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Quelques résultats d'existence, de contrôlabilité et de stabilisation pour des systèmes couplés fluide-structure.

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A ma grand-mère

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Chapitre 1

Introduction générale

Contexte.

Interaction fluide-structure.

Les problèmes d'interaction fluide-structure apparaissent naturellement en aérodynamique, aéroacoustique, biologie. Ils sont de deux types : un solide est immergé dans un fluide. C'est le cas d'un poisson (solide déformable) ou d'un sous-marin (solide indéformable) dans une rivière ou dans un océan. Un fluide est contenu dans un domaine dont tout ou partie de la frontière est déformable. C'est le cas de l'écoulement sanguin dans une artère ou du mécanisme du mouvement respiratoire : l'air arrive dans les poumons qui se gonflent sous l'effet du diaphragme.

L'étude de ces problèmes est pluri-disciplinaire puisqu'il faut d'abord les modéliser, puis en faire l'étude aussi bien théorique que numérique. Présentons très brièvement l'exemple d'un solide indéformable dans une cavité remplie d'un fluide newtonien incompressible et visqueux en deux dimensions. Le solide indéformable est caractérisé par la position de son centre de masse, un point de $\mathbb{R}^2$, et par l'angle dans $[0, 2\pi[$ que fait le solide par rapport à une droite de référence. Le fluide est déterminé par sa vitesse et sa pression. Grâce aux lois de conservation, nous obtenons donc dans cet exemple : deux équations différentielles ordinaires pour le solide et deux équations aux dérivées partielles pour le fluide. De plus, le solide agit sur le fluide et réciproquement, ce qui donne les conditions d'interaction aux points de contact du fluide et du solide. Tout cela, ajouté à des conditions initiales et des conditions aux bords pour le fluide où il n'y a pas contact avec le solide, donne le système total.

Contrôlabilité et stabilisabilité.

L'idée de la contrôlabilité est également naturelle. On considère une équation d'évolution modélisant ou non un phénomène physique, biologique, *etc.* On veut alors, en agissant sur une partie du domaine, amener la solution du problème contrôlé à un objectif prescrit. Il existe différentes notions de contrôlabilité ; celle qui nous intéressera plus loin (voir Chapitre 5) est la contrôlabilité à zéro. La contrôlabilité à zéro en temps $T > 0$ d'un système consiste, pour toute condition initiale, à trouver un contrôle tel que la solution du problème contrôlé est nulle en temps $T > 0$.

La stabilisabilité est une notion très proche de la contrôlabilité. On se donne un taux de décroissance $\omega > 0$. On veut alors, pour toute condition initiale y^0, trouver un contrôle c (dans un certain espace fonctionnel) tel que la solution y du problème associé vérifie la décroissance exponentielle suivante

$$|y(t)| \leq C\mathrm{e}^{-\omega t}|y^0| \text{ pour tout } t \geq 0.$$

La norme $|\cdot|$ est la norme adéquate sur l'espace des conditions initiales.

Le modèle.

Dans l'article de survey [21], les auteurs Quarteroni, Tuveri et Veneziani proposent la modélisation de l'écoulement du sang dans un vaisseau sanguin. Le système sanguin étant complexe, on ne s'intéresse qu'à une section de vaisseau sanguin. Plusieurs problèmes de modélisation se posent alors : pour l'écoulement sanguin, pour les parois membranaires, pour les conditions d'interaction et enfin pour les conditions d'entrée et de sortie du vaisseau. Modéliser le problème, c'est trouver des équations mathématiques qui décrivent le problème de départ. Tout l'enjeu est de trouver des équations suffisamment «simples» pour obtenir des résultats mathématiques mais suffisamment proches du modèle pour que ces résultats soient cohérents avec les données existantes.

Le fluide.

Le sang n'est pas un milieu homogène. Il est formé de plasma à 55% et d'éléments figurés à 45% (les érythrocytes (globules rouges), les leucocytes (globules blancs) et les thrombocytes (plaquettes)). Cependant, dans les vaisseaux larges, le sang peut être considéré comme un fluide newtonien incompressible et visqueux. Un vaisseau large est un vaisseau sanguin dont l'épaisseur h est négligeable devant le rayon de la section R (voir Figure 1.1). Une artère est un vaisseau large par exemple.

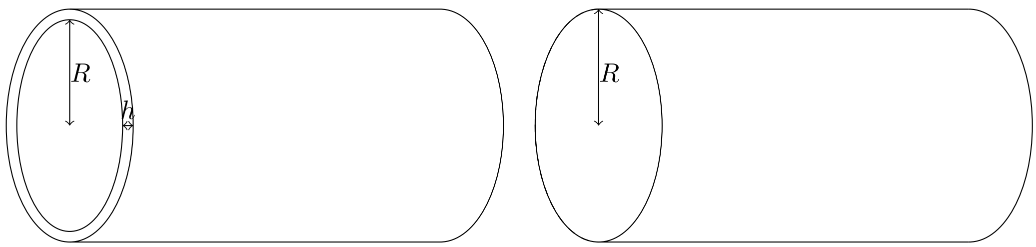

FIGURE 1.1 – Un vaisseau large (gauche) et son approximation mathématique (droite).

Les équations d'évolution de la vitesse $\mathbf{u} = (u_1, u_2, u_3)$ et de la pression p du fluide sont les équations de Navier-Stokes à l'intérieur du domaine :

$$\begin{aligned} \mathbf{u}_t - \operatorname{div}\, \sigma(\mathbf{u}, p) + (\mathbf{u} \cdot \nabla)\mathbf{u} &= \mathbf{0}, \\ \operatorname{div}\, \mathbf{u} &= 0. \end{aligned}$$

Dans la première équation, $\sigma(\mathbf{u}, p)$ est le tenseur des contraintes du fluide. Il est donné (pour un fluide newtonien) par $\sigma(\mathbf{u}, p) = -p\mathbf{I} + \nu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathbf{tr}}\right)$. Le tenseur $\mathbf{I}$ est identifié à la matrice Identité de $\mathbb{R}^3$.

La structure.

La membrane du vaisseau sanguin est modélisée par une équation des coques, c'est une équation dont les coefficients dépendent explicitement de la géométrie du vaisseau sanguin à un instant de référence. Là encore, plusieurs simplifications ont été faites pour modéliser la membrane : le déplacement est considéré *inélastique* et *anisotropique*. Cela donne un tenseur des déformations linéaire et les déformations de la membrane sont plus importantes dans le sens radial. Cela signifie que seul le déplacement radial sera pris en compte dans la suite. Ce déplacement est cherché par rapport à une position de référence, souvent la position au repos, *i.e.* quand le déplacement est nul. Dans la suite, par simplification, nous considérerons des positions de référence plane, *i.e.* en deux dimensions, la membrane reposera sur un segment et nous parlerons alors d'équations des poutres (*beam equation* en anglais), alors qu'en trois dimensions, la membrane reposera sur un domaine plan, par exemple $(0, L_1) \times (0, L_2) \times \{1\}$, et nous parlerons alors d'équations des plaques (*plate equation* en anglais).

Nous introduisons maintenant des notations que nous garderons dans tout le mémoire quelque soit le cadre. A savoir, nous noterons :

- Γ_0^s la position de référence de la poutre/plaque ou Γ_0^κ (avec $\kappa = \pm$) quand il y aura deux poutres (voir Chapitre 6). Le domaine du fluide correspondant à cet état de référence sera noté Ω_0.
- Pour un déplacement η, $\Gamma^s_{\eta(t)}$ sera la position de la poutre/plaque à l'instant t. Quand il y aura deux déplacements, nous noterons de manière analogue $\Gamma^+_{\eta^+(t)}$ et $\Gamma^-_{\eta^-(t)}$ les positions des poutres supérieures et inférieures au temps t. Le domaine du fluide à l'instant t sera alors noté $\Omega_{\eta(t)}$ (avec $\eta = (\eta^+, \eta^-)$ s'il y a deux poutres).
- Nous noterons Γ la partie fixe de la frontière de Ω_0 et Γ_0 la frontière de Ω_0. Ainsi, $\Gamma_0 = \partial\Omega_0$ et $\Gamma = \Gamma_0 \setminus \Gamma_0^s$ ou $\Gamma = \Gamma_0 \setminus (\Gamma_0^+ \cup \Gamma_0^-)$ quand il y aura deux poutres.

Les différentes équations des poutres/plaques que nous considérons dans la suite s'écrivent toutes dans l'état de référence, c'est la représentation Lagrangienne. Plus précisément, que ce soit en une ou deux dimensions ou avec des conditions aux bords différentes (voir plus loin), les équations des poutres/plaques s'écriront

$$\eta_{tt} + \alpha\Delta_s^2\eta - \beta\Delta_s\eta - \gamma\Delta_s\eta_t = \phi.$$

Les coefficients α, β et γ sont des constantes caractéristiques de la poutre/plaque et sont positives avec, de plus, $\gamma > 0$. Elles correspondent respectivement à la rigidité, l'étirement et la friction de la poutre/plaque. Les opérateurs Δ_s et Δ_s^2 sont les opérateurs Laplacien et bilaplacien définis sur Γ_0^s ou Γ_0^κ selon. Le second membre ϕ correspond à la force exercée par le fluide sur la poutre/plaque et dépend explicitement du choix de l'interaction entre les vitesses du fluide et de la poutre/plaque (voir détails plus loin).

Les interactions.

Dans ce mémoire, elles seront de deux sortes :

- L'égalité des vitesses. Rappelons-nous que la vitesse de la poutre/plaque est «radiale». Donc, en deux dimensions, le déplacement n'est que selon l'axe vertical. On obtient ainsi la condition $\mathbf{u} = \eta_t\mathbf{e}_2$ au bord $\Gamma^s_{\eta(t)}$, *i.e.* $\mathbf{u}(t,x,1+\eta(t,x)) = \eta_t(t,x)\mathbf{e}_2$. Ici $\mathbf{e}_1$ et $\mathbf{e}_2$ sont les vecteurs $(1,0)^{\mathbf{tr}}$ et $(0,1)^{\mathbf{tr}}$.
  Cela s'écrit en trois dimensions $\mathbf{u}(t,x,y,1+\eta(t,x,y)) = \eta_t(t,x,y)\mathbf{e}_3$, soit encore $\mathbf{u} = \eta_t\mathbf{e}_3$ sur $\Gamma^s_{\eta(t)}$ où $\mathbf{e}_1$, $\mathbf{e}_2$ et $\mathbf{e}_3$ sont les vecteurs $(1,0,0)^{\mathbf{tr}}$, $(0,1,0)^{\mathbf{tr}}$ et $(0,0,1)^{\mathbf{tr}}$. Cette condition est considérée dans les papiers [21, 4, 11, 7, 23, 16].
- Deux égalités scalaires en deux dimensions (seul cas considéré dans ce mémoire) : l'égalité des vitesses normales et une condition de non-sortie du fluide. Cette condition est sans doute plus physique que la précédente. Elle est utilisée dans le papier [13]. Elle s'écrit

$$\begin{aligned} \mathbf{u}\cdot\widetilde{\mathbf{n}}^+ &= \eta_t^+, \qquad & \sigma(\mathbf{u},p)\mathbf{n}^+\cdot\mathbf{t}^+ &= 0 \quad \text{sur } \Gamma^+_{\eta^+(t)},\\ \mathbf{u}\cdot\widetilde{\mathbf{n}}^- &= -\eta_t^-, \qquad & \sigma(\mathbf{u},p)\mathbf{n}^-\cdot\mathbf{t}^- &= 0 \quad \text{sur } \Gamma^-_{\eta^-(t)}.\end{aligned}$$

Le vecteur $\mathbf{t}$ est le vecteur unitaire tangent à $\Gamma^+_{\eta^+(t)}$ ou $\Gamma^-_{\eta^-(t)}$ alors que le vecteur $\mathbf{n}$ est le vecteur unitaire normal à $\Gamma^+_{\eta^+(t)}$ ou $\Gamma^-_{\eta^-(t)}$ sortant de $\Omega_{\eta(t)}$. Ils sont donnés par

$$\begin{aligned} \mathbf{t}^+ &= \left(1+(\eta_x^+)^2\right)^{-1/2}(\mathbf{e}_1+\eta_x^+\mathbf{e}_2), \qquad & \mathbf{n}^+ &= \left(1+(\eta_x^+)^2\right)^{-1/2}(-\eta_x^+\mathbf{e}_1+\mathbf{e}_2),\\ \mathbf{t}^- &= \left(1+(\eta_x^-)^2\right)^{-1/2}(\mathbf{e}_1+\eta_x^-\mathbf{e}_2), \qquad & \mathbf{n}^- &= -\left(1+(\eta_x^-)^2\right)^{-1/2}(-\eta_x^-\mathbf{e}_1+\mathbf{e}_2).\end{aligned}$$

Finalement, le vecteur $\widetilde{\mathbf{n}}$ est défini par

$$\widetilde{\mathbf{n}}^+ = \left(1+(\eta_x^+)^2\right)^{1/2}\mathbf{n}^+ = -\eta_x^+\mathbf{e}_1+\mathbf{e}_2, \qquad \text{et} \qquad \widetilde{\mathbf{n}}^- = \left(1+(\eta_x^-)^2\right)^{1/2}\mathbf{n}^- = -(-\eta_x^-\mathbf{e}_1+\mathbf{e}_2).$$

Les conditions aux bords.

Les conditions d'entrée et de sortie peuvent être choisies de plusieurs sortes. D'un point de vue physique, on peut imposer un certain profil en entrée, par exemple un écoulement de Poiseuille, et laisser une condition de sortie libre, c'est-à-dire la composante normale du tenseur des contraintes est nulle

$\sigma(\mathbf{u}, p)\mathbf{e}_1 = 0$ en $x = L$. D'un point de vue mathématique, il est assez pratique de considérer des conditions périodiques (ce qui sort en $x = L$ rentre en $x = 0$, *i.e.* $\mathbf{u}(t, L, y) = \mathbf{u}(t, 0, y)$ pour tout t, tout y) ou des conditions de Dirichlet homogènes *i.e.* $\mathbf{u} = \mathbf{0}$. Nous traiterons ces deux dernières conditions dans la suite.

Définitions et notations.

Avec des conditions de Dirichlet homogènes.

Soit Ω_0 un ouvert connexe borné de $\mathbb{R}^d$ $(d = 2, 3)$. On note Γ_0 sa frontière. Dans la suite, Ω_0 pourra être un domaine rectangulaire et sa frontière sera connexe (si on prend des conditions de Dirichlet homogène) ou non. On introduit l'espace de Hilbert $L^2(\Omega_0) = L^2(\Omega_0; \mathbb{R})$ et les espaces de Sobolev classiques $H^s(\Omega_0) = H^s(\Omega_0; \mathbb{R})$ ainsi que $H^s_0(\Omega_0) = H^s(\Omega_0) \cap \ker \gamma_0$ où γ_0 est la fonction trace de $H^s(\Omega_0)$ dans $H^{s-1/2}(\Gamma_0)$ pour $s > 1/2$. On définit les espaces de Sobolev à indices négatifs $H^{-s}(\Omega_0) = H^{-s}(\Omega_0; \mathbb{R}) = (H^s_0(\Omega_0))'$. Pour une fonction à valeur vectorielle, on introduit les espaces $\mathbf{L}^2(\Omega_0) = [L^2(\Omega_0)]^d$ et $\mathbf{H}^s(\Omega_0) = [H^s(\Omega_0)]^d$ (pour $s \in \mathbb{R}$). On peut définir les mêmes espaces sur Γ_0, par exemple, $L^2(\Gamma_0)$ ou $H^s(\Gamma_0)$ ou également $\mathbf{L}^2(\Gamma_0)$, $\mathbf{H}^s(\Gamma_0)$.
On utilisera aussi l'espace des fonctions vectorielles à divergence nulle et l'espace des fonctions vectorielles à divergence nulle et trace normale nulle définis par

$$\begin{aligned} \mathbf{V}^0(\Omega_0) &= \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega_0) \text{ t.q. div } \mathbf{v} = 0 \text{ et } \int_{\Gamma_0} \mathbf{v} \cdot \mathbf{n} = 0 \right\}, \\ \mathbf{V}^0_{\mathbf{n}}(\Omega_0) &= \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega_0) \text{ t.q. div } \mathbf{v} = 0 \text{ et } \mathbf{v} \cdot \mathbf{n} = 0 \right\} \end{aligned}$$

ainsi que, pour $s \geq 0$, $\mathbf{V}^s(\Omega_0) = \mathbf{H}^s(\Omega_0) \cap \mathbf{V}^0(\Omega_0)$ et $\mathbf{V}^s_{\mathbf{n}}(\Omega_0) = \mathbf{H}^s(\Omega_0) \cap \mathbf{V}^0_{\mathbf{n}}(\Omega_0)$.

Pour Γ^s_0 un ouvert de $\mathbb{R}^{d-1}$(avec toujours $d = 2$ ou $d = 3$), on définit également l'espace de Hilbert $L^2_0(\Gamma^s_0)$ par

$$L^2_0(\Gamma^s_0) = \left\{ \mu \in L^2(\Gamma^s_0) \text{ t.q. } \int_{\Gamma^s_0} \mu = 0 \right\}.$$

On introduit alors les espaces de Sobolev

$$H^\sigma_{(0)}(\Gamma^s_0) = \begin{cases} \left\{ \mu \in H^\sigma(\Gamma^s_0) \cap L^2_0(\Gamma^s_0) \text{ t.q. } \mu = \dfrac{\partial \mu}{\partial n} = 0 \text{ sur } \partial\Gamma^s_0 \right\} & \text{pour } 1 + \frac{d-1}{2} < \sigma, \\ \left\{ \mu \in H^\sigma(\Gamma^s_0) \cap L^2_0(\Gamma^s_0) \text{ t.q. } \mu = 0 \text{ sur } \partial\Gamma^s_0 \right\} & \text{pour } \frac{d-1}{2} < \sigma \leq 1 + \frac{d-1}{2}, \\ H^\sigma(\Gamma^s_0) \cap L^2_0(\Gamma^s_0) & \text{for } 0 \leq \sigma \leq \frac{d-1}{2}. \end{cases}$$

Le vecteur n est le vecteur normal unitaire à $\partial\Gamma^s_0$ extérieur à Γ^s_0. Ce sont des espaces d'interpolation entre $L^2_0(\Gamma^s_0)$ et le domaine de l'opérateur des poutres/plaques (voir Chapitres 2 et 4). La pression est déterminée à une constante additive près dans les équations de Navier-Stokes. Pour avoir l'unicité, nous définissons les espaces

$$\mathcal{H}^s(\Omega_0) = \left\{ q \in H^s(\Omega_0) \text{ t.q. } \int_{\Omega_0} q = 0 \right\} \qquad \text{pour } s \geq 0.$$

Dans la suite, nous devrons définir des solutions à des systèmes dont le domaine spatial dépend de la solution. On introduit d'abord les cylindres, pour $T > 0$ et pour une fonction η comme ci-dessus :

$$\begin{aligned} &\Sigma_T = (0, T) \times \Gamma, && \Sigma^{s,0}_T = (0, T) \times \Gamma^s_0, \\ &Q^0_T = (0, T) \times \Omega_0, && Q^\eta_T = \bigcup_{t \in (0,T)} \{t\} \times \Omega_{\eta(t)}, \\ &\Sigma^0_T = (0, T) \times \Gamma_0, && \Sigma^{s,\eta}_T = \bigcup_{t \in (0,T)} \{t\} \times \Gamma^s_{\eta(t)}. \end{aligned}$$

On définit alors les espaces fonctionnels espace-temps pour le cylindre droit Q_T^0 :

$$\begin{aligned}\mathbf{H}^{\sigma,\tau}(Q_T^0) &= L^2(0,T;\mathbf{H}^\sigma(\Omega_0)) \cap H^\tau(0,T;\mathbf{L}^2(\Omega_0)),\\ \mathbf{V}^{\sigma,\tau}(Q_T^0) &= L^2(0,T;\mathbf{V}^\sigma(\Omega_0)) \cap H^\tau(0,T;\mathbf{V}^0(\Omega_0)),\end{aligned}$$

ainsi que les espaces de Sobolev dans le domaine dépendant du temps $\Omega_{\eta(t)}$:

Définition 1.1. *On dit que* $\mathbf{u}$ *appartient à* $H^\tau(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{H}^\sigma(\Omega_{\eta(t)}))$ *(resp. à* $H^\tau(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^\sigma(\Omega_{\eta(t)}))$*) si*
- *pour presque tout* t *dans* $(0,T)$, $\mathbf{u}(t)$ *appartient à* $\mathbf{H}^\sigma(\Omega_{\eta(t)})$ *(resp. à* $\mathbf{V}^\sigma(\Omega_{\eta(t)})$*),*
- $t\mapsto\|\mathbf{u}(t)\|_{\mathbf{H}^\sigma(\Omega_{\eta(t)})}$ *(resp.* $t\mapsto\|\mathbf{u}(t)\|_{\mathbf{V}^\sigma(\Omega_{\eta(t)})}$*) appartient à* $H^\tau(0,T;\mathbb{R})$.

On définit finalement les espaces fonctionnels espace-temps pour le cylindre Q_T^η :

$$\begin{aligned}\mathbf{H}^{\sigma,\tau}(Q_T^\eta) &= L^2\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{H}^\sigma\left(\Omega_{\eta(t)}\right)\right)\bigcap H^\tau\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{L}^2\left(\Omega_{\eta(t)}\right)\right),\\ \mathbf{V}^{\sigma,\tau}(Q_T^\eta) &= L^2\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^\sigma\left(\Omega_{\eta(t)}\right)\right)\bigcap H^\tau\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^0\left(\Omega_{\eta(t)}\right)\right).\end{aligned}$$

De la même manière, nous définissons les espaces fonctionnels espaces-temps pour le cylindre $\Sigma_T^{s,0}$:

$$H_{(0)}^{\sigma,\tau}(\Sigma_T^{s,0}) = L^2(0,T;H_{(0)}^\sigma(\Gamma_0^s)) \cap H^\tau(0,T;L_0^2(\Gamma_0^s)).$$

Avec des conditions périodiques en une ou deux variables.

On utilisera aussi des domaines périodiques en la première variable x (en deux dimensions) ou périodiques en les deux premières variables x et y (en trois dimensions). Les domaines seront alors de la forme rectangulaire ou parallélipédique (voir Figures 1.2 et 1.4). On utilisera la notation # pour ce cadre. Ainsi, pour $\Omega_0 = \mathbb{R}/L_1\times(0,1)$ (avec $L_1>0$) un domaine de $\mathbb{R}^2$ périodique en la première variable, on notera $L^2_\#(\Omega_0) = L^2_\#(\Omega_0;\mathbb{R})$ l'espace de Hilbert des fonctions $L^2_{\mathrm{loc}}(\mathbb{R}\times(0,1);\mathbb{R})$ périodiques en la variable x de période L_1. De même, nous noterons $H^s_\#(\Omega_0)$ l'ensemble des fonctions de $L^2_\#(\Omega_0)$ appartenant à $H^s_{\mathrm{loc}}(\mathbb{R}\times(0,1))$. Nous donnons les mêmes définitions en trois dimensions avec $\Omega_0 = \mathbb{R}/L_1\times\mathbb{R}/L_2\times(0,1)$ (avec L_1, $L_2>0$), nous noterons $L^2_\#(\Omega_0) = L^2_\#(\Omega_0;\mathbb{R})$ l'espace de Hilbert des fonctions $L^2_{\mathrm{loc}}(\mathbb{R}^2\times(0,1);\mathbb{R})$ périodiques en les variables x et y de période respective L_1 et L_2 ainsi que $H^s_\#(\Omega_0)$ l'ensemble des fonctions de $L^2_\#(\Omega_0)$ appartenant également à $H^s_{\mathrm{loc}}(\mathbb{R}^2\times(0,1))$. Dans ces deux cas, nous pouvons définir de la même manière qu'au dessus les espaces $\mathbf{L}^2_\#(\Omega_0)$ ou encore $\mathbf{H}^s_\#(\Omega_0)$.

Pour $d=2$, la frontière Γ_0 d'un tel domaine Ω_0 est donnée par $\Gamma_0 = (0,L_1)\times\{0\}\bigcup(0,L_1)\times\{1\}$. Ainsi, avec $\Gamma_0^s = (0,L_1)\times\{1\}$, on a $\Gamma = (0,L_1)\times\{0\}$. Pour $d=3$, $\Gamma_0 = (0,L_1)\times(0,L_2)\times\{0\}\bigcup(0,L_1)\times(0,L_2)\times\{1\}$. Et là encore, avec $\Gamma_0^s = (0,L_1)\times(0,L_2)\times\{1\}$, on a $\Gamma = (0,L_1)\times(0,L_2)\times\{0\}$. Nous pouvons alors définir les espaces de fonctions vectorielles à divergence nulle et les espaces de celles à divergence et trace normale nulles :

$$\begin{aligned}\mathbf{V}^s_\#(\Omega_0) &= \Big\{\mathbf{v}\in\mathbf{H}^s_\#(\Omega_0) \text{ t.q. div } \mathbf{v}=0 \text{ dans } \Omega_0\Big\},\\ \mathbf{V}^s_{\#,\mathbf{n}}(\Omega_0) &= \Big\{\mathbf{v}\in\mathbf{H}^s_\#(\Omega_0) \text{ t.q. div } \mathbf{v}=0 \text{ dans } \Omega_0 \text{ et } \mathbf{v}\cdot\mathbf{n}=0 \text{ sur } \Gamma_0\Big\}.\end{aligned}$$

L'espace pour la pression est noté $\mathcal{H}^s_\#(\Omega_0) = \Big\{q\in H^s_\#(\Omega_0) \text{ t.q. } \int_{\Omega_0} q = 0\Big\}$.

En utilisant la Définition 1.1, on introduit les espaces fonctionnels espace-temps pour la vitesse du fluide :

$$\mathbf{H}^{\sigma,\tau}_{\#}(Q^{\eta}_T) = L^2\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{H}^{\sigma}_{\#}\left(\Omega_{\eta(t)}\right)\right)\bigcap H^{\tau}\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{L}^{2}_{\#}\left(\Omega_{\eta(t)}\right)\right),$$

$$\mathbf{V}^{\sigma,\tau}_{\#}(Q^{\eta}_T) = L^2\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^{\sigma}_{\#}\left(\Omega_{\eta(t)}\right)\right)\bigcap H^{\tau}\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^{0}_{\#}\left(\Omega_{\eta(t)}\right)\right).$$

Pour la poutre/plaque, il faut considérer des fonctions $L^2_{\#}(\Gamma^s_0)$ et à valeur moyenne nulle. Nous noterons $L^2_{\#,0}(\Gamma^s_0)$ l'espace composé de telles fonctions. Nous noterons aussi $H^{\sigma}_{\#}(\Gamma^s_0) = H^{\sigma}(\Gamma^s_0)\cap L^2_{\#,0}(\Gamma^s_0)$. Les espaces fonctionnels pour le déplacement de la poutre/plaque sont alors notés

$$H^{\sigma,\tau}_{\#}(\Sigma^{s,0}_T) = L^2(0,T;H^{\sigma}_{\#}(\Gamma^s_0))\cap H^{\tau}(0,T;L^2_{\#,0}(\Gamma^s_0)).$$

Maintenant que les différentes notations ont été introduites, nous allons exposer les différents résultats obtenus. Dans la première partie, nous démontrons des résultats d'existence (et d'unicité) pour les systèmes couplant les équations de Navier-Stokes avec un équation des poutres/plaques (dans deux cadres différents) aussi bien en deux dimensions qu'en trois.

Dans la seconde partie, nous nous intéressons à la contrôlabilité à zéro d'un système couplant les équations de Navier-Stokes avec une équation différentielle ordinaire liée à une équation des poutres vue dans la première partie. Ensuite, nous démonstrons la stabilisation avec n'importe quel taux de décroissance d'un système couplant les équations de Navier-Stokes avec une équation des poutres dans le cadre périodique en deux dimensions.

Partie I.

Dans cette partie, nous nous intéressons aux questions d'existence pour des systèmes couplés fluide-structure. Comme déjà annoncé, par simplification, nous ne nous intéressons ici qu'à des systèmes avec une seule poutre/plaque. Elle sera prise sur la partie supérieure de la frontière du domaine du fluide.

