beginning of this subsection. We set

$$\begin{array}{ll}\tilde{u}(y,t)=\mathbf{R}(t)^T u(X(y,t),t), & \tilde{p}(y,t)=p(X(y,t),t), \\ \tilde{h}'(t)=\mathbf{R}(t)^T h'(t), & \tilde{\omega}(t)=\mathbf{R}(t)^T\omega(t).\end{array} \tag{1.29}$$

Remark 1.8. *The aim of this change of unknowns is to rewrite system* (1.1)–(1.7) *in terms of* $(\tilde{u},\tilde{p},\tilde{h}',\tilde{\omega})$*. Thus we want to be able to come back to the unknowns* (u,p,h',ω) *after having obtained* $(\tilde{u},\tilde{p},\tilde{h}',\tilde{\omega})$*.*
Notice that if $\tilde{h}'$ *and* $\tilde{\omega}$ *are given, then using the last equality of* (1.29) *we see that* $\mathbf{R}$ *satisfies the Cauchy problem*

$$\left\{\begin{array}{lllll}\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}) & = & \mathbb{S}\left(\mathbf{R}\tilde{\omega}\right)\mathbf{R} & = & \mathbf{R}\mathbb{S}\left(\tilde{\omega}\right) \\ \mathbf{R}(t=0) & = & \mathrm{I}_{\mathbb{R}^3} & & \end{array}\right., \quad \textit{with } \mathbb{S}(\omega)=\begin{pmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{pmatrix}.$$

So $\mathbf{R}$ *is determined in a unique way. Thus it is obvious to see that, in* (1.29)*,* h' *and* ω *are also determined in a unique way.*
From there, for $(\tilde{u},\tilde{p},\tilde{h}',\tilde{\omega})$ *given, it is also easy to see that* (u,p) *is determined in a unique way.*

We have also set

$$\begin{array}{ll}\tilde{X}(y,t)=\mathbf{R}(t)^T\left(X(y,t)-h(t)\right), & y\in\mathcal{F}, \\ \tilde{Y}(\tilde{x},t)=Y\left(h(t)+\mathbf{R}(t)\tilde{x},t\right), & \tilde{x}\in\mathbf{R}(t)^T\left(\mathcal{F}(t)-h(t)\right).\end{array}$$

Rewriting system (1.1)–(1.7) leads us to study the following system

$$\frac{\partial\tilde{u}}{\partial t}-\nu[\mathbf{L}\tilde{u}]+[\mathbf{M}\tilde{u}]+[\mathbf{N}\tilde{u}]+\tilde{\omega}(t)\wedge\tilde{u}+[\mathbf{G}\tilde{p}] = \tilde{f}, \quad y\in\mathcal{F}, \quad t\in(0,T), \tag{1.30}$$

$$\mathrm{div}\ \tilde{u} = g_{\tilde{u}}, \quad y\in\mathcal{F}, \quad t\in(0,T), \tag{1.31}$$

$$\tilde{u}=0, \qquad y\in\partial\mathcal{O}, \quad t\in(0,T), \tag{1.32}$$

$$\tilde{u}=\tilde{h}'(t)+\tilde{\omega}(t)\wedge(X^*(y,t))+\frac{\partial X^*}{\partial t}(y,t), \qquad y\in\partial\mathcal{S}, \quad t\in(0,T), \tag{1.33}$$

$$M\tilde{h}''(t) = -\int_{\partial\mathcal{S}} \tilde{\Sigma}(\tilde{u},\tilde{p})\mathrm{cof}\left(\nabla\tilde{X}\right) n\mathrm{d}\Gamma - M\tilde{\omega}(t)\wedge\tilde{h}'(t), \qquad t\in(0,T) \tag{1.34}$$

$$I^*(t)\tilde{\omega}'(t) = -\int_{\partial\mathcal{S}} \tilde{X}\wedge\left(\tilde{\Sigma}(\tilde{u},\tilde{p})\mathrm{cof}\left(\nabla\tilde{X}\right)n\right)\mathrm{d}\Gamma - I^{*\prime}(t)\tilde{\omega}(t) + I^*(t)\tilde{\omega}(t)\wedge\tilde{\omega}(t), \qquad t\in(0,T) \tag{1.35}$$

$$\tilde{u}(y,0) = u_0(y),\ y\in\mathcal{F}, \quad \tilde{h}'(0) = h_1\in\mathbb{R}^3, \quad \tilde{\omega}(0) = \omega_0\in\mathbb{R}^3, \tag{1.36}$$

where I^* is given by

$$I^*(t) \quad = \quad \int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0)\left(|X^*(y,t)|^2\,\mathrm{I}_{\mathbb{R}^3} - X^*(y,t)\otimes X^*(y,t)\right)\mathrm{d}y,$$

and, if $[\cdot]_i$ denotes the i-st component of a vector, we have

$$[\mathbf{L}\tilde{u}]_i(y,t) = [\nabla\tilde{u}(y,t)\Delta\tilde{Y}(\tilde{X}(y,t),t)]_i + \nabla^2\tilde{u}_i(y,t) : \left(\nabla\tilde{Y}\nabla\tilde{Y}^T\right)(\tilde{X}(y,t),t), \tag{1.37}$$

$$\mathbf{M}(\tilde{u},\tilde{h}',\tilde{\omega})(y,t) = -\nabla\tilde{u}(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t)\left(\tilde{h}'(t) + \tilde{\omega}\wedge\tilde{X}(y,t) + \frac{\partial\tilde{X}}{\partial t}(y,t)\right), \tag{1.38}$$

$$\mathbf{N}\tilde{u}(y,t) = \nabla\tilde{u}(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t)\tilde{u}(y,t), \tag{1.39}$$

$$\mathbf{G}\tilde{p}(y,t) = \nabla\tilde{Y}(\tilde{X}(y,t),t)^T\nabla\tilde{p}(y,t), \tag{1.40}$$

$$\tilde{f}(y,t) = \mathbf{R}(t)^T f(X(y,t),t)$$

and

$$g_{\tilde{u}}(y,t) \quad = \quad \nabla\tilde{u}(y,t) : \left(\mathrm{I}_{\mathbb{R}^3} - \nabla\tilde{Y}\left(\tilde{X}(y,t),t\right)^T\right). \tag{1.41}$$

Like in Definition 1.2, we can define a strong solution for system (1.30)–(1.36).

Definition 1.9. *Let be $T>0$. A quadruplet $(\tilde{u},\tilde{p},h,\omega)$ is called a strong solution of the system* (1.30)–(1.36) *when*

$$(\tilde{u},\tilde{p},h,\omega)\in\mathcal{U}(0,T;\mathcal{F})\times\mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{F}))\times\mathrm{H}^2(0,T;\mathbb{R}^3)\times\mathrm{H}^1(0,T;\mathbb{R}^3),$$

$\mathrm{dist}(\mathcal{S},\partial\mathcal{O})>0$*, and when $(\tilde{u},\tilde{p},h,\omega)$ satisfies the system* (1.30)–(1.36) *almost everywhere in $\mathcal{F}\times(0,T)$, or in the trace sense.*

Proposition 1.10. *A quadruplet $(u,p,h,\omega)\in\mathcal{U}(0,T;\mathcal{F}(t))\times\mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{F}(t)))\times\mathrm{H}^2(0,T;\mathbb{R}^3)\times\mathrm{H}^1(0,T;\mathbb{R}^3)$ is a strong solution - in the sense of Definition 1.2 - of system* (1.1)–(1.7) *if and only if $(\tilde{u},\tilde{p},h,\omega)\in\mathcal{U}(0,T;\mathcal{F})\times\mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{F}))\times\mathrm{H}^2(0,T;\mathbb{R}^3)\times\mathrm{H}^1(0,T;\mathbb{R}^3)$ is a strong solution - in the sense of Definition 1.9 - of system* (1.30)–(1.36).

Proof. We let to the reader the care of doing the calculations, by using - in particular - the equalities (1.11), (1.9), (1.25) and (1.26), and by using also the expression of $\tilde{\Sigma}$ given by (1.27). The calculations involve mainly derivations of function compositions. □

The nonhomogeneous divergence condition

Using the Piola identity, the nonhomogeneous divergence condition $g_{\tilde{u}}$ that we have obtained in (1.41) can also be expressed as

$$g_{\tilde{u}} = \mathrm{div}(G(\tilde{u})), \tag{1.42}$$

where

$$G(\tilde{u})(y,t) \quad = \quad \left(\mathrm{I}_{\mathbb{R}^3} - \nabla \tilde{Y}\left(\tilde{X}(y,t),t\right)\right)\tilde{u}(y,t). \tag{1.43}$$

Indeed, the mapping Λ constructed in section 1.3.1 satisfies

$$\mathrm{div}\ \Lambda \quad = \quad 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{S}(t)}$$

For the mapping X so deduced from Problem (1.20), this implies the following property

$$\det \nabla X(y,t) = 1, \qquad \forall (y,t) \in \mathcal{F} \times (0,T). \tag{1.44}$$

Then the Piola identity for the matrix field ∇X becomes

$$\mathrm{div}\left(\nabla Y(X(y,t),t)^T\right) \quad = \quad 0.$$

Since the property (1.44) is invariant by translations and rotations, we also have

$$\mathrm{div}\left(\nabla \tilde{Y}(\tilde{X}(y,t),t)^T\right) \quad = \quad 0.$$

That is why we can write (1.43).

Proposition 1.11. *If $\tilde{u} \in \mathcal{U}(0,T;\mathcal{F})$, then*

$$G(\tilde{u}) \in \mathcal{U}(0,T;\mathcal{F}), \quad G(\tilde{u})(\cdot,0) = 0 \text{ in } \mathcal{F}, \quad \text{and } G(\tilde{u}) = 0 \text{ on } \partial\mathcal{O}.$$

Moreover, $g_{\tilde{u}} = \mathrm{div}(G(\tilde{u}))$ satisfies the compatibility condition

$$\int_{\mathcal{F}} g_{\tilde{u}} \quad = \quad \int_{\partial\mathcal{S}} \left(\tilde{h}' + \tilde{\omega} \wedge X^*(y,t) + \frac{\partial X^*}{\partial t}(y,t)\right) \cdot n \mathrm{d}\Gamma. \tag{1.45}$$

Proof. If $\tilde{u} \in \mathcal{U}(0,T;\mathcal{F})$, then it is easy to verify, from the expression (1.43) and the regularity given in Remark 1.6, that $G(\tilde{u})$ lies in $\mathcal{U}(0,T;\mathcal{F})$. Moreover, the equation (1.32) shows that $G_{\tilde{u},\tilde{X},\tilde{Y}} = 0$ on $\partial\mathcal{O}$. And still from (1.43), the equality (1.33)

$$\tilde{u} = \tilde{h}'(t) + \tilde{\omega}(t) \wedge (X^*(y,t)) + \frac{\partial X^*}{\partial t}(y,t) \quad \text{for } y \in \partial\mathcal{S} \text{ and } t \in (0,T)$$

allows us to extend easily $G_{\tilde{u},\tilde{X},\tilde{Y}}$ on $\partial\mathcal{S}$ in order to get $G_{\tilde{u},\tilde{X},\tilde{Y}} \in \mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))$.
The equality (1.45) comes directly from the divergence formula and the conditions (1.31), (1.32) and (1.33). ☐

1.4 The linearized system

1.4.1 Linearization of the right-hand sides

In order to use a fixed point method, we consider the mappings $\tilde{X}$ and $\tilde{Y}$ close to the values they take at time $t = 0$, and we linearize the nonlinear terms of the equations (1.30), (1.33), (1.34) and (1.35) around $(\tilde{u}, \tilde{p}, \tilde{h}', \tilde{\omega}) = (0,0,0,0)$.
For a more clear view in equations (1.34) and (1.35), we will use the following notation

$$\overline{\Sigma}^*(\tilde{u},\tilde{p})(y,\bar{t},t) = \nu\left(\nabla\tilde{u}(y,t)\nabla Y^*(X(y,t),\bar{t}) + \nabla Y^*(X^*(y,t),\bar{t})^T \nabla\tilde{u}(y,t)^T\right) - \tilde{p}(y,t)\mathrm{Id},$$

$$\overline{\mathcal{F}}_M(\tilde{u},\tilde{p})(\bar{t},t) = -\int_{\partial\mathcal{S}} \overline{\Sigma}^*(\tilde{u},\tilde{p})(y,\bar{t},t)\mathrm{cof}\left(\nabla X^*(y,\bar{t})\right) n \mathrm{d}\Gamma,$$

$$\overline{\mathcal{F}}_I(\tilde{u},\tilde{p})(\bar{t},t) = -\int_{\partial\mathcal{S}} X^*(y,\bar{t}) \wedge \left(\overline{\Sigma}^*(\tilde{u},\tilde{p})(y,\bar{t},t)\mathrm{cof}\left(\nabla X^*(y,\bar{t})\right) n\right) \mathrm{d}\Gamma.$$

Proposition 1.12. *The system* (1.30)–(1.36) *can be rewritten as follows*

$$\frac{\partial \tilde{u}}{\partial t} - \nu\Delta\tilde{u} + \nabla\tilde{p} = \mathfrak{F}(\tilde{u},\tilde{p},\tilde{h}',\tilde{\omega}) \qquad \text{in } \mathcal{F}\times(0,T),$$
$$\text{div } \tilde{u} = \text{div } G(\tilde{u}) \qquad \text{in } \mathcal{F}\times(0,T),$$

$$\tilde{u} = 0, \qquad \text{on } \partial\mathcal{O}\times(0,T),$$
$$\tilde{u} = \tilde{h}'(t) + \tilde{\omega}(t)\wedge(y-h_0) + \mathfrak{W}(\tilde{\omega})(\mathrm{y},t), \qquad y\in\partial\mathcal{S}, \quad t\in(0,T),$$

$$M\tilde{h}''(t) = -\int_{\partial\mathcal{S}} \sigma(\tilde{u},\tilde{p})n\mathrm{d}\Gamma + \mathfrak{F}_M(\tilde{u},\tilde{p},\tilde{h}',\tilde{\omega})(t), \qquad t\in(0,T),$$
$$I_0\tilde{\omega}'(t) = -\int_{\partial\mathcal{S}} (y-h_0)\wedge\sigma(\tilde{u},\tilde{p})n\mathrm{d}\Gamma + \mathfrak{F}_I(\tilde{u},\tilde{p},\tilde{\omega})(t), \qquad t\in(0,T),$$

$$\tilde{u}(y,0) = u_0(y),\ y\in\mathcal{F}, \quad \tilde{h}'(0) = h_1\in\mathbb{R}^3, \quad \tilde{\omega}(0) = \omega_0\in\mathbb{R}^3,$$

with

$$\begin{aligned}
\mathfrak{F}(\tilde{u},\tilde{p},\tilde{h}',\tilde{\omega}) &= \nu[(\mathbf{L}-\Delta)\tilde{u}] - [\mathbf{M}\tilde{u}] - [\mathbf{N}\tilde{u}] - \tilde{\omega}\wedge\tilde{u} + [(\nabla-\mathbf{G})\tilde{p}] + \tilde{f},\\
\mathfrak{W}(\tilde{\omega})(\mathrm{y},t) &= \tilde{\omega}(t)\wedge(X^*(y,t)-X^*(y,0)) + \frac{\partial X^*}{\partial t}(y,t),\\
\mathfrak{F}_M(\tilde{u},\tilde{p},\tilde{h}',\tilde{\omega})(t) &= \overline{\mathcal{F}}_M(\tilde{u},\tilde{p})(t,t) - \overline{\mathcal{F}}_M(\tilde{u},\tilde{p})(0,t) - M\tilde{\omega}(t)\wedge\tilde{h}'(t),\\
\mathfrak{F}_I(\tilde{u},\tilde{p},\tilde{\omega})(t) &= \overline{\mathcal{F}}_I(\tilde{u},\tilde{p})(t,t) - \overline{\mathcal{F}}_I(\tilde{u},\tilde{p})(0,t)\\
&\quad - (I^*(t)-I_0)\,\tilde{\omega}'(t) - {I^*}'(t)\tilde{\omega}(t) + I^*(t)\tilde{\omega}(t)\wedge\tilde{\omega}(t).
\end{aligned}$$

Let us keep in mind that $\mathbf{L}$*,* $\mathbf{M}$*,* $\mathbf{N}$ *and* $\mathbf{G}$ *are given by* (1.37)–(1.40) *and depend on* $\tilde{X}$ *and* $\tilde{Y}$*. The quantity* I_0 *denotes* $I(0)$*.*

1.4.2 The linearized system

Let $\mathbb{F}$, g, $\mathbb{W}$, $\mathbb{F}_M$ and $\mathbb{F}_I$ be some data. We assume that g satisfies the condition

$$\int_{\mathcal{F}} g = \int_{\partial\mathcal{S}} \mathbb{W}\cdot n\mathrm{d}\Gamma$$

and can be written as in (1.42)

$$g = \text{div } G,$$

with the regularities on G given by Proposition 1.11. For the other data we assume that

$$\mathbb{F}\in \mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F})), \qquad \mathbb{W}\in \mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S})),$$
$$\mathbb{F}_M\in \mathrm{L}^2(0,T;\mathbb{R}^3), \qquad \mathbb{F}_I\in \mathrm{L}^2(0,T;\mathbb{R}^3).$$

We now consider the following linear system

$$\frac{\partial \tilde{U}}{\partial t} - \nu\Delta\tilde{U} + \nabla\tilde{P} = \mathbb{F}, \qquad \text{in } \mathcal{F}\times(0,T), \tag{1.46}$$
$$\text{div } \tilde{U} = \text{div } G, \qquad \text{in } \mathcal{F}\times(0,T), \tag{1.47}$$

$$\tilde{U} = 0, \qquad \text{on } \partial\mathcal{O} \times (0,T), \tag{1.48}$$
$$\tilde{U} = \tilde{h}'(t) + \tilde{\omega}(t) \wedge (y - h_0) + \mathbb{W}, \qquad y \in \partial\mathcal{S}, \quad t \in (0,T), \tag{1.49}$$

$$M\tilde{h}''(t) = -\int_{\partial\mathcal{S}} \sigma(\tilde{U}, \tilde{P}) n \mathrm{d}\Gamma + \mathbb{F}_M, \quad t \in (0,T), \tag{1.50}$$
$$I_0 \tilde{\omega}'(t) = -\int_{\partial\mathcal{S}} (y - h_0) \wedge \sigma(\tilde{U}, \tilde{P}) n \mathrm{d}\Gamma + \mathbb{F}_I, \quad t \in (0,T), \tag{1.51}$$

$$\tilde{U}(y,0) = u_0(y), \; y \in \mathcal{F}, \quad \tilde{h}'(0) = h_1 \in \mathbb{R}^3, \quad \tilde{\omega}(0) = \omega_0 \in \mathbb{R}^3. \tag{1.52}$$

A lifting method

By setting $U = \tilde{U} - G$, $P = \tilde{P}$, $H' = \tilde{h}'$ and $\Omega = \tilde{\omega}$ we rewrite the system (1.46)–(1.52) as

$$\frac{\partial U}{\partial t} - \nu \Delta U + \nabla P = \hat{F}, \qquad \text{in } \mathcal{F} \times (0,T),$$
$$\text{div } U = 0, \qquad \text{in } \mathcal{F} \times (0,T),$$

$$U = 0, \qquad \text{on } \partial\mathcal{O} \times (0,T),$$
$$U = H'(t) + \Omega(t) \wedge (y - h_0) + \hat{W}, \qquad y \in \partial\mathcal{S}, \; t \in (0,T),$$

$$MH''(t) = -\int_{\partial\mathcal{S}} \sigma(U,P) n \mathrm{d}\Gamma + \hat{F}_M, \quad t \in (0,T),$$
$$I_0 \Omega'(t) = -\int_{\partial\mathcal{S}} (y - h_0) \wedge \sigma(U,P) n \mathrm{d}\Gamma + \hat{F}_I, \quad t \in (0,T),$$

$$U(y,0) = u_0(y), \; y \in \mathcal{F}, \quad H'(0) = h_1 \in \mathbb{R}^3, \quad \Omega(0) = \omega_0 \in \mathbb{R}^3,$$

with

$$\hat{F} = \mathbb{F} - \frac{\partial G}{\partial t} + 2\nu D(G), \qquad \hat{W} = \mathbb{W} - G,$$
$$\hat{F}_M = \mathbb{F}_M - 2\nu \int_{\partial\mathcal{S}} D(G) n \mathrm{d}\Gamma, \qquad \hat{F}_I = \mathbb{F}_I - 2\nu \int_{\partial\mathcal{S}} (y - h_0) \wedge D(G) n \mathrm{d}\Gamma.$$

We now use a lifting method in order to tackle the non-homogeneous Dirichlet condition $\hat{W}$ and establish an existence result for the linear system (1.46)–(1.52). We split this problem into two more simple problems, by setting

$$U = V + \mathrm{w}, \qquad P = Q + \pi,$$

where, for all $t \in (0,T)$, the couple (w, π) satisfies

$$-\nu \Delta \mathrm{w}(t) + \nabla \pi(t) = 0, \qquad \text{in } \mathcal{F}, \tag{1.53}$$
$$\text{div } \mathrm{w}(t) = 0, \qquad \text{in } \mathcal{F}, \tag{1.54}$$
$$\mathrm{w}(t) = W(\cdot, t), \qquad \text{on } \partial\mathcal{S}, \tag{1.55}$$
$$\mathrm{w}(t) = 0, \qquad \text{on } \partial\mathcal{O}, \tag{1.56}$$

and where the couple (V, Q) satisfies

$$\frac{\partial V}{\partial t} - \nu \Delta V + \nabla Q = F, \qquad \text{in } \mathcal{F} \times (0,T), \tag{1.57}$$
$$\operatorname{div} V = 0, \qquad \text{in } \mathcal{F} \times (0,T), \tag{1.58}$$
$$V = 0, \qquad \text{on } \partial\mathcal{O} \times (0,T), \tag{1.59}$$
$$V = H'(t) + \Omega(t) \wedge (y - h_0), \qquad y \in \partial\mathcal{S}, \quad t \in (0,T), \tag{1.60}$$
$$MH''(t) = -\int_{\partial\mathcal{S}} \sigma(V,Q) n \mathrm{d}\Gamma + F_M, \qquad t \in (0,T), \tag{1.61}$$
$$I_0 \Omega'(t) = -\int_{\partial\mathcal{S}} (y - h_0) \wedge \sigma(V,Q) n \mathrm{d}\Gamma + F_I, \qquad t \in (0,T), \tag{1.62}$$

$$V(y,0) = u_0(y) - \mathrm{w}(y,0), \ y \in \mathcal{F}, \quad H'(0) = h_1 \in \mathbb{R}^3, \quad \Omega(0) = \omega_0 \in \mathbb{R}^3, \tag{1.63}$$

with

$$F = \hat{F} - \frac{\partial \mathrm{w}}{\partial t}, \qquad W = \hat{W},$$
$$F_M = \hat{F}_M + \int_{\partial\mathcal{S}} \sigma(\mathrm{w},\pi) n \mathrm{d}\Gamma, \qquad F_I = \hat{F}_I + \int_{\partial\mathcal{S}} (y - h_0) \wedge \sigma(\mathrm{w},\pi) n \mathrm{d}\Gamma.$$

To sum up, we have

$$F = \mathbb{F} - \frac{\partial G}{\partial t} + \nu \Delta G - \frac{\partial \mathrm{w}}{\partial t}, \tag{1.64}$$
$$W = \mathbb{W} - G, \tag{1.65}$$
$$F_M = \mathbb{F}_M - 2\nu \int_{\partial\mathcal{S}} D(G) n \mathrm{d}\Gamma + \int_{\partial\mathcal{S}} \sigma(\mathrm{w},\pi) n \mathrm{d}\Gamma, \tag{1.66}$$
$$F_I = \mathbb{F}_I - 2\nu \int_{\partial\mathcal{S}} (y - h_0) \wedge D(G) n \mathrm{d}\Gamma + \int_{\partial\mathcal{S}} (y - h_0) \wedge \sigma(\mathrm{w},\pi) n \mathrm{d}\Gamma. \tag{1.67}$$

Stokes problem

We now look at the problem (1.53)–(1.56). Let us notice that we have the compatibility condition

$$\int_{\partial\mathcal{S}} (\mathbb{W}(y) - G(y)) \cdot n \mathrm{d}\Gamma \quad = \quad 0.$$

Let us set a result of existence and uniqueness in $\mathcal{U}(0,T;\mathcal{F}) \times \mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{F}))$ for this non-homogeneous problem, which is a consequence of a result stated in [Gal94], Exercise 6.2, Chapter 4.

Proposition 1.13. *There exists a unique couple* $(\mathrm{w},\pi) \in \mathcal{U}(0,T;\mathcal{F}) \times \mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{F}))$ *solution of the system* (1.53)–(1.56) *for almost all* $t \in (0,T)$*. Moreover, there exists a positive constant* C *such that*

$$\|\mathrm{w}\|_{\mathcal{U}(0,T;\mathcal{F})} + \|\nabla \pi\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))} \quad \leq \quad C \left(\|G\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{F}))} + \|\mathbb{W}\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{F}))} \right).$$

Semigroup approach

We solve (1.57)–(1.63) in the same way as it is done in [TT04]. We project V on the space

$$\mathcal{H} = \left\{\phi \in \mathbf{L}^2(\mathcal{O}) \mid \text{ div } \phi = 0 \text{ in } \mathcal{O},\ D(\phi) = 0 \text{ in } \mathcal{S},\ \phi \cdot n = 0 \text{ on } \partial\mathcal{O}\right\},$$

and we consider

$$\mathcal{V} = \left\{\phi \in \mathbf{H}^1(\mathcal{O}) \mid \text{ div } \phi = 0 \text{ in } \mathcal{O},\ D(\phi) = 0 \text{ in } \mathcal{S},\ \phi \cdot n = 0 \text{ on } \partial\mathcal{O}\right\}.$$

Let us recall a lemma stated in [Tem83, page 18].

Lemma 1.14. *For all $\phi \in \mathcal{H}$, there exists $l_\phi \in \mathbb{R}^3$ and $\omega_\phi \in \mathbb{R}$ such that*

$$\phi(y) = l_\phi + \omega_\phi \wedge (y - h_0) \ \textit{ for all } y \in \mathcal{S}.$$

This result allows us to extend V in $\mathcal{S}$ and then consider the system in the whole domain $\mathcal{O}$. Indeed, for $V \in \mathcal{H}$, this lemma gives us H' and Ω

$$\begin{aligned} V &= H_V'(t) + \Omega_V(t) \wedge (y - h_0) \\ &= H'(t) + \Omega(t) \wedge (y - h_0). \end{aligned}$$

Let us now define a new inner product on $\mathbf{L}^2(\mathcal{O})$ by

$$(\psi, \phi)_{\mathbf{L}^2(\mathcal{O})} = \int_{\mathcal{F}} (\psi \cdot \phi)\mathrm{d}y + \rho_{\mathcal{S}} \int_{\mathcal{S}} \psi(y) \cdot \phi(y)\mathrm{d}y. \tag{1.68}$$

We recall that $\rho_{\mathcal{S}}$ is the density of the rigid body $\mathcal{S}$. The corresponding Euclidean norm is equivalent to the usual one in $\mathbf{L}^2(\mathcal{O})$. If ψ et ϕ lie in $\mathcal{H}$, then a simple calculation leads us to

$$(\psi, \phi)_{\mathbf{L}^2(\mathcal{O})} = \int_{\mathcal{F}} (\psi \cdot \phi)\mathrm{d}y + M l_\phi \cdot l_\psi + I_0 \omega_\phi \cdot \omega_\psi.$$

In order to solve (1.57)–(1.63) we use a semigroup approach. We define

$$D(A) \ = \ \left\{\phi \in \mathbf{H}^1(\mathcal{O}) \mid \ \phi_{|\mathcal{F}} \in \mathbf{H}^2(\mathcal{F}),\ \text{div } \phi = 0 \text{ in } \mathcal{O},\ D(\phi) = 0 \text{ in } \mathcal{S},\ \phi \cdot n = 0 \text{ on } \partial\mathcal{O}\right\}.$$

For all $V \in D(A)$ we set

$$\mathcal{A}V = \begin{cases} -\nu\Delta V \ \text{ in } \mathcal{F}, \\ \dfrac{2\nu}{M}\displaystyle\int_{\partial\mathcal{S}} D(V)n\mathrm{d}\Gamma \\ + \left(2\nu {I_0}^{-1} \displaystyle\int_{\partial\mathcal{S}} (y - h_0) \wedge D(V)n\mathrm{d}\Gamma\right) \wedge (y - h_0) \ \text{ in } \mathcal{S}, \end{cases}$$

and

$$AV = \mathbb{P}\mathcal{A}V,$$

where $\mathbb{P}$ is the orthogonal projection from $\mathbf{L}^2(\mathcal{O})$ onto $\mathcal{H}$. Then we get a unique solution in $\mathcal{U}(0, T; \mathcal{F}) \times \mathrm{L}^2(0, T; \mathrm{H}^1(\mathcal{F})) \times \mathrm{H}^2(0, T; \mathbb{R}^3) \times \mathrm{H}^1(0, T; \mathbb{R}^3)$ by following the steps of [TT04].

1.4.3 Result for the linear problem

Definition 1.15. *A quadruplet* $(\tilde{U}, \tilde{P}, \tilde{h}', \tilde{\omega})$ *is a solution of the linear problem* (1.46)–(1.52) *if there exists two couples* (V, Q) *et* (w, π) *such that* $(\mathrm{w}(t), \pi(t))$ *is the solution of the system* (1.53)–(1.56) *(given by Proposition 1.13) for all* $t \in (0, T)$*, such that* $(V, Q, \tilde{h}', \tilde{\omega})$ *is the solution of the problem* (1.57)–(1.63) *(given by the semigroup approach), and such that*

$$(\tilde{U}, \tilde{P}) = (V, Q) + (\mathrm{w}, \pi) + (G, 0).$$

Theorem 1.16. *Let* $\mathbb{F} \in \mathrm{L}^2(0, T; \mathbf{L}^2(\mathcal{F}))$, $\mathbb{F}_M \in \mathrm{L}^2(0, T; \mathbb{R}^3)$, $\mathbb{F}_I \in \mathrm{L}^2(0, T; \mathbb{R}^3)$, $\mathbb{W} \in \mathrm{H}^1(0, T; \mathbf{H}^{3/2}(\partial\mathcal{F}))$ *be given. Let* G *satisfy the results of Proposition 1.11. We assume that* $u_0 \in \mathbf{H}^1(\mathcal{F})$ *with*

$$\mathrm{div}\ u_0 = 0 \ \text{in } \mathcal{F}, \quad u_0 = 0 \ \text{on } \partial\mathcal{O}, \quad u_0(y) = h_1 + \omega_0 \wedge (y - h_0) \ \text{on } \partial\mathcal{S}.$$

Then the system (1.46)–(1.52) *admits a unique solution* $(\tilde{U}, \tilde{P}, \tilde{h}', \tilde{\omega})$ *(in the sense of Definition 1.15) in* $\mathcal{U}(0, T; \mathcal{F}) \times \mathrm{L}^2(0, T; \mathrm{H}^1(\mathcal{F})) \times \mathrm{H}^1(0, T; \mathbb{R}^3) \times \mathrm{H}^1(0, T; \mathbb{R}^3)$.
Moreover there exists a positive constant K *such that*

$$\begin{aligned}
&\|\tilde{U}\|_{\mathcal{U}} + \|\nabla P\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))} + \|h'\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)} + \|\omega\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)} \\
&\leq K \left(\|u_0\|_{\mathbf{H}^1(\mathcal{O})} + \|G\|_{\mathcal{U}(0,T;\mathcal{F})} + \|G\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))} \right. \\
&\qquad \left. + \|\mathbb{W}\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))} + \|\mathbb{F}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2)} + \|\mathbb{F}_M\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} + \|\mathbb{F}_I\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} \right).
\end{aligned}$$

Proof. Proposition 1.13 provides us a solution $(\mathrm{w}, \pi) \in \mathcal{U}(0, T; \mathcal{F}) \times \mathrm{L}^2(0, T; \mathrm{H}^1(\mathcal{F}))$ for the nonhomogeneous Stokes problem (1.53)–(1.56). Let us recall the expressions (1.64)–(1.67) of the quantities which appear in some second members of the system (1.57)–(1.63)

$$\begin{aligned}
&F = \mathbb{F} - \frac{\partial G}{\partial t} + \nu \Delta G - \frac{\partial \mathrm{w}}{\partial t}, \\
&W = \mathbb{W} - G, \\
&F_M = \mathbb{F}_M - 2\nu \int_{\partial\mathcal{S}} D(G) n \mathrm{d}\Gamma + \int_{\partial\mathcal{S}} \sigma(\mathrm{w}, \pi) n \mathrm{d}\Gamma, \\
&F_I = \mathbb{F}_I - 2\nu \int_{\partial\mathcal{S}} (y - h_0) \wedge D(G) n \mathrm{d}\Gamma + \int_{\partial\mathcal{S}} (y - h_0) \wedge \sigma(\mathrm{w}, \pi) n \mathrm{d}\Gamma.
\end{aligned}$$

Then the semigroup approach 1.4.2 gives us a unique solution(V, Q, H', Ω) for the problem (1.57)–(1.63), with

$$V \in \mathrm{L}^2\left(0, T; \mathbf{H}^2(\mathcal{F})\right) \cap C\left([0, T]; \mathbf{H}^1(\mathcal{F})\right) \cap \mathrm{H}^1\left(0, T; \mathbf{L}^2(\mathcal{F})\right),$$
$$Q \in \mathrm{L}^2\left(0, T; \mathrm{H}^1(\mathcal{F})\right), \quad H' \in \mathrm{H}^1\left(0, T; \mathbb{R}^3\right), \quad \Omega \in \mathrm{H}^1\left(0, T; \mathbb{R}^3\right).$$

We get then

$$(U, P) = (V, Q) + (\mathrm{w}, \pi),$$

so in setting $(\tilde{U}, \tilde{P}, \tilde{h}', \tilde{\omega}) = (U + G, P, H', \Omega)$ we get a solution for the problem (1.46)–(1.52) (which is unique). For the wanted estimate, we first write

$$\begin{aligned}
\|\tilde{U}\|_{\mathcal{U}(0,T;\mathcal{F})} &\leq \|U\|_{\mathcal{U}(0,T;\mathcal{F})} + \|G\|_{\mathcal{U}(0,T;\mathcal{F})} \\
&\leq \|V\|_{\mathcal{U}(0,T;\mathcal{F})} + \|\mathrm{w}\|_{\mathcal{U}(0,T;\mathcal{F})} + \|G\|_{\mathcal{U}(0,T;\mathcal{F})}
\end{aligned}$$

and

$$\|\nabla P\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))} \leq \|\nabla Q\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))} + \|\nabla \pi\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))}.$$

Then we use the estimate of Proposition 1.13

$$\begin{aligned}\|\tilde{U}\|_{\mathcal{U}} + \|\nabla P\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))} \quad \leq \quad & \|V\|_{\mathcal{U}} + \|\nabla Q\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))} + \|G\|_{\mathcal{U}} \\ & + C\left(\|G\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))} + \|\mathbb{W}\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))}\right).\end{aligned}$$

It remains to use an estimate of the semigroup theory for estimating $\|V\|_{\mathcal{U}} + \|\nabla Q\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))}$, and to use again the estimate of Proposition 1.13 to conclude. ☐

1.5 A local existence result for the nonlinear system

The main result of this section is the following:

Theorem 1.17. *Assume that w^* provides - through Problem (1.10) - a mapping X^* which satisfies the hypotheses* **H1**–**H4**. *Let be $f \in \mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}(t)))$, $\eta > 0$ such that $0 < \eta \leq \mathrm{dist}(\mathcal{S},\partial\mathcal{O})$, and $u_0 \in \mathbf{H}^1(\mathcal{F})$ such that*

$$\mathrm{div}\ u_0 = 0 \ \text{in } \mathcal{F}, \quad u_0 = 0 \ \text{on } \partial\mathcal{O}, \quad u_0(y) = h_1 + \omega_0 \wedge (y - h_0) \ \text{on } \partial\mathcal{S}.$$

Then there exists $T_0 > 0$ such that the problem (1.1)–(1.7) *admits a unique strong solution (u,p,h,ω) in*

$$\mathcal{U}(0,T_0;\mathcal{F}(t)) \times \mathrm{L}^2(0,T_0;\mathrm{H}^1(\mathcal{F}(t))) \times \mathrm{H}^2(0,T_0;\mathbb{R}^3) \times \mathrm{H}^1(0,T_0;\mathbb{R}^3).$$

Moreover, if we assume that, for all $t \in [0,T_0)$, $\mathrm{dist}(\mathcal{S}(t),\partial\mathcal{O}) \geq \eta$, then we have the alternative

(a) either $T_0 = +\infty$ (that is to say the solution is global in time)
(b) or the function $t \mapsto \|u(t)\|_{\mathbf{H}^1(\mathcal{F}(t))}$ is not bounded in $[0,T_0)$.

In order to prove this theorem, we first denote

$$\mathbb{H} = \mathcal{U}(0,T_0;\mathcal{F}) \times \mathrm{L}^2(0,T_0;\mathrm{H}^1(\mathcal{F})) \times \mathrm{H}^1(0,T_0;\mathbb{R}^3) \times \mathrm{H}^1(0,T_0;\mathbb{R}^3).$$

The solution of the problem (1.1)–(1.7) can be seen through system (1.30)–(1.36), and thus as a fixed point of the mapping

$$\begin{array}{rccc}\mathfrak{N}: & \mathbb{H} & \longrightarrow & \mathbb{H} \\ & (U,P,h',\omega) & \longmapsto & (\tilde{U},\tilde{P},\tilde{h}',\tilde{\omega})\end{array}$$

where $(\tilde{U},\tilde{P},\tilde{h},\tilde{\omega})$ is the solution (in the sense of Definition 1.15) of the system

$$\frac{\partial \tilde{U}}{\partial t} - \nu\Delta\tilde{U} + \nabla\tilde{P} = \mathfrak{F}(U,P,h',\omega), \qquad \text{in } \mathcal{F}\times(0,T),$$
$$\mathrm{div}\ \tilde{U} = g_U, \qquad \text{in } \mathcal{F}\times(0,T),$$

$$\tilde{U}(y,t) = 0, \quad y \in \partial\mathcal{O}, \quad t \in (0,T),$$
$$\tilde{U}(y,t) = \tilde{h}'(t) + \tilde{\omega}(t) \wedge (y - h_0) + \mathfrak{W}(\omega), \quad y \in \partial\mathcal{S}, \quad t \in (0,T),$$

$$M\tilde{h}''(t) = -\int_{\partial\mathcal{S}} \sigma(\tilde{U},\tilde{P})n\mathrm{d}\Gamma + \mathfrak{F}_M(U,P,h',\omega)(t), \quad t \in (0,T),$$
$$I_0\tilde{\omega}'(t) = -\int_{\partial\mathcal{S}} (y - h_0) \wedge \sigma(\tilde{U},\tilde{P})n\mathrm{d}y + \mathfrak{F}_I(U,P,h',\omega), \quad t \in (0,T),$$

$$\tilde{U}(y,0) = u_0(y),\ y \in \mathcal{F}, \quad \tilde{h}'(0) = h_1, \qquad \tilde{\omega}(0) = \omega_0,$$

with

$$\begin{aligned}
\mathfrak{F}(U,P,h',\omega) &= \nu[(\mathbf{L}-\Delta)U] - [\mathbf{M}U] - [\mathbf{N}U] - \omega \wedge U + [(\nabla - \mathbf{G})P] + \tilde{f}, \\
\mathfrak{W}(\omega)(\mathrm{y},t) &= \omega(t) \wedge (X^*(y,t) - X^*(y,0)) + \frac{\partial X^*}{\partial t}(y,t), \\
\mathfrak{F}_M(U,P,h',\omega)(t) &= \overline{\mathcal{F}}_M(U,P)(t,t) - \overline{\mathcal{F}}_M(U,P)(0,t) - M\omega(t) \wedge h'(t), \\
\mathfrak{F}_I(U,P,h',\omega)(t) &= \overline{\mathcal{F}}_I(U,P)(t,t) - \overline{\mathcal{F}}_I(U,P)(0,t) - (I^* - I_0)\,\omega'(t) - I^{*\prime}\omega(t) + I^*\omega(t) \wedge \omega(t), \\
g_U(y,t) &= \mathrm{trace}\left(\nabla U(y,t)\left(\mathrm{I}_{\mathbb{R}^3} - \nabla \tilde{Y}\left(\tilde{X}(y,t),t\right)\right)\right),
\end{aligned}$$

with $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{G}$ given by (1.37)–(1.40) and depending on $\tilde{X}$ and $\tilde{Y}$, with $\tilde{X}$ and $\tilde{Y}$ depending only on the mapping X^* (see Remark 1.6), and with $\mathfrak{F}_M$ and $\mathfrak{F}_I$ given by (1.46)–(1.46).
Let be $T > 0$ and $R > 0$. We define

$$\mathfrak{K} = \left\{(W,Q,h,\omega) \in \mathbb{H} \mid \|W\|_{\mathcal{U}} + \|\nabla Q\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))} + \|h''\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} + \|\omega'\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} \leq R\right\}.$$

Then we follow the steps of the proof which is given in [Tak03], in order to make $\mathfrak{K}$ stable by $\mathfrak{N}$ and $\mathfrak{N}$ contracting, for R large enough and T small enough. The alternative dealing with the explosion in time of the strong solution can be proven classically.

1.6 Global existence

In this part, we aim at proving that the strong solution given by Theorem 1.17 is global in time, under some assumptions on the data.

1.6.1 Statement of a global existence result

Theorem 1.18. *Assume that the hypotheses in Theorem 1.17 hold true. Moreover we assume that* $f \in \mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}(t))) \cap \mathrm{L}^{3/2}(0,\infty;\mathbf{L}^2(\mathcal{F}(t)))$. *If the data* $\|f\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}(t)))}$, $\|f\|_{\mathrm{L}^{3/2}(0,\infty;\mathbf{L}^2(\mathcal{F}(t)))}$, $\|w^*\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S}^*(t)))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S}^*(t)))}$, $\|u_0\|_{\mathbf{H}^1(\mathcal{F})}$, $|h_1|_{\mathbb{R}^3}$ *and* $|\omega_0|_{\mathbb{R}^3}$ *are small enough, then we are in the case of the assertion (a) in Theorem 1.17, that is to say that the strong solution of the problem* (1.1)–(1.7) *is global in time.*

In order to prove this global existence result, we are going to think by absurd. Assume that $T_0 < \infty$. Let us show that the functions

$$t \mapsto \|u(t)\|_{\mathbf{H}^1(\mathcal{F}(t))}, \quad t \mapsto |h'(t)|, \quad t \mapsto |\omega(t)|$$

are bounded in $[0,T_0)$.

1.6.2 Proof of Theorem 1.18

Let us begin by setting an intermediate result.

Proposition 1.19. *Let* (u,p,h,ω) *be a strong solution of the system* (1.1)–(1.7) *defined on* $[0,T_0)$ *with* $T_0 > 0$. *Furthermore assume that there exists* $\eta > 0$ *such that for all* $t \in [0,T_0)$

$$\mathrm{dist}(\mathcal{S}(t), \partial\mathcal{O}) \geq \eta.$$

Then there exists a positive constant C (depending on T_0 and η) such that

$$\|u\|_{\mathrm{L}^\infty(0,T_0;\mathbf{L}^2(\mathcal{F}(t)))} + \|u\|_{\mathrm{L}^2(0,T_0;\mathbf{H}^1(\mathcal{F}(t)))} + \|h'\|_{\mathrm{L}^\infty(0,T_0;\mathbb{R}^3)} + \|\omega\|_{\mathrm{L}^\infty(0,T_0;\mathbb{R}^3)} \quad \leq \quad CC_0^2,$$

with

$$C_0 := \exp\left(C\left(\|w^*\|^2_{\mathrm{L}^2(0,T_0;\mathbf{H}^3(\mathcal{S}^*(t)))} + \|w^*\|^4_{\mathrm{L}^2(0,T_0;\mathbf{L}^2(\mathcal{S}^*(t)))}\right)\right)\times$$
$$\left(\|u_0\|^2_{\mathbf{L}^2(\mathcal{F})} + |h_1|^2 + |\omega_0|^2 + \|w^*\|^2_{\mathrm{H}^1(0,T_0;\mathbf{H}^1(\mathcal{S}^*(t)))}\left(1+\|w^*\|^2_{\mathrm{L}^2(0,T_0;\mathbf{H}^3(\mathcal{S}^*(t)))}\right) + \|f\|^2_{\mathrm{L}^2(0,T_0;\mathbf{L}^2(\mathcal{F}(t)))}\right)^{1/2}.$$

Preliminary estimates In the proof of this proposition, we use the extended velocities $\overline{w}^*$ and $\overline{w}$ given in section 1.3.1. Let us recall the relation

$$\overline{w}(x,t) \quad = \quad \mathbf{R}(t)\overline{w}^*(\mathbf{R}(t)^T(x-h(t)),t), \quad x\in\mathcal{F}(t).$$

This yields the following estimates, for some positive constant C independent of time

$$\begin{aligned}
\|\overline{w}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}(t)))} &= \|\overline{w}^*\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\tilde{\mathcal{F}}(t)))},\\
\|\nabla\overline{w}\|_{\mathbf{L}^2(\mathcal{F}(t))} &\leq C\|\nabla\overline{w}^*\|_{\mathbf{L}^2(\tilde{\mathcal{F}}(t))},\\
\|(\overline{w}\cdot\nabla)\overline{w}\|_{\mathbf{L}^2(\mathcal{F}(t))} &\leq C\|(\overline{w}^*\cdot\nabla)\overline{w}^*\|_{\mathbf{L}^2(\tilde{\mathcal{F}}(t))},\\
\|\overline{w}\|_{\mathbf{W}^{1,\infty}(\mathcal{F}(t))} &\leq C\|\overline{w}^*\|_{\mathbf{W}^{1,\infty}(\tilde{\mathcal{F}}(t))},\\
\left\|\frac{\partial\overline{w}}{\partial t}\right\|_{\mathbf{L}^2(\mathcal{F}(t))} &\leq C\left(\left\|\frac{\partial\overline{w}^*}{\partial t}\right\|_{\mathbf{L}^2(\mathcal{F}(t))} + \|\overline{w}^*\|_{\mathbf{H}^1(\tilde{\mathcal{F}}(t))}(|h'|+|\omega|)\right).
\end{aligned}$$

Note that the extension $\overline{w}^*$ of the datum w^*, defined by the classical Dirichlet problem (1.18), obeys the following estimates for all integer $k\geq 1$ (see [Gal94] for instance):

$$\begin{aligned}
\|\overline{w}^*(\cdot,t)\|_{\mathbf{H}^k(\tilde{\mathcal{F}}(t))} &\leq \overline{C}_{\mathcal{F}}\|w^*(\cdot,t)\|_{\mathbf{H}^{k-1/2}(\partial\mathcal{S}^*(t))}\\
&\leq C_{\mathcal{F}}\|w^*(\cdot,t)\|_{\mathbf{H}^k(\mathcal{S}^*(t))}.
\end{aligned}$$

The constant $C_{\mathcal{F}}$ does not depend on time, since we have assumed that $\mathrm{dist}(\mathcal{S}(t),\partial\mathcal{O})\geq\eta>0$ for all $t\in[0,T_0)$. Thus we control the quantities involving $\overline{w}$ by the datum w^*, as follows

$$\begin{aligned}
\|\overline{w}\|_{\mathbf{H}^1(\mathcal{F}(t))} &\leq C_{\mathcal{F}}\|w^*\|_{\mathbf{H}^1(\mathcal{S}^*(t))}, && (1.69)\\
\left\|\frac{\partial\overline{w}}{\partial t}\right\|_{\mathbf{L}^2(\mathcal{F}(t))} &\leq C\left(\left\|\frac{\partial\overline{w}^*}{\partial t}\right\|_{\mathbf{H}^1(\mathcal{F}(t))} + \|\overline{w}^*\|_{\mathbf{H}^1(\tilde{\mathcal{F}}(t))}(|h'|+|\omega|)\right)\\
&\leq CC_{\mathcal{F}}\left(\left\|\frac{\partial w^*}{\partial t}\right\|_{\mathbf{H}^1(\mathcal{S}^*(t))} + \|w^*\|_{\mathbf{H}^1(\mathcal{S}^*(t))}(|h'|+|\omega|)\right), && (1.70)\\
\|\overline{w}\|_{\mathbf{W}^{1,\infty}(\mathcal{F}(t))} &\leq CC_3\|\overline{w}^*\|_{\mathbf{H}^3(\tilde{\mathcal{F}}(t))}\\
&\leq CC_3C_{\mathcal{F}}\|w^*\|_{\mathbf{H}^3(\mathcal{S}^*(t))}, && (1.71)\\
\|(\overline{w}\cdot\nabla)\overline{w}\|_{\mathbf{L}^2(\mathcal{F}(t))} &\leq C\|\overline{w}^*\|_{\mathbf{W}^{1,\infty}(\tilde{\mathcal{F}}(t))}\|\overline{w}^*\|_{\mathbf{L}^2(\tilde{\mathcal{F}}(t))}\\
&\leq CC_3\|w^*\|_{\mathbf{H}^3(\mathcal{S}^*(t))}\|w^*\|_{\mathbf{H}^1(\mathcal{S}^*(t))}. && (1.72)
\end{aligned}$$

Let us estimate the time-derivative of $I^*(t)$, written as

$$I^{*\prime}(t) \quad = \quad \int_{\mathcal{S}}\rho_{\mathcal{S}}(y,0)\left(2\left(\frac{\partial X^*}{\partial t}\cdot X^*\right)\mathrm{I}_{\mathbb{R}^3} - \frac{\partial X^*}{\partial t}\otimes X^* - X^*\otimes\frac{\partial X^*}{\partial t}\right)(y,t)\mathrm{d}y.$$

We have

$$\begin{aligned}
\left|I^{*\prime}(t)\right|_{\mathbb{R}^9} &\leq C_I \left\| \frac{\partial X^*}{\partial t}(\cdot,t)\right\|_{\mathbf{L}^2(\mathcal{S})} \|X^*(\cdot,t)\|_{\mathbf{L}^2(\mathcal{S})} \\
&\leq \tilde{C}_I \left\| \frac{\partial X^*}{\partial t}(\cdot,t)\right\|_{\mathbf{L}^2(\mathcal{S})} \left(1 + \|X^*(\cdot,t) - \mathrm{Id}_{\mathcal{S}}\|_{\mathbf{L}^2(\mathcal{S})}\right), \\
\left\|I^{*\prime}\right\|_{\mathrm{L}^2(0,T;\mathbb{R}^9)} &\leq \overline{C}_I \left\| \frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{S}))} \left(1 + \sqrt{T}\left\| \frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{S}))}\right).
\end{aligned}$$

Besides, we can verify that

$$\left\| \frac{\partial X^*}{\partial t}(\cdot,t)\right\|_{\mathbf{L}^2(\mathcal{S})} = \|w^*\|_{\mathbf{L}^2(\mathcal{S}^*(t))},$$

so that

$$\left\|I^{*\prime}\right\|_{\mathrm{L}^2(0,T;\mathbb{R}^9)} \leq \overline{C}_I \|w^*\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{S}^*(t)))}\left(1 + \sqrt{T}\|w^*\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{S}^*(t)))}\right). \tag{1.73}$$

Proof of Proposition 1.19 Let us set $v = u - \overline{w}$. The function v satisfies this system

$$\frac{\partial v}{\partial t} + (u\cdot\nabla)v - \nu\Delta u + \nabla p = f - (v\cdot\nabla)\overline{w} - (\overline{w}\cdot\nabla)\overline{w} - \frac{\partial \overline{w}}{\partial t}, \quad x\in\mathcal{F}(t), \quad t\in(0,T), \tag{1.74}$$

$$\mathrm{div}\ u = 0, \quad x\in\mathcal{F}(t), \quad t\in(0,T), \tag{1.75}$$

$$v = 0, \qquad x\in\partial\mathcal{O},\ t\in(0,T), \tag{1.76}$$

$$v = h'(t) + \omega(t)\wedge(x - h(t)), \qquad x\in\ \partial\mathcal{S}(t),\ t\in(0,T), \tag{1.77}$$

$$Mh''(t) = -\int_{\partial\mathcal{S}(t)} \sigma(u,p)n\mathrm{d}\Gamma, \qquad t\in(0,T), \tag{1.78}$$

$$(I\omega)'(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t))\wedge\sigma(u,p)n\mathrm{d}\Gamma, \qquad t\in(0,T), \tag{1.79}$$

$$h(0) = h_0\in\mathbb{R}^3, \quad h'(0) = h_1\in\mathbb{R}^3, \quad \omega(0) = \omega_0\in\mathbb{R}^3, \tag{1.80}$$

$$v(x,0) = v_0(x) := u_0(x) - \overline{w}_0(x), \quad x\in\mathcal{F}. \tag{1.81}$$

In the first equation we take the inner product with v and we integer on $\mathcal{F}(t)$ to get

$$\begin{aligned}
&\int_{\mathcal{F}(t)} \left(\frac{\partial v}{\partial t} + (u\cdot\nabla)v\right)\cdot v\ \mathrm{d}x - \int_{\mathcal{F}(t)} \mathrm{div}\left(\sigma(u,p)\right)\cdot v\ \mathrm{d}x \\
&= \int_{\mathcal{F}(t)} f\cdot v\ \mathrm{d}x - \int_{\mathcal{F}(t)} ((v\cdot\nabla)\overline{w})\cdot v\ \mathrm{d}x - \int_{\mathcal{F}(t)} ((\overline{w}\cdot\nabla)\overline{w})\cdot v\ \mathrm{d}x - \int_{\mathcal{F}(t)} \frac{\partial\overline{w}}{\partial t}\cdot v\ \mathrm{d}x.
\end{aligned} \tag{1.82}$$

- On one hand, we have by using the Reynolds transport theorem

$$\int_{\mathcal{F}(t)} \left(\frac{\partial v}{\partial t} + (u\cdot\nabla)v\right)\cdot v \ \mathrm{d}x \quad = \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{F}(t)} |v|^2 \mathrm{d}x\right).$$

- On the other hand, since div $v = 0$, we have

$$\mathrm{div}\,(\sigma(u,p))\cdot v \quad = \quad \mathrm{div}\,(\sigma(u,p)v) - 2\nu D(u) : D(v),$$

which implies - by using the divergence formula and the fact that v is equal to 0 on $\partial\mathcal{O}$ - that

$$\begin{aligned}
\int_{\mathcal{F}(t)} \mathrm{div}\,(\sigma(u,p))\cdot v \quad &= \quad \int_{\partial\mathcal{S}(t)} \sigma(u,p)v\cdot n\mathrm{d}\Gamma - 2\nu\int_{\mathcal{F}(t)} D(u) : D(v) \\
&= \quad -\int_{\partial\mathcal{S}(t)} \sigma(u,p)v\cdot n\mathrm{d}\Gamma - 2\nu\int_{\mathcal{F}(t)} D(\overline{w}) : D(v) - 2\nu\int_{\mathcal{F}(t)} |D(v)|^2 \ ;
\end{aligned}$$

and yet on $\partial\mathcal{S}(t)$ we have $v(t) = h'(t) + \omega(t)\wedge(x - h(t))$. Thus

$$\begin{aligned}
\int_{\mathcal{F}(t)} \mathrm{div}\,(\sigma(u,p))\cdot v \quad = \quad & h'(t)\cdot\int_{\partial\mathcal{S}(t)} \sigma(u,p)n\mathrm{d}\Gamma + \omega(t)\cdot\int_{\partial\mathcal{S}(t)} (x-h(t))\wedge\sigma(u,p)n\mathrm{d}\Gamma \\
& -2\nu\int_{\mathcal{F}(t)} D(\overline{w}) : D(v) - 2\nu\int_{\mathcal{F}(t)} |D(v)|^2.
\end{aligned}$$

By using the equations (1.78) and (1.79) we deduce

$$\begin{aligned}
\int_{\mathcal{F}(t)} \mathrm{div}\,(\sigma(u,p))\cdot v \quad &= \quad -Mh'(t)\cdot h''(t) - I\omega'(t)\cdot\omega(t) - 2\nu\int_{\mathcal{F}(t)} D(\overline{w}) : D(v) - 2\nu\int_{\mathcal{F}(t)} |D(v)|^2 \\
&= \quad -\frac{M}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|h'(t)|^2\right) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\left(\sqrt{I}\omega\right)(t)\right|^2\right) + \frac{1}{2}{I^*}'\tilde{\omega}(t)\cdot\tilde{\omega}(t) \\
& \quad\ -2\nu\int_{\mathcal{F}(t)} D(\overline{w}) : D(v) - 2\nu\int_{\mathcal{F}(t)} |D(v)|^2,
\end{aligned}$$

by recalling the notation $\tilde{\omega} = \mathbf{R}^T\omega$. We return to the equality (1.82) to obtain

$$\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(t)} |v|^2\mathrm{d}x + 2\nu\int_{\mathcal{F}(t)} |D(v)|^2\mathrm{d}x + \frac{M}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|h'(t)|^2\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\left(\sqrt{I}\omega\right)(t)\right|^2\right) \\
& \quad = \int_{\mathcal{F}(t)} f\cdot v\ \mathrm{d}x + \frac{1}{2}{I^*}'\tilde{\omega}\cdot\tilde{\omega} - 2\nu\int_{\mathcal{F}(t)} D(\overline{w}) : D(v)\mathrm{d}x \\
& \qquad - \int_{\mathcal{F}(t)} ((v\cdot\nabla)\overline{w})\cdot v\ \mathrm{d}x - \int_{\mathcal{F}(t)} ((\overline{w}\cdot\nabla)\overline{w})\cdot v\ \mathrm{d}x - \int_{\mathcal{F}(t)} \frac{\partial\overline{w}}{\partial t}\cdot v\ \mathrm{d}x.
\end{aligned}$$

It follows that there exists $\overline{C} > 0$ (depending only on η) such that

$$\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + 2\nu\|D(v)\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + \frac{M}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|h'(t)|^2\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\left(\sqrt{I}\omega\right)(t)\right|^2\right) \\
& \leq \overline{C}\left(\left\|\frac{\partial\overline{w}}{\partial t}\right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + \|(\overline{w}\cdot\nabla)\overline{w}\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + \|D(\overline{w})\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + \|f\|^2_{\mathbf{L}^2(\mathcal{F}(t))}\right. \\
& \quad \left. + \|v\|^2_{\mathbf{L}^2(\mathcal{F}(t))}\left(1 + \|\nabla\overline{w}\|^2_{\mathbf{L}^\infty(\mathcal{F}(t))}\right) + |\omega|^2(1 + |{I^*}'|^2)\right).
\end{aligned}$$

Using the estimates (1.69)–(1.72), we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + 2\nu\|D(v)\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + \frac{M}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|h'(t)|^2\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\left(\sqrt{I}\omega\right)(t)\right|^2\right) \\
&\leq \overline{\overline{C}}\left(\left\|\frac{\partial w^*}{\partial t}\right\|^2_{\mathbf{L}^2(\mathcal{S}^*(t))} + \|w^*\|^2_{\mathbf{H}^3(\mathcal{S}^*(t))}\|w^*\|^2_{\mathbf{H}^1(\mathcal{S}^*(t))} + \|w^*\|^2_{\mathbf{H}^1(\mathcal{S}^*(t))} + \|f\|^2_{\mathbf{L}^2(\mathcal{F}(t))}\right. \\
&\quad \left. + \left(\|v\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + |\omega|^2 + |h'|^2\right)\left(1 + \|w^*\|^2_{\mathbf{H}^3(\mathcal{S}^*(t))} + |I^{*\prime}|^2\right)\right).
\end{aligned} \tag{1.83}$$

Besides, we can extend the velocity field v into $\mathcal{S}(t)$ by setting $v(x,t) = h'(t) + \omega(t)\wedge(x-h(t))$ for $x \in \mathcal{S}(t)$. Thus we have $v \in \mathbf{H}^1_0(\mathcal{O})$ with div $v = 0$, and the following formula

$$\nabla v : \nabla v - 2D(v) : D(v) \quad = \quad -\mathrm{div}\left((v\cdot\nabla)v - (\mathrm{div}\ v)v\right) - (\mathrm{div}\ v)^2$$

combined to the Poincaré inequality enables us to write

$$\begin{aligned}
\|v\|_{\mathbf{H}^1(\mathcal{F}(t))} &\leq \|v\|_{\mathbf{H}^1(\mathcal{O})} \\
&\leq C_{\mathcal{O}}\|\nabla v\|_{\mathbf{L}^2(\mathcal{O})} = 2C_{\mathcal{O}}\|D(v)\|_{\mathbf{L}^2(\mathcal{O})} = 2C_{\mathcal{O}}\|D(v)\|_{\mathbf{L}^2(\mathcal{F}(t))}.
\end{aligned}$$

Then we can conclude by using inequality (1.73) and Grönwall's lemma on (1.83).

End of the proof of Theorem 1.18

Let (u,p,h,ω) be a strong solution of the system (1.1)–(1.7). In order to get the result, by absurd it is sufficient to show that if there exists $\eta > 0$ such that $\mathrm{dist}(\mathcal{S}(t),\partial\mathcal{O}) \geq \eta$ for all $t \in (0,T_0)$, then there exists a constant $C_{T_0} > 0$ such that

$$\|u(t)\|_{\mathbf{H}^1(\mathcal{F}(t))} \quad \leq \quad C_{T_0}, \quad \forall t \in [0,T_0).$$

We recall that $v = u - \overline{w}$, and that v satisfies the system (1.74)–(1.81). Let us consider the equation (1.74). By taking the inner product of this equality with $\dfrac{\partial v}{\partial t}$ and by integrating on $\mathcal{F}(t)$ we get

$$\begin{aligned}
\left\|\frac{\partial v}{\partial t}\right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} - \int_{\mathcal{F}(t)} \mathrm{div}\ \sigma(u,p)\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x &= -\int_{\mathcal{F}(t)}[(u\cdot\nabla)v]\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x - \int_{\mathcal{F}(t)}[(v\cdot\nabla)\overline{w}]\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x \\
&\quad - \int_{\mathcal{F}(t)}[(\overline{w}\cdot\nabla)\overline{w}]\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x - \int_{\mathcal{F}(t)}\frac{\partial\overline{w}}{\partial t}\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x + \int_{\mathcal{F}(t)} f\cdot\frac{\partial v}{\partial t}\mathrm{d}x,
\end{aligned}$$

for almost all $t \in (0,T)$, and furthermore

$$\begin{aligned}
&\left\|\frac{\partial v}{\partial t}\right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} - \int_{\mathcal{F}(t)} \mathrm{div}\ \sigma(u,p)\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x \quad = \\
&\qquad -\int_{\mathcal{F}(t)}[(v\cdot\nabla)v]\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x - \int_{\mathcal{F}(t)}[(\overline{w}\cdot\nabla)v]\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x - \int_{\mathcal{F}(t)}[(v\cdot\nabla)\overline{w}]\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x \\
&\qquad -\int_{\mathcal{F}(t)}[(\overline{w}\cdot\nabla)\overline{w}]\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x - \int_{\mathcal{F}(t)}\frac{\partial\overline{w}}{\partial t}\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x + \nu\int_{\mathcal{F}(t)}\Delta\overline{w}\cdot\frac{\partial v}{\partial t}\ \mathrm{d}x + \int_{\mathcal{F}(t)} f\cdot\frac{\partial v}{\partial t}\mathrm{d}x
\end{aligned} \tag{1.84}$$

almost everywhere in $(0, T_0)$.
Up to a density argument - as the one which is detailed in [CT08] - and by using the relations $v = h'(t) + \omega(t) \wedge (x - h(t))$ on $\partial\mathcal{S}(t)$ and $v = 0$ on $\partial\mathcal{O}$, we have

$$\begin{aligned} -\int_{\mathcal{F}(t)} \mathrm{div}\,(\sigma(u,p)) \cdot \frac{\partial v}{\partial t} \quad &= \\ \nu \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}(t)} |D(v)|^2 + M|h''|^2 + \left|\sqrt{I}\omega'\right|^2 &- Mh'' \cdot (h' \wedge \omega) + (\omega \wedge (I\omega)) \cdot \omega' + {I^*}'\tilde{\omega} \cdot \tilde{\omega}' \\ +2\nu \int_{\partial\mathcal{S}(t)} D(\overline{w})n \cdot (h'' + h' \wedge \omega + \omega' \wedge (x - h))\,\mathrm{d}\Gamma. & \end{aligned} \tag{1.85}$$

By combining the equalities (1.84) and (1.85) we obtain almost everywhere in $(0, T)$

$$\begin{aligned} \left\|\frac{\partial v}{\partial t}\right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + \nu \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}(t)} |D(v)|^2 \,\mathrm{d}x + M|h''(t)|^2 + \left|\left(\sqrt{I}\omega'\right)(t)\right|^2 \\ = \int_{\mathcal{F}(t)} \mathbb{F}_1 \cdot \frac{\partial v}{\partial t} \,\mathrm{d}x + \mathbb{F}_2 \cdot h''(t) + \mathbb{F}_3 \cdot \omega'(t) + \mathbb{F}_4, \end{aligned}$$

using the notation of [SMSTT08]

$$\begin{aligned} \mathbb{F}_1 &= f - [(v \cdot \nabla)v] + [(\overline{w} \cdot \nabla)v] + [(v \cdot \nabla)\overline{w}] + [(\overline{w} \cdot \nabla)\overline{w}] - \frac{\partial \overline{w}}{\partial t} + \nu\Delta\overline{w}, \\ \mathbb{F}_2 &= Mh'(t) \wedge \omega(t) - 2\nu \int_{\partial\mathcal{S}(t)} D(\overline{w})n\mathrm{d}\Gamma, \\ \mathbb{F}_3 &= (I\omega) \wedge \omega - {I^*}'\tilde{\omega} \cdot \tilde{\omega}' - 2\nu \int_{\partial\mathcal{S}(t)} (x - h) \wedge D(\overline{w})n\mathrm{d}\Gamma, \\ \mathbb{F}_4 &= -2\nu \int_{\partial\mathcal{S}(t)} D(\overline{w})n \cdot (h' \wedge \omega)\,\mathrm{d}\Gamma. \end{aligned}$$

Using the Cauchy-Schwartz inequality combined with the Young inequality and the fact that $\|u\|_{\mathbf{L}^2(\mathcal{F}(t))}$, h' and ω are bounded in $[0, T)$ (by Proposition 1.19), we deduce that there exists a positive constant C_3 such that for almost all $t \in (0, T)$

$$\begin{aligned} \left\|\frac{\partial v}{\partial t}\right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + 2\nu \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}(t)} |D(v)|^2 \,\mathrm{d}x + M|h''(t)|^2 + \left|\left(\sqrt{I}\omega'\right)(t)\right|^2 \\ \leq C_3 \left(C_0^2 + C_0^2\|\nabla v\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + \|(v \cdot \nabla)v\|^2_{\mathbf{L}^2(\mathcal{F}(t))}\right), \end{aligned} \tag{1.86}$$

by recalling that the constant C_0 is the one which appear in Proposition 1.19.
Then, in order to face up with the nonlinear term, we first use the Hölder inequality

$$\|(v \cdot \nabla)v\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^3} \leq \|v\|^2_{[\mathrm{L}^4(\mathcal{F}(t))]^3} \|\nabla v\|^2_{[\mathrm{L}^4(\mathcal{F}(t))]^9}$$

and we recall the continuous embedding of $\mathrm{H}^{3/4}(\mathcal{F}(t))$ in $\mathrm{L}^4(\mathcal{F}(t))$

$$\|(v \cdot \nabla)v\|^2_{\mathbf{L}^2(\mathcal{F}(t))} \leq C\|v\|^2_{[\mathrm{H}^{3/4}(\mathcal{F}(t))]^3} \|\nabla v\|^2_{[\mathrm{H}^{3/4}(\mathcal{F}(t))]^9}.$$

Thus, by using an interpolation inequality (see [LM72], for instance) we obtain

$$\begin{aligned}
\|(v\cdot\nabla)v\|^2_{\mathbf{L}^2(\mathcal{F}(t))} &\leq \tilde{C}\|v\|^{3/2}_{[\mathrm{H}^1(\mathcal{F}(t))]^3}\|v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^3}\|\nabla v\|^{3/2}_{[\mathrm{H}^1(\mathcal{F}(t))]^9}\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9} \\
&\leq C_1\|v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^3}\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\left(\|v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^3}+\|\nabla v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)^{3/2}\times \\
&\quad \left(\|\nabla v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}+\sum_{i=1}^3\|\nabla^2 v_i\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)^{3/2}. \qquad (1.87)
\end{aligned}$$

From Proposition 1.19, $\|v\|_{\mathbf{L}^2}$ is bounded (by some constant K_1 for instance), so we have

$$\|(v\cdot\nabla)v\|^2_{\mathbf{L}^2(\mathcal{F}(t))} \leq \tilde{C}_1\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\left(1+\|\nabla v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)^{3/2}\left(\|\nabla v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}+\sum_{i=1}^3\|\nabla^2 v_i\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)^{3/2}.$$

On the other hand, if we consider the following Stokes resolvent problem at some fixed time $t>0$

$$\begin{aligned}
v-\nu\Delta v+\nabla p &= \overline{f}+v \text{ in } \mathcal{F}(t),\\
\operatorname{div} v &= 0 \text{ in } \mathcal{F}(t),\\
v &= h'+\omega\wedge(x-h) \text{ on } \partial\mathcal{S}(t),\\
v &= 0 \text{ on } \partial\mathcal{O},
\end{aligned}$$

with

$$\overline{f} := f-\frac{\partial v}{\partial t}-(v\cdot\nabla)v-(v\cdot\nabla)\overline{w}-(\overline{w}\cdot\nabla)v,$$

we have the following estimate which is given for instance by [FS94]

$$\sum_{i=1}^3\|\nabla^2 v_i\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9} \leq C_2\left(\|\overline{f}\|_{[\mathrm{L}^2(\mathcal{F}(t))]^3}+\|v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^3}+\|h'\|_{\mathbb{R}^3}+\|\omega\|_{\mathbb{R}^3}\right).$$

Since $\|v\|_{\mathbf{L}^2}$, h' and ω are bounded by Proposition 1.19, we have

$$\sum_{i=1}^3\|\nabla^2 v_i\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9} \leq C_2\left(\|\overline{f}\|_{[\mathrm{L}^2(\mathcal{F}(t))]^3}+K_1\right).$$

Remark 1.20. *Notice that the constant C_2 does not depend on time, since we have* $\mathrm{dist}(\mathcal{S}(t),\partial\mathcal{O})$ *for all $t\in(0,T_0$.*

Consequently, by noticing that

$$\|\overline{f}\|_{[\mathrm{L}^2(\mathcal{F}(t))]^3} \leq \|f\|_{[\mathrm{L}^2(\mathcal{F}(t))]^3}+\left\|\frac{\partial v}{\partial t}\right\|_{[\mathrm{L}^2(\mathcal{F}(t))]^3}+\|(v\cdot\nabla)v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^3}+K_2K_1+K_3\|\nabla v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9},$$

we get this inequality

$$\|(v\cdot\nabla)v\|^2_{\mathbf{L}^2(\mathcal{F}(t))} \leq \hat{C}\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\left(1+\|\nabla v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)^{3/2}\times$$

$$\left(\hat{K}_2K_1+\hat{K}_3\|\nabla v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}+\|f\|_{\mathbf{L}^2(\mathcal{F}(t))}+\left\|\frac{\partial v}{\partial t}\right\|_{\mathbf{L}^2(\mathcal{F}(t))}+\|(v\cdot\nabla)v\|_{\mathbf{L}^2(\mathcal{F}(t))}\right)^{3/2}.$$

By a classical convexity inequality we can develop

$$\left(1+\|\nabla v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)^{3/2} \quad \leq \quad \tilde{C}_1\left(1+\|\nabla v\|^{3/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)$$

and

$$\left(\hat{K}_2 K_1+\hat{K}_3\|\nabla v\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}+\|f\|_{\mathbf{L}^2(\mathcal{F}(t))}+\left\|\frac{\partial v}{\partial t}\right\|_{\mathbf{L}^2(\mathcal{F}(t))}+\|(v\cdot\nabla)v\|_{\mathbf{L}^2(\mathcal{F}(t))}\right)^{3/2}$$
$$\leq \tilde{C}_2\left(1+\|\nabla v\|^{3/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}+\|f\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}+\left\|\frac{\partial v}{\partial t}\right\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}+\|(v\cdot\nabla)v\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}\right).$$