L'étude de ce type de système a déjà été traitée. L'existence de solutions faibles pour des systèmes couplés fluide-poutre/plaque a été prouvée aussi bien en deux qu'en trois dimensions dans [7, 11]. Dans [4], l'auteur prouve l'existence de solutions fortes pour des temps et des conditions initiales petites ainsi que pour un paramètre petit (voir Théorème 1.2). C'est à partir du système traité dans [4] que nous avons travaillé.

Détaillons un peu plus le résultat de [4]. Dans ce papier, l'auteur considère un domaine rectangulaire avec condition périodique pour la première variable. La poutre est sur la partie supérieure du domaine (voir Figure 1.2). Le système est :

$$\begin{aligned}
\mathbf{u}_t - \mathrm{div}\ \sigma(\mathbf{u},p) + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \mathbf{0} && \text{dans } Q^{\eta}_T,\\
\mathrm{div}\ \mathbf{u} &= 0 && \text{dans } Q^{\eta}_T,\\
\mathbf{u} &= \eta_t\mathbf{e}_2 && \text{sur } \Sigma^{s,\eta}_T,\\
\mathbf{u} &= \mathbf{0} && \text{sur } \Sigma_T,\\
\eta_{tt} - \alpha\eta_{xxxx} - \beta\eta_{xx} - \gamma\eta_{txx} &= -\sigma(\mathbf{u},p)\widetilde{\mathbf{n}}\cdot\mathbf{e}_2 && \text{sur } \Sigma^{s,0}_T,\\
(\mathbf{u}(0),\eta(0),\eta_t(0)) &= (\mathbf{u}^0,\eta^{1,0},\eta^{2,0}).
\end{aligned} \tag{1.1}$$

Avec les notations de [4], le second membre de l'équation des poutres est

$$\left(\rho_1 p + \nu\rho_2\left[\eta_x\left(u_{1,y}+u_{2,x}-2\nu u_{2,y}\right)\right]\right)_{\Gamma_t}.$$

L'auteur prouve pour ce système le résultat suivant.

Théorème 1.2 (Beirão da Veiga, Théorème 1.1 dans [4])**.** *Supposons que* $\gamma > 0$ *et que*

$$\mathbf{u}^0 \in \mathbf{V}^1_{\#}(\Omega_{\eta^{1,0}}),\quad \eta^{1,0}\in H^{5/2}_{\#}(\Gamma^s_0),\qquad \eta^{2,0}\in H^{3/2}_{\#}(\Gamma^s_0)$$

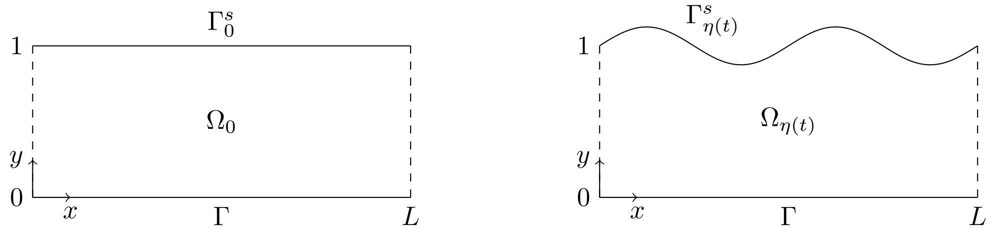

FIGURE 1.2 – Le domaine Ω_0 (à gauche) et $\Omega_{\eta(t)}$ (à droite).

avec les conditions de compatibilité $\mathbf{u}^0 = \mathbf{0}$ *sur* Γ *et* $\mathbf{u}^0 = \eta^{2,0}\mathbf{e}_2$ *sur* $\Gamma_{\eta^{1,0}}$. *Alors, si* ρ_1 *et* $\|\eta^{1,0}\|_{W^{1,\infty}(\Gamma_0^s)}$ *sont assez petits, le système* (1.1) *a une solution* $(\mathbf{u}, p, \eta)$ *pour un temps* T *suffisament petit. De plus*

$$\mathbf{u} \in L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathbf{H}^2_\#(\Omega_{\eta(t)})\right), \quad \mathbf{u}_t \in L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathbf{L}^2_\#(\Omega_{\eta(t)})\right), \quad p \in L^2\left(\bigcup_{t\in(0,T)} \{t\} \times H^1_\#(\Omega_{\eta(t)})\right)$$

et

$$\eta_t \in L^2(0,T; H^{5/2}_\#(\Gamma_0^s)) \cap L^\infty(0,T; H^{3/2}_\#(\Gamma_0^s)), \qquad \eta_{tt} \in L^2(0,T; H^{-1/2}_\#(\Gamma_0^s)).$$

Si $\alpha > 0$*, en supposant que* $\eta^{1,0}$ *appartient à* $H^{7/2}_\#(\Gamma_0^s)$*, on obtient de plus* η *dans* $L^\infty(0,T; H^{7/2}_\#(\Gamma_0^s))$.

Dans les deux premiers chapitres, nous étudions le système (1.1) dans les cas $\alpha > 0$ (voir Chapitre 2) et $\alpha = 0$ (voir Chapitre 3) en deux dimensions. Dans le premier cas, nous considérons des conditions de Dirichlet homogènes au bord du domaine où il n'y a pas la poutre. Nous aurions tout aussi bien pu prendre des conditions périodiques. Par contre, dans le second cas (*i.e.* $\alpha = 0$), nous considérons le cadre périodique (comme dans [4], voir Figure 1.2). Nous ne pouvons pas traiter le cas des conditions de Dirichlet homogènes dans ce cas à cause du semi-groupe associé à l'équation des ondes amorties (voir Chapitre 3 pour les détails).Dans le Chapitre 4, nous étendons les résultats des chapitres précédents à la troisème dimension dans les deux cas $\alpha > 0$ et $\alpha = 0$. Nous allons maintenant décrire précisément les résultats des différents chapitres de cette partie.

Chapitre 2.

Dans ce chapitre, qui a fait l'objet de la publication [16], nous considérons le domaine $\Omega_0 = (0, L) \times (0, 1)$. La poutre repose sur la partie supérieure. L'état de référence est $\Gamma_0^s = (0, L) \times \{1\}$. La frontière fixe Γ est alors donnée par les trois autres segments, à savoir :

$$\Gamma = (0, L) \times \{0\} \bigcup \{0\} \times (0, 1) \bigcup \{L\} \times (0, 1).$$

Nous considérons des conditions de Dirichlet homogénes sur Γ. Le système s'écrit alors

$$\begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div} \sigma(\mathbf{u}, p) &=& \mathbf{0} & \text{dans } Q_T^\eta, \\ \operatorname{div} \mathbf{u} &=& 0 & \text{dans } Q_T^\eta, \\ \mathbf{u} &=& \eta_t \mathbf{e}_2 & \text{sur } \Sigma_T^{s,\eta}, \\ \mathbf{u} &=& \mathbf{0} & \text{sur } \Sigma_T, \\ \mathbf{u}(0) &=& \mathbf{u}^0 & \text{dans } \Omega_{\eta^{1,0}}, \end{array} \tag{1.2}$$

$$\begin{array}{rcll} \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} &=& -\sigma(\mathbf{u}, p)\widetilde{\mathbf{n}}\cdot\mathbf{e}_2 & \text{sur } \Sigma_T^{s,0}, \\ (\eta(0), \eta_t(0)) &=& (\eta^{1,0}, \eta^{2,0}) & \text{dans } \Gamma_0^s. \end{array} \tag{1.3}$$

Le chapitre 2 est consacré à la démonstration des résultats suivants pour le système (1.2)–(1.3) :

Théorème 1.3. *Soit* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *dans* $\mathbf{V}^1(\Omega_{\eta^{1,0}}) \times H^3_{(0)}(\Gamma_0^s) \times H^1_{(0)}(\Gamma_0^s)$. *Il existe* $R > 0$ *tel que pour toute donnée initiale satisfaisant* $\|\mathbf{u}^0\|^2_{\mathbf{V}^1(\Omega_{\eta^{1,0}})} + \|\eta^{1,0}\|^2_{H^3_{(0)}(\Gamma_0^s)} + \|\eta^{2,0}\|^2_{H^1_{(0)}(\Gamma_0^s)} \leq R^2$ *et la condition de compatibilité*

$$\mathbf{u}^0 = \mathbf{0} \text{ sur } \Gamma \qquad \text{et} \qquad \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_2 \text{ sur } \Gamma^s_{\eta^{1,0}}, \tag{1.4}$$

le système (1.2)–(1.3) *admet une unique solution globale* $(\mathbf{u}, p, \eta)$ *dans l'espace*

$$\mathbf{V}^{2,1}(Q_T^\eta) \times L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1\left(\Omega_{\eta(t)}\right)\right) \times H^{4,2}_{(0)}(\Sigma_T^{s,0}).$$

Théorème 1.4. *Soit* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *dans* $\mathbf{V}^1(\Omega_{\eta^{1,0}}) \times H^3_{(0)}(\Gamma_0^s) \times H^1_{(0)}(\Gamma_0^s)$ *satisfaisant la condition de compatibilité* (1.4). *Il existe un temps* $T_0 > 0$ *tel que le système* (1.2)–(1.3) *admet une unique solution* $(\mathbf{u}, p, \eta)$ *dans* $\mathbf{V}^{2,1}(Q_{T_0}^\eta) \times L^2\left(\bigcup_{t\in(0,T_0)} \{t\} \times \mathcal{H}^1\left(\Omega_{\eta(t)}\right)\right) \times H^{4,2}_{(0)}(\Sigma_{T_0}^{s,0})$.

On remarque donc que les Théorèmes 1.3 et 1.4 améliorent le Théorème 1.2. En effet, pour un demi-cran de régularité en moins pour les conditions initiales de la poutre, nous obtenons des solutions continues en temps dans cet espace avec de plus par exemple η_{tt} dans $L^2(\Sigma_T^{s,0})$.

Chapitre 3.

Ce chapitre est consacré à l'étude du système (1.5)–(1.6) couplant les équations de Navier-Stokes

$$\begin{array}{rcll} \mathbf{u}_t - \operatorname{div} \sigma(\mathbf{u}, p) + (\mathbf{u}\cdot\nabla)\mathbf{u} & = & \mathbf{0} & \text{dans } Q_T^\eta, \\ \operatorname{div} \mathbf{u} & = & 0 & \text{dans } Q_T^\eta, \\ \mathbf{u} & = & \eta_t \mathbf{e}_2 & \text{sur } \Sigma_T^{s,\eta}, \\ \mathbf{u} & = & \mathbf{0} & \text{sur } \Sigma_T, \\ \mathbf{u}(0) & = & \mathbf{u}^0 & \text{dans } \Omega_{\eta^{1,0}} \end{array} \tag{1.5}$$

et l'équation des ondes fortement amortie

$$\begin{array}{rcll} \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} & = & -\sigma(\mathbf{u}, p)\widetilde{\mathbf{n}}\cdot\mathbf{e}_2 & \text{sur } \Sigma_T^{s,0}, \\ (\eta(0), \eta_t(0)) & = & (\eta^{1,0}, \eta^{2,0}) & \text{dans } \Gamma_0^s \end{array} \tag{1.6}$$

en deux dimensions dans le cadre périodique. Le domaine Ω_0 est alors le même que celui de [4], voir Figure 1.2. Plus précisément, nous montrons les résultats suivants.

Théorème 1.5. *Soient* $0 < \varepsilon \leq 1/2$ *et* $T > 0$. *Soit* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *dans* $\mathbf{V}^1_\#(\Omega_{\eta^{1,0}}) \times H^{2+\varepsilon}_\#(\Gamma_0^s) \times H^{1+\varepsilon}_\#(\Gamma_0^s)$. *Il existe* $R > 0$ *tel que pour toute condition initiale satisfaisant*

$$\|\mathbf{u}^0\|^2_{\mathbf{V}^1_\#(\Omega_{\eta^{1,0}})} + \|\eta^{1,0}\|^2_{H^{2+\varepsilon}_\#(\Gamma_0^s)} + \|\eta^{2,0}\|^2_{H^{1+\varepsilon}_\#(\Gamma_0^s)} \leq R^2$$

et la condition de compatibilité (1.4), *le système* (1.5)–(1.6) *admet une unique solution forte* $(\mathbf{u}, p, \eta)$ *dans*

$$\mathbf{V}^{2,1}_\#(Q_T^\eta) \times L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1_\#\left(\Omega_{\eta(t)}\right)\right) \times \mathcal{E}_T^\varepsilon.$$

L'espace $\mathcal{E}_T^\varepsilon$, *dépendant de* ε, *est défini par*

$$\mathcal{E}_T^\varepsilon = H^1(0, T; H^{2+\varepsilon}_\#(\Gamma_0^s)) \cap H^2(0, T; H^\varepsilon_\#(\Gamma_0^s)).$$

Théorème 1.6. *Soit* $0 < \varepsilon \leq 1/2$. *Soit* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *dans* $\mathbf{V}^1_\#(\Omega_{\eta^{1,0}}) \times H^{2+\varepsilon}_\#(\Gamma_0^s) \times H^{1+\varepsilon}_\#(\Gamma_0^s)$ *satisfaisant la condition de compatibilité* (1.4). *Il existe un temps* $T_0 > 0$ *tel que le système* (1.5)–(1.6) *admet une unique solution forte* $(\mathbf{u}, p, \eta)$ *dans* $\mathbf{V}^{2,1}_\#(Q_{T_0}^\eta) \times L^2\left(\bigcup_{t\in(0,T_0)} \{t\} \times \mathcal{H}^1_\#\left(\Omega_{\eta(t)}\right)\right) \times \mathcal{E}_{T_0}^\varepsilon$.

On remarque que le système (1.5)–(1.6) correspond au cas $\alpha = 0$ dans [4] (voir Théorème 1.2). On voit ainsi que les Théorèmes 1.5 et 1.6 proposent l'alternative de l'existence globale pour données petites ou existence locale pour des données initiales de taille quelconques. Les conditions initiales pour l'équation (1.6) sont un peu moins régulières ($1/2 - \varepsilon$ cran de régularité en moins) que dans le Théorème 1.2.

Chapitre 4.

Ce chapitre étend les résultats des chapitres précédents à la dimension trois. Dans ce contexte, nous garderons les mêmes notations pour les vecteurs et opérateurs.

On se donne un ouvert borné connexe ω_0 de $\mathbb{R}^2$ à frontière suffisament régulière. On construit alors $\Omega_0 = \omega_0 \times (0,1)$. On définit ensuite $\Gamma_0^s = \omega_0 \times \{1\}$. La partie fixe de la frontière Γ est formée de Γ_b et Γ_l, voir Figure 1.3.

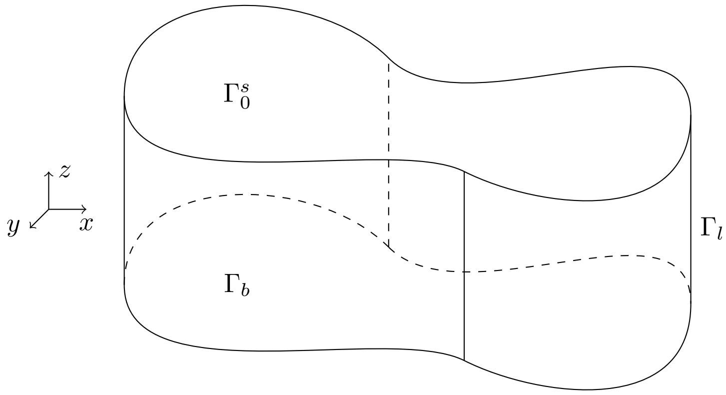

FIGURE 1.3 – Le domaine Ω_0 en trois dimensions dans le cas $\alpha > 0$.

Les équations de Navier-Stokes sont :

$$\begin{array}{rcll} \mathbf{u}_t - \text{div}\ \sigma(\mathbf{u},p) + (\mathbf{u}\cdot\nabla)\mathbf{u} & = & \mathbf{0} & \text{dans } Q_T^\eta, \\ \text{div}\ \mathbf{u} & = & 0 & \text{dans } Q_T^\eta, \\ \mathbf{u} & = & \eta_t \mathbf{e}_3 & \text{sur } \Sigma_T^{s,\eta}, \\ \mathbf{u} & = & \mathbf{0} & \text{sur } \Sigma_T, \\ \mathbf{u}(0) & = & \mathbf{u}^0 & \text{dans } \Omega_{\eta^{1,0}}. \end{array} \tag{1.7}$$

Le déplacement η est maintenant une fonction du temps t et aussi des variables d'espace (x,y). L'équation des plaques devient

$$\begin{array}{rcll} \eta_{tt} + \alpha\Delta_s^2\eta - \beta\Delta_s\eta - \gamma\Delta_s\eta & = & 0 & \text{sur } \Sigma_T^{s,0}, \\ \eta & = & 0 & \text{sur } \sigma_T^{s,0}, \\ \dfrac{\partial\eta}{\partial n_s} & = & 0 & \text{sur } \sigma_T^{s,0}, \\ (\eta(0),\eta_t(0)) & = & (\eta^{1,0},\eta^{2,0}) & \text{dans } \Gamma_0^s. \end{array} \tag{1.8}$$

Ici $\sigma_T^{s,0} = (0,T)\times\partial\Gamma_0^s$ et n_s est le vecteur normal unitaire à $\partial\Gamma_0^s$ extérieur à Γ_0^s. L'opérateur Δ_s^2 est un opérateur à domaine $H^4_{(0)}(\Gamma_0^s)$ dans $L^2(\Gamma_0^s)$ défini par

$$\Delta_s^2\mu = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \mu = \frac{\partial^4\mu}{\partial x^4} + 2\frac{\partial^4\mu}{\partial x^2\partial y^2} + \frac{\partial^4\mu}{\partial y^4}, \qquad \text{pour tout } \mu \in H^4_{(0)}(\Gamma_0^s).$$

Nous démontrons alors les résultats suivants.

Théorème 1.7. *Soit* $(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})$ *dans* $\mathbf{V}^1(\Omega_{\eta^{1,0}}) \times H^3_{(0)}(\Gamma_0^s) \times H^1_{(0)}(\Gamma_0^s)$. *Il existe* $R > 0$ *tel que pour toute donnée initiale satisfaisant* $\|\mathbf{u}^0\|^2_{\mathbf{V}^1(\Omega_{\eta^{1,0}})} + \|\eta^{1,0}\|^2_{H^3_{(0)}(\Gamma_0^s)} + \|\eta^{2,0}\|^2_{H^1_{(0)}(\Gamma_0^s)} \leq R^2$ *et la condition de compatibilité*

$$\mathbf{u}^0 = \mathbf{0} \ \textit{sur}\ \Gamma \qquad \textit{et} \qquad \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_3 \ \textit{sur}\ \Gamma^s_{\eta^{1,0}}, \tag{1.9}$$

le système (1.7)–(1.8) *admet une unique solution globale* $(\mathbf{u}, p, \eta)$ *dans l'espace*

$$\mathbf{V}^{2,1}(Q_T^\eta) \times L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1\left(\Omega_{\eta(t)}\right)\right) \times H^{4,2}_{(0)}(\Sigma_T^{s,0}).$$

Théorème 1.8. *Soit* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *dans* $\mathbf{V}^1(\Omega_{\eta^{1,0}}) \times H^3_{(0)}(\Gamma_0^s) \times H^1_{(0)}(\Gamma_0^s)$ *satisfaisant la condition de compatibilité* (1.9)*. Il existe un temps* $T_0 > 0$ *tel que le système* (1.7)–(1.8) *admet une unique solution* $(\mathbf{u}, p, \eta)$ *dans* $\mathbf{V}^{2,1}(Q_{T_0}^\eta) \times L^2\left(\bigcup_{t\in(0,T_0)} \{t\} \times \mathcal{H}^1\left(\Omega_{\eta(t)}\right)\right) \times H^{4,2}_{(0)}(\Sigma_{T_0}^{s,0})$.

Ainsi, la régularité des conditions initiales est la même que pour le système en deux dimensions (voir Théorèmes 1.3 et 1.4) et donne la même régularité pour les solutions.

Pour le cadre périodique, on introduit deux longueurs L_1 et L_2, et $\omega_0 = \mathbb{R}/L_1 \times \mathbb{R}/L_2$. On définit alors $\Omega_0 = \omega_0 \times (0,1)$. L'état de référence de la plaque est $\Gamma_0^s = \omega_0 \times \{1\}$ et la partie fixe de la frontière Γ est réduite à $\Gamma = \omega_0 \times \{0\}$, voir Figure 1.4.

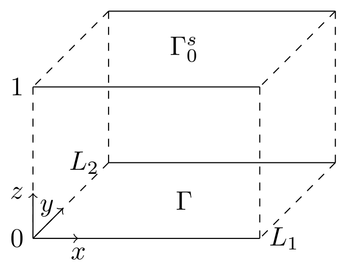

FIGURE 1.4 – Le domaine Ω_0 en trois dimensions dans le cadre périodique ($\alpha = 0$).

Comme dans le chapitre 3, le système est

$$\begin{array}{rcll} \mathbf{u}_t - \operatorname{div} \sigma(\mathbf{u}, p) + (\mathbf{u}\cdot\nabla)\mathbf{u} & = & \mathbf{0} & \text{dans } Q_T^\eta, \\ \operatorname{div} \mathbf{u} & = & 0 & \text{dans } Q_T^\eta, \\ \mathbf{u} & = & \eta_t \mathbf{e}_3 & \text{sur } \Sigma_T^{s,\eta}, \\ \mathbf{u} & = & \mathbf{0} & \text{sur } \Sigma_T, \\ \mathbf{u}(0) & = & \mathbf{u}^0 & \text{dans } \Omega_{\eta^{1,0}} \end{array} \tag{1.10}$$

et l'équation des ondes fortement amortie

$$\begin{array}{rcll} \eta_{tt} - \beta\Delta_s\eta - \gamma\Delta_s\eta_t & = & -\sigma(\mathbf{u}, p)\widetilde{\mathbf{n}}\cdot\mathbf{e}_3 & \text{sur } \Sigma_T^{s,0}, \\ (\eta(0), \eta_t(0)) & = & (\eta^{1,0}, \eta^{2,0}) & \text{dans } \Gamma_0^s. \end{array} \tag{1.11}$$

Nous démontrons, dans la seconde partie du chapitre 4, les résultats suivants.

Théorème 1.9. *Soit* $T > 0$*. Soit* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *dans* $\mathbf{V}^1_\#(\Omega_{\eta^{1,0}}) \times H^{5/2}_\#(\Gamma_0^s) \times H^{3/2}_\#(\Gamma_0^s)$*. Il existe* $R > 0$ *tel que pour toute condition initiale satisfaisant* $\|\mathbf{u}^0\|^2_{\mathbf{V}^1_\#(\Omega_{\eta^{1,0}})} + \|\eta^{1,0}\|^2_{H^{5/2}_\#(\Gamma_0^s)} + \|\eta^{2,0}\|^2_{H^{3/2}_\#(\Gamma_0^s)} \le R^2$ *et la condition de compatibilité* (1.9)*, le système* (1.10)–(1.11) *admet une unique solution forte* $(\mathbf{u}, p, \eta)$ *dans*

$$\mathbf{V}^{2,1}_\#(Q_T^\eta) \times L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1_\#\left(\Omega_{\eta(t)}\right)\right) \times \mathcal{E}_T.$$

L'espace $\mathcal{E}_T$ *est défini par*

$$\mathcal{E}_T = H^1(0,T; H^{5/2}_\#(\Gamma_0^s)) \cap H^2(0,T; H^{1/2}_\#(\Gamma_0^s)).$$

Théorème 1.10. *Soit* $\varepsilon > 0$. *Soit* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *dans* $\mathbf{V}^1(\Omega_{\eta^{1,0}}) \times H^{5/2}_\#(\Gamma^s_0) \times H^{3/2}_\#(\Gamma^s_0)$ *satisfaisant la condition de compatibilité* (1.9). *Il existe un temps* $T_0 > 0$ *tel que le système* (1.10)–(1.11) *a une unique solution forte* $(\mathbf{u}, p, \eta)$ *dans* $\mathbf{V}^{2,1}_\#(Q^\eta_{T_0}) \times L^2(\bigcup_{t\in(0,T_0)} \{t\} \times \mathcal{H}^1_\#(\Omega_{\eta(t)})) \times \mathcal{E}_{T_0}$.

On peut facilement faire un parallèle entre le système en deux dimensions et le même en trois dimensions. Ce dernier donne, pour des conditions initiales $(\eta^{1,0}, \eta^{2,0})$ dans $H^{5/2}_\#(\Gamma^s_0) \times H^{3/2}_\#(\Gamma^s_0) = H^{2+1/2}_\#(\Gamma^s_0) \times H^{1+1/2}_\#(\Gamma^s_0)$, des solutions pour l'équation des «plaques» dans l'espace $\mathcal{E}_T$. Ainsi, le système en trois dimensions correspond au cas limite $\varepsilon = 1/2$ du Chapitre 3. On remarque en effet que $\mathcal{E}_T$ équivaut à $\mathcal{E}^\varepsilon_T$ (voir la définition au Théorème 1.5) pour $\varepsilon = 1/2$. De plus, les conditions initiales des Théorèmes 1.9 et 1.10 sont celles que prend Beirão da Veiga dans le Théorème 1.2.

Quelques mots sur la démonstration des Théorèmes 1.3 à 1.10.

Dans chaque cas, la démonstration repose sur les mêmes étapes. Premièrement, par un changement de variables, nous ramenons les équations posées dans Q^η_T au domaine fixe Q^0_T. Ce changement de variables dépend du déplacement η et sa bonne définition dépend donc de la régularité de ce déplacement. Le système équivalent obtenu est ainsi posé dans le cylindre Q^0_T mais le changement de variables a fait apparaître des non-linéarités. Ensuite, nous traitons le système linéarisé autour de la position d'équilibre $(\mathbf{u}, p, \eta) = (\mathbf{0}, 0, 0)$ en considérant les non-linéarités comme des seconds membres. Cela nous permet de prouver, pour des seconds membres et des conditions initiales suffisamment réguliers, l'existence d'une unique solution globale (*i.e.* sur $[0, T]$ où $T > 0$ est arbitraire mais fini) dans un certain espace de solutions. Finalement, une méthode de point fixe, une fois une estimation des non-linéarités obtenue en fonction des solutions du système linéarisé, permet de conclure.

Partie II.

Dans ce chapitre, nous nous intéressons à des questions de contrôlabilité et de stabilisabilité de systèmes couplés proches de ceux du chapitre précédent. Plus précisément, nous considérons dans le Chapitre 5, la contrôlabilité à zéro d'un système couplant les équations de Navier-Stokes à une équation différentielle ordinaire provenant d'une simplification d'une équation des poutres en deux dimensions. Le contrôle agit sur un domaine à l'intérieur du domaine du fluide.

Dans le chapitre suivant, nous considérons un système proche de [2] où les murs du canal périodique sont modélisés par des poutres. Nous montrons, pour tout taux de décroissance et pour des conditions initiales proches d'une position d'équilibre, la stabilisation de la solution du système avec deux contrôles agissant sur la partie supérieure de la frontière, *i.e.* sur la poutre du dessus.

Chapitre 5.

Dans ce chapitre, nous considérons un domaine au repos Ω_0 dont la frontière contient le segment $\Gamma^s_0 = (0, L) \times \{1\}$ (avec $L > 0$) correspondant à l'état de référence de la poutre. La partie fixe de la frontière Γ est connexe et connectée à Γ^s_0, voir Figure 1.5. L'équation du fluide est toujours l'équation de Navier-Stokes, cette fois avec un contrôle $\mathbf{c}$ agissant dans $\omega \subset \Omega_{\eta(t)}$:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u}, p) &= \mathbf{c}\chi_\omega && \text{dans } Q^\eta_T,\\ \operatorname{div}\mathbf{u} &= 0 && \text{dans } Q^\eta_T,\\ \mathbf{u} &= \eta_t\mathbf{e}_2 && \text{sur } \Sigma^{s,\eta}_T,\\ \mathbf{u} &= \mathbf{0} && \text{sur } \Sigma_T,\\ \mathbf{u}(0) &= \mathbf{u}^0 && \text{dans } \Omega_{\eta^{1,0}}. \end{aligned} \tag{1.12}$$

L'équation des poutres s'écrit dans ce contexte

$$\begin{aligned} q'' + Aq &= -\Pi_N[\sigma(\mathbf{u}, p)\widetilde{\mathbf{n}}\cdot\mathbf{e}_2], \quad \text{dans } (0, T),\\ (q(0), q'(0)) &= (q^{1,0}, q^{2,0}) \end{aligned} \tag{1.13}$$

où N est un entier supérieur à 1, A est une matrice symétrique définie positive carrée de taille $N \times N$ et Π_N est la projection de $L^2_0(\Gamma^s_0)$ sur $\mathbb{R}^N$. L'inconnue q est un vecteur de $\mathbb{R}^{N\times 1}$ et le déplacement de la

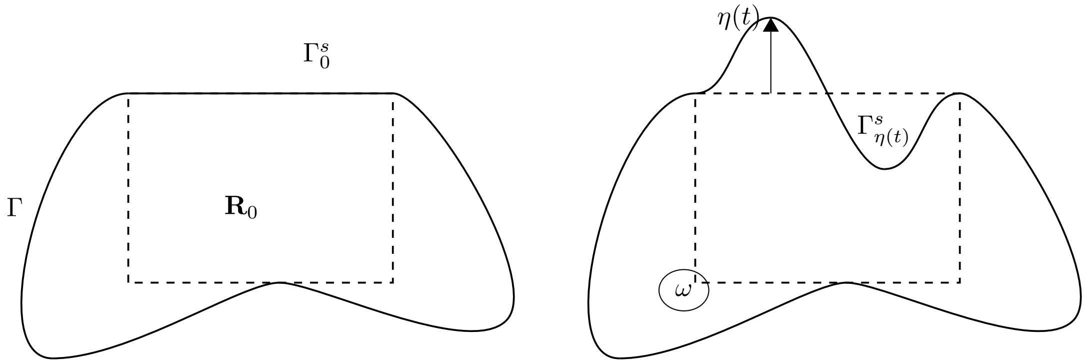

FIGURE 1.5 – Les domaines Ω_0 (à gauche) et $\Omega_{\eta(t)}$ (à droite).

poutre η est donné à partir de q par $\eta = Zq$ où Z est un vecteur ligne $\mathbb{R}^{1\times N}$ formé des N premiers vecteurs propres de l'opérateur des poutres $\mathcal{A}_{\alpha,\beta}$ défini par $D(\mathcal{A}_{\alpha,\beta}) = H^4_{(0)}(\Gamma^s_0)$ et $\mathcal{A}_{\alpha,\beta}\mu = \alpha\mu_{xxxx} - \beta\mu_{xx}$ pour tout μ dans $D(\mathcal{A}_{\alpha,\beta})$.

Nous démontrons alors :

Théorème 1.11. *Soit $T > 0$. Soit $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ dans $\mathbf{V}^1(\Omega_{\eta^{1,0}}) \times \mathbb{R}^N \times \mathbb{R}^N$ satisfaisant les conditions de compatibilité $\mathbf{u} = \mathbf{0}$ sur Γ et $\mathbf{u} = Zq^{2,0}\mathbf{e}_2$ sur $\Gamma^s_{q^{1,0}}$. Alors il existe $R > 0$ tel que si*

$$\|\mathbf{u}^0\|^2_{\mathbf{V}^1(\Omega_{q^{1,0}})} + |q^{1,0}|^2_{\mathbb{R}^N} + |q^{2,0}|^2_{\mathbb{R}^N} < R^2,$$

le système (1.12)–(1.13) est contrôlable à zéro au temps T en les variables $(\mathbf{u}, q, q')$. Cela signifie qu'il existe un contrôle $\mathbf{c}$ dans $L^2(0,T;\mathbf{L}^2(\omega))$ tel que la solution $(\mathbf{u}, p, q)$ du système (1.12)–(1.13) avec $\mathbf{c}$ comme second membre vérifie

$$\mathbf{u}(T) = \mathbf{0}, \quad q(T) = 0 \quad \textit{and} \quad q'(T) = 0.$$

La preuve de ce résultat repose sur une inégalité d'observabilité pour le système adjoint du système linérisé autour de $(\mathbf{u}, p, q) = (\mathbf{0}, 0, 0)$ obtenue grâce à une inégalité de Carleman. Un point fixe dans des espaces à poids (en temps) permet de conclure. Ce travail se place dans la lignée d'une série de papiers de Raymond et Vanninathan [25, 27, 26, 28] où les auteurs considèrent plusieurs types de systèmes couplés (linéaires) fluide-structure où l'équation pour le fluide est modélisée de l'équation d'Helmholtz à l'équation de Stokes.

Chapitre 6.

Nous nous intéressons ici à un domaine avec condition périodique en la variable x en deux dimensions. La frontière est en deux parties et est composée de deux poutres de mêmes caractéristiques. Un fluide visqueux incompressible occupe l'intérieur du domaine. Nous souhaitons stabiliser le système pour n'importe quel taux de décroissance $\omega > 0$ pour des conditions initiales proches de la solution stationnaire nulle.

Dans ce chapitre, nous considérons les conditions d'interaction du papier [13]. Les deux poutres aux repos donnent les deux états de références $\Gamma_0^+ = (0, 2\pi) \times \{1\}$ et $\Gamma_0^- = (0, 2\pi) \times \{-1\}$. Pour ce domaine, il n'y a pas de frontière fixe. A l'instant t, les parties mobiles de la frontière s'écrivent en fonction des déplacements η^+ et η^-, avec la notation $\eta = (\eta^+, \eta^-)$:

$$\begin{aligned}
\Gamma^+_{\eta^+(t)} &= \Big\{(x,y) \in \mathbb{R}^2 \text{ t.q. } x \in (0,2\pi) \text{ et } y = 1 + \eta^+(t,x)\Big\},\\
\Gamma^-_{\eta^-(t)} &= \Big\{(x,y) \in \mathbb{R}^2 \text{ t.q. } x \in (0,2\pi) \text{ et } y = -1 + \eta^-(t,x)\Big\},\\
\Omega_{\eta(t)} &= \Big\{(x,y) \in \mathbb{R}^2 \text{ t.q. } x \in (0,2\pi) \text{ et } -1 + \eta^-(t,x) < y < 1 + \eta^+(t,x)\Big\},
\end{aligned}$$

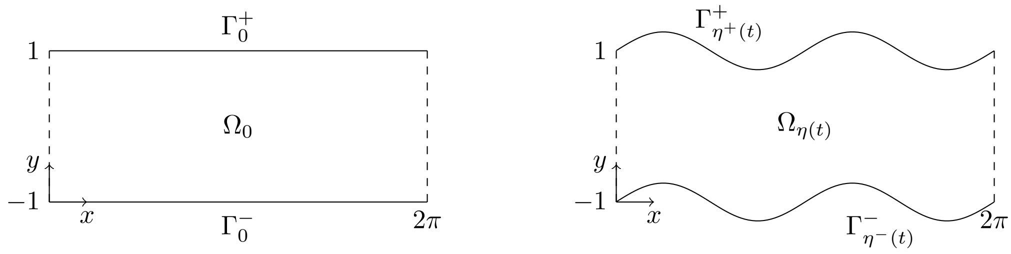

FIGURE 1.6 – Ω_0 (gauche) et $\Omega_{\eta(t)}$ (droite).

voir Figure 1.6.

Suivant les notations du début du chapitre, nous notons, pour $T > 0$

$$\Sigma_T^{+,\eta^+} = \bigcup_{t\in(0,T)} \{t\}\times\Gamma^+_{\eta^+}, \qquad \Sigma_T^{-,\eta^-} = \bigcup_{t\in(0,T)} \{t\}\times\Gamma^-_{\eta^-}, \qquad \Gamma_0 = \Gamma_0^+\times\Gamma_0^-,$$
$$\Sigma_T^{+,0} = (0,T)\times\Gamma_0^+, \qquad \Sigma_T^{-,0} = (0,T)\times\Gamma_0^- \qquad \Sigma_T^0 = (0,T)\times\Gamma_0.$$

Le système s'écrit alors

$$\begin{array}{rcll}
\mathbf{u}_t - \mathrm{div}\ \sigma(\mathbf{u},p) + (\mathbf{u}\cdot\nabla)\mathbf{u} &=& \mathbf{0} & \text{dans } Q_T^\eta,\\
\mathrm{div}\ \mathbf{u} &=& 0 & \text{dans } Q_T^\eta,\\
\mathbf{u}\cdot\widetilde{\mathbf{n}}^\kappa &=& \kappa\eta_t^\kappa & \text{sur } \Sigma_T^{\kappa,\eta^\kappa},\\
\sigma(\mathbf{u},p)\mathbf{n}^\kappa\cdot\mathbf{t}^\kappa &=& f_0^+\chi_+ & \text{sur } \Sigma_T^{\kappa,\eta^\kappa},\\
\eta_{tt}^\kappa + \alpha\eta_{xxxx}^\kappa - \beta\eta_{xx}^\kappa - \gamma\eta_{txx}^\kappa &=& -\kappa\sigma(\mathbf{u},p)\mathbf{n}^\kappa\cdot\mathbf{n}^\kappa + f^+\chi_+ & \text{sur } \Sigma_T^{\kappa,0},\\
(\mathbf{u}(0),\eta^\kappa(0),\eta_t^\kappa(0)) &=& (\mathbf{u}^0,\eta^{1,0},\eta^{2,0}) & \text{dans } \Omega_{\eta^{1,0}}\times\Gamma_0^s\times\Gamma_0^s
\end{array} \tag{1.14}$$

où les fonctions f_0^+ et f^+ sont des contrôles agissant sur la poutre supérieure, χ_+ est la fonction caractéristique de la partie supérieure $\Gamma^+_{\eta^+}$ ou Γ_0^+. De plus, f_0^+ est une fonction du temps seulement.

Nous montrons alors le résultat suivant.

Théorème 1.12. *Pour tout taux de décroissance $\omega > 0$, il existe une constante $r_0 > 0$ et une fonction strictement croissante R de $\mathbb{R}^+$ dans lui-même telles que si r appartient à $(0, r_0)$ et si $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ est un élément de $\mathbf{V}^1_\#(\Omega_{\eta^{1,0}})\times H^3_\#(\Gamma_0)\times H^1_\#(\Gamma_0)$ satisfaisant la condition de compatibilité (avec les notations $\mathbf{u}^0 = (u^{0,1}, u^{0,2})$)*

$$-\eta_x^{1,0,+}u^{0,1} + u^{0,2} = \eta^{2,0,+} \quad \text{on } \Gamma^+_{\eta^{1,0,+}}, \qquad -\eta_x^{1,0,-}u^{0,1} + u^{0,2} = \eta^{2,0,-} \quad \text{on } \Gamma^-_{\eta^{1,0,-}}$$

et l'inégalité

$$\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{\mathbf{V}^1_\#(\Omega_{\eta^{1,0}})\times H^3_\#(\Gamma_0)\times H^1_\#(\Gamma_0)} \le R(r),$$

alors le système

$$\begin{array}{rcll}
\mathbf{u}_t - \mathit{div}\ \sigma(\mathbf{u},p) + (\mathbf{u}\cdot\nabla)\mathbf{u} &=& \mathbf{0} & \textit{dans } Q_T^\eta,\\
\mathit{div}\ \mathbf{u} &=& 0 & \textit{dans } Q_T^\eta,\\
\mathbf{u}\cdot\widetilde{\mathbf{n}}^+ &=& \eta_t^+ & \textit{sur } \Sigma_T^{+,\eta^+},\\
\mathbf{u}\cdot\widetilde{\mathbf{n}}^- &=& -\eta_t^- & \textit{sur } \Sigma_T^{-,\eta^-},\\
\mathbf{S}(\mathbf{u})\mathbf{n}^+\cdot\mathbf{t}^+ &=& f_0^+ & \textit{sur } \Sigma_T^{+,\eta^+},\\
\mathbf{S}(\mathbf{u})\mathbf{n}^-\cdot\mathbf{t}^- &=& 0 & \textit{sur } \Sigma_T^{-,\eta^-},\\
\eta_{tt}^+ + \alpha\eta_{xxxx}^+ - \beta\eta_{xx}^+ - \gamma\eta_{txx}^+ &=& -\sigma(\mathbf{u},p)\mathbf{n}^+\cdot\mathbf{n}^+ + f^+ & \textit{sur } \Sigma_T^{+,0},\\
\eta_{tt}^- + \alpha\eta_{xxxx}^- - \beta\eta_{xx}^- - \gamma\eta_{txx}^- &=& \sigma(\mathbf{u},p)\mathbf{n}^-\cdot\mathbf{n}^- & \textit{sur } \Sigma_T^{-,0},\\
(\mathbf{u}(0),\eta(0),\eta_t(0)) &=& (\mathbf{u}^0,\eta^{1,0},\eta^{2,0}) &
\end{array}$$

avec les contrôles (obtenus par lois de feedback)

$$f_0^+ = -\mathcal{B}_0^*\Pi_0\left[\left(\mathbf{u}\circ\phi_\eta^{-1}\right)_0^1\right] \quad et \quad f^+ = -\sum_{|k|\leq M_\omega;k\neq 0}\Pi_k^{3,+}\left[\left(\mathbf{P}_k\left[\mathbf{u}\circ\phi_\eta^{-1}\right]_k, \eta_k, \eta_{k,t}\right)\right]$$

admet une unique solution $(\mathbf{u}, p, \eta)$ *telle que*

$$\left\|e^{\omega\cdot}\mathbf{u}\circ\phi_\eta^{-1}\right\|_{\mathbf{H}^{2,1}_\#(Q^0_\infty)} + \left\|e^{\omega\cdot}p\circ\phi_\eta^{-1}\right\|_{L^2(0,\infty;\mathcal{H}^1_\#(\Omega_0))}\left\|e^{\omega\cdot}\eta\right\|_{H^{4,2}_\#(\Sigma^0_\infty)} + \left\|e^{\omega\cdot}\eta_t\right\|_{H^{2,1}_\#(\Sigma^0_\infty)} \leq r,$$

où $\phi_{\eta(t)}$ *est le changement de variables qui transforme* $\Omega_{\eta(t)}$ *en* Ω_0 *(voir section 6.4).*

De plus, les lois de Feedback Π_k *(pour* k *dans* $\mathbb{Z}$ *tel que* $|k| \leq M_\omega$ *et* $k \neq 0$*) et* Π_0 *sont obtenus comme unique solutions d'équations de Riccati algébriques de dimension finie (voir sections 6.5.3 et 6.6.2, en particulier les équations* (6.45) *et* (6.54)*).*

La stabilisation de système couplé fluide-poutre a déjà été étudiée dans [23]. Dans ce papier, l'auteur considère un système couplant les équations de Navier-Stokes en deux dimensions avec une équation des poutres dans un domaine rectangulaire avec des conditions de Dirichlet homogènes aux bords où il n'y a pas de poutres. De plus, les conditions d'interaction sont celles de [4].

Les preuves des résultats de [23] et du Théorème 1.12 sont assez proches. Elles reposent sur une réécriture du système dans un domaine fixe grâce à un changement de variables (voir les chapitres précédents également), puis sur la linéarisation du système obtenu autour de la solution nulle. Ensuite, nous prouvons la stabilisation du système linéarisé nonhomogène. Et finalement par une méthode de point fixe, nous montrons la stabilisation du système de départ écrit dans le domaine fixe.

Le système considéré ici peut être vu comme une extension des systèmes étudiés dans [2, 3] ou [29] où les différents auteurs étudient la stabilisation d'un fluide dans un canal périodique autour d'un profil de Poiseuille grâce à des actuateurs agissant sur la composante normale de la vitesse au(x) bord(s) du canal.

Nous nous sommes intéressés aux conditions d'interaction introduites dans [13] parce qu'elles nous permettent de prouver le problème de continuation unique associé à notre problème. Cette étape est souvent la plus compliquée pour ce genre de système, voir par exemple [18, 19]. Pour des conditions aux bords comme dans [4, 23], le problème de continuation unique revient à résoudre l'équation d'Orr-Sommerfeld avec des conditions aux bords différentes et même si nous savons que les solutions de cette équation sont analytiques, nous ne sommes pas capables de prouver le problème de continuation unique associé.

Relation entre les équations et les interactions.

La formule de la divergence appliquée à la vitesse $\mathbf{u}$ où $(\mathbf{u}, p, \eta)$ satisfait, par exemple, le système décrit dans le Chapitre 2 donne

$$0 = \int_{\Omega_{\eta(t)}} \operatorname{div}\mathbf{u} = \int_{\partial\Omega_{\eta(t)}} \mathbf{u}\cdot\mathbf{n} = \int_{\Gamma_0^s} \eta_t.$$

Ainsi, η_t est de moyenne nulle. De plus, avec les conditions d'encastrement de la poutre, les termes de l'équation des poutres η_{tt}, $\beta\Delta_s\eta$ et $\gamma\Delta_s\eta_t$ sont également de moyenne nulle. Ainsi, si le second membre l'est aussi, nous aurons alors nécessairement $\alpha\Delta_s^2\eta$ de moyenne nulle (cela s'applique de la même manière en trois dimensions).

Cette remarque montre que le multiplicateur de Lagrange associé à la contrainte d'incompressibilité qui apparait dans la constante de la pression projette l'équation des poutres/plaques dans l'espace des fonctions à moyenne nulle. Les solutions d'une telle équation sont alors nécessairement cherchées parmi celles de moyenne nulle (il en existe grâce au Théorème de Lax-Milgram).

Dans la suite, au lieu d'écrire l'équation des poutres/plaques avec ce multiplicateur de Lagrange, nous projetons l'équation dans l'espace des fonctions à moyenne nulle. Cela permet en particulier de définir les opérateurs indépendamment de l'équation comme des applications linéaires d'un espace de fonctions à moyenne nulle dans lui-même.

Choix de la langue.

Les cinq chapitres suivants sont écrits en anglais. Les chapitres 2 et 5 ont été écrits bien avant ce mémoire : le premier a été publié électroniquement en février dernier dans *SIAM Journal of Mathematical Analysis* [16] et le second a été soumis il y a quelques mois dans *SIAM Journal of Control and Optimization*. Ces journaux à comités de lecture internationaux sont écrits en anglais. Les autres chapitres (Chapitres 3, 4 et 6) ont également été écrits en anglais dans l'espoir qu'ils puissent être soumis un jour à publication dans ce même type de revue.

Partie I

Existence et unicité de solution pour des systèmes couplés fluide-structure

Chapitre 2

Existence et unicité de solution pour un système couplant les équations de Navier-Stokes et une équation des poutres en deux dimensions

2.1 Introduction.

We study a fluid-structure system coupling the Navier-Stokes equations in a 2D domain with a damped beam equation located on the boundary of a domain occupied by a fluid flow. For similar systems, the existence of weak solutions has been established in [7, 13] for 2D domains and in [7, 11] for 3D domains.

Here we are interested in the existence of local in time strong solutions. In [4], Beirão da Veiga proves the existence of local strong solutions for small data under the assumption $\alpha \geq 0$ (see the beam equation (2.3)). In this paper, we improve this type of result, with $\alpha > 0$, by showing the existence of local strong solutions without any smallness condition (Theorem 2.3) and we also prove the existence of global strong solutions in a given time interval $[0, T]$ for small data (Theorem 2.2).

In the author's knowlegde, this problem has been introduced in [21] by Quarteroni, Tuveri and Veneziani to model cardiovascular systems like blood flow in large vessels, arteries for instance.

Let $L > 0$ and $T > 0$ be respectively a length and a time. Let η be a function from $(0, T) \times (0, L)$ to $(-1, +\infty)$. Let $t \in (0, T)$, we can define a domain $\Omega_{\eta(t)}$ depending on time by

$$\Omega_{\eta(t)} = \Big\{(x, y)\ ;\ 0 \leq x \leq L \text{ and } 0 \leq y \leq 1 + \eta(t, x)\Big\}.$$

Here $\eta(t)$ is the displacement of the beam. We note by $\Gamma_0^s = (0, L) \times \{1\}$ the reference configuration of the beam. The displacement η has to satisfy the following assumption

$$\exists \delta_0 > 0 \text{ such that } \forall t \geq 0\ \forall x \in (0, L) \quad 1 + \eta(t, x) \geq \delta_0 > 0 \tag{2.1}$$

to ensure that, for every time t, $\Omega_{\eta(t)}$ is a connected domain. Let us set $\Omega_0 = (0, L) \times (0, 1)$ and $\Gamma_0 = \partial\Omega_0$, that is

$$\Gamma_0 = \{0\} \times (0, 1) \bigcup \{L\} \times (0, 1) \bigcup (0, L) \times \{0\} \bigcup (0, L) \times \{1\}.$$

We also set $\Gamma = \Gamma_0 \setminus \Gamma_0^s$, the fixed boundary part

$$\Gamma = \{0\} \times (0, 1) \bigcup \{L\} \times (0, 1) \bigcup (0, L) \times \{0\}$$

and

$$\Gamma^s_{\eta(t)} = \Big\{(x, y)\ ;\ 0 \leq x \leq L \text{ and } y = 1 + \eta(t, x)\Big\}.$$

Thus $\partial\Omega_{\eta(t)} = \Gamma \cup \Gamma^s_{\eta(t)}$. We will use other notations:

$$\begin{array}{ll} \Sigma_T = (0,T) \times \Gamma, & \Sigma_T^{s,0} = (0,T) \times \Gamma^s_0, \\ Q_T^0 = (0,T) \times \Omega_0, & Q_T^\eta = \bigcup\limits_{t\in(0,T)} \{t\} \times \Omega_{\eta(t)}, \\ \Sigma_T^0 = (0,T) \times \Gamma_0, & \Sigma_T^{s,\eta} = \bigcup\limits_{t\in(0,T)} \{t\} \times \Gamma^s_{\eta(t)}. \end{array}$$

The velocity $\mathbf{u}$ and the pressure p of the fluid in the domain Q_T^η are described by the Navier-Stokes equations

$$\begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \text{div}\,\sigma(\mathbf{u},p) & = & 0 & \text{in } Q_T^\eta, \\ \text{div}\,\mathbf{u} & = & 0 & \text{in } Q_T^\eta, \\ \mathbf{u} & = & \eta_t \mathbf{e}_2 & \text{on } \Sigma_T^{s,\eta}, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}(0) & = & \mathbf{u}^0 & \text{in } \Omega_{\eta^{1,0}}. \end{array} \tag{2.2}$$

The displacement η satisfies the following beam equation

$$\begin{array}{rcll} \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} & = & \phi[\mathbf{u},p,\eta] & \text{on } \Sigma_T^{s,0}, \\ \eta(0) & = & \eta^{1,0} & \text{in } \Gamma^s_0, \\ \eta_t(0) & = & \eta^{2,0} & \text{in } \Gamma^s_0. \end{array} \tag{2.3}$$

In these equations, σ and ϕ are defined by

$$\begin{array}{rcl} \sigma(\mathbf{u},p) & = & -pI + \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{tr}}), \\ \phi[\mathbf{u},p,\eta] & = & -\sigma(\mathbf{u},p)(-\eta_x\mathbf{e}_1 + \mathbf{e}_2)\cdot\mathbf{e}_2, \end{array}$$

where $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ and $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$, $\nu > 0$ is the viscosity of the fluid; $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$ are constants relative to the structure (see [4] for more details).

2.2 Functional settings.

We have to define the function spaces for the solutions $(\mathbf{u},p,\eta)$ of (2.2)–(2.3). In the fixed domain Ω_0, we define the classical Hilbert space in two dimensions $\mathbf{L}^2(\Omega_0) = L^2(\Omega_0;\mathbb{R}^2)$ and in the same way the Sobolev spaces $\mathbf{H}^s(\Omega_0) = H^s(\Omega_0;\mathbb{R}^2)$. We introduce

$$\begin{array}{rcl} \mathbf{V}^\sigma(\Omega_0) & = & \Big\{\mathbf{u} \in \mathbf{H}^\sigma(\Omega_0)\ ;\ \text{div}\,\mathbf{u} = 0 \text{ in } \Omega_0\Big\}, \\ \mathbf{H}^{\sigma,\tau}(Q_T^0) & = & L^2(0,T;\mathbf{H}^\sigma(\Omega_0)) \cap H^\tau(0,T;\mathbf{L}^2(\Omega_0)), \\ \mathbf{V}^{\sigma,\tau}(Q_T^0) & = & L^2(0,T;\mathbf{V}^\sigma(\Omega_0)) \cap H^\tau(0,T;\mathbf{V}^0(\Omega_0)). \end{array}$$

We need a definition of Sobolev spaces in the time dependent domain $\Omega_{\eta(t)}$:

Definition 2.1. *We say that* $\mathbf{u}$ *belongs to* $H^\tau(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{H}^\sigma(\Omega_{\eta(t)}))$ *(respectively to* $H^\tau(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^\sigma(\Omega_{\eta(t)}))$*) if*
– *for almost every* t *in* $(0,T)$, $\mathbf{u}(t)$ *belongs to* $\mathbf{H}^\sigma(\Omega_{\eta(t)})$ *(resp. in* $\mathbf{V}^\sigma(\Omega_{\eta(t)})$*),*
– $t \mapsto \|\mathbf{u}(t)\|_{\mathbf{H}^\sigma(\Omega_{\eta(t)})}$ *(resp.* $t \mapsto \|\mathbf{u}(t)\|_{\mathbf{V}^\sigma(\Omega_{\eta(t)})}$*) is in* $H^\tau(0,T;\mathbb{R})$.

We finally define

$$\begin{array}{rcl} \mathbf{H}^{\sigma,\tau}(Q_T^\eta) & = & L^2(\bigcup\limits_{t\in(0,T)}\{t\}\times\mathbf{H}^\sigma(\Omega_{\eta(t)}))\bigcap H^\tau(\bigcup\limits_{t\in(0,T)}\{t\}\times\mathbf{L}^2(\Omega_{\eta(t)})), \\ \mathbf{V}^{\sigma,\tau}(Q_T^\eta) & = & L^2(\bigcup\limits_{t\in(0,T)}\{t\}\times\mathbf{V}^\sigma(\Omega_{\eta(t)}))\bigcap H^\tau(\bigcup\limits_{t\in(0,T)}\{t\}\times\mathbf{V}^0(\Omega_{\eta(t)})). \end{array}$$

Solutions $(\mathbf{u}, p, \eta)$ of (2.2)–(2.3) satisfy

$$\begin{aligned} 0 &= \int_{\Omega_{\eta(t)}} \operatorname{div} \mathbf{u}(t) = \int_{\partial\Omega_{\eta(t)}} \mathbf{u}(t)\cdot\mathbf{n}(t) = \int_{\Gamma^s_{\eta(t)}} \mathbf{u}(t)\cdot\mathbf{n}(t) + \int_\Gamma \mathbf{u}(t)\cdot\mathbf{n}_0 \\ &= \int_{\Gamma^s_0} \eta_t(t) + 0 = \int_0^L \eta_t(t,x)\mathrm{d}x, \end{aligned}$$

where $\mathbf{n}(t) = \dfrac{1}{\sqrt{1+\eta_x^2(t)}}\Big(-\eta_x(t)\mathbf{e}_1 + \mathbf{e}_2\Big)$ is the unit normal to $\Gamma^s_{\eta(t)}$ outward $\Omega_{\eta(t)}$ and $\mathbf{n}_0$ is the unit normal to each part of Γ outward $\Omega_{\eta(t)}$, that is

$$\mathbf{n}_0 = \mathbf{e}_1 \text{ on } \{L\}\times(0,1), \qquad \mathbf{n}_0 = -\mathbf{e}_1 \text{ on } \{0\}\times(0,1) \quad \text{or} \quad \mathbf{n}_0 = -\mathbf{e}_2 \text{ on } (0,L)\times\{0\}.$$

Thus we must choose $\eta^{2,0}$ in $L^2_0(\Gamma^s_0) = \Big\{\eta \in L^2(\Gamma^s_0)\ ;\ \int_{\Gamma^s_0}\eta = 0\Big\}$. Furthermore, we can choose $\eta^{1,0} \in L^2_0(\Gamma^s_0)$ and then we shall have

$$\int_{\Gamma^s_0} \eta(t) = 0 \quad \text{and} \quad \int_{\Gamma^s_0} \eta_t(t) = 0 \quad \forall t \geq 0. \tag{2.4}$$

We have to choose boundary conditions for η too. Here, we decide to fix η and η_x on $(0,T)\times\{0,L\}$ as follows:

$$\eta(t,0) = \eta(t,L) = 0 \quad \text{and} \quad \eta_x(t,0) = \eta_x(t,L) = 0 \quad \forall t \in (0,T). \tag{2.5}$$

We could have chosen periodic boundary conditions as in [4]. The result obtained in the following may be directly translated to this situation.