Thus

$$\|(v\cdot\nabla)v\|^2_{\mathbf{L}^2(\mathcal{F}(t))}\leq \hat{C}\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\left(1+\|\nabla v\|^{3/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)\times$$
$$\tilde{C}_2\left(1+\|\nabla v\|^{3/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}+\|f\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}+\left\|\frac{\partial v}{\partial t}\right\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}+\|(v\cdot\nabla)v\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}\right).$$

Then, if we set $\hat{\hat{C}}=\tilde{C}_1\tilde{C}_2\hat{C}$ and

$$\begin{aligned}
\mathbb{A} &= \hat{\hat{C}}\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\left(1+\|\nabla v\|^{3/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)\times\\
&\quad \left(1+\|\nabla v\|^{3/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}+\|f\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}+\left\|\frac{\partial v}{\partial t}\right\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}\right)\\
\mathbb{B} &= \hat{\hat{C}}\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\left(1+\|\nabla v\|^{3/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right),
\end{aligned}$$

we have

$$\|(v\cdot\nabla)v\|^2_{\mathbf{L}^2(\mathcal{F}(t))} \quad\leq\quad \mathbb{A}+\mathbb{B}\|(v\cdot\nabla)v\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))},$$

and thus by the Young inequality we get

$$\|(v\cdot\nabla)v\|^2_{\mathbf{L}^2(\mathcal{F}(t))} \quad\leq\quad 4\mathbb{A}+\mathbb{B}^4,$$

with

$$\mathbb{B}^4 \quad\leq\quad \bar{C}\|\nabla v\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\left(1+\|\nabla v\|^6_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right).$$

Therefore by returning to (1.86) we obtain for $\tilde{C}_3$ large enough

$$\left\|\frac{\partial v}{\partial t}\right\|^2_{\mathbf{L}^2(\mathcal{F}(t))}+2\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(t)}|D(v)|^2\,\mathrm{d}x+M|h''(t)|^2+\left|\left(\sqrt{I}\omega'\right)(t)\right|^2$$
$$\leq \tilde{C}_3\left(1+4\mathbb{A}+\mathbb{B}^4\right).$$

This leads to

$$\begin{aligned}
&\left\|\frac{\partial v}{\partial t}\right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + 2\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(t)}|D(v)|^2\,\mathrm{d}x + M|h''(t)|^2 + \left|\left(\sqrt{I}\omega'\right)(t)\right|^2 \\
\leq\ & \overline{C}_3\Big(C_0^2 + C_0^2\|\nabla v\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^9} \\
&+\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\left(1+\|\nabla v\|^{3/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)\left(1+\|\nabla v\|^{3/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9} + \|f\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))} + \left\|\frac{\partial v}{\partial t}\right\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}\right) \\
&+\|\nabla v\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\left(1+\|\nabla v\|^6_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)\Big) \\
\leq\ & \overline{C}_3\Big(C_0^2 + \left(2 + C_0^2 + \|f\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}\right)\|\nabla v\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^9} + \left(1+\|f\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}\right)\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9} \\
&+\|\nabla v\|^{7/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9} + \|\nabla v\|^8_{[\mathrm{L}^2(\mathcal{F}(t))]^9} + \left(\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9} + \|\nabla v\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)\left\|\frac{\partial v}{\partial t}\right\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}\Big).
\end{aligned}$$

Then, still by the Young inequality, we get

$$\begin{aligned}
&\left\|\frac{\partial v}{\partial t}\right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} + 2\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(t)}|D(v)|^2\,\mathrm{d}x + M|h''(t)|^2 + \left|\left(\sqrt{I}\omega'\right)(t)\right|^2 \\
&\qquad\leq\ \overline{C}_4C_0^2 + \overline{C}_5\Big(\left(\|\nabla v\|^{1/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9} + \|\nabla v\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\right)\left(1+\|f\|^{3/2}_{\mathbf{L}^2(\mathcal{F}(t))}\right) \\
&\qquad\quad +\|\nabla v\|^{7/2}_{[\mathrm{L}^2(\mathcal{F}(t))]^9} + \|\nabla v\|^8_{[\mathrm{L}^2(\mathcal{F}(t))]^9}\Big).
\end{aligned}$$

By integrating this inequality on $[0,t]$, and with the Cauchy-Schwartz inequality we get

$$\begin{aligned}
&\int_0^t\left\|\frac{\partial v}{\partial t}(\bar{t})\right\|^2_{\mathbf{L}^2(\mathcal{F}(\bar{t}))}\mathrm{d}\bar{t} + \nu\|\nabla v(t)\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^9} + M\int_0^t|h''(\bar{t})|^2\mathrm{d}\bar{t} + \int_0^t\left|\left(\sqrt{I}\omega'\right)(\bar{t})\right|^2\mathrm{d}\bar{t} \\
\leq\ & \nu\|\nabla v_0\|^2_{[\mathrm{L}^2(\mathcal{F})]^9} + T_0\overline{C}_4C_0^2 + \overline{C}_5T_0^{1/4}\left(\|f\|_{\mathrm{L}^{3/2}(0,T_0;\mathbf{L}^2(\mathcal{F}(t)))} + T_0^{1/2}\right)\|\nabla v\|^{1/2}_{\mathrm{L}^2(0,T_0;[\mathrm{L}^2(\mathcal{F}(t))])^9} \\
&+\|f\|_{\mathrm{L}^{3/2}(0,T_0;\mathbf{L}^2(\mathcal{F}(t)))}\left(\int_0^t\|\nabla v(\bar{t})\|^4_{[\mathrm{L}^2(\mathcal{F}(\bar{t}))]^9}\mathrm{d}\bar{t}\right)^{1/2} + \|\nabla v\|^2_{\mathrm{L}^2(0,T_0;[\mathrm{L}^2(\mathcal{F}(t))])^9} \\
&+\overline{C}_5\int_0^t\|\nabla v(\bar{t})\|^{7/2}_{[\mathrm{L}^2(\mathcal{F}(\bar{t}))]^9}\mathrm{d}\bar{t} + \overline{C}_5\int_0^t\|\nabla v(\bar{t})\|^8_{[\mathrm{L}^2(\mathcal{F}(s))]^9}\mathrm{d}\bar{t}.
\end{aligned}$$

By Proposition 1.19 we deduce

$$\begin{aligned}
&\int_0^t\left\|\frac{\partial v}{\partial t}(\bar{t})\right\|^2_{\mathbf{L}^2(\mathcal{F}(\bar{t}))}\mathrm{d}\bar{t} + \nu\|\nabla v(t)\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^9} + M\int_0^t|h''(\bar{t})|^2\mathrm{d}\bar{t} + \int_0^t\left|\left(\sqrt{I}\omega'\right)(\bar{t})\right|^2\mathrm{d}\bar{t} \\
&\qquad\leq\ \nu\|\nabla v_0\|^2_{[\mathrm{L}^2(\mathcal{F})]^9} + T_0\overline{C}_4C_0^2 + \overline{C}_5T_0^{1/4}\left(\|f\|_{\mathrm{L}^{3/2}(0,T_0;\mathbf{L}^2(\mathcal{F}(t)))} + T_0^{1/2}\right)C^{1/2}C_0 \\
&\qquad\quad+\|f\|_{\mathrm{L}^{3/2}(0,T_0;\mathbf{L}^2(\mathcal{F}(t)))}\left(\int_0^t\|\nabla v(\bar{t})\|^4_{[\mathrm{L}^2(\mathcal{F}(\bar{t}))]^9}\mathrm{d}\bar{t}\right)^{1/2} + C^2C_0^4 \\
&\qquad\quad+\overline{C}_5\int_0^t\|\nabla v(\bar{t})\|^{7/2}_{[\mathrm{L}^2(\mathcal{F}(\bar{t}))]^9}\mathrm{d}\bar{t} + \overline{C}_5\int_0^t\|\nabla v(\bar{t})\|^8_{[\mathrm{L}^2(\mathcal{F}(\bar{t}))]^9}\mathrm{d}\bar{t}. \qquad (1.88)
\end{aligned}$$

For small data we can have in particular C_0 and $\|\nabla v_0\|^2_{[\mathrm{L}^2(\mathcal{F})]^9}$ small enough to satisfy

$$\nu\|\nabla v_0\|^2_{[\mathrm{L}^2(\mathcal{F})]^9} + T_0\overline{C}_4 C_0^2 + \overline{C}_5 T_0^{1/4}\left(\|f\|_{\mathrm{L}^{3/2}(0,T_0;\mathbf{L}^2(\mathcal{F}(t)))} + T_0^{1/2}\right)C^{1/2}C_0 \\ + CC_0^2\|f\|_{\mathrm{L}^{3/2}(0,T_0;\mathbf{L}^2(\mathcal{F}(t)))} + C^2C_0^4 + 2C^2C_0^4\overline{C}_5 \quad < \quad \nu. \tag{1.89}$$

For such initial data we notice in particular that $\|\nabla v_0\|^2_{[\mathrm{L}^2(\mathcal{F})]^9} < 1$, and then by continuity there exists a maximal time $\tilde{T}_0$ such that for all $t \in [0, \tilde{T}_0]$ we have $\|\nabla v(t)\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^9} \leq 1$. Let us show that $\tilde{T}_0 = T_0$; By absurd, let us assume that $\tilde{T}_0 < T_0$. For all $\bar{t} \in [0, \tilde{T}_0]$ we have for $r \in \left\{4, \dfrac{7}{2}, 8\right\}$

$$\|\nabla v(\bar{t})\|^r_{[\mathrm{L}^2(\mathcal{F}(\bar{t}))]^9} \quad \leq \quad \|\nabla v(\bar{t})\|^2_{[\mathrm{L}^2(\mathcal{F}(\bar{t}))]^9},$$

and by returning to (1.88) we deduce

$$\begin{aligned}\nu\|\nabla v(\tilde{T}_0)\|^2_{[\mathrm{L}^2(\mathcal{F}(\tilde{T}_0))]^9} \quad \leq \quad & \nu\|\nabla v_0\|^2_{[\mathrm{L}^2(\mathcal{F})]^9} + T_0\overline{C}_4 C_0^2 + \overline{C}_5 T_0^{1/4}\left(\|f\|_{\mathrm{L}^{3/2}(0,T_0;\mathbf{L}^2(\mathcal{F}(t)))} + T_0^{1/2}\right)C^{1/2}C_0 \\ & + CC_0^2\|f\|_{\mathrm{L}^{3/2}(0,T_0;\mathbf{L}^2(\mathcal{F}(t)))} + C^2C_0^4 \\ & + 2C^2C_0^4\overline{C}_5.\end{aligned}$$

Then under the hypothesis (1.89) we deduce that $\|\nabla v(\tilde{T}_0)\|^2_{[\mathrm{L}^2(\mathcal{F}(\tilde{T}_0))]^9} < 1$, and by continuity we can find $\epsilon > 0$ such that $\|\nabla v(t)\|^2_{[\mathrm{L}^2(\mathcal{F}(t))]^9} \leq 1$ for $t \in [\tilde{T}_0, \tilde{T}_0 + \epsilon]$. This belies the definition of $\tilde{T}_0$ as an upper bound, and thus $\tilde{T}_0 = T_0$.
This shows that by assuming the initial data small enough, $\|\nabla v(t)\|_{[\mathrm{L}^2(\mathcal{F}(t))]^9}$ is bounded (in $\mathrm{L}^\infty(0, T_0)$), and thus by Proposition 1.19 the norm $\|v\|_{\mathbf{H}^1(\mathcal{F}(t))}$ is bounded. Finally, we just have to recall that $v = u - w$ and we obtain the result from Theorem 1.17 by showing that T_0 can't be finite.

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Chapter 2

Extension to the case where the regularity of the solid's deformation is limited

In this chapter, we reconsider the study which has been done in the previous chapter (in dimension 3), in investigating the case where the solid's deformation is limited in regularity. Indeed, the previous study has been lead in considering the maximum regularity for the Lagrangian mapping X^*, which enables to handle its Eulerian velocity w^* with the maximum regularity.
In a framework where the regularity of X^* is limited, things become more delicate. In particular, its Eulerian velocity w^* - whose definition has to be made clear from now on - is defined on a the domain $\mathcal{S}^*(t)$ which is directly defined by X^*, and thus deducing its regularity is not so obvious. Furthermore, extending w^* to the whole domain, as it is done in the previous chapter, is no more straightforward. We prefer then extending directly the mapping X^* to the whole domain, in order to define a change of unknowns, which enables to rewrite the main system in non-depending time domains. Here again, the equivalence of the two systems is no more so obvious.
We prove afterwards by a fixed point method the local existence result, in estimating the non-linear terms which involve - indirectly - the mapping X^*. We note in particular that the change of unknowns so used is suitable, in the sense that it enables us to uncouple the terms due to the solid's deformation X^* and the terms due to the unknowns. Finally, we have to verify that the class of functions chosen for X^* enables to get the desired regularity on the velocity w^*, in order to conserve the global existence result stated in the previous chapter.
In order to simplify, we assume that the solid's density is constant at time $t = 0$

$$\rho_{\mathcal{S}}(y,0) \quad = \quad \rho_{\mathcal{S}} > 0, \quad y \in \mathcal{S},$$

like in the previous chapter, so that the classical trace theorems hold for the weighted Sobolev spaces in domain $\mathcal{S}^*(t)$. We also impose the initial position of the center of mass, as $h_0 = 0$ (without loss of generality).

2.1 Regularity and properties of the solid's deformation

2.1.1 Functional settings

Let be $T > 0$. We assume that $\mathcal{S}(0)$ is regular enough. We still denote $\mathcal{F} = \mathcal{F}(0)$ and $\mathcal{S} = \mathcal{S}(0)$. We set

$$S_T^0 = \mathcal{S} \times (0,T), \qquad Q_T^0 = \mathcal{F} \times (0,T),$$

and

$$Q_T \quad = \quad \bigcup_{t\in(0,T)} \mathcal{F}(t)\times\{t\}.$$

For all integer $m \geq 3$, we consider deformations X^* lying in a space that we denote by $\mathcal{W}^m(0,T;\mathcal{S})$, and that we define as

$$\mathcal{W}^m(0,T;\mathcal{S}) \quad = \quad \left\{ X^* : \mathcal{S}\times[0,T] \rightarrow \bigcup_{t\in[0,T]} \mathcal{S}^*(t)\times\{t\},\ \frac{\partial X^*}{\partial t} \in \mathrm{H}^{m,m/2}(S_T^0) \right\},$$

where

$$\mathrm{H}^{m,m/2}(S_T^0) \quad \equiv \quad \mathrm{L}^2(0,T;\mathbf{H}^m(\mathcal{S})) \cap \mathrm{H}^{m/2}(0,T;\mathbf{L}^2(\mathcal{S})).$$

We take into account the initial conditions that we assume on X^*, in defining $\mathcal{W}_0^m(0,T;\mathcal{S})$ as follows

$$X^* \in \mathcal{W}_0^m(0,T;\mathcal{S}) \Leftrightarrow \begin{cases} X^* \in \mathcal{W}^m(0,T;\mathcal{S}), \\ X^*(y,0) = y,\ \dfrac{\partial X^*}{\partial t}(y,0) = 0 \quad \forall y \in \mathcal{S}. \end{cases}$$

Notice that, in particular, we have by interpolation

$$\mathcal{W}^m(0,T;\mathcal{S}) \quad \subset \quad \mathrm{H}^1(0,T;\mathbf{H}^m(\mathcal{S})) \cap \mathrm{H}^{m/2+1/2}(0,T;\mathbf{H}^1(\mathcal{S})).$$

Besides, we will be lead in this chapter to consider mappings limited in time regularity, so that the time regularity will not exceed H^2. Thus, for more clarity, we set

$$\begin{aligned}
\mathcal{W}_m(S_T^0) &= \mathrm{H}^1(0,T;\mathbf{H}^m(\mathcal{S})) \cap \mathrm{H}^2(0,T;\mathbf{H}^1(\mathcal{S})),\\
\mathcal{W}_m(Q_T^0) &= \mathrm{H}^1(0,T;\mathbf{H}^m(\mathcal{F})) \cap \mathrm{H}^2(0,T;\mathbf{H}^1(\mathcal{F})),\\
\tilde{\mathcal{W}}_m(Q_T^0) &= \mathrm{L}^\infty(0,T;\mathbf{H}^m(\mathcal{F})) \cap \mathrm{W}^{1,\infty}(0,T;\mathbf{H}^1(\mathcal{F})),\\
\mathcal{H}_m(S_T^0) &= \mathrm{L}^2(0,T;\mathbf{H}^m(\mathcal{S})) \cap \mathrm{H}^1(0,T;\mathbf{H}^1(\mathcal{S})),\\
\mathcal{H}_m(Q_T^0) &= \mathrm{L}^2(0,T;\mathbf{H}^m(\mathcal{F})) \cap \mathrm{H}^1(0,T;\mathbf{H}^1(\mathcal{F})).
\end{aligned}$$

2.1.2 Constraints on the deformation

Let us recall the constraints we assume for the solid's deformation represented by the mapping X^*:

H1 For all $t \geq 0$, $X^*(\cdot,t)$ is a C^1-diffeomorphism from $\mathcal{S}$ onto $\mathcal{S}(t)$.

H2 The solid's volume is constant through the time, which is equivalent to assuming that for all $t \geq 0$ we have

$$\int_{\partial\mathcal{S}} \frac{\partial X^*}{\partial t}(y,t)\cdot(\mathrm{cof}\nabla X^*)\, n \mathrm{d}\Gamma \quad = \quad 0.$$

H3 The linear momentum is preserved through the time, which is equivalent to assuming that for all $t \geq 0$ we have

$$\int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0)\frac{\partial X^*}{\partial t}(y,t)\mathrm{d}y = 0.$$

H4 The angular momentum is preserved through the time, which is equivalent to assuming that for all $t \geq 0$ we have

$$\int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0)X^*(y,t) \wedge \frac{\partial X^*}{\partial t}(y,t)\mathrm{d}y = 0.$$

Relaxing these constraints would be more complicated. Concerning the hypothesis **H1**, the regularity C^1 in space is provided by the functional framework $\mathcal{W}_0^m(0,T;\mathcal{S})$ and the Sobolev embeddings theorem. The hypothesis **H2** corresponds to the conservation of the volume, and could be relaxed in cases where the fluid is compressible or fill an unbounded domain for instance, but the mathematical analysis of the main system in such a framework is another issue that we do not get onto in this thesis. In this chapter, this hypothesis is important for the compatibility condition

$$\int_{\partial\mathcal{S}(t)} w \cdot n\mathrm{d}\Gamma = \int_{\mathcal{F}(t)} \mathrm{div}\ u\mathrm{d}x = 0.$$

Hypotheses **H3** and **H4** come close to a matter of physical modeling: The deformation X^* has to guarantee the *self-propelled* nature of the solid's deformation, that is to say that the solid's momenta are not modified by its deformation. These two last constraints have no importance for the results given in this chapter.

2.2 The change of variables

The goal of this section is to extend to the whole domain $\overline{\mathcal{O}}$ the mapping $X_{\mathcal{S}}(\cdot,t)$ and its inverse $Y_{\mathcal{S}}(\cdot,t)$, initially defined on $\overline{\mathcal{S}}$ and $\overline{\mathcal{S}(t)}$ respectively. The process we use is not the same as the one given in [SMSTT08]. Instead of extending the Eulerian flow induced by the solid's deformation, we directly extend the solid's deformation, because the difference in our case lies in the fact that the regularity - in space - of the Dirichlet data on the time-dependent boundary $\partial\mathcal{S}(t)$ is limited.

2.2.1 Extending the Lagrangian flow instead of the Eulerian velocity

Let be $T > 0$. In section 1.3.1 of the previous chapter, we have constructed the extension of the solid's Lagrangian mapping $X_{\mathcal{S}}$ by extending first the Eulerian velocities. For instance, the velocity w^* had been extended in solving the following divergence problem

$$\begin{cases} \mathrm{div}\ \overline{w}^* = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{S}^*(t)},\ t \in (0,T), \\ \overline{w}^*(x^*,t) = 0 & \text{if } \mathrm{dist}(x^*, \mathcal{S}^*(t)) \geq \eta > 0,\ t \in (0,T), \\ \overline{w}^*(x^*,t) = w^*(x^*,t) & \text{if } x^* \in \mathcal{S}^*(t),\ t \in (0,T). \end{cases}$$

The strength of this approach lies in the simplicity of such a problem. Note that the last equality of this problem is a Dirichlet condition which occurs on $\partial\mathcal{S}^*(t) = \partial X^*(\mathcal{S},t)$, that is to say a boundary which is directly defined by the Lagrangian mapping X^*. Since no restriction on the regularity X^* was assumed in this chapter, we used to have the maximal regularity on the extension $\overline{w}^*$. The extension of X^* could have been obtained as the Lagrangian mapping associated with $\overline{w}^*$, that is to say as being the solution of the Cauchy problem

$$\frac{\partial \overline{X}^*}{\partial t}(y,t) = \overline{w}^*(\overline{X}^*(y,t),t), \quad \tilde{X}(y,0) = y - h_0, \quad y \in \mathcal{F}.$$

In the framework of this chapter, we can not proceed in the same way. Indeed, the boundary $\partial\mathcal{S}^*(t)$ on which we set the Dirichlet condition w^* depends on X^*, which depends itself on w^* (according to Problem (1.10)).
In order to circumvent this limitation, we extend directly the Lagrangian mapping X^*, instead of obtaining this extension through an extended Eulerian velocity. For that, we consider the problem formulated as

$$\text{Find } \tilde{X} \in \mathcal{W}_m(Q_T^0) \text{ such that}$$
$$\begin{cases} \det \nabla \tilde{X} = 1 & \text{in } \mathcal{F} \times (0,T), \\ \tilde{X} = X^* & \text{on } \partial\mathcal{S} \times (0,T), \\ \tilde{X}(y,t) = \mathbf{R}(t)^T(y - h(t)) & (y,t) \in \partial\mathcal{O} \times (0,T), \end{cases} \tag{2.1}$$

where the vector h and the rotation $\mathbf{R}$ lie in $\mathrm{H}^2(0,T;\mathbb{R}^3)$ and $\mathrm{H}^2(0,T;\mathbb{R}^9)$ respectively. It can be viewed as the analogous problem - for Lagrangian mappings - corresponding to the divergence problem above, with the difference that the Dirichlet condition on $\partial\mathcal{O}$ takes into account the change of frame between the extension $\tilde{X}$ so obtained and the change of variables X. Indeed, instead of using a cut-off function for defining the desired extension (like it is done with the cut-off function ξ for the function Λ in section 1.3.1), we can directly set the change of variables

$$X(y,t) \quad = \quad h(t) + \mathbf{R}(t)\tilde{X}(y,t), \quad (y,t) \in \mathcal{F} \times (0,T),$$

which satisfies $X_{|\partial\mathcal{O}}(\cdot,t) \equiv \mathrm{Id}_{\partial\mathcal{O}}$, and thus enables us to preserve the homogeneous Dirichlet condition (1.3) on $\partial\mathcal{O}$ (see the condition (1.32)).
Solving problem (2.1) in $\mathcal{W}_m(Q_T^0)$ requires some preliminary technical results that we state below.

2.2.2 Preliminary results

Let us recall a result stated in the Appendix B of [GS91] (Proposition B.1), which treats of Sobolev regularities for products of functions, and that we state in the particular case of dimension 3 as:

Lemma 2.1. *Let s, μ and κ in $\mathbb{R}$. If $f \in \mathrm{H}^{s+\mu}(\mathcal{F})$ and $g \in \mathrm{H}^{s+\kappa}(\mathcal{F})$, then there exists a positive constant C such that*

$$\|fg\|_{\mathrm{H}^s(\mathcal{F})} \quad \leq \quad C\|f\|_{\mathrm{H}^{s+\mu}(\mathcal{F})}\|g\|_{\mathrm{H}^{s+\kappa}(\mathcal{F})},$$

(i) when $s + \mu + \kappa \geq 3/2$,
(ii) with $\mu \geq 0$, $\kappa \geq 0$, $2s + \mu + \kappa \geq 0$,
(iii) except that $s + \mu + \kappa > 3/2$ if equality holds somewhere in (ii).

For instance, if $m > 5/2$, and a fortiori for all integer $m \geq 3$, we have for f, $g \in \mathrm{H}^{m-1}(\mathcal{F})$ the following estimate

$$\|fg\|_{\mathrm{H}^{m-1}(\mathcal{F})} \quad \leq \quad C\|f\|_{\mathrm{H}^{m-1}(\mathcal{F})}\|g\|_{\mathrm{H}^{m-1}(\mathcal{F})}. \tag{2.2}$$

A consequence of this Lemma is the following useful result.

Lemma 2.2. *Let be $T > 0$. Let $\tilde{X}$ be in $\mathcal{W}_m(Q_T^0)$, with $m \geq 3$. Then*

$$\mathrm{cof}\nabla\tilde{X} \quad \in \quad \mathrm{H}^1(0,T;\mathbf{H}^{m-1}(\mathcal{F})) \cap \mathrm{H}^2(0,T;\mathbf{L}^2(\mathcal{F})), \tag{2.3}$$

and there exists a positive constant C such that

$$\begin{aligned} &\|\mathrm{cof}\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} \\ &\leq C\|\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)}\left(1 + \|\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)}\right) \\ &\leq C\sqrt{T}\left\|\frac{\partial\tilde{X}}{\partial t}\right\|_{\mathcal{H}_m(Q_T^0)}\left(1 + \sqrt{T}\left\|\frac{\partial\tilde{X}}{\partial t}\right\|_{\mathcal{H}_m(Q_T^0)}\right). \end{aligned} \tag{2.4}$$

Moreover, if $\tilde{X}_1,\ \tilde{X}_2 \in \mathcal{W}_m(Q_T^0)$, then

$$\begin{aligned} &\|\mathrm{cof}\nabla\tilde{X}_2 - \mathrm{cof}\nabla\tilde{X}_1\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} \\ &\leq C\|\nabla\tilde{X}_2 - \nabla\tilde{X}_1\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)}\left(1 + \|\nabla\tilde{X}_1\| + \|\nabla\tilde{X}_2\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)}\right) \\ &\leq C\sqrt{T}\left\|\frac{\partial(\tilde{X}_2 - \tilde{X}_1)}{\partial t}\right\|_{\mathcal{H}_m(Q_T^0)}\left(1 + \sqrt{T}\left\|\frac{\partial\tilde{X}_1}{\partial t}\right\|_{\mathcal{H}_m(Q_T^0)} + \sqrt{T}\left\|\frac{\partial\tilde{X}_1}{\partial t}\right\|_{\mathcal{H}_m(Q_T^0)}\right). \end{aligned} \tag{2.5}$$

Proof. For proving (2.3), it is sufficient to show that the space $\mathrm{H}^1(0,T;\mathrm{H}^{m-1}(\mathcal{F}))\cap\mathrm{H}^2(0,T;\mathrm{L}^2(\mathcal{F}))$ is stable by product. For that, let us consider two functions f and g which lie in this space. We write

$$\frac{\partial(fg)}{\partial t} \quad = \quad \frac{\partial f}{\partial t}g + f\frac{\partial g}{\partial t}.$$

Applying Lemma 2.1 with $s = m-1$ and $\mu = \kappa = 0$, we get

$$\begin{aligned} \left\|\frac{\partial(fg)}{\partial t}\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1}(\mathcal{F}))} \quad \leq \quad C\Bigg(&\left\|\frac{\partial f}{\partial t}\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1}(\mathcal{F}))}\|g\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1}(\mathcal{F}))} \\ &+ \left\|\frac{\partial g}{\partial t}\right\|_{\mathrm{L}^2(0,T;\mathrm{H}^{m-1}(\mathcal{F}))}\|f\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1}(\mathcal{F}))}\Bigg) \end{aligned}$$

and thus $fg \in \mathrm{H}^1(0,T;\mathrm{H}^{m-1}(\mathcal{F}))$. For the regularity of fg in $\mathrm{H}^2(0,T;\mathrm{L}^2(\mathcal{F}))$, we consider

$$\frac{\partial^2(fg)}{\partial t^2} \quad = \quad \frac{\partial^2 f}{\partial t^2}g + f\frac{\partial^2 g}{\partial t^2} + 2\frac{\partial f}{\partial t}\frac{\partial g}{\partial t}$$

with

$$\begin{aligned} &\frac{\partial^2 f}{\partial t^2},\ \frac{\partial^2 g}{\partial t^2} \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F})), && f,\ g \in \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{F})), \\ &\frac{\partial f}{\partial t} \in \mathrm{L}^2(0,T;\mathrm{L}^\infty(\mathcal{F})), && \frac{\partial g}{\partial t} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathcal{F})), \end{aligned}$$

because of the embedding $\mathbf{H}^{m-1}(\mathcal{F}) \hookrightarrow \mathbf{L}^\infty(\mathcal{F})$, so that we get

$$\frac{\partial^2(fg)}{\partial t^2} \quad \in \quad \mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))$$

and the desired regularity. This shows in particular that $\mathrm{H}^1(0,T;\mathrm{H}^{m-1}(\mathcal{F}))\cap \mathrm{H}^2(0,T;\mathrm{L}^2(\mathcal{F}))$ is an algebra, and we can show the estimate (2.4) in noticing that the cofactor matrix is made of quadratic terms (in dimension 3), so that

$$\begin{aligned}
&\|\mathrm{cof}\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} \\
&\leq \tilde{C}\|\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} \left(\|\nabla\tilde{X}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} + 1\right), \\
&\leq C\|\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} \left(\|\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} + 1\right).
\end{aligned}$$

The argument for proving (2.5) is the same. □

2.2.3 Extension of the Lagrangian mappings

Let be $T_0 \geq T > 0$. Let $h \in \mathrm{H}^2(0,T_0;\mathbb{R}^3)$ be a vector and $\mathbf{R} \in \mathrm{H}^2(0,T_0;\mathbb{R}^9)$ a rotation which provides the angular velocity $\omega \in \mathrm{H}^1(0,T_0;\mathbb{R}^3)$. We assume that

$$h_0 = 0, \qquad \mathbf{R}(0) = \mathrm{I}_{\mathbb{R}^3},$$

and we set

$$\tilde{h}'(t) = \mathbf{R}(t)^T h(t), \qquad \tilde{\omega}(t) = \mathbf{R}(t)^T \omega(t).$$

Lemma 2.3. *Let $m \geq 3$ be an integer. Let X^* be a mapping which lies in $\mathcal{W}_0^m(0,T_0;\mathcal{S})$ and satisfies for all $t \geq 0$ the equality*

$$\int_{\partial\mathcal{S}} \frac{\partial X^*}{\partial t} \cdot (\mathrm{cof}\nabla X^*)\, n \mathrm{d}\Gamma \quad = \quad 0. \tag{2.6}$$

Then for $T > 0$ small enough, there exists a mapping $\tilde{X} \in \mathcal{W}_m(Q_T^0)$ satisfying

$$\begin{cases} \det\nabla\tilde{X} = 1 & \text{in } \mathcal{F}\times(0,T), \\ \tilde{X} = X^* & \text{on } \partial\mathcal{S}\times(0,T), \\ \tilde{X} = \mathbf{R}^T(\mathrm{Id} - h) & \text{on } \partial\mathcal{O}\times(0,T), \end{cases} \tag{2.7}$$

and the estimate

$$\|\tilde{X} - \mathrm{Id}_{\mathcal{F}}\|_{\mathcal{W}_m(Q_T^0)} \quad \leq \quad C\left(\|X^* - \mathrm{Id}_{\mathcal{S}}\|_{\mathcal{W}_m(S_{T_0}^0)} + \|\tilde{h}'\|_{H^1(0,T_0;\mathbb{R}^3)} + \|\tilde{\omega}\|_{H^1(0,T_0;\mathbb{R}^3)}\right), \tag{2.8}$$

for some independent positive constant C - which in particular does not depend on T. Besides, if $\tilde{X}_1$ and $\tilde{X}_2$ are the solutions of problem (2.7) corresponding to the data $(X^*, h_1, \mathbf{R}_1)$ and $(X^*, h_2, \mathbf{R}_2)$ respectively, with

$$h_1(0) = h_2(0) = 0, \quad \mathbf{R}_1(0) = \mathbf{R}_2(0) = 0, \quad h_1'(0) = h_2'(0), \quad \omega_1(0) = \omega_2(0),$$

then the difference $\tilde{X}_2 - \tilde{X}_1$ satisfies

$$\|\tilde{X}_2 - \tilde{X}_1\|_{\mathcal{W}_m(Q_T^0)} \quad \leq \quad C_{21}\left(\|\tilde{h}_2' - \tilde{h}_1'\|_{H^1(0,T_0;\mathbb{R}^3)} + \|\tilde{\omega}_2 - \tilde{\omega}_1\|_{H^1(0,T_0;\mathbb{R}^3)}\right), \tag{2.9}$$

where the constant C_{21} does not depend on T.

Remark 2.4. *A mapping $\tilde{X}$ given by the result above satisfies in particular*

$$\tilde{X}(\cdot,0) \quad = \quad \mathrm{Id}_{\mathcal{F}}.$$

Proof. Given the initial data $X^*(\cdot,0) = \mathrm{Id}_{\mathcal{S}}$, $h_0 = 0$, $\mathbf{R}(0) = \mathrm{I}_{\mathbb{R}^3}$, $h'(0) = h_1$ and $\omega(0) = \omega_0$, let us consider the system (2.7) derived in time, as

$$\begin{cases} \left(\mathrm{cof}\nabla\tilde{X}\right) : \dfrac{\partial\nabla\tilde{X}}{\partial t} = 0 & \text{in } \mathcal{F}\times(0,T), \\ \dfrac{\partial\tilde{X}}{\partial t} = \dfrac{\partial X^*}{\partial t} & \text{on } \partial\mathcal{S}\times(0,T), \\ \dfrac{\partial\tilde{X}}{\partial t}(y,t) = -\tilde{h}'(t) - \tilde{\omega}(t)\wedge\tilde{X}(y,t) & (y,t)\in\partial\mathcal{O}\times(0,T), \\ \tilde{X}(\cdot,0) = \mathrm{Id}_{\mathcal{F}}. \end{cases}$$

This system can be viewed as a modified nonlinear divergence problem, that we state as

$$\begin{cases} \mathrm{div}\,\dfrac{\partial\tilde{X}}{\partial t} = f(\tilde{X}) & \text{in } \mathcal{F}\times(0,T), \\ \dfrac{\partial\tilde{X}}{\partial t} = \dfrac{\partial X^*}{\partial t} & \text{on } \partial\mathcal{S}\times(0,T), \\ \dfrac{\partial\tilde{X}}{\partial t}(y,t) = -\tilde{h}'(t) - \tilde{\omega}(t)\wedge\tilde{X}(y,t) & (y,t)\in\partial\mathcal{O}\times(0,T), \\ \tilde{X}(\cdot,0) = \mathrm{Id}_{\mathcal{F}}, \end{cases}$$

with

$$f(\tilde{X}) \quad = \quad \left(\mathrm{I}_{\mathbb{R}^3} - \mathrm{cof}\nabla\tilde{X}\right) : \frac{\partial\nabla\tilde{X}}{\partial t}.$$

If we search solutions to this system which are continuous in space, let us notice (in using the Piola identity) that the compatibility condition for this divergence system is nothing else than the equality (2.6).
A solution of this system can be seen as a fixed point of the mapping

$$\begin{array}{rccc} \mathfrak{T}: & \mathcal{W}_m(Q_T^0) & \rightarrow & \mathcal{W}_m(Q_T^0) \\ & \tilde{X}_1 & \mapsto & \tilde{X}_2, \end{array} \tag{2.10}$$

where $\tilde{X}_2$ is a solution of the classical divergence problem

$$\begin{cases} \mathrm{div}\,\dfrac{\partial\tilde{X}_2}{\partial t} = f(\tilde{X}_1) & \text{in } \mathcal{F}\times(0,T), \\ \dfrac{\partial\tilde{X}_2}{\partial t} = \dfrac{\partial X^*}{\partial t} & \text{on } \partial\mathcal{S}\times(0,T), \\ \dfrac{\partial\tilde{X}_2}{\partial t} = -\tilde{h}' - \tilde{\omega}\wedge\tilde{X}_1 & \text{on } \partial\mathcal{O}\times(0,T), \end{cases}$$

in adding the initial condition $\tilde{X}_2(\cdot,0) = \mathrm{Id}_{\mathcal{F}}$. We have the classical estimates (see [Gal94] for instance)

$$\begin{aligned} \left\|\frac{\partial\tilde{X}_2}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^m(\mathcal{F}))} \leq\ & C_{\mathcal{F}}\left(\|f(\tilde{X}_1)\|_{\mathrm{L}^2(\mathbf{H}^{m-1}(\mathcal{F}))} + \left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^{m-1/2}(\partial\mathcal{S}))}\right. \\ & \left. + \|\tilde{h}'\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} + \|\tilde{\omega}\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)}\|\tilde{X}_1\|_{\mathrm{L}^\infty(\mathbf{H}^{m-1/2}(\partial\mathcal{S}))}\right) \\ \leq\ & \tilde{C}_{\mathcal{F}}\left(\|f(\tilde{X}_1)\|_{\mathrm{L}^2(\mathbf{H}^{m-1}(\mathcal{F}))} + \left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^m(\mathcal{S}))}\right. \\ & \left. + \|\tilde{h}'\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} + \|\tilde{\omega}\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)}\|\tilde{X}_1\|_{\mathrm{L}^\infty(\mathbf{H}^m(\mathcal{S}))}\right) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
\left\|\frac{\partial^2 \tilde{X}_2}{\partial t^2}\right\|_{\mathrm{L}^2(\mathbf{H}^1(\mathcal{F}))} & \leq C_{\mathcal{F}}\left(\|f(\tilde{X}_1)\|_{\mathrm{H}^1(\mathbf{L}^2(\mathcal{F}))} + \left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{H}^1(\mathbf{H}^{1/2}(\partial\mathcal{S}))}\right. \\
& \qquad \left. + \|\tilde{h}'\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)} + \|\tilde{\omega}\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)}\|\tilde{X}_1\|_{\mathrm{L}^\infty(\mathbf{H}^{1/2}(\partial\mathcal{S}))} + \|\tilde{\omega}\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)}\left\|\frac{\partial \tilde{X}_1}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^{1/2}(\partial\mathcal{S}))}\right) \\
& \leq \tilde{C}_{\mathcal{F}}\left(\|f(\tilde{X}_1)\|_{\mathrm{H}^1(\mathbf{L}^2(\mathcal{F}))} + \left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{H}^1(\mathbf{H}^1(\mathcal{S}))}\right. \\
& \qquad \left. + \|\tilde{h}'\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)} + \|\tilde{\omega}\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)}\left(\|\tilde{X}_1\|_{\mathrm{L}^\infty(\mathbf{H}^1(\mathcal{S}))} + \left\|\frac{\partial \tilde{X}_1}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^1(\mathcal{S}))}\right)\right).
\end{aligned} \tag{2.12}$$

Indeed, let us verify that for $\tilde{X} \in \mathcal{W}_m(Q_T^0)$, satisfying $\tilde{X}(\cdot,0) = \mathrm{Id}_{\mathcal{F}}$, we have $f(\tilde{X}) \in \mathrm{L}^2(0,T;\mathbf{H}^{m-1}(\mathcal{F}))\cap \mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F}))$. For that, we recall from the previous lemma that $\mathrm{cof}\nabla\tilde{X} \in \mathrm{H}^1(0,T;\mathbf{H}^{m-1}(\mathcal{F}))\cap \mathrm{H}^2(0,T;\mathbf{L}^2(\mathcal{F}))$, and we first use the result of Lemma 2.1 with $s = m-1$ and $\mu = \kappa = 0$ to get

$$\begin{aligned}
\|f(\tilde{X})\|_{\mathrm{L}^2(\mathbf{H}^{m-1}(\mathcal{F}))} & \leq C_1\|\mathrm{I}_{\mathbb{R}^3} - \mathrm{cof}\nabla\tilde{X}\|_{\mathrm{L}^\infty(\mathbf{H}^{m-1}(\mathcal{F}))}\left\|\frac{\partial\nabla\tilde{X}}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^{m-1}(\mathcal{F}))} \\
& \leq C_1\sqrt{T}\|\mathrm{I}_{\mathbb{R}^3} - \mathrm{cof}\nabla\tilde{X}\|_{\mathrm{H}^1(\mathbf{H}^{m-1}(\mathcal{F}))}\left\|\frac{\partial\tilde{X}}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^{m}(\mathcal{F}))}.
\end{aligned}$$

For the regularity in $\mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F}))$, let us first notice that we have by interpolation

$$\mathrm{L}^2(0,T;\mathbf{H}^{m-1}(\mathcal{F}))\cap\mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F})) \hookrightarrow \mathrm{L}^\infty(0,T;\mathbf{H}^{m/2-1/2}(\mathcal{F})).$$

Then we use Lemma 2.1 with $s = 0$ and $\mu = \kappa = m/2 - 1/2$, and the embedding $\mathbf{H}^{m-1}(\mathcal{F}) \hookrightarrow \mathbf{L}^\infty(\mathcal{F})$ in order to get

$$\begin{aligned}
& \left\|\frac{\partial(f(\tilde{X}))}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{L}^2)} \\
\leq\ & C_2\left(\left\|\frac{\partial\mathrm{cof}\nabla\tilde{X}}{\partial t}\right\|_{L^\infty(\mathbf{H}^{m/2-1/2})}\left\|\frac{\partial\nabla\tilde{X}}{\partial t}\right\|_{L^2(\mathbf{H}^{m/2-1/2})} + \left\|I_{\mathbb{R}^3} - \mathrm{cof}\nabla\tilde{X}\right\|_{L^\infty(\mathbf{H}^{m-1})}\left\|\frac{\partial^2\nabla\tilde{X}}{\partial t^2}\right\|_{L^2(\mathbf{L}^2)}\right) \\
\leq\ & C_2\sqrt{T}\left(\left\|\frac{\partial\mathrm{cof}\nabla\tilde{X}}{\partial t}\right\|_{L^\infty(\mathbf{H}^{m/2-1/2})}\left\|\frac{\partial\nabla\tilde{X}}{\partial t}\right\|_{L^\infty(\mathbf{H}^{m/2-1/2})} + \left\|I_{\mathbb{R}^3} - \mathrm{cof}\nabla\tilde{X}\right\|_{H^1(\mathbf{H}^{m-1})}\left\|\frac{\partial^2\nabla\tilde{X}}{\partial t^2}\right\|_{L^2(\mathbf{L}^2)}\right).
\end{aligned}$$

Thus, we finally have

$$\|f(\tilde{X})\|_{\mathrm{L}^2(\mathbf{H}^{m-1})\cap\mathrm{H}^1(\mathbf{L}^2)} \leq C_3\sqrt{T}\|\mathrm{I}_{\mathbb{R}^3} - \mathrm{cof}\nabla\tilde{X}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2(\mathcal{F}))}\left\|\frac{\partial\tilde{X}}{\partial t}\right\|_{\mathcal{H}_m(Q_T^0)}. \tag{2.13}$$

The estimates (2.11) and (2.12) combined to (2.13) and (2.4) show that the mapping $\mathfrak{T}$ is well-defined.
Moreover, if for some $R > 0$ we consider the set

$$\mathfrak{B}_R = \left\{\tilde{X} \in \mathcal{W}_m(Q_T^0) \mid \tilde{X}(\cdot,0) = \mathrm{Id}_{\mathcal{F}},\ \left\|\frac{\partial\tilde{X}}{\partial t}\right\|_{\mathcal{H}_m(Q_T^0)} \leq R\right\},$$

and the inequality

$$\|\tilde{X}_1 - \mathrm{Id}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^m(\mathcal{F}))} \leq \sqrt{1+T^2}\left\|\frac{\partial \tilde{X}_1}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^m(\mathcal{F}))},$$

for $\tilde{X} \in \mathfrak{B}_R$, then the set $\mathfrak{B}_R$ is stable by $\mathfrak{T}$ for T small enough and R large enough. Notice that $\mathfrak{B}_R$ is a closed subset of $\mathcal{W}_m(Q_T^0)$. Let us verify that $\mathfrak{T}$ is a contraction in $\mathfrak{B}_R$.
For $\tilde{X}_1$ and $\tilde{X}_2$ in $\mathfrak{B}_R$, we denote $\tilde{Z} = \mathfrak{T}(\tilde{X}_2) - \mathfrak{T}(\tilde{X}_1)$ which satisfies the divergence system

$$\begin{cases} \operatorname{div} \dfrac{\partial \tilde{Z}}{\partial t} = f(\tilde{X}_2) - f(\tilde{X}_1) & \text{in } \mathcal{F}\times(0,T), \\ \dfrac{\partial \tilde{Z}}{\partial t} = 0 & \text{on } \partial\mathcal{S}\times(0,T), \\ \dfrac{\partial \tilde{Z}}{\partial t} = 0 & \text{on } \partial\mathcal{O}\times(0,T), \end{cases}$$

and thus the estimate

$$\left\|\frac{\partial \tilde{Z}}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^m(\mathcal{F}))\cap\mathrm{H}^1(\mathbf{H}^1(\mathcal{F}))} \leq C\left\|f(\tilde{X}_2) - f(\tilde{X}_1)\right\|_{\mathrm{L}^2(\mathbf{H}^{m-1}(\mathcal{F}))\cap\mathrm{H}^1(\mathbf{L}^2(\mathcal{F}))}.$$

We write

$$f(\tilde{X}_2) - f(\tilde{X}_1) = \left(\mathrm{cof}\nabla\tilde{X}_2 - \mathrm{cof}\nabla\tilde{X}_1\right) : \frac{\partial\nabla X_2}{\partial t} + \left(\mathrm{I}_{\mathbb{R}^3} - \mathrm{cof}\nabla\tilde{X}_1\right) : \frac{\partial\nabla(\tilde{X}_2 - \tilde{X}_1)}{\partial t}.$$

In reconsidering the steps of the proofs of the estimate (2.13), and in using (2.5), we can verify that for T small enough the mapping $\mathfrak{T}$ is a contraction in $\mathfrak{B}_R$. Thus $\mathfrak{T}$ admits a unique fixed point in $\mathfrak{B}_R$.
For the estimate (2.9), let us just notice that the difference $\tilde{Z}$ of two mappings $\tilde{X}_1$ and $\tilde{X}_2$ of $\mathfrak{B}_R$ - corresponding to the data $(X^*, h_1, \mathbf{R}_1)$ and $(X^*, h_2, \mathbf{R}_2)$ respectively - satisfies the system

$$\begin{cases} \operatorname{div} \dfrac{\partial \tilde{Z}}{\partial t} = f(\tilde{X}_2) - f(\tilde{X}_1) & \text{in } \mathcal{F}\times(0,T), \\ \dfrac{\partial \tilde{Z}}{\partial t} = 0 & \text{on } \partial\mathcal{S}\times(0,T), \\ \dfrac{\partial \tilde{Z}}{\partial t} = (\mathbf{R}_2 - \mathbf{R}_1)^T(y - h_2) - \mathbf{R}_1^T(h_2 - h_1) & \text{on } \partial\mathcal{O}\times(0,T), \end{cases}$$

with the estimates

$$\begin{aligned} \|h_2 - h_1\|_{\mathrm{L}^\infty(0,T_0;\mathbb{R}^3)} &\leq \sqrt{T_0}\|h_2' - h_1'\|_{\mathrm{L}^2(0,T_0;\mathbb{R}^3)}, \\ &\leq \sqrt{T_0}\|\mathbf{R}_2 - \mathbf{R}_1\|_{\mathrm{L}^\infty(0,T_0;\mathbb{R}^9)}\|\tilde{h}_2'\|_{\mathrm{L}^2(0,T_0;\mathbb{R}^3)} + \|\tilde{h}_2' - \tilde{h}_1'\|_{\mathrm{L}^2(0,T_0;\mathbb{R}^3)}, \\ \|\mathbf{R}_2 - \mathbf{R}_1\|_{\mathrm{L}^\infty(0,T_0;\mathbb{R}^9)} &\leq C\sqrt{T_0}\|\tilde{\omega}_2 - \tilde{\omega}_1\|_{\mathrm{L}^2(0,T_0;\mathbb{R}^3)}\exp\left(C\sqrt{T_0}\|\tilde{\omega}_2\|_{\mathrm{L}^2(0,T_0;\mathbb{R}^3)}\right), \end{aligned}$$

due to the equality

$$h_2' - h_1' = (\mathbf{R}_2 - \mathbf{R}_1)\tilde{h}_2' + \mathbf{R}_1\left(\tilde{h}_2' - \tilde{h}_1'\right)$$

and the Grönwall's lemma applied to

$$\begin{aligned} &\frac{\partial}{\partial t}(\mathbf{R}_2 - \mathbf{R}_1) = (\mathbf{R}_2 - \mathbf{R}_1)\mathbb{S}(\tilde{\omega}_2) + \mathbf{R}_1\mathbb{S}(\tilde{\omega}_2 - \tilde{\omega}_1), \\ &(\mathbf{R}_2 - \mathbf{R}_1)(0) = 0. \end{aligned}$$

Then we proceed as previously, and the end of the proof for the announced estimate is left to the reader. □

Lemma 2.5. *Let be T_0, and $T > 0$ small enough to define $\tilde{X} \in \mathcal{W}_m(Q_T^0)$ solution of problem (2.7), for $X^* \in \mathcal{W}_m(S_{T_0}^0)$, $h \in \mathrm{H}^2(0,T_0;\mathbb{R}^3)$ and $\mathbf{R} \in \mathrm{H}^2(0,T_0;\mathbb{R}^9)$. Let us denote $\tilde{Y}(\cdot,t)$ the inverse of the mapping $\tilde{X}(\cdot,t)$ - for all $t \in [0,T)$. Then we have*

$$\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} \leq C\left(\|X^* - \mathrm{Id}_{\mathcal{S}}\|_{\mathcal{W}_m(S_{T_0}^0)} + \|\tilde{h}'\|_{\mathrm{H}^1(0,T_0;\mathbb{R}^3)} + \|\tilde{\omega}\|_{\mathrm{H}^1(0,T_0;\mathbb{R}^9)}\right). \tag{2.14}$$

Let $\tilde{X}_1,\ \tilde{X}_2 \in \mathcal{W}_m(Q_T^0)$ be the solutions of problem (2.7), with data $(X^*, h_1, \mathbf{R}_1)$ and $(X^*, h_2, \mathbf{R}_2)$ respectively. Then for T small enough, if we denote by $\tilde{Y}_1(\cdot,t)$ and $\tilde{Y}_2(\cdot,t)$ the inverses of $\tilde{X}_1(\cdot,t)$ and $\tilde{X}_2(\cdot,t)$ respectively, we have

$$\|\nabla\tilde{Y}_2(\tilde{X}_2) - \nabla\tilde{Y}_1(\tilde{X}_1)\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} \leq C\left(\|\tilde{h}_2' - \tilde{h}_1'\|_{\mathrm{H}^1(0,T_0;\mathbb{R}^3)} + \|\tilde{\omega}_2 - \tilde{\omega}_1\|_{\mathrm{H}^1(0,T_0;\mathbb{R}^9)}\right). \tag{2.15}$$

Besides, if $\det\nabla\tilde{X} \equiv 1$ in Q_T^0, then we have

$$\nabla\tilde{Y}(\tilde{X}(\cdot,t),t) = \mathrm{cof}\nabla\tilde{X}(\cdot,t)^T, \tag{2.16}$$

Proof. The estimate (2.14) is obtained in writing

$$\begin{aligned}
\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3} &= \left(\mathrm{I}_{\mathbb{R}^3} - \nabla\tilde{X}\right)\left(\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\right) + \left(\mathrm{I}_{\mathbb{R}^3} - \nabla\tilde{X}\right), \\
\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} &\leq \frac{\|\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)}}{1 - C\|\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)}},
\end{aligned}$$

in using the fact that the space $\mathrm{H}^1(0,T;\mathbf{H}^{m-1}(\mathcal{F})) \cap \mathrm{H}^2(0,T;\mathbf{L}^2(\mathcal{F}))$ is an algebra (as it has been shown in the proof of Lemma 2.3). Then we use the estimate (2.8).
For the estimate (2.15), we write

$$\nabla\tilde{Y}_2(\tilde{X}_2) - \nabla\tilde{Y}_1(\tilde{X}_1) = \left(\nabla\tilde{Y}_2(\tilde{X}_2) - \nabla\tilde{Y}_1(\tilde{X}_1)\right)\left(I_{\mathbb{R}^3} - \nabla\tilde{X}_1\right) - \nabla\tilde{Y}_2(\tilde{X}_2)\left(\nabla\tilde{X}_2 - \nabla\tilde{X}_1\right),$$

so that in the algebra $\mathrm{H}^1(0,T;\mathbf{H}^{m-1}(\mathcal{F})) \cap \mathrm{H}^2(0,T;\mathbf{L}^2(\mathcal{F}))$ we have

$$\|\nabla\tilde{Y}_2(\tilde{X}_2) - \nabla\tilde{Y}_1(\tilde{X}_1)\| \leq \frac{\left(1 + \|\mathrm{I}_{\mathbb{R}^3} - \nabla\tilde{Y}_2\|\right)\|\nabla\tilde{X}_2 - \nabla\tilde{X}_1\|}{1 - C\|\mathrm{I}_{\mathbb{R}^3} - \nabla\tilde{X}_1\|},$$

The estimate (2.9) and what precedes enable us to conclude. The equality (2.16) is due to the formula

$$\nabla\tilde{Y}(\tilde{X}(y,t),t) = \frac{\left(\mathrm{cof}\nabla\tilde{X}(y,t)\right)^T}{\det\nabla\tilde{X}(y,t)}.$$

□

Some of the regularities and the properties of the mapping $\tilde{X}$ are preserved for X. We sum this properties in the following subsection.

2.2.4 Regularities for the change of variables

Lemma 2.3 provides a mapping $\tilde{X}$ that we use in order to define

$$X(y,t) = h(t) + \mathbf{R}(t)\tilde{X}(y,t), \quad y \in \mathcal{F}. \tag{2.17}$$

This mapping satisfies the following properties

$$\begin{cases} \det\nabla X = 1 & \text{in } \mathcal{F}\times(0,T), \\ X(\cdot,t)\equiv \mathrm{Id}_{\partial\mathcal{O}} & t\in(0,T), \\ X(y,t) = h(t)+\mathbf{R}(t)X^*(y,t) & (y,t)\in\partial\mathcal{S}\times(0,T). \end{cases}$$

Since $\tilde{X}(\cdot,t)$ is invertible, we denote by $\tilde{Y}(\cdot,t)$ its inverse, and the inverse of $X(\cdot,t)$ satisfies

$$Y(x,t) \quad = \quad \tilde{Y}(\mathbf{R}(t)^T(x-h(t)),t), \quad x\in\mathcal{F}(t).$$

The mapping X as the same degree of regularity as the mapping $\tilde{X}$.

Proposition 2.6. *Let X^* be a mapping of $\mathcal{W}_0^m(0,T;\mathcal{S})$ which satisfies (2.6), and let $\tilde{X}$ be the extension of X^* provided by Lemma 2.3 (for T small enough). Let X be the mapping given by (2.17). For all $t\in[0,T]$, the mapping $y\mapsto X(y,t)$ is a C^1-diffeomorphism from $\mathcal{O}$ onto $\mathcal{O}$, from $\partial\mathcal{S}$ onto $\partial\mathcal{S}(t)$, and from $\mathcal{F}$ onto $\mathcal{F}(t)$. We denote by $Y(\cdot,t)$ its inverse at some time t. We have*

$$\begin{aligned} (y,t)\mapsto X(y,t) &\in \mathrm{H}^1(0,T;\mathbf{H}^m(\mathcal{F})\cap\mathrm{H}^2(0,T;\mathbf{H}^1(\mathcal{F})), \\ \det\nabla X(y,t) &= 1, \textit{ for all } (y,t)\in\mathcal{F}\times[0,T], \\ (y,t)\mapsto\nabla Y(X(y,t),t) &\in \mathrm{H}^1(0,T;\mathbf{H}^{m-1}(\mathcal{F}))\cap\mathrm{H}^2(0,T;\mathbf{L}^2(\mathcal{F})). \end{aligned}$$

Proof. Notice first that the mapping $(y,t)\mapsto h(t)+\mathbf{R}(t)y$ preserves the regularity in space of $\tilde{X}$. For the regularity in time, let us consider

$$\frac{\partial X}{\partial t}(y,t) \quad = \quad h'(t)+\omega(t)\wedge\big(\mathbf{R}(t)\tilde{X}(y,t)\big)+\mathbf{R}(t)\frac{\partial\tilde{X}}{\partial t}(y,t), \quad y\in\partial\mathcal{S},$$

with $h'\in\mathrm{H}^1(0,T;\mathbb{R}^3)$, and $\mathbf{R}\in\mathrm{H}^2(0,T;\mathbb{R}^9)$ which provides $\omega\in\mathrm{H}^1(0,T;\mathbb{R}^3)$. Then it is easy to prove the regularity in time of $\dfrac{\partial X}{\partial t}$. For the regularity of the mapping $(y,t)\mapsto\nabla Y(X(y,t),t)$, Lemma 2.2 combined to the equality

$$\nabla Y(X(y,t),t) \quad = \quad \nabla\tilde{Y}(\tilde{X}(y,t),t)\mathbf{R}(t)^T$$

enables to get the desired result. ☐

2.3 The change of unknowns

In this section we come back to the rewriting of system (1.1)–(1.7) in domains which does not depend on time, and we aim at proving that the system so obtained is well equivalent to (1.1)–(1.7). Let us recall the change of unknowns we have chosen:

$$\begin{aligned} &\tilde{u}(y,t) = \mathbf{R}(t)^T u(X(y,t),t), \qquad && \tilde{h}'(t) = \mathbf{R}(t)^T h'(t), \\ &\tilde{p}(y,t) = p(X(y,t),t), && \tilde{\omega}(t) = \mathbf{R}(t)^T\omega(t). \end{aligned}$$

Rewriting system (1.1)–(1.7) in terms of the unknowns $(\tilde{u},\tilde{p},\tilde{h}',\tilde{\omega})$ leads - after some calculations - to system (1.30)–(1.36).

Equivalence of systems

Whereas the calculations requires no supplementary subtlety, the delicate point is to prove that the regularities coincide. Proving that

$$
\begin{aligned}
& (u,p,h',\omega) \in \mathrm{H}^{2,1}(Q_T) \times \mathrm{L}^2(0,T,\mathrm{H}^1(\mathcal{F}(t))) \times \mathrm{H}^1(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3) \\
\Leftrightarrow \quad & (\tilde{u},\tilde{p},\tilde{h}',\tilde{\omega}) \in \mathrm{H}^{2,1}(Q_T^0) \times \mathrm{L}^2(0,T,\mathrm{H}^1(\mathcal{F})) \times \mathrm{H}^1(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3)
\end{aligned}
$$

is not so obvious in this framework where the regularities of the mappings X and Y are limited. Then Proposition 1.10 becomes the following one.

Proposition 2.7. *Let be $T > 0$ and $m \geq 3$ an integer. Let $X^* \in \mathcal{W}_0^m(0,T;\mathcal{S}))$ be a deformation which satisfies the hypotheses* **H1**–**H4***. Then a quadruplet $(u,p,h,\omega) \in \mathrm{H}^{2,1}(Q_T) \times \mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{F}(t))) \times \mathrm{H}^2(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3)$ satisfies system* (1.1)–(1.7) *if and only if $(\tilde{u},\tilde{p},\tilde{h}',\tilde{\omega})$ given by* (1.23) *lies in $\mathrm{H}^{2,1}(Q_T^0) \times \mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{F})) \times \mathrm{H}^1(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3)$ and satisfies system* (1.30)–(1.36)*.*

Proof. Since we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}^T) = -\mathbb{S}(\tilde{\omega})\mathbf{R}^T,
$$

the solution of this differential equation can be expressed with the exponential of a Magnus expansion, which involves sums and products of matrices of kind $-\mathbb{S}(\tilde{\omega})$. Notice that the space $\mathrm{H}^1(0,T;\mathbb{R}^9)$ is an algebra, and thus given $\tilde{\omega} \in \mathrm{H}^1(0,T;\mathbb{R}^3)$ it is easy to verify that $\mathbf{R}$ lies in $\mathrm{H}^2(0,T;\mathbb{R}^9)$. It follows that $h' = \mathbf{R}\tilde{h}'$ lies in $\mathrm{H}^1(0,T;\mathbf{R}^d)$.
Let us recall that

$$
\|u\|^2_{\mathrm{H}^{2,1}(Q_T)} = \int_0^T \|u(\cdot,t)\|^2_{\mathbf{H}^2(\mathcal{F}(t))}\mathrm{d}t + \int_0^T \left\| \frac{\partial u}{\partial t}(\cdot,t)\right\|_{\mathbf{L}^2(\mathcal{F}(t))} \mathrm{d}t.
$$

Since $\det \nabla X = 1$, we make the change of variables which leads to

$$
\|u(\cdot,t)\|^2_{\mathbf{L}^2(\mathcal{F}(t))} = \int_{\mathcal{F}(t)} |u(x,t)|^2 \mathrm{d}x = \int_{\mathcal{F}} |u(X(y,t),t)|^2 \mathrm{d}x = \int_{\mathcal{F}} |\tilde{u}(x,t)|^2 \mathrm{d}x,
$$

if we choose for $|\cdot|$ the Euclidean norm in $\mathbb{R}^3$. For the regularity in $\mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F}(t)))$, we calculate

$$
\begin{aligned}
\mathbf{R}(t)^T \frac{\partial u}{\partial t}(X(y,t),t) = {} & \tilde{\omega} \wedge \tilde{u}(y,t) + \frac{\partial \tilde{u}}{\partial t}(y,t) \\
& - \nabla\tilde{u}(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t)\left(\tilde{h}'(t) + \tilde{\omega}(t) \wedge \tilde{X}(y,t) + \frac{\partial \tilde{X}}{\partial t}\right).
\end{aligned}
$$

We apply Lemma 2.1 with $s = 1$, $\mu = 0$ and $\kappa = m - 1$ in order to have

$$
\nabla\tilde{u}(\cdot,t)\nabla\tilde{Y}(\tilde{X}(\cdot,t),t) \in \mathbf{H}^1(\mathcal{F}),
$$

and with $s = 0$, $\mu = 1$ and $\kappa = 1$ in order to have

$$
\begin{aligned}
\nabla\tilde{u}(\cdot,t)\nabla\tilde{Y}(\tilde{X}(\cdot,t),t)\tilde{X}(\cdot,t) & \in \mathbf{L}^2(\mathcal{F}), \\
\nabla\tilde{u}(\cdot,t)\nabla\tilde{Y}(\tilde{X}(\cdot,t),t)\frac{\partial \tilde{X}}{\partial t}(\cdot,t) & \in \mathbf{L}^2(\mathcal{F}),
\end{aligned}
$$

so that there exists a positive constant C such that

$$\int_0^T \left\| \frac{\partial u}{\partial t}(\cdot,t) \right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} \mathrm{d}t = \int_0^T \left\| \mathbf{R}(t)^T \frac{\partial u}{\partial t}(X(\cdot,t),t) \right\|^2_{\mathbf{L}^2(\mathcal{F})} \mathrm{d}t$$

$$\leq C \left(\|\tilde{\omega}\|^2_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)} \|\tilde{u}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \left\| \frac{\partial \tilde{u}}{\partial t} \right\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|\nabla \tilde{u}\|^2_{\mathrm{L}^2(0,T;\mathbf{H}^1(\mathcal{F}))} \|\nabla \tilde{Y}(\tilde{X})\|^2_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m-1}(\mathcal{F}))} \times \right.$$
$$\left. \left(\|\tilde{h}'\|^2_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)} + \|\tilde{\omega}\|^2_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)} \|\tilde{X}\|^2_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\mathcal{F}))} + \left\| \frac{\partial \tilde{X}}{\partial t} \right\|^2_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\mathcal{F}))} \right) \right).$$

For the regularity in $\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}(t)))$, we consider the equalities

$$\begin{aligned} \nabla u(X(y,t),t) &= \mathbf{R}(t) \nabla \tilde{u}(y,t) \nabla \tilde{Y}(\tilde{X}(y,t),t) \mathbf{R}(t)^T, \\ \nabla^2 u_i(X(y,t),t) &= \nabla \left(\nabla u_i(X(y,t),t) \right) \nabla \tilde{Y}(\tilde{X}(y,t),t) \mathbf{R}(t)^T. \end{aligned}$$

Applying Lemma 2.1 with $s=1$, $\mu=0$ and $\kappa = m-1$, we get $\nabla u(X) \in \mathrm{L}^2(0,T;\mathbf{H}^1(\mathcal{F}(t)))$, and it implies through the second equality that $\nabla^2 u_i \in \mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}(t)))$.
Thus we have

$$\begin{aligned} &(\tilde{u},\tilde{h}',\tilde{\omega}) \in \mathrm{H}^{2,1}(Q_T^0) \times \mathrm{H}^1(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3) \\ &\Rightarrow (u,h',\omega) \in \mathrm{H}^{2,1}(Q_T) \times \mathrm{H}^1(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3). \end{aligned}$$

The analogous reverse implication can be obtained similarly. Thus (u,h',ω) lies in $\mathrm{H}^{2,1}(Q_T) \times \mathrm{H}^1(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3)$ if and only if $(\tilde{u},\tilde{h}',\tilde{\omega})$ lies in $\mathrm{H}^{2,1}(Q_T^0) \times \mathrm{H}^1(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3)$. The same equivalence is obtained similarly for the pressures p and $\tilde{p}$, since we have

$$\nabla p(X(y,t),t,t) = \nabla Y(X(y,t),t)^T \nabla \tilde{p}(y,t), \quad y \in \mathcal{F}.$$

The care of verifying the computations - in using the chain rule - which lead to system (1.30)–(1.36) are left to the reader. In particular, the nonhomogeneous divergence condition (1.31), given by (1.41), is due to the Piola identity, which provides in our case

$$\mathrm{div}\left(\nabla \tilde{Y}(\tilde{X}(y,t),t)^T \right) = 0.$$

It remains to recall the remark 1.8 to complete the proof. □

2.4 The local existence result

2.4.1 Statement

Adapting Theorem 1.17 consists in specifying the regularity assumed for the solid's deformation X^*. Thus Theorem 1.17 becomes:

Theorem 2.8. *Assume that $X^* \in \mathcal{W}_0^m(0,\infty;\mathcal{S})$ satisfies the hypotheses $\mathbf{H1}-\mathbf{H4}$, with $m \geq 3$. Let be $f \in \mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}(t)))$. Assume that $0 < \mathrm{dist}(\mathcal{S},\partial\mathcal{O})$, and that $u_0 \in \mathbf{H}^1(\mathcal{F})$ satisfies*

$$\mathrm{div}\ u_0 = 0 \text{ in } \mathcal{F}, \quad u_0 = 0 \text{ on } \partial\mathcal{O}, \quad u_0(y) = h_1 + \omega_0 \wedge y \text{ on } \partial\mathcal{S}.$$

Then there exists $T_0 > 0$ such that problem (1.1)–(1.7) admits a unique strong solution (u,p,h,ω) in

$$\mathcal{U}(0,T_0;\mathcal{F}(t)) \times \mathrm{L}^2(0,T_0;\mathrm{H}^1(\mathcal{F}(t))) \times \mathrm{H}^2(0,T_0;\mathbb{R}^3) \times \mathrm{H}^1(0,T_0;\mathbb{R}^3).$$

Moreover, if we assume that, for all $t \in [0,T)$, $\mathrm{dist}(\mathcal{S}(t), \partial\mathcal{O}) \geq \eta$, then we have the alternative

(a) either $T_0 = +\infty$ (that is to say the solution is global in time)
(b) or the function $t \mapsto \|u(t)\|_{\mathbf{H}^1(\mathcal{F}(t))}$ is not bounded in $[0, T_0)$.

2.4.2 Proof

Let us set

$$\mathbb{H}_T \quad = \quad \mathcal{U}(0,T;\mathcal{F}) \times \mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{F})) \times \mathrm{H}^1(0,T;\mathbb{R}^3) \times \mathrm{H}^1(0,T;\mathbb{R}^3).$$

The equivalence of the solutions of systems (1.1)–(1.7) and (1.30)–(1.36) has been established in Proposition 2.7. A solution of system (1.30)–(1.36) is seen as a fixed point of the mapping

$$\begin{array}{rccc} \mathcal{N}: & \mathbb{H}_T & \to & \mathbb{H}_T \\ & (V,Q,K',\varpi) & \mapsto & (U,P,H',\Omega) \end{array}$$

where (U,P,H',Ω) satisfies

$$\begin{aligned} \frac{\partial U}{\partial t} - \nu\Delta U + \nabla P = F(V,Q,K',\varpi), \qquad & \text{in } \mathcal{F}\times(0,T), \\ \mathrm{div}\ U = \mathrm{div}\ G_{(K',\varpi)}(V), \qquad & \text{in } \mathcal{F}\times(0,T), \end{aligned}$$

$$\begin{aligned} U = 0, \qquad & \text{in } \partial\mathcal{O}\times(0,T), \\ U = H'(t) + \Omega(t)\wedge y + \frac{\partial X^*}{\partial t} + W(\varpi), \qquad & (y,t)\in\partial\mathcal{S}\times(0,T), \end{aligned}$$

$$\begin{aligned} MH'' = -\int_{\partial\mathcal{S}} \sigma(U,P)n\mathrm{d}\Gamma + F_M(V,Q,K',\varpi), \qquad & \text{in } (0,T) \\ I_0\Omega'(t) = -\int_{\partial\mathcal{S}} y\wedge\sigma(U,P)n\mathrm{d}\Gamma + F_I(V,Q,K',\varpi), \qquad & \text{in } (0,T) \end{aligned}$$

$$U(y,0) = u_0(y),\ y\in\mathcal{F}, \quad \hat{h}'(0) = h_1 \in \mathbb{R}^3, \quad \hat{\omega}(0) = \omega_0 \in \mathbb{R}^3.$$

The expressions of the right-hand-side are given by

$$\begin{aligned} F(V,Q,K',\varpi) &= \nu(\mathbf{L}_{(K',\varpi)} - \Delta)V - \mathbf{M}_{(K',\varpi)}(V,K',\varpi) - \mathbf{N}_{(K',\varpi)}V \\ &\quad -(\mathbf{G}_{(K',\varpi)} - \nabla)Q - \varpi\wedge V, && (2.18) \\ G_{(K',\varpi)}(V) &= \left(\mathrm{I}_{\mathbb{R}^3} - \nabla\tilde{Y}(\tilde{X}(y,t),t)\right)V, && (2.19) \\ W(\varpi) &= \varpi\wedge(X^* - \mathrm{Id}), && (2.20) \\ F_M(V,Q,K',\varpi) &= -M\varpi\wedge K'(t) \\ &\quad -\nu\int_{\partial\mathcal{S}} \left(\nabla V\left(\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\right) + \left(\nabla\tilde{Y}(\tilde{X})^T - \mathrm{I}_{\mathbb{R}^3}\right)\nabla V^T\right)\nabla\tilde{Y}(\tilde{X})^T n\mathrm{d}\Gamma \\ &\quad -\int_{\partial\mathcal{S}} \sigma(V,Q)\left(\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\right)^T n\mathrm{d}\Gamma, && (2.21) \\ F_I(V,Q,K',\varpi) &= -(I^* - I_0)\Omega' - {I^*}'\Omega + I^*\Omega\wedge\Omega \\ &\quad -\nu\int_{\partial\mathcal{S}} y\wedge\left(\nabla V\left(\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\right) + (\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3})^T\nabla V^T\right)\nabla\tilde{Y}(\tilde{X})^T n\mathrm{d}\Gamma \\ &\quad -\int_{\partial\mathcal{S}} y\wedge\sigma(V,Q)\left(\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\right)^T n\mathrm{d}\Gamma \\ &\quad +\int_{\partial\mathcal{S}} (X^* - \mathrm{Id})\wedge\left(\tilde{\Sigma}(V,Q)\nabla\tilde{Y}(\tilde{X})^T n\right)\mathrm{d}\Gamma. && (2.22) \end{aligned}$$

The mapping $\tilde{X}$ is given by Lemma 2.3, with (K', ϖ, X^*) as data. We have in particular

$$\nabla\tilde{Y}(\tilde{X}(\cdot,0),0) - \mathrm{I}_{\mathbb{R}^3} \quad = \quad 0.$$

The mapping $\nabla\tilde{Y}(\tilde{X})$ satisfies the estimates stated in Lemma 2.5, which will be useful in order to make $\mathcal{N}$ a contracting mapping. More specifically, the estimates (2.4) and (2.14) are recalled:

$$\begin{aligned}
\|\tilde{X} - \mathrm{Id}_{\mathcal{F}}\|_{\mathrm{H}^1(\mathbf{H}^m)\cap\mathrm{H}^2(\mathbf{H}^1)} &\leq C\left(1 + \|K'\|_{H^1(0,T_0;\mathbb{R}^3)} + \|\varpi\|_{H^1(0,T_0;\mathbb{R}^3)}\right), &(2.23)\\
\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} &\leq C\left(1 + \|K'\|_{\mathrm{H}^1(0,T_0;\mathbb{R}^3)} + \|\varpi\|_{\mathrm{H}^1(0,T_0;\mathbb{R}^9)}\right). &(2.24)
\end{aligned}$$

For the expression of $F(V,Q,K',\varpi)$, let us recall that

$$\begin{aligned}
[\mathbf{L}_{(K',\varpi)}(V)]_i(y,t) &= [\nabla V(y,t)\Delta\tilde{Y}(\tilde{X}(y,t),t)]_i + \nabla^2 V_i(y,t) : \left(\nabla\tilde{Y}\nabla\tilde{Y}^T\right)(\tilde{X}(y,t),t),\\
\mathbf{M}_{(K',\varpi)}(V,K',\varpi)(y,t) &= -\nabla V(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t)\left(K'(t) + \varpi\wedge\tilde{X}(y,t) + \frac{\partial\tilde{X}}{\partial t}(y,t)\right),\\
\mathbf{N}_{(K',\varpi)}V(y,t) &= \nabla V(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t)V(y,t),\\
\mathbf{G}_{(K',\varpi)}Q(y,t) &= \nabla\tilde{Y}(\tilde{X}(y,t),t)^T\nabla Q(y,t).
\end{aligned}$$

For $R > 0$, we set the ball

$$\begin{aligned}
\mathcal{B}_R = \{&(U,P,H',\Omega) \in \mathbb{H}_T \mid \\
&\|U\|_{\mathrm{H}^{2,1}(Q_T^0)} + \|\nabla P\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|H'\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)} + \|\Omega\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)} \leq R\Big\}.
\end{aligned}$$

The preliminary study of the linearized system provides the following estimate

$$\begin{aligned}
\|\mathcal{N}(V,Q,K',\varpi)\|_{\mathbb{H}_T} \leq\ & C_T^{(0)}\Big(1 + \|G_{(K',\varpi)}(V)\|_{\mathrm{H}^{2,1}(Q_T^0)}\\
&+\|F(V,Q,K',\varpi)\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} + \|W(\varpi)\|_{H^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))}\\
&+\|F_M(V,Q,K',\varpi)\|_{L^2(0,T;\mathbb{R}^3)} + \|F_I(V,Q,K',\varpi)\|_{L^2(0,T;\mathbb{R}^3)}\Big), \qquad (2.25)
\end{aligned}$$

where the constant $C_T^{(0)}$ is nondecreasing with respect to T, and depends on the data

$$\|u_0\|_{\mathbf{H}^1(\mathcal{F})}, \quad |h_1|_{\mathbb{R}^3}, \quad |\omega_0|_{\mathbb{R}^3}, \quad \left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{-1/2}(\partial\mathcal{S}))}.$$

The rest of this section is devoted to proving that for R large enough and T small enough

- the ball B_R is stable by $\mathcal{N}$,
- the mapping $\mathcal{N}$ is contracting in B_R.