With (2.4) and (2.5), we get

$$\int_{\Gamma^s_0} \eta_{tt} = 0\ ,\ \int_{\Gamma^s_0} \eta_{xx} = 0 \text{ and } \int_{\Gamma^s_0} \eta_{txx} = 0 \quad \text{for all } t \geq 0.$$

We use M_s the orthogonal projection from $L^2(\Gamma^s_0)$ onto $L^2_0(\Gamma^s_0)$ to rewrite the equation (2.3). We will use a special trace function γ_s defined by

$$\gamma_s p = M_s(p_{|\Gamma^s_0}) = p_{|\Gamma^s_0} - \frac{1}{|\Gamma^s_0|}\int_{\Gamma^s_0} p_{|\Gamma^s_0} \quad \forall p \in H^\sigma(\Omega_0) \text{ with } \sigma > \frac{1}{2}.$$

Equation (2.3) becomes

$$\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha M_s \eta_{xxxx} = \gamma_s p + \overline{\phi}[\mathbf{u},\eta]. \tag{2.6}$$

with $\overline{\phi}[\mathbf{u},\eta] = -\nu\gamma_s\Big(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{tr}}\Big)(-\eta_x\mathbf{e}_1 + \mathbf{e}_2)\cdot\mathbf{e}_2$.

Let us introduce the spaces

$$H^\sigma_{(0)}(\Gamma^s_0) = \left\{\begin{array}{ll} \Big\{\mu \in H^\sigma(\Gamma^s_0)\cap L^2_0(\Gamma^s_0) \text{ s.t. } \mu = \mu_x = 0 \text{ at } x = 0, L\Big\} & \text{for } \frac{3}{2} < \sigma, \\ \Big\{\mu \in H^\sigma(\Gamma^s_0)\cap L^2_0(\Gamma^s_0) \text{ s.t. } \mu = 0 \text{ at } x = 0, L\Big\} & \text{for } \frac{1}{2} < \sigma \leq \frac{3}{2}, \\ H^\sigma(\Gamma^s_0)\cap L^2_0(\Gamma^s_0) & \text{for } 0 \leq \sigma \leq \frac{1}{2}. \end{array}\right. \tag{2.7}$$

Due to (2.4) and (2.5), we look for η in the spaces

$$H^{\sigma,\tau}_{(0)}(\Sigma^{s,0}_T) = L^2(0,T; H^\sigma_{(0)}(\Gamma^s_0)) \cap H^\tau(0,T; L^2_0(\Gamma^s_0)).$$

The pressure term p is defined in the Navier-Stokes equations up to an additive constant. Then, we define the space $\mathcal{H}^\sigma(\Omega_0)$ by

$$\mathcal{H}^\sigma(\Omega_0) = \Big\{p \in H^\sigma(\Omega_0) \text{ such that } \int_{\Omega_0} p = 0\Big\}.$$

We will look for p in $L^2(\bigcup_{t\in(0,T)}\{t\}\times\mathcal{H}^1(\Omega_{\eta(t)}))$ (see Definition 2.1).

2.3 Main results.

We can now state the two main theorems of this paper. First, we consider global strong solutions of the system (2.2)–(2.3) with a condition on the size of the initial data only. Second, we prove the existence of a local strong solution for the same system.

Theorem 2.2. *Let* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0}) \in \mathbf{V}^1(\Omega_{\eta^{1,0}}) \times H^3_{(0)}(\Gamma^s_0) \times H^1_{(0)}(\Gamma^s_0)$. *There exists* $R > 0$ *such that for any initial data satisfying* $\|\mathbf{u}^0\|^2_{\mathbf{V}^1(\Omega_{\eta^{1,0}})} + \|\eta^{1,0}\|^2_{H^3_{(0)}(\Gamma^s_0)} + \|\eta^{2,0}\|^2_{H^1_{(0)}(\Gamma^s_0)} \leq R^2$ *and the compatibility condition*

$$\mathbf{u}^0 = \mathbf{0} \quad \text{on } \Gamma, \qquad \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_2 \quad \text{on } \Gamma^s_{\eta^{1,0}}, \tag{2.8}$$

system (2.2)–(2.3) *has a unique global strong solution* $(\mathbf{u}, p, \eta)$ *in*

$$\mathbf{V}^{2,1}(Q^\eta_T) \times L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1\left(\Omega_{\eta(t)}\right)\right) \times H^{4,2}_{(0)}(\Sigma^{s,0}_T).$$

Theorem 2.3. *Let* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0}) \in \mathbf{V}^1(\Omega_{\eta^{1,0}}) \times H^3_{(0)}(\Gamma^s_0) \times H^1_{(0)}(\Gamma^s_0)$ *satisfying the compatibility condition* (2.8). *There exists a time* $T_0 > 0$ *such that system* (2.2)–(2.3) *has a unique strong solution* $(\mathbf{u}, p, \eta) \in \mathbf{V}^{2,1}(Q^\eta_{T_0}) \times L^2\left(\bigcup_{t\in(0,T_0)} \{t\} \times \mathcal{H}^1\left(\Omega_{\eta(t)}\right)\right) \times H^{4,2}_{(0)}(\Sigma^{s,0}_{T_0})$.

The core of the paper consists in the proof of these theorems. First of all, thanks to a suitable change of variables, we introduce an equivalent problem (2.13) in a cylindrical domain Q^0_T. Due to the change of variables, new nonlinear terms appear in the equations. The proof of existence of solutions for system (2.13) is split into different steps:

(*i*) We study the non homogeneous linearized system (2.21), where the nonlinearities in (2.13) are now considered as right-hand sides. The proof of existence for this system uses a fixed point method for another equivalent system (2.30) introduced in section 2.5.2 thanks to the splitting method due to J. P. Raymond ([22]). Indeed, we see in section 2.5.1 that we cannot apply a fixed point method directly to system (2.21).

(*ii*) From the linearized system, we prove the existence of strong solutions for system (2.13) thanks to another fixed point method in section 2.6.

In section 2.7, we complete the proof by checking that the change of variables defined in section 2.4 is suitable in the sense of Definition 2.4.

2.4 An equivalent problem in the fixed domain Ω_0.

We want to use a change of variables to rewrite system (2.2)–(2.3) in the domain $Q^0_T = (0,T) \times \Omega_0$. This change of variables introduces nonlinear terms in the variables $(\mathbf{u}, p, \eta)$ that we will treat as right-hand sides in section 2.5. As in [4], for a fixed $t \in (0,T)$, we introduce the change of variables:

$$\begin{array}{rcl} \Omega_{\eta(t)} & \longrightarrow & \Omega_0 \\ (x,y) & \longmapsto & (x,z) = \left(x, \dfrac{y}{1+\eta(t,x)}\right). \end{array} \tag{2.9}$$

Setting $\hat{f}(x,z) = f(x,y)$, we have the formulas

$$\hat{f}(x,z) = f(x,(1+\eta(t,x))z) \qquad \text{and} \qquad f(x,y) = \hat{f}\left(x, \frac{y}{1+\eta(t,x)}\right).$$

Then we can calculate the derivatives of $f(x,y)$ using the derivatives of $\hat{f}(x,z)$:

$$\begin{cases} f_t &= \hat{f}_t - z\dfrac{\eta_t}{1+\eta}\hat{f}_z, \\ f_x &= \hat{f}_x - z\dfrac{\eta_x}{1+\eta}\hat{f}_z, \\ f_y &= \dfrac{1}{1+\eta}\hat{f}_z, \\ f_{xx} &= \hat{f}_{xx} - 2z\dfrac{\eta_x}{1+\eta}\hat{f}_{xz} + \left(z\dfrac{\eta_x}{1+\eta}\right)^2 \hat{f}_{zz} - z\dfrac{(1+\eta)\eta_{xx}-\eta_x^2}{(1+\eta)^2}\hat{f}_z, \\ f_{yy} &= \dfrac{1}{(1+\eta)^2}\hat{f}_{zz}. \end{cases}$$

Now, we state the system satisfied by $\hat{\mathbf{u}}(x,z) = \mathbf{u}(x,y)$ and $\hat{p}(x,z) = p(x,y)$:

$$\begin{array}{rcll} \hat{\mathbf{u}}_t - \operatorname{div}\sigma(\hat{\mathbf{u}},\hat{p}) &=& \hat{\mathbf{F}}[\hat{\mathbf{u}},\hat{p},\eta] & \text{in } Q_T^0, \\ \operatorname{div}\hat{\mathbf{u}} &=& \operatorname{div}\hat{\mathbf{w}}[\hat{\mathbf{u}},\eta] & \text{in } Q_T^0, \\ \hat{\mathbf{u}}(0) &=& \hat{\mathbf{u}}^0 & \text{in } \Omega_0, \\ \hat{\mathbf{u}} &=& \eta_t(t,x)\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \hat{\mathbf{u}} &=& \mathbf{0} & \text{on } \Sigma_T \end{array} \tag{2.10}$$

with

$$\begin{aligned} \hat{\mathbf{F}}(t,x,z) &= \hat{\mathbf{F}}[\hat{\mathbf{u}},\hat{p},\eta] \\ &= -\eta\hat{\mathbf{u}}_t + \left[z\eta_t + \nu z\left(\frac{\eta_x^2}{1+\eta} - \eta_{xx}\right)\right]\hat{\mathbf{u}}_z \\ &\quad + \nu\left\{-2z\eta_x\hat{\mathbf{u}}_{xz} + \eta\hat{\mathbf{u}}_{xx} + \frac{z^2\eta_x^2-\eta}{1+\eta}\hat{\mathbf{u}}_{zz}\right\} \\ &\quad + z(\eta_x\hat{p}_z - \eta\hat{p}_x)\mathbf{e}_1 - (1+\eta)\hat{u}_1\hat{\mathbf{u}}_x + (z\eta_x\hat{u}_1 - \hat{u}_2)\hat{\mathbf{u}}_z \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \hat{\mathbf{w}}(t,x) &= \hat{\mathbf{w}}[\hat{\mathbf{u}},\eta] \\ &= -\eta\hat{u}_1\mathbf{e}_1 + z\eta_x\hat{u}_1\mathbf{e}_2 \end{aligned}$$

For instance, to calculate the divergence term, we write $u_{1,x} + u_{2,z}$ in terms of $\hat{\mathbf{u}}$ and taking $1+\eta$ as a multiplier, we get :

$$0 = (1+\eta)\hat{u}_{1,x} - z\eta_x\hat{u}_{1,z} + \hat{u}_{2,z}.$$

Then we see that

$$\hat{u}_{1,x} + \hat{u}_{2,z} = \operatorname{div}\hat{\mathbf{u}} = -\eta\hat{u}_{1,x} + z\eta_x\hat{u}_{1,z} = \operatorname{div}\hat{\mathbf{w}}.$$

The beam equation (2.3) becomes

$$\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha M_s\eta_{xxxx} = \gamma_s\hat{p} - 2\nu\gamma_s\hat{u}_{2,z} + \gamma_s\hat{H}[\hat{\mathbf{u}},\eta]$$

with

$$\hat{H}[\hat{\mathbf{u}},\eta] = \nu\left(\frac{\eta_x}{1+\eta}\hat{u}_{1,z} + \eta_x\hat{u}_{2,x} - \frac{\eta_x^2 - 2\eta}{1+\eta}\hat{u}_{2,z}\right). \tag{2.12}$$

To simplify the notations, we drop out the symbol $\hat{}$ and we obtain the system

$$\begin{array}{rcll} \mathbf{u}_t - \operatorname{div}\sigma(\mathbf{u},p) &=& \mathbf{F}[\mathbf{u},p,\eta] & \text{in } Q_T^0, \\ \operatorname{div}\mathbf{u} &=& \operatorname{div}\mathbf{w}[\mathbf{u},\eta] & \text{in } Q_T^0, \\ \mathbf{u} &=& \mathbf{0} & \text{on } \Sigma_T^0, \\ \mathbf{u} &=& \eta_t\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{u}(0) &=& \mathbf{u}^0 & \text{in } \Omega_0, \\ \eta_{tt} - \gamma\eta_{txx} - \beta\eta_{xx} + \alpha M_s\eta_{xxxx} &=& \gamma_s p - 2\nu\gamma_s u_{2,z} + \gamma_s H[\mathbf{u},\eta] & \text{on } \Sigma_T^{s,0}, \\ \eta(0) &=& \eta^{1,0} & \text{in } \Gamma_0^s, \\ \eta_t(0) &=& \eta^{2,0} & \text{in } \Gamma_0^s. \end{array} \tag{2.13}$$

The previous system is equivalent to system (2.2)–(2.3). More precisely, we state the following definition:

Definition 2.4. $(\mathbf{u}, p, \eta)$ *in* $\mathbf{H}^{2,1}(Q_T^\eta) \times L^2(\bigcup_{t\in(0,T)}\{t\} \times \mathcal{H}^1(\Omega_{\eta(t)})) \times H^{4,2}_{(0)}(\Sigma_T^{s,0})$ *is solution of* (2.2)–(2.3) *when the following conditions are satisfied:*

(i) $(\hat{\mathbf{u}}, \hat{p}, \eta)$ *obtained for the change of variables* $\hat{\mathbf{u}}(x,z) = \mathbf{u}(x,y)$, $\hat{p}(x,z) = p(x,y)$ *with* $z = \frac{y}{1+\eta(t,x)}$ *is a solution of* (2.13),

(ii) for any time t *in* $(0,T)$, *the previous change of variables is a* $\mathcal{C}^1$*-diffeomorphism from* $\Omega_{\eta(t)}$ *into* Ω_0,

(iii) η *satisfies condition* (2.1).

If we set $\mathbf{u} = \mathbf{v} + \mathbf{w}[\mathbf{u}, \eta]$, we notice that div $\mathbf{v} = 0$ and the system satisfied by $(\mathbf{v}, p, \eta)$ is

$$\begin{array}{rcll} \mathbf{v}_t + \text{div }\sigma(\mathbf{v},p) & = & \mathbf{f}[\mathbf{u},p,\eta] & \text{in } Q_T^0, \\ \text{div }\mathbf{v} & = & 0 & \text{in } Q_T^0, \\ \mathbf{v} & = & -\mathbf{w}[\mathbf{u},\eta] & \text{on } \Sigma_T, \\ \mathbf{v} & = & \eta_t\mathbf{e}_2 - \mathbf{w}[\mathbf{u},\eta] & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v}(0) & = & \mathbf{u}^0 - \mathbf{w}[\mathbf{u},\eta](0) & \text{in } \Omega_0, \\ \eta_{tt} - \gamma\eta_{txx} - \beta\eta_{xx} + \alpha M_s \eta_{xxxx} & = & \gamma_s p - 2\nu\gamma_s v_{2,z} - 2\nu\gamma_s w_{2,z}[\mathbf{u},\eta] + \gamma_s H[\mathbf{u},\eta] & \text{on } \Sigma_T^{s,0}, \\ \eta(0) & = & \eta^{1,0} & \text{in } \Gamma_0^s, \\ \eta_t(0) & = & \eta^{2,0} & \text{in } \Gamma_0^s \end{array} \tag{2.14}$$

with

$$\mathbf{f}[\mathbf{u},p,\eta] = \mathbf{F}[\mathbf{u},p,\eta] + \nu\Delta\mathbf{w}[\mathbf{u},\eta] - \partial_t\mathbf{w}[\mathbf{u},\eta], \tag{2.15}$$

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 \quad \text{and} \quad \mathbf{w}[\mathbf{u},\eta] = w_1[\mathbf{u},\eta]\mathbf{e}_1 + w_2[\mathbf{u},\eta]\mathbf{e}_2.$$

The explicit expression of $\mathbf{w}[\mathbf{u},\eta] = -\eta u_1\mathbf{e}_1 + z\eta_x u_1\mathbf{e}_2$ only depends on u_1 and η. Thus, the boundary conditions on Σ_T and $\Sigma_T^{s,0}$ are

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Sigma_T \qquad \mathbf{v} = \eta_t\mathbf{e}_2 \quad \text{on } \Sigma_T^{s,0}.$$

Moreover, the term $-2\nu\gamma_s v_{2,z}$ in $(2.14)_6$ vanishes. Indeed, $v_{1,x} + v_{2,z} = 0$ in Q_T^0 and $v_1 = 0$ on $\Sigma_T^{s,0}$. Furthermore, if $\mathbf{v}$ is in $\mathbf{H}^{2,1}(Q_T^0)$, then $v_{1,x|\Sigma_T^{s,0}} = 0$ and $v_{2,z|\Sigma_T^{s,0}} = 0$. That is why we are considering the following system:

$$\begin{array}{rcll} \mathbf{v}_t + \text{div }\sigma(\mathbf{v},p) & = & \mathbf{f}[\mathbf{u},p,\eta] & \text{in } Q_T^0, \\ \text{div }\mathbf{v} & = & 0 & \text{in } Q_T^0, \\ \mathbf{v} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{v} & = & \eta_t\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v}(0) & = & \mathbf{v}^0 & \text{on } \Omega_0, \\ \eta_{tt} - \gamma\eta_{txx} - \beta\eta_{xx} + \alpha M_s\eta_{xxxx} & = & \gamma_s p + h[\mathbf{u},\eta] & \text{on } \Sigma_T^{s,0}, \\ \eta(0) & = & \eta^{1,0} & \text{in } \Gamma_0^s, \\ \eta_t(0) & = & \eta^{2,0} & \text{in } \Gamma_0^s \end{array} \tag{2.16}$$

where

$$h[\mathbf{u},\eta] = -2\nu\gamma_s w_{2,z}[\mathbf{u},\eta] + \gamma_s H[\mathbf{u},\eta] \tag{2.17}$$

and

$$\mathbf{v}^0 = \mathbf{u}^0 - \mathbf{w}[\mathbf{u},\eta](0) = \mathbf{u}^0 + \eta^{1,0}u_1^0\mathbf{e}_1 - z\eta_x^{1,0}u_1^0\mathbf{e}_2. \tag{2.18}$$

In an other hand, to have continuity on [0,T), the previous conditions on $\mathbf{v}$ must be checked at time $t = 0$. Thus, we have to add a compatibility condition at time $t = 0$:

$$\text{div }\mathbf{v}^0 = 0 \quad \text{in } \Omega_0, \qquad \mathbf{v}^0 = \mathbf{0} \quad \text{on } \Gamma \qquad \text{and} \qquad \mathbf{v}^0 = \eta^{2,0}\mathbf{e}_2 \quad \text{on } \Gamma_0^s \tag{2.19}$$

which is written in terms of $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ as follows:

$$\text{div }(\mathbf{u}^0 + \eta^{1,0}u_1^0\mathbf{e}_1 - z\eta_x^{1,0}u_1^0\mathbf{e}_2) = 0 \quad \text{in } \Omega_0, \qquad \mathbf{u}^0 = \mathbf{0} \quad \text{on } \Gamma \qquad \text{and} \qquad \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_2 \quad \text{on } \Gamma_0^s. \tag{2.20}$$

2.5 Study of an auxiliary linear system.

In this section, we prove existence and uniqueness of solutions to the following system

$$\begin{array}{rcll} \mathbf{v}_t + \mathrm{div}\ \sigma(\mathbf{v},p) & = & \mathbf{f} & \text{in } Q_T^0, \\ \mathrm{div}\ \mathbf{v} & = & 0 & \text{in } Q_T^0, \\ \mathbf{v} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{v} & = & \eta_t \mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v}(0) & = & \mathbf{v}^0 & \text{in } \Omega_0, \\ \eta_{tt} - \gamma\eta_{txx} - \beta\eta_{xx} + \alpha M_s \eta_{xxxx} & = & \gamma_s p + h & \text{on } \Sigma_T^{s,0}, \\ \eta(0) & = & \eta^{1,0} & \text{in } \Gamma_0^s, \\ \eta_t(0) & = & \eta^{2,0} & \text{in } \Gamma_0^s \end{array} \tag{2.21}$$

for a right-hand side

$$(\mathbf{f}, h) \in Z_T = \mathbf{L}^2(Q_T^0) \times L^2(0,T; H^{1/2}_{(0)}(\Gamma_0^s)), \tag{2.22}$$

and initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ in $X^0_{\mathbf{cc}}$ where

$$X^0 = \mathbf{H}^1(\Omega_0) \times H^3_{(0)}(\Gamma_0^s) \times H^1_{(0)}(\Gamma_0^s)$$

and

$$X^0_{\mathbf{cc}} = \Big\{ (\mathbf{z}^0, \mu^{1,0}, \mu^{2,0}) \in X^0 \text{ such that } (\mathbf{z}^0, \mu^{1,0}, \mu^{2,0}) \text{ satisfies (2.19)} \Big\}$$

The space X^0 will be equipped of the norm

$$\|(\mathbf{z}^0, \mu^{1,0}, \mu^{2,0})\|_{X^0} = \Big(\|\mathbf{z}^0\|^2_{\mathbf{H}^1(\Omega_0)} + \|\mu^{1,0}\|^2_{H^3(\Gamma_0^s)} + \|\mu^{2,0}\|^2_{H^1(\Gamma_0^s)} \Big)^{1/2}.$$

The main result of this section is the following theorem:

Theorem 2.5. *Let $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ be in $X^0_{\mathbf{cc}}$ and $(\mathbf{f}, h)$ be in Z_T. Then, system (2.21) admits one and only one solution $(\mathbf{v}, p, \eta)$ in*

$$\begin{array}{rcl} X_T & = & \Big\{ (\mathbf{z}, q, \mu) \in \mathbf{H}^{2,1}(Q_T^0) \times L^2(0,T; \mathcal{H}^1(\Omega_0)) \times H^{4,2}_{(0)}(\Sigma_T^{s,0}) \\ & & \textit{such that } \mathbf{z} = \mathbf{0} \textit{ on } \Sigma_T \textit{ and } \mathbf{z} = \mu_t \mathbf{e}_2 \textit{ on } \Sigma_T^{s,0} \Big\} \end{array}. \tag{2.23}$$

Moreover, we get the estimate

$$\|(\mathbf{v}, p, \eta)\|_{X_T} \le C(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T}). \tag{2.24}$$

2.5.1 Why a fixed point method on the pressure term p does not work?

A way to find solutions of the coupled system (2.21) is to use a fixed point method. For a given $\overline{p}$ in $L^2(0,T;\mathcal{H}^1(\Omega_0))$, we consider the following system

$$\begin{array}{rcll} \mathbf{v}_t + \mathrm{div}\ \sigma(\mathbf{v},p) & = & \mathbf{f} & \text{in } Q_T^0, \\ \mathrm{div}\ \mathbf{v} & = & 0 & \text{in } Q_T^0, \\ \mathbf{v} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{v} & = & \eta_t \mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v}(0) & = & \mathbf{v}^0 & \text{in } \Omega_0, \\ \eta_{tt} - \gamma\eta_{txx} - \beta\eta_{xx} + \alpha M_s \eta_{xxxx} & = & \gamma_s \overline{p} + h & \text{on } \Sigma_T^{s,0}, \\ \eta(0) & = & \eta^{1,0} & \text{in } \Gamma_0^s, \\ \eta_t(0) & = & \eta^{2,0} & \text{in } \Gamma_0^s. \end{array} \tag{2.25}$$

For fixed given $(\eta^{1,0}, \eta^{2,0})$ and h, we can solve the beam equation. Next, knowning η, we can find solutions to the Stokes system with right-hand side $\mathbf{f}$, initial data $\mathbf{v}^0$ and a boundary condition depending on η_t.

This idea cannot be applied directly with isomorphism theorems for the two equations separately. Indeed, we get first

Proposition 2.6. *Let* $(\eta^{1,0}, \eta^{2,0})$ *be in* $H_s = H^3_{(0)}(\Gamma^s_0) \times H^1_{(0)}(\Gamma^s_0)$. *For* $\gamma_s \overline{p}$, h *in* $L^2(0,T; L^2_0(\Gamma^s_0))$, *equation*

$$\begin{array}{rcll} \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha M_s \eta_{xxxx} & = & \gamma_s \overline{p} + h & \text{on } \Sigma^{s,0}_T, \\ \eta(0) & = & \eta^{1,0} & \text{in } \Gamma^s_0, \\ \eta_t(0) & = & \eta^{2,0} & \text{in } \Gamma^s_0 \end{array}$$

admits a solution η *in* $H^{4,2}_{(0)}(\Sigma^{s,0}_T)$ *satisfying the estimate*

$$\|\eta\|_{H^{4,2}_{(0)}(\Sigma^{s,0}_T)} \leq C\Big(\|(\eta^{1,0}, \eta^{2,0})\|_{H^s} + \|h\|_{L^2(\Sigma^{s,0}_T)} + \|\gamma_s \overline{p}\|_{L^2(\Sigma^{s,0}_T)} \Big).$$

Then, the result for the Stokes equations is the following.

Proposition 2.7. *Let* $\mathbf{v}^0$ *be in* $\mathbf{V}^1(\Omega_0)$. *For* $\mathbf{f}$ *and* g *respectively in* $\mathbf{L}^2(Q^0_T)$ *and* $H^{2,1}_{(0)}(\Sigma^{s,0}_T)$ *with the compatibility condition* $\mathbf{v}^0 = g(0)\mathbf{e}_2$ *on* Γ^s_0 *and* $\mathbf{v}^0 = \mathbf{0}$ *on* Γ, *then the system*

$$\begin{array}{rcll} \mathbf{v}_t - \mathit{div}\, \sigma(\mathbf{v}, p) & = & \mathbf{f} & \text{in } Q^0_T, \\ \mathit{div}\, \mathbf{v} & = & 0 & \text{in } Q^0_T, \\ \mathbf{v} & = & g\mathbf{e}_2 & \text{on } \Sigma^{s,0}_T, \\ \mathbf{v} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{v}(0) & = & \mathbf{v}^0 & \text{in } \Omega_0 \end{array} \tag{2.26}$$

admits a unique solution $(\mathbf{v}, p)$ *in* $\mathbf{V}^{2,1}(Q^0_T) \times L^2(0,T; \mathcal{H}^1(\Omega_0))$ *and furthermore*

$$\|(\mathbf{v}, p)\|_{\mathbf{V}^{2,1}(Q^0_T) \times L^2(0,T;\mathcal{H}^1(\Omega_0))} \leq C\Big(\|\mathbf{v}^0\|_{\mathbf{V}^1(\Omega_0)} + \|g\|_{H^{2,1}_{(0)}(\Sigma^{s,0}_T)} + \|\mathbf{f}\|_{\mathbf{L}^2(Q^0_T)} \Big).$$

The first proposition comes from regularity results for the beam equation proved in Proposition 2.13. The second proposition is a result from [22] in the case when g belongs to $H^{2,1}_{(0)}(\Sigma^{s,0}_T)$.
To conclude, the solution $(\mathbf{v}, p, \eta)$ of system (2.25) obeys

$$\|(\mathbf{v}, p, \eta)\|_{X_T} \leq C\Big(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T} + \|\overline{p}\|_{L^2(0,T;\mathcal{H}^1(\Omega_0))} \Big).$$

Thus this method gives directly the solution of system (2.25) in the expected spaces (thanks to the isomorphism theorems) but we cannot act on the constant C to get a contraction. That is why we have to consider a new equivalent system.

2.5.2 New equivalent system.

Let us define the so-called Leray projection P from $\mathbf{L}^2(\Omega_0)$ in $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$ where

$$\mathbf{V}^0_{\mathbf{n}}(\Omega_0) = \Big\{ \mathbf{u} \in \mathbf{L}^2(\Omega_0) \text{ such that } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_0 \text{ and div } \mathbf{u} = 0 \text{ in } \Omega_0 \Big\}.$$

We want to split system (2.21) into two parts in order to construct a contraction mapping acting on a part of the pressure term only. More precisely, following the idea of [22, 23], the Stokes system can be expressed in terms of $\mathbf{v}_e = P\mathbf{v}$, $\mathbf{v}_s = (I - P)\mathbf{v}$ and their associated pressures p_e, p_s, then we will construct a contraction mapping acting on p_e to obtain the expected result.

To express simply the Stokes system in the variables $(\mathbf{v}_e, \mathbf{v}_s, p_e, p_s)$, we have to introduce some operators. Let us note N the operator defined from $H^\sigma(\Gamma_0)$ to $H^{\sigma+3/2}(\Omega_0)$ (for $\sigma \geq -1/2$) by $q = N(g)$ (for g in $H^\sigma(\Gamma_0)$) if

$$\Delta q(t) = 0 \quad \text{in } \Omega_0 \qquad \text{and} \qquad \frac{\partial q(t)}{\partial \mathbf{n}} = g \quad \text{on } \Gamma_0. \tag{2.27}$$

Then, for $\mathbf{z}$ in $\mathbf{L}^2(\Omega_0)$, the solution π of

$$\Delta \pi = \text{div } \mathbf{z} \quad \text{in } \Omega_0 \qquad \text{and} \qquad \frac{\partial \pi}{\partial \mathbf{n}} = \mathbf{z} \cdot \mathbf{n} \quad \text{on } \Gamma_0$$

is a sum of two terms π_1 and π_2 in $H^1(\Omega_0)$ satisfying

$$\pi_1 \in H_0^1(\Omega_0) \quad \text{and} \quad \Delta\pi_1 = \text{div}\,\mathbf{z} \quad \text{in } \Omega_0 \qquad \text{and} \qquad \pi_2 = N\Big((\mathbf{z} - \nabla\pi_1)\cdot\mathbf{n}\Big).$$

Setting $\pi_1 = -(-\Delta_D)^{-1}(\text{div}\,\mathbf{z})$, we get $\pi_2 = N((\mathbf{z} + \nabla(-\Delta_D)^{-1}(\text{div}\,\mathbf{z}))\cdot\mathbf{n})$. Thus, we can define the operator π from $\mathbf{L}^2(\Omega_0)$ into $H^1(\Omega_0)$ by

$$\pi(\mathbf{z}) = -(-\Delta_D)^{-1}(\text{div}\,\mathbf{z}) + N((\mathbf{z} + \nabla(-\Delta_D)^{-1}(\text{div}\,\mathbf{z}))\cdot\mathbf{n}) \quad \text{for } \mathbf{z} \in \mathbf{L}^2(\Omega_0). \tag{2.28}$$

Finally we note N_s the restriction on $H^\sigma(\Gamma_0^s)$ of N, that is $N_s(g) = N(g\chi_s)$ for any g in $H^\sigma(\Gamma_0^s)$ ($\sigma \geq -1/2$).

With these notations, system $(2.21)_{1-5}$ is equivalent to

$$\begin{array}{rcll} \mathbf{v}_{e,t} - \nu\Delta\mathbf{v}_e + \nabla p_e &=& P\mathbf{f} & \text{in } Q_T^0, \\ \mathbf{v}_e &=& -\gamma_\tau\mathbf{v}_s & \text{on } \Sigma_T^0, \\ \mathbf{v}_e(0) &=& P\mathbf{v}^0 & \text{in } \Omega_0, \\ \mathbf{v}_s &=& \nabla N_s(\eta_t) & \text{in } {}_T^0, \\ p_s &=& \pi(\mathbf{f}) - N_s(\eta_{tt}) & \text{in } Q_T^0, \\ p &=& p_e + p_s & \text{in } Q_T^0. \end{array} \tag{2.29}$$

The explications to obtain this system are detailed in [22].

The pressure term in the right-hand side of the beam equation is

$$\gamma_s p = \gamma_s p_e + \gamma_s\pi(\mathbf{f}) - \gamma_s N_s(\eta_{tt}).$$

System (2.21) is equivalent to the following system in terms of $(\mathbf{v}_e, p_e, \mathbf{v}_s, p_s, \eta)$:

$$\begin{array}{rcll} \mathbf{v}_{e,t} - \nu\Delta\mathbf{v}_e + \nabla p_e &=& P\mathbf{f} & \text{in } Q_T^0, \\ \mathbf{v}_e &=& -\gamma_\tau\mathbf{v}_s & \text{on } \Sigma_T^0, \\ \mathbf{v}_e(0) &=& P\mathbf{v}^0 & \text{in } \Omega_0, \\ \mathbf{v}_s &=& \nabla N_s(\eta_t) & \text{in } Q_T^0, \\ (I + \gamma_s N_s)\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha M_s\eta_{xxxx} &=& \gamma_s p_e + \tilde{h} & \text{on } \Sigma_T^{s,0}, \\ \eta(0) &=& \eta^{1,0} & \text{in } \Gamma_0^s, \\ \eta_t(0) &=& \eta^{2,0} & \text{in } \Gamma_0^s, \\ p &=& p_e + p_s & \text{in } Q_T^0, \\ \mathbf{v} &=& \mathbf{v}_e + \mathbf{v}_s & \text{in } Q_T^0, \\ p_s &=& \pi(\mathbf{f}) - N_s(\eta_{tt}) & \text{in } Q_T^0 \end{array} \tag{2.30}$$

with

$$\tilde{h} = h + \gamma_s\pi(\mathbf{f}). \tag{2.31}$$

We want to find solutions to system (2.30). With $(\mathbf{f}, h)$ and $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ fixed, our method is to set the pressure term $\overline{p}_e \in L^2(0,T;\mathcal{H}^1(\Omega_0))$ only in the right-hand side of the beam equation. Then, considering $\overline{p}_e$ only in $L^{2-\varepsilon}(0,T,\mathcal{H}^1(\Omega_0))$ (for a small parameter $\varepsilon > 0$), we find a solution η of the modified beam equation in a space E_T^ε. The next step is, with η in E_T^ε, to get $\mathbf{v}_e$, $\mathbf{v}_s$ and p_e respectively in $\mathbf{V}^{2,1}(Q_T^0)$, $L^2(0,T;\mathbf{H}^2(\Omega_0)) \cap H^{3/4}(0,T;\mathbf{H}^{1/2}(\Omega_0))$ and $L^2(0,T;\mathcal{H}^1(\Omega_0))$. All this results will allow us to define a contraction mapping from a ball of the space of pressure term $L^2(0,T;\mathcal{H}^1(\Omega_0))$ into itself for a small time T_0 in $(0,T)$. Then, because of the linearity of system (2.30), we will have the existence and uniqueness of a strong solution in $(0,T)$ corresponding with fixed initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ in $X_{\mathbf{cc}}^0$ and right-hand members $(\mathbf{f}, h)$ in Z_T.

2.5.3 Existence of solutions for each part of (2.21) and estimates.

We begin this loop by fixing a pressure term $\overline{p}_e$ in the beam equation. We will suppose that $\overline{p}_e$ belongs to $L^2(0,T;\mathcal{H}^1(\Omega_0))$. By a classic embedding theorem, we get $\gamma_s\overline{p}_e \in L^{2-\varepsilon}\Big(0,T;H_{(0)}^{1/2}(\Gamma_0^s)\Big)$ for any $0 < \varepsilon < 1$. Then we have the estimate

$$\|\gamma_s\overline{p}_e\|_{L^{2-\varepsilon}\left(0,T;H_{(0)}^{1/2}(\Gamma_0^s)\right)} \leq CT^\theta\|\gamma_s\overline{p}_e\|_{L^2\left(0,T;H_{(0)}^{1/2}(\Gamma_0^s)\right)} \quad \text{for } \theta = \frac{1}{2-\varepsilon} - \frac{1}{2}. \tag{2.32}$$

Thus, we can prove the following proposition:

Proposition 2.8. *Let $0<\varepsilon<1$, $(\eta^{1,0},\eta^{2,0})$ in H_s and $(\mathbf{f},h)$ in Z_T defined in Proposition 2.6 and in (2.22). Then, first $\tilde{h}$ defined by (2.31) is in $L^2(0,T;H^{1/2}_{(0)}(\Gamma^s_0))$ and Second, with $\overline{p}_e$ in $L^{2-\varepsilon}(0,T;\mathcal{H}^1(\Omega_0))$, the equation*

$$\begin{array}{rcll}(I+\gamma_s N_s)\eta_{tt}-\beta\eta_{xx}-\gamma\eta_{txx}+\alpha M_s\eta_{xxxx} & = & \gamma_s\overline{p}_e+\tilde{h} & \text{on } \Sigma^{s,0}_T,\\ \eta(0) & = & \eta^{1,0} & \text{in } \Gamma^s_0,\\ \eta_t(0) & = & \eta^{2,0} & \text{in } \Gamma^s_0\end{array} \tag{2.33}$$

admits a unique solution η in

$$E^\varepsilon_T=L^{2-\varepsilon}(0,T;H^{4+\varepsilon/2}_{(0)}(\Gamma^s_0))\cap W^{2,2-\varepsilon}(0,T;H^{\varepsilon/2}_{(0)}(\Gamma^s_0)), \tag{2.34}$$

satisfying

$$\|\eta\|_{E^\varepsilon_T}\leq C\Big(\|(\eta^{1,0},\eta^{2,0})\|_{H_s}+\|\overline{p}_e\|_{L^{2-\varepsilon}(0,T;\mathcal{H}^1(\Omega_0))}+\|\tilde{h}\|_{L^{2-\varepsilon}(0,T;H^{1/2}_{(0)}(\Gamma^s_0))}\Big). \tag{2.35}$$

Furthermore, η_t belongs to $H^{3/2,3/4}_{(0)}(\Sigma^{s,0}_T)$.

Proof. First, $\tilde{h}$ is in $L^2(0,T;H^{1/2}_{(0)}(\Gamma^s_0))$ thanks to the regularity of $\mathbf{f}$ and h *via* formula (2.31).
We want to rewrite (2.33) as a first order system. For that, we set

$$Y(t)=\begin{pmatrix}\eta(t)\\ \eta_t(t)\end{pmatrix},\quad Y^0=\begin{pmatrix}\eta^{1,0}\\ \eta^{2,0}\end{pmatrix}$$

and

$$\mathcal{A}=\begin{pmatrix}0 & I\\ (I+\gamma_s N_s)^{-1}(-\alpha M_s\Delta^2+\beta\Delta) & \gamma(I+\gamma_s N_s)^{-1}\Delta\end{pmatrix}. \tag{2.36}$$

Then $D(-\mathcal{A})=H^4_{(0)}(\Gamma^s_0)\times H^2_{(0)}(\Gamma^s_0)$. We note $\mathbb{H}=H^2_{(0)}(\Gamma^s_0)\times L^2_0(\Gamma^s_0)$. Y is the solution of the equation

$$\begin{array}{rcll}Y'(t) & = & \mathcal{A}Y(t)+\begin{pmatrix}0\\ (I+\gamma_s N_s)^{-1}(\gamma_s\overline{p}_e+\tilde{h})\end{pmatrix} & \text{on } \Sigma^{s,0}_T,\\ Y(0) & = & Y_0 & \text{in } \Gamma^s_0.\end{array} \tag{2.37}$$

We use the well-known Duhamel's formula, with $\mathcal{B}=\begin{pmatrix}0\\ (I+\gamma_s N_s)^{-1}(\gamma_s\overline{p}_e+\tilde{h})\end{pmatrix}$,

$$Y(t)=\mathrm{e}^{t\mathcal{A}}Y_0+\int_0^t \mathrm{e}^{(t-\tau)\mathcal{A}}\mathcal{B}(\tau)\mathrm{d}\tau.$$

For $\kappa>0$, we have formally

$$(-\mathcal{A})^\kappa Y(t)=(-\mathcal{A})^\kappa\mathrm{e}^{t\mathcal{A}}Y_0+\int_0^t(-\mathcal{A})^\kappa\mathrm{e}^{(t-\tau)\mathcal{A}}\mathcal{B}(\tau)\mathrm{d}\tau$$

and because Y_0 is in $[D(-\mathcal{A}),\mathbb{H}]_{1/2}$ and $\mathcal{B}(\tau)$ is in $[D(-\mathcal{A}),\mathbb{H}]_{3/4}$, we get

$$(-\mathcal{A})^\kappa Y(t)=(-\mathcal{A})^{\kappa-1/2}\mathrm{e}^{t\mathcal{A}}(-\mathcal{A})^{1/2}Y_0+\int_0^t(-\mathcal{A})^{\kappa-1/4}\mathrm{e}^{(t-\tau)\mathcal{A}}(-\mathcal{A})^{1/4}\mathcal{B}(\tau)\mathrm{d}\tau.$$

Now using triangle inequality in $\mathbb{H}$ we have for $r>1$

$$\begin{array}{rl} & \|(-\mathcal{A})^\kappa Y(t)\|^r_{\mathbb{H}}\\ \leq & c\Big(\Big\|(-\mathcal{A})^{\kappa-1/2}\mathrm{e}^{t\mathcal{A}}\Big\|^r_{\mathcal{L}(\mathbb{H})}\Big\|(-\mathcal{A})^{1/2}Y_0\Big\|^r_{\mathbb{H}}+\Big\|\int_0^t(-\mathcal{A})^{\kappa-1/4}\mathrm{e}^{(t-\tau)\mathcal{A}}(-\mathcal{A})^{1/4}\mathcal{B}(\tau)\mathrm{d}\tau\Big\|^r_{\mathbb{H}}\Big).\end{array}$$

Because $(-\mathcal{A})$ is a generator of an analytic semigroup (see the proof in [23] which relies on a result in [8]), we get the estimates (see [20]):

$$\left\|(-\mathcal{A})^{\kappa}\mathrm{e}^{t\mathcal{A}}\right\|_{\mathcal{L}(\mathbb{H})} \leq \frac{c}{t^{\kappa}} \quad \text{for } \kappa > 0.$$

With the Young's formula, we have:

$$\begin{aligned}
&\int_0^T \left\|(-\mathcal{A})^{\kappa}Y(t)\right\|_{\mathbb{H}}^r \mathrm{d}t \\
\leq\ & \int_0^T \left\|(-\mathcal{A})^{\kappa-1/2}\mathrm{e}^{t\mathcal{A}}\right\|_{\mathcal{L}(\mathbb{H})}^r \mathrm{d}t \left\|(-\mathcal{A})^{1/2}Y_0\right\|_{\mathbb{H}}^r \\
&\qquad + \left(\int_0^T \left\|(-\mathcal{A})^{\kappa-1/4}\mathrm{e}^{(\cdot)\mathcal{A}}\right\|_{\mathcal{L}(\mathbb{H})}^p\right)^{r/p} \left(\int_0^T \left\|(-\mathcal{A})^{1/4}\mathcal{B}(\cdot)\right\|_{\mathbb{H}}^q\right)^{r/q} \\
\leq\ & c\int_0^T \frac{\mathrm{d}t}{t^{(\kappa-\frac{1}{2})r}} \left\|(-\mathcal{A})^{1/2}Y_0\right\|_{\mathbb{H}}^r + c\left\|t \mapsto \frac{1}{t^{(\kappa-\frac{1}{4})p}}\right\|_{L^p(0,T)}^r \left\|(-\mathcal{A})^{1/4}\mathcal{B}(\cdot)\right\|_{L^q(0,T;\mathbb{H})}^r
\end{aligned}$$

with $1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$ and κ has to satisfy

$$\left\{\begin{array}{lcl} (\kappa-\frac{1}{2})r & < & 1 \\ (\kappa-\frac{1}{4})p & < & 1 \end{array}\right. .$$

Then the triplet $(p,q,r) = (1, 2-\varepsilon, 2-\varepsilon)$ is suitable. For this choice, κ has only to obey $\kappa < 1+\frac{\varepsilon}{4-2\varepsilon}$ and thus $\kappa = 1+\varepsilon/4$ is convenient. This gives us first

$$Y \in L^{2-\varepsilon}(0,T;[D((-\mathcal{A})^2), D(-\mathcal{A})]_{1-\varepsilon/4}) = L^{2-\varepsilon}(0,T;H^{4+\varepsilon/2}_{(0)}(\Gamma^s_0)\times H^{2+\varepsilon/2}_{(0)}(\Gamma^s_0))$$

and second

$$Y' \in L^{2-\varepsilon}(0,T;[D(-\mathcal{A}),\mathbb{H}]_{1-\varepsilon/4}) = L^{2-\varepsilon}(0,T;H^{2+\varepsilon/2}_{(0)}(\Gamma^s_0)\times H^{\varepsilon/2}_{(0)}(\Gamma^s_0)).$$

Thus, η solution of (2.33) belongs to $L^{2-\varepsilon}(0,T;H^{4+\varepsilon/2}_{(0)}(\Gamma^s_0)) \cap W^{2,2-\varepsilon}(0,T;H^{\varepsilon/2}_{(0)}(\Gamma^s_0))$. The estimate comes from the Duhamel's formula and the different calculations above.
The last part is to prove that η_t is in $H^{3/2,3/4}_{(0)}(\Sigma^{s,0}_T)$. We use different interpolation formulas: η belongs to E^{ε}_T thus η_t is in $L^{2-\varepsilon}(0,T;H^{2+\varepsilon/2}_{(0)}(\Gamma^s_0))\cap W^{1,2-\varepsilon}(0,T;H^{\varepsilon/2}_{(0)}(\Gamma^s_0))$ which can be embedded continuously in $W^{\lambda,2-\varepsilon}(0,T;[H^{2+\varepsilon/2}_{(0)}(\Gamma^s_0), H^{\varepsilon/2}_{(0)}(\Gamma^s_0)]_{\lambda})$ for $0<\lambda<1$. A quick calculation gives us

$$[H^{2+\varepsilon/2}_{(0)}(\Gamma^s_0), H^{\varepsilon/2}_{(0)}(\Gamma^s_0)]_{\lambda} = H^{2+\varepsilon/2-2\lambda}_{(0)}(\Gamma^s_0).$$

An embedding formula in Sobolev spaces of fractional order (see [1]) gives:

$$W^{\lambda,2-\varepsilon}(0,T) \hookrightarrow W^{0,2}(0,T) \quad \text{when } \lambda = \frac{1}{2-\varepsilon}-\frac{1}{2}.$$

So $W^{\lambda,2-\varepsilon}(0,T;[H^{2+\varepsilon/2}_{(0)}(\Gamma^s_0), H^{\varepsilon/2}_{(0)}(\Gamma^s_0)]_{\lambda}) \hookrightarrow L^2(0,T;H^{3/2}_{(0)}(\Gamma^s_0))$. In the same way we can prove that $W^{\lambda,2-\varepsilon}(0,T;[H^{2+\varepsilon/2}_{(0)}(\Gamma^s_0), H^{\varepsilon/2}_{(0)}(\Gamma^s_0)]_{\lambda}) \hookrightarrow H^{3/4}(0,T;L^2_0(\Gamma^s_0))$. □

We use a new definition of solutions for the Stokes system (2.26). Indeed, we look for a solution $(\mathbf{v}_e, \mathbf{v}_s, p_e)$ of the equivalent system (see section 2.5.2)

$$\begin{array}{rcll}
\mathbf{v}_{e,t} - \nu\Delta\mathbf{v}_e + \nabla p_e & = & P\mathbf{f} & \text{in } Q^0_T, \\
\mathbf{v}_e & = & -\gamma_\tau \mathbf{v}_s & \text{on } \Sigma_T, \\
\mathbf{v}_e(0) & = & P\mathbf{v}^0 & \text{in } \Omega_0, \\
\mathbf{v}_s & = & \nabla N_s(g) & \text{in } Q^0_T, \\
\mathbf{v} & = & \mathbf{v}_e + \mathbf{v}_s & \text{in } Q^0_T, \\
p & = & p_s + p_e & \text{in } Q^0_T, \\
p_s & = & \pi(\mathbf{f}) - N_s(g_t) & \text{in } Q^0_T
\end{array} \qquad (2.38)$$

where $\pi(\mathbf{f})$ is given in (2.28). We now can state the following result on solutions of the Stokes equivalent system (2.38):

Proposition 2.9. *Let g be in $H^{3/2,3/4}_{(0)}(\Sigma^{s,0}_T)$, $\mathbf{f}$ in $\mathbf{L}^2(Q^0_T)$ and $\mathbf{v}^0$ in $\mathbf{V}^1(\Omega_0)$ with the compatibility condition $\mathbf{v}^0 = \mathbf{0}$ on Γ and $\mathbf{v}^0 = g(0)\mathbf{e}_2$ on Γ^s_0. Then,* (2.38) *admits a unique solution $(\mathbf{v}_e, \mathbf{v}_s, p_e)$ in $X^{e,s}_T = \mathbf{V}^{2,1}(Q^0_T) \times L^2(0,T;\mathbf{H}^2(\Omega_0)) \cap H^{3/4}(0,T;\mathbf{H}^{1/2}(\Omega_0)) \times L^2(0,T;\mathcal{H}^1(\Omega_0))$. We have the estimate*

$$\|(\mathbf{v}_e, \mathbf{v}_s, p_e)\|_{X^{e,s}_T} \leq c\Big(\|\mathbf{v}^0\|_{\mathbf{V}^1(\Omega_0)} + \|g\|_{H^{3/2,3/4}_{(0)}(\Sigma^{s,0}_T)} + \|\mathbf{f}\|_{\mathbf{L}^2(Q^0_T)}\Big).$$

Proof. The regularity of the Stokes system can be treated following [22]. Because we are in a rectangular domain, we have to use Lemma 3.11 in [23] to get the optimal spatial regularity of the Stokes operator. Thanks to this lemma, we know that the lifting operator $\mathbf{D}$ defined for $\mathbf{a}$ in $\mathbf{L}^2(\Omega_0)$ and r in $H^{3/2}_{(0)}(\Gamma^s_0)$ by $\mathbf{w} = \mathbf{D}(\mathbf{a}, r)$ in $\mathbf{H}^2(\Omega_0)$ if and only if there exists a function π in $\mathcal{H}^1(\Omega_0)$ such that $\mathbf{w}$ satisfies (for $\theta_0 > 0$ large)

$$\begin{array}{rcll} \theta_0\mathbf{w} - \nu\Delta\mathbf{w} + \nabla\pi & = & \mathbf{a} & \text{in } \Omega_0, \\ \text{div } \mathbf{w} & = & 0 & \text{in } \Omega_0, \\ \mathbf{w} & = & r\mathbf{e}_2 & \text{on } \Gamma^s_0, \\ \mathbf{w} & = & \mathbf{0} & \text{on } \Gamma \end{array}$$

is a continuous operator. Thus, following [22], we get that $\mathbf{v}_e$ satisfies

$$\mathbf{v}_{e,t} = \nu P\Delta\mathbf{v}_e + (\theta_0 Id + (-\nu P\Delta))P\mathbf{D}(\mathbf{f}, g), \quad \mathbf{v}_e(0) = P\mathbf{v}^0. \tag{2.39}$$

Here the Stokes operator $\nu P\Delta$ defined in $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$ with domain $\mathbf{V}^2(\Omega_0) \cap \mathbf{V}^1_0(\Omega_0)$ has been extended by transposition as an operator in $(\mathbf{V}^2(\Omega_0) \cap \mathbf{V}^1_0(\Omega_0))'$ with domain $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$. Furthermore, the Stokes operator is the generator of an analytic semigroup on $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$.

For $\mathbf{f}$ in $\mathbf{L}^2(Q^0_T)$ and g in $L^2(0,T;H^{3/2}_{(0)}(\Gamma^s_0))$ the right-hand side of (2.39) belongs to $L^2(0,T;\mathbf{V}^0_{\mathbf{n}}(\Omega_0))$ thanks to the regularity of $\mathbf{D}$. Thus, we get with $\mathbf{v}^0_e = P\mathbf{v}^0$ in $\mathbf{V}^1_0(\Omega_0)$, that $\mathbf{v}_e$ belongs to $\mathbf{V}^{2,1}(Q^0_T)$ and satisfies the estimate

$$\|\mathbf{v}_e\|_{\mathbf{V}^{2,1}(Q^0_T)} \leq C\Big(\|P\mathbf{v}^0\|_{\mathbf{V}^1(\Omega_0)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega_0)} + \|g\|_{L^2(0,T;H^{3/2}_{(0)}(\Gamma^s_0))}\Big).$$

From the expression of p_e, we directly get that p_e belongs to $L^2(0,T;\mathcal{H}^1(\Omega_0))$ and satisfies the expected estimate.

The regularity of the operator N_s gives that $\mathbf{v}_s$ belongs to $L^2(0,T;\mathbf{H}^2(\Omega_0)) \cap H^{3/4}(0,T;\mathbf{H}^{1/2}(\Omega_0))$ and satisfies the estimate

$$\|\mathbf{v}_s\|_{L^2(0,T;\mathbf{H}^2(\Omega_0))} + \|\mathbf{v}_s\|_{H^{3/4}(0,T;\mathbf{H}^{1/2}(\Omega_0))} \leq C\|g\|_{H^{3/2,3/4}_{(0)}(\Sigma^{s,0}_T)}.$$

All these results together give that $(\mathbf{v}_e, \mathbf{v}_s, p_e)$ belongs to $X^{e,s}_T$ and satisfies the estimate

$$\|(\mathbf{v}_e, \mathbf{v}_s, p_e)\|_{X^{e,s}_T} \leq c\Big(\|\mathbf{v}^0\|_{\mathbf{V}^1(\Omega_0)} + \|g\|_{H^{3/2,3/4}_{(0)}(\Sigma^{s,0}_T)} + \|\mathbf{f}\|_{\mathbf{L}^2(Q^0_T)}\Big).$$

□

2.5.4 Construction of a solution of system (2.21).

In order to prove the existence of solutions for the system (2.21), we have to construct a contraction mapping for the equivalent system (2.30). Initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ in $X^0_{\mathbf{cc}}$ and right-hand sides $(\mathbf{f}, h)$ in Z_T are fixed in this section. For $\overline{p}_e$, we consider the mapping $\mathcal{G}$ defined by

$$\begin{array}{rccc} \mathcal{G}: & L^2(0,T;\mathcal{H}^1(\Omega_0)) & \longrightarrow & X^\varepsilon_T = \Big\{(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta) \in X^{e,s}_T \times E^\varepsilon_T\Big\} \\ & \overline{p}_e & \longmapsto & (\mathbf{v}_e, \mathbf{v}_s, p_e, \eta) \text{ the solution of system (2.40)} \end{array}$$

$$\begin{array}{rcll}
\mathbf{v}_{e,t} - \nu\Delta\mathbf{v}_e + \nabla p_e &=& P\mathbf{f} & \text{in } Q_T^0,\\
\mathbf{v}_e &=& -\gamma_\tau \mathbf{v}_s & \text{on } \Sigma_T,\\
\mathbf{v}_e(0) &=& P\mathbf{v}^0 & \text{in } \Omega_0,\\
\mathbf{v}_s &=& \nabla N_s(\eta_t) & \text{in } Q_T^0,\\
(I+\gamma_s N_s)\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha M_s \eta_{xxxx} &=& \gamma_s \overline{p}_e + \tilde{h} & \text{on } \Sigma_T^{s,0},\\
\eta(0) &=& \eta^{1,0} & \text{in } \Gamma_0^s,\\
\eta_t(0) &=& \eta^{2,0} & \text{in } \Gamma_0^s,\\
p &=& p_e + p_s & \text{in } Q_T^0,\\
\mathbf{v} &=& \mathbf{v}_e + \mathbf{v}_s & \text{in } Q_T^0,\\
p_s &=& \pi(\mathbf{f}) - N_s(\eta_{tt}) & \text{in } Q_T^0
\end{array} \tag{2.40}$$

where $\tilde{h}$ is defined from $\mathbf{f}$ and h in (2.31), E_T^ε and $X_T^{e,s}$ are defined respectively in (2.34) and in Proposition 2.9.