Preliminary estimates

The estimates given in the lemmas below are not sharp, but they are sufficient to prove the desired result.

Lemma 2.9. *There exists a positive constant C such that for all (V,Q,K',ϖ) in $\mathbb{H}_T$ we have*

$$\begin{aligned}
&\|(\Delta - \mathbf{L})V\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \leq C\|V\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))}\times\\
&\left(\|\nabla\tilde{Y}(\tilde{X})\nabla\tilde{Y}(\tilde{X})^T - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m-1}(\mathcal{F}))} + \|\Delta\tilde{Y}_i(\tilde{X}(\cdot,t),t)\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m-2}(\mathcal{F}))}\right),\\
&\|\Delta\tilde{Y}_i(\tilde{X}(\cdot,t),t)\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m-2}(\mathcal{F}))} \leq C\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m-1}(\mathcal{F}))}\|\nabla\tilde{Y}(\tilde{X})\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m-1}(\mathcal{F}))},\\
&\|(\nabla - \mathbf{G})Q\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \leq C\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^{m-1}(\mathcal{F}))}\|\nabla Q\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}))}.
\end{aligned}$$

Proof. Given the regularities stated in Lemma 2.5, the only delicate point that has to be verified is $\Delta\tilde{Y}(\tilde{X}) \in \mathrm{L}^\infty(0,T;\mathbf{H}^{m-2}(\mathcal{F}))$. For that, let us consider the i-th component of $\Delta Y(X)$; We write

$$\Delta\tilde{Y}_i(X(\cdot,t),t) \quad = \quad \mathrm{trace}\left(\nabla^2\tilde{Y}_i(X(\cdot,t),t)\right)$$

with

$$\begin{aligned}\nabla^2\tilde{Y}_i(\tilde{X}(\cdot,t),t) \quad &= \quad \left(\nabla\left(\nabla\tilde{Y}_i(\tilde{X}(\cdot,t),t)\right)\right)\nabla\tilde{Y}(\tilde{X}(\cdot,t),t) \\ &= \quad \left(\nabla\left(\nabla\tilde{Y}_i(\tilde{X}(\cdot,t),t) - \mathrm{I}_{\mathbb{R}^3}\right)\right)\nabla\tilde{Y}(\tilde{X}(\cdot,t),t),\end{aligned}$$

and we apply Lemma 2.1 with $s = m-2$, $\mu = 0$ and $\kappa = 1$ to obtain

$$\|\Delta\tilde{Y}_i(\tilde{X}(\cdot,t),t)\|_{\mathbf{H}^{m-2}(\mathcal{F})} \quad \leq \quad C\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathbf{H}^{m-1}(\mathcal{F})}\|\nabla\tilde{Y}(\tilde{X})\|_{\mathbf{H}^{m-1}(\mathcal{F})}.$$

□

Corollary 2.10. *There exists a positive constant C such that for all (V,Q,K',ϖ) in $\mathbb{H}_T$ we have*

$$\begin{aligned}&\|(\Delta - \mathbf{L})\,V\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \leq C\sqrt{T}\,\|V\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))}\times \\ &\left(\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(0,T;\mathbf{H}^{m-1}(\mathcal{F}))}\left(1 + \|\nabla\tilde{Y}(\tilde{X})\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m-1}(\mathcal{F}))}\right)\right), \qquad (2.26)\\ &\|(\nabla - \mathbf{G})Q\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \leq C\sqrt{T}\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(0,T;\mathrm{H}^{m-1}(\mathcal{F}))}\,\|\nabla Q\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}))}. \qquad (2.27)\end{aligned}$$

Proof. Since $\nabla\tilde{Y}(\tilde{X}(\cdot,0),0) - \mathrm{I}_{\mathbb{R}^3} = 0$, we have

$$\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m-1}(\mathcal{F}))} \quad \leq \quad \sqrt{T}\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(0,T;\mathbf{H}^{m-1}(\mathcal{F}))}.$$

The following quadratic term is treated as

$$\nabla\tilde{Y}(\tilde{X})\nabla\tilde{Y}(\tilde{X})^T - \mathrm{I}_{\mathbb{R}^3} \quad = \quad \left(\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\right)\nabla\tilde{Y}(\tilde{X})^T + \left(\nabla\tilde{Y}(\tilde{X})^T - \mathrm{I}_{\mathbb{R}^3}\right).$$

□

Lemma 2.11. *There exists a positive constant C such that for all (V,Q,K',ϖ) in $\mathbb{H}_T$ we have*

$$\begin{aligned}\|\mathbf{M}(V,K',\varpi)\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \quad &\leq \quad CT^{1/10}\|\nabla\tilde{Y}(\tilde{X})\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m-1}(\mathcal{F}))}\left\|K' + \varpi\wedge\tilde{X} + \frac{\partial\tilde{X}}{\partial t}\right\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\mathcal{F}))}\times \\ &\qquad \|V\|^{1/5}_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\mathcal{F}))}\|V\|^{4/5}_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))}, \qquad (2.28)\\ \|\mathbf{N}V\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \quad &\leq \quad CT^{1/10}\|\nabla\tilde{Y}(\tilde{X})\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m-1}(\mathcal{F}))}\|V\|^{6/5}_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\mathcal{F}))}\|V\|^{4/5}_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))}, \qquad (2.29)\\ \|\varpi\wedge V\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} \quad &\leq \quad C\sqrt{T}\|\varpi\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)}\|V\|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{F}))}. \qquad (2.30)\end{aligned}$$

Proof. Let us recall an estimate proved in [TT04] (Lemma 5.2) which is still true in dimension 3; There exists a positive constant C such that for all v, w in $\mathrm{H}^{2,1}(Q_T^0)$ we have

$$\|(w\cdot\nabla)v\|_{\mathrm{L}^{5/2}(0,T,\mathbf{L}^2(\mathcal{F}))} \quad \leq \quad C\|w\|_{\mathrm{L}^\infty(0,T,\mathbf{H}^1(\mathcal{F}))}\|v\|^{1/5}_{\mathrm{L}^\infty(0,T,\mathbf{H}^1(\mathcal{F}))}\|v\|^{4/5}_{\mathrm{L}^2(0,T,\mathbf{H}^2(\mathcal{F}))}. \qquad (2.31)$$

In applying the estimate (2.31) with $v = U$ and $w = -\nabla\tilde{Y}(\tilde{X})\left(H' + \Omega\wedge\tilde{X} + \dfrac{\partial\tilde{X}}{\partial t}\right)$, combined to the Hölder inequality which gives

$$\|(w\cdot\nabla)v\|_{\mathrm{L}^2(0,T,\mathbf{L}^2(\mathcal{F}))} \quad\leq\quad T^{1/10}\|(w\cdot\nabla)v\|_{\mathrm{L}^{5/2}(0,T,\mathbf{L}^2(\mathcal{F}))},$$

we get

$$\begin{aligned}&\|\mathbf{M}V\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))}\leq\\ &CT^{1/10}\left\|\nabla\tilde{Y}(\tilde{X})\left(K' + \varpi\wedge\tilde{X} + \frac{\partial\tilde{X}}{\partial t}\right)\right\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\mathcal{F}))}\|V\|^{1/5}_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\mathcal{F}))}\|V\|^{4/5}_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))}.\end{aligned}$$

We apply Lemma 2.1 on w with $s = 1$, $\mu = m-2$ and $\kappa = 0$, and then we obtain (2.28). For the estimate (2.29), we proceed similarly; We use the inequality (2.31) with $v = U$ and $w = \nabla\tilde{Y}(\tilde{X})U$, and we apply Lemma 2.1 on w with $s = 1$, $\mu = m-2$ and $\kappa = 0$.
For the estimate (2.30), we simply write

$$\begin{aligned}\|\varpi\wedge V\|_{L^2(0,T;\mathbf{L}^2(\mathcal{F}))} &\leq C\|\varpi\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)}\wedge V\|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{F}))}\\ &\leq C\sqrt{T}\|\varpi\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)}\|V\|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{F}))}.\end{aligned}$$

□

Lemma 2.12. *There exists a positive constant C such that for all $((V,K',\varpi)$ in $\mathrm{H}^{2,1}(Q_T^0)\times\mathrm{H}^1(0,T;\mathbb{R}^3)\times\mathrm{H}^1(0,T;\mathbb{R}^3)$ we have*

$$\begin{aligned}\|G(V)\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))} &\leq C\sqrt{T}\|V\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))}\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(0,T;\mathbf{H}^2(\mathcal{F}))},\\ \|G(V)\|_{\mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F}))} &\leq C\sqrt{T}\left(\|V\|_{\mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F}))}\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(0,T;\mathbf{H}^2(\mathcal{F}))}\right.\\ &\quad\left.+\|V\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))}\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^2(0,T;\mathbf{L}^2(\mathcal{F}))\cap\mathrm{H}^1(0,T;\mathbf{H}^2(\mathcal{F}))}\right).\end{aligned}$$

Proof. For $m\geq 3$, $\nabla\tilde{Y}(\tilde{X})$ lies in $\mathrm{H}^1(0,T;\mathbf{H}^2(\mathcal{F}))$. We apply Lemma 2.1 with $s = 2$, $\mu = 0$ and $\kappa = 0$, and we get

$$\begin{aligned}\|G(V)(\cdot,t)\|_{\mathbf{H}^2(\mathcal{F})} &\leq C\,\|V\|_{\mathbf{H}^2(\mathcal{F})}\,\|\nabla\tilde{Y}(\tilde{X}(\cdot,t),t) - I\|_{\mathbf{H}^2(\mathcal{F})},\\ \|G(V)\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))} &\leq C\,\|V\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))}\,\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\mathcal{F}))}\\ &\leq C\sqrt{T}\,\|V\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))}\,\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{H}^1(0,T;\mathbf{H}^2(\mathcal{F}))}.\end{aligned}$$

For proving the regularity $\mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F}))$, we first write

$$\frac{\partial G(V)}{\partial t} \quad=\quad \left(\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\right)\frac{\partial V}{\partial t} + \frac{\partial}{\partial t}\left(\nabla\tilde{Y}(\tilde{X})\right)V.$$

Notice that we have the embedding

$$\mathrm{L}^2(0,T;\mathbf{H}^{m-1}(\mathcal{F}))\cap\mathrm{H}^1(0,T;\mathbf{L}^2(\mathcal{F}))\hookrightarrow\mathrm{L}^\infty(0,T;\mathbf{H}^{m/2-1/2}(\mathcal{F})),$$

and thus - in applying Lemma 2.1 with $s = 0$, $\mu = m/2-1/2$ and $\kappa = 1$ - the estimate

$$\begin{aligned}\left\|\frac{\partial G(V)}{\partial t}\right\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}))} &\leq C\left(\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^\infty(\mathcal{F}))}\left\|\frac{\partial V}{\partial t}\right\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}))}\right.\\ &\quad\left.+\sqrt{T}\left\|\frac{\partial}{\partial t}\left(\nabla\tilde{Y}(\tilde{X})\right)\right\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m/2-1/2}(\mathcal{F}))}\|V\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\mathcal{F}))}\right).\end{aligned}$$

□

Lemma 2.13. *There exists a positive constant C such that for all (V, Q, K', ϖ) in $\mathbb{H}_T$ we have*

$$\|W(\varpi)\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))} \leq C\sqrt{T}\|\varpi\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)},$$
$$\|F_M(V,Q,K',\varpi)\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} \leq$$
$$C\left(\sqrt{T}\|K'\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)}\|\varpi\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)} + \left(\|V\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))} + \|Q\|_{\mathrm{L}^2(0,T;\mathbf{H}^1(\mathcal{F}))}\right) \times\right.$$
$$\left.\sqrt{T}\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{H^1(0,T;\mathbf{L}^\infty(\partial\mathcal{S}))}\left(\|\nabla\tilde{Y}(\tilde{X})\|_{L^\infty(\partial\mathcal{S}\times(0,T))} + 1\right)\right),$$

$$\|F_I(V,Q,K',\varpi)\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} \leq$$
$$C\left(\|I^* - I_0\|_{L^\infty(0,T;\mathbb{R}^9)}\|\varpi\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)}\right.$$
$$+\sqrt{T}\|{I^*}'\|_{L^\infty(0,T;\mathbb{R}^9)}\|\varpi\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)} + \sqrt{T}\|I^*\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^9)}\|\varpi\|^2_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)}$$
$$+\sqrt{T}\left(\|V\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathcal{F}))} + \|Q\|_{\mathrm{L}^2(0,T;\mathbf{H}^1(\mathcal{F}))}\right) \times$$
$$\left.\left(1 + \|\nabla\tilde{Y}(\tilde{X})\|_{L^\infty(\partial\mathcal{S}\times(0,T))}\right)\left(\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^3}\|_{H^1(0,T;\mathbf{L}^\infty(\partial\mathcal{S}))} + \|\nabla X^* - \mathrm{Id}_{\partial\mathcal{S}}\|_{H^1(0,T;\mathbf{L}^\infty(\partial\mathcal{S}))}\right)\right),$$

$$\|{I^*}'\|_{L^\infty(0,T;\mathbb{R}^9)} \leq C, \qquad \|I^* - I_0\|_{L^\infty(0,T;\mathbb{R}^9)} \leq CT.$$

Proof. For the first estimate, we write (for $m \geq 3$)

$$\begin{aligned}
\left\|\frac{\partial W(\varpi)}{\partial t}\right\|_{\mathrm{L}^2(0,T;\mathbf{H}^{3/2}(\mathcal{F}))} &\leq C\|\varpi'\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)}\|X^* - \mathrm{Id}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))} \\
&\quad + C\|\varpi\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)}\left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m/2}(\partial\mathcal{S}))}, \\
&\leq C\sqrt{T}\|\varpi'\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)}\|X^* - \mathrm{Id}\|_{\mathrm{H}^1(0,T;\mathbf{H}^2(\mathcal{S}))} \\
&\quad + C\sqrt{T}\|\varpi\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^3)}\left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{m/2+1/2}(\mathcal{S}))},
\end{aligned}$$

in recalling that

$$\mathrm{L}^2(0,T;\mathbf{H}^m(\mathcal{S})) \cap \mathrm{H}^1(0,T;\mathbf{H}^1(\mathcal{S})) \hookrightarrow \mathrm{L}^\infty(0,T;\mathbf{H}^{m/2+1/2}(\mathcal{S})).$$

There is no particular difficulty for proving the other two estimates, if we refer to the respective expressions of F_M and F_I given by (2.21) and (2.22). However, let us detail the terms due to the inertia matrices. We have

$${I^*}'(t) = \int_{\mathcal{S}} \rho_{\mathcal{S}}\left(2\left(\frac{\partial X^*}{\partial t}\cdot X^*\right)\mathrm{I}_{\mathbb{R}^3} - \frac{\partial X^*}{\partial t}\otimes X^* - X^*\otimes\frac{\partial X^*}{\partial t}\right)(y,t)\mathrm{d}y,$$

and thus

$$\begin{aligned}
\left|{I^*}'(t)\right|_{\mathbb{R}^9} &\leq C_I\left\|\frac{\partial X^*}{\partial t}(\cdot,t)\right\|_{\mathbf{L}^2(\mathcal{S})}\|X^*(\cdot,t)\|_{\mathbf{L}^2(\mathcal{S})}, \\
\left\|{I^*}'\right\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^9)} &\leq C_I\left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\mathcal{S}))}\|X^*\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\mathcal{S}))}, \\
\|I^* - I_0\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^9)} &\leq T\left\|{I^*}'\right\|_{\mathrm{L}^\infty(0,T;\mathbb{R}^9)}.
\end{aligned}$$

□

Stability of the set B_R by the mapping $\mathcal{N}$

We are now in position to claim that, for R large enough and T small enough, the ball $\mathcal{B}_R$ is stable by $\mathcal{N}$.

Lemma 2.14. *Let us assume that $T \leq 1$ and $R \geq 1$. There exists a positive constant C_0, which does not depend on T or R, such that for $(V, Q, K', \varpi) \in B_R$ we have*

$$\begin{aligned}
\|F(V,P,K',\varpi)\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}))} &\leq C_0 T^{1/10} R^3,\\
\|G(V)\|_{\mathrm{H}^{2,1}(Q_T^0)} &\leq C_0\sqrt{T}R^2,\\
\|W(\varpi)\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))} &\leq C_0\sqrt{T}R,\\
\|F_M(V,P,K',\varpi)\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} &\leq C_0\sqrt{T}R^3,\\
\|F_I(V,P,K',\varpi)\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} &\leq C_0\sqrt{T}R^3.
\end{aligned}$$

Proof. These estimates follow from Corollary 2.10 and Lemmas 2.11, 2.12, 2.13 combined with the estimates (2.23) and (2.24). ☐

Combining Lemma 2.14 and the estimate (2.25), we obtain that, for R large enough ($R > C_T^{(0)}$) and T small enough, we have

$$\mathcal{N}(B_R) \quad \subset \quad B_R.$$

Lipschitz stability for the mapping $\mathcal{N}$

Let $(V_1, P_1, K_1', \varpi_1)$ and $(V_2, P_2, K_2', \varpi_2)$ be in B_R. We set

$$\begin{aligned}
(U_1, P_1, H_1', \Omega_1) &= \mathcal{N}(V_1, Q_1, K_1', \varpi_1),\\
(U_2, P_2, H_2', \Omega_2) &= \mathcal{N}(V_2, Q_2, K_2', \varpi_2),
\end{aligned}$$

and

$$\begin{aligned}
&U = U_2 - U_1, && \overline{P} = P_2 - P_1, && H' = H_2' - H_1', && \Omega = \Omega_2 - \Omega_1,\\
&V = V_2 - V_1, && \overline{Q} = Q_2 - Q_1, && K' = K_2' - K_1', && \varpi = \varpi_2 - \varpi_1.
\end{aligned}$$

We also denote $\tilde{X}_1$, $\nabla\tilde{Y}_1(\tilde{X}_1)$ the mappings provided by Lemma 2.3 with (K_1', ϖ_1, X^*) as data, and similarly $\tilde{X}_2$, $\nabla\tilde{Y}_2(\tilde{X}_2)$ the mappings provided by (K_2', ϖ_2, X^*).
The quadruplet (U, P, H', Ω) satisfies the system

$$\begin{aligned}
\frac{\partial U}{\partial t} - \nu\Delta U + \nabla P = \overline{F}, &\qquad \text{in } \mathcal{F}\times(0,T),\\
\operatorname{div} U = \operatorname{div} \overline{G}, &\qquad \text{in } \mathcal{F}\times(0,T),
\end{aligned}$$

$$\begin{aligned}
U = 0, &\qquad \text{in } \partial\mathcal{O}\times(0,T),\\
U = H'(t) + \Omega(t)\wedge y + W_\varpi, &\qquad (y,t)\in\partial\mathcal{S}\times(0,T),
\end{aligned}$$

$$\begin{aligned}
MH'' &= -\int_{\partial\mathcal{S}} \sigma(U,P)n\mathrm{d}\Gamma + \overline{F}_M, \text{ in } (0,T)\\
I_0\Omega'(t) &= -\int_{\partial\mathcal{S}} y\wedge\sigma(U,P)n\mathrm{d}\Gamma + \overline{F}_I, \text{ in } (0,T)
\end{aligned}$$

$$U(y,0) = 0, \text{ in } \mathcal{F}, \quad H'(0) = 0 \in \mathbb{R}^3, \quad \Omega(0) = 0 \in \mathbb{R}^3,$$

with

$$\begin{array}{rcl} \overline{F} & = & F(V_2, Q_2, K_2', \Omega_2) - F(V_1, Q_1, K_1', \Omega_1), \\ \overline{G} & = & G_{(K_2', \varpi_2)} V_2 - G_{(K_1', \varpi_1)} V_1 \\ \overline{W} & = & W(\varpi) \\ \overline{F}_M & = & F_M(V_2, Q_2, K_2', \varpi_2) - F_M(V_1, Q_1, K_1', \varpi_1), \\ \overline{F}_I & = & F_I(V_2, Q_2, K_2', \varpi_2) - F_I(V_1, Q_1, K_1', \varpi_1). \end{array}$$

In particular, the study of this nonhomogeneous linear system provides the estimate

$$\begin{array}{rcl} \|(U, P, H', \Omega)\|_{\mathbb{H}_T} & \leq & C_T^{(0)} \Big(\|\overline{F}\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} + \|\overline{G}\|_{\mathrm{H}^{2,1}(Q_T^0)} + \|W_\varpi\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))} \\ & & + \|\overline{F}_M\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} + \|\overline{F}_I\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} \Big) . \end{array} \quad (2.32)$$

Notice that the right-hand-sides $\overline{F}$, $\overline{G}$, $\overline{F}_M$ and $\overline{F}_I$ can be written as polynomial differential forms, multiplicative of one of the quantities

$$V, \quad Q, \quad K', \quad \varpi, \quad (\tilde{X}_2 - \tilde{X}_1), \quad \left(\nabla\tilde{Y}_2(\tilde{X}_2) - \nabla\tilde{Y}_1(\tilde{X}_1)\right).$$

For instance, the nonhomogeneous divergence condition $\overline{G}$ can be written as

$$\overline{G} \;=\; \left(\nabla\tilde{Y}_2(\tilde{X}_2) - \nabla\tilde{Y}_1(\tilde{X}_1)\right) V_2 + (\nabla\tilde{Y}_1(\tilde{X}_1) - \mathrm{I}_{\mathbb{R}^3}) V.$$

We have in particular

$$\begin{array}{rcl} \tilde{X}_2(\cdot,0) - \tilde{X}_1(\cdot,0) & = & 0, \\ \nabla\tilde{Y}_2(\tilde{X}_2(\cdot,0),0) - \nabla\tilde{Y}_1(\tilde{X}_1(\cdot,0),0) & = & 0. \end{array}$$

The mapping $\nabla\tilde{Y}_2(\tilde{X}_2) - \nabla\tilde{Y}_1(\tilde{X}_1)$ satisfies the estimate (2.15) stated in Lemma 2.5, which is useful in order to make $\mathcal{N}$ a contraction. More specifically, the estimates (2.9) and (2.15) are rewritten as

$$\begin{array}{rcl} \|\tilde{X}_2 - \tilde{X}_1\|_{\mathrm{H}^1(\mathbf{H}^m)\cap\mathrm{H}^2(\mathbf{H}^1)} & \leq & C\left(\|K'\|_{H^1(0,T_0;\mathbb{R}^3)} + \|\varpi\|_{H^1(0,T_0;\mathbb{R}^3)}\right), \\ \|\nabla\tilde{Y}(\tilde{X})_2 - \nabla\tilde{Y}_1(\tilde{X}_1)\|_{\mathrm{H}^1(\mathbf{H}^{m-1})\cap\mathrm{H}^2(\mathbf{L}^2)} & \leq & C\left(\|K'\|_{\mathrm{H}^1(0,T_0;\mathbb{R}^3)} + \|\varpi\|_{\mathrm{H}^1(0,T_0;\mathbb{R}^9)}\right). \end{array}$$

Then we state the following result, which can be proven with the same techniques that have been used for obtaining Lemma 2.14.

Lemma 2.15. *For R large enough and T small enough, there exists a positive constant C_0 - which does not depend on T or R - such that*

$$\begin{array}{rcl} \left\|\overline{F}\right\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathcal{F}))} & \leq & C_0 T^{1/10} R^2 \|(V,Q,K',\varpi)\|_{\mathbb{H}_T}, \\ \left\|\overline{G}\right\|_{\mathrm{H}^{2,1}(Q_T^0)} & \leq & C_0\sqrt{T} R \left(\|V\|_{\mathrm{H}^{2,1}(Q_T^0)} + \|K'\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)} + \|\varpi\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)}\right), \\ \left\|\overline{W}\right\|_{\mathrm{H}^1(0,T;\mathbf{L}^{3/2}(\partial\mathcal{S}))} & \leq & C_0\sqrt{T}\|\varpi\|_{\mathrm{H}^1(0,T;\mathbb{R}^3)}, \\ \left\|\overline{F}_M\right\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} & \leq & C_0\sqrt{T} R^2 \|(V,Q,K',\varpi)\|_{\mathbb{H}_T}, \\ \left\|\overline{F}_I\right\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} & \leq & C_0\sqrt{T} R^2 \|(V,Q,K',\varpi)\|_{\mathbb{H}_T}. \end{array}$$

With regards to the estimate (2.32), we deduce from this lemma that for R large enough and T small enough the mapping $\mathcal{N}$ is a contraction in B_R.

2.5 Verification of the assumptions for the global existence result

For adapting the results of Chapter 1 when the regularity of the solid's deformation is limited, it remains us to adapt Theorem 1.18:

Theorem 2.16. *Assume that the hypotheses in Theorem 2.8 hold true. Moreover we assume that f is small enough in $\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}(t)))\cap \mathrm{L}^{3/2}(0,\infty;\mathbf{L}^2(\mathcal{F}(t)))$. Assume that $\|u_0\|_{\mathbf{H}^1(\mathcal{F})}$, $|h_1|_{\mathbb{R}^3}$ and $|\omega_0|_{\mathbb{R}^3}$ are small enough, and that the deformation X^* is close enough to the identity $\mathrm{Id}_{\mathcal{S}}$ in $\mathcal{W}_0^m(0,T;\mathcal{S})$, for $m\geq 3$. Then we are in the case of the assertion (a) in Theorem 2.8, that is to say that the strong solution of problem (1.1)–(1.7) is global in time.*

With regards to the hypotheses of Theorem 1.18, it is sufficient to consider $X^*-\mathrm{Id}_{\mathcal{S}}$ in $\mathcal{W}_0^3(0,T;\mathcal{S})$ and to verify that we can still define the Eulerian velocity w^* in $\mathcal{S}^*(t)$ which satisfies

$$w^* \quad \in \quad \mathrm{L}^2(0,T;\mathbf{H}^3(\mathcal{S}^*(t)))\cap \mathrm{H}^1(0,T;\mathbf{H}^1(\mathcal{S}^*(t))),$$

and which is small in this space when $X^*-\mathrm{Id}_{\mathcal{S}}$ is small in $\mathcal{W}_0^3(0,T;\mathcal{S})$. Then the steps of the proof of Theorem 1.18 can be straightforwardly applied.

Lemma 2.17. *Let $X^*\in\mathcal{W}_0^m(0,T;\mathcal{S})$ such that for all $t\in[0,T)$ the mapping $X^*(\cdot,t)$ is a C^1-diffeomorphism from $\mathcal{S}$ onto $\mathcal{S}^*(t)$. Then the function defined by*

$$w^*(x^*,t) \quad = \quad \frac{\partial X^*}{\partial t}(Y^*(x^*,t),t), \quad x^*\in\mathcal{S}^*(t), t\in[0,T),$$

satisfies

$$w^* \quad \in \quad \mathrm{L}^2(0,T;\mathbf{H}^3(\mathcal{S}^*(t)))\cap \mathrm{H}^1(0,T;\mathbf{H}^1(\mathcal{S}^*(t))).$$

Moreover, $\|w^*\|_{\mathrm{L}^2(0,T;\mathbf{H}^3(\mathcal{S}^*(t)))\cap \mathrm{H}^1(0,T;\mathbf{H}^1(\mathcal{S}^*(t)))}$ is an increasing function of

$$\left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^3(\mathcal{S}))\cap\mathrm{H}^1(\mathbf{H}^1(\mathcal{S}))}, \quad \|\nabla Y^*(X^*)\|_{\mathrm{L}^\infty(\mathbf{H}^2(\mathcal{S}))}, \quad \|\mathrm{det}\nabla X^*(\cdot,t)\|_{\mathrm{L}^\infty(\mathbf{L}^\infty(\mathcal{S}))},$$

and tends to 0 when $\left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^3(\mathcal{S}))\cap\mathrm{H}^1(\mathbf{H}^1(\mathcal{S}))}$ goes to 0.

Proof. Let us use a result given in the Appendix of [BB74] (Lemma A.4), which treats of regularity in Sobolev spaces for composition of functions: There exists a positive constant C such that for all $t\in(0,T)$ we have

$$\begin{aligned}\|w^*(\cdot,t)\|_{\mathbf{H}^3(\mathcal{S}^*(t))} &\leq C\left\|\frac{\partial X^*}{\partial t}(\cdot,t)\right\|_{\mathbf{H}^3(\mathcal{S})}\frac{\|Y^*(\cdot,t)\|^3_{\mathbf{H}^3(\mathcal{S}^*(t))}+1}{\inf\limits_{x^*\in\mathcal{S}^*(t)}|\mathrm{det}\nabla Y^*(x^*,t)|^{1/2}}\\ &\leq C\left\|\frac{\partial X^*}{\partial t}(\cdot,t)\right\|_{\mathbf{H}^3(\mathcal{S})}\left(\|Y^*(\cdot,t)\|^3_{\mathbf{H}^3(\mathcal{S}^*(t))}+1\right)\|\mathrm{det}\nabla X^*(\cdot,t)\|^{1/2}_{\mathbf{L}^\infty(\mathcal{S})}.\end{aligned}$$

Let us notice that in using the change of variables induced by $X^*(\cdot,t)$, we have

$$\begin{aligned}\|Y^*(\cdot,t)\|^2_{\mathbf{L}^2(\mathcal{S}^*(t))} &= \int_{\mathcal{S}^*(t)}|Y^*(x^*,t)|^2_{\mathbb{R}^3}\,\mathrm{d}x^*\\ &= \int_{\mathcal{S}}|y|^2\mathrm{det}\nabla X^*(y,t)\mathrm{d}y,\\ \|\nabla Y^*(\cdot,t)\|_{\mathbf{L}^2(\mathcal{S}^*(t))} &\leq \|\nabla Y^*(X^*(y,t),t)\|_{\mathbf{L}^2(\mathcal{S})}\|\mathrm{det}\nabla X^*(\cdot,t)\|^{1/2}_{\mathrm{L}^\infty(\mathcal{S})}.\end{aligned}$$

The following equality

$$\nabla^2 Y^*(X^*(\cdot,t),t) = \left(\nabla\left(\nabla Y^*(X^*(\cdot,t),t)\right)\right)\nabla Y^*(X^*(\cdot,t),t) \tag{2.33}$$

yields

$$\|\nabla^2 Y^*(\cdot,t)\|_{\mathbf{L}^2(\mathcal{S}^*(t))} \leq C\|\nabla Y^*(X^*(\cdot,t),t)\|_{\mathbf{H}^1(\mathcal{S})}\|\nabla Y^*(X^*(\cdot,t),t)\|_{\mathbf{L}^\infty(\mathcal{S})}\|\mathrm{det}\nabla X^*(\cdot,t)\|^{1/2}_{\mathrm{L}^\infty(\mathcal{S})}.$$

Moreover, in applying Lemma 2.1 with $s=1$, $\mu=0$ and $\kappa=1$, the equality (2.33) implies

$$\|\nabla^2 Y^*(X^*(\cdot,t),t)\|_{\mathbf{H}^1(\mathcal{S})} \leq C\|\nabla Y^*(X^*(\cdot,t),t)\|^2_{\mathbf{H}^2(\mathcal{S})}.$$

The following equality

$$\begin{aligned}\nabla^3 Y^*(X^*(\cdot,t),t) &= \left(\nabla^2\left(\nabla Y^*(X^*(\cdot,t),t)\right)\right)\left(\nabla Y^*(X^*(\cdot,t),t)\right)^2 \\ &\quad + \left(\nabla\left(\nabla Y^*(X^*(\cdot,t),t)\right)\right)\left(\nabla^2 Y^*(X^*(\cdot,t),t)\right)\end{aligned}$$

combined with the previous estimate enables us to obtain

$$\|\nabla^3 Y^*(\cdot,t)\|_{\mathbf{L}^2(\mathcal{S}^*(t))} \leq C\|\nabla Y^*(X^*(\cdot,t),t)\|^3_{\mathbf{H}^2(\mathcal{S})}\|\mathrm{det}\nabla X^*(\cdot,t)\|^{1/2}_{\mathrm{L}^\infty(\mathcal{S})}.$$

Finally we get

$$\begin{aligned}\|w^*\|_{\mathrm{L}^2(\mathbf{H}^3(\mathcal{S}^*(t)))} &\leq \tilde{C}\left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{H}^3(\mathcal{F}))}\|\mathrm{det}\nabla X^*(\cdot,t)\|^{1/2}_{\mathrm{L}^\infty(\mathbf{L}^\infty(\mathcal{S}))}\times \\ &\quad \left(1+\left(\|\mathrm{det}\nabla X^*(\cdot,t)\|_{\mathrm{L}^\infty(\mathbf{L}^\infty(\mathcal{S}))}\sum_{k=1}^{3}\|\nabla Y^*(X^*)\|^{2k}_{\mathrm{L}^\infty(\mathbf{H}^2(\mathcal{S}))}\right)^{3/2}\right).\end{aligned}$$

For the regularity of w^* in $\mathrm{H}^1(0,T;\mathbf{H}^1(\mathcal{S}^*(t)))$, we estimate

$$\left\|\frac{\partial w^*}{\partial t}(\cdot,t)\right\|_{\mathbf{L}^2(\mathcal{S}^*(t))} \leq C\|\mathrm{det}\nabla X^*(\cdot,t)\|^{1/2}_{\mathbf{L}^\infty(\mathcal{S})}\left\|\frac{\partial w^*}{\partial t}(X^*(\cdot,t),t)\right\|_{\mathbf{L}^2(\mathcal{S})},$$

and we calculate

$$\begin{aligned}\frac{\partial w^*}{\partial t}(x^*,t) &= \frac{\partial^2 X^*}{\partial t^2}(Y^*(x^*,t),t)+\frac{\partial \nabla X^*}{\partial t}(Y^*(x^*,t),t)\frac{\partial Y^*}{\partial t}(x^*,t), \quad x^*\in\mathcal{S}^*(t), \\ \frac{\partial w^*}{\partial t}(X^*(y,t),t) &= \frac{\partial^2 X^*}{\partial t^2}(y,t)-\frac{\partial \nabla X^*}{\partial t}(y,t)\nabla Y^*(X^*(y,t),t)\frac{\partial X^*}{\partial t}(y,t), \quad y\in\mathcal{S}.\end{aligned}$$

Thus we have

$$\begin{aligned}\left\|\frac{\partial w^*}{\partial t}(\cdot,t)\right\|_{\mathrm{L}^2(\mathbf{L}^2(\mathcal{S}^*(t)))} &\leq C\|\mathrm{det}\nabla X^*(\cdot,t)\|^{1/2}_{\mathrm{L}^\infty(\mathbf{L}^\infty(\mathcal{S}))}\times\left(\left\|\frac{\partial^2 X^*}{\partial t^2}\right\|_{\mathrm{L}^2(\mathbf{L}^2(\mathcal{S}))}\right. \\ &\quad \left.+\left\|\frac{\partial \nabla X^*}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{L}^2(\mathcal{S}))}\|\nabla Y^*(X^*)\|_{\mathrm{L}^\infty(\mathbf{H}^2(\mathcal{S}))}\left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty(\mathcal{S}))}\right).\end{aligned}$$

□

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Chapter 3

Stabilization of the fluid-solid system, by the deformation of the self-propelled solid

In this chapter, we prove the stabilization to zero of a fluid-solid system modeling a deformable solid which propels itself in a viscous incompressible fluid. The physical nature of the control is a velocity of deformation. The initial data are assumed to be small enough. The strategy we follows is globally the same as the one used in [Ray10a]. It first consists in rewriting the main system in cylindrical domains, and in linearizing the nonlinear system so obtained. Then we prove the stabilization of the linear system by controls which are chosen in feedback form, and which correspond to solid's deformations that satisfy some linearized constraints. After that, we prove the stabilization of the full nonlinear system with deformations which satisfy nonlinear physical constraints.

3.1 Introduction

In this introduction, we first complete and present again the fluid-solid model which has been first detailed in the introduction of Chapter 1, so the reader can skip this presentation.

3.1.1 Presentation of the model

In this chapter, we are interested in the stabilization of a deformable solid in a viscous incompressible fluid. The domain representing the solid at time t is denoted by $\mathcal{S}(t)$. We assume that $\mathcal{S}(t) \subset \mathcal{O}$ where $\mathcal{O}$ is a bounded regular domain of $\mathbb{R}^d$ (with $d = 2$ or 3). The fluid surrounding the structure occupies the domain $\mathcal{O} \setminus \overline{\mathcal{S}(t)} = \mathcal{F}(t)$.

The solid's motion can be represented by a Lagrangian mapping $X_{\mathcal{S}}$ that we decompose into a rigid part and - what is called - an undulatory deformation X^*, as follows

$$X_{\mathcal{S}}(y,t) = X^R(X^*(y,t),t) = h(t) + \mathbf{R}(t)X^*(y,t), \quad y \in \mathcal{S}(0).$$

The mapping $X^R(y,t) = h(t) + \mathbf{R}(t)y$ is unknown. It is given by the position of the solid's center of mass $h(t)$ and the rotation $\mathbf{R}(t)$ resulting from the angular velocity $\omega(t)$. We denote by $Y^R(x,t) = \mathbf{R}(t)^T(x - h(t))$ the inverse of $X^R(\cdot,t)$. We define

$$\mathcal{S}^*(t) = X^*(\mathcal{S}(0),t), \qquad \mathcal{F}^*(t) = \mathcal{O} \setminus \overline{\mathcal{S}^*(t)},$$

and we have

$$\mathcal{S}(t) = X^R(\mathcal{S}^*(t), t) = X_{\mathcal{S}}(\mathcal{S}(0), t), \qquad \mathcal{F}(t) = \mathcal{O} \setminus \overline{\mathcal{S}(t)}.$$

The Eulerian velocity of the solid, corresponding to the decomposition above, is written in $\mathcal{S}(t)$ and can be split into two parts. One part corresponds to a rigid displacement and the other part corresponds to an undulatory velocity w, as follows

$$u_{\mathcal{S}}(x,t) \quad = \quad h'(t) + \omega(t) \wedge (x - h(t)) + w(x,t), \quad x \in \overline{\mathcal{S}(t)}.$$

The velocity field w which appears in this equality is expressed in terms of w^*, h and ω, through the following change of frame

$$w(x,t) \quad = \quad \mathbf{R}(t)\ w^*\left(\mathbf{R}(t)^T(x - h(t)), t\right), \quad x \in \mathcal{S}(t). \tag{3.1}$$

The field w^* is the Eulerian velocity related to the Lagrangian flow X^* through the following Cauchy problem

$$\frac{\partial X^*}{\partial t}(y,t) = w^*(X^*(y,t),t), \quad X^*(y,0) = y - h(0), \quad y \in \mathcal{S}(0). \tag{3.2}$$

The mapping $X^*(\cdot, t)$ represents the deformation of the solid in its own frame of reference, that is to say its shape. We can choose it as a control function. It is equivalent to assuming that the solid is strong enough to impose its own shape, in spite of the fluid's forces at the fluid-structure interface.
The fluid flow is described by its velocity u and its pressure p. The quadruplet (u, p, h, ω) satisfies the following coupled system

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p \quad = \quad 0, \quad x \in \mathcal{F}(t), \quad t \in (0,\infty), \tag{3.3}$$

$$\operatorname{div} u \quad = \quad 0, \quad x \in \mathcal{F}(t), \quad t \in (0,\infty), \tag{3.4}$$

$$u = 0, \qquad x \in \partial\mathcal{O}, \quad t \in (0,\infty), \tag{3.5}$$

$$u = h'(t) + \omega(t) \wedge (x - h(t)) + w(x,t), \qquad x \in \partial\mathcal{S}(t), \quad t \in (0,\infty), \tag{3.6}$$

$$Mh''(t) = -\int_{\partial\mathcal{S}(t)} \sigma(u,p)n\mathrm{d}\Gamma, \qquad t \in (0,\infty), \tag{3.7}$$

$$\left(I\omega\right)'(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t)) \wedge \sigma(u,p)n\mathrm{d}\Gamma, \qquad t \in (0,\infty), \tag{3.8}$$

$$u(y,0) = u_0(y),\ y \in \mathcal{F}(0), \quad h(0) = h_0 \in \mathbb{R}^d, \quad h'(0) = h_1 \in \mathbb{R}^d, \quad \omega(0) = \omega_0 \in \mathbb{R}^3. \tag{3.9}$$

Without loss of generality, we can assume that $h_0 = 0$. The symbol $\wedge$ denotes the cross product in $\mathbb{R}^2$ or $\mathbb{R}^3$. The result of a cross product in $\mathbb{R}^3$ is a vector, whereas it is a scalar in $\mathbb{R}^2$. However $\mathbb{R}^2$ can be immersed in $\mathbb{R}^3$, and the result of a cross product in $\mathbb{R}^2$ can be read on the third component of a 3D-vector, as follows

$$\begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 0 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

In the same idea, since the angular velocity $\omega(t)$ is a scalar function in dimension 2 and a 3D-vector in dimension 3, the calculations on vectors hold if we consider that in dimension 2 the angular velocity is seen as

$$\begin{pmatrix} 0 \\ 0 \\ \omega(t) \end{pmatrix}.$$

That is why in all cases we consider the angular velocity as a vector of $\mathbb{R}^3$. In dimension 2, $\omega(t)$ is the time derivative of an angle $\theta(t)$. We associate with this angle the classical rotation

$$\mathbf{R}(t) \quad = \quad \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) & 0 \\ \sin(\theta(t)) & \cos(\theta(t)) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In dimension 3, associated with the angular velocity vector $\omega(t)$, we introduce the rotation $\mathbf{R}(t)$ obtained as being the solution of the following Cauchy problem

$$\left\{ \begin{array}{rcl} \dfrac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} & = & \mathbb{S}(\omega)\,\mathbf{R} \\ \mathbf{R}(0) & = & \mathrm{I}_{\mathbb{R}^3}, \end{array} \right. \qquad \text{with } \mathbb{S}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{3.10}$$

The linear map $\omega \wedge \cdot$ can be represented by the matrix $\mathbb{S}(\omega)$.

In equations (3.7) and (3.8) - which result from the Newton's laws - the solid's mass M is constant, whereas the inertia moment depends *a priori* on time. In dimension 2 the inertia moment is a scalar function which can be read on the inertia matrix given by

$$I(t) \quad = \quad \left(\int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t) \left| x - h(t) \right|^2 \mathrm{d}x \right) \mathrm{I}_{\mathbb{R}^3}.$$

In dimension 3 it is a tensor written as

$$I(t) \quad = \quad \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t) \left(|x-h(t)|^2 \mathrm{I}_{\mathbb{R}^3} - (x-h(t)) \otimes (x-h(t)) \right) \mathrm{d}x.$$

The quantity $\rho_{\mathcal{S}}$ denotes the solid's density, and obeys the principle of mass conservation

$$\rho_{\mathcal{S}}(X_{\mathcal{S}}(y,t),t) = \frac{\rho_{\mathcal{S}}(y,0)}{\det \nabla X_{\mathcal{S}}(y,t)} = \frac{\rho_{\mathcal{S}}(y,0)}{\det \nabla X^*(y,t)}.$$

The mapping $X^*(\cdot,t)$ represents a deformation in the solid's frame of reference. If $X^*(\cdot,t)$ is a C^1-diffeomorphism, then the change of variables given by the mapping $X_{\mathcal{S}}(\cdot,t)$ enables us to express more simply

$$I(t) = \left(\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) |y|^2 \mathrm{d}y \right) \mathrm{I}_{\mathbb{R}^3} \qquad \text{in dimension 2,}$$

$$I(t) = \mathbf{R}(t) I_0 \mathbf{R}(t)^T \qquad \text{in dimension 3,}$$

where I_0 denotes the moment of inertia tensor at time $t = 0$, given by

$$I_0 \quad = \quad \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(y,0) \left(|y|^2 \mathrm{I}_{\mathbb{R}^3} - y \otimes y \right) \mathrm{d}y.$$

Notice that in dimension 2 the inertia moment does not depend on time, and here again the angular momentum can be read on the third component of the 3D-vector $I\omega$.
In system (3.3)–(3.9), ν is the kinematic viscosity of the fluid, and the normalized vector n is the normal at $\partial\mathcal{S}(t)$ exterior to $\mathcal{F}(t)$. It is a coupled system between the incompressible Navier-Stokes equations and the Newton's laws. The coupling is in particular made in the fluid-structure interface, through the equality of velocities (3.6) and through the Cauchy stress tensor

$$\sigma(u,p) = 2\nu D(u) - p \ \mathrm{Id} = \nu \left(\nabla u + (\nabla u)^T\right) - p \ \mathrm{Id},$$

representing the fluid's forces which lead to the exchange of momenta between the fluid and the solid.

3.1.2 Results and methods

For the full nonlinear system (3.3)–(3.9), the equations are written in the Eulerian configuration, and thus we are lead to think that w^* is the more suitable function to be chosen as a control (instead of X^*). But such a mapping is defined on the domain $\mathcal{S}^*(t)$, which is itself defined by $X^*(\cdot,t)$.
Moreover, the study of such a nonlinear system is based on the preliminary study of the corresponding linearized system which is

$$\frac{\partial \hat{u}}{\partial t} - \lambda\hat{u} - \mathrm{div}\ \sigma(\hat{u},\hat{p}) = 0, \quad \text{in } \mathcal{F}(0)\times(0,\infty), \tag{3.11}$$

$$\mathrm{div}\ \hat{u} = 0, \quad \text{in } \mathcal{F}(0)\times(0,\infty), \tag{3.12}$$

$$\hat{u} = 0, \qquad \text{in } \partial\mathcal{O}\times(0,\infty), \tag{3.13}$$

$$\hat{u} = \hat{h}'(t) + \hat{\omega}(t)\wedge y + \zeta(y,t), \qquad y\in\ \partial\mathcal{S}(0), \quad t\in(0,\infty), \tag{3.14}$$

$$M\hat{h}''(t) = -\int_{\partial\mathcal{S}} \sigma(\hat{u},\hat{p})n\mathrm{d}\Gamma, \quad t\in(0,\infty), \tag{3.15}$$

$$I_0\hat{\omega}'(t) = -\int_{\partial\mathcal{S}} y\wedge\sigma(\hat{u},\hat{p})n\mathrm{d}\Gamma, \quad t\in(0,\infty), \tag{3.16}$$

$$\hat{u}(y,0) = u_0(y),\ y\in\mathcal{F}, \quad \hat{h}'(0) = h_1\in\mathbb{R}^d, \quad \hat{\omega}(0) = \omega_0\in\mathbb{R}^3, \tag{3.17}$$

and where the more suitable control to be chosen is the function ζ, related to the Lagrangian velocity $\dfrac{\partial X^*}{\partial t}$ by

$$\zeta = e^{\lambda t}\frac{\partial X^*}{\partial t}_{|\partial\mathcal{S}}.$$

If we assume that $X^*(y,0) = y - h_0 = y$, it is equivalent to searching for X^*. For the nonlinear system (3.3)–(3.9), we are looking for mappings X^* satisfying a set of hypotheses given by :

H1 For all $t\geq 0$, $X^*(\cdot,t)$ is a C^1-diffeomorphism from $\mathcal{S}(0)$ onto $\mathcal{S}^*(t)$.

H2 In order to respect the incompressibility condition given by (3.4), the volume of the whole solid is preserved through the time. That is equivalent to say that

$$\int_{\partial\mathcal{S}(0)} \frac{\partial X^*}{\partial t} \cdot (\mathrm{cof}\nabla X^*)\, n \mathrm{d}\Gamma = 0. \tag{3.18}$$

H3 The linear momentum of the solid is preserved through the time, which leads to

$$\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t) \mathrm{d}y \quad = \quad 0. \tag{3.19}$$

H4 The angular momentum of the solid is preserved through the time, which leads to

$$\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t) \wedge \frac{\partial X^*}{\partial t}(y,t) \mathrm{d}y \quad = \quad 0. \tag{3.20}$$

Imposing constraints (3.19) and (3.20) enables us to get the two following constraints on the undulatory velocity w

$$\int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t) w(x,t) \mathrm{d}y \quad = \quad 0, \tag{3.21}$$

$$\int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)(x-h(t)) \wedge w(x,t) \mathrm{d}y \quad = \quad 0. \tag{3.22}$$

As equations (3.7) and (3.8) are written, constraints (3.21) and (3.22) are implicitly satisfied in system (3.3)–(3.9). Hypotheses **H3** and **H4** are made to guarantee the *self-propelled* nature of the solid's motion, that means no other help than its own deformation enables it to interact and to move in the surrounding fluid. The solid interacts with the environing fluid through the Dirichlet condition imposed on $\partial\mathcal{S}$, which determine the fluid's behavior and thus the trace of $\sigma(u,p)n$ on $\partial\mathcal{S}$, that is to say the fluid's forces that act on the solid's position through equations (3.7)–(3.8). Notice that constraints (3.18) and (3.20) are nonlinear with respect to the mapping X^*. We linearize them when we consider the linear system (3.11)–(3.17) (see Definition 3.2). For this linear system, the constraint induced by Hypothesis **H1** can be relaxed, since we only consider mappings $X^*(\cdot,t)$ continuous in time and such that $X^*(\cdot,0) = \mathrm{Id}_{\mathcal{S}}$.

The exact null controllability of system (3.3)–(3.9) is an open and complicated problem, and is quite debatable from a physical point of view. The question that we set in this chapter is the following :
Is system (3.3)–(3.9) *stabilizable to zero with an arbitrarily exponential decay rate, with such mappings X^*?*
The answer we give to this question is Theorem 3.24, and requires that the initial conditions u_0, h_1 and ω_0 are small enough. We state it as follows:

Theorem. *For (u_0, h_1, ω_0) small enough in $\mathbf{H}^1_{cc}$, system* (3.3)–(3.9) *is stabilizable with an arbitrary exponential decay rate $\lambda > 0$, by solid's displacements $X^* - \mathrm{Id}_{\mathcal{S}} \in \mathcal{W}_\lambda(S^0_\infty)$ satisfying the constraints* **H1**–**H4** *given above. That is to say there exists a constant C, depending only on (u_0, h_1, ω_0), such that*

$$\|(u(\cdot,t), h'(t), \omega(t))\|_{\mathbf{H}^1(\mathcal{F}(t))\times\mathbb{R}^d\times\mathbb{R}^3} \quad \leq \quad C\mathrm{e}^{-\lambda t}.$$

Many control problems involving the motion of a solid body in a fluid have been investigated before. Let us cite for instance the results given in [Kha07] and [Kha08], which treats of the swim obtained by a model of particular deformations. Let us also cite the results of [SMTT07] and [LST11], which deal with swimming at low Reynolds number. A local controllability result is given in [GRar] for the case of an inviscid incompressible fluid. Studies in the case of perfect fluids are given for instance in [CM11]. The stabilization of a coupled parabolic-hyperbolic Stokes-Lamé system has been investigated in [AT09]. The strategy we follow is similar to the one adopted in [Ray10a] (for stabilizing an other fluid-structure system), and consists in showing that the linearized system (3.11)–(3.17) defines an analytic semigroup of contractions. Thus, the approximate controllability of this system enables us to stabilize the unstable modes which are in finite number. Our main result for this linear system is stated in Theorem 3.17. The link with the nonlinear system is made through the feedback stabilization of the associated nonhomogeneous linear system (Proposition 3.23), so that a fixed point method leads us to get the stabilization of system (3.32)–(3.38) (see section 3.8), which is nothing else than system (3.3)–(3.9) rewritten in cylindrical domains.
One of the main difficulties for treating this nonlinear system lies in the fact that we limit the regularity of the solid's deformation. In particular, the change of variables we use in order to rewrite system (3.3)–(3.9) in cylindrical domains is limited in regularity, so that a new approach - different from the one given in [SMSTT08] for instance - is required for constructing it (see section 3.9). This rewriting makes appear nonlinearities due to the geometry. The regularity of the control X^* is partly chosen to make it at least C^1 in space. Besides, its regularity is adjusted in order to choose an appropriate control in the proof of Theorem 3.15. It leads us to consider $X^* \in \mathcal{W}_\lambda(S^0_\infty)$, where $\mathcal{W}_\lambda(S^0_\infty)$ is defined in section 3.2.2.
Another difficulty is due to the constraints that must be imposed on the control X^*. They are linear or not, depending on wether we treat a linear system or a nonlinear one. The control X^*_ζ obtained in section 3.6.1 from ζ chosen in form of a feedback operator, in solving a modified Stokes resolvent system, satisfies the linearized constraints. Provided that this deformation X^*_ζ is close to the identity (and thus that the feedback is small enough), we show in section 3.6.3 that we can associate with it a unique control X^* which satisfies the nonlinear constraints, and thus which can be chosen as a deformation for the nonlinear system (3.32)–(3.38). The control X^* is seen through a projection of the displacement $X^*_\zeta - \mathrm{Id}_{\mathcal{S}}$ on a set of nonlinear constraints. One of the strength of this method lies in the fact that the residual term $X^* - X^*_\zeta$ has some good Lipschitz properties, so that it can be considered as a right-hand-side which can be tackled in a fixed point method. Another strength of this approach lies in the fact that it could be generalized to other problems involving other sets of nonlinear constraints.
However, the projection used in section 3.6.3 acts on functions defined in $\mathcal{S} \times (0, \infty)$. Thus the nonlinear systems studied to stabilize system (3.3)–(3.9) (namely systems (3.89)–(3.97) and (3.45)–(3.51)) are non causal, in the sense that their right-hand-sides depends on the full interval $(0, \infty)$. The interest of our result is to prove the existence of a stabilizing control. The limitation is that we do not characterize this control by a causal nonlinear system. This is a first step in studying such a nonlinear stabilization problem. We expect to be able to find this control in a feedback form in a future work (this is still under investigation).

Definitions and notations are given in section 3.2. In particular, we define the notion of *admissible* controls, and we precise the definitions of the exponential stabilization for systems (3.3)–(3.9) and (3.11)–(3.17). In section 3.3 we set the linearized problem. For that we first rewrite system (3.3)–(3.9) in a cylindrical domain, and we linearize the one obtained with respect to the unknowns and the data. The change of variable used to rewrite system (3.3)–(3.9) in a cylindrical domain is constructed in the Appendix A. Section 3.4 is devoted to the study of the linearized system. We rewrite the latter in a formal operator formulation where the pressure is eliminated, which enables us to deal with an analytic semigroup of contractions. In this operator formulation, a *mass-added* affect appears (see (3.59)). In section 3.5, we prove that system (3.11)–(3.17) is approximately controllable and exponentially stabilizable by boundary controls on $\partial\mathcal{S}$. Then in section 3.5.2 the boundary control is considered as a feedback operator, in order to shift the spectrum of the operator governing the linear system. After that we show in section 3.6 that this boundary feedback operator can define an *admissible* solid's deformation. This process allows us to stabilize the nonhomogeneous linear system in section 3.7. Section 3.8 is devoted to the stabilization of system (3.3)–(3.9).

3.2 Definitions and notation

We denote by $\mathcal{F} = \mathcal{F}(0)$ the domain occupied by the fluid at time $t = 0$, and by $\mathcal{S} = \mathcal{S}(0)$ the domain occupied by the solid at $t = 0$. We assume that $\mathcal{S}$ is regular enough. We set $\mathcal{S}^*(t) = X^*(\mathcal{S}, t)$, $\mathcal{F}^*(t) = \mathcal{O} \setminus \overline{\mathcal{S}^*(t)}$,

$$S^0_\infty = \mathcal{S} \times (0, \infty), \qquad Q^0_\infty = \mathcal{F} \times (0, \infty),$$

and

$$S^*_\infty = \bigcup_{t\geq 0} \mathcal{S}^*(t) \times \{t\}, \quad Q^*_\infty = \bigcup_{t\geq 0} \mathcal{F}^*(t) \times \{t\}, \quad Q_\infty = \bigcup_{t\geq 0} \mathcal{F}(t) \times \{t\}.$$

Let us introduce some functional spaces. We use the notation

$$\mathbf{H}^s(\Omega) = [\mathrm{H}^s(\Omega)]^d \text{ or } [\mathrm{H}^s(\Omega)]^{d^k}, \text{ for some integer } k, \text{ for all bounded domain } \Omega \text{ of } \mathbb{R}^d.$$

We classically define

$$\begin{array}{rcl} \mathbf{V}^0_n(\mathcal{F}) & = & \left\{\phi \in \mathbf{L}^2(\mathcal{F}) \mid \operatorname{div} \phi = 0 \text{ in } \mathcal{F},\ \phi \cdot n = 0 \text{ on } \partial\mathcal{O}\right\}, \\ \mathbf{V}^1_n(\mathcal{F}) & = & \left\{\phi \in \mathbf{H}^1(\mathcal{F}) \mid \operatorname{div} \phi = 0 \text{ in } \mathcal{F},\ \phi \cdot n = 0 \text{ on } \partial\mathcal{O}\right\}, \\ \mathrm{H}^{2,1}(Q^0_\infty) & = & \mathrm{L}^2(0, \infty; \mathbf{H}^2(\mathcal{F})) \cap \mathrm{H}^1(0, \infty; \mathbf{L}^2(\mathcal{F})), \end{array}$$

and also

$$\mathrm{H}^{2,1}(Q_\infty) \quad = \quad \mathrm{L}^2(0, \infty; \mathbf{H}^2(\mathcal{F}(t))) \cap \mathrm{H}^1(0, \infty; \mathbf{L}^2(\mathcal{F}(t)))$$

that we define as

$$f \in \mathrm{H}^{2,1}(Q_\infty) \quad \Leftrightarrow \quad \int_0^\infty \|f(\cdot, t)\|^2_{\mathbf{H}^2(\mathcal{F}(t))} \,\mathrm{d}t + \int_0^\infty \left\|\frac{\partial f}{\partial t}(\cdot, t)\right\|^2_{\mathbf{L}^2(\mathcal{F}(t))} \mathrm{d}t \ < \ +\infty.$$

Recall the continuous embedding for the cylindrical domain Q^0_∞:

$$\mathrm{H}^{2,1}(Q^0_\infty) \quad \hookrightarrow \quad \mathrm{L}^\infty(0, \infty; \mathbf{H}^1(\mathcal{F})).$$

We finally set - for the fluid - the spaces dealing with compatibility conditions

$$\begin{array}{rcl} \mathbf{H}^0_{cc} & = & \left\{(u_0, h_1, \omega_0) \in \mathbf{V}^0_n(\mathcal{F}) \times \mathbb{R}^d \times \mathbb{R}^3 \mid\ u_0 = h_1 + \omega_0 \wedge y \text{ on } \partial\mathcal{S}\right\}, \\ \mathbf{H}^1_{cc} & = & \left\{(u_0, h_1, \omega_0) \in \mathbf{V}^1_n(\mathcal{F}) \times \mathbb{R}^d \times \mathbb{R}^3 \mid\ u_0 = h_1 + \omega_0 \wedge y \text{ on } \partial\mathcal{S}\right\}. \end{array}$$

For more simplicity, we assume that the density $\rho_{\mathcal{S}}$ at time $t = 0$ is constant with respect to the space:

$$\rho_{\mathcal{S}}(y,0) = \rho_{\mathcal{S}} > 0.$$

We assume without loss of generality that $h_0 = 0$. This implies in particular

$$\int_{\mathcal{S}} y \mathrm{d}y \quad = \quad 0.$$

Let us state the conditions we shall assume on mappings chosen as controls.

3.2.1 Definitions of the changes of variables in the fluid part

Let us introduce some functional spaces for the changes of variables which will be defined in $\mathcal{F}$. The mapping which has the most important role (in the rewriting of system (3.3)–(3.9) in cylindrical domains, and in the final fixed point method) is denoted by $\tilde{X}$. Let us describe the classes of functions we will consider for these mappings and their displacements. First, for $\lambda > 0$ and for some integer $m \geq 3$, we define the set $\mathcal{W}_\lambda^m(Q_\infty^0)$ as

$$\mathcal{W}_\lambda^m(Q_\infty^0) = \left\{ \tilde{X} : Q_\infty^0 \rightarrow Q_\infty^* \mid e^{\lambda t}\frac{\partial \tilde{X}}{\partial t} \in \mathrm{L}^2(0,\infty;\mathbf{H}^m(\mathcal{F})) \cap \mathrm{H}^{m/2-1/2}(0,\infty;\mathbf{H}^1(\mathcal{F})) \right\}.$$

Remark 3.1. *The space* $\mathrm{L}^2(0,\infty;\mathbf{H}^m(\mathcal{F}))\cap\mathrm{H}^{m/2-1/2}(0,\infty;\mathbf{H}^1(\mathcal{F}))$ *can be obtained by embedding, as follows*

$$\mathrm{H}^{m,m/2}(Q_\infty^0) \quad \hookrightarrow \quad \mathrm{L}^2(0,\infty;\mathbf{H}^m(\mathcal{F})) \cap \mathrm{H}^{m/2-1/2}(0,\infty;\mathbf{H}^1(\mathcal{F})),$$

where

$$\mathrm{H}^{m,m/2}(Q_\infty^0) \quad \equiv \quad \mathrm{L}^2(0,\infty;\mathbf{H}^m(\mathcal{Q})) \cap \mathrm{H}^{m/2}(0,\infty;\mathbf{L}^2(\mathcal{Q})).$$

Besides, for a sake of simplicity, we will consider in the rest of this chapter that $m = 3$. Some results - in particular technical lemmas - are still true in the general case where $m > 2$ if $d = 2$ and $m > 5/2$ if $d = 3$. But the main results are obtained only for $m \geq 3$, if we follow the methods used for their proofs. Besides, the proofs of some technical Lemmas require sometimes to distinguish the cases $m \geq 3$ and $m \leq 3$.

In the following, we will mainly consider mappings $\tilde{X}$ satisfying $\tilde{X}(\cdot,0) = \mathrm{Id}_{\mathcal{F}}$, and thus we will consider the displacements

$$\tilde{Z} = \tilde{X} - \mathrm{Id}_{\mathcal{F}} \quad \in \quad \mathcal{W}_\lambda(Q_\infty^0).$$

where the space $\mathcal{W}_\lambda(Q_\infty^0)$ is defined as follows

$$\tilde{Z} \in \mathcal{W}_\lambda(Q_\infty^0) \Leftrightarrow \left\{ \begin{array}{l} \tilde{Z} \in \mathcal{W}_\lambda^3(Q_\infty^0), \\ \\ \tilde{Z}(y,0) = 0, \ \dfrac{\partial \tilde{Z}}{\partial t}(y,0) = 0 \quad \forall y \in \mathcal{F}. \end{array} \right.$$

We endow it with the scalar product

$$\langle \tilde{Z}_1 ; \tilde{Z}_2 \rangle_{\mathcal{W}_\lambda(Q_\infty^0)} \quad := \quad \left\langle e^{\lambda t}\frac{\partial \tilde{Z}_1}{\partial t} ; e^{\lambda t}\frac{\partial \tilde{Z}_2}{\partial t} \right\rangle_{\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{F}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{F}))}$$

which makes it a Hilbert space, because of the continuous embedding

$$\mathcal{W}_\lambda(Q^0_\infty) \quad \hookrightarrow \quad \mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\mathcal{F})) \cap \mathrm{W}^{1,\infty}(0,\infty;\mathbf{H}^1(\mathcal{F})). \tag{3.23}$$

Indeed, for $\tilde{X} - \mathrm{Id}_\mathcal{F} \in \mathcal{W}_\lambda(Q^0_\infty)$, we have the following estimates

$$\begin{aligned}
\|\tilde{X} - \mathrm{Id}_\mathcal{F}\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\mathcal{F}))} &\leq \int_0^\infty e^{-\lambda s}\left\|e^{\lambda s}\frac{\partial \tilde{X}}{\partial t}(\cdot,s)\right\|_{\mathbf{H}^3(\mathcal{F})} \mathrm{d}s,\\
&\leq \frac{1}{\sqrt{2\lambda}}\left\|e^{\lambda t}\frac{\partial \tilde{X}}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{F}))},\\
\left\|\frac{\partial \tilde{X}}{\partial t}\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^1(\mathcal{F}))} &\leq \int_0^\infty e^{-\lambda s}\left\|e^{\lambda s}\frac{\partial^2 \tilde{X}}{\partial t^2}(\cdot,s)\right\|_{\mathbf{H}^1(\mathcal{F})} \mathrm{d}s,\\
&\leq \frac{1+\lambda}{\sqrt{2\lambda}}\left\|e^{\lambda t}\frac{\partial \tilde{X}}{\partial t}\right\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{F}))}.
\end{aligned}$$

Thus for more clarity we set

$$\begin{aligned}
\mathcal{H}_3(Q^0_\infty) &= \mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{F})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{F})),\\
\tilde{\mathcal{W}}_3(Q^0_\infty) &= \mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\mathcal{F})) \cap \mathrm{W}^{1,\infty}(0,\infty;\mathbf{H}^1(\mathcal{F})).
\end{aligned}$$

The changes of variables $\tilde{X}$ will be - indirectly - obtained through extensions of solid's deformations X^*. In order to consider displacements $\tilde{X} - \mathrm{Id}_\mathcal{F}$ which lie in the spaces given above, we will consider solid's deformations which satisfy at least

$$e^{\lambda t}\frac{\partial X^*}{\partial t}_{|\partial\mathcal{S}} \quad \in \quad \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S})).$$

3.2.2 Definitions for the linear system (3.11)–(3.17)

The solid's displacements $Z^* = X^* - \mathrm{Id}_\mathcal{S}$ we will consider lie in the space $\mathcal{W}_\lambda(S^0_\infty)$, which can be defined analogously as $\mathcal{W}_\lambda(Q^0_\infty)$. We endow it with the scalar product

$$\langle Z_1^*; Z_2^*\rangle_{\mathcal{W}_\lambda(S^0_\infty)} \quad := \quad \left\langle e^{\lambda t}\frac{\partial Z_1^*}{\partial t}; e^{\lambda t}\frac{\partial Z_2^*}{\partial t}\right\rangle_{\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S}))},$$

Then we have the following continuous embedding (which could be obtained with the same kind of estimates as the one given for $\tilde{W}_3(Q^0_\infty)$ above)

$$\mathcal{W}_\lambda(S^0_\infty) \quad \hookrightarrow \quad \mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\mathcal{S})) \cap \mathrm{W}^{1,\infty}(0,\infty;\mathbf{H}^1(\partial\mathcal{S})).$$

Thus for more clarity we also set

$$\begin{aligned}
\mathcal{H}_3(S^0_\infty) &= \mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S})),\\
\tilde{\mathcal{W}}_3(S^0_\infty) &= \mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\mathcal{S})) \cap \mathrm{W}^{1,\infty}(0,\infty;\mathbf{H}^1(\mathcal{S})).
\end{aligned}$$

Note that in the linearized system (3.11)–(3.17) the solid's deformation appears only through the boundary velocity

$$\zeta \quad = \quad e^{\lambda t}\frac{\partial X^*}{\partial t}_{|\partial\mathcal{S}}.$$

The chosen control function for this linear homogeneous system is then a velocity ζ, which will satisfy

$$\zeta \quad \in \quad \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S})).$$

Definition 3.2. *A deformation $X^* \in \mathcal{W}_\lambda(S^0_\infty)$ is said admissible for the linear system* (3.11)–(3.17) *if it satisfies the following hypotheses*

$$\int_{\partial\mathcal{S}} \frac{\partial X^*}{\partial t}(y,t)\cdot n\mathrm{d}\Gamma(y) \quad = \quad 0, \tag{3.24}$$

$$\int_{\mathcal{S}} X^*(y,t)\mathrm{d}y \quad = \quad 0, \tag{3.25}$$

$$\int_{\mathcal{S}} y\wedge \frac{\partial X^*}{\partial t}(y,t)\mathrm{d}y \quad = \quad 0 \tag{3.26}$$

for all $t\geq 0$.

Definition 3.3. *We say that the linear system* (3.11)–(3.17) *is stabilizable with an arbitrary exponential decay rate if for all $\lambda>0$ there exists a velocity of deformation $\zeta\in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))$ and a positive constant C - depending only on u_0, h_1 and ω_0 - such that the solution $(\hat{u},\hat{p},\hat{h}',\hat{\omega})$ of system* (3.11)–(3.17) *satisfies for all $t\geq 0$*

$$\|(\hat{u}(\cdot,t),\hat{h}'(t),\hat{\omega}(t))\|_{\mathbf{H}^0_{cc}} \quad \leq \quad C.$$

3.2.3 Definitions for the nonlinear system (3.3)–(3.9)

Definition 3.4. *A deformation $X^* \in \mathcal{W}_\lambda(S^0_\infty)$ is said admissible for the nonlinear system* (3.3)–(3.9) *if $X^*(\cdot,t)$ if $X^*(\cdot,t)_{|\partial\mathcal{S}}$ is a C^1-diffeomorphism from $\mathcal{S}$ onto $\mathcal{S}^*(t)$ for all $t\geq 0$, and if it satisfies the following hypotheses*

$$\int_{\partial\mathcal{S}} \frac{\partial X^*}{\partial t}(y,t)\cdot \left(\mathrm{cof}\nabla X^*(y,t)\right) n\mathrm{d}\Gamma(y) = 0, \tag{3.27}$$

$$\int_{\mathcal{S}^*(t)} X^*(y,t)\mathrm{d}y = 0, \tag{3.28}$$

$$\int_{\mathcal{S}^*(t)} X^*(y,t)\wedge \frac{\partial X^*}{\partial t}(y,t)\mathrm{d}y = 0 \tag{3.29}$$

for all $t\geq 0$.

Remark 3.5. *First, the constraint which forces X^* to be a C^1-diffeomorphism can be relaxed if X^* stays close enough to the identity $\mathrm{Id}_{\mathcal{S}}$. Since in this work we consider small data, we are lead to consider small solid's deformations, and assuming the smallness of $X^*-\mathrm{Id}_{\mathcal{S}}$ in $\mathcal{W}_\lambda(S^0_\infty)$, and thus in particular in $\mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\mathcal{S}))$, enables us to assume that this constraint is always satisfied.*
The constraint (3.27) *satisfied by an admissible control X^* - in the sense of the previous definition - can be expressed as*

$$\int_{\partial\mathcal{S}^*(t)} w^*(x^*,t)\cdot n\mathrm{d}\Gamma(x^*) = 0,$$

for all $t \geq 0$. The quantity $\mathrm{cof}\nabla X^*$ *denotes the cofactor matrix of* ∇X^**. We have the classical relation*

$$\mathrm{cof}\nabla X^*(y,t) \quad = \quad (\det\nabla X^*(y,t))\, \nabla Y^*(X^*(y,t),t)^T,$$

if $Y^*(\cdot,t)$ *denotes the inverse of* $X^*(\cdot,t)$*. Assuming that the equation* (3.27) *- where this quantity appears - is satisfied is strictly equivalent to assuming that the solid's volume is constant through the time. Indeed, we can show that*

$$\mathrm{div}\ w^*(x^*,t) = \left.\frac{\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\det\nabla X^*(y,t)\right)}{\det\nabla X^*(y,t)}\right|_{y=Y^*(x^*,t)}$$

and thus that the equation (3.27) *can be rewritten, for all $t \geq 0$*

$$|\mathcal{S}(t)| = \int_{\mathcal{S}} \det\nabla X^*(y,t)\mathrm{d}y = |\mathcal{S}|\,.$$

Constraints (3.27) *and* (3.29) *are nonlinear with respect to the mapping* X^**. The corresponding linearized constraints are considered in the definition 3.2.*

Definition 3.6. *We say that system* (3.3)–(3.9) *is stabilizable with an arbitrary exponential decay rate if for all $\lambda > 0$ there exists an admissible deformation X^* (in the sense of Definition 3.4) and a positive constant C - depending only on u_0, h_1 and ω_0 - such that the solution (u,p,h',ω) of system* (3.3)–(3.9) *satisfies for all $t \geq 0$*

$$\|(u(\cdot,t),h'(t),\omega(t))\|_{\mathbf{H}^1(\mathcal{F}(t))\times\mathbb{R}^d\times\mathbb{R}^3} \quad \leq \quad C\mathrm{e}^{-\lambda t}.$$

3.3 The linearized system

3.3.1 The change of variables

In order to make a change of unknowns which enables to rewrite the main system in cylindrical domains, we extend to the whole domain $\overline{\mathcal{O}}$ the mappings $X_{\mathcal{S}}(\cdot,t)$ and $Y_{\mathcal{S}}(\cdot,t)$, initially defined respectively on $\mathcal{S}$ and $\mathcal{S}(t)$.
For a vector $h \in \mathrm{H}^2(0,\infty;\mathbb{R}^d)$ and a rotation $\mathbf{R} \in \mathrm{H}^2(0,\infty;\mathbb{R}^9)$ which provides an angular velocity $\omega \in \mathrm{H}^1(0,\infty;\mathbb{R}^3)$, and for an *admissible* deformation X^* - in the sense of Definition 3.4 - we construct a mapping X such that

$$\begin{cases} \det\nabla X = 1, & \text{in } \mathcal{F}\times(0,\infty), \\ X = h + \mathbf{R}X^*, & \text{on } \partial\mathcal{S}\times(0,\infty), \\ X = \mathrm{Id}_{\partial\mathcal{O}}, & \text{on } \partial\mathcal{O}\times(0,\infty), \end{cases}$$

and such that for all $t \geq 0$ the function $X(\cdot,t)$ map $\mathcal{F}$ onto $\mathcal{F}(t)$, $\partial\mathcal{S}$ onto $\partial\mathcal{S}(t)$, and let invariant the boundary $\partial\mathcal{O}$. The details of this construction are given in Appendix A, in the same time as the regularity deduced on X.

3.3.2 Rewriting system (3.3)–(3.9) in a cylindrical domain

Let us transform system (3.3)–(3.9) into a system which deals with non-depending time domains. For that we make the change of unknowns

$$\begin{array}{ll} \tilde{u}(y,t) = \mathbf{R}(t)^T u(X(y,t),t), & u(x,t) = \mathbf{R}(t)\tilde{u}(Y(x,t),t), \\ \tilde{p}(y,t) = p(X(y,t),t), & p(x,t) = \tilde{p}(Y(x,t),t), \end{array} \tag{3.30}$$

for $x \in \overline{\mathcal{F}(t)}$ and $y \in \overline{\mathcal{F}}$, and

$$\tilde{h}'(t) = \mathbf{R}(t)^T h'(t), \qquad \tilde{\omega}(t) = \mathbf{R}(t)^T \omega(t). \tag{3.31}$$

Remark 3.7. *In $\mathbb{R}^3$, let us notice that if $\tilde{h}'$ and $\tilde{\omega}$ are given, then by using the second equality of* (3.31) *we see that $\mathbf{R}$ satisfies the Cauchy problem*

$$\begin{array}{rcccl} \dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}) & = & \mathbb{S}\left(\mathbf{R}\tilde{\omega}\right)\mathbf{R} & = & \mathbf{R}\mathbb{S}\left(\tilde{\omega}\right) \\ \mathbf{R}(t=0) & = & \mathrm{I}_{\mathbb{R}^3}, & & \end{array} \qquad \text{with } \mathbb{S}(\tilde{\omega}) = \begin{pmatrix} 0 & -\tilde{\omega}_3 & \tilde{\omega}_2 \\ \tilde{\omega}_3 & 0 & -\tilde{\omega}_1 \\ -\tilde{\omega}_2 & \tilde{\omega}_1 & 0 \end{pmatrix}.$$

So $\mathbf{R}$ is determined in a unique way. Thus it is obvious to see that, in (3.31)*, h' and ω are also determined in a unique way. Moreover, since we have*

$$u(x,t) = \mathbf{R}(t)\tilde{u}(Y(x,t),t), \qquad p(x,t) = \tilde{p}(Y(x,t),t),$$

and since the mapping Y depends only on h, ω and the control X^*, we finally see that if $(\tilde{u}, \tilde{p}, \tilde{h}', \tilde{\omega})$ is given, then (u, p, h', ω) is determined in a unique way.

For a sake of clarity, let us also define the mappings

$$\tilde{X}(y,t) = \mathbf{R}(t)^T (X(y,t) - h(t)), \qquad \tilde{Y}(\tilde{x},t) = Y(h(t) + \mathbf{R}(t)\tilde{x}, t).$$

Using the change of unknowns given above by (3.30) and (3.31), system (3.3)–(3.9) is rewritten in the cylindrical domain $\mathcal{F} \times (0,\infty)$ as follows

$$\frac{\partial \tilde{u}}{\partial t} - \nu \mathbf{L}\tilde{u} + \mathbf{M}(\tilde{u}, \tilde{h}', \tilde{\omega}) + \mathbf{N}\tilde{u} + \tilde{\omega}(t) \wedge \tilde{u} + \mathbf{G}\tilde{p} = 0, \qquad y \in \mathcal{F}, \quad t \in (0,\infty), \tag{3.32}$$

$$\operatorname{div} \tilde{u} = g, \qquad y \in \mathcal{F}, \quad t \in (0,\infty), \tag{3.33}$$

$$\tilde{u} = 0, \qquad y \in \partial\mathcal{O}, \quad t \in (0,\infty), \tag{3.34}$$

$$\tilde{u} = \tilde{h}'(t) + \tilde{\omega}(t) \wedge X^*(y,t) + \frac{\partial X^*}{\partial t}(y,t), \qquad y \in \partial\mathcal{S}, \quad t \in (0,\infty), \tag{3.35}$$

$$M\tilde{h}''(t) = -\int_{\partial\mathcal{S}} \tilde{\sigma}(\tilde{u},\tilde{p}) \nabla\tilde{Y}(\tilde{X})^T n \mathrm{d}\Gamma - M\tilde{\omega}(t) \wedge \tilde{h}'(t), \tag{3.36}$$

$$I^*(t)\tilde{\omega}'(t) = -\int_{\partial\mathcal{S}} X^*(y,t) \wedge \left(\tilde{\sigma}(\tilde{u},\tilde{p})\nabla\tilde{Y}(\tilde{X})^T n\right) \mathrm{d}\Gamma - {I^*}'(t)\tilde{\omega}(t) + I^*(t)\tilde{\omega}(t) \wedge \tilde{\omega}(t), \tag{3.37}$$

$$\tilde{u}(y,0) = u_0(y), \ y \in \mathcal{F}, \quad \tilde{h}'(0) = h_1 \in \mathbb{R}^d, \quad \tilde{\omega}(0) = \omega_0 \in \mathbb{R}^3, \tag{3.38}$$

where $[\cdot]_i$ specifies the i-th component of a vector

$$[\mathbf{L}\tilde{u}]_i(y,t) = [\nabla\tilde{u}(y,t)\Delta\tilde{Y}(\tilde{X}(y,t),t)]_i + \nabla^2\tilde{u}_i(y,t) : \left(\nabla\tilde{Y}\nabla\tilde{Y}^T\right)(\tilde{X}(y,t),t), \tag{3.39}$$

$$\mathbf{M}(\tilde{u},\tilde{h}',\tilde{\omega})(y,t) = -\nabla\tilde{u}(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t)\left(\tilde{h}'(t) + \tilde{\omega}(t) \wedge \tilde{X}(y,t) + \frac{\partial\tilde{X}}{\partial t}(y,t)\right), \tag{3.40}$$

$$\mathbf{N}\tilde{u}(y,t) = \nabla\tilde{u}(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t)\tilde{u}(y,t), \tag{3.41}$$

$$\mathbf{G}\tilde{p}(y,t) = \nabla\tilde{Y}(\tilde{X}(y,t),t)^T \nabla\tilde{p}(y,t), \tag{3.42}$$

$$\tilde{\sigma}(\tilde{u},\tilde{p})(y,t) = \nu\left(\nabla\tilde{u}(y,t)\nabla\tilde{Y}(\tilde{X}(y,t),t) + \nabla\tilde{Y}(\tilde{X}(y,t),t)^T\nabla\tilde{u}(y,t)^T\right) - \tilde{p}(y,t)\mathrm{I}_{\mathbb{R}^d}$$

and

$$\begin{aligned} g(y,t) &= \operatorname{trace}\left(\nabla\tilde{u}(y,t)\left(\mathrm{I}_{\mathbb{R}^d} - \nabla\tilde{Y}\left(\tilde{X}(y,t),t\right)\right)\right) \\ &= \nabla\tilde{u}(y,t) : \left(\mathrm{I}_{\mathbb{R}^d} - \nabla\tilde{Y}\left(\tilde{X}(y,t),t\right)^T\right). \end{aligned} \tag{3.43}$$

Notice that from the Piola identity we can actually express this nonhomogeneous divergence term as $g = \operatorname{div} G$, where

$$G(y,t) = \left(\mathrm{I}_{\mathbb{R}^d} - \nabla\tilde{Y}(\tilde{X}(y,t),t)\right)\tilde{u}(y,t).$$

For $\lambda > 0$, we make the following change of unknowns:

$$\hat{u} = e^{\lambda t}\tilde{u}, \quad \hat{p} = e^{\lambda t}\tilde{p}, \quad \hat{h}' = e^{\lambda t}\tilde{h}', \quad \hat{\omega} = e^{\lambda t}\tilde{\omega}, \quad \zeta = e^{\lambda t}\frac{\partial X^*}{\partial t}. \tag{3.44}$$

The system (3.32)–(3.38) is transformed into

$$\begin{aligned} \frac{\partial\hat{u}}{\partial t} - \lambda\hat{u} - \nu\mathbf{L}\hat{u} + e^{-\lambda t}\mathbf{M}(\tilde{u},\tilde{h}',\tilde{\omega}) + e^{-\lambda t}\mathbf{N}\hat{u} + \mathbf{G}\hat{p} = \hat{F}, &\qquad y\in\mathcal{F}, \quad t\in(0,\infty), \\ \operatorname{div}\hat{u} = \operatorname{div} G, &\qquad y\in\mathcal{F}, \quad t\in(0,\infty), \end{aligned}$$

$$\begin{aligned} \hat{u} &= 0, & y\in\partial\mathcal{O}, \quad t\in(0,\infty), \\ \hat{u} &= \hat{h}'(t) + \hat{\omega}(t)\wedge X^*(y,t) + \zeta(y,t), & y\in\partial\mathcal{S}, \quad t\in(0,\infty), \end{aligned}$$

$$\begin{aligned} M\hat{h}''(t) - \lambda M\hat{h}' &= -\int_{\partial\mathcal{S}} \tilde{\sigma}(\hat{u},\hat{p})\nabla\tilde{Y}(\tilde{X})^T n \,\mathrm{d}\Gamma + \hat{F}_M, & t\in(0,\infty) \\ I^*(t)\hat{\omega}'(t) - \lambda I^*(t)\hat{\omega}(t) &= -\int_{\partial\mathcal{S}} X^*\wedge\left(\tilde{\sigma}(\hat{u},\hat{p})\nabla\tilde{Y}(\tilde{X})^T n\right)\mathrm{d}\Gamma + \hat{F}_I, & t\in(0,\infty) \end{aligned}$$

$$\hat{u}(y,0) = u_0(y), \ y\in\mathcal{F}, \quad \hat{h}'(0) = h_1\in\mathbb{R}^d, \quad \hat{\omega}(0) = \omega_0\in\mathbb{R}^3,$$

with

$$\hat{F} = -e^{-\lambda t}\hat{\omega}\wedge\hat{u}, \quad \hat{F}_M = -Me^{-\lambda t}\hat{\omega}\wedge\hat{h}', \quad \hat{F}_I = -I^{*\prime}\hat{\omega} + e^{-\lambda t}I^*\hat{\omega}\wedge\hat{\omega}.$$

We rewrite this system as follows

$$\frac{\partial\hat{u}}{\partial t} - \lambda\hat{u} - \nu\Delta\hat{u} + \nabla\hat{p} = F(\hat{u},\hat{p},\hat{h}',\hat{\omega}), \qquad \text{in } \mathcal{F}\times(0,\infty), \tag{3.45}$$

$$\operatorname{div}\hat{u} = \operatorname{div} G(\hat{u}), \qquad \text{in } \mathcal{F}\times(0,\infty), \tag{3.46}$$

$$\hat{u} = 0, \qquad \text{on } \partial\mathcal{O}\times(0,\infty), \tag{3.47}$$

$$\hat{u} = \hat{h}'(t) + \hat{\omega}(t)\wedge y + \zeta + W(\hat{\omega}), \qquad (y,t)\in\partial\mathcal{S}\times(0,\infty), \tag{3.48}$$

$$M\hat{h}'' - \lambda M\hat{h}' = -\int_{\partial\mathcal{S}}\sigma(\hat{u},\hat{p})n\mathrm{d}\Gamma + F_M(\hat{u},\hat{p},\hat{h}',\hat{\omega}), \qquad \text{in } (0,\infty), \tag{3.49}$$

$$I_0\hat{\omega}'(t) - \lambda I_0\hat{\omega} = -\int_{\partial\mathcal{S}} y\wedge\sigma(\hat{u},\hat{p})n\mathrm{d}\Gamma + F_I(\hat{u},\hat{p},\hat{\omega}), \qquad \text{in } (0,\infty), \tag{3.50}$$

$$\hat{u}(y,0) = u_0(y),\ y \in \mathcal{F}, \quad \hat{h}'(0) = h_1 \in \mathbb{R}^d, \quad \hat{\omega}(0) = \omega_0 \in \mathbb{R}^3, \tag{3.51}$$

with

$$\begin{aligned}
F(\hat{u},\hat{p},\hat{h}',\hat{\omega}) &= \nu(\mathbf{L}-\Delta)\hat{u} - e^{-\lambda t}\mathbf{M}(\hat{u},\hat{h}',\hat{\omega}) - e^{-\lambda t}\mathbf{N}\hat{u} - (\mathbf{G}-\nabla)\hat{p} - e^{-\lambda t}\hat{\omega}\wedge\hat{u},\\
G(\hat{u}) &= \left(\mathrm{I}_{\mathbb{R}^d} - \nabla\tilde{Y}(\tilde{X}(y,t),t)\right)\hat{u},\\
W(\hat{\omega}) &= \hat{\omega}\wedge(X^* - \mathrm{Id}),\\
F_M(\hat{u},\hat{p},\hat{h}',\hat{\omega}) &= -Me^{-\lambda t}\hat{\omega}\wedge\hat{h}'(t)\\
&\quad -\nu\int_{\partial\mathcal{S}}\left(\nabla\hat{u}\left(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d}\right) + \left(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d}\right)^T\nabla\hat{u}^T\right)\nabla\tilde{Y}(\tilde{X})^T n\mathrm{d}\Gamma\\
&\quad -\int_{\partial\mathcal{S}}\sigma(\hat{u},\hat{p})\left(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d}\right)^T n\mathrm{d}\Gamma,\\
F_I(\hat{u},\hat{p},\hat{\omega}) &= -(I^*-I_0)\hat{\omega}' + \lambda(I^*-I_0)\hat{\omega} - {I^*}'\hat{\omega} + e^{-\lambda t}I^*\hat{\omega}\wedge\hat{\omega}\\
&\quad -\nu\int_{\partial\mathcal{S}} y\wedge\left(\nabla\hat{u}\left(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d}\right) + (\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d})^T\nabla\hat{u}^T\right)\nabla\tilde{Y}(\tilde{X})^T n\mathrm{d}\Gamma\\
&\quad -\int_{\partial\mathcal{S}} y\wedge\left(\sigma(\hat{u},\hat{p})(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d})^T n\right)\mathrm{d}\Gamma\\
&\quad +\int_{\partial\mathcal{S}}(X^*-\mathrm{Id})\wedge\left(\overline{\sigma}^*(\hat{u},\hat{p})\nabla\tilde{Y}(\tilde{X})^T n\right)\mathrm{d}\Gamma.
\end{aligned}$$

3.3.3 Linearization

Statement

We linearize system (3.45)–(3.51) around $(\hat{u},\hat{p},\hat{h}',\hat{\omega},X^*) = (0,0,0,0,\mathrm{Id})$, and we obtain formally

$$\frac{\partial U}{\partial t} - \lambda U - \nu\Delta U + \nabla P = 0, \quad \text{in } \mathcal{F}\times(0,\infty), \tag{3.52}$$

$$\operatorname{div} U = 0, \quad \text{in } \mathcal{F}\times(0,\infty), \tag{3.53}$$

$$U = 0, \qquad \text{in } \partial\mathcal{O}\times(0,\infty), \tag{3.54}$$

$$U = H'(t) + \Omega(t)\wedge y + \zeta(y,t), \qquad y\in\partial\mathcal{S}, \quad t\in(0,\infty), \tag{3.55}$$

$$MH''(t) - \lambda MH'(t) = -\int_{\partial\mathcal{S}}\sigma(U,P)n\mathrm{d}\Gamma, \quad t\in(0,\infty), \tag{3.56}$$

$$I_0\Omega'(t) - \lambda I_0\Omega(t) = -\int_{\partial\mathcal{S}} y\wedge\sigma(U,P)n\mathrm{d}\Gamma, \quad t\in(0,\infty), \tag{3.57}$$

$$U(y,0) = u_0(y),\ y\in\mathcal{F}, \quad H'(0) = h_1\in\mathbb{R}^d, \quad \Omega(0) = \omega_0\in\mathbb{R}^3, \tag{3.58}$$

with

$$\sigma(U,P)(y,t) = 2\nu D(U)(y,t) - P(y,t)\mathrm{Id}.$$

Note that in this linear system the control (initially chosen as the solid's deformation) appears only on the boundary $\partial\mathcal{S}$ through the function $\zeta = e^{\lambda t}\dfrac{\partial X^*}{\partial t}$. Thus we reduce the problem of stabilizability of this linear system to a problem of boundary stabilization.

Constraints adapted to the linear system

Let us observe that constraints (3.18) and (3.20) are nonlinear, with respect to the mapping X^*. These constraints are written

$$\int_{\partial\mathcal{S}} \frac{\partial X^*}{\partial t} \cdot (\mathrm{cof}\nabla X^*(y,t)n)\,\mathrm{d}\Gamma = 0, \qquad \int_{\mathcal{S}} X^* \wedge \frac{\partial X^*}{\partial t}\mathrm{d}y = 0.$$

The linearization of this equalities leads us to consider the following constraints

$$\int_{\partial\mathcal{S}} \frac{\partial X^*}{\partial t} \cdot n\mathrm{d}\Gamma = 0, \qquad \int_{\mathcal{S}} y \wedge \frac{\partial X^*}{\partial t}\mathrm{d}y = 0.$$

Constraint (3.19) is already linear. This leads us to consider constraints stated in Definition 3.2 for controlling the linear system (3.11)–(3.17).

3.4 Definition of an analytic semigroup

In this section we study the following linear system, which is nothing else than the system (3.11)–(3.17) introduced in the introduction of this chapter:

$$\begin{aligned} \frac{\partial u}{\partial t} - \lambda u - \mathrm{div}\ \sigma(u,p) &= 0, \quad \text{in } \mathcal{F}\times(0,\infty),\\ \mathrm{div}\ u &= 0, \quad \text{in } \mathcal{F}\times(0,\infty), \end{aligned}$$

$$\begin{aligned} u = 0, &\qquad \text{in } \partial\mathcal{O}\times(0,\infty),\\ u = h'(t) + \omega(t)\wedge y + \zeta(y,t), &\qquad y\in\partial\mathcal{S}, \quad t\in(0,\infty), \end{aligned}$$

$$\begin{aligned} Mh''(t) - \lambda M h'(t) &= -\int_{\partial\mathcal{S}} \sigma(u,p)n\mathrm{d}\Gamma, \quad t\in(0,\infty),\\ I_0\omega'(t) - \lambda I_0\omega(t) &= -\int_{\partial\mathcal{S}} y\wedge\sigma(u,p)n\mathrm{d}\Gamma, \quad t\in(0,\infty), \end{aligned}$$

$$u(y,0) = u_0(y),\ y\in\mathcal{F}, \quad h'(0) = h_1 \in \mathbb{R}^d, \quad \omega(0) = \omega_0 \in \mathbb{R}^3.$$

3.4.1 Introduction of some operators

Let us introduce some operators. Let us first recall the stress tensor

$$D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right).$$

and let us denote the Hessian matrix operator as

$$H = \nabla^2.$$

For $h'\in\mathbb{R}^d$ and $\omega\in\mathbb{R}^3$, we define $N(h')$ and $\hat{N}(\omega)$ as being the respective solutions q and $\hat{q}$ of the Neumann problems

$$\Delta q = 0 \text{ in } \mathcal{F}, \quad \frac{\partial q}{\partial n} = h'\cdot n \text{ on } \partial\mathcal{S}, \quad \frac{\partial q}{\partial n} = 0 \text{ on } \partial\mathcal{O},$$
$$\Delta \hat{q} = 0 \text{ in } \mathcal{F}, \quad \frac{\partial \hat{q}}{\partial n} = (\omega\wedge y)\cdot n \text{ on } \partial\mathcal{S}, \quad \frac{\partial \hat{q}}{\partial n} = 0 \text{ on } \partial\mathcal{O}.$$

For $\varphi \in \mathbf{H}^{1/2}(\partial\mathcal{S})$, we define $L_0\varphi = \mathrm{w}$ as being the solution of the Stokes problem

$$\begin{aligned} -\nu\Delta\mathrm{w} + \nabla\psi &= 0 && \text{in } \mathcal{F},\\ \operatorname{div}\,\mathrm{w} &= 0 && \text{in } \mathcal{F},\\ \mathrm{w} &= \varphi && \text{on } \partial\mathcal{S},\\ \mathrm{w} &= 0 && \text{on } \partial\mathcal{O}. \end{aligned}$$

Similarly we define $\hat{L}_0\varphi = \hat{\mathrm{w}}$ as being the solution of

$$\begin{aligned} -\nu\Delta\hat{\mathrm{w}} + \nabla\hat{\psi} &= 0 && \text{in } \mathcal{F},\\ \operatorname{div}\,\hat{\mathrm{w}} &= 0 && \text{in } \mathcal{F},\\ \hat{\mathrm{w}} &= \varphi \wedge y && \text{on } \partial\mathcal{S},\\ \hat{\mathrm{w}} &= 0 && \text{on } \partial\mathcal{O}. \end{aligned}$$

We also define the following circulations

$$\mathcal{C}\varphi = \int_{\partial\mathcal{S}} \varphi n \mathrm{d}\Gamma, \qquad \hat{\mathcal{C}}\varphi = \int_{\partial\mathcal{S}} y \wedge \varphi n \mathrm{d}\Gamma.$$

Let us denote by $\mathbb{P} : \mathbf{L}^2(\mathcal{F}) \mapsto \mathbf{V}^0_n(\mathcal{F})$ the so-called Leray or Helmholtz operator, which is the orthogonal projection induced by the decomposition

$$\mathbf{L}^2(\mathcal{F}) = \mathbf{V}^0_n(\mathcal{F}) \oplus \nabla\mathbf{H}^1(\mathcal{F}).$$

Then we define in $\mathbf{V}^0_n$ the Stokes operator

$$A_0 \quad = \quad \nu\mathbb{P}\Delta,$$

with domain $D(A_0) = \mathbf{H}^2(\mathcal{F}) \cap \mathbf{H}^1_0(\mathcal{F}) \cap \mathbf{V}^0_n(\mathcal{F})$.

3.4.2 Operator formulation

Let us first consider system (3.11)–(3.17) only when $\zeta = 0$. (v, p, h', ω) are the unknowns, but this system can be transformed into a system whose unknowns are only (v, h', ω). Indeed, by following the method which is used in [Ray07] for instance, the pressure p can be eliminated in equations (3.11), (3.15) and (3.16). By this mean we obtain that p can be written

$$p \quad = \quad \pi - \frac{\partial q}{\partial t},$$

where π is solution of the following Neumann problem

$$\Delta\pi(t) = 0 \text{ in } \mathcal{F}, \qquad \frac{\partial\pi(t)}{\partial n} = \nu\mathbb{P}\Delta v(t)\cdot n \text{ on } \partial\mathcal{F},$$

and q is solution of this other Neumann problem which involves the boundary conditions

$$\Delta q(t) = 0 \text{ in } \mathcal{F}, \quad \frac{\partial q(t)}{\partial n} = (h'(t) + \omega(t)\wedge y)\cdot n \text{ on } \partial\mathcal{S}, \quad \frac{\partial q(t)}{\partial n} = 0 \text{ on } \partial\mathcal{O}.$$

Moreover, ∇p can be expressed through a lifting. More precisely, we have

$$\nabla p \quad = \quad (-A_0)\mathbb{P}L_0(h' + \omega\wedge y) \quad \text{in } \mathcal{F}.$$

Thus, we can split system (3.11)–(3.17) into two systems, one satisfied by $(\mathbb{P}v, h', \omega)$, and the other one by $(\mathrm{Id} - \mathbb{P})v$. Explicitly, by denoting $V = (\mathbb{P}v, h', \omega)^T$, system (3.11)–(3.17) can be rewritten (for $\zeta = 0$) as follows

$$\begin{aligned} (\mathbb{M}_0 + \mathbb{M}_{add})\, V' &= \mathbb{A}V + \lambda \mathbb{M}_0 V, && (3.59)\\ (\mathrm{Id} - \mathbb{P})\, v &= (\mathrm{Id} - \mathbb{P}) \Big(L_0(h') + \hat{L}_0(\omega) \Big), && (3.60) \end{aligned}$$

with

$$\mathbb{M}_0 = \begin{bmatrix} \mathrm{Id} & 0 & 0 \\ 0 & M\mathrm{Id} & 0 \\ 0 & 0 & I_0 \end{bmatrix}, \qquad \mathbb{M}_{add} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathcal{C}N & \mathcal{C}\hat{N} \\ 0 & \hat{\mathcal{C}}N & \hat{\mathcal{C}}\hat{N} \end{bmatrix},$$

and

$$\mathbb{A} = \begin{bmatrix} A_0 & (-A_0)\mathbb{P}L_0 & (-A_0)\mathbb{P}\hat{L}_0 \\ \mathcal{C}(-2\nu D + NA_0) & -\mathcal{C}HN & -\mathcal{C}H\hat{N} \\ \hat{\mathcal{C}}(-2\nu D + NA_0) & -\hat{\mathcal{C}}HN & -\hat{\mathcal{C}}H\hat{N} \end{bmatrix}.$$

3.4.3 Main properties of the operator $\mathbb{A}$

We set

$$\mathbb{M} = \mathbb{M}_0 + \mathbb{M}_{add}.$$

Lemma 3.8. *$\mathbb{M}$ is self-adjoint and positive.*

Proof. Observe that $\mathbb{M}_0$ is self-adjoint and positive. Then it is sufficient to show that $\mathbb{M}_{add}$ is self-adjoint and non-negative. Let us begin with noticing that $\mathcal{C}N$ is self-adjoint. Indeed, if q_1 and q_2 denote respectively $N(h_1')$ and $N(h_2')$, by using twice the Green formula we get

$$\begin{aligned} \textstyle\int_{\mathcal{F}} \nabla q_1 \cdot \nabla q_2 \mathrm{d}y &= \int_{\partial\mathcal{S}} (h_1' \cdot n) q_2 \mathrm{d}\Gamma &&= \int_{\partial\mathcal{S}} (h_2' \cdot n) q_1 \mathrm{d}\Gamma \\ &= h_1' \cdot \mathcal{C}N(h_2') &&= h_2' \cdot \mathcal{C}N(h_1'). \end{aligned}$$

We can also see that $\hat{\mathcal{C}}\hat{N}$ is self-adjoint. If $\hat{q}_1$ and $\hat{q}_2$ denote respectively $\hat{N}(\omega_1)$ and $\hat{N}(\omega_2)$, by using twice the Green formula we get

$$\begin{aligned} \textstyle\int_{\mathcal{F}} \nabla \hat{q}_1 \cdot \nabla \hat{q}_2 \mathrm{d}y &= \int_{\partial\mathcal{S}} \omega_1 \cdot (y \wedge n) \hat{q}_2 \mathrm{d}\Gamma &&= \int_{\partial\mathcal{S}} \omega_2 \cdot (y \wedge n) \hat{q}_1 \mathrm{d}\Gamma \\ &= \omega_1 \cdot \hat{\mathcal{C}}\hat{N}(\omega_2) &&= \omega_2 \cdot \hat{\mathcal{C}}\hat{N}(\omega_1). \end{aligned}$$

Likewise, let us show that $(\hat{\mathcal{C}}N)^T = \mathcal{C}\hat{N}$. First, we denote $\hat{q} = \hat{N}(\omega)$, and by using twice the Greene formula, we have

$$\begin{aligned} \mathcal{C}\hat{N}(\omega) &= \int_{\mathcal{F}} \nabla \hat{q} \mathrm{d}y &&= \int_{\partial\mathcal{S}} \frac{\partial \hat{q}}{\partial n} y \mathrm{d}\Gamma \\ &= \int_{\partial\mathcal{S}} y \otimes y (n \wedge \omega) \mathrm{d}\Gamma &&= \left(\int_{\partial\mathcal{S}} y \otimes (y \wedge n) \mathrm{d}\Gamma \right) \omega. \end{aligned}$$

On the other hand, if we denote $q = \hat{\mathcal{C}}N(h')$, let us notice that $q = h' \cdot y$. And then

$$\hat{\mathcal{C}}N(h') = \int_{\partial\mathcal{S}} (y \cdot h')(y \wedge n) \mathrm{d}\Gamma = \left(\int_{\partial\mathcal{S}} (y \wedge n) \otimes y \mathrm{d}\Gamma \right) h'.$$

Then we can conclude that $\mathbb{M}_{add}$ is self-adjoint. In order to prove that $\mathbb{M}_{add}$ is non-negative, let us see that the corresponding quadratic term can be written

$$V^T \mathbb{M}_{add} V = \int_{\mathcal{F}} |\nabla q + \nabla \hat{q}|^2 \, \mathrm{d}y \geq 0$$

for $V = (\mathbb{P}v, h', \omega)^T$, $q = N(h')$ and $\hat{q} = \hat{N}(\omega)$. □

Remark 3.9. *Notice that in the case where $\mathcal{O} = \mathbb{R}^d$, the solution q of the following Neumann problem*

$$\Delta q = 0 \text{ in } \mathcal{F}, \quad \frac{\partial q}{\partial n} = h' \cdot n \text{ on } \partial\mathcal{S}$$

is in particular such that $\nabla q = h'$, and thus from the divergence formula

$$\begin{aligned} \mathcal{C}N(h') &= \int_{\partial\mathcal{S}} q n \mathrm{d}\Gamma = \int_{\mathcal{F}} \nabla q \mathrm{d}y = \int_{\mathcal{F}} h' \mathrm{d}y \\ &= h' \int_{\mathcal{F}} \rho_{\mathcal{F}} \mathrm{d}y = M_{\mathcal{F}} h'. \end{aligned}$$

Similarly, for $\hat{\mathcal{C}}\hat{N}$ the inertia moment $I_{\mathcal{F}}$ of $\mathcal{F}$ appears, as follows

$$\hat{\mathcal{C}}\hat{N} = I_{\mathcal{F}} \omega.$$

Thus, in exterior domain, $\mathbb{M}_{add}$ corresponds to the inertia elements (mass, inertia moment) of the fluid.

In the following, we will denote

$$\mathcal{A} = (\mathbb{M}_0 + \mathbb{M}_{add})^{-1} \mathbb{A}.$$

Proposition 3.10.

$$D(\mathcal{A}) = \mathbf{H}^1_{cc} \cap \left(\mathbf{H}^2(\mathcal{F}) \times \mathbb{R}^d \times \mathbb{R}^3\right),$$

and $\mathbb{A} = \mathbb{M}\mathcal{A}$ is self-adjoint.

Proof. Let $V_1 = (v_1, h_1', \omega_1)^T$ and $V_2 = (v_2, h_2', \omega_2)^T$ lie in $D(\mathcal{A})$. We set $(F_1, F_{M_1}, F_{I_1})^T = (\lambda\mathbb{M} - \mathbb{A})V_1$ and $(F_2, F_{M_2}, F_{I_2})^T = (\lambda\mathbb{M} - \mathbb{A})V_2$, that is to say that we have for $i \in \{1, 2\}$

$$\begin{aligned} \lambda v_i - \mathrm{div}\, \sigma(v_i, p_i) &= F_i, \quad \text{in } \mathcal{F}, \\ \mathrm{div}\, v_i &= 0, \quad \text{in } \mathcal{F}, \end{aligned}$$

$$\begin{aligned} v_i = 0, & \qquad \text{in } \partial\mathcal{O}, \\ v_i = h_i'(t) + \omega_i(t) \wedge y, & \qquad y \in \partial\mathcal{S}, \end{aligned}$$

$$\lambda M h_i'(t) = -\int_{\partial\mathcal{S}} \sigma(v_i, p_i) n \mathrm{d}\Gamma + F_{M_i},$$

$$\lambda I_0 \omega_i(t) = -\int_{\partial\mathcal{S}} y \wedge \sigma(v_i, p_i) n \mathrm{d}\Gamma + F_{I_i},$$

with

$$p_i \quad = \quad N(A_0 v_i) - \lambda N(h_i' + \omega_i \wedge y).$$

We calculate

$$\begin{aligned}\langle V_2; (F_1, F_{M_1}, F_{I_1})^T\rangle_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^d\times\mathbb{R}^3} &= \lambda\langle V_1, V_2\rangle_{\mathbf{L}^2(\mathcal{F})} - \int_{\mathcal{F}} v_1 \cdot \text{div } \sigma(v_1, p_1) \\ &\quad + h_2' \cdot \int_{\partial\mathcal{S}} \sigma(v_1, p_1) n \mathrm{d}\Gamma + \omega_2 \cdot \int_{\partial\mathcal{S}} y \wedge \sigma(v_1, p_1) n \mathrm{d}\Gamma,\end{aligned}$$

and by integration by parts we get

$$\begin{aligned}&-\int_{\mathcal{F}} v_1 \cdot \text{div } \sigma(v_1, p_1) + h_2' \cdot \int_{\partial\mathcal{S}} \sigma(v_1, p_1) n \mathrm{d}\Gamma + \omega_2 \cdot \int_{\partial\mathcal{S}} y \wedge \sigma(v_1, p_1) n \mathrm{d}\Gamma \\ =\ & -\int_{\partial\mathcal{S}} v_2 \cdot \sigma(v_1, p_1) n \mathrm{d}\Gamma + 2\nu \int_{\mathcal{S}} D(v_1) : D(v_2) + h_2' \cdot \int_{\partial\mathcal{S}} \sigma(v_1, p_1) n \mathrm{d}\Gamma + \omega_2 \cdot \int_{\partial\mathcal{S}} y \wedge \sigma(v_1, p_1) n \mathrm{d}\Gamma \\ =\ & 2\nu \int_{\mathcal{S}} D(v_1) : D(v_2).\end{aligned}$$

Then, by swapping the roles of V_1 and V_2, it is easy to see that

$$\langle V_2; (F_1, F_{M_1}, F_{I_1})^T\rangle_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^d\times\mathbb{R}^3} \quad = \quad \langle (F_2, F_{M_2}, F_{I_2})^T; V_1\rangle_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^d\times\mathbb{R}^3}.$$

It shows that $\lambda\mathbb{M} - \mathbb{A}$ is self-adjoint. Since $\mathbb{M}$ is self-adjoint, the proof is complete. □

Proposition 3.11. *The resolvent of $\mathcal{A}$ is compact.*

Proof. For $\mathbb{F} = (F, F_M, F_I)^T \in$, we consider the following system

$$\mathbb{M}\left(\lambda \mathrm{Id} - \mathcal{A}\right) V \quad = \quad \mathbb{F},$$

where $V = (v, h', \omega)^T$ is the unknown. This system can be rewritten

$$\begin{aligned}\lambda v - \text{div } \sigma(v, p) &= F, \quad \text{in } \mathcal{F}, \\ \text{div } v &= 0, \quad \text{in } \mathcal{F},\end{aligned}$$

$$\begin{aligned}v = 0, &\qquad \text{in } \partial\mathcal{O}, \\ v = h'(t) + \omega(t) \wedge y, &\qquad y \in \partial\mathcal{S},\end{aligned}$$

$$\begin{aligned}\lambda M h'(t) &= -\int_{\partial\mathcal{S}} \sigma(v, p) n \mathrm{d}\Gamma + F_M, \\ \lambda I_0 \omega(t) &= -\int_{\partial\mathcal{S}} y \wedge \sigma(v, p) n \mathrm{d}\Gamma + F_I,\end{aligned}$$

with

$$p \quad = \quad N(A_0 v) - \lambda N(h' + \omega \wedge y).$$

Up to the calculations made in the proof of Proposition 3.10, this problem is equivalent to this variational problem

$$\text{Find } V \in \mathbf{V}_n^1 \times \mathbb{R}^d \times \mathbb{R}^3 \text{ such that} \quad a(V,W) = l(W) \tag{3.61}$$

with $W = (w, k', \alpha)^T$, and

$$\begin{aligned} a(V,W) &= \lambda\left(\int_{\mathcal{F}} v\cdot w + Mh'\cdot k' + I_0\omega\cdot\alpha\right) + 2\nu\int_{\mathcal{F}} D(v) : D(w), \\ l(W) &= \int_{\mathcal{F}} F\cdot w + F_M\cdot k' + F_I\cdot\alpha. \end{aligned}$$

By using the Lax-Milgram theorem in choosing $\lambda = 1$, we can prove that problem (3.61) has a unique solution. Thus $\mathbb{M}(\lambda\mathrm{Id} - \mathcal{A})$ is invertible, and, since $\mathbb{M}$ is positive, $\lambda\mathrm{Id} - \mathcal{A}$ is invertible. □

The results of the two last propositions yield the following theorem (see [Kat95] for instance).

Theorem 3.12. *The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}_n^0(\mathcal{F}) \times \mathbb{R}^d \times \mathbb{R}^3$, and the resolvent of $\mathcal{A}$ is compact.*

3.4.4 Formulating the control problem

By denoting $U = (\mathbb{P}u, h', \omega)^T$, we consider system (3.11)–(3.17) transformed into

Proposition 3.13. *The triplet (u, h', ω) is solution of system (3.11)–(3.17) if and only if $U = (\mathbb{P}u, h', \omega)^T$ and $(\mathrm{Id} - \mathbb{P})u$ satisfy the operator formulation*

$$U' = \mathcal{A}_\lambda U + \mathcal{B}_\lambda \zeta, \tag{3.62}$$

$$(\mathrm{Id} - \mathbb{P})\, u = (\mathrm{Id} - \mathbb{P})\left(L_0(h') + \hat{L}_0(\omega)\right). \tag{3.63}$$

with $\mathcal{A}_\lambda = \mathcal{A} + \lambda\mathbb{M}^{-1}\mathbb{M}_0$, $\mathcal{B}_\lambda = \mathbb{M}^{-1}\mathbb{B}_\lambda = \mathbb{B}_\lambda$, and

$$\mathbb{B}_\lambda = \begin{bmatrix} (\lambda\mathrm{Id} - A_0)\, L_0 \\ 0 \\ 0 \end{bmatrix}.$$

Proof. Let us first recall that ζ must obey to an incompressibility constraint given by

$$\int_{\partial\mathcal{S}} \zeta\cdot n\mathrm{d}\Gamma = 0.$$

Thus we can formally extend ζ in the whole domain $\mathcal{O}$ while assuming that div $\zeta = 0$ in $\mathcal{F}$. That is why we can make the control appear only in the equation (3.62), the one which deals with $\mathbb{P}u$.

Let us recall the linearized system (3.11)–(3.17) rewritten as

$$\begin{aligned} \frac{\partial u}{\partial t} - \lambda u - \mathrm{div}\ \sigma(u,p) &= 0, \quad \text{in } \mathcal{F}\times(0,\infty), \\ \mathrm{div}\ u &= 0, \quad \text{in } \mathcal{F}\times(0,\infty), \end{aligned}$$

$$\begin{aligned} u = 0, &\qquad \text{in } \partial\mathcal{O}\times(0,\infty), \\ u = h'(t) + \omega(t)\wedge y + \zeta(y,t), &\qquad y\in\partial\mathcal{S}, \quad t\in(0,\infty), \end{aligned}$$

$$
\begin{aligned}
Mh''(t) - \lambda Mh'(t) &= -\int_{\partial\mathcal{S}} \sigma(u,p)n\mathrm{d}\Gamma, \quad t \in (0,\infty),\\
I_0\omega'(t) - \lambda I_0\omega(t) &= -\int_{\partial\mathcal{S}} y \wedge \sigma(u,p)n\mathrm{d}\Gamma, \quad t \in (0,\infty),
\end{aligned}
$$

$$
u(y,0) = u_0(y),\ y \in \mathcal{F}, \quad h'(0) = h_1 \in \mathbb{R}^d, \quad \omega(0) = \omega_0 \in \mathbb{R}^3.
$$

The boundary condition on $\partial\mathcal{S}$ can be tackled by using a lifting method, as in Chapter 1. It consists in splitting the velocity $u = v + \mathrm{w}$ and the pressure $p = q + \pi$, so that we have

$$
\begin{aligned}
-\nu\Delta \mathrm{w} + \nabla\pi &= 0, && \text{in } \mathcal{F},\\
\operatorname{div}\, \mathrm{w} &= 0, && \text{in } \mathcal{F},\\
\mathrm{w} &= \zeta, && \text{on } \partial\mathcal{S},\\
\mathrm{w} &= 0, && \text{on } \partial\mathcal{O},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial v}{\partial t} - \lambda v - \operatorname{div}\, \sigma(v,q) &= -\frac{\partial \mathrm{w}}{\partial t} + \lambda \mathrm{w}, && \text{in } \mathcal{F}\times(0,\infty),\\
\operatorname{div}\, v &= 0, && \text{in } \mathcal{F}\times(0,\infty),
\end{aligned}
$$

$$
\begin{aligned}
v &= 0, && \text{in } \partial\mathcal{O}\times(0,\infty),\\
v &= h'(t) + \omega(t)\wedge y, && y \in \partial\mathcal{S}, \quad t \in (0,\infty),
\end{aligned}
$$

$$
\begin{aligned}
Mh''(t) - \lambda Mh'(t) &= -\int_{\partial\mathcal{S}} \sigma(v,q)n\mathrm{d}\Gamma - \int_{\partial\mathcal{S}} \sigma(\mathrm{w},\pi)n\mathrm{d}\Gamma, \quad t \in (0,\infty),\\
I_0\omega'(t) - \lambda I_0\omega(t) &= -\int_{\partial\mathcal{S}} y\wedge\sigma(v,q)n\mathrm{d}\Gamma - \int_{\partial\mathcal{S}} y\wedge\sigma(\mathrm{w},\pi)n\mathrm{d}\Gamma, \quad t \in (0,\infty),
\end{aligned}
$$

$$
v(y,0) \;=\; u_0(y) - \mathrm{w}(y,0),\ y \in \mathcal{F}, \quad h'(0) = h_1 \in \mathbb{R}^d, \quad \omega(0) = \omega_0 \in \mathbb{R}^3.
$$

This system can be formulated as follows

$$
\begin{aligned}
\mathbb{M}V' &= \mathbb{A}V + \lambda\mathbb{M}_0 V + \mathbb{B}_1\dot{\zeta} + \mathbb{B}_{\lambda,0}\zeta,\\
(\mathrm{Id} - \mathbb{P})\, u &= (\mathrm{Id} - \mathbb{P})\left(L_0(h') + \hat{L}_0(\omega)\right),
\end{aligned}
$$

with $V = (\mathbb{P}v, h', \omega)$, and

$$
\mathbb{B}_1 = \begin{bmatrix} -L_0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbb{B}_{\lambda,0} = \begin{bmatrix} \lambda L_0 \\ \mathcal{C}(-2\nu D + NA_0)L_0 \\ \hat{\mathcal{C}}(-2\nu D + NA_0)L_0 \end{bmatrix}.
$$

Notice that $\mathbb{M}^{-1}\mathbb{B}_{\lambda,1} = \mathbb{B}_{\lambda,1}$. The Duhamel's formula gives

$$
V(t) \;=\; e^{t\mathcal{A}_\lambda}\left(U(0) + \mathbb{B}_1\zeta(\cdot,0)\right) + \int_0^t e^{(t-s)\mathcal{A}_\lambda}\left(\mathbb{M}^{-1}\mathbb{B}_{\lambda,0}\zeta + \mathbb{B}_1\dot{\zeta}\right)\mathrm{d}s,
$$

and an integration part leads to

$$\int_0^t e^{(t-s)\mathcal{A}_\lambda}\mathbb{B}_1\dot{\zeta}\mathrm{d}s \quad = \quad \mathbb{B}_1\zeta(\cdot,t) - e^{t\mathcal{A}_\lambda}\mathbb{B}_1\zeta(\cdot,0) + \int_0^t e^{(t-s)\mathcal{A}_\lambda}\mathcal{A}_\lambda\mathbb{B}_1\zeta\mathrm{d}s.$$

Notice that $\mathbb{M}_0\mathbb{B}_1 = 0$ and that

$$\mathbb{A}\mathbb{B}_1 \quad = \quad \begin{bmatrix} -A_0L_0 \\ -\mathcal{C}(-2\nu D + NA_0)L_0 \\ -\hat{\mathcal{C}}(-2\nu D + NA_0)L_0 \end{bmatrix}.$$

Thus we have

$$\begin{aligned} U(t) &= e^{t\mathcal{A}_\lambda}U(0) + \int_0^t e^{(t-s)\mathcal{A}_\lambda}\mathbb{M}^{-1}\left(\mathbb{B}_{\lambda,0} + \mathbb{A}\mathbb{B}_1\right)\zeta\mathrm{d}s, \\ U'(t) &= \mathcal{A}_\lambda U(t) + \mathcal{B}_\lambda\zeta(\cdot,t), \end{aligned}$$

and finally system (3.11)–(3.17) can be expressed formally in the form given by (3.62)–(3.63). □

3.4.5 Regularity of solutions of the linearized system

Proposition 3.14. *For $\lambda > 0$, $(u_0, h_1, \omega_0) \in \mathbf{H}^1_{cc}$ and $\zeta \in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))$, system* (3.11)–(3.17) *admits a unique solution (v, h', ω) such that*

$$v \in \mathrm{H}^{2,1}(Q_0^\infty), \quad h' \in \mathrm{H}^1(0,\infty;\mathbb{R}^d), \quad \omega \in \mathrm{H}^1(0,\infty;\mathbb{R}^3).$$

Proof. With regards to the formulation (3.62)–(3.63) and Theorem 3.12, the proof of this proposition can be deduced from Proposition 3.3 of [TT04]. □

3.5 Approximate controllability and stabilization

In order to prove that system (3.11)–(3.17) is exponentially stabilizable (in the sense of Definition 3.3), let us first show that it is approximatively controllable.

3.5.1 Approximate controllability of the homogeneous linear system

Theorem 3.15. *Let us set $\lambda = 0$. System* (3.11)–(3.17) *is approximately controllable, in the space $\mathbf{H}^0_{cc}$ by velocities $\zeta \in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))$ satisfying*

$$\int_{\partial\mathcal{S}} \zeta\cdot n\mathrm{d}\Gamma \quad = \quad 0.$$

Proof. Note that the operator formulation given by Proposition 3.13 does not enable us to write system (3.11)–(3.17) as an evolution equation. Instead of exploiting this operator formulation, we directly use the writing (3.11)–(3.17) and the definition of approximate controllability, as in [Ray10a].

Let us show that if $(u_0, h_1, \omega_0) = (0,0,0)$ then the reachable set $R(T)$ at time T, when the control ζ describes $\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))$ with the compatibility condition

$$\int_{\partial\mathcal{S}} \zeta\cdot n\mathrm{d}\Gamma \quad = \quad 0$$

is dense in the space $\mathbf{L}^2(\mathcal{F}) \times \mathbb{R}^d \times \mathbb{R}^3$ that we endow with the scalar product

$$\langle (u, h', \omega); (\phi, k', r)\rangle \quad = \quad \int_{\mathcal{F}} u \cdot \phi + M h' \cdot k' + I_0 \omega \cdot r.$$

For that, let (ϕ^T, k'^T, r^T) be in $R(T)^{\perp}$. We want to show that $(\phi^T, k'^T, r^T) = (0, 0, 0)$.
Let us introduce the adjoint system

$$-\frac{\partial \phi}{\partial t} - \operatorname{div} \sigma(\phi, \psi) \quad = \quad 0, \quad \text{in } \mathcal{F} \times (0, T), \tag{3.64}$$

$$\operatorname{div} \phi \quad = \quad 0, \quad \text{in } \mathcal{F} \times (0, T), \tag{3.65}$$

$$\phi = 0, \qquad \text{in } \partial\mathcal{O} \times (0, T), \tag{3.66}$$

$$\phi = k'(t) + r(t) \wedge y, \qquad y \in \partial\mathcal{S}, \quad t \in (0, T), \tag{3.67}$$

$$-M k''(t) = -\int_{\partial\mathcal{S}} \sigma(\phi, \psi) n \mathrm{d}\Gamma, \quad t \in (0, T), \tag{3.68}$$

$$-I_0 r'(t) = -\int_{\partial\mathcal{S}} y \wedge \sigma(\phi, \psi) n \mathrm{d}\Gamma, \quad t \in (0, T), \tag{3.69}$$

$$\phi(y, T) \quad = \quad \phi^T(y), \; y \in \mathcal{F}, \quad k'(T) = k'^T \in \mathbb{R}^d, \quad r(T) = r^T \in \mathbb{R}^3. \tag{3.70}$$

By integrations by parts from systems (3.11)–(3.17) and (3.64)–(3.70), we obtain

$$\int_{\mathcal{F}} u(T) \cdot \phi^T + M h'(T) \cdot k'^T + I_0 \omega(T) \cdot r^T \quad = \quad -\int_0^T \int_{\partial\mathcal{S}} \zeta \cdot \sigma(\phi, \psi) n \mathrm{d}\Gamma \mathrm{d}t.$$

Thus we deduce that if $(\phi^T, k'^T, r^T) \in R(T)^{\perp}$, then we have

$$\int_0^T \int_{\partial\mathcal{S}} \zeta \cdot \sigma(\phi, \psi) n \mathrm{d}\Gamma \mathrm{d}t \quad = \quad 0, \tag{3.71}$$

for all $\zeta \in \mathrm{L}^2(0, \infty; \mathbf{H}^{5/2}(\partial\mathcal{S})) \cap \mathrm{H}^1(0, \infty; \mathbf{H}^{1/2}(\partial\mathcal{S}))$ such that $\int_{\partial\mathcal{S}} \zeta \cdot n \mathrm{d}\Gamma = 0$.
This is equivalent to say that there exists a constant $C(t)$ such that

$$\sigma(\phi, \psi) n = C(t) n, \qquad \text{on } \partial\mathcal{S}. \tag{3.72}$$

We can consider $\overline{\psi} = \psi - C$ instead of ψ, which does not modify the system (3.64)–(3.70). Thus we get $\sigma(\phi, \psi) n = 0$ in $\mathrm{L}^2(0, T; \mathbf{L}^2(\partial\mathcal{S}))$. Then equations (3.68) and (3.69) become

$$k'' = 0, \qquad r' = 0.$$

Now, let us look at $\phi_t = \dfrac{\partial \phi}{\partial t}$ and $\psi_t = \dfrac{\partial \psi}{\partial t}$ with the condition

$$\sigma(\phi_t, \psi_t) n = 0 \qquad \text{on } \partial\mathcal{S}, \tag{3.73}$$

and the homogeneous Dirichlet condition

$$\phi_t \quad = \quad 0, \quad \text{on } \partial\mathcal{F} = \partial\mathcal{O} \cup \partial\mathcal{S}.$$

We use an expansion of the solution ϕ_t of system

$$\begin{aligned} -\frac{\partial \phi_t}{\partial t} - \mathrm{div}\ \sigma(\phi_t, \psi_t) &= 0, \quad \text{in } \mathcal{F} \times (0,T), \\ \mathrm{div}\ \phi_t &= 0, \quad \text{in } \mathcal{F} \times (0,T), \\ \phi_t &= 0, \quad \text{in } \partial\mathcal{F} \times (0,T), \end{aligned}$$

in terms of the eigenfunctions of the Stokes operator, similarly as it is done in [OP99]. Then the approximate controllability problem reduces to showing that if

$$\begin{aligned} -\nu\Delta v + \nabla p = \lambda v \qquad & \text{in } \mathcal{F}, \\ \mathrm{div}\ v = 0 \qquad & \text{in } \mathcal{F}, \\ v = 0 \qquad & \text{on } \partial\mathcal{F}, \\ \sigma(v,p)n = 0 \qquad & \text{on } \partial\mathcal{S}, \end{aligned}$$

with $\lambda \in \mathbb{R}$, then $v = 0$ in $\mathcal{F}$. Thus we get (see [FL96])

$$\phi_t = 0 \qquad \text{in } \mathcal{F}.$$

Then we have

$$\begin{aligned} -\mathrm{div}\ \sigma(\phi,\psi) = 0, \qquad & \text{in } \mathcal{F} \times (0,T), \\ \mathrm{div}\ \phi = 0, \qquad & \text{in } \mathcal{F} \times (0,T), \\ \phi = 0, \qquad & \text{in } \partial\mathcal{O} \times (0,T), \\ \phi = k' + r \wedge y, \qquad & (y,t) \in \partial\mathcal{S} \times (0,T). \end{aligned}$$

An energy estimate leads us to

$$\begin{aligned} 2\nu \int_{\mathcal{F}} |D(\phi)|^2 \,\mathrm{d}y &= \int_{\partial\mathcal{S}} \phi \cdot \sigma(\phi,\psi) n \mathrm{d}\Gamma \\ &= k' \cdot \int_{\partial\mathcal{S}} \sigma(\phi,\psi) n \mathrm{d}\Gamma + r \cdot \int_{\partial\mathcal{S}} y \wedge \sigma(\phi,\psi) n \mathrm{d}\Gamma. \end{aligned}$$

Combined to (3.72), we get

$$\int_{\mathcal{F}} |D(\phi)|^2 \,\mathrm{d}y = 0,$$

and thus $D(\phi) = 0$. Using a result from [Tem83], we have

$$\phi = k' + r \wedge y, \qquad \text{in } \mathcal{F}.$$

The condition (3.66) enables us to conclude

$$k' = 0, \quad r = 0, \text{ and } \phi = 0 \text{ in } \mathcal{F}.$$

Then the proof is completed. □

Remark 3.16. *The adjoint system introduced in* (3.64)-(3.70) *can be written in terms of operators; Indeed, denoting* $\boldsymbol{\phi} = (\mathbb{P}\phi, k', r)^T$*, we can formulate this system as follows*

$$\begin{aligned} -\boldsymbol{\phi}' &= \mathcal{A}_0^* \boldsymbol{\phi}, \\ \boldsymbol{\phi}(T) &= (\mathbb{P}\phi^T, k'^T, r^T)^T, \\ (\mathrm{Id} - \mathbb{P})\phi &= (\mathrm{Id} - \mathbb{P}) \left(L_0(k') + \hat{L}_0(r) \right), \end{aligned}$$

where $\mathcal{A}_0$ is self-adjoint, so that we can write

$$\boldsymbol{\phi}(t) \quad = \quad e^{(T-t)\mathcal{A}_0}(\mathbb{P}\phi^T, k'^T, r^T)^T.$$

The adjoint operator of operator $\mathcal{B}_0$ (whose expression is given in Proposition 3.13) can be expressed as follows

$$\mathcal{B}_0^*\boldsymbol{\phi} \quad = \quad -\sigma(z,\pi)n,$$

where (z,π) is the solution of

$$\begin{aligned} -\nu\Delta z + \nabla\pi = (-A_0)\phi \qquad & \text{in } \mathcal{F}, \\ \operatorname{div} z = 0 \qquad & \text{in } \mathcal{F}, \\ z = 0 \qquad & \text{on } \partial\mathcal{F} = \partial\mathcal{O} \cup \partial\mathcal{S} \end{aligned}$$

(see Lemma A.4 of [Ray07] for more details). Then, proving the result stated above by using the classical characterization of approximate controllability instead of using directly the definition would lead to other difficulties.

3.5.2 Stabilizability of the homogeneous linear system

Theorem 3.17. *For all $(u_0, h_1, \omega_0) \in \mathbf{H}^1_{cc}$, system (3.11)–(3.17) is stabilizable with an arbitrary exponential decay rate $\lambda > 0$, in the sense of Definition 3.3.*

Proof. Without loss of generality, we can choose λ in the resolvent of $\mathcal{A}$. Due to theorem 3.12, we know that the spectrum of $-\mathcal{A}$ is only a pointwise spectrum constituted of a countable number distinct eigenvalues, that we can order as follows

$$\Re\lambda_1 \geq \Re\lambda_2 \geq \cdots \geq \Re\lambda_N > -\lambda > \Re\lambda_{N+1} \geq \ldots.$$

Moreover the generalized eigenspace of each eigenvalue is of finite dimension (see [Kat95], page 187). Let us denote by $\Lambda(\lambda_i)$ the real generalized eigenspace associated with λ_i if $\lambda_i \in \mathbb{R}$ and with the pair $(\lambda_i, \overline{\lambda_i})$ if $\Im\lambda_i \neq 0$, and let us set

$$\mathbf{H}_u = \bigoplus_{i=1}^{N} \Lambda(\lambda_i), \qquad \mathbf{H}_s = \bigoplus_{i=N+1}^{\infty} \Lambda(\lambda_i).$$

If $E(\lambda_i)$ denotes the complex generalized eigenspace associated with λ_i and if $(e_j(\lambda_i))_{1\leq j\leq m(\lambda_i)}$ (where $m(\lambda_i)$ denoting the geometric multiplicity of λ_i) is a basis of $E(\lambda_i)$, then $\Lambda(\lambda_i)$ is nothing else than the space generated by the family $\{\Re e_j(\lambda_i), \Im e_j(\lambda_i) \mid 1 \leq j \leq m(\lambda_j)\}$. Let us observe that $\mathbf{H}_u$ is the unstable space of system (3.11)–(3.17) while $\mathbf{H}_s$ is the stable space. Let us denote by P_λ the projection onto the finite-dimensional unstable subspace $\mathbf{H}_u$ (parallel to the stable subspace $\mathbf{H}_s$). If we project system (3.62) on $\mathbf{H}_u$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} P_\lambda \begin{pmatrix} \mathbb{P}u \\ h' \\ \omega \end{pmatrix} = \mathcal{A}P_\lambda \begin{pmatrix} \mathbb{P}u \\ h' \\ \omega \end{pmatrix} + P_\lambda \mathcal{B}_0\zeta_0, \qquad P_\lambda \begin{pmatrix} \mathbb{P}u(\cdot,0) \\ h'(0) \\ \omega(0) \end{pmatrix} = P_\lambda \begin{pmatrix} u_0 \\ h_1 \\ \omega_0 \end{pmatrix}. \tag{3.74}$$

Due to Theorem 3.15, system (3.11)–(3.17) is approximately controllable in time $T > 0$. Thus the projected system (3.74) is also approximately controllable. Since it is of finite dimension, it is also controllable. Let ζ_0 be a control such that $P_\lambda(u,h',\omega)(T) = (0,0,0)$, and let us still denote by ζ_0 its extension by zero to (T,∞). Now, we notice that $P_\lambda(\mathbb{P}u,h',\omega)$ is the solution of system (3.74) corresponding to ζ_0 if and only if $P_\lambda(\mathbb{P}\hat{u},\hat{h}',\hat{\omega}) = e^{\lambda t}P_\lambda(\mathbb{P}u,h',\omega)$ is the solution of system

$$\frac{\mathrm{d}}{\mathrm{d}t}P_\lambda\begin{pmatrix}\mathbb{P}u\\ h'\\ \omega\end{pmatrix} = \mathcal{A}_\lambda P_\lambda\begin{pmatrix}\mathbb{P}u\\ h'\\ \omega\end{pmatrix} + P_\lambda\mathcal{B}_\lambda\zeta, \qquad P_\lambda\begin{pmatrix}\mathbb{P}u(\cdot,0)\\ h'(0)\\ \omega(0)\end{pmatrix} = P_\lambda\begin{pmatrix}u_0\\ h_1\\ \omega_0\end{pmatrix}, \tag{3.75}$$

corresponding to the control $\zeta = e^{\lambda t}\zeta_0$.
Thus system (3.75) is stabilizable - in the sense of Definition 3.3. System (3.75) is the projection of system (3.11)–(3.17) onto its unstable space. Due to [Tri75] and [MT78], system (3.11)–(3.17) is stabilizable by a control $\zeta \in \mathrm{L}^2(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S}))$ if and only if its projection onto its finite-dimensional unstable subspace is stabilizable. The proof is complete. ☐

We know that system (3.75) is stabilizable by a control $\zeta \in \mathrm{L}^2(0,\infty;\Xi)$ satisfying

$$\int_{\partial\mathcal{S}} \zeta\cdot n\mathrm{d}\Gamma \quad = \quad 0,$$

where Ξ is a subspace of finite dimension included in $\mathbf{H}^{5/2}(\partial\mathcal{S})$ (see [BT11] for instance). Moreover such a stabilizing control can be found in a feedback form. In other words, there exists

$$\mathcal{K}_\lambda \quad \in \quad \mathcal{L}(\mathbf{H}_u;\Xi)$$

such that the operator

$$\mathcal{A}_\lambda P_\lambda + \mathcal{B}_\lambda\mathcal{K}_\lambda P_\lambda$$

is exponentially stabilizable on $\mathbf{H}_u$. Denoting $\mathbf{U} = (\mathbb{P}u,h',\omega)$, let us now consider the system

$$P_\lambda\mathbf{U}' \quad = \quad (\mathcal{A}_\lambda + \mathcal{B}_\lambda\mathcal{K}_\lambda P_\lambda)\,P_\lambda\mathbf{U}. \tag{3.76}$$

When $\mathcal{K}_\lambda$ is determined through an infinite time horizon control problem, in using the optimality system, it is easy to prove that the solution of (3.76) is such that $\mathcal{K}_\lambda\mathbf{U} \in \mathrm{H}^1(0,\infty;\Xi)$. Choosing $\zeta = \mathcal{K}_\lambda P_\lambda\mathbf{U}$ in (3.62), this system becomes (3.76), and we have the estimate

$$\|\mathbf{U}\|_{\mathrm{H}^{2,1}(Q^0_\infty)\times\mathrm{H}^1(0,\infty;\mathbb{R}^d)\times\mathrm{H}^1(0,\infty;\mathbb{R}^3)} \quad \leq \quad C\|(u_0,h_1,\omega_0)\|_{\mathbf{H}^1_{cc}}.$$

We will still denote $\mathcal{K}_\lambda$ the extension of $\mathcal{K}_\lambda P_\lambda$ to the space $\mathbf{H}_s$, by setting

$$\mathcal{K}_\lambda(\mathrm{Id}-P_\lambda) \quad = \quad 0.$$

The continuity of $\mathcal{K}_\lambda P_\lambda$ provides

$$\|\mathcal{K}_\lambda P_\lambda\mathbf{U}\|_{\mathrm{H}^1(0,\infty;\Xi)} \quad \leq \quad C_{\mathcal{K}_\lambda}\|\mathbf{U}\|_{\mathbf{H}_u},$$

and thus we also have

$$\|\mathcal{K}_\lambda\mathbf{U}\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))} \quad \leq \quad C_{\mathcal{K}_\lambda}\|\mathbf{U}\|_{\mathrm{H}^{2,1}(Q^0_\infty)\times\mathrm{H}^1(0,\infty;\mathbb{R}^d)\times\mathrm{H}^1(0,\infty;\mathbb{R}^3)},$$

Above all, such a choice changes the unstable operator $\mathcal{A}_\lambda$ into $\mathcal{A}_\lambda + \mathcal{B}_\lambda\mathcal{K}_\lambda P_\lambda$ which is stable. In the following, we will only need the regularity

$$\zeta \quad \in \quad \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\mathcal{S})).$$

3.6 Defining an admissible deformation from a feedback boundary velocity

As system (3.3)–(3.9) is written, the control considered has to satisfy the nonlinear constraints imposed in Definition 3.4. Given a control $\zeta \in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\mathcal{S}))$ with the compatibility condition

$$\int_{\partial\mathcal{S}} \zeta\cdot n \mathrm{d}\Gamma \quad = \quad 0,$$

chosen in a feedback form, as explained in the previous section, we first want to define an internal solid's deformation, denoted by X^*_ζ, which is *admissible* in the sense of Definition 3.2. In particular we look for X^*_ζ which satisfies the linearized constraints given by (3.24)–(3.26), and such that

$$\frac{\partial X^*_\zeta}{\partial t}(y,t) \quad = \quad e^{-\lambda t}\zeta(y,t), \quad (y,t)\in\partial\mathcal{S}\times(0,\infty).$$

We also want that the norm of $X^*_\zeta - \mathrm{Id}_\mathcal{S}$ in $\mathcal{W}_\lambda(S^0_\infty)$ is controlled by the norm of ζ in $\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\mathcal{S}))\cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\mathcal{S}))$.
After that we want to deduce from X^*_ζ an *admissible* deformation X^* in the sense of Definition 3.4. The way we proceed has to provide good properties for the difference $X^* - X^*_\zeta$, because the nonhomogeneous terms induced by this deviance have to be tackled in a contracting fixed point method.

3.6.1 Defining a deformation satisfying the linearized constraints

We search for the deformation X^*_ζ in writing it as

$$X^*_\zeta(y,t) \quad = \quad y + \int_0^t e^{-\lambda s}\varphi(y,s)\mathrm{d}s,$$

where the velocity $\varphi(\cdot,t)$ is the solution of the following elliptic system

$$\mu\varphi - 2\mathrm{div}\ D(\varphi) = F(\varphi) \qquad \text{in } \mathcal{S}, \tag{3.77}$$
$$\varphi = \zeta \qquad \text{on } \partial\mathcal{S}, \tag{3.78}$$

with

$$D(\varphi) \quad = \quad \frac{1}{2}\left(\nabla\varphi + \nabla\varphi^T\right),$$
$$F(\varphi)(y,t) \quad = \quad \frac{\rho_\mathcal{S}}{M}\left(\int_{\partial\mathcal{S}} 2D(\varphi)n\mathrm{d}\Gamma\right) + \rho_\mathcal{S} I_0^{-1}\left(\int_{\partial\mathcal{S}} y\wedge 2D(\varphi)n\mathrm{d}\Gamma\right)\wedge y.$$

Lemma 3.18. *For $\zeta \in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\mathcal{S}))$, system (3.77)–(3.78) admits a unique solution φ in $\mathcal{H}_3(S^0_\infty)$, for $\mu > 0$ large enough. Moreover, there exists a positive constant $C > 0$ such that the deformation X^*_ζ obtained by (3.77) satisfies*

$$\|X^*_\zeta - \mathrm{Id}_\mathcal{S}\|_{\mathcal{W}_\lambda(S^0_\infty)} \quad \leq \quad C\|\zeta\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\mathcal{S}))}. \tag{3.79}$$

Besides, if $\zeta_1, \zeta_2 \in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\mathcal{S}))\cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\mathcal{S}))$*, and if* $X^*_{\zeta_1}$ *and* $X^*_{\zeta_2}$ *denote the solutions associated with* ζ_1 *and* ζ_2 *respectively, then*

$$\|X^*_{\zeta_2} - X^*_{\zeta_1}\|_{\mathcal{W}_\lambda(\mathcal{S}^0_\infty)} \leq C\|\zeta_2-\zeta_1\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\mathcal{S}))}. \tag{3.80}$$

Proof. The proof of this lemma is given in section 3.6.2. □

Remark 3.19. *The compatibility condition assumed for the datum* ζ *is useless for the proof of Lemma 3.18, but contributes to making the mapping* X^*_ζ *so obtained an admissible control (in the sense of Definition 3.2.*

Let us see that the mapping X^*_ζ so chosen is admissible.