Proposition 2.10. *The mapping $\mathcal{G}$ is well-defined from $L^2(0,T;\mathcal{H}^1(\Omega_0))$ into X_T^ε. We have moreover the estimate, for $\theta > 0$ defined in* (2.32)*:*

$$\|\mathcal{G}(\overline{p}_e)\| \le C\Big(\|(\mathbf{v}^0,\eta^{1,0},\eta^{2,0})\|_{X^0} + \|(\mathbf{f},h)\|_{Z_T} + T^\theta \|\overline{p}_e\|_{L^2(0,T;\mathcal{H}^1(\Omega_0))}\Big). \tag{2.41}$$

Furthermore, for two pressures $\overline{p}_{e,1}$ and $\overline{p}_{e,2}$ in $L^2(0,T;\mathcal{H}^1(\Omega_0))$, we have $\mathcal{G}(\overline{p}_{e,1}) - \mathcal{G}(\overline{p}_{e,2}) = (\mathbf{v}_{e,1} - \mathbf{v}_{e,2}, \mathbf{v}_{s,1} - \mathbf{v}_{s,2}, p_{e,1} - p_{e,2}, \eta_1 - \eta_2)$ solution corresponding with $\mathcal{G}(\overline{p}_{e,1} - \overline{p}_{e,2})$ in (2.40) *with zero for initial data and right-hand sides. Moreover, $\mathcal{G}(\overline{p}_{e,1}) - \mathcal{G}(\overline{p}_{e,2})$ satisfies the estimate*

$$\|\mathcal{G}(\overline{p}_{e,1}) - \mathcal{G}(\overline{p}_{e,2})\|_{X_T^\varepsilon} \le cT^\theta \|\overline{p}_{e,1} - \overline{p}_{e,2}\|_{L^2(0,T;\mathcal{H}^1(\Omega_0))}.$$

Proof. Thanks to section 2.5.3, in Proposition 2.8, we get η in E_T^ε and η_t in $H^{3/2,3/4}_{(0)}(\Sigma_T^{s,0})$; Together with Proposition 2.9 (for $g = \eta_t$), it follows that $(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta)$ belongs to X_T^ε and satisfies estimate (2.41).

The proof of the second part of this proposition relies on the linearity of the system and the same propositions. □

We now are able to construct a contraction mapping from a ball of $L^2(0,T;\mathcal{H}^1(\Omega_0))$ into itself. Let us consider the linear operator $\mathcal{F}$ from $L^2(0,T;\mathcal{H}^1(\Omega_0))$ into itself defined by

$$\mathcal{F} = \mathcal{P} \circ \mathcal{G}$$

where $\mathcal{P}$ is the projection from X_T^ε into $L^2(0,T;\mathcal{H}^1(\Omega_0))$ defined obviously by

$$\mathcal{P}(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta) = p_e.$$

We detail some properties on $\mathcal{F}$ in the proposition:

Proposition 2.11. *$\mathcal{F}$ is well-defined from $L^2(0,T;\mathcal{H}^1(\Omega_0))$ into itself and, for any $R > 0$, there exists a time $T_0 > 0$ such that $\mathcal{F}$ is a contraction in*

$$\mathcal{B}_{L^2(0,T_0;\mathcal{H}^1(\Omega_0))}(R) = \Big\{ q_e \in L^2(0,T_0;\mathcal{H}^1(\Omega_0)) \text{ such that } \|q_e\|_{L^2(0,T_0;\mathcal{H}^1(\Omega_0))} \le R \Big\}.$$

Proof. **Step 1:** The well-posedness of $\mathcal{F}$ comes from Proposition 2.10.
Step 2: Furthermore, from estimates

$$\|(\mathbf{v}_e, \mathbf{v}_s, p_e)\|_{X_T^{e,s}} \le c\Big(\|\mathbf{v}^0\|_{\mathbf{V}^1(\Omega_0)} + \|\eta\|_{E_T^\varepsilon} + \|\mathbf{f}\|_{\mathbf{L}^2(Q_T^0)}\Big)$$

and

$$\|\eta\|_{E_T^\varepsilon} \le C\Big(\|(\eta^{1,0},\eta^{2,0})\|_{H_s} + \|\gamma_s \overline{p}_e\|_{L^{2-\varepsilon}(0,T;H^{1/2}_{(0)}(\Gamma_0^s))}\Big)$$

we get

$$\|(\mathbf{v}_e, p_e, \eta)\|_{X_T^\varepsilon} \le C\Big(\|(\mathbf{v}^0,\eta^{1,0},\eta^{2,0})\|_{X^0} + \|(\mathbf{f},h)\|_{Z_T} + T^\theta \|\overline{p}_e\|_{L^2(0,T;\mathcal{H}^1(\Omega_0))}\Big)$$

and thanks to

$$\|p_e\|_{L^2(0,T;\mathcal{H}^1(\Omega_0))} \leq C\|(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta)\|_{X_T^\varepsilon},$$

we have finally

$$\begin{array}{ll} & \|p_e\|_{L^2(0,T;\mathcal{H}^1(\Omega_0))} \\ \leq & C\Big(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T} + T^\theta \|\overline{p}_e\|_{L^2(0,T;\mathcal{H}^1(\Omega_0))}\Big). \end{array} \tag{2.42}$$

Thus, we now introduce $R > 0$ such that $C(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T}) \leq R/2$. If we take $\overline{p}_e$ in $\mathcal{B}_{L^2(0,T;\mathcal{H}^1(\Omega_0))}(R)$ then, for any time T_0, we get

$$\|p_e\|_{L^2(0,T_0;\mathcal{H}^1(\Omega_0))} \leq R/2 + CT_0^\theta R$$

which gives

$$\|p_e\|_{L^2(0,T_0;\mathcal{H}^1(\Omega_0))} < R$$

for T_0 such that $CT_0^\theta < 1/2$ for instance.

Step 3: The contraction is obtained for two pressure terms $\overline{p}_{e,1}$,$\overline{p}_{e,2}$ thanks to the second part of Proposition 2.10. Indeed, we have for two pressures $\overline{p}_{e,1}$ and $\overline{p}_{e,2}$ in $L^2(0,T;\mathcal{H}^1(\Omega_0))$, the estimate:

$$\|\mathcal{F}(\overline{p}_{e,1}) - \mathcal{F}(\overline{p}_{e,2})\|_{L^2(0,T;\mathcal{H}^1(\Omega_0))} \leq cT^\theta \|\overline{p}_{e,1} - \overline{p}_{e,2}\|_{L^2(0,T;\mathcal{H}^1(\Omega_0))}.$$

Thus, for T_0 such that $cT_0^\theta < 1/2$, we get the contraction. □

We have now all the arguments to prove Theorem 2.5.

Proof of Theorem 2.5. By the Banach fixed point theorem, Proposition 2.11 is equivalent to the existence of a unique solution $(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta)$ in $X_{T_0}^\varepsilon$ of system (2.40) on $(0, T_0)$. To get the existence of solutions on $(0, T)$, we use the same idea that in Proposition 2.11 but initializing with $\overline{p}_e$ on $(0, 2T_0)$ defined by $\overline{p}_e = p_e$ on $(0, T_0)$ (with p_e coming from the solution $(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta)$ obtained above) and $\overline{p}_e = 0$ on $(T_0, 2T_0)$. By linearity of the system, the same estimates occur and we have the existence and uniqueness on $(0, 2T_0)$ in $X_{2T_0}^\varepsilon$. Step by step, we get the existence of a solution $(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta)$ of (2.30) in X_T^ε.

To conclude the proof of Theorem 2.5, we need to prove the regularity of the solution $(\mathbf{v}, p, \eta)$ of system (2.21) with $\mathbf{v} = \mathbf{v}_e + \mathbf{v}_s$ and $p = p_e + p_s$. We already have $(\mathbf{v}_e, \mathbf{v}_s, p_e, \eta) \in X_T^\varepsilon = \mathbf{V}^{2,1}(Q_T^0) \times L^2(0, T; \mathbf{H}^2(\Omega_0)) \cap H^{3/4}(0, T; \mathbf{H}^{1/2}(\Omega_0)) \times L^2(0, T; \mathcal{H}^1(\Omega_0)) \times E_T^\varepsilon$. Now, we use the theorem

Theorem 2.12. *Assume that $\mathcal{A}$ is the generator of a analytic semigroup, $\mathcal{B} \in L^2(0, T; \mathbb{H})$ and $Y^0 \in [D(\mathcal{A}), \mathbb{H}]_{1/2}$. Then the problem*

$$\begin{array}{rcl} Y'(t) &=& \mathcal{A}Y(t) + \mathcal{B}(t) \\ Y(0) &=& Y^0 \end{array}$$

has a unique solution in $H^1(0, T; \mathbb{H}) \cap L^2(0, T; D(-\mathcal{A}))$.

In our case, remember that $D(-\mathcal{A}) = H^4_{(0)}(\Gamma_0^s) \times H^2_{(0)}(\Gamma_0^s)$ where $\mathcal{A}$ is defined in (2.36), $\mathbb{H} = H^2_{(0)}(\Gamma_0^s) \times L^2_0(\Gamma_0^s)$ and $\mathcal{B} = (0, (I + \gamma_s N_s)^{-1}(\gamma_s p_e + \tilde{h}))^T$. Then, we have $[D(-\mathcal{A}), \mathbb{H}]_{1/2} = H_s$ and the following proposition

Proposition 2.13. *Let $(\eta^{1,0}, \eta^{2,0})$ be in H_s. For p_e in $L^2(0, T; \mathcal{H}^1(\Omega_0))$ and $\tilde{h}$ in $L^2(0, T; L^2_0(\Gamma_0^s))$, equation*

$$\begin{array}{rcll} (I + \gamma_s N_s)\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha M_s \eta_{xxxx} &=& \gamma_s p_e + \tilde{h} & \text{on } \Sigma_T^{s,0}, \\ \eta(0) &=& \eta^{1,0} & \text{in } \Gamma_0^s, \\ \eta_t(0) &=& \eta^{2,0} & \text{in } \Gamma_0^s \end{array}$$

admits a solution η in $H^{4,2}_{(0)}(\Sigma_T^{s,0})$ satisfying the estimate

$$\|\eta\|_{H^{4,2}_{(0)}(\Sigma_T^{s,0})} \leq C\Big(\|(\eta^{1,0}, \eta^{2,0})\|_{H^s} + \|\tilde{h}\|_{L^2(0,T;L^2_0(\Gamma_0^s))} + \|p_e\|_{L^2(0,T;\mathcal{H}^1(\Omega_0))}\Big).$$

The regularity of η gives η_t in $H^1(0, T; L^2_0(\Gamma_0^s))$ and then $\mathbf{v}_s$ in $H^1(0, T; \mathbf{H}^{1/2}(\Omega_0))$. Consequently, we have $(\mathbf{v}_s, p_s)$ in $\mathbf{H}^{2,1}(Q_T^0) \times L^2(0, T; \mathcal{H}^1(\Omega_0))$.

Thus, the solution $(\mathbf{v}, p, \eta)$ of (2.21) belongs to X_T. The estimate of $(\mathbf{v}, p, \eta)$ in X_T comes from all the previous one. □

2.6 Proof of Theorems 2.2 and 2.3 in the fixed domain Q_T^0.

In this section, we want to prove Theorems 2.2 and 2.3 in the fixed domain in the sense of Definition 2.4. That is we will find solution $(\mathbf{u}, p, \eta)$ of system (2.13). We will use a fixed point method from a space of solutions of system (2.21) into itself. We begin the proof by an estimate on $(\mathbf{F}, \mathbf{w}, h)$ where $(\mathbf{F}, \mathbf{w})$ are defined in (2.11) and $h = \gamma_s H$ with H defined in (2.12):

Proposition 2.14. *For $(\mathbf{u}, p, \eta)$ in X_T, $(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], h[\mathbf{u}, \eta])$ belongs to*

$$W_T = \Big\{(\mathbf{G}, \mathbf{z}, K) \in \mathbf{L}^2(Q_T^0) \times \mathbf{H}^{2,1}(Q_T^0) \times L^2(0,T; H^{1/2}_{(0)}(\Gamma_0^s)) \text{ such that } \mathbf{z} = 0 \text{ on } \Gamma_0\Big\} \tag{2.43}$$

and there exists $\delta > 0$ such that

$$\|(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], h[\mathbf{u}, \eta])\|_{W_T} \le CT^\delta (1 + \|(\mathbf{u}, p, \eta)\|_{X_T}) \|(\mathbf{u}, p, \eta)\|^2_{X_T}. \tag{2.44}$$

Let $(\mathbf{u}_1, p_1, \eta_1)$ and $(\mathbf{u}_2, p_2, \eta_2)$ be two triplets in X_T such that for $i = 1, 2$

$$\|(\mathbf{u}_i, p_i, \eta_i)\|_{X_T} \le R$$

for some $R > 0$, we get

$$\|(\mathbf{F}_1, \mathbf{w}_1, h_1) - (\mathbf{F}_2, \mathbf{w}_2, h_2)\|_{W_T} \le C(1 + R) R T^\delta \|(\mathbf{u}_1, p_1, \eta_1) - (\mathbf{u}_2, p_2, \eta_2)\|_{X_T} \tag{2.45}$$

with the notations $(\mathbf{F}_i, \mathbf{w}_i, h_i) = (\mathbf{F}[\mathbf{u}_i, p_i, \eta_i], \mathbf{w}[\mathbf{u}_i, \eta_i], h[\mathbf{u}_i, \eta_i])$.

To prove Proposition 2.14, we use two lemmas.

Lemma 2.15. *For $0 < \varepsilon' < \varepsilon$, we get $H^{1/2+\varepsilon}(0,T) \hookrightarrow H^{1/2+\varepsilon'}(0,T)$ and if a belongs to $H^{1/2+\varepsilon}(0,T)$ then*

$$\|a\|_{H^{1/2+\varepsilon'}(0,T)} \le cT^{(1-\theta)/2} \|a\|_{H^{1/2+\varepsilon}(0,T)} \quad \text{where } \theta = \frac{1/2 + \varepsilon'}{1/2 + \varepsilon}.$$

Lemma 2.16. *Let $\mathbf{b}$ and a be respectively in $\mathbf{H}^{1,1/2}(Q_T^0)$ and $H^{2,1}(Q_T^0)$, then $a\mathbf{b}$ belongs to $\mathbf{L}^2(Q_T^0)$ and there exists $\delta > 0$ such that*

$$\|a\mathbf{b}\|_{\mathbf{L}^2(Q_T^0)} \le CT^\delta \|a\|_{H^{2,1}(Q_T^0)} \|\mathbf{b}\|_{\mathbf{H}^{1,1/2}(Q_T^0)}.$$

Proof of Lemma 2.15. By interpolation,

$$H^{1/2+\varepsilon'}(0,T) = [H^{1/2+\varepsilon}(0,T), L^2(0,T)]_{1-\theta} \quad \text{where } \theta = \frac{1/2 + \varepsilon'}{1/2 + \varepsilon} \quad (0 < \theta < 1).$$

and then if a is in $H^{1/2+\varepsilon}(0,T)$ then a is in the interpolated space $H^{1/2+\varepsilon'}(0,T)$ with the estimate

$$\|a\|_{H^{1/2+\varepsilon'}(0,T)} \le C \|a\|^\theta_{H^{1/2+\varepsilon}(0,T)} \|a\|^{1-\theta}_{L^2(0,T)}.$$

On the other hand, the embedding $L^\infty(0,T) \hookrightarrow L^2(0,T)$ and an Hlder inequality in $(0,T)$ of finite mass gives $\|a\|_{L^2(0,T)} \le CT^{1/2} \|a\|_{L^\infty(0,T)}$. The embedding $H^{1/2+\varepsilon}(0,T) \hookrightarrow L^\infty(0,T)$ concludes. □

Proof of Lemma 2.16. By Theorem B.3 in [12], for $\mathbf{b} \in \mathbf{H}^{1,1/2}(Q_T^0)$ and $a \in H^{2,1}(Q_T^0)$, then $a\mathbf{b}$ belongs to $\mathbf{H}^{1-2\kappa, 1/2-\kappa}(Q_T^0)$ for $0 \le \kappa < 1/2$. We now use the two following classical embeddings:
– $H^{1/2-\kappa}(0,T;\mathbb{R}) \hookrightarrow L^{1/\kappa}(0,T;\mathbb{R})$ (see [1]),
– $L^{1/\kappa}(0,T;\mathbb{R}) \hookrightarrow L^2(0,T;\mathbb{R})$ (because $2 < 1/\kappa \le +\infty$) with the estimate

$$\|c\|_{L^2(0,T;\mathbb{R})} \le T^{1/2-\kappa} \|c\|_{L^{1/\kappa}(0,T;\mathbb{R})} \quad \text{for } c \in L^{1/\kappa}(0,T;\mathbb{R}).$$

These two estimates give together that $c = \|a\mathbf{b}\|_{\mathbf{L}^2(\Omega_0)}$ which is in $H^{1/2-\kappa}(0,T;\mathbb{R})$ belongs to $L^2(0,T;\mathbb{R})$ with the estimate (for $1/2-\kappa>0$)

$$\begin{aligned}\|a\mathbf{b}\|_{\mathbf{L}^2(Q_T^0)} &\leq CT^{1/2-\kappa}\|a\mathbf{b}\|_{H^{1/2-\kappa}(0,T;\mathbf{L}^2(\Omega_0))}\\ &\leq C'T^{1/2-\kappa}\|a\mathbf{b}\|_{\mathbf{H}^{1-2\kappa,1/2-\kappa}(Q_T^0)}\\ &\leq C''T^{1/2-\kappa}\|a\|_{H^{2,1}(Q_T^0)}\|\mathbf{b}\|_{\mathbf{H}^{1,1/2}(Q_T^0)}.\end{aligned}$$

□

We can now prove Proposition 2.14.

Proof of Proposition 2.14. Thanks to Lemmas 2.15 and 2.16, we can estimate the norms of the right-hand sides. We use the strong regularity of η and $\mathbf{u}$. Indeed, η in $H^{4,2}_{(0)}(\Sigma_T^{s,0})$ gives:

$$\eta \in H^{2\kappa}(0,T;H^{4(1-\kappa)}_{(0)}(\Gamma_0^s)) \quad \text{for} \quad 0<\kappa<1.$$

This gives us directly that

$$\begin{aligned}&\eta \in H^{7/4-\varepsilon/2}(0,T;H^{1/2+\varepsilon}_{(0)}(\Gamma_0^s)),\\ &\eta \in H^{5/4-\varepsilon/2}(0,T;H^{3/2+\varepsilon}_{(0)}(\Gamma_0^s)),\\ &\eta \in H^{3/4-\varepsilon/2}(0,T;H^{5/2+\varepsilon}_{(0)}(\Gamma_0^s)),\\ &\eta \in H^{1/4-\varepsilon/2}(0,T;H^{7/2+\varepsilon}_{(0)}(\Gamma_0^s)).\end{aligned} \tag{2.46}$$

The first three equations of (2.46) gives respectively η, η_x and η_{xx} in $L^\infty(\Sigma_T^{s,0})$ with the following estimates

$$\|\eta\|_{\mathbf{L}^\infty(\Sigma_T^{s,0})} + \|\eta_x\|_{\mathbf{L}^\infty(\Sigma_T^{s,0})} + \|\eta_{xx}\|_{\mathbf{L}^\infty(\Sigma_T^{s,0})} \leq cT^\chi\|\eta\|_{H^{4,2}_{(0)}(\Sigma_T^{s,0})} \quad \text{for} \quad \chi>0.$$

From the last equation in (2.46), we only get $\eta_{xxx} \in L^2(0,T;L^\infty(\Gamma_0^s))$.
Let us check some terms of $\mathbf{F}[\mathbf{u},p,\eta]$,$\mathbf{w}[\mathbf{u},\eta]$ or $h[\mathbf{u},\eta]$.
– For $\mathbf{F}[\mathbf{u},p,\eta]$, we only need to check that all the terms are in $\mathbf{L}^2(Q_T^0)$. The first term in $\mathbf{F}[\mathbf{u},p,\eta]$ is $-\eta\mathbf{u}_t$:

$$\|-\eta\mathbf{u}_t\|_{\mathbf{L}^2(Q_T^0)} \quad\leq\quad \|\eta\|_{L^\infty(\Sigma_T^{s,0})}\|\mathbf{u}_t\|_{\mathbf{L}^2(Q_T^0)}.$$

Then, *via* the embeddings $H^{\frac12+\varepsilon}(0,T) \hookrightarrow H^{\frac12+\varepsilon'}(0,T) \hookrightarrow \mathcal{C}(0,T)$ and the smoothness of η, we get

$$\|\eta\|_{L^\infty(\Sigma_T^{s,0})} \leq cT^{\frac12(1-\theta)}\|\eta\|_{H^{1/2+\varepsilon'}(0,T;H^{3-2\varepsilon'}(\Gamma_0^s))} \;\text{ for } \varepsilon'<\varepsilon \text{ and } \varepsilon \text{ such that } 0\leq\varepsilon<1.$$

Thus $\|\eta\|_{L^\infty(\Sigma_T^{s,0})} \leq cT^{\frac12(1-\theta)}\|\eta\|_{H^{4,2}_{(0)}(\Sigma_T^{s,0})}$ and finally

$$\|-\eta\mathbf{u}_t\|_{L^2(Q_T^0)} \leq cT^{\frac12(1-\theta)}\|\eta\|_{H^{4,2}_{(0)}(\Sigma_T^{s,0})}\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T^0)}.$$

Another term is $\frac{\eta_x^2}{1+\eta}\mathbf{u}_z$ which satisfies

$$\left\|\frac{\eta_x^2}{1+\eta}\mathbf{u}_z\right\|_{\mathbf{L}^2(Q_T^0)} \leq \left\|\frac{1}{1+\eta}\right\|_{L^\infty(\Sigma_T^{s,0})}\|\eta_x\|^2_{L^\infty(\Sigma_T^{s,0})}\|\mathbf{u}_z\|_{\mathbf{L}^2(Q_T^0)},$$

and becomes thanks to Lemma 2.15

$$\left\|\frac{\eta_x^2}{1+\eta}\mathbf{u}_z\right\|_{\mathbf{L}^2(Q_T^0)} \leq cT^{1-\theta}\|\eta\|^2_{H^{4,2}_{(0)}(\Sigma_T^{s,0})}\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T^0)}.$$

Terms with a product of $\mathbf{u}$ and a derivative of $\mathbf{u}$ like $(1+\|\eta\|)u_1\mathbf{u}_x$, $\eta_x u_1\mathbf{u}_z$ or $u_2\mathbf{u}_z$ must be carefully studied. Thanks to Lemma 2.16, because $\mathbf{u}$ belongs to $\mathbf{H}^{2,1}(Q_T^0)$ and then $\mathbf{u}_x$ and $\mathbf{u}_z$ are in $\mathbf{H}^{1,1/2}(Q_T^0)$, we get that $u_1\mathbf{u}_x$, $u_1\mathbf{u}_z$ and $u_2\mathbf{u}_z$ belong to $\mathbf{L}^2(Q_T^0)$ with, for $0\leq\kappa<1/2$,

$$\begin{aligned}&\|-(1+\eta)u_1\mathbf{u}_x + (z\eta_x u_1 - u_2)\mathbf{u}_z\|_{\mathbf{L}^2(Q_T^0)}\\ \leq\;& CT^{1/2-\kappa}\Big(1+\|\eta\|_{L^\infty(\Sigma_T^{s,0})}+\|\eta_x\|_{L^\infty(\Sigma_T^{s,0})}\Big)\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T^0)}\|\nabla\mathbf{u}\|_{\mathbf{H}^{1,1/2}(Q_T^0)}\\ \leq\;& CT^{1/2-\kappa}\Big(1+\|\eta\|_{H^{4,2}_{(0)}(\Sigma_T^{s,0})}\Big)\|\mathbf{u}\|^2_{\mathbf{H}^{2,1}(Q_T^0)}.\end{aligned}$$

– For $\mathbf{w}[\mathbf{u},\eta]$, we have to prove that all the terms belong to $\mathbf{H}^{2,1}(Q_T^0)$ with the expected estimate. First of all, the calculations of the derivatives of $\mathbf{w}[\mathbf{u},\eta]$ are

$$\begin{array}{rcl} \mathbf{w}_x &=& -\eta_x u_1\mathbf{e}_1 - \eta u_{1,x}\mathbf{e}_1 + z\eta_{xx}u_1\mathbf{e}_2 + z\eta_x u_{1,x}\mathbf{e}_2, \\ \mathbf{w}_z &=& -\eta u_{1,z}\mathbf{e}_1 + \eta_x u_1\mathbf{e}_2 + z\eta_x u_{1,z}\mathbf{e}_2, \\ \mathbf{w}_{xx} &=& -\eta_{xx}u_1\mathbf{e}_1 - 2\eta_x u_{1,x}\mathbf{e}_1 - \eta u_{1,xx}\mathbf{e}_1 + z\eta_{xxx}u_1\mathbf{e}_2 + 2z\eta_{xx}u_{1,x}\mathbf{e}_2 + z\eta_x u_{1,xx}\mathbf{e}_2, \\ \mathbf{w}_{zz} &=& -\eta u_{1,zz}\mathbf{e}_1 + 2\eta_x u_{1,z}\mathbf{e}_2 + \eta_x u_{1,zz}\mathbf{e}_2, \\ \mathbf{w}_t &=& -\eta_t u_1\mathbf{e}_1 - \eta u_{1,t}\mathbf{e}_1 + z\eta_{x,t}u_1\mathbf{e}_2 + z\eta_x u_{1,t}\mathbf{e}_2. \end{array} \tag{2.47}$$

Then the estimates of the derivatives in $\mathbf{L}^2(Q_T^0)$ are obtained almost all as for $\mathbf{F}[\mathbf{u},p,\eta]$. Others terms like $\eta_{xxx}\mathbf{u}_1\mathbf{e}_1$ are estimated as follows

$$\|\eta_{xxx}u_1\|_{L^2(Q_T^0)} \leq CT^\theta \|\eta_{xxx}\|_{L^2(0,T;L^\infty(\Gamma_0^s))}\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_0))}. \tag{2.48}$$

– For $h[\mathbf{u},\eta]$. We can remark that h defined in Proposition 2.14 is the trace of function H on Γ_0^s, we can prove that the lifting H of h belongs to $L^2(0,T;H^1(\Omega_0))$. We have to calculate the different terms of H and their derivatives.

$$\begin{array}{rcl} H &=& \nu\left[\dfrac{\eta_x}{1+\eta}u_{1,z} + \eta_x u_{2,x} - \dfrac{\eta_x^2-2\eta}{1+\eta}u_{2,z}\right], \\ H_x &=& \nu\left[\left(\dfrac{\eta_{xx}(1+\eta)-\eta_x^2}{(1+\eta)^2}\right)u_{1,z} + \dfrac{\eta_x}{1+\eta}u_{1,xz} + \eta_{xx}u_{2,x} + \eta_x u_{2,xx}\right. \\ && \left. - \left(2\eta_x\dfrac{\eta_{xx}-1}{1+\eta} - \eta_x\dfrac{\eta_x^2-2\eta}{(1+\eta)^2}\right)u_{2,z} - \dfrac{\eta_x^2-2\eta}{1+\eta}u_{2,xz}\right], \\ H_z &=& \nu\left[\dfrac{\eta_x}{1+\eta}u_{1,zz} + \eta_x u_{2,xz} - \dfrac{\eta_x^2-2\eta}{1+\eta}u_{2,zz}\right]. \end{array} \tag{2.49}$$

Always because of the regularity of η we get the expected estimates.

The second point comes from the at least quadratic nonlinearity of the right-hand sides with respect to $(\mathbf{u},p,\eta)$. Some calculations give estimates (2.45). □

Proposition 2.17. *For a given triplet $(\overline{\mathbf{u}},\overline{p},\overline{\eta})$ in X_T, system* (2.13) *with right-hand sides $(\overline{\mathbf{F}},\overline{\mathbf{w}},\overline{H}) = (\mathbf{F}[\overline{\mathbf{u}},\overline{p},\overline{\eta}],\mathbf{w}[\overline{\mathbf{u}},\overline{\eta}],H[\overline{\mathbf{u}},\overline{\eta}])$ and initial data $(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})$ in X^0 satisifying* (2.20) *admits a unique solution $(\mathbf{u},p,\eta)$ in X_T with the estimate*

$$\|(\mathbf{u},p,\eta)\|_{X_T} \leq c_1(\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{X^0} + c_2T^\delta(1+\|(\overline{\mathbf{u}},\overline{p},\overline{\eta})\|_{X_T})\|(\overline{\mathbf{u}},\overline{p},\overline{\eta})\|_{X_T}^2) \tag{2.50}$$

where $\delta > 0$ is defined in Proposition 2.14*. In other terms, we can construct a mapping*

$$\begin{array}{cccc} \mathcal{X}: & X_T & \longrightarrow & X_T \\ & (\overline{\mathbf{u}},\overline{p},\overline{\eta}) & \longmapsto & \begin{array}{c}\mathcal{X}(\overline{\mathbf{u}},\overline{p},\overline{\eta}) = (\mathbf{u},p,\eta) \text{ is a solution of the system } (2.13) \\ \text{with } (\mathbf{F}[\overline{\mathbf{u}},\overline{p},\overline{\eta}],\mathbf{w}[\overline{\mathbf{u}},\overline{\eta}],H[\overline{\mathbf{u}},\overline{\eta}]) \text{ for right-hand sides.}\end{array} \end{array} \tag{2.51}$$

which satisfies

$$\begin{array}{rl} & \|\mathcal{X}(\overline{\mathbf{u}},\overline{p},\overline{\eta})\|_{X_T} \\ \leq & c_1\Big(\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{X^0} + c_2T^\delta(1+\|(\overline{\mathbf{u}},\overline{p},\overline{\eta})\|_{X_T})\|(\overline{\mathbf{u}},\overline{p},\overline{\eta})\|_{X_T}^2\Big). \end{array} \tag{2.52}$$

Proof. Let us notice that $(\mathbf{u},p,\eta)$ is solution of (2.13) with right-hand sides $(\mathbf{F}[\overline{\mathbf{u}},\overline{p},\overline{\eta}],\mathbf{w}[\overline{\mathbf{u}},\overline{\eta}],H[\overline{\mathbf{u}},\overline{\eta}])$ if and only if $(\mathbf{v},p,\eta) = (\mathbf{u}-\mathbf{w}[\overline{\mathbf{u}},\overline{\eta}],p,\eta)$ is solution of (2.16) with $(\mathbf{f}[\overline{\mathbf{u}},\overline{p},\overline{\eta}],h[\overline{\mathbf{u}},\overline{\eta}])$ as right-hand sides and $(\mathbf{v}^0,\eta^{1,0},\eta^{2,0})$ for initial data (see (2.15), (2.17) and (2.18) for the definitions of $\mathbf{f}$, h and $\mathbf{v}^0$). Then this proposition relies first on the result of existence of solutions for the system (2.21) in Theorem 2.5 and Second on Proposition 2.14 for the estimate. □

We can conclude this section showing existence of solutions in the fixed domain:

Proposition 2.18. *Let* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *be in* X^0 *satisifying* (2.20).

(i) *There exists a time* $T_0 > 0$ *such that system* (2.13) *admits a unique local strong solution* $(\mathbf{u}, p, \eta)$ *in* X_{T_0}.

(ii) *There exists* r *small enough such that, under condition* $\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} \leq r$, *system* (2.13) *admits a unique global strong solution* $(\mathbf{u}, p, \eta)$ *in* X_T.

Proof. Let $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ be in X^0 satisifying (2.20). We note $r = \|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0}$ and we set $R = 2c_1 r$ (where c_1 is the constant in (2.52)).

(i) Let us define $T_0 = \left(\frac{1}{2c_1c_2R(R+1)}\right)^\delta$ and

$$B_{X_{T_0}}(R) = \Big\{(\mathbf{u}, p, \eta) \in X_{T_0} \text{ with } \|(\mathbf{u}, p, \eta)\|_{X_{T_0}} \leq R\Big\}.$$

Then, $\mathcal{X}$ is a contraction mapping in $B_{X_{T_0}}(R)$. Indeed, let $(\mathbf{u}_1, p_1, \eta_1)$ and $(\mathbf{u}_2, p_2, \eta_2)$ be two triplets in $B_{X_{T_0}}(R)$. With the previous notations, we get solutions $\mathcal{X}(\mathbf{u}_i, p_i, \eta_i)$ $(i = 1, 2)$ of system (2.13) corresponding with right-hand sides $(\mathbf{F}[\mathbf{u}_i, p_i, \eta_i], \mathbf{w}[\mathbf{u}_i, \eta_i], H[\mathbf{u}_i, \eta_i])$ $(i = 1, 2)$ and initial data $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$. Each solution obeys the estimate (2.52) thanks to Proposition 2.17 with gives for R and T_0 as above

$$\|\mathcal{X}(\mathbf{u}_i, p_i, \eta_i)\|_{X_{T_0}} \leq \frac{R}{2} + \frac{R}{2} = R.$$

Second, the difference satisfies

$$\begin{aligned} & \|\mathcal{X}(\mathbf{u}_1, p_1, \eta_1) - \mathcal{X}(\mathbf{u}_2, p_2, \eta_2)\|_{X_{T_0}} \\ \leq\ & c_1c_2T_0^\delta(1 + R)R\|(\mathbf{u}_1, p_1, \eta_1) - (\mathbf{u}_2, p_2, \eta_2)\|_{X_{T_0}} \end{aligned} \tag{2.53}$$

thanks to (2.45), that is

$$\|\mathcal{X}(\mathbf{u}_1, p_1, \eta_1) - \mathcal{X}(\mathbf{u}_2, p_2, \eta_2)\|_{X_T} \leq \frac{1}{2}\|(\mathbf{u}_1, p_1, \eta_1) - (\mathbf{u}_2, p_2, \eta_2)\|_{X_T}.$$

(ii) We choose r such that $c_2T^\delta r(1 + 2c_1 r) = 1$, that is

$$r = \frac{1}{c_1^2c_2T^\delta(1 + \sqrt{1 + \frac{2}{c_1c_2T^\delta}})}.$$

Then, $\mathcal{X}$ is a contraction mapping in $B_{X_T}(R)$ (see *(i)* for the details).

□

2.7 Back to the moving domain.

Thanks to Definition 2.4, the proof of Theorems 2.2 and 2.3 in the moving domain consists in proving that the change of variables

$$\begin{array}{rccc} \phi_t : & \Omega_0 & \longrightarrow & \Omega_{\eta(t)} \\ & (x, z) & \longmapsto & (x, y) \end{array}$$

is well-defined as a $\mathcal{C}^1-$diffeomorphism from Ω_0 into $\Omega_{\eta(t)}$ for every $t \in [0, T)$ and that condition (2.1) is checked for the solution $(\mathbf{u}, p, \eta)$ of (2.13). Then, we will have $(\tilde{\mathbf{u}}, \tilde{p}, \eta) = (\phi_t(\mathbf{u}), \phi_t(p), \eta)$ solution of (2.2)–(2.3) in $\mathbf{V}^{2,1}(Q_T^\eta) \times L^2(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1(\Omega_{\eta(t)})) \times H^{4,2}_{(0)}(\Sigma_T^{s,0})$. Furthermore, by the change of variables, we will be able to check which compatibility condition corresponds in Q_T^η to (2.20).

We have to show that condition (2.1) is checked. In the case of the existence of solutions for small data, because we have then

$$\|(\mathbf{u}, p, \eta)\|_{X_T} \leq r,$$

we easily get from $\|\eta\|_{L^\infty(\Sigma_T^{s,0})} \leq \|\eta\|_{H^{4,2}_{(0)}(\Sigma_T^{s,0})}$ (due to the continuous embedding $H^{4,2}_{(0)}(\Sigma_T^{s,0}) \hookrightarrow L^\infty(\Sigma_T^{s,0})$) that

$$\|\eta\|_{L^\infty(\Sigma_T^{s,0})} \leq r \leq 1 - \delta_0 \quad \text{for } r \text{ small enough.}$$

Condition (2.1) is checked for local solutions too thanks to the continuity of the embeddings, for $0 < \varepsilon < 1$, (see the proof of Proposition 2.14)

$$H^{4,2}_{(0)}(\Sigma^{s,0}_{T_0}) \quad \hookrightarrow \quad H^{1/2+\varepsilon}(0,T_0; H^{3-2\varepsilon}_{(0)}(\Gamma^s_0)) \quad \hookrightarrow \quad L^\infty(\Sigma^{s,0}_{T_0})$$

which gives $\|\eta\|_{L^\infty(\Sigma^{s,0}_{T_0})} \leq cT_0^\theta \|\eta\|_{H^{4,2}_{(0)}(\Sigma^{s,0}_{T_0})}$ (for $\theta > 0$) and then $\|\eta\|_{L^\infty(\Sigma^{s,0}_{T_0})} \leq cT_0^\theta R \leq 1 - \delta_0$ for T_0 small enough.
The embedding $H^{4,2}_{(0)}(\Sigma^{s,0}_T) \hookrightarrow \mathcal{C}([0,T];\mathcal{C}^1(\Gamma^s_0))$ together with the condition $1 + \eta \geq \delta_0 > 0$ show that ϕ_t is $\mathcal{C}^1$ diffeomorphism from Ω_0 into $\Omega_{\eta(t)}$.
All the derivatives of the solutions written in the variable (x,y) are combinations of those in the variable (x,z) multiplied at most by η or one of its derivatives which are smooth enough to get $(\tilde{\mathbf{u}}, \tilde{p})$ in $\mathbf{H}^{4,2}(Q^\eta_T) \times L^2(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1(\Omega_{\eta(t)}))$ (the calculations are exactely the ones proving that $\mathbf{F}[\mathbf{u},p,\eta]$ belongs to $\mathbf{L}^2(Q^0_T)$ for $(\mathbf{u},p,\eta)$ in X_T).
The compatibility conditions became after the change of variables

$$\operatorname{div} \mathbf{u}^0 = 0 \quad \text{in } \Omega_{\eta^{1,0}}, \qquad \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_2 \quad \text{on } \Gamma^s_{\eta^{1,0}} \qquad \text{and} \qquad \mathbf{u}^0 = \mathbf{0} \quad \text{on } \Gamma^0.$$

Chapitre 3

Existence et unicité de solution pour un système couplant les équations de Navier-Stokes et une équation des ondes amorties en deux dimensions

3.1 Introduction.

In this chapter, we still consider the system introduced in [21] but in a different setting. More precisely, we treat here the case $\alpha = 0$ in the beam equation (see equations (1.3), with $\sigma = 0$ too, in [4] or (2.3) in Chapter 2) in the periodic case. We prove the existence and uniqueness of strong solutions either for small initial data or for a small time of existence (see Theorems 3.2 and 3.3). This follows the different steps of the previous chapter.

First of all, as we treat in this part the periodic case (see [4]), all the functions in this chapter will be periodic in the x-variable of period $L > 0$, the length of the domain.

Let η be a function *a priori* from $(0,T) \times (0,L)$ into $\mathbb{R}$ satisfying the assumption:

$$\exists \delta_0 > 0 \text{ such that } \forall t \geq 0 \ \forall x \in (0,L) \quad 1 + \eta(t,x) \geq \delta_0 > 0. \tag{3.1}$$

The function η models the displacement of the beam in the upper part of the boundary of the domain. Assumption (3.1) ensures that the domain $\Omega_{\eta(t)}$ (see Figure 3.1) defined by

$$\Omega_{\eta(t)} = \Big\{ (x,y) \in \mathbb{R}^2 \text{ s.t. } x \in (0,L) \text{ and } 0 < y < 1 + \eta(t,x) \Big\}$$

is a connected domain for any time $t \geq 0$. We introduce the moving boundary $\Gamma^s_{\eta(t)}$ defined by

$$\Gamma^s_{\eta(t)} = \Big\{ (x,y) \in \mathbb{R}^2 \text{ s.t. } x \in (0,L) \text{ and } y = 1 + \eta(t,x) \Big\}.$$

The other part of the boundary is denoted by Γ, that is $\Gamma = (0,L) \times \{0\}$. Finally, we introduce $\Omega_0 = (0,L) \times (0,1)$ and $\Gamma^s_0 = (0,L) \times \{1\}$ respectively the reference domain and reference state of the beam corresponding with the case $\eta = 0$, that is when the beam is «at rest». We define also $\Gamma_0 = \partial\Omega_0 = \Gamma \cup \Gamma^s_0$ and give some other notations:

$$\begin{array}{lll} Q^\eta_T = \displaystyle\bigcup_{t \in (0,T)} \{t\} \times \Omega_{\eta(t)}, & \Sigma^{s,\eta}_T = \displaystyle\bigcup_{t \in (0,T)} \{t\} \times \Gamma^s_{\eta(t)}, & \Sigma_T = (0,T) \times \Gamma, \\ Q^0_T = (0,T) \times \Omega, & \Sigma^{s,0}_T = (0,T) \times \Gamma^s_0, & \Sigma^0_T = (0,T) \times \Gamma_0. \end{array}$$

We introduce here the two partial differential equations of our system. First the Navier-Stokes equa-

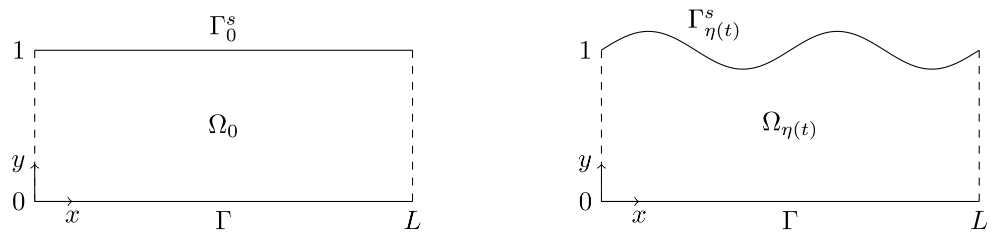

Figure 3.1: The domains Ω_0 (left) and $\Omega_{\eta(t)}$ (right).

tions in the variables $(\mathbf{u}, p)$ (respectively the velocity and the pressure of the fluid)

$$\begin{array}{rcll} \mathbf{u}_t - \text{div } \sigma(\mathbf{u}, p) + (\mathbf{u}\cdot\nabla)\mathbf{u} & = & \mathbf{0} & \text{in } Q_T^\eta, \\ \text{div } \mathbf{u} & = & 0 & \text{in } Q_T^\eta, \\ \mathbf{u} & = & \eta_t \mathbf{e}_2 & \text{on } \Sigma_T^{s,\eta}, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}(0) & = & \mathbf{u}^0 & \text{in } \Omega_{\eta^{1,0}} \end{array} \tag{3.2}$$

and second, the beam equation:

$$\begin{array}{rcll} \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} & = & -\sigma(\mathbf{u}, p)(-\eta_x \mathbf{e}_1 + \mathbf{e}_2)\cdot \mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ (\eta(0), \eta_t(0)) & = & (\eta^{1,0}, \eta^{2,0}) & \text{in } \Gamma_0^s. \end{array} \tag{3.3}$$

In equations (3.2) and (3.3), $\sigma(\mathbf{u}, p)$ is the Cauchy stress tensor defined by $\sigma(\mathbf{u}, p) = \nu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{tr}}\right) - p\mathbf{I}$ where $\mathbf{I}$ is the identity 2×2 matrix. The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are defined by $\mathbf{e}_1 = (1, 0)^{\mathbf{tr}}$ and $\mathbf{e}_2 = (0, 1)^{\mathbf{tr}}$. The coefficient $\nu > 0$ is the viscosity of the fluid and $\beta \geq 0$, $\gamma > 0$ are constants relative to the structure, namely the stretching and the friction of the beam.

3.2 Functional settings.

As in the previous chapter, we need to give a definition for functions in time dependent domains. Furthermore, the different functions are periodic in the first variable.

We introduce the classic Hilbert space $L^2_\#(\Omega_0)$ as the space of $L^2_{\text{loc}}(\mathbb{R}\times(0,1))$ which are $L-$periodic. In the same way, we set $\mathbf{L}^2_\#(\Omega_0) = L^2_\#(\Omega_0;\mathbb{R}^2)$ and $\mathbf{H}^\sigma_\#(\Omega_0) = H^\sigma_\#(\Omega_0)$. We introduce

$$\mathbf{V}^0_\#(\Omega_0) = \left\{\mathbf{z}\in\mathbf{L}^2_\#(\Omega_0) \text{ s.t. div } \mathbf{u} = 0 \text{ in } \Omega_0\right\}$$

and

$$\begin{array}{rcl} \mathbf{H}^{\sigma,\tau}_\#(Q_T^0) & = & L^2(0,T;\mathbf{H}^\sigma_\#(\Omega_0)) \cap H^\tau(0,T;\mathbf{L}^2_\#(\Omega_0)), \\ \mathbf{V}^{\sigma,\tau}_\#(Q_T^0) & = & L^2(0,T;\mathbf{V}^\sigma_\#(\Omega_0)) \cap H^\tau(0,T;\mathbf{V}^0_\#(\Omega_0)). \end{array}$$

We define functions in the time dependent cylinder Q_T^η as follows

Definition 3.1. *We say that* $\mathbf{u}$ *belongs to* $H^\tau(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{H}^\sigma_\#(\Omega_{\eta(t)}))$ *(respectively to* $H^\tau(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^\sigma_\#(\Omega_{\eta(t)})))$ *if*

– *for almost every* t *in* $(0,T)$, $\mathbf{u}(t)$ *belongs to* $\mathbf{H}^\sigma_\#(\Omega_{\eta(t)})$ *(resp. in* $\mathbf{V}^\sigma_\#(\Omega_{\eta(t)})$*),*
– $t\mapsto\|\mathbf{u}(t)\|_{\mathbf{H}^\sigma_\#(\Omega_{\eta(t)})}$ *(resp.* $t\mapsto\|\mathbf{u}(t)\|_{\mathbf{V}^\sigma_\#(\Omega_{\eta(t)})}$*) is in* $H^\tau(0,T;\mathbb{R})$.

Because of the divergence free condition in (3.2), the solutions $(\mathbf{u}, p, \eta)$ of (3.2)–(3.3) have to satisfy

$$\int_{\Omega_0} \text{div } \mathbf{u} = \int_{\Gamma^s_{\eta_0}} \mathbf{u}\cdot\left(1+\eta_x^2\right)^{-1/2}(-\eta_x\mathbf{e}_1 + \mathbf{e}_2) - \int_\Gamma \mathbf{u}\cdot\mathbf{e}_2 = \int_{\Gamma_0^s}\eta_t = 0.$$

Thus, η_t has to satisfy $\int_{\Gamma_0^s} \eta_t = 0$. In the same way, we will consider displacements η satisfying $\int_{\Gamma_0^s} \eta = 0$. Thus, we must take the initial data for the beam $\eta^{1,0}$ and $\eta^{2,0}$ in $L^2_{\#,0}(\Gamma_0^s)$ the space of $x-$periodic function in $L^2_{\text{loc}}(\mathbb{R})$ of period L and of zero mean value in Γ_0^s

$$L^2_{\#,0}(\Gamma_0^s) = \left\{ \mu \in L^2_{\#}(\Gamma_0^s) \text{ s.t. } \int_{\Gamma_0^s} \mu = 0 \right\}.$$

We introduce the projection $M^s_{\#}$ from $L^2_{\#}(\Gamma_0^s)$ onto $L^2_{\#,0}(\Gamma_0^s)$ defined by

$$M^s_{\#}(\mu) = \mu - \frac{1}{|\Gamma_0^s|} \int_{\Gamma_0^s} \mu \quad \text{for all } \mu \in L^2_{\#}(\Gamma_0^s).$$

Then, we define a new trace function $\gamma^s_{\#}$ to set the right-hand side of the beam equation on the space $L^2_{\#,0}(\Gamma_0^s)$ as follows

$$\gamma^s_{\#}(q) = M^s_{\#}(q_{|\Gamma_0^s}) = q_{|\Gamma_0^s} - \frac{1}{|\Gamma_0^s|} \int_{\Gamma_0^s} q_{|\Gamma_0^s} \quad \text{for all } q \in H^\sigma_{\#}(\Omega_0^s)(\text{with } \sigma > 1/2).$$

The beam equation (3.3) becomes

$$\begin{array}{rcll} \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} & = & -\gamma^s_{\#}\left[\sigma(\mathbf{u},p)(-\eta_x\mathbf{e}_1 + \mathbf{e}_2)\cdot\mathbf{e}_2)\right] & \text{on } \Sigma_T^{s,0}, \\ (\eta(0), \eta_t(0)) & = & (\eta^{1,0}, \eta^{2,0}) & \text{in } \Gamma_0^s. \end{array} \tag{3.4}$$

Then, we define the Sobolev spaces for the displacement as $H^\sigma_{\#}(\Gamma_0^s) = H^\sigma(\Gamma_0^s) \cap L^2_{\#,0}(\Gamma_0^s)$ and the spaces on $\Sigma_T^{s,0}$ as follow

$$H^{\sigma,\tau}_{\#}(\Sigma_T^{s,0}) = L^2(0,T; H^\sigma_{\#}(\Gamma_0^s)) \cap H^\tau(0,T; L^2_{\#,0}(\Gamma_0^s)).$$

The pressure is defined in equations (3.2) and (3.4) up to an additive constant. Thus, to obtain the uniqueness of the pressure, we look for pressures in Sobolev spaces with zero mean value on Ω_0. That is, we introduce the spaces

$$\mathcal{H}^\sigma_{\#}(\Omega_0) = \left\{ q \in H^\sigma_{\#}(\Omega_0) \text{ s.t. } \int_{\Omega_0} q = 0 \right\} \quad \text{for } \sigma \geq 0$$

and we will loook for p with $(\mathbf{u}, p, \eta)$ solution of (3.2)–(3.4) in $L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1_{\#}\left(\Omega_{\eta(t)}\right)\right)$.

3.3 Main results.

The only difference between this chapter and the previous one is that here the coefficient $\alpha = 0$ in the beam equation. We will see that the proof is slightly different here because with $\alpha = 0$, the beam equation becomes a strongly damped wave equation which gives less regularity for the solutions (see Proposition 3.7 to compare with Proposition 2.8).

We want to prove the following results:

Theorem 3.2. *Let $\varepsilon > 0$ and $T > 0$. Let $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ be in $\mathbf{V}^1_{\#}(\Omega_{\eta^0}) \times H^{2+\varepsilon}_{\#}(\Gamma_0^s) \times H^{1+\varepsilon}_{\#}(\Gamma_0^s)$. There exists $R > 0$ such that for any initial data satisfying*

$$\|\mathbf{u}^0\|^2_{\mathbf{V}^1_{\#}(\Omega_{\eta^0})} + \|\eta^0\|^2_{H^{2+\varepsilon}_{\#}(\Gamma_0^s)} + \|\eta^1\|^2_{H^{1+\varepsilon}_{\#}(\Gamma_0^s)} \leq R^2$$

and the compatibility condition

$$\mathbf{u}^0 = \mathbf{0} \quad \text{on } \Gamma \qquad \text{and} \quad \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_2 \quad \text{on } \Gamma^s_{\eta^{1,0}}, \tag{3.5}$$

system (3.2)–(3.4) *has a unique global strong solution* $(\mathbf{u}, p, \eta)$ *in*

$$\mathbf{V}^{2,1}_{\#}(Q^{\eta}_T) \times L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1_{\#}\left(\Omega_{\eta(t)}\right)\right) \times \mathcal{E}^{\varepsilon}_T.$$

The space $\mathcal{E}^{\varepsilon}_T$ *depends on* ε *and is defined by*

$$\mathcal{E}^{\varepsilon}_T = H^1(0,T; H^{2+\varepsilon}_{\#}(\Gamma^s_0)) \cap H^2(0,T; H^{\varepsilon}_{\#}(\Gamma^s_0)).$$

Theorem 3.3. *Let* $\varepsilon > 0$. *Let* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *be in* $\mathbf{V}^1_{\#}(\Omega_{\eta^0}) \times H^{2+\varepsilon}_{\#}(\Gamma^s_0) \times H^{1+\varepsilon}_{\#}(\Gamma^s_0)$ *satisfying the compatibility condition* (3.5). *There exists a time* $T_0 > 0$ *such that system* (3.2)–(3.4) *has a unique strong solution* $(\mathbf{u}, p, \eta) \in \mathbf{V}^{2,1}_{\#}(Q^{\eta}_{T_0}) \times L^2(\bigcup_{t\in(0,T_0)} \{t\} \times \mathcal{H}^1_{\#}(\Omega_{\eta(t)})) \times \mathcal{E}^{\varepsilon}_{T_0}$.

These results improve the one in [4] where the author proves existence of strong solutions for small initial data, time of existence and smallness of a parameter.

The different steps of the proof are detailed in Chapter 2. First, thanks to a change of variables, we set the problem in the fixed cylinder $Q^0_T = (0,T) \times \Omega_0$. Then, we study the linearized system with nonhomogeneous right-hand sides. Finally, by a fixed point procedure, we are able to prove existence for the nonlinear system set in the fixed cylinder. The regularity of the change of variables concludes the proof.

More precisely, the proof in the case $\alpha > 0$ in Chapter 2 and this result are quite the same. Indeed, only the lifting of the nonzero divergence term and the nonlinear estimates are different. Thus, we will refer to the previous chapter when the proofs of the different results will be the same.

3.4 Change of variables.

We introduce the change of variables

$$\begin{array}{rccl} \phi_{\eta(t)}: & \Omega_{\eta(t)} & \longrightarrow & \Omega_0 \\ & (x,y) & \longmapsto & (x,z) = \left(x, \dfrac{y}{1+\eta(t,x)}\right). \end{array}$$

Following the previous chapter (see in particular section 2.4), system (3.2)–(3.4) becomes

$$\begin{array}{rcll} \mathbf{u}_t - \operatorname{div} \sigma(\mathbf{u}, p) & = & \mathbf{F}[\mathbf{u}, p, \eta] & \text{in } Q^0_T, \\ \operatorname{div} \mathbf{u} & = & \operatorname{div} \mathbf{w}[\mathbf{u}, \eta] & \text{in } Q^0_T, \\ \mathbf{u} & = & \eta_t \mathbf{e}_2 & \text{on } \Sigma^{s,0}_T, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} & = & \gamma^s_{\#}(p - 2\nu u_{2,z}) + H[\mathbf{u}, \eta] & \text{on } \Sigma^{s,0}_T, \\ (\mathbf{u}(0), \eta(0), \eta_t(0)) & = & (\mathbf{u}^0, \eta^{1,0}, \eta^{2,0}) & \end{array} \tag{3.6}$$

where

$$\begin{array}{rcl} \mathbf{F}[\mathbf{u}, p, \eta] & = & -\eta \mathbf{u}_t + \left[z\eta_t + \nu z\left(\dfrac{\eta_x^2}{1+\eta} - \eta_{xx}\right)\right]\mathbf{u}_z + \nu\left\{-2z\eta_x \mathbf{u}_{xz} + \eta \mathbf{u}_{xx} + \dfrac{z^2\eta_x^2 - \eta}{1+\eta}\mathbf{u}_{zz}\right\} \\ & & + z(\eta_x p_z - \eta p_x)\mathbf{e}_1 - (1+\eta)u_1 \mathbf{u}_x + (z\eta_x u_1 - u_2)\mathbf{u}_z, \\ \mathbf{w}[\mathbf{u}, \eta] & = & -\eta u_1 \mathbf{e}_1 + z\eta_x u_1 \mathbf{e}_2, \\ H[\mathbf{u}, \eta] & = & \nu\gamma^s_{\#}\left(\dfrac{\eta_x}{1+\eta}u_{1,z} + \eta_x u_{2,x} - \dfrac{\eta_x^2 - 2\eta}{1+\eta}u_{2,z}\right). \end{array} \tag{3.7}$$

System (3.6) is equivalent to system (3.2)–(3.4) in the sens of

Definition 3.4. $(\mathbf{u}, p, \eta)$ *in* $\mathbf{H}^{2,1}_{\#}(Q^{\eta}_T) \times L^2(\bigcup_{t\in(0,T)}\{t\} \times \mathcal{H}^1_{\#}(\Omega_{\eta(t)})) \times \mathcal{E}^{\varepsilon}_T$ *is solution of* (3.2)–(3.4) *when the following conditions are satisfied:*

(i) $(\hat{\mathbf{u}}, \hat{p}, \eta)$ *obtained for the change of variables* $\hat{\mathbf{u}}(x,z) = \mathbf{u}(x,y)$, $\hat{p}(x,z) = p(x,y)$ *with* $z = \frac{y}{1+\eta(t,x)}$ *is a solution of* (3.6),

(ii) for any time t in $(0,T)$, the previous change of variables is a $\mathcal{C}^1$-diffeomorphism from $\Omega_{\eta(t)}$ into Ω_0,

(iii) η satisfies condition (3.1).