Lemma 3.20. *For* $\zeta \in \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\mathcal{S}))\cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\mathcal{S}))$*, the deformation* X^*_ζ *provided by Lemma 3.18 and equation* (3.77) *as the solution of system* (3.77)–(3.78) *is admissible for the linear system* (3.11)–(3.17) *in the sense of Definition 3.2.*

Proof. The constraints imposed in Definition 3.2 are equivalent to the following ones expressed in term of φ:

$$\int_{\partial\mathcal{S}} \varphi\cdot n \mathrm{d}\Gamma = 0, \quad \int_{\mathcal{S}} \varphi \mathrm{d}y = 0, \quad \int_{\mathcal{S}} y\wedge\varphi \mathrm{d}y = 0.$$

Thus we have to verify that the mapping φ solution of (3.77)–(3.78) satisfies these constraints. The first constraint, which corresponds to (3.24), is satisfied thanks to the compatibility condition assumed for ζ. Let us recall that we have

$$\int_{\mathcal{S}} y\mathrm{d}y = 0,$$

and since the tensor $D(\varphi)$ is symmetric, we also have

$$y\wedge \mathrm{div} D(\varphi) = \mathrm{div}\left(\mathbb{S}(y)D(\varphi)\right).$$

Then Equation (3.77) leads us to

$$\begin{aligned} \mu\int_{\mathcal{S}}\varphi\mathrm{d}y &= 2\left(\int_{\partial\mathcal{S}} D(\varphi)n\mathrm{d}\Gamma\right) - 2\int_{\mathcal{S}}\mathrm{div}\left(D(\varphi)\right)\mathrm{d}y,\\ \mu\int_{\mathcal{S}} y\wedge\varphi\mathrm{d}y &= 2\left(\int_{\partial\mathcal{S}} y\wedge D(\varphi)n\mathrm{d}\Gamma\right) - 2\int_{\mathcal{S}}\mathrm{div}\left(\mathbb{S}(y)D(\varphi)\right)\mathrm{d}y, \end{aligned}$$

and thus in using the divergence formula we get the two other constraints. □

3.6.2 Proof of Lemma 3.18

Instead of solving system directly (3.77)–(3.78), let us consider a lifting of the nonhomogeneous Dirichlet condition. We set w the solution of the following Dirichlet problem

$$\begin{aligned} \mathrm{div}\ \mathrm{w} = 0 &\quad \text{in } \mathcal{S},\\ \mathrm{w} = \zeta &\quad \text{on } \partial\mathcal{S}, \end{aligned}$$

with the classical estimates (see [Gal94] for instance)

$$
\begin{aligned}
\|\mathrm{w}\|_{\mathbf{H}^2(\mathcal{S})} &\leq C\|\zeta\|_{\mathbf{H}^{3/2}(\mathcal{S})},\\
\|\mathrm{w}\|_{\mathbf{H}^3(\mathcal{S})} &\leq C\|\zeta\|_{\mathbf{H}^{5/2}(\mathcal{S})},\\
\|\mathrm{w}_t\|_{\mathbf{H}^1(\mathcal{S})} &\leq C\|\zeta_t\|_{\mathbf{H}^{1/2}(\mathcal{S})}.
\end{aligned}
$$

Then in setting $\phi = \varphi - \mathrm{w}$, we are interested in solving the following system

$$
\begin{aligned}
\mu\phi - 2\mathrm{div}\ D(\phi) = F(\phi) - \mu\mathrm{w} + \Delta\mathrm{w} + F(\mathrm{w}) \qquad & \text{in } \mathcal{S},\\
\phi = 0 \qquad & \text{on } \partial\mathcal{S},
\end{aligned}
$$

for some $\mu > 0$ large enough, in the space $\mathbf{H}^2(\mathcal{S})$ in a first time. A solution of this system can be obtained as a fixed point of the following mapping

$$
\begin{aligned}
\mathbf{N}\ :\quad \mathbf{H}^2(\mathcal{S}) &\rightarrow \mathbf{H}^2(\mathcal{S})\\
\psi &\mapsto \phi,
\end{aligned}
$$

where ϕ is the solution of the classical elliptic system

$$
\mu\phi - 2\mathrm{div}\ D(\phi) = F(\psi) - \mu\mathrm{w} + 2\mathrm{div}\ D(\mathrm{w}) + F(\mathrm{w}) \qquad \text{in } \mathcal{S}, \tag{3.81}
$$

$$
\phi = 0 \qquad \text{on } \partial\mathcal{S}. \tag{3.82}
$$

For proving that this mapping is well-defined, let us give some preliminary estimates. The equality

$$
2D(\phi) : D(\phi) - \nabla\phi : \nabla\phi \quad = \quad \mathrm{div}\left((\phi\cdot\nabla)\phi - (\mathrm{div}\ \phi)\phi\right) + (\mathrm{div}\ \phi)^2
$$

leads in $\mathbf{H}^1_0(\mathcal{S})$ to

$$
\|\nabla\phi\|^2_{\mathbf{L}^2(\mathcal{S})} \leq 2\|D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})},
$$

and then the Poincaré inequality provides a positive constant C_p such that

$$
\|\phi\|_{\mathbf{H}^1(\mathcal{S})} \leq C_p\|D(\phi)\|_{\mathbf{L}^2(\mathcal{S})}.
$$

We can estimate the norm $\mathbf{H}^2(\mathcal{F})$ as follows

$$
\begin{aligned}
\|\phi\|^2_{\mathbf{H}^2(\mathcal{S})} &\leq C_1\left(\|\mathrm{div}D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})} + \|\phi\|^2_{\mathbf{H}^1(\mathcal{S})}\right),\\
&\leq C_1\left(\|\mathrm{div}D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})} + C_p^2\|D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})}\right).
\end{aligned}
$$

The trace of $D(\psi)$ on $\partial\mathcal{S}$ which appears in the expression of $F(\psi)$ can be estimated as follows

$$
\begin{aligned}
\|D(\psi)n\|_{\mathbf{L}^2(\partial\mathcal{S})} &\leq C_2\|\psi\|_{\mathbf{H}^{3/2+\varepsilon}(\mathcal{S})},\\
&\leq C_2\|\psi\|^{\alpha}_{\mathbf{H}^2(\mathcal{S})}\|\psi\|^{1-\alpha}_{\mathbf{H}^1(\mathcal{S})},\\
&\leq C_2C_p\|\psi\|^{\alpha}_{\mathbf{H}^2(\mathcal{S})}\|D(\psi)\|^{1-\alpha}_{\mathbf{L}^2(\mathcal{S})},
\end{aligned}
$$

with $\alpha = 1/2 + \varepsilon$, for some $\varepsilon > 0$ which can be chosen small enough. Thus, by taking the inner product of the equality (3.81) by $\mathrm{div}D(\phi)$, we obtain

$$
\begin{aligned}
\mu\|D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})} + 2\|\mathrm{div}D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})} &\leq C\left(\|F(\psi)\|_{\mathbf{L}^2(\mathcal{S})} + \|\mathrm{w}\|_{\mathbf{H}^2(\mathcal{S})}\right)\|\mathrm{div}D(\phi)\|_{\mathbf{L}^2(\mathcal{S})}\\
&\leq \tilde{C}\left(\|D(\psi)n\|^2_{\mathbf{L}^2(\partial\mathcal{S})} + \|\mathrm{w}\|^2_{\mathbf{H}^2(\mathcal{S})}\right) + \frac{1}{2}\|\mathrm{div}D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})},\\
\mu\|D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})} + \frac{3}{2}\|\mathrm{div}D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})} &\leq C_3\left(\|\psi\|^{2\alpha}_{\mathbf{H}^2(\mathcal{S})}\|D(\psi)\|^{2-2\alpha}_{\mathbf{L}^2(\mathcal{S})} + \|\mathrm{w}\|^2_{\mathbf{H}^2(\mathcal{S})}\right),\\
(2\mu - 3C_p^2)C_1\|D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})} + 3\|\phi\|^2_{\mathbf{H}^2(\mathcal{S})} &\leq 2C_1C_3\left(\|\psi\|^{2\alpha}_{\mathbf{H}^2(\mathcal{S})}\|D(\psi)\|^{2-2\alpha}_{\mathbf{L}^2(\mathcal{S})} + \|\mathrm{w}\|^2_{\mathbf{H}^2(\mathcal{S})}\right).
\end{aligned}
$$

We use the Young inequality, in introducing some $\delta > 0$ which can be chosen as small as desired, as follows

$$2C_1C_3\|D(\psi)\|^{2-2\alpha}_{\mathbf{L}^2(\mathcal{S})}\|\psi\|^{2\alpha}_{\mathbf{H}^2(\mathcal{S})} \leq \frac{\delta^p}{p}\|\psi\|^2_{\mathbf{H}^2(\mathcal{S})} + \left(\frac{2C_1C_3}{\delta}\right)^q \frac{1}{q}\|D(\psi)\|^2_{\mathbf{L}^2(\mathcal{S})}$$

with $p = 2/(1+2\varepsilon)$ and $q = 2/(1-2\varepsilon)$. Thus, in setting

$$\mathbf{B}_R = \left\{\phi \in \mathbf{H}^2(\mathcal{S}) \mid (2\mu - 3C_p^2)C_1\|D(\phi)\|^2_{\mathbf{L}^2(\mathcal{S})} + \|\phi\|^2_{\mathbf{H}^2(\mathcal{S})} \leq R\right\},$$

and in choosing $\delta > 0$ small enough, and $\mu > 0$ and $R > 0$ large enough, we can see that the ball $\mathbf{B}_R$ is stable by the mapping $\mathbf{N}$. By the same inequalities we can see that $\mathbf{N}$ is contracting in $\mathbf{B}_R$, and thus $\mathbf{N}$ admits a unique fixed point in $\mathbf{H}^2(\mathcal{S})$.
The same method can be applied in order to prove the regularity in $\mathbf{H}^3(\mathcal{S})$. Indeed, since we have the equality $\nabla(\mathrm{div}D(\phi)) = \mathrm{div}D(\nabla\phi)$, the gradient satisfies a similar equality, as follows

$$\mu\nabla\phi - 2\mathrm{div}D(\nabla\phi) = \nabla F(\phi) - \mu\nabla\mathrm{w} + 2\mathrm{div}D(\nabla\mathrm{w}) + \nabla F(\mathrm{w}) \qquad \text{in } \mathcal{S},$$

so that we can show that $\nabla\phi$ lies in $\mathbf{H}^2(\mathcal{S})$. Then we have the estimate

$$\|\phi\|_{\mathbf{H}^3(\mathcal{S})} \leq C\|\mathrm{w}\|_{\mathbf{H}^3(\mathcal{S})},$$

and since $\varphi = \phi + \mathrm{w}$, we have

$$\begin{aligned}\|\varphi\|_{\mathbf{H}^3(\mathcal{S})} &\leq \tilde{C}\|\mathrm{w}\|_{\mathbf{H}^3(\mathcal{S})} \leq \overline{C}\|\zeta\|_{\mathbf{H}^{5/2}(\mathcal{S})},\\ \|\varphi\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S}))} &\leq \overline{C}\|\zeta\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\mathcal{S}))}.\end{aligned}$$

The estimate which deals with the time-derivative of ϕ can be obtained easily. Indeed, taking the inner scalar product of the equality

$$\mu\phi_t - \Delta\phi_t = F(\phi_t) + \mathrm{w}_t$$

by ϕ_t, we notice that the contribution of the right-hand-side force vanishes, as follows

$$\int_{\mathcal{S}} F(\phi_t)\cdot\phi_t = 0,$$

since ϕ_t satisfies the constraints

$$\int_{\mathcal{S}} \phi_t = 0, \qquad \int_{\mathcal{S}} y \wedge \phi_t = 0.$$

By this mean we get easily

$$\|\varphi\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S}))} \leq \hat{C}\|\zeta\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\mathcal{S}))}.$$

3.6.3 Projection of the displacement

Let us consider a control $X^*_\zeta \in \mathcal{W}_\lambda(S^0_\infty)$ which is *admissible* in the sense of Definition 3.2. Such an *admissible* control has been obtained in the previous subsection from a velocity ζ which will be chosen as a feedback operator in order to stabilize the linear part of the main system. Instead of projecting X^*_ζ on a set of controls satisfying the nonlinear constraints, we prefer projecting the displacements $X^*_\zeta - \mathrm{Id}_{\mathcal{S}}$, because we choose the space $\mathcal{W}_\lambda(S^0_\infty)$ as an Hilbertian framework. We denote the displacement by

$$Z^*_\zeta \quad = \quad X^*_\zeta - \mathrm{Id}_{\mathcal{S}}.$$

The goal of this subsection is to define (in a suitable way) a mapping $X^* \in \mathcal{W}_\lambda(S^0_\infty)$ which satisfies the nonlinear constraints. We associate with it the displacement

$$Z^* \quad = \quad X^* - \mathrm{Id}_{\mathcal{S}},$$

so that the wanted mapping is now Z^*. We can decompose such a mapping as follows

$$\begin{aligned} Z^* &= Z^*_\zeta + \left(Z^* - Z^*_\zeta\right) \\ &= Z^*_\zeta + \left(X^* - X^*_\zeta\right). \end{aligned}$$

Let us define the differentiable mapping

$$\begin{array}{rcccl} \mathfrak{F} & : & \mathcal{W}_\lambda(S^0_\infty) & \rightarrow & \mathrm{H}^1_0(0,\infty;\mathbb{R}^d) \times \mathrm{H}^1_0(0,\infty;\mathbb{R}^3) \times \mathrm{H}^1_0(0,\infty;\mathbb{R}) \\ & & Z^* & \mapsto & \mathfrak{F}(Z^*) \end{array}$$

$$\mathfrak{F}(Z^*)(y,t) = \left(\int_{\mathcal{S}} \frac{\partial Z^*}{\partial t}; \ \int_{\mathcal{S}} (Z^* + \mathrm{Id}_{\mathcal{S}}) \wedge \frac{\partial Z^*}{\partial t}; \ \int_{\partial\mathcal{S}} (\mathrm{cof}(\nabla Z^* + \mathrm{I}_{\mathbb{R}^d}))^T \frac{\partial Z^*}{\partial t} \cdot n \right),$$

and the spaces

$$\begin{aligned} \mathcal{E}_{nl} &= \left\{ Z^* \in \mathcal{W}_\lambda(S^0_\infty) \mid \mathfrak{F}(Z^*) = 0 \right\}, \\ \mathcal{E}_{l} &= \left\{ Z^*_\zeta \in \mathcal{W}_\lambda(S^0_\infty) \mid D_0\mathfrak{F}(Z^*_\zeta) = 0 \right\}, \end{aligned}$$

where

$$D_0\mathfrak{F}(Z^*_\zeta) \quad = \quad t \mapsto \left(\int_{\mathcal{S}} \frac{\partial Z^*_\zeta}{\partial t}; \ \int_{\mathcal{S}} \mathrm{Id}_{\mathcal{S}} \wedge \frac{\partial Z^*_\zeta}{\partial t}; \ \int_{\partial\mathcal{S}} \frac{\partial Z^*_\zeta}{\partial t} \cdot n \right).$$

Note that $\mathcal{E}_l$ is a space where lie $\mathrm{Id}_{\mathcal{S}}$ and X^*_ζ. That is why the constraints satisfied by Z^*_ζ and X^*_ζ are the same.
The purpose of this paragraph is to project any displacement $Z^*_\zeta \in \mathcal{E}_l$ on the set $\mathcal{E}_{nl}$, provided that the displacement Z^*_ζ is close enough to 0.

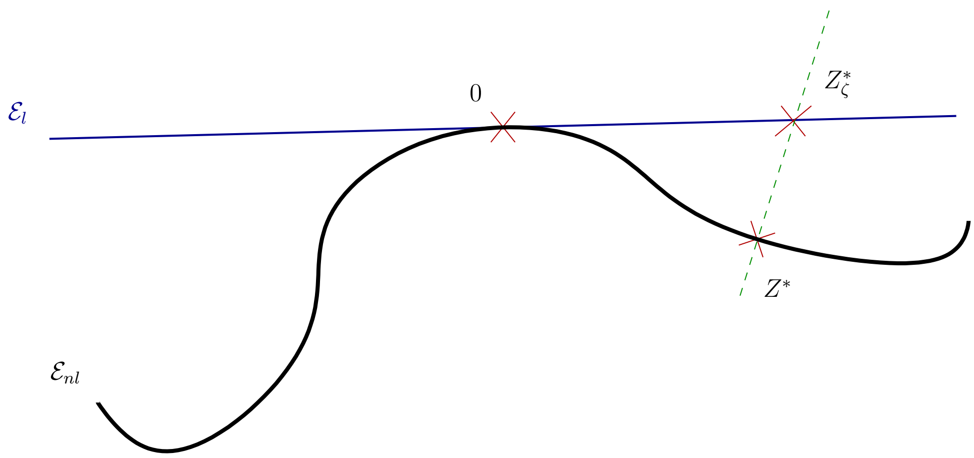

Figure 3.1: Projection of a given mapping on the set of *admissible* deformations

Theorem 3.21. *Let be $Z^*_\zeta \in \mathcal{W}_\lambda(S^0_\infty)$. If Z^*_ζ is small enough in $\mathcal{W}_\lambda(S^0_\infty)$, then there exists a unique mapping $Z^* \in \mathcal{E}_{nl}$ such that*

$$\|Z^* - Z^*_\zeta\|^2_{\mathcal{W}_\lambda(S^0_\infty)} = \min_{\mathcal{Z}^* \in \mathcal{E}_{nl}} \|\mathcal{Z}^* - X^*_\zeta\|^2_{\mathcal{W}_\lambda(S^0_\infty)}.$$

Moreover, we have that

$$\|Z^* - Z^*_\zeta\|_{\mathcal{W}_\lambda(S^0_\infty)} = o\left(\|Z^*_\zeta\|_{\mathcal{W}_\lambda(S^0_\infty)}\right). \tag{3.83}$$

Thus we denote by $\mathcal{P} : Z^*_\zeta \mapsto Z^*$ the projection so obtained.
If the displacements $Z^*_{\zeta_1}$ and $Z^*_{\zeta_2}$ are close enough to 0 in $\mathcal{W}_\lambda(S^0_\infty)$, then

$$\begin{aligned} &\|(Z^*_2 - Z^*_{\zeta_2}) - (Z^*_1 - Z^*_{\zeta_1})\|_{\mathcal{W}_\lambda(S^0_\infty)} \leq \\ &K^*\left(\|Z^*_{\zeta_1}\|_{\mathcal{W}_\lambda(S^0_\infty)} + \|Z^*_{\zeta_2}\|_{\mathcal{W}_\lambda(S^0_\infty)}\right) \times \|Z^*_{\zeta_2} - Z^*_{\zeta_1}\|_{\mathcal{W}_\lambda(S^0_\infty)}, \end{aligned} \tag{3.84}$$

with $Z^*_1 = \mathcal{P}Z^*_{\zeta_1}$ and $Z^*_2 = \mathcal{P}Z^*_{\zeta_2}$, and $K^*(r) \to 0$ when r goes to 0.

Proof. The proof of this theorem is an application of Theorem 3.33 of [Bon06] (page 74), that we state as follows :

Theorem 3.22. *Let $\mathcal{W}$ be a Hilbert space, $\mathcal{R}$ a Banach space, and g a mapping of class C^2 from $\mathcal{W}$ to $\mathcal{R}$, such that $g^{-1}(\{0\}) \neq \emptyset$. Let be $Z^*_\zeta \in \mathcal{W}$, and $Z^*_0 \in g^{-1}(\{0\})$ such that $D_{X^*_0}g$ is surjective. Then there exists $\varepsilon > 0$ such that if $\|Z^*_\zeta - Z^*_0\|_{\mathcal{W}} \leq \varepsilon$, then the following optimization problem under equality constraints*

$$\min_{\mathcal{Z}^* \in \mathcal{W},\ g(\mathcal{Z}^*)=0} \|\mathcal{Z}^* - Z^*_\zeta\|^2_{\mathcal{W}}$$

admits a unique solution Z^*. Moreover, the mapping $Z^*_\zeta \mapsto Z^*$ so obtained is C^1.

In order to apply this theorem with

$$\begin{aligned} \mathcal{W} &= \mathcal{W}_\lambda(S^0_\infty), & \mathcal{R} &= \mathrm{H}^1_0(0,\infty;\mathbb{R}^d) \times \mathrm{H}^1_0(0,\infty;\mathbb{R}^3) \times \mathrm{H}^1_0(0,\infty;\mathbb{R}), \\ g &= \mathfrak{F}, & Z^*_0 &= 0, \end{aligned}$$

the only nontrivial assumption to be verified is that the mapping $D_0\mathfrak{F}$ is surjective. For that, let us consider

$$(a,b,c) \quad \in \quad \mathrm{H}^1_0(0,\infty;\mathbb{R}^d)\times\mathrm{H}^1_0(0,\infty;\mathbb{R}^3)\times\mathrm{H}^1_0(0,\infty;\mathbb{R}).$$

An antecedent $\mathcal{Z}^*$ of this triplet (a,b,c) can be obtained as

$$\mathcal{Z}^*(y,t) \quad = \quad \int_0^t e^{-\lambda s}\varphi(y,s)\mathrm{d}s,$$

where φ is the solution of the following system

$$\begin{aligned} \mu\varphi - 2\mathrm{div}\ D(\varphi) = F(\varphi) + F_{a,b} \qquad & \text{in } \mathcal{S}, \\ \varphi = \frac{c}{|\partial\mathcal{S}|}n \qquad & \text{on } \partial\mathcal{S}, \end{aligned}$$

for $\mu > 0$ large enough, with

$$\begin{aligned} F_{a,b}(y,t) \quad &= \quad \frac{\rho_{\mathcal{S}}}{M}a(t) + \rho_{\mathcal{S}}\left(I_0^{-1}b(t)\right)\wedge y, \\ F(\mathcal{Z}^*)(y,t) \quad &= \quad \frac{\rho_{\mathcal{S}}}{M}\left(\int_{\partial\mathcal{S}} 2D(\mathcal{Z}^*)n\mathrm{d}\Gamma\right) + \left(I_0^{-1}\int_{\partial\mathcal{S}}\mathrm{Id}_{\mathcal{S}}\wedge 2D(\mathcal{Z}^*)n\mathrm{d}\Gamma\right)\wedge y. \end{aligned}$$

The previous study (given in subsection 3.6.2) can be straightforwardly adapted to get the existence of a solution in $\varphi \in \mathcal{H}_3(S^0_\infty)$ for such a system, and thus a displacement $\mathcal{Z}^* \in \mathcal{W}_\lambda(S^0_\infty)$. Since the projection $\mathcal{P} : Z^*_\zeta \mapsto Z^*$ so obtained (for Z^*_ζ close to 0) is C^1, we can notice that its differential at 0 is the identity $\mathrm{Id}_{\mathcal{W}_\lambda(S^0_\infty)}$, and thus a Taylor development shows that

$$\begin{aligned} &Z^* = Z^*_\zeta + o\left(\|Z^*_\zeta\|_{\mathcal{W}_\lambda(S^0_\infty)}\right), \\ &\|Z^* - Z^*_\zeta\|_{\mathcal{W}_\lambda(S^0_\infty)} = o\left(\|Z^*_\zeta\|_{\mathcal{W}_\lambda(S^0_\infty)}\right). \end{aligned}$$

For $Z^*_{\zeta_1}$ and $Z^*_{\zeta_2}$ close to 0, the estimate (3.84) is obtained in considering a Taylor development around $Z^*_{\zeta_1}$ for the mapping $\mathcal{P} - \mathrm{Id}_{\mathcal{W}_\lambda(S^0_\infty)}$:

$$\begin{aligned} \left(Z^*_2 - Z^*_{\zeta_2}\right) - \left(Z^*_1 - Z^*_{\zeta_1}\right) \quad = \quad & \left[D_{Z^*_{\zeta_1}}\mathcal{P} - \mathrm{Id}_{\mathcal{W}_3(S^0_\infty)}\right]\left(Z^*_{\zeta_2} - Z^*_{\zeta_1}\right) \\ & + o\left(\|Z^*_{\zeta_2} - Z^*_{\zeta_1}\|_{\mathcal{W}_\lambda(S^0_\infty)}\right). \end{aligned}$$

Since $\mathcal{Z}^* \mapsto D_{\mathcal{Z}^*}\mathcal{P}$ is continuous at 0, we have

$$D_{Z^*_{\zeta_1}}\mathcal{P} - \mathrm{Id}_{\mathcal{W}_\lambda(S^0_\infty)} \to 0 \qquad \text{when } \|Z^*_{\zeta_1}\|_{\mathcal{W}_\lambda(S^0_\infty)} \text{ goes to } 0,$$

and thus we obtain the announced estimate. □

Then, from the displacement $Z^*_\zeta = X^*_\zeta - \mathrm{Id}_{\mathcal{S}}$ we can define a deformation X^* as follows

$$X^* \quad = \quad \mathcal{P}(X^*_\zeta - \mathrm{Id}_{\mathcal{S}}) + \mathrm{Id}_{\mathcal{S}}.$$

This deformation is *admissible* in the sense of Definition 3.4.
The interest of such a decomposition (that is to say $X^* = X^*_\zeta + (X^* - X^*_\zeta)$ with X^* given by Theorem 3.21) lies in the fact that the *admissible* - in the sense of Definition 3.4 - control X^* so decomposed will enable us to stabilize the nonhomogeneous linear part of system (3.3)–(3.9)

thanks to the term X^*_ζ (see the previous section), whereas the residual term $(X^* - X^*_\zeta)$ satisfies the property (3.83), which leads to

$$\|X^* - X^*_\zeta\|_{\mathcal{W}_\lambda(S^0_\infty)} = o\left(\|X^*_\zeta - \mathrm{Id}_\mathcal{S}\|_{\mathcal{W}_\lambda(S^0_\infty)}\right), \quad (3.85)$$

$$\left\| e^{\lambda t}\frac{\partial X^*}{\partial t} - e^{\lambda t}\frac{\partial X^*_\zeta}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))} = o\left(\|X^*_\zeta - \mathrm{Id}_\mathcal{S}\|_{\mathcal{W}_\lambda(S^0_\infty)}\right). \quad (3.86)$$

With regards to X^*_ζ which satisfies only the linearized constraints, $(X^* - X^*_\zeta)$ can be seen as a corrective term which enables X^* to satisfy the nonlinear constraints that all *admissible* control of system (3.3)–(3.9) must satisfy.
Lastly, the estimate (3.84) is reformulated as follows

$$\begin{aligned}&\|(X^*_2 - X^*_{\zeta_2}) - (X^*_1 - X^*_{\zeta_1})\|_{\mathcal{W}_\lambda(S^0_\infty)} \leq \\ &K^*\left(\|X^*_{\zeta_1} - \mathrm{Id}_\mathcal{S}\|_{\mathcal{W}_\lambda(S^0_\infty)} + \|X^*_{\zeta_2} - \mathrm{Id}_\mathcal{S}\|_{\mathcal{W}_\lambda(S^0_\infty)}\right) \times \|X^*_{\zeta_2} - X^*_{\zeta_1}\|_{\mathcal{W}_\lambda(S^0_\infty)},\end{aligned} \quad (3.87)$$

$$\begin{aligned}&\left\| e^{\lambda t}\frac{\partial(X^*_2 - X^*_{\zeta_2})}{\partial t} - e^{\lambda t}\frac{\partial(X^*_1 - X^*_{\zeta_1})}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))} \leq \\ &K^*\left(\|X^*_{\zeta_1} - \mathrm{Id}_\mathcal{S}\|_{\mathcal{W}_\lambda(S^0_\infty)} + \|X^*_{\zeta_2} - \mathrm{Id}_\mathcal{S}\|_{\mathcal{W}_\lambda(S^0_\infty)}\right) \times \|X^*_{\zeta_2} - X^*_{\zeta_1}\|_{\mathcal{W}_\lambda(S^0_\infty)},\end{aligned} \quad (3.88)$$

where $K^*(r) \to 0$ when r goes to 0, and where

$$\begin{aligned}X^*_1 &= \mathcal{P}(X^*_{\zeta_1} - \mathrm{Id}_\mathcal{S}) + \mathrm{Id}_\mathcal{S},\\ X^*_2 &= \mathcal{P}(X^*_{\zeta_2} - \mathrm{Id}_\mathcal{S}) + \mathrm{Id}_\mathcal{S}.\end{aligned}$$

3.7 Stabilization of a nonhomogeneous nonlinear system, by admissible deformations

Let be $\lambda > 0$. Let us consider the following nonhomogeneous nonlinear system:

$$\frac{\partial U}{\partial t} - \lambda U - \nu\Delta U + \nabla P = \mathbb{F}, \qquad \text{in } \mathcal{F}\times(0,\infty), \quad (3.89)$$

$$\mathrm{div}\ U = \mathrm{div}\ \mathbb{G}, \qquad \text{in } \mathcal{F}\times(0,\infty), \quad (3.90)$$

$$U = 0, \qquad \text{in } \partial\mathcal{O}\times(0,\infty), \quad (3.91)$$

$$U = H'(t) + \Omega(t)\wedge y + e^{\lambda t}\frac{\partial X^*}{\partial t}(y,t) + \mathbb{W}(y,t), \qquad \text{in } \partial\mathcal{S}\times(0,\infty), \quad (3.92)$$

$$MH''(t) - \lambda MH'(t) = -\int_{\partial\mathcal{S}} \sigma(U,P)n\mathrm{d}\Gamma + \mathbb{F}_M, \quad t\in(0,\infty), \quad (3.93)$$

$$I_0\Omega'(t) - \lambda I_0\Omega(t) = -\int_{\partial\mathcal{S}} y\wedge\sigma(U,P)n\mathrm{d}\Gamma + \mathbb{F}_I, \quad t\in(0,\infty), \quad (3.94)$$

$$U(y,0) = u_0(y),\ y\in\mathcal{F}, \quad H'(0) = h_1\in\mathbb{R}^d, \quad \Omega(0) = \omega_0\in\mathbb{R}^3, \quad (3.95)$$

where

$$X^* = \mathcal{P}(X^*_\zeta - \mathrm{Id}_\mathcal{S}) + \mathrm{Id}_\mathcal{S}, \tag{3.96}$$

(see the subsection 3.6.3) and where $X^*_\zeta - \mathrm{Id}_\mathcal{S}$ is a displacement obtained such that

$$e^{\lambda t}\frac{\partial X^*_\zeta}{\partial t}_{|\mathcal{S}} = \zeta = \mathcal{K}_\lambda(U - \mathbb{G}, H', \Omega), \tag{3.97}$$

$$\tag{3.98}$$

as it is explained in the subsection 3.6.1. In this system the right-hand-sides $\mathbb{F}$, $\mathbb{G}$, $\mathbb{W}$, $\mathbb{F}_M$ and $\mathbb{F}_I$ are given, and where X^* is an *admissible* deformation, in the sense of Definition 3.4.
Then in equation (3.92) we replace

$$e^{\lambda t}\frac{\partial X^*}{\partial t} = \zeta + e^{\lambda t}\left(\frac{\partial X^*}{\partial t} - \frac{\partial X^*_\zeta}{\partial t}\right),$$

with

$$\zeta = \mathcal{K}_\lambda(U - \mathbb{G}, H', \Omega)$$

and where the estimate (3.86) combined to (3.79) gives

$$\left\| e^{\lambda t}\frac{\partial X^*}{\partial t} - e^{\lambda t}\frac{\partial X^*_\zeta}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))} = o\left(\|\zeta\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap \mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))}\right).$$

We then obtain a closed-loop system, where the chosen deformation X^* is expressed in terms of a feedback law, in order to shift the spectrum of the operator $\mathcal{A}_\lambda$, and thus to transform it into an operator $\mathcal{A}_\lambda + \mathcal{B}_\lambda\mathcal{K}_\lambda$ that generates a semigroup of contraction.

Proposition 3.23. *Let be $\lambda > 0$. Assume that the following data are small enough, that is to say*

$$\begin{aligned}
(u_0, h_1, \omega_0) &\in \mathbf{H}^1_{cc}, \\
\mathbb{F} &\in \mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F})), \\
\mathbb{F}_M &\in \mathrm{L}^2(0,\infty;\mathbb{R}^d), \\
\mathbb{F}_I &\in \mathrm{L}^2(0,\infty;\mathbb{R}^3), \\
\mathbb{W} &\in \mathrm{L}^2(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^{-1/2}(\partial\mathcal{S}))
\end{aligned}$$

and $\mathbb{G} \in \mathrm{H}^{2,1}(Q^0_\infty)$ - satisfying $\mathbb{G}_{|\partial\mathcal{O}} = 0$ and the ad-hoc compatibility condition - are small enough. Then system (3.89)–(3.97) admits a unique solution (U, P, H', Ω) in $\mathrm{H}^{2,1}(Q^0_\infty)\times\mathrm{L}^2(0,\infty;\mathrm{H}^1(\mathcal{F}))\times \mathrm{H}^1(0,\infty;\mathbb{R}^d)\times\mathrm{H}^1(0,\infty;\mathbb{R}^3)$.
Moreover, there exists a constant $C > 0$ such that

$$\begin{aligned}
&\|U\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|\nabla P\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathcal{F}))} + \|H'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} + \|\Omega\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)} \leq \\
&C\Big(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3} + \|\mathbb{G}\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|\mathbb{W}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{-1/2}(\partial\mathcal{S}))} \\
&+ \|\mathbb{F}\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \|\mathbb{F}_M\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)} + \|\mathbb{F}_I\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)}\Big).
\end{aligned} \tag{3.99}$$

Proof. Let us set $\overline{U} := U - \mathbb{G}$ to cope with the nonhomogeneous divergence condition. The system so obtained deals with $(\overline{U}, P, H', \Omega)$ and has got some additional nonhomogeneous terms due to $\mathbb{G}$. The quadruplet $(\overline{U}, P, H', \Omega)$ satisfies

$$\frac{\partial \overline{U}}{\partial t} - \lambda \overline{U} - \nu \Delta \overline{U} + \nabla P = \mathbb{F} + \mathbb{F}_{\mathbb{G}}, \qquad \text{in } \mathcal{F} \times (0,\infty), \tag{3.100}$$

$$\text{div } \overline{U} = 0, \qquad \text{in } \mathcal{F} \times (0,\infty), \tag{3.101}$$

$$\overline{U} = 0, \qquad \text{in } \partial\mathcal{O} \times (0,\infty), \tag{3.102}$$

$$\begin{aligned}\overline{U} = H'(t) + \Omega(t) \wedge y + \mathcal{K}(\overline{U}, H', \Omega)(y,t) \\ + e^{\lambda t}\frac{\partial (X^* - X^*_\zeta)}{\partial t} + \mathbb{W} - \mathbb{G}, \qquad (y,t) \in \partial\mathcal{S} \times (0,\infty),\end{aligned} \tag{3.103}$$

$$MH'' - \lambda M H' = -\int_{\partial\mathcal{S}} \sigma(\overline{U}, P) n \mathrm{d}\Gamma + \mathbb{F}_M + \mathbb{F}_{M,\mathbb{G}} \quad \text{in } (0,\infty), \tag{3.104}$$

$$I_0 \Omega'(t) - \lambda I_0 \Omega(t) = -\int_{\partial\mathcal{S}} y \wedge \sigma(\overline{U}, P) n \mathrm{d}\Gamma + \mathbb{F}_I + \mathbb{F}_{I,\mathbb{G}} \quad \text{in } (0,\infty), \tag{3.105}$$

$$V(y,0) = u_0(y) - \mathbb{G}(y,0), \ y \in \mathcal{F}, \quad H'(0) = h_1 \in \mathbb{R}^d, \quad \Omega(0) = \omega_0 \in \mathbb{R}^3, \tag{3.106}$$

with

$$\begin{aligned}\mathbb{F}_{\mathbb{G}} &= -\frac{\partial \mathbb{G}}{\partial t} + \lambda \mathbb{G} + \nu \Delta \mathbb{G}, \\ \mathbb{F}_{M,\mathbb{G}} &= -2\nu \int_{\partial\mathcal{S}} D(\mathbb{G}) n \mathrm{d}\Gamma, \\ \mathbb{F}_{I,\mathbb{G}} &= -2\nu \int_{\partial\mathcal{S}} y \wedge D(\mathbb{G}) n \mathrm{d}\Gamma.\end{aligned}$$

System (3.100)–(3.106) can be formally rewritten

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = (\mathcal{A}_\lambda + \mathcal{B}_\lambda \mathcal{K}_\lambda)\mathbf{U} + \mathbf{F} + \mathcal{B}_\lambda(\mathbb{W} - \mathbb{G}) + \mathcal{B}_\lambda\left(e^{\lambda t}\frac{\partial (X^*(\mathbf{U}) - X^*_\zeta(\mathbf{U}))}{\partial t}\right),$$

with $\mathbf{U} = \left(\mathbb{P}\overline{U}, H', \Omega\right)^T$,

$$\mathbf{F} = \begin{pmatrix} \mathbb{F} + \mathbb{F}_{\mathbb{G}} \\ \mathbb{F}_M + \mathbb{F}_{M,\mathbb{G}} \\ \mathbb{F}_I + \mathbb{F}_{I,\mathbb{G}} \end{pmatrix}.$$

w, and $e^{\lambda t}\dfrac{\partial (X^*(\mathbf{U}) - X^*_\zeta(\mathbf{U}))}{\partial t}$ is entirely defined by $\zeta = \mathcal{K}_\lambda(\mathbf{U})$; More precisely, in this equality the mapping X^*_ζ is chosen such that $e^{\lambda t}\dfrac{\partial X^*_\zeta}{\partial t}$ is an extension to $\mathcal{S}$ of the function $\zeta = \mathcal{K}_\lambda \mathbf{U}$ (see the subsection 3.6.1), and $X^* - \mathrm{Id}_{\mathcal{S}}$ is defined as the projection of $X^*_\zeta - \mathrm{Id}_{\mathcal{S}}$ by $\mathcal{P}$ (see the subsection 3.6.3).
A solution of this system can be seen as a fixed point of the mapping

$$\mathcal{N} : \mathbf{V} \longrightarrow \mathbf{U},$$

where

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = (\mathcal{A}_\lambda + \mathcal{B}_\lambda\mathcal{K}_\lambda)\,\mathbf{U} + \mathbf{F} + \mathcal{B}_\lambda\,(\mathbb{W} - \mathbb{G}) + \mathcal{B}_\lambda\left(e^{\lambda t}\frac{\partial(X^*(\mathbf{V}) - X^*_\zeta(\mathbf{V}))}{\partial t}\right),$$

and where $(X^*(\mathbf{V}) - X^*_\zeta(\mathbf{V}))$ is entirely defined by $\mathcal{K}_\lambda\mathbf{V}$; More precisely, in this equality the mapping X^*_ζ is chosen such that $e^{\lambda t}\dfrac{\partial X^*_\zeta}{\partial t}$ is an extension to $\mathcal{S}$ of the function $\zeta_{\mathbf{V}} = \mathcal{K}_\lambda\mathbf{V}$ (see the subsection 3.6.1), and $X^* - \mathrm{Id}_{\mathcal{S}}$ is defined as the projection of $X^*_\zeta - \mathrm{Id}_{\mathcal{S}}$ by $\mathcal{P}$ (see the subsection 3.6.3).
There exists a positive constant C such that

$$\begin{aligned}
&\|\overline{U}\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|H'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} + \|\Omega\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)} \leq \\
&C\left(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3} + \|\mathbb{G}(\cdot,0)\|_{\mathbf{H}^1(\mathcal{F})}\right. \qquad (3.107)\\
&+ \left\|e^{\lambda t}\frac{\partial(X^*(\mathbf{V}) - X^*_\zeta(\mathbf{V}))}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))} \\
&+ \|\mathbb{F}\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \|\mathbb{F}_\mathbb{G}\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \|\mathbb{G}\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|\mathbb{W}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{-1/2}(\partial\mathcal{S}))} \\
&\left.+ \|\mathbb{F}_M\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)} + \|\mathbb{F}_{M,\mathbb{G}}\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)} + \|\mathbb{F}_I\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)} + \|\mathbb{F}_{I,\mathbb{G}}\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)}\right). \qquad (3.108)
\end{aligned}$$

The estimates (3.86) and (3.88) combined to (3.79) and (3.80) respectively enable us to see that for small data the mapping $\mathcal{N}$ is a contraction in the ball

$$B = \left\{\|\overline{U}\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|H'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} + \|\Omega\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)} \leq 2C\right\},$$

where the constant C appears in the estimate (3.108).
Thus, system (3.100)–(3.106) admits a unique solution $(\overline{U}, P, H', \Omega)$ in $\mathrm{H}^{2,1}(Q^0_\infty)\times\mathrm{L}^2(0,\infty;\mathrm{H}^1(\mathcal{F}))\times \mathrm{H}^1(0,\infty;\mathbb{R}^d)\times\mathrm{H}^1(0,\infty;\mathbb{R}^3)$, and there exists a positive constant C_2 such that

$$\begin{aligned}
&\|\overline{U}\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|\nabla P\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathcal{F}))} + \|H'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} + \|\Omega\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)} \leq \\
&C_2\left(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3} + \|\mathbb{G}(\cdot,0)\|_{\mathbf{H}^1(\mathcal{F})}\right.\\
&+ \|\mathbb{F}\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \|\mathbb{F}_\mathbb{G}\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \|\mathbb{G}\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|\mathbb{W}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{-1/2}(\partial\mathcal{S}))} \\
&\left.+ \|\mathbb{F}_M\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)} + \|\mathbb{F}_{M,\mathbb{G}}\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)} + \|\mathbb{F}_I\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)} + \|\mathbb{F}_{I,\mathbb{G}}\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)}\right). \qquad (3.109)
\end{aligned}$$

Then, the desired estimate (3.99) is obtained in combining the estimates (3.109) to the following ones

$$\begin{aligned}
\|U\|_{\mathrm{H}^{2,1}(Q^0_\infty)} &\leq \|\overline{U}\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|\mathbb{G}\|_{\mathrm{H}^{2,1}(Q^0_\infty)}, \\
\|\nabla P\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} &\leq C\left(\|U\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|\mathbb{F}\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}\right).
\end{aligned}$$

□

3.8 Stabilization of system (3.3)–(3.9)

3.8.1 Back to the nonlinear system written in a cylindrical domain

System (3.3)–(3.9) is transformed into system (3.45)–(3.51), so that proving the stabilization to zero, with an arbitrary exponential decay rate $\lambda > 0$, of system (3.3)–(3.9) is equivalent to

proving the wellposedness of system (3.45)–(3.51) for all $\lambda > 0$, for some well-chosen deformation X^*.
In system (3.45)–(3.51), the mapping X^* is *admissible* (in the sense of Definition 3.2). It has to be chosen also in order to stabilize the linear part of this system. For that, we use the decomposition provided by Theorem 3.21, as follows

$$\begin{aligned}
e^{\lambda t}\frac{\partial X^*}{\partial t} &= e^{\lambda t}\frac{\partial X^*_\zeta}{\partial t} + e^{\lambda t}\frac{\partial}{\partial t}\left(\mathcal{P}(X^*_\zeta - \mathrm{Id}_\mathcal{S}) - (X^*_\zeta - \mathrm{Id}_\mathcal{S})\right), \\
e^{\lambda t}\frac{\partial X^*}{\partial t} &= e^{\lambda t}\frac{\partial X^*_\zeta}{\partial t} + e^{\lambda t}\frac{\partial}{\partial t}\left(X^* - X^*_\zeta\right),
\end{aligned}$$

where $e^{\lambda t}\dfrac{\partial X^*_\zeta}{\partial t} = \breve{\zeta}$ is chosen as a the extension to $\mathcal{S}$ of a boundary feedback operator $\zeta = \mathcal{K}_\lambda(\hat{u} - G(\hat{u}, \hat{h}', \hat{\omega}), \hat{h}', \hat{\omega})$ (see the subsection 3.6.1), and it depends on the unknowns. This term will enable to stabilize the linear part of the system, whereas the residual term $X^* - X^*_\zeta$ satisfies the properties (3.86) and (3.88).
Let us then rewrite system (3.45)–(3.51) as

$$\begin{aligned}
\frac{\partial \hat{u}}{\partial t} - \lambda\hat{u} - \nu\Delta\hat{u} + \nabla\hat{p} &= F(\hat{u}, \hat{p}, \hat{h}', \hat{\omega}), && \text{in } \mathcal{F}\times(0,\infty), \\
\mathrm{div}\ \hat{u} &= \mathrm{div}\ G(\hat{u}, \hat{h}', \hat{\omega}), && \text{in } \mathcal{F}\times(0,\infty),
\end{aligned}$$

$$\begin{aligned}
\hat{u} &= 0, && \text{on } \partial\mathcal{O}\times(0,\infty), \\
\hat{u} = \hat{h}'(t) + \hat{\omega}(t)\wedge y &+ \mathcal{K}_\lambda(\hat{u} - G(\hat{u}, \hat{h}', \hat{\omega}), \hat{h}', \hat{\omega}) && \\
&+ e^{\lambda t}\frac{\partial(X^* - X^*_\zeta)}{\partial t} + W(\hat{u}, \hat{h}', \hat{\omega}), && (y,t)\in\partial\mathcal{S}\times(0,\infty),
\end{aligned}$$

$$\begin{aligned}
M\hat{h}'' - \lambda M\hat{h}' &= -\int_{\partial\mathcal{S}} \sigma(\hat{u}, \hat{p})n\mathrm{d}\Gamma + F_M(\hat{u}, \hat{p}, \hat{h}', \hat{\omega}), && \text{in } (0,\infty), \\
I_0\hat{\omega}'(t) - \lambda I_0\hat{\omega} &= -\int_{\partial\mathcal{S}} y\wedge\sigma(\hat{u}, \hat{p})n\mathrm{d}\Gamma + F_I(\hat{u}, \hat{p}, \hat{\omega}), && \text{in } (0,\infty),
\end{aligned}$$

$$\hat{u}(y,0) = u_0(y),\ y\in\mathcal{F}, \quad \hat{h}'(0) = h_1\in\mathbb{R}^d, \quad \hat{\omega}(0) = \omega_0\in\mathbb{R}^3,$$

with

$$
\begin{aligned}
F(\hat{u},\hat{p},\hat{h}',\hat{\omega}) &= \nu(\mathbf{L}-\Delta)\hat{u} - e^{-\lambda t}\mathbf{M}(\hat{u},\hat{h}',\hat{\omega}) - e^{-\lambda t}\mathbf{N}\hat{u} - (\mathbf{G}-\nabla)\hat{p} - e^{-\lambda t}\hat{\omega}\wedge\hat{u}, && (3.110)\\
G(\hat{u},\hat{h}',\hat{\omega}) &= \left(\mathrm{I}_{\mathbb{R}^d} - \nabla\tilde{Y}(\tilde{X}(y,t),t)\right)\hat{u}, && (3.111)\\
W(\hat{\omega}) &= \hat{\omega}\wedge(X^* - \mathrm{Id}), && (3.112)\\
F_M(\hat{u},\hat{p},\hat{h}',\hat{\omega}) &= -Me^{-\lambda t}\hat{\omega}\wedge\hat{h}'(t)\\
&\quad -\nu\int_{\partial\mathcal{S}}\left(\nabla\hat{u}\left(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d}\right)+\left(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d}\right)^T\nabla\hat{u}^T\right)\nabla\tilde{Y}(\tilde{X})^T n\mathrm{d}\Gamma\\
&\quad -\int_{\partial\mathcal{S}}\sigma(\hat{u},\hat{p})\left(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d}\right)^T n\mathrm{d}\Gamma, && (3.113)\\
F_I(\hat{u},\hat{p},\hat{\omega}) &= -(I^*-I_0)\hat{\omega}' + \lambda(I^*-I_0)\hat{\omega} - {I^*}'\hat{\omega} + e^{-\lambda t}I^*\hat{\omega}\wedge\hat{\omega}\\
&\quad -\nu\int_{\partial\mathcal{S}} y\wedge\left(\nabla\hat{u}\left(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d}\right)+(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d})^T\nabla\hat{u}^T\right)\nabla\tilde{Y}(\tilde{X})^T n\mathrm{d}\Gamma\\
&\quad -\int_{\partial\mathcal{S}} y\wedge\left(\sigma(\hat{u},\hat{p})(\nabla\tilde{Y}(\tilde{X})-\mathrm{I}_{\mathbb{R}^d})n\right)^T\mathrm{d}\Gamma\\
&\quad +\int_{\partial\mathcal{S}}(X^*-\mathrm{Id})\wedge\left(\tilde{\sigma}(\hat{u},\hat{p})\nabla\tilde{Y}(\tilde{X})^T n\right)\mathrm{d}\Gamma. && (3.114)
\end{aligned}
$$

Note that the mappings $\tilde{X}$ and $\tilde{Y}$ depend on the deformation X^* and also on the unknowns $\hat{h}'$ and $\hat{\omega}$.

3.8.2 Statement

Theorem 3.24. *For (u_0,h_1,ω_0) small enough in $\mathbf{H}^1_{cc}$, system (3.3)–(3.9) is stabilizable with an arbitrary exponential decay rate $\lambda>0$, in the sense of Definition 3.6.*

With regards to the changes of unknowns made in (3.30)–(3.31) and (3.44), since we have

$$\|(\hat{u},\hat{h}',\hat{\omega})\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^d\times\mathbb{R}^3} = \|e^{\lambda t}(\tilde{u},\tilde{h}',\tilde{\omega})\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^d\times\mathbb{R}^3} = \|e^{\lambda t}(u,h',\omega)\|_{\mathbf{L}^2(\mathcal{F}(t))\times\mathbb{R}^d\times\mathbb{R}^3},$$

proving the theorem above is equivalent to proving that for (u_0,h_1,ω_0) small enough in $\mathbf{H}^1_{cc}$ the solution of system (3.45)–(3.51) satisfies

$$\|(\hat{u},\hat{h}',\hat{\omega})\|_{\mathbf{L}^2(\mathcal{F})\times\mathbb{R}^d\times\mathbb{R}^3} < \infty.$$

Then the Datko theorem will enable us to conclude.

3.8.3 Proof of Theorem 3.24

Let us set

$$\mathbb{H} = \mathrm{H}^{2,1}(Q^0_\infty)\times\mathrm{L}^2(0,\infty;\mathrm{H}^1(\mathcal{F}))\times\mathrm{H}^1(0,\infty;\mathbb{R}^d)\times\mathrm{H}^1(0,\infty;\mathbb{R}^3).$$

A solution of system (3.45)–(3.51) can be seen as a fixed point of the mapping

$$
\begin{array}{cccc}
\mathcal{N}: & \mathbb{H} & \to & \mathbb{H}\\
& (\hat{v},\hat{q},\hat{k}',\hat{\varpi}) & \mapsto & (\hat{u},\hat{p},\hat{h}',\hat{\omega})
\end{array}
$$

where $(\hat{u}, \hat{p}, \hat{h}', \hat{\omega})$ satisfies

$$\frac{\partial \hat{u}}{\partial t} - \lambda \hat{u} - \nu \Delta \hat{u} + \nabla \hat{p} = F(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi}) \qquad \text{in } \mathcal{F} \times (0, \infty),$$
$$\text{div } \hat{u} = \text{div } G(\hat{v}, \hat{k}', \hat{\varpi}) \qquad \text{in } \mathcal{F} \times (0, \infty),$$

$$\hat{u} = 0 \qquad \text{on } \partial\mathcal{O} \times (0, \infty),$$
$$\hat{u} = \hat{h}'(t) + \hat{\omega}(t) \wedge y + e^{\lambda t} \frac{\partial X^*_{\text{dec}}}{\partial t} + W(\hat{v}, \hat{k}', \hat{\varpi}) \qquad \text{on } \partial\mathcal{S} \times (0, \infty),$$

$$M \hat{h}'' - \lambda M \hat{h}' = - \int_{\partial \mathcal{S}} \sigma(\hat{u}, \hat{p}) n \mathrm{d}\Gamma + F_M(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi}) \qquad \text{in } (0, \infty),$$
$$I_0 \hat{\omega}'(t) - \lambda I_0 \hat{\omega} = - \int_{\partial \mathcal{S}} y \wedge \sigma(\hat{u}, \hat{p}) n \mathrm{d}\Gamma + F_I(\hat{v}, \hat{k}', \hat{\varpi}) \qquad \text{in } (0, \infty),$$

$$\hat{u}(y, 0) = u_0(y), \ y \in \mathcal{F}, \quad \hat{h}'(0) = h_1 \in \mathbb{R}^d, \quad \hat{\omega}(0) = \omega_0 \in \mathbb{R}^3,$$

with

$$\begin{aligned} e^{\lambda t} \frac{\partial X^*_{\text{dec}}}{\partial t} &= \breve{\zeta} + e^{\lambda t} \frac{\partial}{\partial t} \Big(\mathcal{P} \left(X^*_\zeta - \text{Id}_\mathcal{S} \right) - (X^*_\zeta - \text{Id}_\mathcal{S}) \Big), \\ \breve{\zeta}_{|\partial \mathcal{S}} &= \mathcal{K}_\lambda (\hat{u} - G(\hat{v}, \hat{k}', \hat{\varpi}), \hat{h}', \hat{\omega}) = \zeta, \end{aligned}$$

and X^*_ζ is the extension to $\mathcal{S}$ (see the subsection 3.6.1) such that

$$e^{\lambda t} \frac{\partial X^*_\zeta}{\partial t}_{|\partial \mathcal{S}} = \mathcal{K}_\lambda (\hat{v} - G(\hat{v}, \hat{k}', \hat{\varpi}), \hat{k}', \hat{\varpi}).$$

Notice that in that decomposition of the deformation X^*, only the term $\breve{\zeta}$ depends on the unknowns $(\hat{u}, \hat{h}', \hat{\omega})$; It is defined as an extension to $\mathcal{S}$ of the boundary control ζ (see the subsection 3.6.1), chosen in a feedback operator form.
The remaining term depends exclusively on the data $(\hat{v}, \hat{k}', \hat{\varpi})$; In particular, the displacement $\mathcal{P}(X^*_\zeta - \text{Id}_\mathcal{S})$ induces a solid's deformation X^* which defines (with $\hat{k}'$ and $\hat{\varpi}$) the mapping $\tilde{X}$. This mapping $\tilde{X}$ appears in the expressions of the right-hand-sides $F(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi})$, $G(\hat{v}, \hat{k}', \hat{\varpi})$, $W(\hat{v}, \hat{k}', \hat{\varpi})$, $F_M(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi})$ and $F_I(\hat{v}, \hat{k}', \hat{\varpi})$.
Then the proof of the estimate (3.99) of Theorem 3.23 can be applied in that case, and we have

$$\begin{aligned} & \|\hat{u}\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|\nabla \hat{p}\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathcal{F}))} + \left\| \hat{h}' \right\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} + \|\hat{\omega}\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)} \leq \\ & C_0 \Bigg(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3} + \left\| G(\hat{v}, \hat{k}', \hat{\varpi}) \right\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \left\| W(\hat{v}, \hat{k}', \hat{\varpi}) \right\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S}))} \\ & + \left\| F(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi}) \right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \left\| F_M(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi}) \right\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)} + \left\| F_I(\hat{v}, \hat{k}', \hat{\varpi}) \right\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)} \Bigg). \end{aligned} \tag{3.115}$$

provided that the quantities in the right-hand-side of this inequality are small enough, that is to say that (u_0, h_1, ω_0), $W(\hat{\varpi})$, $G(\hat{v}, \hat{k}', \hat{\varpi})$, $F(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi})$, $F_M(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi})$ and $F_I(\hat{v}, \hat{k}', \hat{\varpi})$ are small enough in $\mathbf{H}^1_{cc}$, $\mathrm{L}^2(0, \infty; \mathbf{H}^{3/2}(\partial\mathcal{S})) \cap \mathrm{H}^1(0, \infty; \mathbf{H}^{-1/2}(\partial\mathcal{S}))$, $\mathrm{H}^{2,1}(Q^0_\infty)$, $\mathrm{L}^2(0, \infty; \mathbf{L}^2(\mathcal{F}))$, $\mathrm{L}^2(0, \infty; \mathbb{R}^d)$ and $\mathrm{L}^2(0, \infty; \mathbb{R}^3)$ respectively.

The expressions of the right-hand-sides F, G, W, F_M and F_I are given by (3.110)–(3.114).

Preliminary estimates

Lemma 3.25. *There exists a positive constant C such that for all $(\hat{u}, \hat{p}, \hat{h}', \hat{\omega})$ in $\mathrm{H}^{2,1}(Q^0_\infty) \times \mathrm{L}^2(0,\infty;\mathrm{H}^1(\mathcal{F})) \times \mathrm{H}^1(0,\infty;\mathbb{R}^d) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3)$ we have*

$$\|(\Delta - \mathbf{L})\,\hat{u}\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \leq C\,\|\hat{u}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} \times \tag{3.116}$$

$$\left(\|\nabla\tilde{Y}(\tilde{X})\nabla\tilde{Y}(\tilde{X})^T - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))} + \|\Delta\tilde{Y}(\tilde{X})\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^1(\mathcal{F}))}\right), \tag{3.117}$$

$$\|(\nabla - \mathbf{G})\hat{p}\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \leq C\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(0,\infty;\mathrm{H}^2(\mathcal{F}))}\,\|\nabla\hat{p}\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}\,. \tag{3.118}$$

Proof. The only delicate point that consists in verifying that $\Delta\tilde{Y}(\tilde{X})$ in $\mathrm{L}^\infty(0,\infty;\mathbf{H}^1(\mathcal{F}))$. For that, let us consider the i-th component of $\Delta\tilde{Y}(\tilde{X})$; We write

$$\Delta\tilde{Y}_i(\tilde{X}(\cdot,t),t) \quad = \quad \mathrm{trace}\left(\nabla^2\tilde{Y}_i(\tilde{X}(\cdot,t),t)\right)$$

with

$$\begin{aligned}\nabla^2\tilde{Y}_i(\tilde{X}(\cdot,t),t) &= \left(\nabla\left(\nabla\tilde{Y}_i(\tilde{X}(\cdot,t),t)\right)\right)\nabla\tilde{Y}(\tilde{X}(\cdot,t),t)\\ &= \left(\nabla\left(\nabla\tilde{Y}_i(\tilde{X}(\cdot,t),t) - \mathrm{I}_{\mathbb{R}^d}\right)\right)\nabla\tilde{Y}(\tilde{X}(\cdot,t),t),\end{aligned}$$

and we apply Lemma 3.29 with $s=1$, $\mu=0$ and $\kappa=1$ to obtain

$$\|\Delta\tilde{Y}_i(\tilde{X}(\cdot,t),t)\|_{\mathbf{H}^1(\mathcal{F})} \quad \leq \quad C\|\nabla\tilde{Y}(\tilde{X}(\cdot,t),t) - \mathrm{I}_{\mathbb{R}^d}\|_{\mathbf{H}^2(\mathcal{F})}\|\nabla\tilde{Y}(\tilde{X}(\cdot,t),t)\|_{\mathbf{H}^2(\mathcal{F})}.$$

□

Lemma 3.26. *There exists a positive constant C such that for all $(\hat{u}, \hat{p}, \hat{h}', \hat{\omega})$ in $\mathrm{H}^{2,1}(Q^0_\infty) \times \mathrm{L}^2(0,\infty;\mathrm{H}^1(\mathcal{F})) \times \mathrm{H}^1(0,\infty;\mathbb{R}^d) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3)$ we have*

$$\begin{aligned}\|\mathbf{M}\hat{u}\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \quad &\leq \quad C\|\hat{u}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}\left(\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))} + 1\right) \times\\ &\left(\|\hat{h}'\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^d)} + \|\hat{\omega}\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^d)}\|\tilde{X}\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^1(\mathcal{F}))} + \left\|\frac{\partial\tilde{X}}{\partial t}\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^1(\mathcal{F}))}\right),\end{aligned} \tag{3.119}$$

$$\begin{aligned}\|\mathbf{N}\hat{u}\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \quad &\leq \quad C\,\|\hat{u}\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^1(\mathcal{F}))}\,\|\hat{u}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} \times\\ &\left(\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))} + 1\right),\end{aligned} \tag{3.120}$$

$$\|\hat{\omega}\wedge\hat{u}\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \quad \leq \quad C\|\hat{\omega}\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)}\|\hat{u}\|_{L^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}. \tag{3.121}$$

Proof. There is no particular difficulty for obtaining these estimates. □

Lemma 3.27. *There exists a positive constant C such that for all $\hat{u} \in \mathrm{H}^{2,1}(Q^0_\infty)$ we have*

$$\begin{aligned}\|G(\hat{u})\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} \quad &\leq \quad C\|\hat{u}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))},\\ \|G(\hat{u})\|_{\mathrm{H}^1(0,\infty;\mathbf{L}^2(\mathcal{F}))} \quad &\leq \quad C\left(\|\hat{u}\|_{\mathrm{H}^1(0,\infty;\mathbf{L}^2(\mathcal{F}))}\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))}\right.\\ &\qquad \left. + \|\hat{u}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{W}^{1,\infty}(0,\infty;\mathbf{L}^2(\mathcal{F}))}\right).\end{aligned}$$

Proof. The quantity $\nabla\tilde{Y}(\tilde{X})$ lies in $\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))$. We apply Lemma 3.29 with $s=2$, $\mu=0$ and $\kappa=0$ in order to get

$$\begin{aligned}\|G(\hat{u})(\cdot,t)\|_{\mathbf{H}^2(\mathcal{F})} \quad &\leq \quad C\,\|\hat{u}\|_{\mathbf{H}^2(\mathcal{F})}\,\|\nabla\tilde{Y}(\tilde{X}(\cdot,t),t) - I\|_{\mathbf{H}^2(\mathcal{F})},\\ \|G(\hat{u})\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} \quad &\leq \quad C\,\|\hat{u}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}\,\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))}.\end{aligned}$$

For proving the regularity $\mathrm{H}^1(0,\infty;\mathbf{L}^2(\mathcal{F}))$, we first write

$$\frac{\partial G(\hat{u})}{\partial t} \quad = \quad \left(\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\right)\frac{\partial \hat{u}}{\partial t} + \left(\frac{\partial}{\partial t}\left(\nabla\tilde{Y}(\tilde{X}) - I\right)\right)\hat{u}.$$

The quantity $\nabla\tilde{Y}(\tilde{X})$ lies in $\mathrm{H}^2(0,\infty;\mathbf{L}^2(\mathcal{F})) \hookrightarrow \mathrm{W}^{1,\infty}(0,\infty;\mathbf{L}^2(\mathcal{F}))$, so that we have the estimates

$$\begin{aligned}\left\|\frac{\partial G(\hat{u})}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \quad &\leq \quad C\left(\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\mathcal{F}))}\left\|\frac{\partial \hat{u}}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))}\right. \\ &\quad \left. + \|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{W^{1,\infty}(0,\infty;\mathbf{L}^2(\mathcal{F}))}\|\hat{u}\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^\infty(\mathcal{F}))}\right).\end{aligned}$$

□

Lemma 3.28. *There exists a positive constant C such that for all $(\hat{u},\hat{p},\hat{h}',\hat{\omega})$ in $\mathrm{H}^{2,1}(Q^0_\infty) \times \mathrm{L}^2(0,\infty;\mathrm{H}^1(\mathcal{F})) \times \mathrm{H}^1(0,\infty;\mathbb{R}^d) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3)$ we have*

$$\|W(\hat{\omega})\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S}))} \leq C\|\hat{\omega}\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)}\|X^* - \mathrm{Id}\|_{\tilde{\mathcal{W}}_3(Q^0_\infty)}, \tag{3.122}$$

$$\begin{aligned}&\left\|F_M(\hat{u},\hat{p},\hat{h}',\hat{\omega})\right\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)} \leq \\ &C\left(\|\hat{h}'\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)}\|\hat{\omega}\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)} + \left(\|\hat{u}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} + \|\hat{p}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^1(\mathcal{F}))}\right)\times\right. \\ &\left.\left(\|\nabla\tilde{Y}(\tilde{X})\|_{L^\infty(\partial\mathcal{S}\times(0,\infty))}\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{R^d}\|_{L^\infty(\partial\mathcal{S}\times(0,\infty))} + \|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{L^\infty(\partial\mathcal{S}\times(0,\infty))}\right)\right),\end{aligned} \tag{3.123}$$

$$\begin{aligned}&\|F_I(\hat{u},\hat{p},\hat{\omega})\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)} \leq \\ &C\left((1+\lambda)\|I^* - I_0\|_{L^\infty(0,\infty;\mathbb{R}^9)}\|\hat{\omega}\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)}\right. \\ &+\|{I^*}'\|_{L^2(0,\infty;\mathbb{R}^9)}\|\hat{\omega}\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^3)} + \|I^*\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^9)}\|\hat{\omega}\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^3)}\|\hat{\omega}\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)} \\ &+\left(\|\hat{u}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} + \|\hat{p}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^1(\mathcal{F}))}\right)\times\left(\|\nabla\tilde{Y}(\tilde{X}) - \mathrm{I}_{\mathbb{R}^d}\|_{L^\infty(\partial\mathcal{S}\times(0,\infty))}\right. \\ &\left.\left.+\|\nabla\tilde{Y}(\tilde{X})\|_{L^\infty(\partial\mathcal{S}\times(0,\infty))}\left(\|\nabla\tilde{Y}(\tilde{X}) - I\|_{L^\infty(\partial\mathcal{S}\times(0,\infty))} + \|X^* - \mathrm{Id}\|_{L^\infty(\partial\mathcal{S}\times(0,\infty))}\right)\right)\right),\end{aligned} \tag{3.124}$$

with

$$\begin{aligned}\|{I^*}'\|_{L^2(0,\infty;\mathbb{R}^9)} \quad &\leq \quad C\|X^*\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^2(\mathcal{S}))}\left\|\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{S}))}, \\ \|I^* - I_0\|_{L^\infty(0,\infty;\mathbb{R}^9)} \quad &\leq \quad C\|X^*\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^2(\mathcal{S}))}\left\|e^{\lambda t}\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{S}))}, \\ \|I^*\|_{L^\infty(0,\infty;\mathbb{R}^9)} \quad &\leq \quad C\|X^*\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^2(\mathcal{S}))}.\end{aligned}$$

Proof. There is no particular difficulty for proving the other two estimates, if we refer to the respective expressions of W, F_M and F_I given by (3.112), (3.113) and (3.114). □

For some radius $R > 0$, let us define the ball

$$\begin{aligned}B_R = \Big\{&(\hat{u},\hat{p},\hat{h}',\hat{\omega}) \in \mathrm{H}^{2,1}(Q^0_\infty) \times \mathrm{L}^2(0,\infty;\mathbf{H}^1(\mathcal{F})) \times \mathrm{H}^1(0,\infty;\mathbb{R}^d) \times \mathrm{H}^1(0,\infty;\mathbb{R}^3) \mid \\ &\|\hat{u}\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + \|\hat{p}\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^1(\mathcal{F}))} + \|\hat{h}'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} + \|\hat{\omega}\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)} \leq 2RC_0\Big\}\end{aligned}$$

with

$$R = \left(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3}\right),$$

and where the constant C_0 appears in the estimate (3.115).

Stability of the set B_R by the mapping $\mathcal{N}$

Let be $(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi}) \in B_R$, for R small enough. In the estimates provided by the previous lemmas 3.25, 3.26, 3.27, 3.28, note that from the estimate (3.139), (3.140), (3.141) and (3.138) combined to (3.85) and (3.79) we can deduce

$$\begin{aligned}
\|F(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi})\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} &= o(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3}),\\
\|G(\hat{v}, \hat{k}', \hat{\varpi})\|_{\mathrm{H}^{2,1}(Q_\infty^0)} &= o(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3}),\\
\|W(\hat{v}, \hat{k}', \hat{\varpi})\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial\mathcal{S}))} &= o(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3}),\\
\|F_M(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi})\|_{\mathrm{L}^2(0,T;\mathbb{R}^d)} &= o(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3}),\\
\|F_I(\hat{v}, \hat{k}', \hat{\varpi})\|_{\mathrm{L}^2(0,T;\mathbb{R}^3)} &= o(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3}).
\end{aligned}$$

In particular, for R small enough we can define the mapping $\mathcal{N}$, and we can write the estimate (3.115) that we recall

$$\begin{aligned}
&\left\|\mathcal{N}(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi})\right\|_{\mathrm{H}^{2,1}(Q_\infty^0)\times\mathrm{L}^2(0,\infty;\mathrm{L}^2(\mathcal{F}))\times\mathrm{H}^1(0,\infty;\mathbb{R}^d)\times\mathrm{H}^1(0,\infty;\mathbb{R}^3)} \leq\\
&C_0\Bigg(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3} \left\|G(\hat{v}, \hat{k}', \hat{\varpi})\right\|_{\mathrm{H}^{2,1}(Q_\infty^0)} + \left\|W(\hat{v}, \hat{k}', \hat{\varpi})\right\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S}))}\\
&\quad + \left\|F(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi})\right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \left\|F_M(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi})\right\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)} + \left\|F_I(\hat{v}, \hat{k}', \hat{\varpi})\right\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)}\Bigg),
\end{aligned}$$

and thus

$$\left\|\mathcal{N}(\hat{v}, \hat{q}, \hat{k}', \hat{\varpi})\right\|_{\mathrm{H}^{2,1}(Q_\infty^0)\times\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}))\times\mathrm{H}^1(0,T;\mathbb{R}^d)\times\mathrm{H}^1(0,T;\mathbb{R}^3)} \leq C_0\left(R + o(R)\right).$$

This shows that for $R = \left(\|u_0\|_{\mathbf{H}^1(\mathcal{F})} + |h_1|_{\mathbb{R}^d} + |\omega_0|_{\mathbb{R}^3}\right)$ small enough the ball B_R is stable by the mapping $\mathcal{N}$.

Lipschitz stability for the mapping $\mathcal{N}$

Let $(\hat{v}_1, \hat{q}_1, \hat{k}'_1, \hat{\varpi}_1)$ and $(\hat{v}_2, \hat{q}_2, \hat{k}'_2, \hat{\varpi}_2)$ be in B_R. We set

$$(\hat{u}_1, \hat{p}_1, \hat{h}'_1, \hat{\omega}_1) = \mathcal{N}(\hat{v}_1, \hat{q}_1, \hat{k}'_1, \hat{\varpi}_1), \qquad (\hat{u}_2, \hat{p}_2, \hat{h}'_2, \hat{\omega}_2) = \mathcal{N}(\hat{v}_2, \hat{q}_2, \hat{k}'_2, \hat{\varpi}_2),$$

and

$$\begin{aligned}
\hat{u} &= \hat{u}_2 - \hat{u}_1, & \hat{p} &= \hat{p}_2 - \hat{p}_1, & \hat{h}' &= \hat{h}'_2 - \hat{h}'_1, & \hat{\omega} &= \hat{\omega}_2 - \hat{\omega}_1,\\
\hat{v} &= \hat{v}_2 - \hat{v}_1, & \hat{q} &= \hat{q}_2 - \hat{q}_1, & \hat{k}' &= \hat{k}'_2 - \hat{k}'_1, & \hat{\varpi} &= \hat{\varpi}_2 - \hat{\varpi}_1.
\end{aligned}$$

The quadruplet $(\hat{u}, \hat{p}, \hat{h}', \hat{\omega})$ satisfies the system

$$\begin{aligned}
\frac{\partial \hat{u}}{\partial t} - \lambda\hat{u} - \nu\Delta\hat{u} + \nabla\hat{p} = \overline{F}, &\qquad \text{in } \mathcal{F}\times(0,\infty),\\
\operatorname{div} \hat{u} = \operatorname{div} \overline{G}, &\qquad \text{in } \mathcal{F}\times(0,\infty),
\end{aligned}$$

$$\hat{u} = 0, \qquad \text{on } \partial\mathcal{O} \times (0,\infty),$$
$$\hat{u} = \hat{h}'(t) + \hat{\omega}(t) \wedge y + \overline{W} + e^{\lambda t}\left(\frac{\partial X^*_{\mathrm{dec},2}}{\partial t} - \frac{\partial X^*_{\mathrm{dec},1}}{\partial t}\right), \qquad (y,t) \in \partial\mathcal{S} \times (0,\infty),$$

$$M\hat{h}'' - \lambda M\hat{h}' = -\int_{\partial\mathcal{S}} \sigma(\hat{u},\hat{p})n\mathrm{d}\Gamma + \overline{F}_M, \qquad \text{in } (0,\infty)$$
$$I_0\hat{\omega}'(t) - \lambda I_0\hat{\omega} = -\int_{\partial\mathcal{S}} y \wedge \sigma(\hat{u},\hat{p})n\mathrm{d}\Gamma + \overline{F}_I, \qquad \text{in } (0,\infty)$$

$$\hat{u}(y,0) = 0, \text{ in } \mathcal{F}, \quad \hat{h}'(0) = 0 \in \mathbb{R}^d, \quad \hat{\omega}(0) = 0 \in \mathbb{R}^3,$$

with

$$\begin{aligned}
\overline{F} &= F(\hat{v}_2,\hat{q}_2,\hat{k}'_2,\hat{\varpi}_2) - F(\hat{v}_1,\hat{q}_1,\hat{k}'_1,\hat{\varpi}_1),\\
\overline{G} &= G(\hat{v}_2,\hat{k}'_2,\hat{\varpi}_2) - G(\hat{v}_1,\hat{k}'_1,\hat{\varpi}_1),\\
\overline{W} &= W(\hat{v}_2,\hat{k}'_2,\hat{\varpi}_2) - W(\hat{v}_1,\hat{k}'_1,\hat{\varpi}_1),\\
\overline{F}_M &= F_M(\hat{v}_2,\hat{q}_2,\hat{k}'_2,\hat{\varpi}_2) - F_M(\hat{v}_1,\hat{q}_1,\hat{k}'_1,\hat{\varpi}_1),\\
\overline{F}_I &= F_I(\hat{v}_2,\hat{q}_2,\hat{k}'_2,\hat{\varpi}_2) - F_I(\hat{v}_1,\hat{q}_1,\hat{k}'_1,\hat{\varpi}_1),
\end{aligned}$$

and

$$\begin{aligned}
e^{\lambda t}\left(\frac{\partial X^*_{\mathrm{dec},2}}{\partial t} - \frac{\partial X^*_{\mathrm{dec},1}}{\partial t}\right) &= \left(\breve{\zeta}_2 - \breve{\zeta}_1\right) + e^{\lambda t}\frac{\partial}{\partial t}\Big(\mathcal{P}Z^*_{\zeta_2} - \mathcal{P}Z^*_{\zeta_1} + (Z^*_{\zeta_2} - Z^*_{\zeta_1})\Big),\\
(\breve{\zeta}_2 - \breve{\zeta}_1)_{|\partial\mathcal{S}} &= \mathcal{K}_\lambda(\hat{u} - \overline{G}, \hat{h}', \hat{\omega}),\\
e^{\lambda t}\frac{\partial}{\partial t}\Big(Z^*_{\zeta_2} - Z^*_{\zeta_1}\Big)_{|\partial\mathcal{S}} &= \mathcal{K}_\lambda(\hat{v} - \overline{G}, \hat{k}', \hat{\varpi}).
\end{aligned}$$

The displacements $Z^*_{\zeta_1}$ and $Z^*_{\zeta_2}$ induce the solid's deformations

$$\begin{aligned}
X^*_1 &= \mathrm{Id}_{\mathcal{S}} + \mathcal{P}Z^*_{\zeta_1},\\
X^*_2 &= \mathrm{Id}_{\mathcal{S}} + \mathcal{P}Z^*_{\zeta_2},
\end{aligned}$$

respectively, which define the mappings $\tilde{X}_1$ and $\tilde{X}_2$ respectively (with the help of $(\hat{k}'_1,\hat{\varpi}_1)$ and $(\hat{k}'_2,\hat{\varpi}_2)$ respectively). These mappings $\tilde{X}_1$ and $\tilde{X}_2$ appears in the expressions of the right-hand-sides.
The right-hand-sides $\overline{F}$, $\overline{G}$, $\overline{W}$, $\overline{F}_M$ and $\overline{F}_I$ can be expressed as quantities which are multiplicative of the differences

$$\begin{aligned}
&\hat{v}, \quad \hat{q}, \quad \hat{k}', \quad \hat{\omega}, \quad (X^*_2 - X^*_1), \quad \left(\tilde{X}_2 - \tilde{X}_1\right),\\
&\left(\nabla\tilde{Y}(\tilde{X})_2 - \nabla\tilde{Y}(\tilde{X})_1\right), \quad \left(\nabla\tilde{Y}(\tilde{X})_2 - \nabla\tilde{Y}(\tilde{X})_1\right), \quad \left(\Delta\tilde{Y}(\tilde{X})_2 - \Delta\tilde{Y}(\tilde{X})_1\right).
\end{aligned}$$

For instance, the nonhomogeneous divergence condition $\overline{G}$ can be written as

$$\overline{G} = \left(\nabla\tilde{Y}_2(\tilde{X}_2) - \nabla\tilde{Y}_1(\tilde{X}_1)\right)\hat{v}_2 + (\nabla\tilde{Y}_1(\tilde{X}_1) - \mathrm{I}_{\mathbb{R}^3})\hat{v}.$$

Then the estimates of Lemmas 3.25, 3.26, 3.27, 3.28 can be adapted for this right-hand-sides, so that the estimates (3.139), (3.140), (3.141) and (3.138) combined to (3.88) and (3.80) enable us to prove that for R small enough we have

$$\begin{array}{rcl}
\|\overline{F}\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} & = & o(\|(\hat{v},\hat{q},\hat{k}',\hat{\varpi})\|_{\mathbb{H}}), \\
\|\overline{G}\|_{\mathrm{H}^{2,1}(Q^0_\infty)} & = & o(\|(\hat{v},\hat{k}',\hat{\varpi})\|_{\mathrm{H}^{2,1}(Q^0_\infty)\times\mathrm{H}^1(0,\infty;\mathbb{R}^d)\times\mathrm{H}^1(0,\infty;\mathbb{R}^3)}), \\
\|\overline{W}\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S}))} & = & o(\|(\hat{v},\hat{k}',\hat{\varpi})\|_{\mathrm{H}^{2,1}(Q^0_\infty)\times\mathrm{H}^1(0,\infty;\mathbb{R}^d)\times\mathrm{H}^1(0,\infty;\mathbb{R}^3)}), \\
\|\overline{F}_M\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)} & = & o(\|(\hat{v},\hat{q},\hat{k}',\hat{\varpi})\|_{\mathbb{H}}), \\
\|\overline{F}_I\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)} & = & o(\|(\hat{v},\hat{k}',\hat{\varpi})\|_{\mathrm{H}^{2,1}(Q^0_\infty)\times\mathrm{H}^1(0,\infty;\mathbb{R}^d)\times\mathrm{H}^1(0,\infty;\mathbb{R}^3)}).
\end{array}$$

Then the estimate (3.99) can be applied for $(\hat{u},\hat{p},\hat{h}',\hat{\omega})$ (for R small enough):

$$\begin{array}{rcl}
\|(\hat{u},\hat{p},\hat{h}',\hat{\omega})\|_{\mathbb{H}} & \leq & C_0\Big(+ \left\|\overline{G}\right\|_{\mathrm{H}^{2,1}(Q^0_\infty)} + o(\|\hat{v},\hat{k}',\hat{\varpi}\|_{\mathrm{H}^{2,1}(Q^0_\infty)\times\mathrm{L}^2(0,\infty;\mathbb{R}^d)\times\mathrm{L}^2(0,\infty;\mathbb{R}^3)}) \\
& & + \left\|\overline{W}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{3/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{-1/2}(\partial\mathcal{S}))} \\
& & + \left\|\overline{F}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \left\|\overline{F}_M\right\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)} + \left\|\overline{F}_I\right\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)}\Big).
\end{array}$$

Then we have for R small enough

$$\begin{array}{rcl}
\|(\hat{u},\hat{p},\hat{h}',\hat{\omega})\|_{\mathbb{H}} & \leq & C_0 \times o(\|(\hat{v},\hat{q},\hat{k}',\hat{\varpi})\|_{\mathbb{H}}) \\
& \leq & \dfrac{1}{2}\|(\hat{v},\hat{q},\hat{k}',\hat{\varpi})\|_{\mathbb{H}},
\end{array}$$

and thus the mapping $\mathcal{N}$ is a contraction in B_R.

3.9 Appendix A: The change of variables

Let us consider an *admissible* deformation $X^* \in \mathcal{W}_\lambda(S^0_\infty)$ - in the sense of Definition 3.4 - which satisfies, in particular, for all $t > 0$ the following condition

$$\int_{\partial\mathcal{S}} \mathrm{cof}\nabla X^*(y,t)^T \frac{\partial X^*}{\partial t}(y,t)\cdot n \mathrm{d}\Gamma(y) \quad = \quad 0. \tag{3.125}$$

The regularity considered for $e^{\lambda t}\dfrac{\partial X^*}{\partial t}$ in this section is

$$\mathcal{H}_{5/2}(\ S^0_\infty) \quad = \quad \mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S})).$$

The goal of this subsection is to extend to the whole domain $\overline{\mathcal{O}}$ the mappings $X_{\mathcal{S}}(\cdot,t)$ and $Y_{\mathcal{S}}(\cdot,t)$, initially defined respectively on $\mathcal{S}$ and $\mathcal{S}(t)$. The process we use is not the same as the one given in [SMSTT08]. Instead of extending the Eulerian flow given by the solid's deformation, we directly extend the solid's deformation, because the difference in our case lies in the fact that the regularity of the Dirichlet data - written in Eulerian formulation on the time-dependent boundary $\partial\mathcal{S}(t)$ - is limited.
The goal is to construct a mapping X such that

$$\left\{\begin{array}{ll}
\mathrm{det}\nabla X = 1, & \text{in } \mathcal{F}\times(0,\infty), \\
X = X_{\mathcal{S}}, & \text{on } \partial\mathcal{S}\times(0,\infty), \\
X = \mathrm{Id}_{\partial\mathcal{O}}, & \text{on } \partial\mathcal{O}\times(0,\infty).
\end{array}\right.$$

3.9.1 Preliminary results

Let us recall a result stated in the Appendix B of [GS91] (Proposition B.1), which treats of Sobolev regularities for products of functions, and that we state as:

Lemma 3.29. *Let s, μ, and κ in $\mathbb{R}$. If $f \in \mathrm{H}^{s+\mu}(\mathcal{F})$ and $g \in \mathrm{H}^{s+\kappa}(\mathcal{F})$, then there exists a positive constant C such that*

$$\|fg\|_{\mathrm{H}^s(\mathcal{F})} \leq C\|f\|_{\mathrm{H}^{s+\mu}(\mathcal{F})}\|g\|_{\mathrm{H}^{s+\kappa}(\mathcal{F})},$$

(i) when $s+\mu+\kappa \geq d/2$,
(ii) with $\mu \geq 0$, $\kappa \geq 0$, $2s+\mu+\kappa \geq 0$,
(iii) except that $s+\mu+\kappa > d/2$ if equality holds somewhere in (ii).

A consequence of this Lemma is the following result.

Lemma 3.30. *Let $\overline{X}^*$ be in $\tilde{\mathcal{W}}_3(Q_\infty^0)$. Then*

$$\mathrm{cof}\nabla\overline{X}^* \in \mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F})) \cap \mathrm{W}^{1,\infty}(0,\infty;\mathbf{L}^2(\mathcal{F})), \tag{3.126}$$

and, if $\overline{X}^* - \mathrm{Id}_{\mathcal{F}}$ is small enough in $\tilde{\mathcal{W}}_3(Q_\infty^0)$, there exists a positive constant C such that

$$\|\mathrm{cof}\nabla\overline{X}^* - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)} \leq C\|\nabla\overline{X}^* - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)}. \tag{3.127}$$

Besides, if $\overline{X}_1^* - \mathrm{Id}_{\mathcal{F}}$ and $\overline{X}_2^* - \mathrm{Id}_{\mathcal{F}}$ are small enough $\tilde{\mathcal{W}}_3(Q_\infty^0)$, there exists a positive constant C such that

$$\|\mathrm{cof}\nabla\overline{X}_2^* - \mathrm{cof}\nabla\overline{X}_1^*\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)} \leq C\|\nabla\overline{X}_2^* - \nabla\overline{X}_1^*\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)}. \tag{3.128}$$

Proof. For proving (3.126), the case $d=2$ is obvious. For the general case, let us show that the space $\mathrm{L}^\infty(0,\infty;\mathrm{H}^2(\mathcal{F})) \cap \mathrm{W}^{1,\infty}(0,\infty;\mathrm{L}^2(\mathcal{F}))$ is stable by product. For that, let us consider two functions f and g which lie in this space. Applying Lemma 3.29 with $s=2$ and $\mu=\kappa=0$, we get

$$\|fg\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))} \leq C\|f\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))}\|g\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))}.$$

For the regularity in $\mathrm{W}^{1,\infty}(0,\infty;\mathbf{L}^2(\mathcal{F})$, we write

$$\frac{\partial(fg)}{\partial t} = \frac{\partial f}{\partial t}g + f\frac{\partial g}{\partial t}.$$

Using the continuous embedding $\mathbf{H}^2(\mathcal{F}) \hookrightarrow \mathbf{L}^\infty(\mathcal{F})$, we get

$$\left\|\frac{\partial(fg)}{\partial t}\right\|_{\mathrm{L}^\infty(0,\infty;\mathrm{L}^2(\mathcal{F}))} \leq C\left\|\frac{\partial f}{\partial t}\right\|_{\mathrm{L}^\infty(0,\infty;\mathrm{L}^2(\mathcal{F}))}\|g\|_{\mathrm{L}^\infty(0,\infty;\mathrm{H}^2(\mathcal{F}))} + \left\|\frac{\partial g}{\partial t}\right\|_{\mathrm{L}^\infty(0,\infty;\mathrm{L}^2(\mathcal{F}))}\|f\|_{\mathrm{L}^\infty(0,\infty;\mathrm{H}^2(\mathcal{F}))},$$

and thus the desired regularity. Thus the space $\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F})) \cap \mathrm{W}^{1,\infty}(0,\infty;\mathbf{L}^2(\mathcal{F}))$ is an algebra. The estimate (3.127) is obtained by the differentiability of the mapping $\nabla X^* \mapsto \mathrm{cof}\nabla X^*$ (see [All07] for instance); More precisely, we have

$$\mathrm{I}_{\mathbb{R}^d} - \mathrm{cof}\nabla\overline{X}^* = \left(\nabla\overline{X}^* - \mathrm{I}_{\mathbb{R}^d}\right)^T - \mathrm{div}\left(\overline{X}^* - \mathrm{Id}_{\mathcal{F}}\right) + o\left(\|\overline{X}^* - \mathrm{Id}_{\mathcal{F}}\|_{\tilde{\mathcal{W}}_3(Q_\infty^0)}\right),$$

so that we get

$$\|\mathrm{cof}\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)} \quad \leq \quad C\|\nabla\tilde{X} - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)}.$$

The estimate (3.128) can be obtained by the mean-value theorem. □

3.9.2 Extension of the Lagrangian mappings

Let us first extend the solid's deformation X^* to the fluid domain $\mathcal{F}$.

Proposition 3.31. *Let $X^* - \mathrm{Id}_{\mathcal{S}} \in \mathcal{W}_\lambda(S^0_\infty)$ be an admissible deformation, in the sense of Definition 3.4. Let us assume that $X^* - \mathrm{Id}_{\mathcal{S}}$ is small enough in $\mathcal{W}_\lambda(S^0_\infty)$, that is to say that the function*

$$(y,t) \mapsto e^{\lambda t}\frac{\partial X^*}{\partial t}$$

is small enough in $\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{S})) \cap \mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{S}))$. Then there exists a mapping $\overline{X}^* \in \tilde{\mathcal{W}}_3(Q^0_\infty)$ satisfying

$$\begin{cases} \det\nabla\overline{X}^* = 1 & \text{in } \mathcal{F}\times(0,\infty), \\ \overline{X}^* = X^* & \text{on } \partial\mathcal{S}\times(0,\infty), \\ \overline{X}^* = \mathrm{Id}_{\partial\mathcal{O}} & \text{on } \partial\mathcal{O}\times(0,\infty), \end{cases} \tag{3.129}$$

and such that

$$\|\overline{X}^* - \mathrm{Id}_{\mathcal{F}}\|_{\tilde{\mathcal{W}}_3(Q^0_\infty)} \quad \leq \quad C\left\|e^{\lambda t}\frac{\partial X^*}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))} \tag{3.130}$$

for some positive constant C independent of X^*. Besides, if $X_1^* - \mathrm{Id}_{\mathcal{S}}$ and $X_2^* - \mathrm{Id}_{\mathcal{S}}$ are two displacements small enough in $\mathcal{W}_\lambda(S^0_\infty)$, then the solutions $\overline{X}_1^*$ and $\overline{X}_2^*$ of problem (3.129), corresponding to X_1^* and X_2^* as data respectively, satisfy

$$\|\overline{X}_2^* - \overline{X}_1^*\|_{\tilde{\mathcal{W}}_3(Q^0_\infty)} \quad \leq \quad C\left\|e^{\lambda t}\frac{\partial X_2^*}{\partial t} - e^{\lambda t}\frac{\partial X_1^*}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))\cap\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))}. \tag{3.131}$$

Proof. Given the initial datum $X^*(y,0) = y$ for $y \in \overline{\mathcal{S}}$, let us consider the system (3.129) derived in time, as follows

$$\begin{cases} \left(\mathrm{cof}\nabla\overline{X}^*\right) : \dfrac{\partial\nabla\overline{X}^*}{\partial t} = 0 & \text{in } \mathcal{F}\times(0,\infty), \\ \dfrac{\partial\overline{X}^*}{\partial t} = \dfrac{\partial X^*}{\partial t} & \text{on } \partial\mathcal{S}\times(0,\infty), \\ \dfrac{\partial\overline{X}^*}{\partial t} = 0 & \text{on } \partial\mathcal{O}\times(0,\infty). \end{cases}$$

This system can be viewed as a modified nonlinear divergence problem, that we state as

$$\begin{cases} \mathrm{div}\,\dfrac{\partial\overline{X}^*}{\partial t} = f(\overline{X}^*) & \text{in } \mathcal{F}\times(0,\infty), \\ \dfrac{\partial\overline{X}^*}{\partial t} = \dfrac{\partial X^*}{\partial t} & \text{on } \partial\mathcal{S}\times(0,\infty), \\ \dfrac{\partial\overline{X}^*}{\partial t} = 0 & \text{on } \partial\mathcal{O}\times(0,\infty), \\ \overline{X}^*(\cdot,0) = \mathrm{Id}_{\mathcal{F}}, \quad \dfrac{\partial\overline{X}^*}{\partial t}(\cdot,0) = 0, & \end{cases} \tag{3.132}$$

with

$$f(\overline{X}^*) \quad = \quad \left(\mathrm{I}_{\mathbb{R}^d} - \mathrm{cof}\nabla\overline{X}^*\right) : \frac{\partial \nabla \overline{X}^*}{\partial t}.$$

If we search solutions to this system which are continuous in space, in using the Piola identity we can verify that the compatibility condition for this divergence system is nothing else than the equality (3.125).
A solution of this system can be viewed as a fixed point of the mapping

$$\begin{array}{rccc} \mathfrak{T}: & \mathcal{W}_\lambda(Q_\infty^0) & \rightarrow & \mathcal{W}_\lambda(Q_\infty^0) \\ & \overline{X}_1^* - \mathrm{Id}_{\mathcal{F}} & \mapsto & \overline{X}_2^* - \mathrm{Id}_{\mathcal{F}}, \end{array} \tag{3.133}$$

where $\overline{X}_2^*$ satisfies the classical divergence problem

$$\begin{cases} \mathrm{div}\, \dfrac{\partial \overline{X}_2^*}{\partial t} = f(\overline{X}_1^*) & \text{in } \mathcal{F}\times(0,\infty), \\ \dfrac{\partial \overline{X}_2^*}{\partial t} = \dfrac{\partial X^*}{\partial t} & \text{on } \partial\mathcal{S}\times(0,\infty), \\ \dfrac{\partial \overline{X}_2^*}{\partial t} = 0 & \text{on } \partial\mathcal{O}\times(0,\infty), \\ \overline{X}_2^*(\cdot,0) = \mathrm{Id}_{\mathcal{F}}, \quad \dfrac{\partial \overline{X}_2^*}{\partial t}(\cdot,0) = 0. & \end{cases}$$

Indeed, let us first verify that for $\overline{X}^* - \mathrm{Id}_{\mathcal{F}} \in \mathcal{W}_\lambda(Q_\infty^0)$ we have $e^{\lambda t} f(\overline{X}^*) \in \mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F})) \cap \mathrm{H}^1(0,\infty;\mathbf{L}^2(\mathcal{F}))$. For that, we recall from the previous lemma that $\mathrm{cof}\nabla\overline{X}^* \in \mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))\cap \mathrm{W}^{1,\infty}(0,\infty;\mathbf{L}^2(\mathcal{F}))$, and we first use the result of Lemma 3.29 with $s=2$ and $\mu=\kappa=0$ to get

$$\|e^{\lambda t} f(\overline{X}^*)\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} \quad \leq \quad C\|\mathrm{I}_{\mathbb{R}^d} - \mathrm{cof}\nabla\overline{X}^*\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))} \left\| e^{\lambda t}\frac{\partial\nabla\overline{X}^*}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))}.$$

For the regularity in $\mathrm{H}^1(0,\infty;\mathbf{L}^2(\mathcal{F}))$, we write

$$\begin{aligned} e^{\lambda t}\frac{\partial f(\overline{X}^*)}{\partial t} \quad &= \quad (\mathrm{I}_{\mathbb{R}^d} - \mathrm{cof}\nabla\overline{X}^*) : \left(e^{\lambda t}\frac{\partial^2\nabla\overline{X}^*}{\partial t^2}\right) - \frac{\partial \mathrm{cof}\nabla\overline{X}^*}{\partial t} : \left(e^{\lambda t}\frac{\partial\nabla\overline{X}^*}{\partial t}\right), \\ \left\| e^{\lambda t}\frac{\partial f(\overline{X}^*)}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathcal{F}))} \quad &\leq \quad C\|\mathrm{I}_{\mathbb{R}^d} - \mathrm{cof}\nabla\overline{X}^*\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))} \left\| e^{\lambda t}\frac{\partial\nabla\overline{X}^*}{\partial t}\right\|_{\mathrm{H}^1(0,\infty;\mathbf{L}^2(\mathcal{F}))} \\ & \quad +C\left\|\frac{\partial \mathrm{cof}\nabla\overline{X}^*}{\partial t}\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^2(\mathcal{F}))} \left\| e^{\lambda t}\frac{\partial\nabla\overline{X}^*}{\partial t}\right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} \end{aligned}$$

in using the continuous embedding $\mathbf{H}^2(\mathcal{F}) \hookrightarrow \mathbf{L}^\infty(\mathcal{F})$. Thus there exists a positive constant C_0 such that

$$\left\|e^{\lambda t} f(\overline{X}^*)\right\|_{\mathrm{H}^{2,1}(Q_\infty^0)} \quad \leq \quad C_0\|\mathrm{I}_{\mathbb{R}^d} - \mathrm{cof}\nabla\overline{X}^*\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))\cap \mathrm{W}^{1,\infty}(0,\infty;\mathbf{L}^2(\mathcal{F}))} \left\| e^{\lambda t}\frac{\partial\nabla\overline{X}^*}{\partial t}\right\|_{\mathrm{H}^{2,1}(Q_\infty^0)}. \tag{3.134}$$

The estimate (3.134) shows in particular that the mapping $\mathfrak{T}$ is well-defined. Moreover, for the divergence problem (3.134) there exists a positive constant $C_{\mathcal{F}}$ (see [Gal94] for instance) such that

$$\left\| e^{\lambda t} \frac{\partial \overline{X}_2^*}{\partial t} \right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^3(\mathcal{F}))} \leq C_{\mathcal{F}} \left(\| e^{\lambda t} f(\overline{X}_1^*) \|_{\mathrm{L}^2(0,\infty;\mathbf{H}^2(\mathcal{F}))} + \left\| e^{\lambda t} \frac{\partial X^*}{\partial t} \right\|_{\mathrm{L}^2(0,\infty;\mathbf{H}^{5/2}(\partial\mathcal{S}))} \right),$$

and also

$$\left\| e^{\lambda t} \frac{\partial \overline{X}_2^*}{\partial t} \right\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^1(\mathcal{F}))} \leq C_{\mathcal{F}} \left((1+\lambda) \| e^{\lambda t} f(\overline{X}_1^*) \|_{\mathrm{H}^1(0,\infty;\mathbf{L}^2(\mathcal{F}))} + \left\| e^{\lambda t} \frac{\partial X^*}{\partial t} \right\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^{1/2}(\partial\mathcal{S}))} \right).$$

Thus there exists a positive constant $\hat{C}_{\mathcal{F}}$ such that

$$\left\| e^{\lambda t} \frac{\partial \overline{X}_2^*}{\partial t} \right\|_{\mathcal{H}_3(Q_\infty^0)} \leq \hat{C}_{\mathcal{F}} \left(\| e^{\lambda t} f(\overline{X}_1^*) \|_{\mathrm{H}^{2,1}(Q_\infty^0)} + \left\| e^{\lambda t} \frac{\partial X^*}{\partial t} \right\|_{\mathcal{H}_3(S_\infty^0)} \right). \tag{3.135}$$

Let us consider the set

$$\mathfrak{B}_R = \left\{ \overline{Z}^* \in \mathcal{W}_\lambda(Q_\infty^0), \ \| \overline{Z}^* \|_{\mathcal{W}_\lambda(Q_\infty^0)} \leq R \right\}$$

with

$$R = 2\hat{C}_{\mathcal{F}} \left\| e^{\lambda t} \frac{\partial X^*}{\partial t} \right\|_{\mathcal{H}_3(S_\infty^0)}.$$

Notice that a mapping $\overline{X}_1^* - \mathrm{Id}_{\mathcal{F}} \in \mathfrak{B}_R$ satisfies in particular the following inequality, obtained in the same way we have proceeded to get the embedding (3.23):

$$\begin{aligned} \| \overline{X}_1^* - \mathrm{Id}_{\mathcal{F}} \|_{\tilde{\mathcal{W}}_3(Q_\infty^0)} &\leq C_1 \left\| e^{\lambda t} \frac{\partial \overline{X}_1^*}{\partial t} \right\|_{\mathcal{H}_3(Q_\infty^0)} = C_1 \| \overline{X}_1^* - \mathrm{Id}_{\mathcal{F}} \|_{\mathcal{W}_\lambda(Q_\infty^0)} \\ &\leq C_1 R. \end{aligned}$$

Then the inequality (3.135) combined to the estimates (3.134) and (3.127) show that for $\overline{X}_1^* - \mathrm{Id}_{\mathcal{F}} \in \mathfrak{B}_R$ we have

$$\| \overline{X}_2^* - \mathrm{Id}_{\mathcal{F}} \|_{\mathcal{W}_\lambda(Q_\infty^0)} \leq \hat{C}_{\mathcal{F}} \left(C_0 C C_1 R^2 (C_1 R + 1) + \frac{R}{2\hat{C}_{\mathcal{F}}} \right),$$

and thus for R small enough, $\mathfrak{B}_R$ is stable by $\mathfrak{T}$. Notice that $\mathfrak{B}_R$ is a closed subset of $\tilde{\mathcal{W}}_3(Q_\infty^0)$. Let us verify that $\mathfrak{T}$ is a contraction in $\mathfrak{B}_R$.
For $\overline{X}_1^* - \mathrm{Id}_{\mathcal{F}}$ and $\overline{X}_2^* - \mathrm{Id}_{\mathcal{F}}$ in $\mathfrak{B}_R$, we denote $\overline{Z}^* = \mathfrak{T}(\overline{X}_2^* - \mathrm{Id}_{\mathcal{F}}) - \mathfrak{T}(\overline{X}_1^* - \mathrm{Id}_{\mathcal{F}})$ which satisfies the divergence system

$$\begin{cases} \operatorname{div} \dfrac{\partial \overline{Z}^*}{\partial t} = f(\overline{X}_2^*) - f(\overline{X}_1^*) & \text{in } \mathcal{F} \times (0,\infty), \\ \dfrac{\partial \overline{Z}^*}{\partial t} = 0 & \text{on } \partial\mathcal{S} \times (0,\infty), \\ \dfrac{\partial \overline{Z}^*}{\partial t} = 0 & \text{on } \partial\mathcal{O} \times (0,\infty). \end{cases}$$

and thus the estimate

$$\left\| e^{\lambda t} \frac{\partial \overline{Z}^*}{\partial t} \right\|_{\mathcal{H}_3(Q_\infty^0)} \leq C_{\mathcal{F}} \left\| e^{\lambda t} \left(f(\overline{X}_2^*) - f(\tilde{X}_1) \right) \right\|_{\mathrm{H}^{2,1}(Q_\infty^0)}.$$

For tackling the Lipschitz property of the nonlinearity, we write

$$f(\overline{X}_2^*) - f(\overline{X}_1^*) = \left(\mathrm{cof} \nabla \overline{X}_2^* - \mathrm{cof} \nabla \overline{X}_1^* \right) : \frac{\partial \nabla \overline{X}_2^*}{\partial t} + \left(\mathrm{I}_{\mathbb{R}^d} - \mathrm{cof} \nabla \overline{X}_1^* \right) : \frac{\partial \nabla (\overline{X}_2^* - \overline{X}_1^*)}{\partial t}.$$

In reconsidering the steps of the proof of the estimate (3.134) and in using (3.128), we can verify that for R small enough the mapping $\mathfrak{T}$ is a contraction in $\mathfrak{B}_R$. Thus $\mathfrak{T}$ admits a unique fixed point in $\mathfrak{B}_R$.
For the estimate (3.131), if $\overline{X}_1^*$ and $\overline{X}_2^*$ are two solutions corresponding to X_1^* and X_2^* respectively, let us just write the system satisfied by the difference $\overline{Z}^* = \overline{X}_2^* - \overline{X}_1^*$:

$$\begin{cases} \mathrm{div} \, \dfrac{\partial \overline{Z}^*}{\partial t} = f(\overline{X}_2^*) - f(\overline{X}_1^*) & \text{in } \mathcal{F} \times (0, \infty), \\ \dfrac{\partial \overline{Z}^*}{\partial t} = \dfrac{\partial X_2^*}{\partial t} - \dfrac{\partial X_1^*}{\partial t} & \text{on } \partial \mathcal{S} \times (0, \infty), \\ \dfrac{\partial \overline{Z}^*}{\partial t} = 0 & \text{on } \partial \mathcal{O} \times (0, \infty). \end{cases}$$

Then the methods used above can be similarly applied to this system in order to deduce from it the announced result. □

Let us now consider $h \in \mathrm{H}^2(0, \infty; \mathbb{R}^d)$, and $\mathbf{R} \in \mathrm{H}^2(0, \infty; \mathbb{R}^9)$ which provides $\omega \in \mathrm{H}^1(0, \infty; \mathbb{R}^3)$. Let us construct a mapping X such that $X(\cdot, 0) = \mathrm{Id}_{\mathcal{F}}$ and

$$\begin{cases} \det \nabla X = 1 & \text{in } \mathcal{F} \times (0, \infty), \\ X = h(t) + \mathbf{R}(t) X^* & \text{on } \partial \mathcal{S} \times (0, \infty), \\ X = \mathrm{Id}_{\partial \mathcal{O}} & \text{on } \partial \mathcal{S} \times (0, \infty). \end{cases}$$

We cannot solve this problem as we have done for problem (3.129), because the proof would require the unknowns h and ω arbitrarily small enough, a thing that we cannot provide. Instead of that, we utilize the mapping $\overline{X}^*$ provided by Proposition 3.31, and we search for a mapping $\overline{X}^R$ such that

$$X(\cdot, t) = \overline{X}^R(\cdot, t) \circ \overline{X}^*(\cdot, t).$$

Such a mapping $\overline{X}^R$ has to satisfy

$$\begin{cases} \det \nabla \overline{X}^R = 1 & \text{in } \mathcal{F} \times (0, \infty), \\ \overline{X}^R = h + \mathbf{R} \mathrm{Id}_{\partial \mathcal{S}} & \text{on } \partial \mathcal{S} \times (0, \infty), \\ \overline{X}^R = \mathrm{Id}_{\partial \mathcal{O}} & \text{on } \partial \mathcal{O} \times (0, \infty). \end{cases}$$

For that, let us proceed as in [Tak03]: We consider a cut-off function $\xi \in C^\infty(\mathcal{F})$, such that $\xi \equiv 1$ in a vicinity of $\partial \mathcal{S}$ and $\xi \equiv 0$ in a vicinity of $\partial \mathcal{O}$. We define the function

$$\mathfrak{F}_R(x, t) = \frac{1}{2} h'(t) \wedge (x - h(t)) - \frac{1}{2} |x - h(t)|^2 \omega(t),$$

so that $\mathrm{curl}(\mathfrak{F}_R)(x,t) = h'(t) + \omega(t) \wedge (x - h(t))$, and we construct $\overline{X}^R$ as the solution of the following Cauchy problem

$$\frac{\partial \overline{X}^R}{\partial t}(\overline{x}^*, t) = \mathrm{curl}(\xi \mathfrak{F}_R)(\overline{X}^R(\overline{x}^*, t), t), \quad \overline{X}^R(\overline{x}^*, 0) = \overline{x}^*, \quad \overline{x}^* \in \overline{X}^*(\mathcal{F}, t) = \mathcal{F}^*. \tag{3.136}$$

We can verify (see [Tak03] for instance) that the mapping $\overline{X}^R$ so obtained has the desired properties, and thus we can set

$$X(y,t) \quad = \quad \overline{X}^R(\overline{X}^*(y,t), t), \quad (y,t) \in \mathcal{F} \times (0, \infty). \tag{3.137}$$

Since $\overline{X}^*(\cdot, t)$ and $\overline{X}^R(\cdot, t)$ are invertible, the mapping $X(\cdot, t)$ is invertible, and we denote by $Y(\cdot, t)$ its inverse. The mapping X presents the same type of regularity as the mapping $\overline{X}^*$. We sum its properties in the following proposition.

Proposition 3.32. *Let X^* be an admissible control - in the sense of Definition 3.4 - and $\overline{X}^*$ the extension of X^* provided by Proposition 3.31 (for $e^{\lambda t} \dfrac{\partial X^*}{\partial t}$ small enough in $\mathcal{H}_3(S^0_\infty)$). Let X be the mapping given by* (3.137)*. For all $t \geq 0$, the mapping $X(\cdot, t)$ is a C^1-diffeomorphism from $\mathcal{O}$ onto $\mathcal{O}$, from $\partial \mathcal{S}$ onto $\partial \mathcal{S}(t)$, and from $\mathcal{F}$ onto $\mathcal{F}(t)$. We denote by $Y(\cdot, t)$ its inverse at some time t. We have*

$$\begin{aligned} & (y,t) \mapsto X(y,t) \in \tilde{\mathcal{W}}_3(Q^0_\infty), \\ & \det \nabla X(y,t) = 1, \text{ for all } (y,t) \in \mathcal{F} \times (0, \infty). \end{aligned}$$

The proof for the regularity of X can be straightforwardly deduced from Lemma 3.35 in the Appendix B of this chapter. We do not give more detail in this section, because the aim of the latter is only to get a change of variables which enables us rewrite the main system as an equivalent one written in fixed domains.

3.10 Appendix B: Proofs of estimates for the changes of variables

Let us recall that for $X^* - \mathrm{Id}_{\mathcal{S}} \in \mathcal{W}_\lambda(S^0_\infty)$, Proposition 3.31 enables us to define the extension $\overline{X}^* \in \mathcal{W}_\lambda(Q^0_\infty) + \mathrm{Id}_{\mathcal{F}}$ satisfying

$$\begin{cases} \det \nabla \overline{X}^* = 1 & \text{in } \mathcal{F} \times (0, \infty), \\ \overline{X}^* = X^* & \text{on } \partial \mathcal{S} \times (0, \infty), \\ \overline{X}^* = \mathrm{Id}_{\partial \mathcal{O}} & \text{on } \partial \mathcal{O} \times (0, \infty), \end{cases}$$

and

$$\|\overline{X}^* - \mathrm{Id}_{\mathcal{F}}\|_{\tilde{\mathcal{W}}_3(Q^0_\infty)} \quad \leq \quad C \left\| e^{\lambda t} \frac{\partial X^*}{\partial t} \right\|_{\mathcal{H}_3(Q^0_\infty)}.$$

For $h \in \mathrm{H}^2(0, \infty; \mathbb{R}^d)$ and $\mathbf{R} \in \mathrm{H}^2(0, \infty; \mathbb{R}^9)$ which provides $\omega \in \mathrm{H}^1(0, \infty; \mathbb{R}^3)$ such that

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} & = & \mathbb{S}(\omega)\,\mathbf{R} \\ \mathbf{R}(0) & = & \mathrm{I}_{\mathbb{R}^3}, \end{cases} \qquad \text{with } \mathbb{S}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

we can define $\overline{X}^R$ through the problem

$$\frac{\partial \overline{X}^R}{\partial t}(\overline{x}^*,t) = \mathrm{curl}(\xi \mathfrak{F}_R)(\overline{X}^R(\overline{x}^*,t),t), \quad \overline{X}^R(\overline{x}^*,0) = \overline{x}^*, \quad \overline{x}^* \in \overline{X}^*(\mathcal{F},t) = \mathcal{F}^*(t),$$

where ξ is a regular cut-off function, and

$$\mathfrak{F}_R(x,t) = \frac{1}{2}h'(t)\wedge(x-h(t)) - \frac{1}{2}|x-h(t)|^2\omega(t).$$

Then we define

$$X = \overline{X}^R \circ \overline{X}^*,$$

and

$$\tilde{X} = \mathbf{R}^T(X-h).$$

Lemma 3.33. *For $X_1^* - \mathrm{Id}_{\mathcal{S}}$ and $X_2^* - \mathrm{Id}_{\mathcal{S}}$ small enough in $\mathcal{W}_\lambda(S_\infty^0)$, let $\overline{X}_1^*$ and $\overline{X}_2^*$ be the solutions of Problem (3.129) (see Proposition 3.31) corresponding to the data X_1^* and X_2^* respectively. If we denote by $\overline{Y}_1^*(\cdot,t)$ and $\overline{Y}_2^*(\cdot,t)$ the inverses of $\overline{X}_1^*$ and $\overline{X}_2^*$ respectively, we have*

$$\|\nabla\overline{Y}_2^*(\overline{X}_2^*) - \nabla\overline{Y}_1^*(\overline{X}_1^*)\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))\cap \mathrm{W}^{1,\infty}(0,\infty;\mathbf{L}^2(\mathcal{F}))} \leq C\left\| e^{\lambda t}\frac{\partial X_2^*}{\partial t} - e^{\lambda t}\frac{\partial X_1^*}{\partial t}\right\|_{\mathcal{H}_3(S_\infty^0)}. \tag{3.138}$$

Proof. Let us recall the estimate (3.131), obtained for $X_1^* - \mathrm{Id}_{\mathcal{S}}$ and $X_2^* - \mathrm{Id}_{\mathcal{S}}$ small enough in $\mathcal{W}_\lambda(S_\infty^0)$:

$$\|\overline{X}_2^* - \overline{X}_1^*\|_{\tilde{\mathcal{W}}_3(Q_\infty^0)} \leq C\left\| e^{\lambda t}\frac{\partial X_2^*}{\partial t} - e^{\lambda t}\frac{\partial X_1^*}{\partial t}\right\|_{\mathcal{H}_3(S_\infty^0)}.$$

The estimate (3.138) can be obtained in considering the two following intermediate estimates

$$\|\nabla\overline{X}_1^* - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)} \leq \left\| e^{\lambda t}\frac{\partial X_1^*}{\partial t}\right\|_{\mathcal{H}_3(S_\infty^0)}$$

and

$$\begin{aligned}
\nabla\overline{Y}_2^*(\overline{X}_2^*) - \mathrm{I}_{\mathbb{R}^d} &= \left(\mathrm{I}_{\mathbb{R}^d} - \nabla\overline{X}_2^*\right)\left(\nabla\overline{Y}_2^*(\overline{X}_2^*) - \mathrm{I}_{\mathbb{R}^d}\right) + \left(\mathrm{I}_{\mathbb{R}^d} - \nabla\overline{X}_2^*\right), \\
\|\nabla\overline{Y}_2^*(\overline{X}_2^*) - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)} &\leq \frac{\|\nabla\overline{X}_2^* - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)}}{1 - C\|\nabla\overline{X}_2^* - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)}} \\
&\leq 2\|\nabla\overline{X}_2^* - \mathrm{I}_{\mathbb{R}^d}\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap\mathrm{W}^{1,\infty}(\mathbf{L}^2)},
\end{aligned}$$

since $\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\mathcal{F}))\cap\mathrm{W}^{1,\infty}(0,\infty;\mathbf{L}^2(\mathcal{F}))$ is an algebra, and then in writing

$$\begin{aligned}
\nabla\overline{Y}_2^*(\overline{X}_2^*) - \nabla\overline{Y}_1^*(\overline{X}_1^*) &= \left(\nabla\overline{Y}_2^*(\overline{X}_2^*) - \nabla\overline{Y}_1^*(\overline{X}_1^*)\right)\left(\mathrm{I}_{\mathbb{R}^d} - \nabla\overline{X}_1^*\right) - \left(\nabla\overline{X}_2^* - \nabla\overline{X}_1^*\right)\nabla\overline{Y}_2^*(\overline{X}_2^*), \\
\|\nabla\overline{Y}_2^*(\overline{X}_2^*) - \nabla\overline{Y}_1^*(\overline{X}_1^*)\| &= \frac{\left(1 + \|\nabla\overline{Y}_2^*(\overline{X}_2^*) - \mathrm{I}_{\mathbb{R}^d}\|\right)\|\nabla\overline{X}_2^* - \nabla\overline{X}_1^*\|}{1 - C\|\nabla\overline{X}_1^* - \mathrm{I}_{\mathbb{R}^d}\|}.
\end{aligned}$$

□

Lemma 3.34. *Let $\overline{X}_1^R$ and $\overline{X}_2^R$ be the extensions defined by problem* (3.136)*, with data $(h_1, \mathbf{R}_1) \in \mathrm{H}^2(0,\infty;\mathbb{R}^d) \times \mathrm{H}^2(0,\infty;\mathbb{R}^9)$ and $(h_2, \mathbf{R}_2) \in \mathrm{H}^2(0,\infty;\mathbb{R}^d) \times \mathrm{H}^2(0,\infty;\mathbb{R}^9)$ respectively. Then we have*

$$\|\overline{X}_2^R - \overline{X}_1^R\|_{\mathrm{H}^2(0,\infty;W^{4,\infty}(\mathbb{R}^d))} \quad \leq \quad O\left(\|\hat{h}_2' - \hat{h}_1'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} + \|\hat{\omega}_2 - \hat{\omega}_1\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)}\right),$$

where we recall that we recall that

$$\begin{array}{ll} \hat{h}_1' = e^{\lambda t}\mathbf{R}_1^T h_1', & \hat{h}_2' = e^{\lambda t}\mathbf{R}_2^T h_2', \\ \hat{\omega}_1 = e^{\lambda t}\mathbf{R}_1^T \omega_1, & \hat{\omega}_2 = e^{\lambda t}\mathbf{R}_2^T \omega_2. \end{array}$$

Proof. The change of variables given by a mapping $\overline{X}^R$ is slightly the same as the one utilized in [Tak03]; In considering the writing

$$\mathbf{R}(t)^T \mathfrak{F}_R(x,t) = \frac{1}{2}\tilde{h}'(t) \wedge \mathbf{R}(t)^T(x - h(t)) - \frac{1}{2}|x - h(t)|^2 \tilde{\omega}(t),$$

the steps of the proofs of Lemmas 6.11 and 6.12 of [Tak03] can be then repeated, with the difference that in infinite time horizon we rather have

$$\begin{aligned} \|\overline{X}_2^R - \overline{X}_1^R\|_{\mathrm{H}^2(0,\infty;W^{4,\infty}(\mathbb{R}^d))} \quad \leq \quad & K_0^R \left(\|\tilde{h}_2' - \tilde{h}_1'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} + \|\tilde{\omega}_2 - \tilde{\omega}_1\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)} \right. \\ & \left. + \|h_2 - h_1\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^d)} + \|\mathbf{R}_2 - \mathbf{R}_1\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^9)}\right), \end{aligned}$$

where K_0^R is bounded when h_1, h_2 are close to 0 and $\mathbf{R}_1$, $\mathbf{R}_2$ are close to $\mathrm{I}_{\mathbb{R}^3}$. In order to estimate $\|h_2 - h_1\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^d)}$ and $\|\mathbf{R}_2 - \mathbf{R}_1\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^9)}$, we first apply the Grönwall's lemma on

$$\begin{aligned} \frac{\partial}{\partial t}(\mathbf{R}_2 - \mathbf{R}_1) &= (\mathbf{R}_2 - \mathbf{R}_1)\mathbb{S}(\tilde{\omega}_2) + \mathbf{R}_1\mathbb{S}(\tilde{\omega}_2 - \tilde{\omega}_1) \\ (\mathbf{R}_2 - \mathbf{R}_1)(0) &= 0 \end{aligned}$$

in order to get

$$\|\mathbf{R}_2 - \mathbf{R}_1\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^9)} \quad \leq \quad C\|\tilde{\omega}_2 - \tilde{\omega}_1\|_{\mathrm{L}^1(0,\infty;\mathbb{R}^3)} \exp\left(C\|\tilde{\omega}_2\|_{\mathrm{L}^1(0,\infty;\mathbb{R}^3)}\right).$$

Besides, it is easy to see that

$$\|\tilde{\omega}_2 - \tilde{\omega}_1\|_{\mathrm{L}^1(0,\infty;\mathbb{R}^3)} \quad \leq \quad \frac{1}{\sqrt{2\lambda}}\|\hat{\omega}_2 - \hat{\omega}_1\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)},$$

so that $\|\mathbf{R}_2 - \mathbf{R}_1\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^9)}$ is controlled by $\|\hat{\omega}_2 - \hat{\omega}_1\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^3)}$. Then the term $\|h_2 - h_1\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^d)}$ can be treated in writing

$$\begin{aligned} \|h_2 - h_1\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^d)} &\leq \|h_2' - h_1'\|_{\mathrm{L}^1(0,\infty;\mathbb{R}^d)}, \\ h_2' - h_1' &= (\mathbf{R}_2 - \mathbf{R}_1)\tilde{h}_2' + \mathbf{R}_1\left(\tilde{h}_2' - \tilde{h}_1'\right), \\ \|h_2' - h_1'\|_{\mathrm{L}^1(0,\infty;\mathbb{R}^d)} &\leq \|\mathbf{R}_2 - \mathbf{R}_1\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^9)}\|\tilde{h}_2'\|_{\mathrm{L}^1(0,\infty;\mathbb{R}^d)} + \|\tilde{h}_2' - \tilde{h}_1'\|_{\mathrm{L}^1(0,\infty;\mathbb{R}^d)} \\ \|\tilde{h}_2'\|_{\mathrm{L}^1(0,\infty;\mathbb{R}^d)} &\leq \frac{1}{\sqrt{2\lambda}}\|\hat{h}_2'\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)}, \\ \|\tilde{h}_2' - \tilde{h}_1'\|_{\mathrm{L}^1(0,\infty;\mathbb{R}^d)} &\leq \frac{1}{\sqrt{2\lambda}}\|\hat{h}_2' - \hat{h}_1'\|_{\mathrm{L}^2(0,\infty;\mathbb{R}^d)}. \end{aligned}$$

Finally, it is easy to verify that

$$\begin{aligned} \|\tilde{h}_2' - \tilde{h}_1'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} &\leq (1+\lambda)\|\hat{h}_2' - \hat{h}_1'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)}, \\ \|\tilde{\omega}_2 - \tilde{\omega}_1\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} &\leq (1+\lambda)\|\hat{\omega}_2 - \hat{\omega}_1\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)}. \end{aligned}$$

□

Lemma 3.35. *Let X_1 and X_2 be defined by*

$$X_1 = \overline{X}_1^R \circ \overline{X}_1^*, \qquad X_2 = \overline{X}_2^R \circ \overline{X}_2^*,$$

where $\overline{X}_1^*$, $\overline{X}_2^*$, $\overline{X}_1^R$ and $\overline{X}_2^R$ are given in the assumptions of the previous lemmas. Then we have

$$\|X_2 - X_1\|_{\tilde{\mathcal{W}}_3(Q_\infty^0)} = O\left(\|\hat{h}_2' - \hat{h}_1'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} + \|\hat{\omega}_2 - \hat{\omega}_1\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)} + \left\| e^{\lambda t}\frac{\partial X_2^*}{\partial t} - e^{\lambda t}\frac{\partial X_1^*}{\partial t}\right\|_{\mathcal{H}_3(S_\infty^0)} \right).$$

Proof. Let us write

$$X_2 - X_1 = \overline{X}_2^R \circ \overline{X}_2^* - \overline{X}_2^R \circ \overline{X}_1^* + \left(\overline{X}_2^R - \overline{X}_1^R\right) \circ \overline{X}_1^*.$$

For tackling the difference $\overline{X}_2^R \circ \overline{X}_2^* - \overline{X}_2^R \circ \overline{X}_1^*$, let us apply Lemma A.3 of the Appendix of [BB74]; We get

$$\begin{aligned}\|\overline{X}_2^R \circ \overline{X}_2^* - \overline{X}_2^R \circ \overline{X}_1^*\|_{\mathbf{H}^3(\mathcal{F})} \leq\ & C\|\overline{X}_2^R\|_{C^4(\overline{\mathcal{F}})}\|\overline{X}_2^* - \overline{X}_1^*\|_{\mathbf{H}^3(\mathcal{F})} \times \\ & \left(\|\overline{X}_1^*\|^3_{\mathbf{H}^3(\mathcal{F})} + \|\overline{X}_2^*\|^3_{\mathbf{H}^3(\mathcal{F})} + 1\right),\end{aligned}$$

and thus the regularity in $\mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\mathcal{F}))$. The regularity in $\mathrm{W}^{1,\infty}(0,\infty;\mathbf{H}^1(\mathcal{F}))$ can be also obtained in applying Lemma A.3 of [BB74] for the time derivative of $\overline{X}_2^R \circ \overline{X}_2^* - \overline{X}_2^R \circ \overline{X}_1^*$.
For the term $\left(\overline{X}_2^R - \overline{X}_1^R\right) \circ \overline{X}_1^*$, we apply Lemma A.2 of the Appendix of [BB74]; We get

$$\|\left(\overline{X}_2^R - \overline{X}_1^R\right) \circ \overline{X}_1^*\|_{\mathbf{H}^3(\mathcal{F})} \leq \|\overline{X}_2^R - \overline{X}_1^R\|_{C^3(\overline{\mathcal{F}})}\left(\|\overline{X}_1^*\|^3_{\mathbf{H}^3(\mathcal{F})} + 1\right),$$

and thus the regularity in $\mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\mathcal{F}))$. Here again the regularity in $\mathrm{W}^{1,\infty}(0,\infty;\mathbf{H}^1(\mathcal{F}))$ is obtained in applying the same lemma on the time derivative. ☐

Proposition 3.36. *Let $\tilde{X}_1$ and $\tilde{X}_2$ be defined by*

$$\tilde{X}_1 = \mathbf{R}_1^T(X_1 - h_1), \qquad \tilde{X}_2 = \mathbf{R}_2^T(X_1 - h_2),$$

where X_1 and X_2 are given in the assumptions of the previous lemma. Then

$$\|\tilde{X}_2 - \tilde{X}_1\|_{\tilde{\mathcal{W}}_3(Q_\infty^0)} \leq r\tilde{K}(r), \tag{3.139}$$

$$\|\nabla\tilde{Y}_2(\tilde{X}_2) - \nabla\tilde{Y}_1(\tilde{X}_1)\|_{\mathrm{L}^\infty(\mathbf{H}^2)\cap \mathrm{W}^{1,\infty}(\mathbf{L}^2)} \leq r\tilde{K}(r), \tag{3.140}$$

$$\|\Delta\tilde{Y}_2(\tilde{X}_2) - \Delta\tilde{Y}_1(\tilde{X}_1)\|_{\mathrm{L}^\infty(\mathbf{H}^1)} \leq r\tilde{K}(r), \tag{3.141}$$

where

$$r = \|\hat{h}_2' - \hat{h}_1'\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^d)} + \|\hat{\omega}_2 - \hat{\omega}_1\|_{\mathrm{H}^1(0,\infty;\mathbb{R}^3)} + \left\| e^{\lambda t}\frac{\partial X_2^*}{\partial t} - e^{\lambda t}\frac{\partial X_1^*}{\partial t}\right\|_{\mathcal{H}_3(S_\infty^0)},$$

and $\tilde{K}(r)$ is bounded when r goes to 0.

Proof. For proving (3.139), it is sufficient to write

$$\tilde{X}_2 - \tilde{X}_1 = \mathbf{R}_2^T(X_2 - X_1) + \left(\mathbf{R}_2^T - \mathbf{R}_1^T\right)(X_1 - h_1) - \mathbf{R}_2^T(h_2 - h_1)$$

and to apply the previous lemma. The estimate (3.140) can be proven exactly like the estimate (3.138). Finally, for the estimate (3.141) we denote by $\tilde{Y}_{i,1}$ and $\tilde{Y}_{i,2}$ the i-th component of $\tilde{Y}_1$ and $\tilde{Y}_2$ respectively, and we write

$$\begin{aligned}\nabla^2\tilde{Y}_{i,2}(\tilde{X}_2) - \nabla^2\tilde{Y}_{i,1}(\tilde{X}_1) &= \left(\nabla\left(\nabla\tilde{Y}_{i,2}(\tilde{X}_2) - \nabla\tilde{Y}_{i,1}(\tilde{X}_1)\right)\right)\nabla\tilde{Y}_2(\tilde{X}_2)\\ &\quad + \left(\nabla\left(\tilde{Y}_{i,1}(\tilde{X}_1)\right)\right)\left(\nabla\tilde{Y}_2(\tilde{X}_2) - \nabla\tilde{Y}_1(\tilde{X}_1)\right),\end{aligned}$$

and we apply Lemma 3.29. □

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Chapter 4

Practical means for describing solid's deformations

In this chapter, we propose some practical means in order to generate a solid's deformation $X_{\mathcal{S}}(\cdot, t)$ which satisfy the decomposition

$$X_{\mathcal{S}}(y,t) \quad = \quad h(t) + \mathbf{R}(t)X^*(y,t),$$

where X^* is a datum of the problem. Instead of considering an arbitrary deformation X^* - which however has to verify some physical constraints - like in the previous chapters, we search to generate a family of deformations by imposing only one or several restrictive parameters. For instance we can view the deformable solid as a fish, or an eel or a tadpole, that is to say an organism which can act on its spine bone in order to change its whole shape. The spine bone can be modelized as a curve of constant length. We develop in a first part a 2-dimensional classical model (called the Kirchhoff model), already used in [LV05] or [SMSTT08], which consists in considering the curvature of the spine bone as a control function, and in describing - with the help of a tubular neighborhood around this spine bone - the way the whole solid's shape is affected by this control function. Next, we extend this model to the 3D case, by considering as control functions the metric parameters of the Darboux frame whose the spine bone is endowed with.

4.1 The eel-like swimming model

The model we study in this section takes place in 2-dimension.

4.1.1 Presentation

Let us consider an injective curve of length L, parameterized by its arc-length coordinate s (which is a parameter that does not depend on time), and whose the curvature denoted by $\gamma(s,t)$ depends on time. This curve is supposed to represent the spine bone of a solid structure. We assume that at time $t = 0$ this spine bone is a straight line, which is equivalent to assuming that

$$\gamma(s,0) \quad = \quad 0, \quad s \in [0, L].$$

The tangent and normal vectors of this curve, in the inertial frame, denoted by $\mathcal{T}(s,t)$ and $\mathcal{N}(s,t)$ respectively, satisfies the following Frenet-Serret formulas

$$\frac{\partial \mathbf{C}}{\partial s}(s,t) \quad = \quad \mathbf{A}(s,t)\mathbf{C}(s,t),$$

where we denote

$$\mathbf{C}(s,t) = \left(\ \mathcal{T}(s,t) \mid \mathcal{N}(s,t) \ \right)^T, \qquad \mathbf{A}(s,t) = \begin{pmatrix} 0 & \gamma(s,t) \\ -\gamma(s,t) & 0 \end{pmatrix}.$$

Since the set of matrices

$$\left\{ \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \ \gamma \in \mathbb{R} \right\}$$

is an Abelian group, solving this differential system is easy. If we set

$$\alpha(s,t) \quad = \quad \int_0^s \gamma(\varsigma,t)\mathrm{d}\varsigma, \quad s \in [0,L],$$

the resolvent of $\mathbf{A}(\cdot,t)$, that we denote by $\mathbf{B}(\cdot,t)$, has the simple expression

$$\mathbf{B}(s,t) \quad = \quad \begin{pmatrix} \cos\alpha(s,t) & \sin\alpha(s,t) \\ -\sin\alpha(s,t) & \cos\alpha(s,t) \end{pmatrix}.$$

Thus the tangent and normal vectors are determined, up to an initial condition, as

$$\begin{aligned} \mathbf{C}(s,t) \quad &= \quad \mathbf{B}(s,t)\mathbf{C}(0,t), \\ \left(\ \mathcal{T}(s,t) \mid \mathcal{N}(s,t) \ \right) \quad &= \quad \left(\ \mathcal{T}(0,t) \mid \mathcal{N}(0,t) \ \right) \begin{pmatrix} \cos\alpha(s,t) & -\sin\alpha(s,t) \\ \sin\alpha(s,t) & \cos\alpha(s,t) \end{pmatrix}. \end{aligned}$$

The matrix $\mathbf{C}(0,t)$ contains the value of the tangent and normal vectors at the abscissa $s = 0$. This can be viewed as the orientation of the head of the structure, that we will treat later. The arc-parameter of the curve is given by

$$F_{\mathbf{C}}(s,t) \quad = \quad F_{\mathbf{C}}(0,t) + \int_0^s \mathcal{T}(\varsigma,t)\mathrm{d}\varsigma.$$

Here again, the vector $F_{\mathcal{C}}(0,t)$ is - for the moment - undetermined. It can represent the position of the structure's head (in the inertial frame), and will be chosen later.

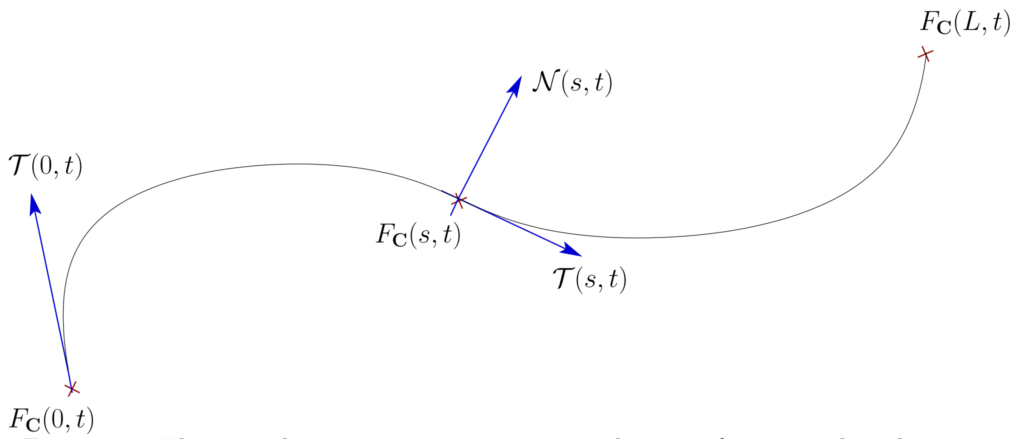

Figure 4.1: The spine bone seen as a parameterized curve of constant length.

We now endow this curve with a tubular neighborhood, defined by a regular function ε representing the radius of the tubular neighborhood, as described in the picture below:

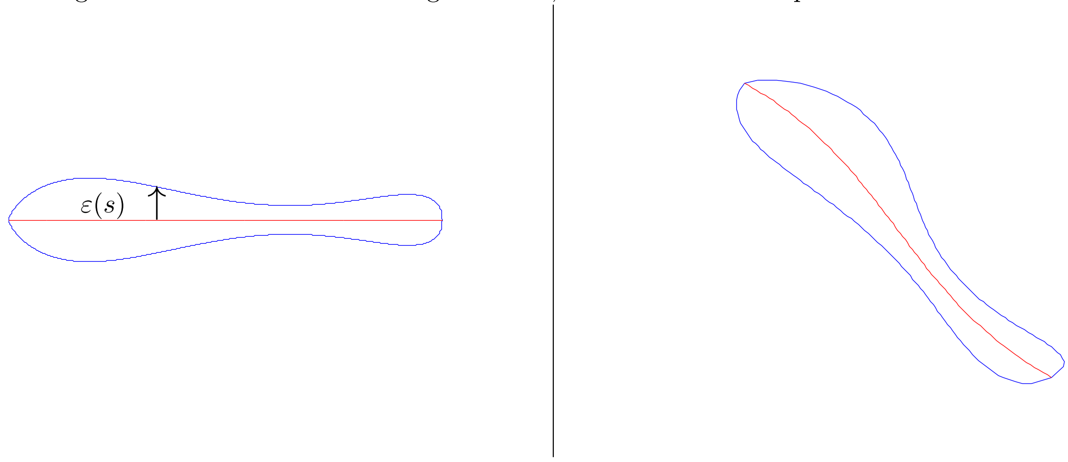

Figure 4.2: Deformation of the structure for a prescribed spine bone curvature.

On the pictures above, we have chosen

$$\begin{aligned}\varepsilon(s) &= 0.01 * s^{0.75}(L-s)^{0.40}(((L-s)-0.35*L)^2+5),\\ \gamma(s,t) &= \frac{-\pi}{2L^2}s(L-s)-\frac{\pi}{4} \text{ for the right picture.}\end{aligned}$$

In general, the function ε is assumed to be positive on $(0,L)$, and must satisfy

$$\begin{aligned}&\varepsilon(0)=\varepsilon(L)=0,\\ &\lim_{s\mapsto 0}\varepsilon'(s)=+\infty, \quad \lim_{s\mapsto L}\varepsilon'(s)=-\infty.\end{aligned}$$

The conditions on ε' are given in order to get some regularity on the structure's shape so obtained. Supplementary conditions can be given on the higher order derivatives for this function, if we desire more regularity. Notice that this approach provides a parametrization of the solid's boundary, given as :

$$\begin{cases} y_1 &= s,\\ y_2 &= \pm\varepsilon(s).\end{cases}$$

Then the Lagrangian mapping so obtained for the description of the structure is the following

$$X_{\mathcal{S}}(y,t) \quad = \quad F_{\mathbf{C}}(y_1,t)+y_2\mathcal{N}(y_1,t), \quad y_2\in[-\varepsilon(y_1),\varepsilon(y_1)],\ y_1\in[0,L].$$

We can rewrite this mapping as follows

$$X_{\mathcal{S}}(y,t) \quad = \quad F_{\mathbf{C}}(0,t)+\mathbf{C}(0,t)^T B(y,t),$$

with

$$B(y,t) \quad = \quad \int_0^{y_1} \begin{pmatrix} \cos\alpha(s,t) \\ \sin\alpha(s,t) \end{pmatrix} \mathrm{d}s + y_2 \begin{pmatrix} -\sin\alpha(y_1,t) \\ \cos\alpha(y_1,t) \end{pmatrix}, \quad y_2 \in [-\varepsilon(y_1), \varepsilon(y_1)], \ y_1 \in [0, L]. \tag{4.1}$$

Let us make explicit right now the vector $F_{\mathbf{C}}(0,t)$ and the rotation $\mathbf{C}(0,t)$.

4.1.2 Conservation of the momenta

The vector $F_{\mathbf{C}}(0,t)$ and the rotation $\mathbf{C}(0,t)$ have to be determined, and cannot be chosen arbitrarily. More precisely, in the referential associated with the solid, we can only choose these two quantities

$$\tilde{F}_{\mathbf{C}}(0,t) = \mathbf{R}^T \left(F_{\mathbf{C}}(0,t) - h(t) \right), \qquad \tilde{\mathbf{C}}(0,t) = \mathbf{R}^T \mathbf{C}(0,t)^T, \tag{4.2}$$

where we recall that the rotation $\mathbf{R}$ is associated with the angular velocity ω of the whole solid. These two quantities can be expressed only in terms of our control parameter, γ or α. Actually, the hypotheses **H3** and **H4**, that guarantee the *self-propelled* nature of the solid, impose the choice of these quantities. Indeed, writing

$$\begin{aligned} Mh(t) &= \int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0) X_{\mathcal{S}}(y,t) \mathrm{d}y, \\ I(t)\omega(t) &= \int_{\mathcal{S}} \rho_{\mathcal{S}} \rho_{\mathcal{S}}(y,0)(X_{\mathcal{S}}(y,t) - h(t)) \wedge \left(\frac{\partial X_{\mathcal{S}}}{\partial t}(y,t) - h'(t) \right) \mathrm{d}y, \end{aligned}$$

is equivalent to choosing

$$\tilde{F}_{\mathbf{C}}(0,t) = -\mathbf{R}_B(t)^T h_B(t), \qquad \tilde{\mathbf{C}}(0,t) = \mathbf{R}_B(t)^T,$$

where

$$h_B(t) \quad = \quad \frac{1}{M} \int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0) B(y,t) \mathrm{d}y,$$

and where the rotation $\mathbf{R}_B(t)$ is obtained as the solution of the Cauchy problem

$$\begin{aligned} \frac{\mathrm{d}\mathbf{R}_B}{\mathrm{d}t} &= \mathbb{S}(\omega_B)\,\mathbf{R}_B \\ \mathbf{R}_B(0) &= \mathrm{I}_{\mathbb{R}^3}, \end{aligned} \qquad \text{with } \mathbb{S}(\omega_B) = \begin{pmatrix} 0 & -\omega_{B3} & \omega_{B2} \\ \omega_{B3} & 0 & -\omega_{B1} \\ -\omega_{B2} & \omega_{B1} & 0 \end{pmatrix}$$

and

$$\begin{aligned} \omega_B(t) &= I_B(t)^{-1} \int_{\mathcal{S}} \rho_{\mathcal{S}} B(y,t) \wedge \frac{\partial B}{\partial t} \mathrm{d}y, \\ I_B(t) &= \int_{\mathcal{S}} \rho_{\mathcal{S}} \left| B(y,t) - h_B(t) \right|^2 \mathrm{d}y. \end{aligned}$$

Notice that it is possible to choose the constants $\tilde{F}_{\mathbf{C}}(0,t)$ and $\tilde{\mathbf{C}}(0,t)$ in such a way, because they depend only on the mapping B, and the latter depends only on α, and thus on γ which is our control parameter.
The solid's deformation in the inertial frame is decomposed as follows

$$\begin{aligned} X_{\mathcal{S}}(y,t) &= F_{\mathbf{C}}(0,t) + \mathbf{C}(0,t)^T B(y,t) \\ &= h(t) + \mathbf{R}(t) \left(\tilde{F}_{\mathbf{C}}(0,t) + \tilde{\mathbf{C}}(0,t) B(y,t) \right) \\ &= h(t) + \mathbf{R}(t) \mathbf{R}_B(t)^T (B(y,t) - h_B(t)). \end{aligned}$$

Thus we can identify the deformation X^* as

$$X^*(y,t) \quad = \quad \mathbf{R}_B(t)^T(B(y,t) - h_B(t)).$$

It depends only on the mapping B, which is entirely defined by the curvature of the structure's spine bone γ chosen as a control function. By consequence, the mapping X^* so obtained is well a datum of the problem.

4.1.3 Invertibility of the deformation so generated

Let us study the invertibility of the mapping $X_{\mathcal{S}}$ so obtained. Since this property is invariant by translations and rotations, let us do it for the mapping B. The determinant of its gradient can be easily calculated, and it gives

$$\det \nabla B(y,t) \quad = \quad 1 - y_2\gamma(y_1,t).$$

Thus this determinant remains positive if we have the following condition satisfied for all time t:

$$\sup_{s\in(0,L)} |\varepsilon(s)\gamma(s,t)| \quad < \quad 1. \tag{4.3}$$

Then it is easy to see that under this condition the mapping $B(\cdot,t)$ is a $C^1 - diffeomorphism$, provided that the function ε is regular enough. Assuming this condition is equivalent to considering small curvatures γ, that is to say small deformations of the spine bone, compared to the radius of the tubular neighborhood.

4.1.4 Conservation of the volume

Let us observe that, under the condition (4.3), the deformation generated in this model preserves the structure's volume through the time. Indeed, we can calculate

$$\begin{aligned} V(t) = \int_{\mathcal{S}(t)} 1\mathrm{d}x \quad &= \quad \int_{\mathcal{S}(0)} |\det \nabla X_{\mathcal{S}}(y,t)|\, \mathrm{d}y \\ &= \quad \int_0^L \int_{-\varepsilon(y_1)}^{\varepsilon(y_1)} (1 - y_2\gamma(y_1,t))\, \mathrm{d}y_2 \mathrm{d}y_1 \\ &= \quad 2\int_0^L \varepsilon(s)\mathrm{d}s. \end{aligned}$$

In this 2D-case, this condition - corresponding to the hypothesis **H2** - is satisfied without assuming more hypothesis on ε. We will see that it is not the case anymore for the analogous 3D model.

4.1.5 Linearization for small deformations

In this paragraph, let us take a look at the expressions of the kinematic quantities of the model, when the curvature γ is close to 0. First, the Lagrangian velocity associated to the mapping $X^*(\cdot,t)$ generated by this model can be calculated as

$$\frac{\partial X^*}{\partial t}(y,t) \quad = \quad \mathbf{R}_B(t)^T\left(\frac{\partial B}{\partial t}(y,t) - h_B'(t) - \omega_B(t)\wedge(B(y,t) - h_B(t))\right),$$

with

$$\frac{\partial B}{\partial t}(y,t) = \int_0^{y_1} \dot{\alpha}(s,t) \left(\begin{array}{c} -\sin\alpha(s,t) \\ \cos\alpha(s,t) \end{array} \right) \mathrm{d}s - \dot{\alpha}(y_1,t) y_2 \left(\begin{array}{c} \cos\alpha(y_1,t) \\ \sin\alpha(y_1,t) \end{array} \right).$$

Let us denote

$$\beta(s,t) = \int_0^s \alpha(\varsigma,t)\mathrm{d}\varsigma.$$

The mappings $\frac{\partial B}{\partial t}$, h_B, h_B', $\mathbf{R}_B$ and ω_B are nonlinear with respect to the control functions α and $\dot{\alpha}$). Let us linearize their expressions around $(\alpha,\dot{\alpha}) = (0,0)$ in the simple case where the density $\rho_{\mathcal{S}}(\cdot,0)$ at time $t=0$ is constant equal to $\rho_{\mathcal{S}} > 0$:

$$\begin{aligned}
\frac{\partial B}{\partial t}(y,t) &= \left(\begin{array}{c} -y_2\dot{\alpha}(y_1,t) \\ \dot{\beta}(y_1,t) \end{array} \right) + o(\alpha,\dot{\alpha}), \\
h_B(t) &= \frac{\rho_{\mathcal{S}}}{M}\int_{\mathcal{S}} (y-h_0)\mathrm{d}y + o(\alpha) = o(\alpha), \\
h_B'(t) &= \frac{\rho_{\mathcal{S}}}{M}\int_{\mathcal{S}} \left(\begin{array}{c} -y_2\dot{\alpha}(s,t) \\ \dot{\beta}(s,t) \end{array} \right) \mathrm{d}y + o(\alpha,\dot{\alpha}) \\
&= \frac{\displaystyle\int_0^L \varepsilon(s)\dot{\beta}(s,t)\mathrm{d}s}{\displaystyle\int_0^L \varepsilon(s)\mathrm{d}s} \left(\begin{array}{c} 0 \\ 1 \end{array} \right) + o(\alpha,\dot{\alpha}), \\
\omega_B(t) &= \rho_{\mathcal{S}} I_0^{-1} \int_{\mathcal{S}} y \wedge \left(\begin{array}{c} -y_2\dot{\alpha}(y_1,t) \\ \dot{\beta}(y_1,t) \end{array} \right) \mathrm{d}y + o(\alpha,\dot{\alpha}) \\
&= \frac{\displaystyle\int_0^L \left(2s\varepsilon(s)\dot{\beta}(s,t) + \frac{2}{3}\varepsilon(s)^3\dot{\alpha}(s,t) \right) \mathrm{d}s}{\displaystyle\int_0^L \left(2s^2\varepsilon(s) + \frac{2}{3}\varepsilon(s)^3 \right) \mathrm{d}s} + o(\alpha,\dot{\alpha}).
\end{aligned}$$

Thus, combining these linearized expressions with equality (4.4), we obtain

$$\frac{\partial X^*}{\partial t}(y,t) = v^*(y,t) + o(\alpha,\dot{\alpha}),$$

with

$$\begin{aligned}
v^*(y,t) &= \left(\begin{array}{c} -y_2\dot{\alpha}(y_1,t) \\ \dot{\beta}(y_1,t) \end{array} \right) - H_B'(t) - \Omega_B(t) \wedge y, \\
H_B'(t) &= \frac{\displaystyle\int_0^L \varepsilon(s)\dot{\beta}(s,t)\mathrm{d}s}{\displaystyle\int_0^L \varepsilon(s)\mathrm{d}s} \left(\begin{array}{c} 0 \\ 1 \end{array} \right), \\
\Omega_B(t) &= \frac{\displaystyle\int_0^L \left(2s\varepsilon(s)\dot{\beta}(s,t) + \frac{2}{3}\varepsilon(s)^3\dot{\alpha}(s,t) \right) \mathrm{d}s}{\displaystyle\int_0^L \left(2s^2\varepsilon(s) + \frac{2}{3}\varepsilon(s)^3 \right) \mathrm{d}s}.
\end{aligned}$$

Remark 4.1. *We can easily verify that the linearized velocity* v^* *so obtained is admissible (in the sense of Definition 3.2) for the linear system* (3.11)–(3.17).

4.2 Extension to the 3D case.

The model we propose in this section is the extension to the 3D framework of the 2-dimensional model studied previously.

4.2.1 Presentation

For this 3-dimensional model, we still consider the structure as a spine bone surrounded by a tubular neighborhood. The spine bone is seen as a curve parameterized by its arc-coordinate s, and the tubular neighborhood is seen as a surface that we will describe further. The time-depending parameters chosen to describe this curve are the one of the Darboux frame, that is to say $\kappa_g(s,t)$ the geodesic curvature, $\kappa_n(s,t)$ the normal curvature and $\tau_r(s,t)$ the relative torsion. The spine bone has a constant length equal to L, and is assumed to be straight at time $t=0$, so that

$$\kappa_g(s,0) = \kappa_n(s,0) = \tau_r(s,0) = 0, \quad s \in [0,L].$$

Denoting $(\mathcal{T},\mathcal{N},\mathcal{B})$ the orthonormal basis provided by the Darboux frame, and

$$\mathbf{C} = \left(\begin{array}{c|c|c} \mathcal{T} & \mathcal{N} & \mathcal{B} \end{array} \right)^T, \qquad \mathbf{A} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_r \\ -\kappa_n & -\tau_r & 0 \end{pmatrix},$$

let us recall the Frenet-Serret formulas that we write as follows

$$\frac{\partial \mathbf{C}}{\partial s}(s,t) \quad = \quad \mathbf{A}(s,t)\mathbf{C}(s,t).$$

The resolvent $\mathbf{B}(\cdot,t)$ of the matrix $\mathbf{A}(\cdot,t)$ satisfies

$$\mathbf{C}(s,t)^T = \mathbf{C}(0,t)^T\mathbf{B}(s,t)^T, \quad \text{with} \quad \mathbf{C}(0,t)^T = \left(\begin{array}{c|c|c} \mathcal{T}(0,t) & \mathcal{N}(0,t) & \mathcal{B}(0,t) \end{array} \right).$$

The matrix $\mathbf{C}(0,t)^T$ can represent the orientation of the head of the structure, it will be made explicit further. The position of a point of the spine bone (in the inertial frame), located by its abscissa s, is given by the arc-parameter $F_{\mathbf{C}}(s,t)$, whose expression is

$$F_{\mathbf{C}}(s,t) \quad = \quad F_{\mathbf{C}}(0,t) + \int_0^s \mathcal{T}(\varsigma,t)\mathrm{d}\varsigma.$$

Here again the vector $F_{\mathbf{C}}(0,t)$ - which can represent the position of the head of the structure - remains to be determined.

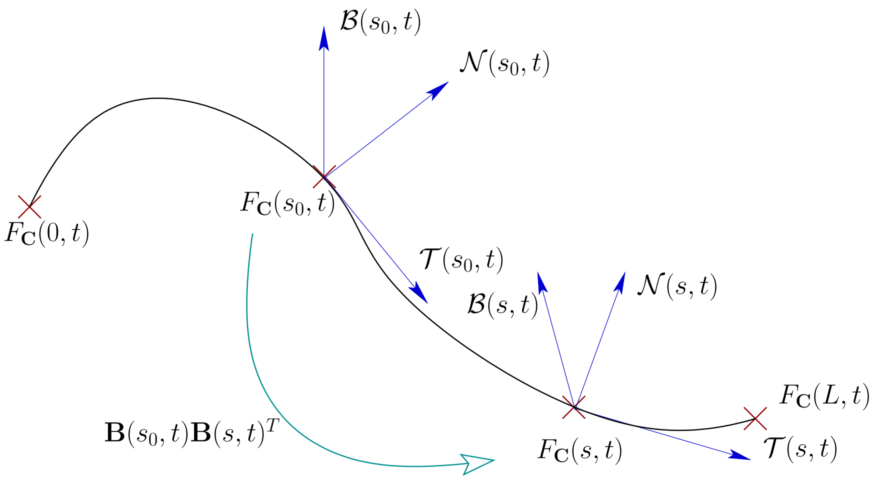

Figure 4.3: Evolution of the Darboux frame on the spine bone

We now endow the curve with a surrounding surface, described as a tubular neighborhood defined by a function ε which represents the radius of this surface. A point on this surrounding surface is located with the arc-coordinate s of the corresponding point on the spine bone (see the picture above), and with an angle φ.

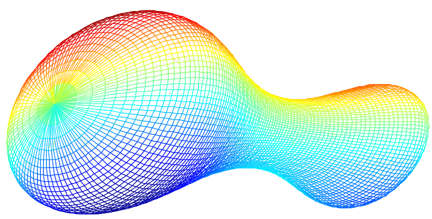

Figure 4.4: Example of a surface obtained with $\varepsilon(s,\varphi) = 0.01 * (L-s)^{0.75} s^{0.40}((s-0.35L)^2+5)$.

Such a point has for Cartesian coordinates

$$y_1 = s, \qquad y_2 = \varepsilon(s,\varphi)\cos\varphi, \qquad y_3 = \varepsilon(s,\varphi)\sin\varphi.$$

Like for the 2D-case, let us assume that

$$\begin{aligned} &\varepsilon(0,\varphi) = \varepsilon(L,\varphi) = 0, \\ &\lim_{s\mapsto 0} \varepsilon'(s,\varphi) = +\infty, \quad \lim_{s\mapsto L} \varepsilon'(s,\varphi) = -\infty. \end{aligned}$$

The structure so obtained can be parameterized in cylindrical coordinates, as follows

$$y_1 = s, \qquad y_2 = r\cos\kappa, \qquad y_3 = r\sin\kappa,$$

with $r \in [0, \varepsilon(s,\kappa)]$. The corresponding Lagrangian mapping is defined as

$$X_{\mathcal{S}}(y,t) \quad = \quad F_{\mathbf{C}}(y_1,t) + y_2\mathcal{N}(y_1,t) + y_3\mathcal{B}(y_1,t),$$

for $y_2^2 + y_3^2 \leq \varepsilon(y_1,\varphi)$, $\varphi \in [0,2\pi]$ and $s \in [0,L]$, with

$$B(y,t) \quad = \quad \int_0^{y_1} \tilde{\mathcal{T}}(\sigma,t)\mathrm{d}\sigma + y_2\tilde{\mathcal{N}}(y_1,t) + y_3\tilde{\mathcal{B}}(y_1,t).$$

The vectors $\mathcal{T}$, $\mathcal{N}$ and $\mathcal{B}$ are the columns of the matrix $\mathbf{B}^T$ which depends only on κ_g, κ_n and τ_r, so that the mapping B depends only on the control functions κ_g, κ_n and τ_r.
We now proceed as for the 2-dimensional model, in decomposing

$$\begin{aligned} X_{\mathcal{S}}(y,t) &= F_{\mathbf{C}}(0,t) + \mathbf{C}(0,t)^T B(y,t) \\ &= h(t) + \mathbf{R}(t)\left(\tilde{F}_{\mathbf{C}}(0,t) + \tilde{\mathbf{C}}(0,t)B(y,t)\right), \end{aligned}$$

where the two quantities $\tilde{F}_{\mathbf{C}}(0,t)$ and $\tilde{\mathbf{C}}(0,t)$ have to be chosen in terms of the data.

4.2.2 Conservation of the momenta

In order to satisfy the hypotheses **H3** and **H4**, the steps of the process detailed for the 2-dimensional model can be straightforwardly repeated. Thus we obtain the decomposition

$$X_{\mathcal{S}}(y,t) \quad = \quad h(t) + \mathbf{R}(t)X^*(y,t)$$

with

$$\begin{aligned} X^*(y,t) &= \mathbf{R}_B(t)^T\left(B(y,t) - h_B(t)\right), \\ h_B(t) &= \frac{1}{M}\int_{\mathcal{S}} B(y,t)\mathrm{d}y, \\ \omega_B(t) &= I_B(t)^{-1}\int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0)(B(y,t) - h_B(t)) \wedge \left(\frac{\partial B}{\partial t}(y,t) - h_B'(t)\right)\mathrm{d}y, \end{aligned}$$

but with the difference that the inertia matrix associated with the mapping $B(\cdot,t)$ has not the same expression in 3-dimension:

$$I_B(t) \quad = \quad \int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0)\left(|B(y,t) - h_B(t)|^2 \mathrm{I}_{\mathbb{R}^3} - (B(y,t) - h_B(t)) \otimes (B(y,t) - h_B(t))\right)\mathrm{d}y.$$

4.2.3 Invertibility of the deformation so generated

As in dimension 2, we can caluclate

$$\det\nabla B(y,t) \quad = \quad 1 - y_2\kappa_g(y_1,t) - y_3\kappa_n(y_1,t).$$

Thus, the following condition is required in order to make the mapping $B(\cdot,t)$, and so $X^*(\cdot,t)$, a C^1-diffeormorphism:

$$\sup_{(s,\varphi)\in(0,L)\times(0,2\pi)} \varepsilon(s,\varphi)\sqrt{\kappa_g(s,t)^2 + \kappa_n(s,t)^2} \quad < \quad 1. \tag{4.4}$$

Notice that this condition requires only that κ_g and κ_n (representing the pitching and the hook of the structure) have to be small enough. The relative torsion τ_r which represents the rolling of the structure (around the axe associated with its spine bone) does not affect its volume.

4.2.4 Conservation of the volume

In order to prove that - under the condition (4.4) - the volume of the whole solid is preserved through the time, we shall assume that ε has a symmetry, given by

$$\varepsilon\left(s, \frac{\pi}{2} - \kappa\right) = \varepsilon\left(s, \frac{\pi}{2} + \kappa\right). \tag{4.5}$$

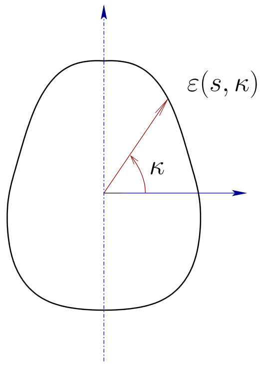

Figure 4.5: Cross section of the structure.

This symmetry on ε can be found in the shape of many locomotive animals, like the shape of a fish or a snake. Actually, swimming animals that do not present this symmetry are quite rare in nature. Let us verify that assuming this hypothesis enables us to prove that the volume of the whole solid obtained by this model is constant. We calculate

$$\begin{aligned}
V(t) &= \int_{\mathcal{S}(t)} 1\mathrm{d}x = \int_{\mathcal{S}} |\det \nabla X_{\mathcal{S}}(y,t)| \mathrm{d}y \\
&= \int_{\mathcal{S}} \left(1 - y_2\kappa_g(y_1,t) - y_3\kappa_n(y_1,t)\right) \mathrm{d}y \\
&= \int_0^L \int_0^{2\pi} \int_0^{\varepsilon(s,\varphi)} \left(1 - r(\cos\varphi)\kappa_g(s,t) - r(\sin\varphi)\kappa_n(s,t)\right) r\mathrm{d}r\mathrm{d}\varphi\mathrm{d}s \\
&= \int_0^L \int_0^{2\pi} \int_0^{\varepsilon(s,\varphi)} \left(\frac{\varepsilon(s,\varphi)^2}{2} - \frac{\varepsilon(s,\varphi)^3}{3}\left((\cos\varphi)\kappa_g(s,t) - (\sin\varphi)\kappa_n(s,t)\right)\right) \mathrm{d}\varphi\mathrm{d}s.
\end{aligned}$$

Then we notice that under the condition (4.5) we have

$$\int_0^{2\pi} \varepsilon(s,\varphi)^3 \cos\varphi \mathrm{d}\varphi = \int_0^{2\pi} \varepsilon(s,\varphi)^3 \sin\varphi \mathrm{d}\varphi = 0.$$

Thus

$$V(t) = \frac{1}{2}\int_0^L \int_0^{2\pi} \varepsilon(s,\varphi)^2 \mathrm{d}\varphi\mathrm{d}s.$$

4.2.5 Linearization of the 3D model

Recall that the matrix $\mathbf{B}$ is the resolvent of the matrix $\mathbf{A}$, and thus can be expressed with the exponential a Magnus expansion. If we linearize the terms of this expansion for small κ_g, τ_r and κ_n, only the first term of this expansion remains, so that the matrix $\mathbf{B}$ can be approximated as

$$\mathbf{B}(s,t) \quad \simeq \quad \exp\left(\int_0^s \mathbf{A}(\varsigma,t)\mathrm{d}\varsigma\right).$$

Let us denote

$$\mathfrak{K}_g(s,t) = \int_0^s \kappa_g(\varsigma,t)\mathrm{d}\varsigma, \quad \mathfrak{K}_n(s,t) = \int_0^s \kappa_n(\varsigma,t)\mathrm{d}\varsigma, \quad \vartheta(s,t) = \int_0^s \tau_r(\varsigma,t)\mathrm{d}\varsigma,$$

and

$$\zeta = \sqrt{\mathfrak{K}_g^2 + \mathfrak{K}_n^2 + \vartheta^2}.$$

We can calculate $\exp\left(\int_0^s \mathbf{A}(\varsigma,t)\mathrm{d}\varsigma\right)$ as

$$\mathbf{B} \quad \simeq \quad \begin{pmatrix} \frac{\vartheta^2}{\zeta^2} + \frac{\mathfrak{K}_g^2+\mathfrak{K}_n^2}{\zeta^2}\cos\zeta & \frac{\mathfrak{K}_g}{\zeta}\cos\zeta + \frac{\vartheta\mathfrak{K}_n}{\zeta^2}(\cos\zeta - 1) & \frac{\mathfrak{K}_n}{\zeta}\sin\zeta + \frac{\mathfrak{K}_g\vartheta}{\zeta^2}(1-\cos\zeta) \\ -\frac{\mathfrak{K}_g}{\zeta}\sin\zeta - \frac{\vartheta\mathfrak{K}_n}{\zeta^2}(1-\cos\zeta) & \frac{\mathfrak{K}_n^2}{\zeta^2} + \frac{\mathfrak{K}_g^2+\vartheta^2}{\zeta^2}\cos\zeta & \frac{\vartheta}{\zeta}\sin\zeta - \frac{\mathfrak{K}_g\mathfrak{K}_n}{\zeta^2}(1-\cos\zeta) \\ -\frac{\mathfrak{K}_n}{\zeta}\sin\zeta + \frac{\mathfrak{K}_g\vartheta}{\zeta^2}(1-\cos\zeta) & -\frac{\vartheta}{\zeta}\sin\zeta - \frac{\mathfrak{K}_g\mathfrak{K}_n}{\zeta^2}(1-\cos\zeta) & \frac{\mathfrak{K}_g^2}{\zeta^2} + \frac{\vartheta^2+\mathfrak{K}_n^2}{\zeta^2}\cos\zeta \end{pmatrix}.$$

And here again, for small $\mathfrak{K}_g$, $\mathfrak{K}_n$ and ϑ, we approximate

$$\mathbf{B} \quad \simeq \quad \begin{pmatrix} 1 & \mathfrak{K}_g & \mathfrak{K}_n \\ -\mathfrak{K}_g & 1 & \vartheta \\ -\mathfrak{K}_n & -\vartheta & 1 \end{pmatrix}.$$

Thus we identify

$$\tilde{\mathcal{T}} \simeq \begin{pmatrix} 1 \\ \mathfrak{K}_g \\ \mathfrak{K}_n \end{pmatrix}, \ \tilde{\mathcal{N}} \simeq \begin{pmatrix} -\mathfrak{K}_g \\ 1 \\ \vartheta \end{pmatrix}, \ \tilde{\mathcal{B}} \simeq \begin{pmatrix} -\mathfrak{K}_n \\ -\vartheta \\ 1 \end{pmatrix},$$

and the Lagrangian mapping B can be approximated as

$$B(y,t) \quad \simeq \quad \begin{pmatrix} y_1 - \mathfrak{K}_g(y_1,t)y_2 - \mathfrak{K}_n(y_1,t)y_3 \\ K_g(y_1,t) + y_2 - \vartheta y_3 \\ K_n(y_1,t) + \vartheta y_2 + y_3 \end{pmatrix},$$

if we denote

$$K_g(y_1,t) = \int_0^{y_1} \mathfrak{K}_g(s,t)\mathrm{d}s, \qquad K_n(y_1,t) = \int_0^{y_1} \mathfrak{K}_n(s,t)\mathrm{d}s.$$

Similarly, we approximate its Lagrangian velocity as

$$\frac{\partial B}{\partial t}(y,t) \quad \simeq \quad \begin{pmatrix} -\dot{\mathfrak{K}}_g(y_1,t)y_2 - \dot{\mathfrak{K}}_n(y_1,t)y_3 \\ \dot{K}_g(y_1,t) - \dot{\vartheta}y_3 \\ \dot{K}_n(y_1,t) + \dot{\vartheta}y_2 \end{pmatrix}.$$

Recall that the velocity v^* that we can consider as control is expressed as

$$v^*(y,t) \quad = \quad \frac{\partial B}{\partial t}(y,t) - H_B'(t) - \Omega_B(t) \wedge y, \tag{4.6}$$

with

$$H_B'(t) = \frac{1}{M}\int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0)\frac{\partial B}{\partial t}(y,t)\mathrm{d}y, \qquad \Omega_B(t) = I_0^{-1}\int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0) y \wedge \frac{\partial B}{\partial t}(y,t)\mathrm{d}y.$$

In the simple case where the density $\rho_{\mathcal{S}}(\cdot,0)$ at time $t=0$ is constant equal to $\rho_{\mathcal{S}} > 0$, and where the function ε does not depend on the angular parameter φ, but only on the curvilinear abscissa s, as follows

$$\varepsilon(s,\varphi) \quad = \quad \varepsilon(s),$$

we can calculate

$$H_B'(t) \quad = \quad \left(\int_0^L \varepsilon(s)^2 \mathrm{d}s\right)^{-1} \begin{pmatrix} 0 \\ \displaystyle\int_0^L \varepsilon(s)^2 \dot{K}_g(s,t)\mathrm{d}s \\ \displaystyle\int_0^L \varepsilon(s)^2 \dot{K}_n(s,t)\mathrm{d}s \end{pmatrix},$$

$$\Omega_B(t) \quad = \quad \begin{pmatrix} \displaystyle\left(\int_0^L \varepsilon(s)^4\mathrm{d}s\right)^{-1} \int_0^L \dot{\vartheta}(s,t)\varepsilon(s)^4\mathrm{d}s \\ \displaystyle -\left(\int_0^L \left(s^2\varepsilon(s)^2 + \frac{\varepsilon(s)^2}{4}\right)\mathrm{d}s\right)^{-1} \int_0^L \left(\dot{K}_n(s,t)s\varepsilon(s)^2 + \dot{\mathfrak{K}}_n(s,t)\frac{\varepsilon(s)^4}{4}\right)\mathrm{d}s \\ \displaystyle\left(\int_0^L \left(s^2\varepsilon(s)^2 + \frac{\varepsilon(s)^2}{4}\right)\mathrm{d}s\right)^{-1} \int_0^L \left(\dot{K}_g(s,t)s\varepsilon(s)^2 + \dot{\mathfrak{K}}_g(s,t)\frac{\varepsilon(s)^4}{4}\right)\mathrm{d}s \end{pmatrix}.$$

4.3 Back on the problem of approximate controllability for the linearized model

Given this family of deformations X^* generated by these models, let us see if acting on the shape of the spine bone enables us to get some information on the approximate controllability of the linearized fluid-solid system. Since the 2-dimensional model is more simple than the 3-dimensional one, let us consider in this section only the 3-dimensional model. Let us recall the equality (3.71) that we obtain in considering the corresponding adjoint system (3.64)–(3.70):

$$\int_0^T \int_{\partial\mathcal{S}} v^* \cdot \sigma(\phi,\psi)n\mathrm{d}\Gamma \quad = \quad 0. \tag{4.7}$$

Let us recall too the adjoint equations corresponding to the Newton's laws:

$$Mk''(t) = \int_{\partial\mathcal{S}} \sigma(\phi,\psi)n\mathrm{d}\Gamma, \quad t \in (0,T),$$

$$I_0 r'(t) = \int_{\partial\mathcal{S}} y \wedge \sigma(\phi,\psi)n\mathrm{d}\Gamma, \quad t \in (0,T),$$

Notice that in the expression of v^* - given in (4.6) - obtained for small deformations of the spine bone, only the quantities $(\dot{K}_g, \dot{K}_n)$ and $(\dot{\mathfrak{K}}_g, \dot{\mathfrak{K}}_n, \dot{\vartheta})$ appear, knowing that we have the relations

$$\dot{K}_g(y_1,t) = \int_0^{y_1} \dot{\mathfrak{K}}_g(s,t)\mathrm{d}s, \qquad \dot{K}_n(y_1,t) = \int_0^{y_1} \dot{\mathfrak{K}}_n(s,t)\mathrm{d}s.$$

Thus in this section we consider as control functions the quantities $\dot{\mathfrak{K}}_g$, $\dot{\mathfrak{K}}_n$ and $\dot{\vartheta}$. They represent some angular velocities indicating the orientation of the spine bone. Then the question we set is the following: what quantities of the adjoint system can we cancel in acting only on $\dot{\mathfrak{K}}_g$, $\dot{\mathfrak{K}}_n$ and $\dot{\vartheta}$ in the equality (4.7)?

Let us notice that if $\dot{\mathfrak{K}}_g$, $\dot{\mathfrak{K}}_n$ and $\dot{\vartheta}$ are chosen constant with respect to the curvilinear abscissa $y_1 = s$, that is to say these functions depend only on time, as follows

$$\dot{\mathfrak{K}}_g(s,t) = \dot{\mathfrak{K}}_g(t), \quad \dot{\mathfrak{K}}_n(s,t) = \dot{\mathfrak{K}}(t), \quad \dot{\vartheta}(s,t) = \dot{\vartheta}(t),$$

then we can verify that the velocity v^* has the simple expression.

$$\begin{aligned} v^*(y,t) &= \begin{pmatrix} -\dot{\mathfrak{K}}_g(t)y_2 - \dot{\mathfrak{K}}_n(t)y_3 \\ \dot{\mathfrak{K}}_g(t)y_1 - \dot{\vartheta}(t)y_3 \\ \dot{\mathfrak{K}}_n(t)y_1 + \dot{\vartheta}(t)y_2 \end{pmatrix} - C_\varepsilon \begin{pmatrix} 0 \\ \dot{\mathfrak{K}}_g(t) \\ \dot{\mathfrak{K}}_n(t) \end{pmatrix} - \begin{pmatrix} \dot{\vartheta}(t) \\ -\dot{\mathfrak{K}}_n(t) \\ \dot{\mathfrak{K}}_g(t) \end{pmatrix} \wedge y, \\ v^*(y,t) &= -C_\varepsilon \begin{pmatrix} 0 \\ \dot{\mathfrak{K}}_g(t) \\ \dot{\mathfrak{K}}_n(t) \end{pmatrix}. \end{aligned}$$

where

$$C_\varepsilon = \left(\int_0^L \varepsilon(s)^2 \mathrm{d}s\right)^{-1} \left(\int_0^L s\varepsilon(s)^2 \mathrm{d}s\right).$$

Then, with this expression of v^*, the equality (4.7) becomes

$$\int_0^T \left(k_2''(t)\dot{\mathfrak{K}}_g(t) + k_3''(t)\dot{\mathfrak{K}}_n(t)\right)\mathrm{d}t = 0,$$

and by integration by parts we get

$$\int_0^T \left(k_2'(t)\ddot{\mathfrak{K}}_g(t) + k_3'(t)\ddot{\mathfrak{K}}_n(t)\right)\mathrm{d}t = 0,$$

so that we obtain $k_2' = k_3' = 0$. In other words, we are able to control (approximately) the radial velocity of the structure to zero.

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Chapter 5

A fictitious domain approach for the Stokes problem based on the extended finite element method

In this chapter, we propose to extend to the Stokes problem a fictitious domain approach inspired by eXtended Finite Element Method [MDB99] and studied for Poisson problem in [HR09]. The method allows computations in domains whose boundaries are independent of the mesh. A mixed finite element method is used for fluid flow. For Fluid-structure interactions, at the interface localized by level-set, Dirichlet boundary conditions are taken into account using Lagrange multiplier. A stabilization term is introduced to improve the approximation of the normal constraint tensor at the interface and avoid to have to impose a discrete inf-sup condition. Convergence analysis are given and several numerical tests are performed to illustrate the capabilities of the method.

The contents of this chapter have been submitted as an article, in collaboration with Michel Fournié and Alexei Lozinski.

5.1 Introduction

In this chapter, we are interested in applying on a Stokes problem the fictitious domain approach introduced in [HR09] for Poisson problem and based on the ideas of the eXtended Finite Element Method introduced by Moës, Dolbow and Belytschko in [MDB99] (Xfem in abbreviated form). Xfem was developed in many papers such as [CLR08, HR09, LPRS05, MGB02, SBCB03, SMMB00]. The enrichment of a finite element space with a singular function has been studied earlier by Nitsche in [Nit71], and Strang and Fix in [SF73]. The first application of Xfem was done in structural mechanics when dealing with cracked domains, as in [SCMB01a]. The specificity of the method is that it combines a level-set representation of the geometry of the crack with an enrichment of a finite element space by singular and discontinuous functions. Several strategies can be considered in order to improve the original Xfem. Some of these strategies are mathematically analyzed in [LPRS05]. An *a priori* error estimate of a variant of Xfem for cracked domains is presented in [CLR08]. The approach enables computations in domains whose boundaries are independent of the mesh. A similar attempt was done in [MBT06, SCMB01b]. The purpose of [HR09] was to develop a fully optimal method for the Laplacian problem. More recently, the extension to the contact problems in elastostatics [HR10] was realized and convergence analysis is performed with no-discrete inf-sup condition requirement. In this work, we consider a context of Fluid-structure interactions. The underlying aim consists in imposing a velocity or a force on a moving boundary. The main difficulty of this purpose is that domains have to be reconsidered at each time step. A first approach consists in noticing that the more suitable ways of considering the fluid and the solid correspond to the use of the Eulerian and Lagrangian formulations respectively. It has been investigated in [LT08], [SMSTT05] and [SMST09] in ALE formulation. The main difficulty of these methods is the way of remeshing the domains at each time-step. A mixed formulation is given in [SMSTT08], for deformable bodies, where they consider a velocity field in the whole fixed domain, and in changing only the basis functions at each time step, whether we are in the solid or the fluid region. Some issues are proposed, in order to squeeze the mesh or the basis functions, by the Immersed Boundary Method [Pes02], for instance. We propose an other method to obtain a good numerical approximation of the normal constraint tensor $\sigma(u,p)n$ at the interface which is crucial in models that involve viscous incompressible fluids.

The outline of the chapter is as follows. In section 5.2 we introduce the continuous Stokes problem in the context of Fluid-structure interactions. We recall the corresponding variational formulation with the introduction of a Lagrange multiplier to impose the boundary condition at the interface (represented by a level-set function). In section 5.3, we introduce the fictitious domain method. The enrichment of a finite element space by discontinuous functions is presented. Under this form, the stability of the multiplier is not assumed. To recover optimal error estimates in section 5.3.2, we analyze the convergence of the discrete solutions. In section 5.4, the augmented Lagrangian method is developed in order to have not to impose an inf-sup condition. The convergence analysis for the stabilized method is given in section 5.4.2 and optimal error estimates are proved. Numerical experiments for the fictitious domain is given in section 5.5 (without stabilization in section 5.5.1 and with stabilization in section 5.5.2). Finally, the last section 5.6 is devoted to some practical aspects and comments of the implementation.

5.2 Setting of the problem

In a bounded domain of $\mathbb{R}^2$, denoted by $\mathcal{O}$, we consider a full solid immersed in a viscous incompressible fluid. The domain occupied by the solid is denoted by $\mathcal{S}$, and we denote by Γ its boundary. The fluid surrounding the structure occupies the domain $\mathcal{O} \setminus \overline{\mathcal{S}} = \mathcal{F}$ (see Fig. 5.1).

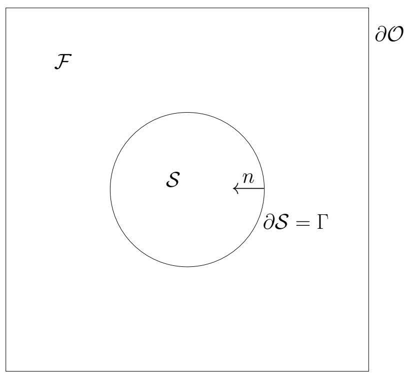

Figure 5.1: Domain for fluid and structure.

We denote by u and p respectively the velocity field and the pressure of the fluid. In this chapter, we are interested in the following Stokes problem

$$\begin{aligned} -\nu\Delta u + \nabla p &= f \quad \text{in } \mathcal{F}, && (5.1)\\ \operatorname{div} u &= 0 \quad \text{in } \mathcal{F}, && (5.2)\\ u &= 0 \quad \text{on } \partial\mathcal{O}, && (5.3)\\ u &= g \quad \text{on } \Gamma, && (5.4) \end{aligned}$$

where $f \in \mathbf{L}^2(\mathcal{F})$, $g \in \mathbf{H}^{1/2}(\Gamma)$. The boundary conditions on Γ is nonhomogeneous. The homogeneous Dirichlet condition we consider on $\partial\mathcal{O}$ has a physical sense, but can be replaced by a nonhomogeneous one, without more difficulty.
With regard to the incompressibility condition, the boundary datum $\mathbf{g}$ must obey

$$\int_\Gamma g \cdot n \mathrm{d}\Gamma = 0.$$

We consider this nonhomogeneous condition as a Dirichlet one imposed on Γ. Notice that other boundary conditions are possible on Γ, such as Neumann conditions, as it is done in [HR09] where mixed boundary conditions are considered. Equation (5.1) is the linearized form, in the stationary case, of the underlying incompressible Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = f \quad \text{in } \mathcal{F}.$$

The scalar constant ν denotes the dynamic viscosity of the fluid. In our presentation, for more simplicity, we only consider the stationary case, and the solid is supposed to be fixed.
The solution of (5.1)–(5.4) can be viewed as the stationary point of the Lagrangian

$$L_0(u,p,\lambda) = \nu\int_\mathcal{F} |D(u)|^2 \,\mathrm{d}\mathcal{F} - \int_\mathcal{F} p\operatorname{div} u \mathrm{d}\mathcal{F} - \int_\mathcal{F} f\cdot u \mathrm{d}\Gamma - \langle \lambda; u-g\rangle_{\mathbf{H}^{-1/2}(\Gamma);\mathbf{H}^{1/2}(\Gamma)}. \qquad (5.5)$$

Note that we should assume some additional smoothness in order for (5.5) to make sense, for example $u \in \mathbf{H}^2(\mathcal{F})$, $p \in \mathrm{H}^1(\mathcal{F})$, $\lambda \in \mathbf{L}^2(\Gamma)$. The exact solution normally has this smoothness provided that $f \in \mathbf{L}^2(\mathcal{F})$ and $g \in \mathbf{H}^{3/2}(\Gamma)$.
The multiplier λ, associated with the Dirichlet condition (5.4), represents the normal constraint tensor on Γ. Its expression is given by

$$\lambda(u,p) = \sigma(u,p)n = 2\nu D(u)n - pn,$$

where

$$D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right).$$

The vector n denotes the outward unit normal vector to $\partial\mathcal{F}$ (see fig. 5.1).

Remark 5.1. *Notice that if we have the incompressibility condition* (5.2)*, then, as a multiplier for the Dirichlet condition on* Γ*, considering* $\sigma(u,p)n$ *is equivalent to considering* $\nu\frac{\partial u}{\partial n} - pn$*, as it is shown in [GH92] or [GR86]. It is mainly due to the equality*

$$\operatorname{div}\left(\nabla u + \nabla u^T\right) = \Delta u,$$

when $\operatorname{div} u = 0$.

A finite element method based on the weak formulation derived from (5.5) does not guarantee, *a priori*, the convergence for the quantity $\sigma(u,p)n$ in $\mathbf{L}^2(\Gamma)$. As it has been done in [BH91, BH92], our approach consists in considering an augmented Lagrangian in adding a quadratic term to the one given in (5.5), as follows

$$L(u,p,\lambda) \quad = \quad L_0(u,p,\lambda) - \frac{\gamma}{2}\int_\Gamma |\lambda - \sigma(u,p)n|^2 \,\mathrm{d}\Gamma. \tag{5.6}$$

The goal is to recover the optimal rate of convergence for the multiplier λ. The constant γ represents a stabilization parameter (see numerical investigations in section 5.5.2). It has to be chosen judiciously.

Let us give the functional spaces we use for the continuous problem (5.1)-(5.4). For the velocity u we consider the following spaces

$$\mathbf{V} = \left\{v \in \mathbf{H}^1(\mathcal{F}) \mid v = 0 \text{ on } \partial\mathcal{O}\right\}, \qquad \mathbf{V}_0 = \mathbf{H}^1_0(\mathcal{F}),$$
$$\mathbf{V}^{\#} = \{v \in \mathbf{V} \mid \text{div } v = 0 \text{ in } \mathcal{F}\}, \qquad \mathbf{V}_0^{\#} = \left\{v \in \mathbf{H}^1_0(\mathcal{F}) \mid \text{div } v = 0 \text{ in } \mathcal{F}\right\}.$$

The pressure p is viewed as a multiplier for the incompressibility condition div $u = 0$, and belongs to $\mathrm{L}^2(\mathcal{F})$. It is determined up to a constant that we fix such that p belongs to

$$Q = \mathrm{L}^2_0(\mathcal{F}) = \left\{p \in \mathrm{L}^2(\mathcal{F}) \mid \int_{\mathcal{F}} p \,\mathrm{d}\mathcal{F} = 0\right\}.$$

The functional space for the multiplier is chosen as

$$\mathbf{W} = \mathbf{H}^{-1/2}(\Gamma) = \left(\mathbf{H}^{1/2}(\Gamma)\right)'.$$

Remark 5.2. *If we want to impose other boundary conditions, as in [HR09] for instance, the functional spaces* $\mathbf{V}_0$ *and* $\mathbf{H}^{1/2}(\Gamma)$ *must be adapted, but there is no particular difficulty.*

The weak formulation of problem (5.1)–(5.4) is given by:

$$\begin{array}{l}\text{Find } (u,p,\lambda) \in \mathbf{V}\times Q \times \mathbf{W} \text{ such that}\\ \left\{\begin{array}{ll} a(u,v) + b(v,p) + c(v,\lambda) = \mathcal{L}(v) & \forall v \in \mathbf{V},\\ b(u,q) = 0 & \forall q \in Q,\\ c(u,\mu) = \mathcal{G}(\mu), & \forall \mu \in \mathbf{W},\end{array}\right.\end{array} \tag{5.7}$$

where, $D(u) : D(v) = \text{trace}\left(D(u)D(v)^T\right)$ denoting the classical inner product for matrices. We set

$$a(u,v) \quad = \quad 2\nu\int_{\mathcal{F}} D(u) : D(v)\mathrm{d}\mathcal{F}, \tag{5.8}$$

$$b(u,q) \quad = \quad -\int_{\mathcal{F}} q\text{div } u\mathrm{d}\mathcal{F}, \tag{5.9}$$

$$c(u,\mu) \quad = \quad -\int_\Gamma \mu\cdot u\mathrm{d}\Gamma, \tag{5.10}$$

$$\mathcal{L}(v) \quad = \quad \int_{\mathcal{F}} f\cdot v\mathrm{d}\mathcal{F}, \tag{5.11}$$

$$\mathcal{G}(\mu) \quad = \quad -\int_\Gamma \mu\cdot g\mathrm{d}\Gamma. \tag{5.12}$$

Let us note that Problem (5.7) is well-posed (see [GH92] for instance). The solution of Problem (5.1)–(5.4) can be viewed as the stationary point of the Lagrangian on $\mathbf{V} \times Q \times \mathbf{W}$

$$L_0(u,p,\lambda) \quad = \quad \nu \int_{\mathcal{F}} |D(u)|^2 \, \mathrm{d}\mathcal{F} - \int_{\mathcal{F}} p\mathrm{div}\ u\mathrm{d}\mathcal{F} - \int_{\mathcal{F}} f \cdot u\mathrm{d}\mathcal{F} - \langle \lambda; u - g \rangle_{\mathbf{H}^{-1/2}(\Gamma);\mathbf{H}^{1/2}(\Gamma)}. \tag{5.13}$$

5.3 The fictitious domain method without stabilization

5.3.1 Presentation of the method

The fictitious domain for the fluid is considered on the whole domain $\mathcal{O}$. Let us introduce three discrete finite element spaces, $\tilde{\mathbf{V}}^h \subset \mathbf{H}^1(\mathcal{O})$ and $\tilde{Q}^h \subset \mathrm{L}^2_0(\mathcal{O})$ on the fictitious domain, and $\tilde{\mathbf{W}}^h \subset \mathbf{L}^2(\mathcal{O})$. Since $\mathcal{O}$ can be a rectangular domain, this spaces can be defined on the same structured mesh, that can be chosen uniform (see Fig. 5.2). The construction of the mesh is highly simplified (no particular mesh is required). We set

$$\tilde{\mathbf{V}}^h \quad = \quad \left\{ v^h \in C(\overline{\mathcal{O}}) \mid v^h_{|\partial\mathcal{O}} = 0,\ v^h_{|T} \in P(T),\ \forall T \in \mathcal{T}^h \right\}, \tag{5.14}$$

where $P(T)$ is a finite dimensional space of regular functions such that $P(T) \supseteq P_k(T)$ for some integer $k \geq 1$. For more details, see [EG04] for instance. The mesh parameter stands for $h = \max_{T \in \mathcal{T}^h} h_T$, where h_T is the diameter of T.

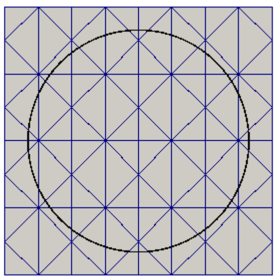

Figure 5.2: An example of a mesh on a fictitious domain.

Then we define

$$\mathbf{V}^h := \tilde{\mathbf{V}}^h_{|\mathcal{F}}, \quad Q^h := \tilde{Q}^h_{|\mathcal{F}}, \quad \mathbf{W}^h := \tilde{\mathbf{W}}^h_{|\Gamma},$$

which are natural discretizations of $\mathbf{V}$, $\mathrm{L}^2(\mathcal{F})$ and $\mathbf{H}^{-1/2}(\Gamma)$, respectively. Similarly to Xfem [LPRS05], where the shape functions of the finite element space is multiplied with an Heaviside function, this corresponds here to the multiplication of the shape functions with the characteristic function of $\mathcal{F}$.

An approximation of problem (5.7) is defined as follows:

$$
\text{Find } (u^h, p^h, \lambda^h) \in \mathbf{V}^h \times Q^h \times \mathbf{W}^h \text{ such that}
\left\{
\begin{array}{ll}
a(u^h, v^h) + b(v^h, p^h) + c(v^h, \lambda^h) = \mathcal{L}(v^h) & \forall v^h \in \mathbf{V}^h, \\
b(u^h, q^h) = 0 & \forall q^h \in Q^h, \\
c(u^h, \mu^h) = \mathcal{G}(\mu^h) & \forall \mu^h \in \mathbf{W}^h.
\end{array}
\right. \tag{5.15}
$$

In matrix notation, the previous formulation corresponds to

$$
\begin{pmatrix} A^0_{uu} & A^0_{up} & A^0_{u\lambda} \\ A^{0^T}_{up} & 0 & 0 \\ A^{0^T}_{u\lambda} & 0 & 0 \end{pmatrix}
\begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{P} \\ \boldsymbol{\Lambda} \end{pmatrix}
=
\begin{pmatrix} \boldsymbol{F^0} \\ 0 \\ \boldsymbol{G^0} \end{pmatrix},
$$

where $\boldsymbol{U}$, $\boldsymbol{P}$ and $\boldsymbol{\Lambda}$ are the degrees of freedom of u^h, p^h and λ^h respectively. As it is done in [BF91] or [EG04] for instance, these matrices A^0_{uu}, A^0_{up}, $A^0_{u\lambda}$ and vectors $\boldsymbol{F^0}$, $\boldsymbol{G^0}$ are the discretization of (5.8)-(5.12), respectively. Denoting $\{\varphi_i\}$, $\{\chi_i\}$ and $\{\psi_i\}$ the selected basis functions of spaces $\tilde{\mathbf{V}}^h$, $\tilde{Q}^h$ and $\tilde{\mathbf{W}}^h$ respectively, we have

$$
\left(A^0_{uu}\right)_{ij} = 2\nu \int_{\mathcal{F}} D(\varphi_i) : D(\varphi_j) \mathrm{d}\mathcal{F}, \quad \left(A^0_{up}\right)_{ij} = -\int_{\mathcal{F}} \chi_j \mathrm{div}\ \varphi_i \mathrm{d}\mathcal{F}, \quad \left(A^0_{u\lambda}\right)_{ij} = -\int_{\Gamma} \varphi_i \cdot \psi_j \mathrm{d}\Gamma,
$$
$$
\left(\boldsymbol{F^0}\right)_i = \int_{\mathcal{F}} f \cdot \varphi_i \mathrm{d}\mathcal{F}, \quad \left(\boldsymbol{G^0}\right)_i = -\int_{\Gamma} g \cdot \psi_i \mathrm{d}\Gamma.
$$

5.3.2 Convergence analysis

Let us define

$$
\begin{array}{rcl}
\mathbf{V}^h_0 & = & \left\{ v^h \in \mathbf{V}^h \mid c(v^h, \mu^h) = 0\ \forall \mu^h \in \mathbf{W}^h \right\}, \\
\mathbf{V}^h_g & = & \left\{ v^h \in \mathbf{V}^h \mid c(v^h, \mu^h) = c(v^h, g)\ \forall \mu^h \in \mathbf{W}^h \right\}, \\
\mathbf{V}^{\#,h} & = & \left\{ v^h \in \mathbf{V}^h \mid b(v^h, q^h) = 0\ \forall q^h \in Q^h \right\}, \\
\mathbf{V}^{\#,h}_0 & = & \left\{ v^h \in \mathbf{V}^h \mid b(v^h, q^h) = 0\ \forall q^h \in Q^h, c(v^h, \mu^h) = 0\ \forall \mu^h \in \mathbf{W}^h \right\}.
\end{array}
$$

The spaces $\mathbf{V}^h_0$, $\mathbf{V}^{\#,h}$ and $\mathbf{V}^{\#,h}_0$ can be viewed as the respective discretizations of the spaces $\mathbf{V}_0$, $\mathbf{V}^{\#}$ and $\mathbf{V}^{\#}_0$.
Let us assume that the following inf-sup condition is satisfied, for some constant $\beta > 0$ independent of h:

H1
$$
\inf_{0 \neq q^h \in Q^h} \sup_{0 \neq v^h \in \mathbf{V}^h_0} \frac{b(v, q)}{\|v^h\|_{\mathbf{V^h}} \|q^h\|_{Q^h}} \geq \beta.
$$

Note that this inf-sup condition concerns only the couple (u, p), and it implies the following property

$$
\overline{q}^h \in Q^h : \ b(v^h, \overline{q}^h) = 0\ \forall v^h \in \mathbf{V}^h_0 \Longrightarrow \overline{q}^h = 0. \tag{5.16}
$$

We shall further assume that the spaces $\tilde{\mathbf{V}}^h$, $\tilde{Q}^h$ and $\tilde{\mathbf{W}}^h$ are chosen in such a way that the following condition is satisfied, for all $h > 0$

H2 $\overline{\mu}^h \in \mathbf{W}^h : \ c(v^h, \overline{\mu}^h) = 0 \ \forall v^h \in \mathbf{V}^h \Longrightarrow \overline{\mu}^h = 0.$

Note that this hypothesis is not as strong as an inf-sup condition for the couple *velocity/multiplier*. It only demands that the space $\mathbf{V}^h$ is rich enough with respect to the space $\mathbf{W}^h$.

Remark 5.3. *We assume only the inf-sup condition for the couple velocity/pressure, not the one for the couple velocity/multiplier. Indeed, the purpose of our work is to stabilize the multiplier associated with the Dirichlet condition on Γ, not the multiplier associated with the incompressibility condition. The stabilization of the pressure - on the domain $\mathcal{F}$ - would be another issue (see page 424 of [Qua09] for instance).*

Lemma 5.4. *The bilinear form a, introduced in* (5.8)*, is*

$$a : (u, v) \mapsto 2\nu \int_{\mathcal{F}} D(u) : D(v) \mathrm{d}\mathcal{F}.$$

It is uniformly $\mathbf{V}^h$-elliptic, that is to say there exists $\alpha > 0$ independent of h such that for all $v^h \in \mathbf{V}^h$

$$a(v^h, v^h) \quad \geq \quad \alpha \left\| v^h \right\|_{\mathbf{V}}^2 .$$

Proof. Notice that $\mathbf{V}^h \subset \mathbf{V}$. Then it is sufficient to prove that the bilinear form a is coercive on the space $\mathbf{V}$, that is to say there exists $\alpha > 0$ such that for all $v \in \mathbf{V}$

$$a(v, v) \quad \geq \quad \alpha \|v\|_{\mathbf{V}}^2.$$

By absurd, suppose that for all $n \in \mathbb{N}$ there exists $(v_n)_n$ such that

$$n\|D(v_n)\|_{[\mathrm{L}^2(\mathcal{F})]^4} \quad < \quad \|v_n\|_{\mathbf{V}}.$$

Without loss of generality, we can assume that $\|v_n\|_{\mathbf{V}} = 1$. In particular, $D(v_n)$ converges to 0 in $[\mathrm{L}^2(\mathcal{F})]^4$. Then, from the Rellich's theorem, we can extract a subsequence v_m which converges in $\mathbf{L}^2(\mathcal{F})$. Using the fact that div $v_m = 0$, the Korn inequality (see [EG04] for instance) enables us to write

$$\|v_m - v_p\|_{\mathbf{H}^1(\mathcal{F})}^2 \quad \leq \quad C \left(\|v_m - v_p\|_{\mathbf{L}^2(\mathcal{F})}^2 + \|D(v_m) - D(v_p)\|_{[\mathrm{L}^2(\mathcal{F})]^4}^2 \right),$$

where C denotes a positive constant [1]. This implies that $(v_m)_m$ is a Cauchy sequence in $\mathbf{H}^1(\mathcal{F})$. Thus it converges to some v_∞ which satisfies $\|D(v_\infty)\|_{\mathbf{L}^2(\mathcal{F})} = 0$. The trace theorem implies that we have also $v_\infty = 0$ on $\partial\mathcal{O}$. Let us notice that $v \mapsto \|D(v)\|_{[\mathrm{L}^2(\mathcal{F})]^4}$ is a norm on $\mathbf{V}$. Indeed, if $\|D(v_\infty)\|_{[\mathrm{L}^2(\mathcal{F})]^4} = 0$, then v_∞ is reduced to a rigid displacement, that is to say $v_\infty = l + \omega \wedge x$ in $\mathcal{F}$. Then, the condition $v_\infty = 0$ on $\partial\mathcal{O}$ leads us to $v_\infty = 0$. It belies the fact that $\|v_m\|_{\mathbf{V}} = 1$. □

1. In the following, the symbol C will denote a generic positive constant which does not depend on the mesh size h. It can depend, however, on the geometry of $\mathcal{F}$ and Γ, on the physical parameters, on the mesh regularity and on other quantities clear from the context. It can take different values at different places.

Proposition 5.5. *Assume that the properties* **H1** *and* **H2** *are satisfied. Then there exists a unique solution* (u^h, p^h, λ^h) *to Problem* (5.15).

Proof. Since Problem (5.15) is of finite dimension, existence of the solution will follow from its uniqueness. To prove uniqueness, it is sufficient to consider the case $f = 0$ and $g = 0$, and to prove that it leads to $(u^h, p^h, \lambda^h) = (0, 0, 0)$. The last two equations in (5.15) show then immediately that $u^h \in \mathbf{V}_0^{\#,h}$, so that taking $v^h = u^h$ in the first equation leads to $u^h = 0$ by Lemma 5.4. Taking any test function from $\mathbf{V}_0^h$ in the first equation of (5.15) shows now that $p^h = 0$, by condition (5.16). And finally the same equation yields $\lambda^h = 0$ by Hypothesis **H2**. □

We recall the following basic result from the theory of saddle point problems [EG04, GG95].

Lemma 5.6. *Let* X *and* M *be Hilbert spaces and* $A(\cdot,\cdot) : X \times X \to \mathbb{R}$ *and* $B(\cdot,\cdot) : X \times M \to \mathbb{R}$ *be bounded bilinear forms such that* A *is coercive*

$$A(u, u) \geq \alpha \|u\|_X^2, \quad \forall u \in X$$

and B *has the following inf-sup property*

$$\inf_{0 \neq q \in M} \sup_{0 \neq u \in X} \frac{B(u, q)}{\|u\|_X \|q\|_M} \quad \geq \quad \beta,$$

with some $\alpha, \beta > 0$*. Then, for all* $\phi \in X'$ *and* $\psi \in M'$*, the problem:*

Find $u \in X$ *and* $p \in M$ *such that*

$$\begin{cases} A(u, v) + B(v, p) = \langle \phi, v \rangle, & \forall v \in X \\ B(u, q) = \langle \psi, q \rangle, & \forall q \in M \end{cases}$$

has a unique solution which satisfies

$$\|u\|_X + \|q\|_M \leq C(\|\phi\|_{X'} + \|\psi\|_{M'})$$

with a constant $C > 0$ *that depends only on* α, β *and on the norms of* A *and* B.

We can now prove the abstract error estimate for velocity and pressure.

Proposition 5.7. *Assume Hypothesis* **H1***. Let* (u, p, λ) *and* (u^h, p^h, λ^h) *be solutions to Problems* (5.7) *and* (5.15) *respectively. There exists a constant* $C > 0$ *independent of* h *such that*

$$\begin{aligned} \|u - u^h\|_{\mathbf{V}} + \|p - p^h\|_{\mathrm{L}^2(\mathcal{F})} \quad \leq \quad & C \Bigg(\inf_{v^h \in \mathbf{V}_g^h} \|u - v^h\|_{\mathbf{V}} \\ & + \inf_{q^h \in Q^h} \|p - q^h\|_{\mathrm{L}^2(\mathcal{F})} + \inf_{\mu^h \in W^h} \|\lambda - \mu^h\|_{\mathbf{H}^{-1/2}(\Gamma)} \Bigg). \end{aligned} \tag{5.17}$$

Proof. Take any $v^h \in \mathbf{V}_g^h$, $q^h \in Q^h$ and $\mu^h \in \mathbf{W}^h$. Comparing the first lines in systems (5.7) and (5.15), we can write

$$a(u^h - v^h, w^h) + b(w^h, p^h - q^h) = a(u - v^h, w^h) + b(w^h, p - q^h) + c(\lambda - \mu^h, w^h) \quad \forall w^h \in \mathbf{V}_0^h. \tag{5.18}$$

We have used here the fact that $c(\lambda^h, w^h) = c(\mu^h, w^h) = 0$ for all $w^h \in \mathbf{V}_0^h$. Similarly, the second lines in systems (5.7) and (5.15) imply

$$b(u^h - v^h, s^h) = b(u - v^h, s^h) \quad \forall s^h \in Q^h. \tag{5.19}$$

Now consider the problem:

Find $\mathbf{x}^h \in \mathbf{V}_0^h$ and $t^h \in Q^h$ such that

$$\begin{cases} a(\mathbf{x}^h, w^h) + b(w^h, t^h) = a(u - v^h, w^h) + b(w^h, p - q^h) + c(\lambda - \mu^h, w^h) & \forall w^h \in \mathbf{V}_0^h, \\ b(\mathbf{x}^h, s^h) = b(u - v^h, s^h) & \forall s^h \in Q^h. \end{cases}$$

Using Lemma 5.6 with $A = a$, $B = b$, $X = \mathbf{V}_0^h$ and $M = Q^h$, the solution $(\mathbf{x}^h, t^h)$ exists and is unique. Moreover, it satisfies

$$\|\mathbf{x}^h\|_{\mathbf{V}} + \|t^h\|_{\mathrm{L}^2(\mathcal{F})} \leq C\left(\|u - v^h\|_{\mathbf{V}} + \|p - q^h\|_{\mathrm{L}^2(\mathcal{F})} + \|\lambda - \mu^h\|_{\mathbf{H}^{-1/2}(\Gamma)}\right).$$

Comparing the system of equations for $(\mathbf{x}^h, t^h)$ with (5.18)–(5.19) and noting that $u - v^h \in \mathbf{V}_0^h$, we can identify

$$\mathbf{x}^h = u^h - v^h, \quad t^h = p^h - q^h.$$

In combination with the triangle inequality, this gives

$$\|u - u^h\|_{\mathbf{V}} + \|p - p^h\|_{\mathrm{L}^2(\mathcal{F})} \leq C\left(\|u - v^h\|_{\mathbf{V}} + \|p - q^h\|_{\mathrm{L}^2(\mathcal{F})} + \|\lambda - \mu^h\|_{\mathbf{H}^{-1/2}(\Gamma)}\right).$$

Since $v^h \in \mathbf{V}_g^h$, $q^h \in Q^h$ and $\mu^h \in \mathbf{W}^h$ are arbitrary, this is equivalent to the desired result. □

In summary, the results of this section tell us that, under Hypotheses **H1** and **H2**, Problem (5.15) has a unique solution which satisfies the a priori estimate (5.17). However, we have no estimate for the multiplier λ^h.

5.3.3 The theoretical order of convergence

The estimation of the convergence rate proposed for the Poisson problem in [HR09] can be straightforwardly transposed to the Stokes problem. Proposition 3 of [HR09] ensures an order of convergence at least equal to $\sqrt{h}$. It can be adapted to our case as follows.

Proposition 5.8. *Assume Hypotheses* **H1**, **H2**. *Let* (u, p, λ) *be the solution of Problem* (5.7) *for* $g = 0$, *such that* $u \in \mathbf{H}^{2+\varepsilon}(\mathcal{F}) \cap \mathbf{H}_0^1(\mathcal{F})$ *for some* $\varepsilon > 0$. *Assume that*

$$\inf_{q^h \in Q^h} \|p - q^h\|_Q \leq Ch^\delta,$$
$$\inf_{\mu^h \in W^h} \|\lambda - \mu^h\|_{\mathbf{W}} \leq Ch^\delta,$$

for some $\delta \geq 1/2$. *Then*

$$\left\|u - u^h\right\|_{\mathbf{V}} + \|p - p^h\|_{\mathrm{L}^2(\mathcal{F})} \leq C\sqrt{h}.$$

Proof. As is shown in [HR09], Section 3, for any $u \in \mathbf{H}^{2+\varepsilon}(\mathcal{F}) \cap \mathbf{H}^1_0(\mathcal{F})$ there exists a finite element interpolating function $v^h \in \mathbf{V}_0^h$ such that

$$\|u - v^h\|_{\mathbf{V}} \leq C\sqrt{h}. \tag{5.20}$$

In fact, v^h is constructed as a standard interpolating vector of $(1-\eta_h)u$ where η_h is a cut-off function equal to 1 in a vicinity of the boundary Γ, more precisely in a band of width $\frac{3h}{2}$, so that v^h vanishes on all the triangles cut by Γ. This ensures that v^h vanishes on Γ so that $v^h \in \mathbf{V}_0^h$. Now, the estimate of the present proposition follows from (5.17) combined with (5.20) (note that $\mathbf{V}_g^h = \mathbf{V}_0^h$ under our assumptions) and the hypotheses on the interpolating functions q^h and μ^h. □

Let us quote other references that treat of this kind of phenomena, as [GG95, RAB07, Ram08, Mau09]. We note, however, that the estimate of the order of convergence in $\sqrt{h}$ seems too pessimistic in view of the numerical tests presented in [HR09] for the Poisson problem (with the possible exception of the lowest order finite elements). In our numerical experiments for the Stokes problem, we do not observe the order of convergence as slow as $\sqrt{h}$.

5.4 The fictitious domain method with stabilization

5.4.1 Presentation of the method

The main purpose of the stabilization method we introduce consists in recovering the convergence on the multiplier λ. For that, the idea is to insert in our formulation a term which takes into account this requirement. Following the idea used in [BH91, BH92], we extend the classical Lagrangian L_0 given in (5.13), as

$$\begin{aligned} L(u,p,\lambda) &= \nu \int_{\mathcal{F}} |D(u)|^2 \,\mathrm{d}\mathcal{F} - \int_{\mathcal{F}} p\mathrm{div}\ u\mathrm{d}\mathcal{F} - \int_{\mathcal{F}} f\cdot u\mathrm{d}\mathcal{F} - \int_{\Gamma} \lambda\cdot(u-g)\mathrm{d}\Gamma \\ &\quad -\frac{\gamma}{2}\int_{\Gamma} |\lambda - \sigma(u,p)n|^2 \,\mathrm{d}\Gamma. \end{aligned}$$

Note that this extended Lagrangian coincides with the previous one on an exact regular solution. The quadratic term so added enables us to take into account an additional cost. Minimizing L leads to forcing λ to reach the desired value corresponding to $\sigma(u,p)n$. The constant $\gamma > 0$ represents the importance we give to this demand. However, notice that this additional term affects the positivity of L; That is why we cannot choose γ too large, and so this approach is not a penalization method. We discuss on this choice of γ in section 5.5.2.
The computations of the first variations leads us to

$$\begin{aligned} \frac{\delta L}{\delta u}(v) &= 2\nu\int_{\mathcal{F}} D(u):D(v)\mathrm{d}\mathcal{F} - \int_{\mathcal{F}} p\mathrm{div}\ v\mathrm{d}\mathcal{F} - \int_{\mathcal{F}} f\cdot v\mathrm{d}\mathcal{F} - \int_{\Gamma}\lambda\cdot v\mathrm{d}\Gamma \\ &\quad +2\nu\gamma\int_{\Gamma}(\lambda-\sigma(u,p)n)\cdot(D(v)n)\,\mathrm{d}\Gamma, \\ \frac{\delta L}{\delta p}(q) &= -\int_{\mathcal{F}} q\mathrm{div}\ u\mathrm{d}\mathcal{F} - \gamma\int_{\Gamma} q\,(\lambda-\sigma(u,p)n)\cdot n\mathrm{d}\Gamma, \\ \frac{\delta L}{\delta\lambda}(\mu) &= -\int_{\Gamma}\mu\cdot(u-g)\mathrm{d}\Gamma - \gamma\int_{\Gamma}(\lambda-\sigma(u,p)n)\cdot\mu\mathrm{d}\Gamma. \end{aligned}$$

Thus the stabilized formulation is:

$$
\text{Find } (u,p,\lambda) \in \mathbf{V} \times Q \times \mathbf{W} \text{ such that}
\begin{cases}
\mathcal{A}((u,p,\lambda);v) = \mathcal{L}(v) & \forall v \in \mathbf{V}, \\
\mathcal{B}((u,p,\lambda);q) = 0 & \forall q \in Q, \\
\mathcal{C}((u,p,\lambda);\mu) = \mathcal{G}(\mu), & \forall \mu \in \mathbf{W},
\end{cases}
\tag{5.21}
$$

where

$$
\begin{aligned}
\mathcal{A}((u,p,\lambda);v) &= 2\nu \int_{\mathcal{F}} D(u) : D(v) \mathrm{d}\mathcal{F} - \int_{\mathcal{F}} p \mathrm{div}\ v \mathrm{d}\mathcal{F} - \int_{\Gamma} \lambda \cdot v \mathrm{d}\Gamma \\
&\quad -4\nu^2\gamma \int_{\Gamma} (D(u)n) \cdot (D(v)n)\, \mathrm{d}\Gamma + 2\nu\gamma \int_{\Gamma} p\, (D(v)n \cdot n)\, \mathrm{d}\Gamma + 2\nu\gamma \int_{\Gamma} \lambda \cdot (D(v)n)\, \mathrm{d}\Gamma, \\
\mathcal{B}((u,p,\lambda);q) &= -\int_{\mathcal{F}} q \mathrm{div}\ u \mathrm{d}\mathcal{F} + 2\nu\gamma \int_{\Gamma} q\, (D(u)n \cdot n)\, \mathrm{d}\Gamma - \gamma \int_{\Gamma} pq \mathrm{d}\Gamma - \gamma \int_{\Gamma} q\lambda \cdot n \mathrm{d}\Gamma, \\
\mathcal{C}((u,p,\lambda);\mu) &= -\int_{\Gamma} \mu \cdot u \mathrm{d}\Gamma + 2\nu\gamma \int_{\Gamma} \mu \cdot (D(u)n) \mathrm{d}\Gamma - \gamma \int_{\Gamma} p(\mu \cdot n) \mathrm{d}\Gamma - \gamma \int_{\Gamma} \lambda \cdot \mu \mathrm{d}\Gamma.
\end{aligned}
$$

In matrix notation, the previous formulation corresponds to

$$
\begin{pmatrix} A_{uu} & A_{up} & A_{u\lambda} \\ A_{up}^T & A_{pp} & A_{p\lambda} \\ A_{u\lambda}^T & A_{p\lambda}^T & A_{\lambda\lambda} \end{pmatrix}
\begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{P} \\ \boldsymbol{\Lambda} \end{pmatrix}
=
\begin{pmatrix} \boldsymbol{F} \\ 0 \\ \boldsymbol{G} \end{pmatrix},
$$

where $\boldsymbol{U}$, $\boldsymbol{P}$ and $\boldsymbol{\Lambda}$ are already introduced in section 5.3.1. As it is done in [BF91] or [EG04] for instance, these matrices are discretizations of the following bilinear forms

$$
\begin{aligned}
\mathcal{A}_{uu} : (u,v) &\longmapsto 2\nu \int_{\mathcal{F}} D(u) : D(v) \mathrm{d}\mathcal{F} - 4\nu^2\gamma \int_{\Gamma} (D(u)n) \cdot (D(v)n)\, \mathrm{d}\Gamma, \\
\mathcal{A}_{up} : (v,p) &\longmapsto -\int_{\mathcal{F}} p \mathrm{div}\ v \mathrm{d}\mathcal{F} + 2\nu\gamma \int_{\Gamma} p\, (D(v)n \cdot n)\, \mathrm{d}\Gamma, \\
\mathcal{A}_{u\lambda} : (u,\lambda) &\longmapsto -\int_{\Gamma} \lambda \cdot v \mathrm{d}\Gamma + 2\nu\gamma \int_{\Gamma} \lambda \cdot (D(v)n)\, \mathrm{d}\Gamma, \\
\mathcal{A}_{pp} : (p,q) &\longmapsto -\gamma \int_{\Gamma} pq \mathrm{d}\Gamma, \\
\mathcal{A}_{p\lambda} : (q,\lambda) &\longmapsto -\gamma \int_{\Gamma} q\lambda \cdot n \mathrm{d}\Gamma, \\
\mathcal{A}_{\lambda\lambda} : (\lambda,\mu) &\longmapsto -\gamma \int_{\Gamma} \lambda \cdot \mu \mathrm{d}\Gamma,
\end{aligned}
$$

and the vectors $\boldsymbol{F}$ and $\boldsymbol{G}$ are the discretization of the following linear forms

$$
\begin{aligned}
\mathcal{L} : v &\longmapsto \int_{\mathcal{F}} f \cdot v \mathrm{d}\Gamma, \\
\mathcal{G} : \mu &\longmapsto -\int_{\Gamma} \mu \cdot g \mathrm{d}\Gamma.
\end{aligned}
$$

Denoting $\{\varphi_i\}$, $\{\chi_i\}$ and $\{\psi_i\}$ the selected basis functions of spaces $\tilde{\mathbf{V}}^h$, $\tilde{Q}^h$ and $\tilde{\mathbf{W}}^h$ respectively,

we have

$$\begin{aligned}
(A_{uu})_{ij} &= 2\nu \int_{\mathcal{F}} D(\varphi_i) : D(\varphi_j) \mathrm{d}\mathcal{F} - 4\nu^2\gamma \int_{\Gamma} (D(\varphi_i)n) \cdot (D(\varphi_j n)) \mathrm{d}\Gamma, \\
(A_{up})_{ij} &= -\int_{\mathcal{F}} \chi_j \mathrm{div}\ \varphi_i \mathrm{d}\mathcal{F} + 2\nu\gamma \int_{\Gamma} \chi_j (D(\varphi_i)n \cdot n) \mathrm{d}\Gamma, \\
(A_{u\lambda})_{ij} &= -\int_{\Gamma} \varphi_i \cdot \psi_j \mathrm{d}\Gamma + 2\nu\gamma \int_{\Gamma} (D(\varphi_i)n) \cdot \psi_j \mathrm{d}\Gamma, \\
(A_{pp})_{ij} &= -\gamma \int_{\Gamma} \chi_i \chi_j \mathrm{d}\Gamma, \\
(A_{p\lambda})_{ij} &= -\gamma \int_{\Gamma} \chi_i (\psi_j \cdot n) \mathrm{d}\Gamma, \\
(A_{\lambda\lambda})_{ij} &= -\gamma \int_{\Gamma} \psi_i \psi_j \mathrm{d}\Gamma, \\
(\boldsymbol{F})_i &= \int_{\mathcal{F}} f \cdot \varphi_i \mathrm{d}\mathcal{F}, \quad (\boldsymbol{G})_i = -\int_{\Gamma} g \cdot \psi_i \mathrm{d}\Gamma.
\end{aligned}$$

5.4.2 A theoretical analysis of the stabilized method

Let us define γ_0 by setting $\gamma = \gamma_0 h$. We first observe that the discrete problem can be rewritten in the following compact form:

$$\begin{aligned}
&\text{Find } (u^h, p^h, \lambda^h) \in \mathbf{V}^h \times Q^h \times \mathbf{W}^h \text{ such that} \\
&\mathcal{M}((u^h, p^h, \lambda^h); (v^h, q^h, \mu^h)) = \mathcal{H}(v^h, q^h, \mu^h), \quad \forall (v^h, q^h, \mu^h) \in \mathbf{V}^h \times Q^h \times \mathbf{W}^h,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}((u,p,\lambda);(v,q,\mu)) \;=\;& 2\nu \int_{\mathcal{F}} D(u) : D(v) \mathrm{d}\mathcal{F} - \int_{\mathcal{F}} (p \mathrm{div}\ v + q \mathrm{div}\ u) \mathrm{d}\mathcal{F} - \int_{\Gamma} (\lambda \cdot v + \mu \cdot u) \mathrm{d}\Gamma \\
& - \gamma_0 h \int_{\Gamma} (2\nu D(u)n - pn - \lambda) \cdot (2\nu D(v)n - qn - \mu)\, \mathrm{d}\Gamma,
\end{aligned}$$

and

$$\mathcal{H}(v,q,\mu) = \int_{\mathcal{F}} f \cdot v \mathrm{d}\Gamma - \int_{\Gamma} \mu \cdot g \mathrm{d}\Gamma.$$

In the following, we will need some assumptions for our theoretical analysis:

A1 For all $v^h \in \mathbf{V}^h$ one has

$$h \| D(v^h) n \|^2_{\mathbf{L}^2(\Gamma)} \quad \leq \quad C \| v^h \|^2_{\mathbf{V}}.$$

A2 For all $q^h \in Q^h$ one has

$$h \| q^h \|^2_{\mathrm{L}^2(\Gamma)} \quad \leq \quad C \| q^h \|^2_{\mathrm{L}^2(\mathcal{F})}.$$

A3 For all $v \in \mathbf{H}^{1/2}(\Gamma)$ one has

$$\| P^h v - v \|_{\mathbf{L}^2(\Gamma)} \quad \leq \quad C h^{1/2} \| v \|_{\mathbf{H}^{1/2}(\Gamma)},$$

where P^h denotes the L^2-orthogonal projector from $\mathbf{H}^{1/2}(\Gamma)$ to $\mathbf{W}^h$.

A4 One has the following inf-sup condition for the velocity-pressure pair of finite element spaces

$$\inf_{q^h\in Q^h}\sup_{v^h\in \mathbf{V}_0^h}\frac{b(v^h,q^h)}{\|q^h\|_{\mathrm{L}^2(\mathcal{F})}\|v^h\|_{\mathbf{V}}}\quad\geq\quad\beta,$$

with $\beta>0$ independent of h.

Note that assumptions **A1** and **A3** are the same as those introduced in [HR09] (cf. equations (5.1) and (5.5) respectively) in the study of the fictitious domain approach for the Laplace equation stabilized à la Barbosa-Hughes. Our assumption **A2** is also similar in nature to those two, and all these three assumptions can be in fact established if one assumes that the intersections of $\mathcal{F}$ with the triangles of the mesh are not "too small" (see Appendix B of [HR09] and section 5.6). Although all these assumptions can be violated in practice if a mesh triangle is cut by the boundary Γ so that only its tiny portion happens to be inside of $\mathcal{F}$, we can argue that such accidents occur rather rarely and their impact on the overall behavior of the method is practically negligible, as confirmed by the numerical experiments for the Laplace equation in [HR09]. This conclusion can be safely transposed to the case of Stokes problem. However, we have now the additional difficulty in the form of the inf-sup condition **A4**. Of course this condition is verified if one chooses the classical stable pair of finite element spaces, like for instance the Taylor-Hood elements P2/P1 pair for velocity/pressure, and if the boundary Γ does not cut the edges of the triangles of the mesh. However, in the general case of an arbitrary geometry, we have by now no evidence of the fulfillment of the inf-sup condition **A4**.
We prove in this subsection the following inf-sup result, which is an adaptation of Lemma 3 from [HR09].

Lemma 5.9. *Under assumptions* **A1**–**A4***, there exists for γ_0 small enough a constant $c>0$ such that*

$$\inf_{(u^h,p^h,\lambda^h)\in\mathbf{V}^h\times Q^h\times\mathbf{W}^h}\ \sup_{(v^h,q^h,\mu^h)\in\mathbf{V}^h\times Q^h\times\mathbf{W}^h}\frac{\mathcal{M}((u^h,p^h,\lambda^h);(v^h,q^h,\mu^h))}{|||u^h,p^h,\lambda^h|||\ |||v^h,q^h,\mu^h|||}\geq c,$$

where the triple norm is defined by

$$|||u,p,\lambda|||=\left(\|u\|_{\mathbf{V}}^2+\|p\|_{\mathrm{L}^2(\mathcal{F})}^2+h\|D(u)n\|_{\mathbf{L}^2(\Gamma)}^2+h\|p\|_{\mathrm{L}^2(\Gamma)}^2+h\|\lambda\|_{\mathbf{L}^2(\Gamma)}^2+\frac{1}{h}\|u\|_{\mathbf{L}^2(\Gamma)}^2\right)^{1/2},$$

and where c is a mesh-independent constant.

Proof. We observe that

$$\begin{aligned}\mathcal{M}((u^h,p^h,\lambda^h);(u^h,-p^h,-\lambda^h)) &= 2\nu\|u^h\|_{\mathbf{V}}^2-\gamma_0 h\int_\Gamma 4\nu^2|D(u^h)n|^2\mathrm{d}\Gamma+\gamma_0 h\int_\Gamma|p^hn-\lambda^h|^2\mathrm{d}\Gamma\\ &\geq \nu\|u^h\|_{\mathbf{V}}^2+\gamma_0 h\|p^hn+\lambda^h\|_{\mathbf{L}^2(\Gamma)}^2\end{aligned}$$

where we have used assumption **A1** and the fact that γ_0 can be taken sufficiently small. More precisely, we can choose γ_0 such that $4\nu^2\gamma_0C\leq\nu$, where C is the constant of assumption **A1**. The inf-sup condition **A4** implies that for all $p^h\in Q^h$ there exists $v_p^h\in\mathbf{V}_0^h$ such that

$$-\int_{\mathcal{F}}p^h\mathrm{div}\ v_p^h\mathrm{d}\mathcal{F}=\|p^h\|_{\mathrm{L}^2(\mathcal{F})}^2\qquad\text{and}\qquad\|v_p^h\|_{\mathbf{V}}\leq C\|p^h\|_{\mathrm{L}^2(\mathcal{F})}.\tag{5.22}$$

Now let us observe that

$$\begin{aligned}
\mathcal{M}((u^h,p^h,\lambda^h);(v_p^h,0,0)) &= 2\nu\int_{\mathcal{F}} D(u^h):D(v_p^h)\mathrm{d}\mathcal{F} + \|p^h\|^2_{\mathrm{L}^2(\mathcal{F})} \\
&\quad -2\nu\gamma_0 h\int_\Gamma (2\nu D(u^h)n - p^h n - \lambda^h)\cdot D(v_p^h)n\mathrm{d}\Gamma \\
&\geq \|p^h\|^2_{\mathrm{L}^2(\mathcal{F})} - \nu\alpha\|u^h\|^2_{\mathbf{V}} - \frac{\nu}{\alpha}\|v_p^h\|^2_{\mathbf{V}} \\
&\quad -\nu\gamma_0 h\alpha\|2\nu D(u^h)n - p^h n - \lambda^h\|^2_{\mathbf{L}^2(\Gamma)} - \frac{\nu\gamma_0 h}{\alpha}\|D(v_p^h)n\|^2_{\mathbf{L}^2(\Gamma)}.
\end{aligned}$$

We have used here the Young inequality which is valid for any $\alpha > 0$. In particular, we can choose α large enough so that we can conclude with the aid of assumptions **A1** and **A2** (the constant C here will be independent of α and h, but dependent on γ_0 and on the constants in the inequalities **A1** and **A2**). We get

$$\begin{aligned}
\mathcal{M}((u^h,p^h,\lambda^h);(v_p^h,0,0)) &\geq \|p^h\|^2_{\mathrm{L}^2(\mathcal{F})} - \nu\alpha\|u^h\|^2_{\mathbf{V}} - \frac{C}{\alpha}\|p^h\|^2_{\mathrm{L}^2(\mathcal{F})} - C\alpha h\|D(u^h)n\|^2_{\mathrm{L}^2(\Gamma)} \\
&\quad -\nu\gamma_0 h\alpha\|p^h n + \lambda^h\|^2_{\mathbf{L}^2(\Gamma)} - \frac{C}{\alpha}\|p^h\|^2_{\mathrm{L}^2(\Gamma)} \\
&\geq \frac{1}{2}\|p^h\|^2_{\mathrm{L}^2(\mathcal{F})} - C\alpha\|u^h\|^2_{\mathbf{V}} - \nu\gamma_0 h\alpha\|p^h n + \lambda^h\|^2_{\mathbf{L}^2(\Gamma)}.
\end{aligned}$$

Let us now take $\bar{\mu}_h = -\frac{1}{h}P^h u^h$ where P^h is the projector from $\mathbf{H}^{1/2}(\Gamma)$ to $\mathbf{W}^h$. Observe that, in using assumption **A1**, we have

$$\begin{aligned}
\mathcal{M}((u^h,p^h,\lambda^h);(0,0,\bar{\mu}^h)) &= \frac{1}{h}\|P^h u^h\|^2_{\mathbf{L}^2(\Gamma)} - \gamma_0\int_\Gamma (2\nu D(u^h)n - p^h n - \lambda^h)\cdot P^h u^h \mathrm{d}\Gamma \\
&\geq \frac{1}{h}\|P^h u^h\|^2_{\mathbf{L}^2(\Gamma)} \\
&\quad -\gamma_0\left(\sqrt{h}\|D(u^h)n\|_{\mathbf{L}^2(\Gamma)} + \sqrt{h}\|p^h n + \lambda^h\|_{\mathbf{L}^2(\Gamma)}\right)\frac{1}{\sqrt{h}}\|P^h u^h\|_{\mathbf{L}^2(\Gamma)} \\
&\geq \frac{1}{2h}\|P^h u^h\|^2_{\mathbf{L}^2(\Gamma)} - C\|u^h\|^2_{\mathbf{V}} - Ch\|p^h n + \lambda^h\|^2_{\mathbf{L}^2(\Gamma)}.
\end{aligned}$$

Combining the above inequalities and taking some small enough numbers $\kappa > 0$ and $\eta > 0$, we can obtain

$$\begin{aligned}
&\mathcal{M}((u^h,p^h,\lambda^h);(u^h+\kappa v_p^h,-p^h,-\lambda^h+\eta\bar{\mu}^h)) \\
&\geq \nu\|u^h\|^2_{\mathbf{V}} + \gamma_0 h\|p^h n + \lambda^h\|^2_{\mathbf{L}^2(\Gamma)} + \frac{\kappa}{2}\|p^h\|^2_{\mathrm{L}^2(\mathcal{F})} - C\alpha\kappa\|u^h\|^2_{\mathbf{V}} - \nu\gamma_0 h\alpha\kappa\|p^h n + \lambda^h\|^2_{\mathbf{L}^2(\Gamma)} \\
&\quad +\frac{\eta}{2h}\|P^h u^h\|^2_{\mathbf{L}^2(\Gamma)} - C\eta\|u^h\|^2_{\mathbf{V}} - C\eta h\|p^h n + \lambda^h\|^2_{\mathbf{L}^2(\Gamma)} \\
&\geq \frac{\nu}{2}\|u^h\|^2_{\mathbf{V}} + \frac{\kappa}{2}\|p^h\|^2_{\mathrm{L}^2(\mathcal{F})} + \frac{\gamma_0}{2}h\|p^h n + \lambda^h\|^2_{\mathbf{L}^2(\Gamma)} + \frac{\eta}{2h}\|P^h u^h\|^2_{\mathbf{L}^2(\Gamma)} \\
&\geq \frac{\nu}{4}\|u^h\|^2_{\mathbf{V}} + \frac{\eta}{2h}\|P^h u^h\|^2_{\mathbf{L}^2(\Gamma)} + \frac{\nu}{4C}h\|D(u^h)n\|^2_{\mathbf{L}^2(\Gamma)} \\
&\quad +\frac{\kappa}{4}\|p^h\|^2_{\mathrm{L}^2(\mathcal{F})} + \frac{\kappa}{4C}h\|p^h\|^2_{\mathrm{L}^2(\Gamma)} + \frac{\gamma_0}{2}h\|p^h n + \lambda^h\|^2_{\mathrm{L}^2(\Gamma)}.
\end{aligned}$$

In the last inequality above, we have used again assumptions **A1** and **A2** (with the corresponding constant C). We now rework the last two terms in order to split p^h and λ^h. Denoting $t = \dfrac{\kappa}{2C\gamma_0}$,

we have

$$\begin{aligned}\frac{\kappa}{4C}h\|p^h\|^2_{\mathrm{L}^2(\Gamma)}+\frac{\gamma_0}{2}h\|p^hn+\lambda^h\|^2_{\mathbf{L}^2(\Gamma)} &= \frac{\gamma_0}{2}h\left((t+1)\|p^h\|^2_{\mathrm{L}^2(\Gamma)}+\|\lambda^h\|^2_{\mathbf{L}^2(\Gamma)}+2\int_\Gamma p^hn\cdot\lambda^h\mathrm{d}\Gamma\right)\\ &\geq \frac{\gamma_0}{2}h\left((t+1)\|p^h\|^2_{\mathrm{L}^2(\Gamma)}+\|\lambda^h\|^2_{\mathbf{L}^2(\Gamma)}\right.\\ &\qquad \left.-(t/2+1)\|p^h\|^2_{\mathrm{L}^2(\Gamma)}-\frac{1}{t/2+1}\|\lambda^h\|^2_{\mathbf{L}^2(\Gamma)}\right)\\ &= \frac{\gamma_0}{2}h\left(\frac{t}{2}\|p^h\|^2_{\mathrm{L}^2(\Gamma)}+\frac{t/2}{t/2+1}\|\lambda^h\|^2_{\mathbf{L}^2(\Gamma)}\right).\end{aligned}$$

So we finally have

$$\begin{aligned}&\mathcal{M}((u^h,p^h,\lambda^h);(u^h+\kappa v_p^h,-p^h,-\lambda^h+\eta\mu^h))\\ &\quad\geq c\left(\|u^h\|^2_{\mathbf{V}}+\|p^h\|^2_{\mathrm{L}^2(\mathcal{F})}+h\|D(u^h)n\|^2_{\mathbf{L}^2(\Gamma)}+h\|p^h\|^2_{\mathrm{L}^2(\Gamma)}+h\|\lambda^h\|^2_{\mathbf{L}^2(\Gamma)}+\frac{1}{h}\|P^hu^h\|^2_{\mathbf{L}^2(\Gamma)}\right).\end{aligned}$$

We can now eliminate the projector P^h in this estimate by the following calculation, which is valid for some $\beta>0$ small enough:

$$\begin{aligned}\|u^h\|^2_{\mathbf{V}}+\frac{1}{h}\|P^hu^h\|^2_{\mathbf{L}^2(\Gamma)} &\geq \|u^h\|^2_{\mathbf{V}}+\frac{\beta}{h}\|P^hu^h\|^2_{\mathbf{L}^2(\Gamma)}=\|u^h\|^2_{\mathbf{V}}+\frac{\beta}{h}\left(\|u^h\|^2_{\mathbf{L}^2(\Gamma)}-\|u^h-P^hu^h\|^2_{\mathbf{L}^2(\Gamma)}\right)\\ &\geq \|u^h\|^2_{\mathbf{V}}+\frac{\beta}{h}\|u^h\|^2_{\mathbf{L}^2(\Gamma)}-C\beta\|u^h\|^2_{\mathbf{H}^{1/2}(\Gamma)}\geq\|u^h\|^2_{\mathbf{V}}+\frac{\beta}{h}\|u^h\|^2_{\mathbf{L}^2(\Gamma)}-C\beta\|u^h\|^2_{\mathbf{V}}\\ &\geq \frac{1}{2}\|u^h\|^2_{\mathbf{V}}+\frac{\beta}{h}\|u^h\|^2_{\mathbf{L}^2(\Gamma)}.\end{aligned}$$

We have used here assumption **A3** and the trace inequality.

In summary, we have obtained that taking

$$(v^h,q^h,\mu^h) \quad = \quad (u^h+\kappa v_p^h,-p^h,-\lambda^h+\eta\bar{\mu}^h)$$

one has

$$\mathcal{M}((u^h,p^h,\lambda^h);(v^h,q^h,\mu^h)) \quad\geq\quad c|||u^h,p^h,\lambda^h|||^2. \tag{5.23}$$

On the other hand,

$$|||v^h,q^h,\mu^h||| \quad\leq\quad M|||u^h,p^h,\lambda^h||| \tag{5.24}$$

with some $M>0$ independent of h. Indeed, we have

$$\begin{aligned}|||v^h,q^h,\mu^h||| &\leq |||u^h,p^h,\lambda^h|||+\kappa|||v_p^h,0,0|||+\eta|||0,0,\bar{\mu}^h|||\\ &\leq |||u^h,p^h,\lambda^h|||+\kappa\left(\|v_p^h\|^2_{\mathbf{V}}+h\|D(v_p^h)n\|^2_{\mathbf{L}^2(\Gamma)}+\frac{1}{h}\|v_p^h\|^2_{\mathbf{L}^2(\Gamma)}\right)^{1/2}+\eta\sqrt{h}\|\bar{\mu}^h\|_{\mathbf{L}^2(\Gamma)}.\end{aligned}$$

Now, by assumption **A1** and the fact that $v_p^h\in\mathbf{V}_0^h$ so that $P^hv_p^h=0$, we have

$$\|v_p^h\|^2_{\mathbf{V}}+h\|D(v_p^h)n\|^2_{\mathbf{L}^2(\Gamma)}+\frac{1}{h}\|v_p^h\|^2_{\mathbf{L}^2(\Gamma)} \quad\leq\quad C\|v_p^h\|^2_{\mathbf{V}}+\frac{1}{h}\|v_p^h-P^hv_p^h\|^2_{\mathbf{L}^2(\Gamma)}.$$

Furthermore, by assumption **A3** and by the definition of $v_p^h \in \mathbf{V}_0^h$ given in (5.22), we have

$$\begin{aligned}\|v_p^h\|_{\mathbf{V}}^2 + h\|D(v_p^h)n\|_{\mathbf{L}^2(\Gamma)}^2 + \frac{1}{h}\|v_p^h\|_{\mathbf{L}^2(\Gamma)}^2 &\leq C\|v_p^h\|_{\mathbf{V}}^2 + C\|v_p^h\|_{\mathbf{H}^{1/2}(\Gamma)}^2 \\ &\leq C\|v_p^h\|_{\mathbf{V}}^2 \leq C\|p^h\|_{\mathrm{L}^2(\mathcal{F})}^2 \leq C|||u^h, p^h, \lambda^h|||.\end{aligned}$$

We have also

$$\sqrt{h}\|\bar{\mu}^h\|_{\mathbf{L}^2(\Gamma)} = \frac{1}{\sqrt{h}}\|P^h u^h\|_{\mathbf{L}^2(\Gamma)} \leq \frac{1}{\sqrt{h}}\|u^h\|_{\mathbf{L}^2(\Gamma)} \leq |||u^h, p^h, \lambda^h|||,$$

hence the inequality (5.24). Dividing (5.23) by (5.24) yields

$$\frac{\mathcal{M}((u^h, p^h, \lambda^h); (v^h, q^h, \mu^h))}{|||v^h, q^h, \mu^h|||} \geq \frac{c}{M}|||u^h, p^h, \lambda^h|||,$$

which is the desired result. □

The lemma above, combined with the fact that the bilinear form $\mathcal{M}$ is bounded in the triple norm on $\mathbf{V} \times Q \times \mathbf{W}$ uniformly with respect to h, leads us by a Céa type lemma (cf. [EG04] or Theorem 5.2 in [HR09]) to the following abstract error estimate

$$|||u - u^h, p - p^h, \lambda - \lambda^h||| \leq C \inf_{(v^h, q^h, \mu^h) \in \mathbf{V}^h \times Q^h \times \mathbf{W}^h} |||u - v^h, p - q^h, \lambda - \mu^h|||.$$

Using the extension theorem for the Sobolev spaces, the standard estimates for the nodal (or Clément if necessary) finite element interpolation operators, and the trace inequality $\|w\|_{\mathrm{L}^2(\Gamma)} \leq C\left(h^{-1}\|w\|_{\mathrm{L}^2(T)} + h\|w\|_{\mathrm{L}^2(T)}\right)$ for any $w \in \mathrm{H}^1(T)$ on any triangle $T \in \mathcal{T}_h$ (which is valid provided Γ is sufficiently smooth - see Appendix A of [HR09] for a proof), we obtain the following error estimate

$$\begin{aligned}\max(\|u - u^h\|_{\mathbf{V}}, \|p - p^h\|_{\mathrm{L}^2(\mathcal{F})}, h\|\lambda - \lambda^h\|_{\mathbf{L}^2(\Gamma)}) \leq |||u - u^h, p - p^h, \lambda - \lambda^h||| \\ \leq C(h^{k_u}\|u\|_{\mathbf{H}^{k_u+1}(\mathcal{F})} + h^{k_p+1}\|p\|_{\mathrm{H}^{k_p+1}(\mathcal{F})} + h^{k_\lambda+1}\|\lambda\|_{\mathbf{H}^{k_\lambda+1/2}(\Gamma)}),\end{aligned}$$

where k_u, k_p and k_λ are the degrees of finite elements used for velocity, pressure and multiplier λ respectively. The proof of this result is rather tedious but can be easily reproduced following the ideas of [HR09] (see, in particular, the proofs of Theorem 5.3 and Lemma 5.4 there).

5.5 Numerical experiments

For numerical experiments, we consider the square $[0,1] \times [0,1]$ and choose as Γ the circle whose level-set representation is

$$(x - 0.5)^2 + (y - 0.5)^2 = R^2,$$

with $R = 0.21$ (see Fig. 5.2). The exact solutions are chosen equal to

$$\begin{aligned}u_{ex}(x_1, x_2) &= \begin{pmatrix} \cos(\pi x_1)\sin(\pi x_2) \\ -\sin(\pi x_1)\cos(\pi x_2) \end{pmatrix}, \\ p_{ex}(x_1, x_2) &= (x_2 - 1/2)\cos(2\pi x_1) + (x_1 - 1/2)\sin(2\pi x_2).\end{aligned}$$

The meshes and all the computations have been obtained with the C++ finite element library GETFEM++ [RP]. In the numerical tests, we compare the discrete solutions with the exact solutions for different meshes (six imbricated uniform meshes).
We denote $\boldsymbol{U}_{ex}$, $\boldsymbol{P}_{ex}$ and $\boldsymbol{\Lambda}_{ex}$ the discrete forms of functions u_{ex}, p_{ex} and $\lambda_{ex} = \sigma(u_{ex}, p_{ex})n$ respectively. For practical purposes, the error introduced by the approximation of the exact vector $\boldsymbol{\Lambda}_{ex}$ by $\boldsymbol{\Lambda}$ is given by the square root of

$$\|\boldsymbol{\Lambda}_{ex} - \boldsymbol{\Lambda}\|^2_{\mathbf{L}^2(\Gamma)} = \int_\Gamma |\sigma(\boldsymbol{U}_{ex}, \boldsymbol{P}_{ex})n - \boldsymbol{\Lambda}|^2 \, \mathrm{d}\Gamma.$$

This scalar product is developed and using the assembling matrices we compute

$$\begin{aligned}\|\boldsymbol{\Lambda}_{ex} - \boldsymbol{\Lambda}\|^2_{\mathbf{L}^2(\Gamma)} &= \langle A_{uu}\boldsymbol{U}_{ex}, \boldsymbol{U}_{ex}\rangle + 2\langle A_{up}\boldsymbol{P}_{ex}, \boldsymbol{U}_{ex}\rangle + 2\langle A_{u\lambda}\boldsymbol{\Lambda}, \boldsymbol{U}_{ex}\rangle + \\ &\quad \langle A_{pp}\boldsymbol{P}_{ex}, \boldsymbol{P}_{ex}\rangle - 2\langle A_{p\lambda}\boldsymbol{\Lambda}, \boldsymbol{P}_{ex}\rangle + \langle A_{\lambda\lambda}\boldsymbol{\Lambda}, \boldsymbol{\Lambda}\rangle,\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the classical Euclidean scalar product in finite dimension. Then, the relative error is given by

$$\frac{\|\boldsymbol{\Lambda}_{ex} - \boldsymbol{\Lambda}\|_{\mathbf{L}^2(\Gamma)}}{\|\boldsymbol{\Lambda}_{ex}\|_{\mathbf{L}^2(\Gamma)}} = \frac{\|\boldsymbol{\Lambda}_{ex} - \boldsymbol{\Lambda}\|_{\mathbf{L}^2(\Gamma)}}{\left(\langle A_{uu}\boldsymbol{U}_{ex}, \boldsymbol{U}_{ex}\rangle + \langle A_{pp}\boldsymbol{P}_{ex}, \boldsymbol{P}_{ex}\rangle + 2\langle A_{up}\boldsymbol{P}_{ex}, \boldsymbol{U}_{ex}\rangle\right)^{1/2}}.$$

5.5.1 Numerical experiments for the method without stabilization

We present numerical computations of errors when no stabilization are imposed. We consider several choices of the finite element spaces $\tilde{\mathbf{V}}^h$, $\tilde{Q}^h$ and $\tilde{\mathbf{W}}^h$. Four couples of spaces are studied (for $\mathbf{u}/p/\lambda$), P1+/P1/P0 (a standard continuous P1 element for $\mathbf{u}$ enriched by a cubic bubble function, standard continuous P1 for the pressure p and discontinuous P0 for the multiplier λ element on a triangle), P2/P1/P0, for triangular meshes and Q1/Q0/Q0, Q2/Q1/Q0 for quadrangular meshes. The elements chosen between velocity and pressure are the ones which ensure the discrete mesh-independent inf-sup condition **H1** in the case of uncut functions (except for the Q1/Q0 pair), that is to say the classical case where regular meshes are considered. Low degrees are selected to control the memory (CPU time) which plays a crucial role in numerical simulations for fluid-structure interactions, specially in an unsteady framework. For the multiplier introduced for the interface, since the stabilization is not used, a discrete mesh-independent inf-sup condition must be satisfied. For instance, the couple of spaces Q1/Q0/Q0 does not satisfy this condition. The error curves between the discrete solution and the exact one are given in Fig. 5.3 for different norms. The rates of convergence are reported.

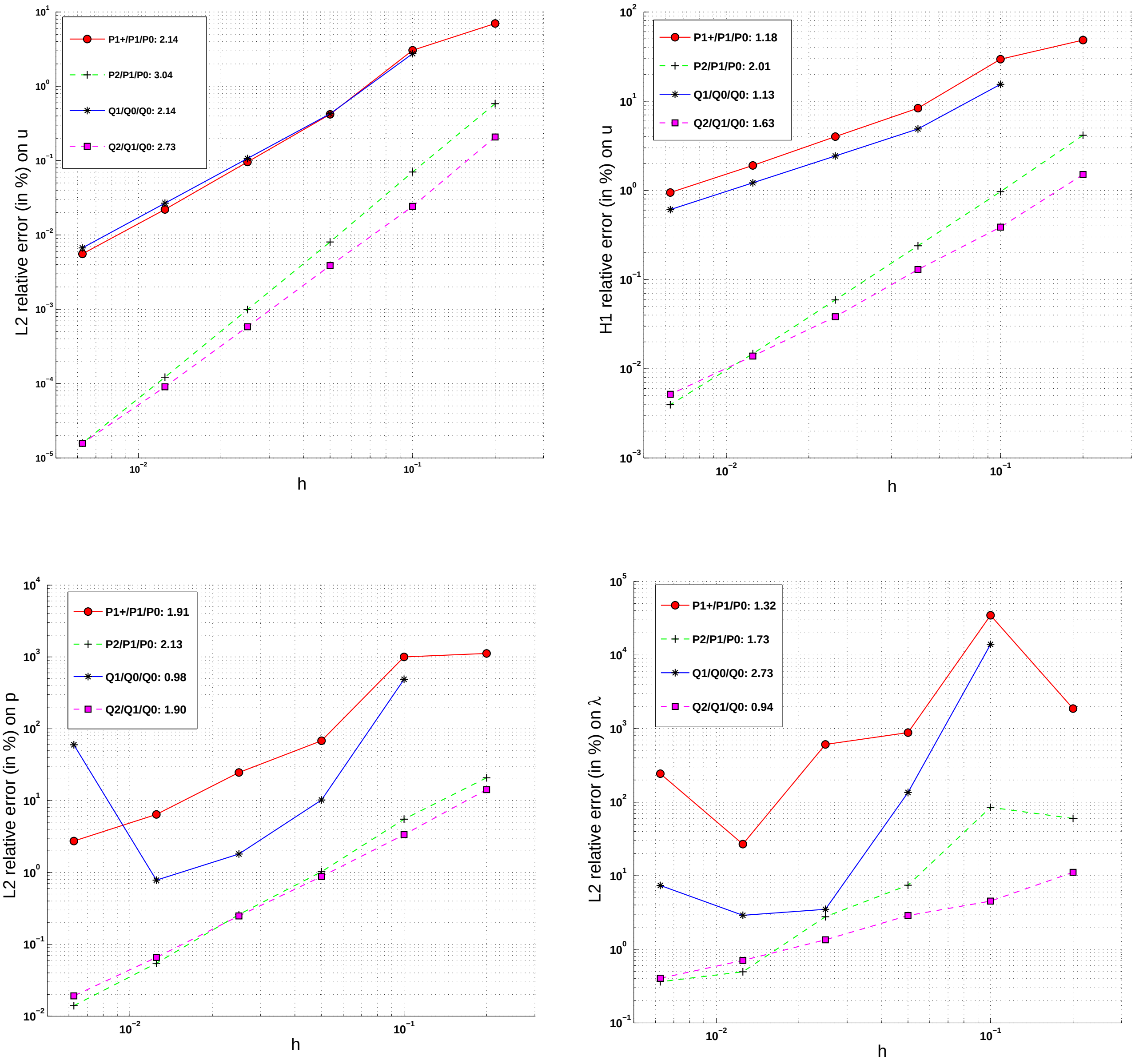

Figure 5.3: Rates of convergence without stabilization for some couples of finite element spaces.

The convergence for the fluid velocity is highlighted, whereas the convergence for the multiplier seems to not occur, in all cases. We get the convergence for the pressure, but not for the test Q1/Q0/Q0 which does anyway not satisfy the inf-sup condition. The rates of convergence are better than what we can expect by the theory for u and p. The results are not so good for the multiplier. Indeed, without stabilization, the the order of magnitude for the relative errors lets us think that the multiplier is not well computed.

5.5.2 Numerical experiments with stabilization

In this part, we consider the method with stabilization terms. Additional terms depending on the positive constant γ are considered in the variational formulation (5.21). In the following, we fix $\gamma = h\gamma_0$, as it is suggested in the proof of Lemma 5.9 (γ is supposed to be constant, which is natural when uniform meshes are considered). The parameter γ (or γ_0) has to respond to a compromise between the coercivity of the system and the weight of the stabilization term. First, the choice of γ is discussed. We choose the P2/P1/P0 couple of spaces with the space step $h = 0.025$. To characterize a good range of values, we present the condition number (of the whole system) in Fig. 5.4, and the relative errors on the multiplier λ for $\gamma_0 \in [10^{-14}; 10^4]$ and more precisely for $\gamma_0 \in [0.001; 0.200]$ in Fig. 5.5.

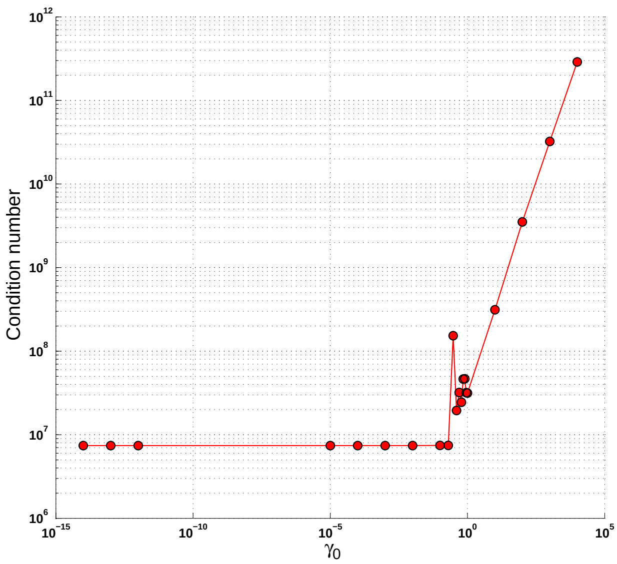

Figure 5.4: The condition number for $\gamma_0 \in [10^{-14}; 10^4]$.

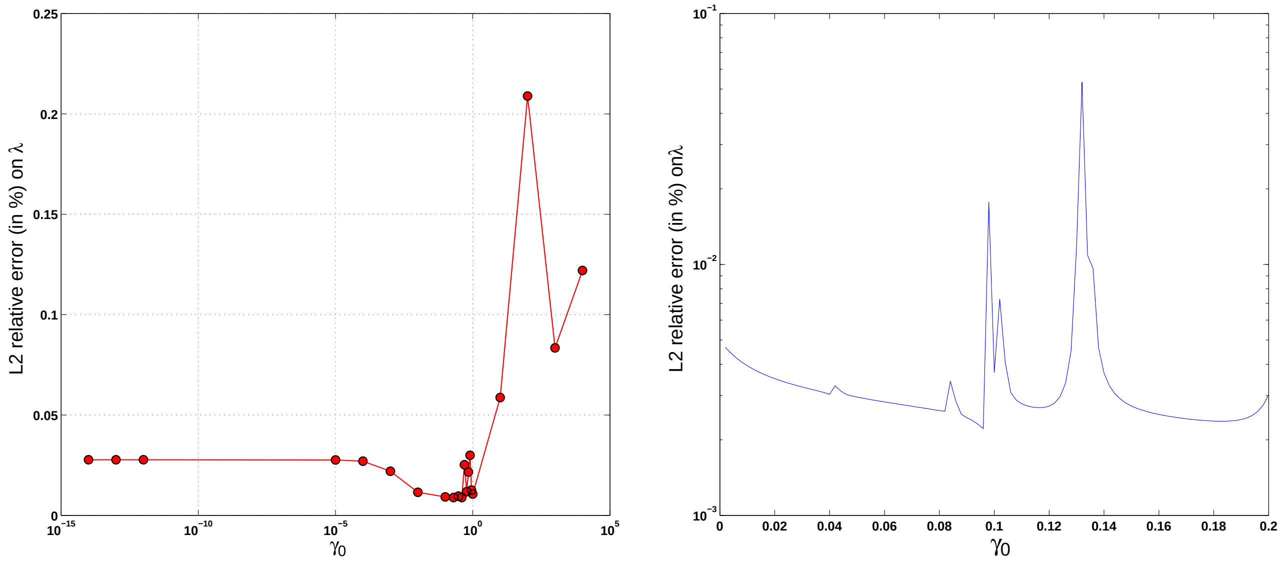

Figure 5.5: The relative errors $\|\lambda - \lambda^h\|_{\mathbf{L}^2(\Gamma)}$ for $\gamma_0 \in [10^{-14}; 10^4]$ (left), $\gamma_0 \in [0.001; 0.200]$ (right).

The condition number given for some very small γ_0 corresponds to the condition number of the system when no stabilization is used. For all situations, the condition number is degraded when stabilization terms are considered and can explode when γ_0 is too large. With regard to the errors on the multiplier λ, there is no improvement for the relative errors on the multiplier when γ_0 is too small. When γ_0 increases, the errors on the multiplier becomes interesting even if some peaks can appear (transition zone where the coercivity property is very poor). Similar observations (same values for γ_0) are observed on the relative errors for the velocity.

With regard to the previous experiments, in the following, we choose $\gamma_0 = 0.05$ (so $\gamma = 0.05 \times h$) and we study the numerical convergence analysis of the method when stabilization is used. The following numerical experiments have been made in the same conditions as the one given in section 5.3. The results are reported in Fig. 5.6.

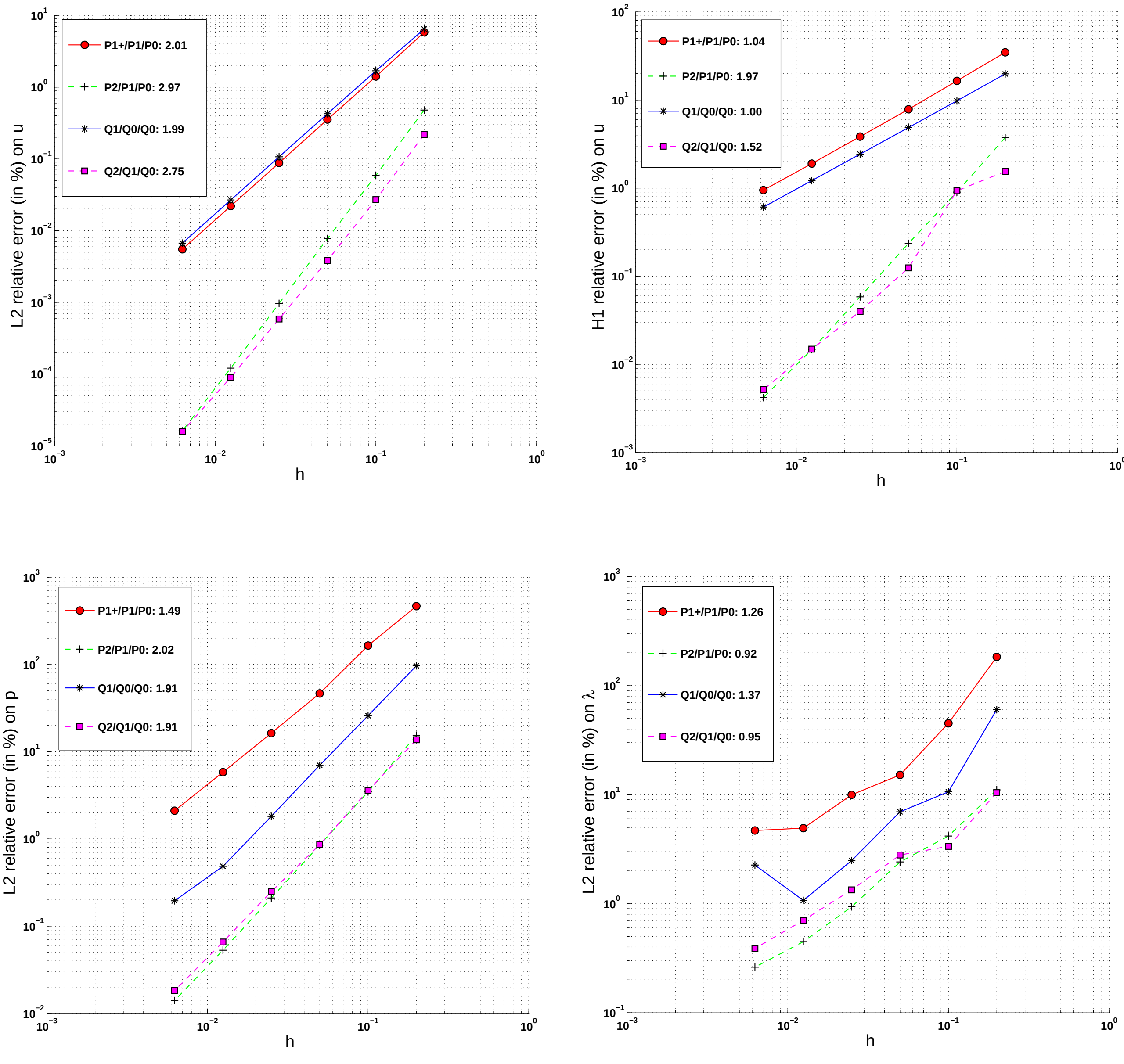

Figure 5.6: Rates of convergence with stabilization for some couples of finite element spaces.

We notice that we do not see any difference on the rate of convergence for the errors on the fluid velocity. As regards to the pressure, we notice a better behavior (compared to the first method without stabilization) is observed for the couple of spaces Q1/Q0/Q0 that do not satisfy the inf-sup condition.In all cases, the improvements appear for the multiplier. The method enables to recover the convergence for the multiplier. Besides, the orders of magnitude let us think that the multiplier is there well computed.

5.6 Some practical remarks on the numerical implementation

The numerical implementation of the method for Stokes problem is based on the code developed under GETFEM++ Library [RP] for Poisson problem. The system is solved in using the library SuperLU [DGL]. The advantages of using the GETFEM++ library (besides its simplicity of developing finite element codes) is that several specific difficulties have been already resolved. Notably,

- to define basis functions of $\mathbf{W}^h$ from traces on Γ of the basis functions of $\tilde{\mathbf{W}}^h$. Indeed, their independence is not ensured and numerical manipulations must be done in order to eliminate possible redundant functions (and avoid to manipulate singular systems),
- to localize the interface between the fluid and the structure, a level-set function which is already implemented (as it is done in [SCMB01a] for instance),
- to compute properly the integrals over elements at the interface (during assembling) external call to QHULL Library [BDH96] is realized (see Fig. 5.7).

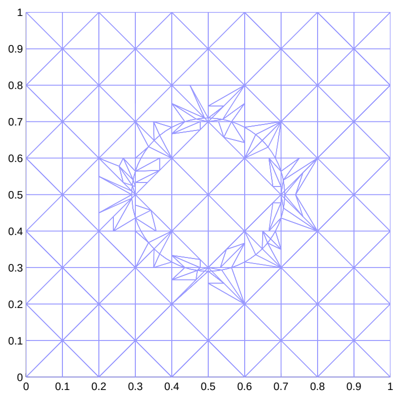

Figure 5.7: Local treatment at the interface using QHULL Library.

As mentioned in the paper [HR09], it is possible to define a reinforced stability to prevent difficulties that can occur when the intersection of the solid and the mesh over the whole domain introduce "very small" elements. The technique is based on a strategy to select elements which are better to deduce the normal derivative on Γ. A similar approach is given in [Pit80]. We has tested this method for Stokes problem but in our numerical experimentations we do not show substantial improvements with this enriched stabilization, compared to the results obtained with the stabilization method detailed in this chapter. However, we expect to take benefits of this second stabilization method when the boundary Γ is led to move through the time, in particular in unsteady framework and fluid-structure interactions. In that case, we consider a moving rigid solid which then occupies a time-depending domain $\mathcal{S}(t)$. The displacement of a rigid solid is given by

$$\begin{aligned} X(y,t) &= h(t) + \mathbf{R}(t)y, \quad y \in \mathcal{S}(0), \\ \mathcal{S}(t) &= h(t) + \mathbf{R}(t)\mathcal{S}(0), \end{aligned}$$

where $h(t)$ denotes the coordinates of the center of mass of the solid, and $\mathbf{R}(t)$ is the rotation which describes the orientation of the solid with respect to its reference configuration. In dimension 2, this orientation can be given by a single angle $\theta(t)$, and we have

$$\mathbf{R}(t) = \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix}.$$

In dimension 2, the angular velocity $\omega(t) = \theta'(t)$ is a scalar function. The fluid domain is given by $\mathcal{O} \setminus \overline{\mathcal{S}(t)} = \mathcal{F}(t)$. The state of the corresponding full system is then defined by the fluid velocity and pressure, $\mathbf{u}$ and p, and the position of the solid given by the coordinates of its center of mass $\mathbf{h}(t)$ and its angular velocity $\omega(t)$. The coupling between the fluid and the structure is mainly made at the interface Γ, through the Dirichlet condition

$$u(x,t) = h'(t) + \omega(t) \wedge (\mathbf{x} - h(t)), \quad \mathbf{x} \in \Gamma(t),$$

and through two differential equations which link the position of the solid and the forces that the fluid applies on its boundary, as follows

$$\begin{aligned} Mh''(t) &= -\int_{\Gamma(t)} \sigma(u,p)n \mathrm{d}\Gamma, \\ I\omega'(t) &= -\int_{\Gamma(t)} (x - h(t)) \wedge \sigma(u,p)n \mathrm{d}\Gamma. \end{aligned}$$

Thus, obtaining a good approximation for $\sigma(\mathbf{u},p)\mathbf{n}$ is essential for simulating the trajectories of the solid. We do not discuss this point furthermore which is in progress and refer to further publications. In short, the quantity $\sigma(u,p)n$ is crucial in fluid-structure models that involve viscous incompressible fluids. Getting a good numerical approximation - which is the main interest of this chapter - is a key point.

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Prospects

Stabilization around a nontrivial stationary solution

With regards to the result obtained in Chapter 3, that is to say the stabilization to zero of the fluid-solid system, we can take an interest to the stabilizability of this system around nontrivial stationary solutions. In order to consider a stationary flow, let us put the deformable solid in a channel where we impose at the entrance a Poiseuille flow, for instance, whose the velocity is denoted u_0. We have homogeneous Dirichlet conditions on the sides of the channel, and a homogeneous Neumann-type condition at the exit of the channel.

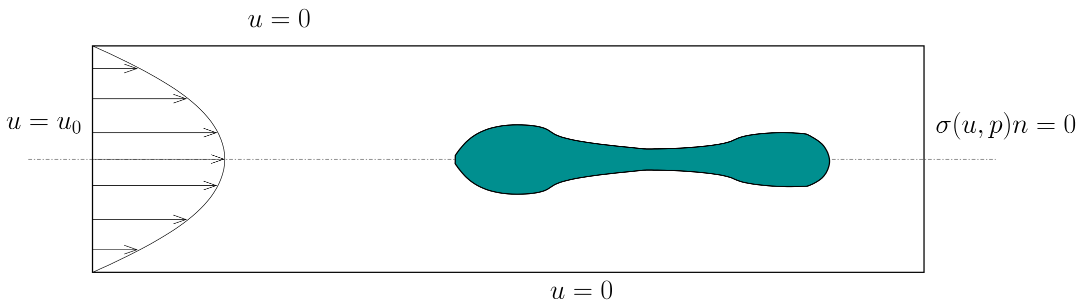

Figure 6.8: A deformable solid in a channel flow.

First, the existence of stationary solutions for the fluid-solid system (with these nonhomogeneous boundary conditions on $\partial\mathcal{O}$) is *a priori* nontrivial. The study and the methods which have been lead in [Gra02] for instance may be useful for such a problem.
Secondly, the results of Chapter 3 could be adapted for the stabilization of the fluid-solid system around possible nontrivial stationary flows, with some differences like the consideration of an Oseen operator instead of the classical Laplace operator used in this chapter.
Besides, the method of projection (see the subsection 3.6.3) which has been carried out - in order to consider deformations which satisfy the physical nonlinear constraints - has been made in $\mathcal{S} \times (0, \infty)$. This approach induces the following limitation: The full stabilized nonlinear system is not causal, because the chosen deformation is necessarily anticipative (in time) with respect to the solution. Thus we expect in a future work to define a projection which enables us to recover a causal system. It would be more convenient in a perspective focused on numerical simulations.
Finally, the more interesting point consists in performing numerical experiments, in order to illustrate the possible results. The numerical method developed in Chapter 5 can be easily adapted. Actually, computations involving a moving solid in a viscous incompressible fluid are currently performed. They are directly based on this numerical method, by it is still a work in progress.

A transmission problem related to the Immersed Boundary Method

The physical nature of the control considered in this thesis, and more specifically in Chapter 3, corresponds to a deformation velocity of a solid. We can wonder if we could consider another type of control, a force for instance. Of course, a solid's deformation could be obtained thanks to internal forces in the solid, through a mechanical system for instance. But we can also study a model where the force is applied on the boundary.
Let us consider the model of the Immersed Boundary Method (see [Pes02]), in dimension 2. In this model a level-set splits the fluid domain into two connected components, and a force is applied on this level-set.

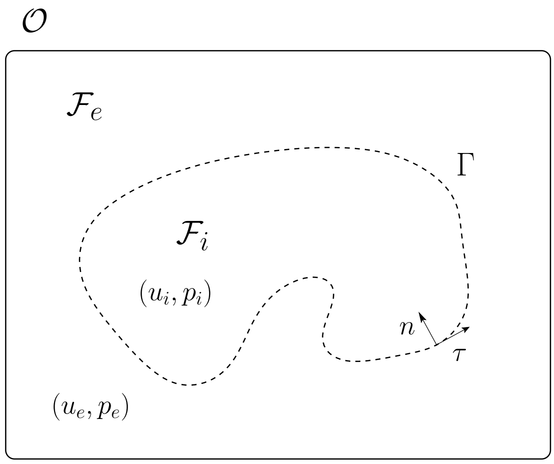

Figure 6.9: A boundary force applied on a level-set, separating the fluid into two parts.

The system considered is then the following

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f, \qquad \text{in } \mathcal{O}, \tag{6.1}$$

$$\mathrm{div} u = 0, \qquad \text{in } \mathcal{O}, \tag{6.2}$$

$$u = 0, \qquad \text{on } \partial\mathcal{O}, \tag{6.3}$$

$$u(\cdot, 0) = u_0, \qquad \text{in } \mathcal{O}, \tag{6.4}$$

where f is defined on $\Gamma(t)$ through the expression

$$f(x,t) \quad = \quad \int_{\Gamma(0)} \tilde{f}(y,t) \delta\left(x - X(y,t)\right) \mathrm{d}\Gamma(y,0), \tag{6.5}$$

and where the mapping $X(\cdot, t)$ is the Lagrangian mapping determined by the fluid's velocity through the problem

$$\frac{\partial X}{\partial t}(y,t) = u(X(y,t),t), \quad X(y,0) = y, \quad y \in \mathcal{O}. \tag{6.6}$$

The change of variables induced by this problem has a physical meaning, since it is the correspondence between the Eulerian description and the Lagrangian description of the fluid's state. We can first rewrite system (6.1)–(6.4) in taking into account the separation made by the level-set on which the force is applied. It leads us to consider the following transmission problem

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \operatorname{div} \sigma(u,p) = 0 & \qquad \text{in } \mathcal{F}_e(t) \text{ and } \mathcal{F}_i(t), \\ [\sigma(u,p)n] = f & \qquad \text{on } \Gamma(t), \\ u_e = u_i & \qquad \text{on } \Gamma(t), \\ u_e = 0 & \qquad \text{on } \partial\mathcal{O}, \end{aligned}$$

with the appropriate initial conditions. In this system, the force induces the jump $[\sigma(u,p)n]$ at the interface between the two fluid parts. The response of the fluid is a velocity on $\Gamma(t)$, which determines through problem (6.6) the value of $X(\cdot,t)$ on $\Gamma(0)$, and thus the shape of the $\Gamma(t)$. The study of such a model has been recently lead in [CCS08], at high Reynolds number, that is to say without viscosity.
From a modeling point of view, instead of considering an arbitrary boundary force f, we can choose a force which is determined by the surface tension on $\Gamma(t)$, like the following one

$$\tilde{f}(s,t) \quad = \quad \frac{\partial}{\partial s}\left(T(s,t)\tau(s,t)\right),$$

with the tension $T(s,t)$ given by

$$T(s,t) \quad = \quad \lambda\left(\left|\frac{\partial X}{\partial s_0}(s,t)\right| - 1\right).$$

In these expressions, λ is a coefficient, s denotes the arc-coordinate of the curve and τ denotes its tangent vector. Such forces have been considered in [HLS94] and [HLS94] for instance, in order to perform numerical simulations.

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Some useful formulas

Tensor calculus and integration formulas

Notation

In the formulas given below, we will denote

$A,\ B$	:	regular tensor fields of order two, seen as matrices of $\mathbb{R}^{d\times d}$, whose components are denoted A_{ij}, B_{ij}
$a,\ b,\ c$	:	regular vector fields of R^d, whose components are respectively denoted a_i, b_i and c_i,
α	:	a regular scalar field of $\mathbb{R}$.

For a given vector a, we introduce the skew-symmetric matrix

$$\mathbb{S}(a) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

Let us define some operators:

$$a \cdot b = \sum_i a_i b_i \quad a \wedge b = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \quad a \otimes b = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}$$

$$A : B = \text{trace}(AB^T) = \sum_{i,j} A_{ij}B_{ij}$$

$$\nabla\alpha = \begin{pmatrix} \dfrac{\partial\alpha}{\partial x_1} \\ \dfrac{\partial\alpha}{\partial x_2} \\ \dfrac{\partial\alpha}{\partial x_3} \end{pmatrix} \quad \nabla a = \begin{pmatrix} \dfrac{\partial a_1}{\partial x_1} & \dfrac{\partial a_1}{\partial x_2} & \dfrac{\partial a_1}{\partial x_3} \\ \dfrac{\partial a_2}{\partial x_1} & \dfrac{\partial a_2}{\partial x_2} & \dfrac{\partial a_2}{\partial x_3} \\ \dfrac{\partial a_3}{\partial x_1} & \dfrac{\partial a_3}{\partial x_2} & \dfrac{\partial a_3}{\partial x_3} \end{pmatrix}$$

$$\text{div } a = \nabla \cdot a = \text{trace}\nabla a = \sum_i \frac{\partial a_i}{\partial x_i}$$

$$\text{curl } a = \nabla \wedge a = \begin{pmatrix} \dfrac{\partial a_3}{\partial x_2} - \dfrac{\partial a_2}{\partial x_3} \\ \dfrac{\partial a_1}{\partial x_3} - \dfrac{\partial a_3}{\partial x_1} \\ \dfrac{\partial a_2}{\partial x_1} - \dfrac{\partial a_1}{\partial x_2} \end{pmatrix} \quad \text{div } A = \begin{pmatrix} \displaystyle\sum_j \frac{\partial A_{1j}}{\partial x_j} \\ \displaystyle\sum_j \frac{\partial A_{2j}}{\partial x_j} \\ \displaystyle\sum_j \frac{\partial A_{3j}}{\partial x_j} \end{pmatrix}$$

Formulas

$a \wedge (b \wedge c) = (a \cdot c)b - (a \cdot b)c$	$\mathrm{div}(\alpha b) = \alpha \mathrm{div}\ b + \nabla\alpha \cdot b$
$a \wedge (b \wedge a) = \left(\lvert a\rvert^2 \mathrm{I}_{\mathbb{R}^d} - a \otimes a\right) b$	$\mathrm{curl}(\alpha b) = \alpha \mathrm{curl}\ b + \nabla\alpha \wedge b$
$a \wedge b = \mathbb{S}(a) b$	$\mathrm{div}\ (a \otimes b) = (a \cdot \nabla) b + b(\mathrm{div}\ a)$
$\mathbb{S}(a)\mathbb{S}(b) = b \otimes a_(a \cdot b)\mathrm{I}_{\mathbb{R}^d}$	$\mathrm{div}(\nabla a) = \Delta a = \sum_i \frac{\partial^2 a_i}{\partial x_i^2}$
$\nabla(a \cdot b) = (\nabla a^T) b + (\nabla b^T) a$	$\mathrm{div}(\nabla a^T) = \nabla(\mathrm{div}\ a)$
$\nabla(\alpha b) = \alpha \nabla b + b \otimes \nabla\alpha$	$\mathrm{div}(Ab) = \mathrm{div}(A^T) \cdot b + A^T : \nabla b$
	$\mathrm{div}(\mathbb{S}(a) B) = \mathbb{S}(a) \mathrm{div}\ B$, if a is constant
	$\mathrm{div}(\mathbb{S}(y) B) = \mathbb{S}(y) \mathrm{div}\ B$, if B is symmetric

Properties of the cofactor matrix

Let us denote $\mathrm{cof}(A)$ the cofactor matrix associated with some matrix A. Whether A is invertible or not, we have the formula

$$A^T \mathrm{cof}(A) \quad = \quad \det(A)\mathrm{I}_{\mathbb{R}^d};$$

It gives an expression for the inverse of the matrix A, when the latter is invertible.
The Piola identity is the following equality, valid for all regular matrix field A:

$$\mathrm{div}(\mathrm{cof} A) \quad = \quad 0$$

The following formula is quite useful, especially when A is a rotation for instance:

$$(Ab) \wedge (Ac) \quad = \quad \mathrm{cof}(A)(b \wedge c).$$

Finally, the cofactor matrix appears in the differential of the determinant, which is

$$D_A(\det) \ : \ B \quad \mapsto \quad \mathrm{cof}(A) : B.$$

The divergence theorem

Theorem ([Gur], p. 16)**.** *Let Ω be a bounded or unbounded regular region of $\mathbb{R}^d$. Let φ be a scalar field, u a vector field and A a tensor field. We assume that φ, u and A are continuous on $\overline{\Omega}$, differentiable almost everywhere in Ω, and of bounded support. Then we have*

$$\int_{\partial\Omega} \varphi n \mathrm{d}\Gamma = \int_\Omega \nabla\varphi, \qquad \int_{\partial\Omega} u \otimes n \mathrm{d}\Gamma = \int_\Omega \nabla u,$$
$$\int_{\partial\Omega} u \cdot n \mathrm{d}\Gamma = \int_\Omega \mathrm{div}\ u, \qquad \int_{\partial\Omega} A n \mathrm{d}\Gamma = \int_\Omega \mathrm{div}\ A,$$

whenever the integrand on the right is piecewise continuous on $\overline{\Omega}$.

Change of variables formulas

Let us consider a regular bounded domain Ω_0 of $\mathbb{R}^d$, and a C^1-diffeomorphism X of $\mathbb{R}^d$. We assume that

$$X - \mathrm{Id} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d), \qquad X^{-1} - \mathrm{Id} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d).$$

Lemma ([All07], Lemma 6.23, p. 135). *Let $f \in \mathbf{L}^1(\partial X(\Omega_0))$. Then $f \circ X \in \mathbf{L}^1(\partial\Omega_0)$, and we have*

$$\int_{\partial X(\Omega_0)} f \mathrm{d}\Gamma = \int_{\partial\Omega_0} (f \circ X) \left|\det\nabla X\right| \left|(\nabla X)^{-T} \circ X n\right|_{\mathbb{R}^d} \mathrm{d}\Gamma.$$

Proposition ([Gur81], p. 51). *Let φ be a continuous scalar function on $X(\Omega_0)$. Then*

$$\begin{aligned} \int_{X(\Omega_0)} \varphi(x)\mathrm{d}x &= \int_{\Omega_0} (\varphi \circ X)(y) \det\nabla X(y)\mathrm{d}y, \\ \int_{\partial X(\Omega_0)} \varphi(x) n \mathrm{d}\Gamma &= \int_{\Omega_0} (\varphi \circ X)(y) \mathrm{cof}\nabla X(y) n \mathrm{d}\Gamma, \end{aligned}$$

where $\mathrm{cof}\nabla X$ denotes the cofactor matrix of ∇X, given by $\mathrm{cof}\nabla X = (\det\nabla X)\left(\nabla X^{-1}\right)^T$.

Derivation with respect to the domain

Theorem (The Reynolds transport theorem). *Let Ω an open bounded subset of $\mathbb{R}^d$ (with $d = 2$ or 3). Let $(X(\cdot,t))_{t\in\mathbb{R}}$ be a family of diffeomorphisms from Ω onto $X(\Omega,t) = \Omega(t)$. We denote $Y(\cdot,t)$ the inverse of $X(\cdot,t)$. We assume that the mapping $t \mapsto X(\cdot,t)$ is of class C^1. Let f be a vector field of class C^1 in variables (X,t), defined on $\mathbb{R} \times \mathbb{R}^d$. Then we have the following formula,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} f(x,t)\mathrm{d}x = \int_{\Omega(t)} \left(\frac{\partial f}{\partial t} + \mathrm{div}\,(f \otimes u) \right) \mathrm{d}x,$$

where u is called the Eulerian velocity field associated with the Lagrangian flow X:

$$u(x,t) = \frac{\partial X}{\partial t}(Y(x,t),t).$$

Results in relation with the Cauchy stress tensor

Let us recall the expression of the symmetric Cauchy stress tensor, for some vector field u, given as

$$D(u) = \frac{1}{2}(\nabla u + \nabla u^T).$$

The Temam lemma

Lemma ([Tem83], Lemma 1.1 p. 18). *The kernel of the operator D consists of functions u of the form*

$$u(x) = a + Bx,$$

where $a \in \mathbb{R}^d$ is a vector, and $B \in \mathbb{R}^{d\times d}$ is a skew-symmetric matrix. In the particular case $d = 3$, the matrix B can be represented by a vector b such that

$$B = \mathbb{S}(b) = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}, \qquad u(x) = a + b \wedge x.$$

Equivalence of norms

Lemma (The Korn's inequality)**.** *Let* Ω *a bounded domain of* $\mathbb{R}^d$. *Then there exists a positive constant* C *such that for all* $v \in \mathbf{H}_0^1(\Omega)$ *we have*

$$\|v\|_{\mathbf{H}^1(\Omega)} \quad \leq \quad C\|D(v)\|_{[\mathrm{L}^2(\Omega)]^{d\times d}}.$$

Lemma. *Let* Ω *a bounded domain of* $\mathbb{R}^d$. *For all* $v \in \mathbf{H}_0^1(\Omega)$ *we have*

$$\|v\|^2_{\mathbf{H}^1(\Omega)} + (\mathrm{div}\ v)^2 \quad = \quad 2\|D(v)\|^2_{[\mathrm{L}^2(\Omega)]^{d\times d}}.$$

This result can be obtained thanks to the Poincaré inequality, combined with the estimate

$$\|\nabla v\|^2_{\mathbf{L}^2(\Omega)} \quad \leq \quad \|D(v)\|^2_{\mathbf{L}^2(\Omega)}$$

which can be obtained thanks to the following formula

$$2D(v) : D(v) - \nabla v : \nabla v \quad = \quad \mathrm{div}\left((v\cdot\nabla)v - (\mathrm{div}\ v)v\right) + (\mathrm{div}\ v)^2 .$$

Solid mechanics in the deformable case

In an inertial frame of reference, the solid's deformation at some time t can be represented by a Lagrangian mapping $X_{\mathcal{S}}(\cdot,t)$. The domain occupied by the solid at this time t is

$$\mathcal{S}(t) \quad = \quad X_{\mathcal{S}}(\mathcal{S}(0),t).$$

Let us assume that the mapping $X_{\mathcal{S}}(\cdot,t)$ is invertible at each time t. We denote $Y_{\mathcal{S}}(\cdot,t)$ its inverse, and we can define the Eulerian velocity vector field in the solid domain:

$$u_{\mathcal{S}}(x,t) \quad = \quad \frac{\partial X_{\mathcal{S}}}{\partial t}(Y_{\mathcal{S}}(x,t),t), \quad x \in \mathcal{S}(t).$$

Kinematic properties for a deformable solid

The vector $h(t)$ denotes the Cartesian coordinates of the solid's center of mass. If M denotes the solid's mass (which is assumed to be constant through the time), the vector $h(t)$ is defined by the equality

$$Mh(t) = \int_{\mathcal{S}} \rho_{\mathcal{S}}(x,t) x \mathrm{d}x \quad \Leftrightarrow \quad \int_{\mathcal{S}} \rho_{\mathcal{S}}(x,t)(x-h(t))\mathrm{d}x = 0.$$

The angular velocity of the solid, denoted by $\omega(t)$, can be defined as follows

$$I(t)\omega(t) \quad = \quad \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)(x-h(t)) \wedge (u_{\mathcal{S}}(x,t) - h'(t))\mathrm{d}x.$$

In this formula, $I(t)$ is the inertia matrix of the solid **at its center of mass**, at time t. This matrix is always symmetric and invertible.
The result of the cross product in $\mathbb{R}^3$ is a vector, whereas it is a scalar in $\mathbb{R}^2$. However $\mathbb{R}^2$ can be immersed in $\mathbb{R}^3$, and the scalar result of a cross product in $\mathbb{R}^2$ can be read on the third component of a 3D-vector, as follows

$$\left(\begin{array}{l} a_1 \\ a_2 \\ 0 \end{array}\right) \wedge \left(\begin{array}{l} b_1 \\ b_2 \\ 0 \end{array}\right) \quad = \quad \left(\begin{array}{l} 0 \\ 0 \\ a_1 b_2 - a_2 b_1 \end{array}\right).$$

In the same idea, since the angular velocity $\omega(t)$ is a scalar function in dimension 2 and a 3D-vector in dimension 3, the calculations on vectors hold if we consider that in dimension 2 the angular velocity is seen as

$$\left(\begin{array}{l} 0 \\ 0 \\ \omega(t) \end{array}\right).$$

That is why in all cases we consider the angular velocity as a vector of $\mathbb{R}^3$.
In dimension 3, the inertia matrix can be expressed in two equivalent forms, as follows

$$\begin{aligned} I(t) \quad &= \quad \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t) \left(|x-h(t)|^2 \mathrm{I}_{\mathbb{R}^d} - (x-h(t)) \otimes (x-h(t)) \right) \mathrm{d}x, \\ I(t)\omega(t) \quad &= \quad \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)(x-h(t)) \wedge (\omega(t) \wedge (x-h(t)))\mathrm{d}x. \end{aligned}$$

In dimension 2, it can be given as

$$I(t) \quad = \quad \left(\int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t) |x - h(t)|^2 \mathrm{d}x \right) \mathrm{I}_{\mathbb{R}^3}.$$

Let us recall that the local form of the mass conservation, given by

$$\frac{\partial \rho_{\mathcal{S}}}{\partial t} + \mathrm{div}(\rho_{\mathcal{S}} u_{\mathcal{S}}) \quad = \quad 0,$$

is nothing else than the following formula written in Lagrangian coordinates

$$\frac{\partial}{\partial t} \left(\rho_{\mathcal{S}}(X_{\mathcal{S}}(y,t),t) \mathrm{det} \nabla X_{\mathcal{S}}(y,t) \right) \quad = \quad 0,$$

so that we have the equality

$$\rho_{\mathcal{S}}(X_{\mathcal{S}}(y,t),t) \quad = \quad \frac{\rho_{\mathcal{S}}(y,0)}{\mathrm{det} \nabla X_{\mathcal{S}}(y,t)}.$$

Thus, if $X_{\mathcal{S}}(\cdot,t)$ is a C^1-diffeomorphism, we can rewrite

$$\begin{aligned} M \quad &= \quad \int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) \mathrm{d}y, \\ I(t) \quad &= \quad \int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) \left(|X_{\mathcal{S}}(y,t) - h(t)| - (X_{\mathcal{S}}(y,t) - h(t)) \otimes (X_{\mathcal{S}}(y,t) - h(t)) \right) \mathrm{d}y \end{aligned}$$

in dimension 3,

$$I(t) \quad = \quad \left(\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) \left| X_{\mathcal{S}}(y,t) - h(t) \right| \mathrm{d}y \right) \mathrm{I}_{\mathbb{R}^3},$$

in dimension 2, and

$$\omega(t) \quad = \quad I(t)^{-1} \int_{\mathcal{S}(0)} (X_{\mathcal{S}}(y,t) - h(t)) \wedge \left(\frac{\partial u_{\mathcal{S}}}{\partial t}(y,t) - h'(t) \right) \mathrm{d}y.$$

The angular velocity ω is related to a rotation $\mathbf{R}$, which can be deduced from the following system of differential equations

$$\left\{ \begin{array}{rcl} \dfrac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} & = & \mathbb{S}(\omega) \mathbf{R} \\ \mathbf{R}(0) & = & \mathrm{I}_{\mathbb{R}^3}, \end{array} \right. \qquad \text{with } \mathbb{S}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Decomposition of the solid's deformation

We can decompose the solid's motion in splitting the mapping $X_{\mathcal{S}}$ as follows

$$X_{\mathcal{S}}(y,t) \quad = \quad h(t) + \mathbf{R}(t) X^*(y,t), \quad y \in \mathcal{S}(0). \tag{7.1}$$

The mapping $X^*(\cdot,t)$ can be seen as the deformation of the solid in its own frame of reference. Its time derivative is expressed as

$$\frac{\partial X^*}{\partial t}(y,t) \quad = \quad h'(t) + \omega(t) \wedge (\mathbf{R}(t) X^*(y,t)) + \mathbf{R}(t) \frac{\partial X^*}{\partial t}(y,t), \quad y \in \mathcal{S}(0). \tag{7.2}$$

As this decomposition is formulated in (7.1), if $h(t)$ and $\omega(t)$ are well the center of mass position and the angular velocity of the whole solid respectively, then it implies that this deformation X^* satisfies the two following equalities

$$\left\{ \begin{array}{l} \displaystyle\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t) \mathrm{d}y = 0, \\ \displaystyle\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0) X^*(y,t) \wedge \frac{\partial X^*}{\partial t}(y,t) \mathrm{d}y = 0. \end{array} \right.$$

Eulerian formulation

The inverse of $X^*(\cdot,t)$ is given by

$$Y^*(x^*,t) \quad = \quad Y(h(t) + \mathbf{R}(t)x^*, t).$$

Then we can associate to this Lagrangian mapping an Eulerian velocity, expressed as

$$w^*(x^*,t) \quad = \quad \frac{\partial X^*}{\partial t}(Y^*(x^*,t),t), \quad x^* \in \mathcal{S}^*(t) := X^*(\mathcal{S}(0),t).$$

We can easily verify that we have an analogous decomposition for the Eulerian velocity $u_{\mathcal{S}}$, given as

$$u_{\mathcal{S}}(x,t) \quad = \quad h'(t) + \omega(t) \wedge (x - h(t)) + w(x,t), \quad x \in \mathcal{S}(t),$$

where w is related to w^* through the following change of frame

$$w(x,t) \quad = \quad \mathbf{R}(t) w^*(\mathbf{R}(t)^T(x - h(t)), t), \quad x \in \mathcal{S}(t).$$

Suitable change of functions for the Lagrangian representation

With regards to the expression of the Lagrangian velocity given in (7.2), let us make an appropriate change of functions. We denote

$$\tilde{u}_{\mathcal{S}}(y,t) = \mathbf{R}(t)^T u_{\mathcal{S}}(X_{\mathcal{S}}(y,t),t) = \mathbf{R}(t)^T \frac{\partial X_{\mathcal{S}}}{\partial t}, \quad \tilde{h}'(t) = \mathbf{R}(t)^T h'(t), \quad \tilde{\omega}(t) = \mathbf{R}(t)^T \omega(t).$$

Then the solid's velocity can be expressed in a more compact form as follows

$$\tilde{u}_{\mathcal{S}}(y,t) \quad = \quad \tilde{h}'(t) + \tilde{\omega}(t) \wedge X^*(y,t) + \frac{\partial X^*}{\partial t}(y,t). \tag{7.3}$$

Optimal decomposition

Let us see that the decomposition (7.1) is optimal, in the sense that the velocity generated by the deformation X^* comes from a minimal kinetic energy. Let us first notice that if the functions $X_{\mathcal{S}}(\cdot,t)$, $h(t)$ and $\mathbf{R}(t)$ are given, then the mapping $X^*(\cdot,t)$ is determined in a unique way, as being

$$X^*(\cdot,t) \quad = \quad \mathbf{R}(t)\left(X_{\mathcal{S}}(\cdot,t) - h(t)\right).$$

Let us consider that the deformation the solid's deformation in the inertia frame of reference - that is to say the mapping $X_{\mathcal{S}}(\cdot,t)$ - is given, and let us see that there exists a unique couple $(h(t), \mathbf{R}(t))$ such that the deformation entirely determined by

$$X^*(\cdot,t) \quad = \quad \mathbf{R}(t)\left(X_{\mathcal{S}}(\cdot,t) - h(t)\right)$$

corresponds to a minimal kinetic energy, and that this deformation satisfies necessarily

$$\int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0)\frac{\partial X^*}{\partial t}(y,t)\mathrm{d}y = 0, \qquad \int_{\mathcal{S}} \rho_{\mathcal{S}}(y,0)X^*(y,t) \wedge \frac{\partial X^*}{\partial t}(y,t)\mathrm{d}y = 0. \tag{7.4}$$

For that, we use an approach used in [Ḿ03] (section 1.1.1), which consists in considering an optimization problem. We write the kinetic energy associated with the deformation $X^*(\cdot,t)$, as

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(x,t)|w(x,t)|^2\mathrm{d}x &= \frac{1}{2}\int_{\mathcal{S}^*(t)} \rho^*_{\mathcal{S}}(x^*,t)|w^*(x^*,t)|^2\mathrm{d}x^* = \frac{1}{2}\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0)\left|\frac{\partial X^*}{\partial t}(y,t)\right|^2\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(y,0)\left|\frac{\partial X_{\mathcal{S}}}{\partial t}(y,t) - h'(t) - \omega(t)\wedge(X_{\mathcal{S}}(y,t)-h(t))\right|^2\mathrm{d}y.\end{aligned}$$

For $X_{\mathcal{S}}(\cdot,t)$ given, we consider the minimization problem

Find $(h,h',\omega) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^3$ such that

$$(h,h',\omega) = \operatorname*{arg\,min}_{(k,k',r)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^3} \left(\mathcal{J}(k,k',r) = \frac{1}{2}\left\|\frac{\partial X_{\mathcal{S}}}{\partial t} - k' - r\wedge(X_{\mathcal{S}}-k)\right\|^2_{\mathbf{L}^2(\mathcal{S}(0))}\right),$$

with the additional condition $h(t) = h(0) + \int_0^t h'(s)\mathrm{d}s$.

The first-order optimal conditions are

$$\begin{aligned} 0 &= \omega \wedge \int_{\mathcal{S}(0)} \rho_{\mathcal{S}}\left(\frac{\partial X_{\mathcal{S}}}{\partial t} - h'\right), \\ 0 &= \omega\wedge\int_{\mathcal{S}(0)} \rho_{\mathcal{S}}(X_{\mathcal{S}} - h') - \int_{\mathcal{S}(0)} \rho_{\mathcal{S}}\left(\frac{\partial X_{\mathcal{S}}}{\partial t} - h'\right), \\ 0 &= I\omega - \int_{\mathcal{S}} \rho_{\mathcal{S}}(X_{\mathcal{S}} - h)\wedge\left(\frac{\partial X_{\mathcal{S}}}{\partial t} - h'\right),\end{aligned}$$

where I denotes the inertia matrix of the solid at its center of mass. We recall its expression in dimension 3

$$I(t) = \int_{\mathcal{S}} \rho_{\mathcal{S}}\left(|X_{\mathcal{S}}(y,t)-h(t)|^2\mathrm{I}_{\mathbb{R}^d} - (X_{\mathcal{S}}(y,t)-h(t))\otimes(X_{\mathcal{S}}(y,t)-h(t))\right)\mathrm{d}y,$$

and in dimension 2

$$I(t) = \left(\int_{\mathcal{S}} \rho_{\mathcal{S}}|X_{\mathcal{S}}(y,t)-h(t)|^2\mathrm{I}_{\mathbb{R}^d}\mathrm{d}y\right)\mathrm{I}_{\mathbb{R}^3}.$$

From the first-order optimality conditions we easily deduce

$$\begin{aligned} h' &= \frac{1}{M}\int_{\mathcal{S}} \rho_{\mathcal{S}}\frac{\partial X_{\mathcal{S}}}{\partial t} \\ \omega &= I^{-1}\int_{\mathcal{S}} \rho_{\mathcal{S}}(X_{\mathcal{S}}-h)\wedge\left(\frac{\partial X_{\mathcal{S}}}{\partial t} - h'\right).\end{aligned}$$

The second-order optimality conditions are

$$0 < M\mathrm{I}_{\mathbb{R}^d}, \quad 0 < I \text{ in the sense of the symmetric matrices}, \quad 0 \leq 0.$$

The first-order optimality conditions for this problem are strictly equivalent to the constraints (7.4).

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Résumé :

Dans cette thèse, nous étudions un système fluide-solide qui modélise les interactions entre une structure déformable, et un fluide visqueux et incompressible qui l'entoure. Il couple les équations de Navier-Stokes incompressibles (pour l'état du fluide) avec les lois de Newton (pour la dynamique du solide). L'existence de solutions fortes est étudiée dans les deux premiers chapitres, pour des déformations du solide limitées ou non en régularité.
Puis nous prouvons la stabilisation à zéro de ce système couplé, pour des perturbations extérieures petites, par des déformations du solide soumises à des contraintes physiques qui lui garantissent en particulier d'être *autopropulsé*. Ensuite nous décrivons des moyens pratiques de générer de telles déformations.
Enfin nous développons une méthode numérique pour un problème de Stokes avec conditions de Dirichlet non homogènes. Elle nous permet d'obtenir une bonne approximation de la trace normale du tenseur des contraintes de Cauchy, pour des frontières qui ne dépendent pas du maillage. Cette méthode combine une approche de type *domaines fictifs* basée sur les idées de Xfem, et une méthode de Lagrangien augmenté. Du point de vue des interactions fluide-structure, l'intérêt de cette méthode réside dans l'importance du rôle joué par les forces du fluide à l'interface fluide-solide.

Mots clés : Interactions fluide-structure, équations de Navier-Stokes, existence et unicité de solutions fortes, mécanique des solides déformables, théorie du contrôle, stabilisation par feedback, Xfem, méthodes de domaines fictifs, stabilisation numérique.

Title: Interaction problems between a deformable structure and a viscous incompressible fluid.

Abstract:

In this thesis, we study a fluid-solid system which is a model for the interactions between a deformable structure, and a viscous incompressible fluid surrounding it. It couples the incompressible Navier-Stokes equations (for the fluid flow) with the Newton's laws (for the solid's dynamics). The existence of strong solutions is studied in the first two chapters, for solid's deformations which are limited or not in regularity.
Then we prove the stabilization to zero of this coupled system, for small external perturbations, by solid's deformations submitted to physical constraints which guarantee its *self-propelled* nature. After that we describe practical means of generating such deformations.
Finally we develop a numerical method for a Stokes problem with nonhomogeneous Dirichlet conditions. It enables us to get a good approximation of the normal trace of the Cauchy stress tensor, for boundaries which does not depend on the mesh. This method combines a *fictitious domain* type approach based on the ideas of Xfem, and an augmented Lagrangian method. In a fluid-structure interaction perspective, the interest of this method lies in the importance of the role played by the fluid's forces at the fluid-solid interface.

Keywords: Fluid-structure interactions, Navier-Stokes equations, strong solutions, mechanics of deformable solids, control theory, feedback stabilization, Xfem, fictitious domain methods, numerical stabilization.

AMS subject classifications (2010): 93C20, 35Q30, 35Q35, 35Q74, 35Q93, 76D03, 76D05, 76D07, 74B99, 74F10, 76D55, 93B52, 93D15, 65M60, 65M85.