In [24], the author considers a lifting of both the divergence condition and the nonhomogeneous Dirichlet condition. We introduce here his notations (with some modifications due to the periodic boundary conditions). For $-1/2 \leq \sigma_1 \leq 2$ and $\sigma_2 \geq 0$, we define

$$\mathbf{H}^{\sigma_1,\sigma_2}_{\Gamma_0,\Omega_0} = \left\{ (\mathbf{g}, h) \in \mathbf{H}^{\sigma_1}_{\#}(\Gamma_0) \times H^{\sigma_2}_{\#}(\Omega_0) \text{ s.t. } \langle \mathbf{g}\cdot\mathbf{n}, 1\rangle_{\mathbf{H}^{\sigma_1}_{\#}(\Gamma_0),\mathbf{H}^{-\sigma_1}_{\#}(\Gamma_0)} = \int_{\Omega_0} h \right\}$$

and for $-1/2 \leq \sigma_1 \leq 2$ and $-1 \leq \sigma_2 \leq 0$

$$\mathbf{H}^{\sigma_1,\sigma_2}_{\Gamma_0,\Omega_0} = \left\{ (\mathbf{g}, h) \in \mathbf{H}^{\sigma_1}_{\#}(\Gamma_0) \times \left(H^{-\sigma_2}_{\#}(\Omega_0)\right)' \text{ s.t. } \langle \mathbf{g}\cdot\mathbf{n}, 1\rangle_{\mathbf{H}^{\sigma_1}_{\#}(\Gamma_0),\mathbf{H}^{-\sigma_1}_{\#}(\Gamma_0)} = \langle h, 1\rangle_{\left(H^{-\sigma_2}_{\#}(\Omega_0)\right)', H^{-\sigma_2}_{\#}(\Omega_0)} \right\}$$

Then, for $(\mathbf{g}, h)$ in $\mathbf{H}^{3/2,1}_{\Gamma_0,\Omega_0}$, there exists a unique solution $(\mathbf{z}, \pi) = (L(\mathbf{g}, h), L_p(\mathbf{g}, h))$ in $\mathbf{H}^2_{\#}(\Omega_0) \times \mathcal{H}^1_{\#}(\Omega_0)$ of the following equation:

$$-\nu\Delta\mathbf{z} + \nabla\pi = \mathbf{0} \quad \text{and} \quad \text{div } \mathbf{z} = h \quad \text{in } \Omega_0 \quad \text{and} \quad \mathbf{z} = \mathbf{g} \quad \text{on } \Gamma_0.$$

The liftings L and L_p define two linear operators with more general regularity:

Proposition 3.5 (Corollary 8.4 in [24])**.** *The operator L is linear and continuous from $\mathbf{H}^{s+1/2,s}_{\Gamma_0,\Omega_0}$ into $\mathbf{H}^{s+1}_{\#}(\Omega_0)$ for all $-1 \leq s \leq 1$ and the operator L_p is linear and continuous from $\mathbf{H}^{s+1/2,s}_{\Gamma_0,\Omega_0}$ into $\mathcal{H}^s_{\#}(\Omega_0)$ for all $-1 \leq s \leq 1$.*

This result will be used in particular for $(\mathbf{g}, h) = (\mathbf{0}, \text{div } \mathbf{w}[\mathbf{u}, \eta])$ with $\mathbf{w}[\mathbf{u}, \eta]$ defined in (3.7). We will see in section 3.6 that, for $(\mathbf{u}, p, \eta)$ smooth enough, $\mathbf{w}[\mathbf{u}, \eta]$ belongs to

$$G_T = \left\{ \mathbf{k} \in \mathbf{L}^2_{\#}(Q^0_T) \text{ s.t. div } \mathbf{k} \in L^2(0,T;\mathbf{H}^1_{\#}(\Omega_0)),\ \mathbf{k}_t \in \mathbf{L}^2_{\#}(Q^0_T) \text{ and } \mathbf{k} = 0 \text{ on } \Sigma^0_T \right\}.$$

Thus, with $\mathbf{w}[\mathbf{u}, \eta]$ in G_T, we get that div $\mathbf{w}_t[\mathbf{u}, \eta]$ belongs to $L^2(0,T;H^{-1}_{\#}(\Omega))$ thanks to first $\mathbf{w}[\mathbf{u}, \eta] = \mathbf{0}$ on $\partial\Omega_0$ and second the property

$$\langle \text{div } \mathbf{w}_t[\mathbf{u}, \eta], h\rangle_{H^{-1}_{\#}(\Omega_0),H^1_{\#}(\Omega_0)} = -\langle \mathbf{w}_t[\mathbf{u}, \eta], \nabla h\rangle_{\mathbf{L}^2_{\#}(\Omega_0),\mathbf{L}^2_{\#}(\Omega_0)} + \langle \mathbf{w}_t[\mathbf{u}, \eta]\cdot\mathbf{n}, h\rangle_{H^{-1/2}_{\#}(\Gamma_0),H^{1/2}_{\#}(\Gamma_0)}$$

for any h in $H^1_{\#}(\Omega_0)$.

This gives, for $\mathbf{w}[\mathbf{u}, \eta]$ in G_T, that $(\mathbf{0}, \text{div } \mathbf{w}[\mathbf{u}, \eta])$ belongs to $L^2\left(0,T;\mathbf{H}^{3/2,1}_{\Gamma_0,\Omega_0}\right) \cap H^1\left(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma_0,\Omega_0}\right)$. Thus, we can lift this nonzero divergence condition by a couple $(\mathbf{z}[\mathbf{u}, \eta], \pi[\mathbf{u}, \eta]) = (\tilde{L}\mathbf{w}[\mathbf{u}, \eta], \tilde{L}_p\mathbf{w}[\mathbf{u}, \eta])$ in $\mathbf{H}^{2,1}_{\#}(Q^0_T) \times L^2(0,T;\mathcal{H}^1_{\#}(\Omega_0))$ solution of

$$-\nu\Delta\mathbf{z}[\mathbf{u}, \eta] + \nabla\pi[\mathbf{u}, \eta] = \mathbf{0} \quad \text{and} \quad \text{div } \mathbf{z}[\mathbf{u}, \eta] = \text{div } \mathbf{w}[\mathbf{u}, \eta] \quad \text{in } \Omega_0 \quad \text{and} \quad \mathbf{z}[\mathbf{u}, \eta] = \mathbf{0} \quad \text{on } \Gamma_0. \tag{3.8}$$

Futhermore, the continuity of the operators L and L_p gives the estimate:

$$\|(\widetilde{L}\mathbf{w}[\mathbf{u}, \eta], \widetilde{L}_p\mathbf{w}[\mathbf{u}, \eta])\|_{\mathbf{H}^{2,1}_{\#}(Q^0_T)\times L^2(0,T;\mathcal{H}^1_{\#}(\Omega_0))} \leq C\|\mathbf{w}[\mathbf{u}, \eta]\|_{G_T}$$

where the norm on G_T is

$$\|\mathbf{k}\|_{G_T} = \left(\|\mathbf{k}\|^2_{\mathbf{L}^2_{\#}(Q^0_T)} + \|\text{div } \mathbf{k}\|^2_{L^2(0,T;H^1_{\#}(\Omega_0))} + \|\mathbf{k}_t\|^2_{\mathbf{L}^2_{\#}(Q^0_T)}\right)^{1/2} \qquad \text{for all } \mathbf{k} \in G_T.$$

Thanks to the liftings $\widetilde{L}$ and $\widetilde{L}_p$, we look for solution of system (3.6) under the form $(\mathbf{u}, p, \eta) = (\mathbf{v}+\mathbf{z}[\mathbf{u},\eta], q+\pi[\mathbf{u},\eta], \eta)$ where $(\mathbf{z}[\mathbf{u},\eta], \pi[\mathbf{u},\eta])$ are defined in (3.8). The system in the variables $(\mathbf{v}, q, \eta)$ is the following:

$$\begin{array}{rcll} \mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},q) & = & \mathbf{f}[\mathbf{u},p,\eta] & \text{in } Q_T^0, \\ \operatorname{div}\mathbf{v} & = & 0 & \text{in } Q_T^0, \\ \mathbf{v} & = & \eta_t\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} & = & q + h[\mathbf{u},\eta] & \text{in } \Sigma_T^{s,0}, \\ (\mathbf{v}(0), \eta(0), \eta_t(0)) & = & (\mathbf{v}^0, \eta^{1,0}, \eta^{2,0}) & \end{array} \tag{3.9}$$

where

$$\mathbf{f}[\mathbf{v},p,\eta] = \mathbf{F}[\mathbf{u},p,\eta] - \mathbf{z}_t[\mathbf{u},\eta], \tag{3.10}$$

$$h[\mathbf{u},\eta] = \gamma_{\#}^s\left[\pi[\mathbf{u},\eta] - 2\nu\Big(\mathbf{z}[\mathbf{u},\eta]\Big)_{2,z}\right] + H[\mathbf{u},\eta] \tag{3.11}$$

and

$$\mathbf{v}^0 = \mathbf{u}^0 - \mathbf{z}[\mathbf{u}^0, \eta^0]. \tag{3.12}$$

Note that the term $-2\nu v_{2,z}$ vanishes in the right-hand side of the beam equation, because $\operatorname{div}\mathbf{v} = v_{1,x} + v_{2,z} = 0$ in Q_T^0 and $v_1 = 0$ on $\Sigma_T^{s,0}$ and for $\mathbf{v}$ in $\mathbf{H}_{\#}^{2,1}(Q_T^0)$, $v_{1,x|\Sigma_T^{s,0}} = 0$. Thus $v_{2,z|\Sigma_T^{s,0}} = 0$.

The compatibility conditions in terms of $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ are

$$\operatorname{div}\mathbf{v}^0 = 0 \text{ in } \Omega_0, \qquad \mathbf{v}^0 = \eta^{2,0}\mathbf{e}_2 \text{ on } \Gamma_0^s \qquad \text{and} \qquad \mathbf{v}^0 = \mathbf{0} \text{ on } \Gamma. \tag{3.13}$$

That is, in terms of $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$:

$$\operatorname{div}\Big(\mathbf{u}^0 - \mathbf{z}[\mathbf{u}^0, \eta^{1,0}]\Big) = 0 \text{ in } \Omega_0, \qquad \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_2 \text{ on } \Gamma_0^s \qquad \text{and} \qquad \mathbf{u}^0 = \mathbf{0} \text{ on } \Gamma. \tag{3.14}$$

From now on, we can follow the different steps of the previous chapter. We will have to adapt the functional space for η (from $H_{(0)}^{4,2}(\Sigma_T^{s,0})$ when $\alpha > 0$ to $\mathcal{E}_T^{\varepsilon}$ here) and the proof of existence of solution for the beam equation.

Let us begin by proving existence and uniqueness of strong solutions for the linearized system.

3.5 Study of an auxiliary linear system.

In this section, we prove existence and uniqueness of solutions to the system

$$\begin{array}{rcll} \mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},q) & = & \mathbf{f} & \text{in } Q_T^0, \\ \operatorname{div}\mathbf{v} & = & 0 & \text{in } Q_T^0, \\ \mathbf{v} & = & \eta_t\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} & = & \gamma_{\#}^s q + h & \text{on } \Sigma_T^{s,0}, \\ (\mathbf{v}(0), \eta(0), \eta_t(0)) & = & (\mathbf{v}^0, \eta^{1,0}, \eta^{2,0}). & \end{array} \tag{3.15}$$

In system (3.15), the initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ belongs to $X_{\mathbf{cc}}^{0,\varepsilon}$ (for $0 < \varepsilon \leq 1/2$) where

$$X^{0,\varepsilon} = \mathbf{H}_{\#}^1(\Omega_0) \times H_{\#}^{2+\varepsilon}(\Gamma_0^s) \times H_{\#}^{1+\varepsilon}(\Gamma_0^s)$$

and

$$X_{\mathbf{cc}}^{0,\varepsilon} = \Big\{(\mathbf{z}^0, \mu^{1,0}, \mu^{2,0}) \in X^{0,\varepsilon} \text{ s.t. } (\mathbf{z}^0, \mu^{1,0}, \mu^{2,0}) \text{ satisfies (3.13)}\Big\}. \tag{3.16}$$

The right-hand side $(\mathbf{f}, h)$ in system (3.15) belongs to

$$Z_T = \mathbf{L}_{\#}^2(Q_T^0) \times L^2(0,T; H_{\#}^{1/2}(\Gamma_0^s)).$$

The main result of this section is the following.

Theorem 3.6. *Let* $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ *be in* $X^{0,\varepsilon}_{\mathbf{cc}}$ *and* $(\mathbf{f}, h)$ *be in* Z_T*.Then, system* (3.15) *admits a unique solution* $(\mathbf{v}, q, \eta)$ *in*

$$X_T^\varepsilon \;=\; \Big\{(\mathbf{z}, r, \mu) \in \mathbf{H}^{2,1}_{\#}(Q_T^0) \times L^2(0,T; \mathcal{H}^1_{\#}(\Omega_0)) \times \mathcal{E}_T^\varepsilon \text{ s.t. } \mathbf{z} = \mathbf{0} \text{ on } \Sigma_T \text{ and } \mathbf{z} = \mu_t \mathbf{e}_2 \text{ on } \Sigma_T^{s,0}\Big\}. \tag{3.17}$$

We have the estimate

$$\|(\mathbf{v}, q, \eta)\|_{X_T^\varepsilon} \le C\Big(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^{0,\varepsilon}} + \|(\mathbf{f}, h)\|_{Z_T}\Big). \tag{3.18}$$

To prove Theorem 3.6, we act as in Chapter 2. That is, we rewrite system (3.15) using the Leray projection from $\mathbf{L}^2_{\#}(\Omega_0)$ onto

$$\mathbf{V}^0_{\#,\mathbf{n}}(\Omega_0) = \left\{\mathbf{z} \in \mathbf{L}^2_{\#}(\Omega_0) \text{ s.t. div } \mathbf{z} = 0 \text{ in } \Omega_0 \text{ and } \mathbf{z}\cdot\mathbf{n} = 0 \text{ on } \Gamma_0\right\}.$$

More precisely, we split the velocity $\mathbf{v}$ into two parts, namely $\mathbf{v}_e = P\mathbf{v}$ and $\mathbf{v}_s = (I-P)\mathbf{v}$. The velocity $\mathbf{v}_e$ is solution of an evolutionary partial differential equation associated with a pressure term q_e and the velocity $\mathbf{v}_s$ is solution of a stationary partial differential equation assicated to another pressure term q_s.

Thanks to the splitting of system (3.6), we are able to prove the existence of a unique solution to the equivalent system. Then, using the equivalence between the two systems, we can conclude the proof (see section 3.5.3 for details).

3.5.1 Equivalent system.

The Leray projection maps $\mathbf{L}^2_{\#}(\Omega_0)$ onto $\mathbf{V}^0_{\#,\mathbf{n}}(\Omega_0)$ along $\nabla H^1_{\#}(\Omega_0)$, that is for every $\mathbf{z}$ in $\mathbf{L}^2_{\#}(\Omega_0)$, there exists a function $\pi(\mathbf{z})$ in $H^1_{\#}(\Omega_0)$ such that $(I-P)\mathbf{z} = \nabla\pi(\mathbf{z})$. Furthermore, we can calculate $\pi(\mathbf{z})$ from $\mathbf{z}$. Indeed, taking the divergence and the normal trace in the identity $(I-P)\mathbf{z} = \nabla\pi(\mathbf{z})$, we get

$$\begin{aligned} \text{div } ((I-P)\mathbf{z}) = \text{div } \mathbf{z} = \text{div } (\nabla\pi(\mathbf{z})) = \Delta\pi(\mathbf{z}) \qquad & \text{in } \Omega_0, \\ ((I-P)\mathbf{z})\cdot\mathbf{n} = \mathbf{z}\cdot\mathbf{n} = \nabla\pi(\mathbf{z})\cdot\mathbf{n} = \frac{\partial\pi(\mathbf{z})}{\partial\mathbf{n}} \qquad & \text{on } \Gamma_0. \end{aligned}$$

The previous system in $\pi(\mathbf{z})$ is ill-posed because, for $\mathbf{z}$ in $\mathbf{L}^2_{\#}(\Omega_0)$, the normal trace of $\mathbf{z}$ is not necessary defined. But, we can decompose $\pi(\mathbf{z})$ into $\pi_1(\mathbf{z})$ and $\pi_2(\mathbf{z})$ defined by

$$\Delta\pi_1(\mathbf{z}) = \text{div } \mathbf{z} \text{ in } \Omega_0 \quad \text{ and } \quad \pi_1(\mathbf{z}) \in H^1_{\#,0}(\Omega_0) = \left\{r \in H^1_{\#}(\Omega_0) \text{ s.t. } r = 0 \text{ on } \Gamma_0\right\}$$

and

$$\Delta\pi_2(\mathbf{z}) = 0 \text{ in } \Omega_0 \quad \text{ and } \quad \frac{\partial\pi(\mathbf{z})}{\partial\mathbf{n}} = (\mathbf{z} - \nabla\pi_1(\mathbf{z}))\cdot\mathbf{n} \text{ on } \Gamma_0.$$

We denote by N the operator from $H^\sigma_{\#}(\Gamma_0)$ into $H^{\sigma+3/2}_{\#}(\Omega_0)$ (for $\sigma \ge -1/2$) defined for g in $H^\sigma_{\#}(\Gamma_0)$ (with $\sigma \ge -1/2$) by $Ng = r$ if and only if

$$\Delta r = 0 \text{ in } \Omega_0 \quad \text{ and } \quad \frac{\partial r}{\partial\mathbf{n}} = g \text{ on } \Gamma_0.$$

Then, with $\pi_1(\mathbf{z}) = -(-\Delta_D)^{-1}(\text{div } \mathbf{z})$ and $\pi_2(\mathbf{z}) = N\left((\mathbf{z} - \nabla\pi_1(\mathbf{z}))\cdot\mathbf{n}\right)$, we get first that

$$\pi_2(\mathbf{z}) = N\left(\left(\mathbf{z} + \nabla(-\Delta_D)^{-1}(\text{div } \mathbf{z})\right)\cdot\mathbf{n}\right)$$

and second that

$$\pi(\mathbf{z}) = -(-\Delta_D)^{-1}(\text{div } \mathbf{z}) + N\left(\left(\mathbf{z} + \nabla(-\Delta_D)^{-1}(\text{div } \mathbf{z})\right)\cdot\mathbf{n}\right). \tag{3.19}$$

In the case of $\mathbf{v}$, with $(\mathbf{v}, p, \eta)$ solution of system (3.15), we know that div $\mathbf{v} = 0$, thus if q is defined by $\nabla q = (I-P)\mathbf{v}$, then $q = N(\mathbf{v}\cdot\mathbf{n}) = N(\eta_t\chi_{\Gamma_0^s})$. We define by N_s the restriction of N to Γ_0^s, that is $N_s = N(\cdot\chi_{\Gamma_0^s})$ defined from $H^\sigma_{\#}(\Gamma_0^s)$ into $H^{\sigma+3/2}_{\#}(\Omega_0)$ (for $\sigma \ge -1/2$).

Finally, we get that system (3.15) is equivalent to the following one:

$$\begin{array}{rcll} \mathbf{v}_{e,t} - \mathrm{div}\,(\mathbf{v}_e, q_e) & = & P\mathbf{f} & \text{in } Q_T^0, \\ \mathbf{v}_e & = & -\gamma_\tau \nabla N_s(\eta_t) & \text{on } \Sigma_T^0, \\ \mathbf{v}_e(0) & = & P\mathbf{v}^0 & \text{in } \Omega_0, \\ \mathbf{v}_s & = & \nabla N_s(\eta_t) & \text{in } Q_T^0, \\ (I + \gamma_\#^s N_s)\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} & = & \gamma_\#^s q_e + \widetilde{h} & \text{on } \Sigma_T^{s,0}, \\ (\eta(0), \eta_t(0)) & = & (\eta^{1,0}, \eta^{2,0}) & \\ q & = & q_e + q_s & \text{in } Q_T^0, \\ \mathbf{v} & = & \mathbf{v}_e + \mathbf{v}_s & \text{in } Q_T^0, \\ q_s & = & \pi(\mathbf{f}) - N_s(\eta_{tt}) & \text{in } Q_T^0 \end{array} \tag{3.20}$$

where

$$\widetilde{h} = h + \gamma_\#^s \pi(\mathbf{f})$$

with the operator π from $\mathbf{L}^2_\#(\Omega_0)$ into $H^1_\#(\Omega_0)$ is defined in (3.19). The whole decomposition of system (3.15) into system (3.20) can be found in the previous chapter or in [23] and the decomposition for the Stokes system in [22].

3.5.2 Existence of solution and regularity of each equation separately.

The next proposition gives existence of solution of the beam equation with a right-hand side in $L^{2-\kappa}(0,T; H^{1/2}_\#(\Gamma_0^s))$, for $0 < \kappa < 1$:

Proposition 3.7. *Let $0 < \kappa < 1$. Let $(\eta^{1,0}, \eta^{2,0})$ be in $H^2_\#(\Gamma_0^s) \times H^1_\#(\Gamma_0^s)$ and $(\mathbf{f}, h)$ be in Z_T. Then, first $\widetilde{h}$ belongs to $L^2(0,T; H^{1/2}_\#(\Gamma_0^s))$ and second, with $\overline{q}_e$ in $L^{2-\kappa}(0,T; \mathcal{H}^1_\#(\Omega_0))$, equation*

$$\begin{array}{rcll} (I + \gamma_\#^s N_s)\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} & = & \gamma_\#^s \overline{q}_e + \widetilde{h} & \text{on } \Sigma_T^{s,0}, \\ (\eta(0), \eta_t(0)) & = & (\eta^{1,0}, \eta^{2,0}) & \text{in } \Gamma_0^s \end{array} \tag{3.21}$$

admits a unique solution η in

$$E_T^\kappa = W^{1,2-\kappa}(0,T; H^2_\#(\Gamma_0^s)) \cap W^{2,2-\kappa}(0,T; H^{2\kappa}_\#(\Gamma_0^s)).$$

Furthermore, η_t belongs to $H^{3/2,3/4}_\#(\Sigma_T^{s,0})$.

Proof. The first point is obvious by definition of $\widetilde{h}$. Second, we write equation (3.21) as a first order system.

We define $H_s = H^2_\#(\Gamma_0^s) \times L^2_{\#,0}(\Gamma_0^s)$ endowed with the norm

$$\|(\mu_1, \mu_2)\|^2_{H_s} = \|(-\Delta_s)\mu_1\|^2_{L^2_{\#,0}(\Gamma_0^s)} + \|\mu_2\|^2_{L^2_{\#,0}(\Gamma_0^s)} \qquad \forall (\mu_1, \mu_2) \in H_s.$$

The operator Δ_s is a operator with domain $H^2_\#(\Gamma_0^s)$ on $L^2_{\#,0}(\Gamma_0^s)$ defined by

$$\Delta_s \mu = \mu_{xx} \text{ for all } \mu \in D(-\Delta_s) = H^2_\#(\Gamma_0^s).$$

Setting

$$Y = \begin{pmatrix} \eta \\ \eta_t \end{pmatrix}, \qquad Y^0 = \begin{pmatrix} \eta^0 \\ \eta^1 \end{pmatrix}$$

and defining the operator $\mathcal{A}_{\beta,\gamma}$ with domain $D(\mathcal{A}_{\beta,\gamma}) = H^2_\#(\Gamma_0^s) \times H^2_\#(\Gamma_0^s)$ on H_s by

$$\mathcal{A}_{\beta,\gamma} = \begin{pmatrix} I & 0 \\ 0 & (I + \gamma_\#^s N_s)^{-1} \end{pmatrix} \begin{pmatrix} 0 & I \\ \beta\Delta_s & \gamma\Delta_s \end{pmatrix},$$

equation (3.21) becomes

$$\begin{array}{rcl} Y' &=& \mathcal{A}_{\beta,\gamma}Y + \left(\begin{array}{c} 0 \\ (I+\gamma^s_{\#}N_s)^{-1}\Big[\gamma^s_{\#}\overline{q}_e + \widetilde{h}\Big] \end{array} \right) \\ Y(0) &=& Y^0. \end{array} \tag{3.22}$$

First, using Proposition 2.2 in [30] and Lemma 3.1 in [23], we get that $\mathcal{A}_{\beta,\gamma}$ is a generator of an analytic semigroup on H_s. Then, following exactly the proof of Proposition 2.8 in Chapter 2, we get that the solution of system (3.22) can be written with the Duhamel formula:

$$Y(t) = \mathrm{e}^{t\mathcal{A}_{\beta,\gamma}}Y^0 + \int_0^t \mathrm{e}^{(t-\tau)\mathcal{A}_{\beta,\gamma}}\mathcal{B}(\tau)\mathrm{d}\tau$$

with $\mathcal{B} = \left(\begin{array}{c} 0 \\ (I+\gamma^s_{\#}N_s)^{-1}\Big[\gamma^s_{\#}\overline{q}_e + \widetilde{h}\Big] \end{array} \right)$ in $L^{2-\kappa}(0,T;D(\mathcal{A}^{1/4}_{\beta,\gamma}))$ and that Y belongs to

$$L^{2-\kappa}(0,T;[D(\mathcal{A}^2_{\beta,\gamma}),D(\mathcal{A}_{\beta,\gamma})]_{1-\kappa}) \cap W^{1,2-\kappa}(0,T;[D(\mathcal{A}_{\beta,\gamma}),H_s]_{1-\kappa}).$$

A simple calculation gives the interpolated spaces (see [17]):

$$\begin{array}{rcl} \big[D(\mathcal{A}^2_{\beta,\gamma}),D(\mathcal{A}_{\beta,\gamma})\big]_{1-\kappa} &=& \left\{(\mu_1,\mu_2)\in H^2_{\#}(\Gamma^s_0)\times H^2_{\#}(\Gamma^s_0) \text{ s.t. } \beta\mu_1+\gamma\mu_2 \in H^{2\kappa}_{\#}(\Gamma^s_0)\right\}, \\ \big[D(\mathcal{A}_{\beta,\gamma}),H_s\big]_{1-\kappa} &=& H^2_{\#}(\Gamma^s_0)\times H^{2\kappa}_{\#}(\Gamma^s_0). \end{array}$$

That is, η belongs to

$$W^{1,2-\kappa}(0,T;H^2_{\#}(\Gamma^s_0)) \cap W^{2,2-\kappa}(0,T;H^{2\kappa}_{\#}(\Gamma^s_0))$$

and then η_t belongs to

$$L^{2-\kappa}(0,T;H^2_{\#}(\Gamma^s_0)) \cap W^{1,2-\kappa}(0,T;H^{2\kappa}_{\#}(\Gamma^s_0)).$$

The same calculation as in Proposition 2.8 in Chapter 2 gives that η_t belongs to $H^{3/2,3/4}_{\#}(\Sigma^{s,0}_T)$. Furthermore, we get the expected estimates from the Duhamel formula. □

The regularity of solution of the Stokes problem stays the same. Namely, we look for solution of the Stokes equivalent system:

$$\begin{array}{rcll} \mathbf{v}_{e,t} - \nu\Delta\mathbf{v}_e + \nabla q_e &=& P\mathbf{f} & \text{in } Q^0_T, \\ \mathbf{v}_e &=& -\gamma_\tau\mathbf{v}_s & \text{on } \Sigma^0_T, \\ \mathbf{v}_e(0) &=& P\mathbf{v}^0 & \text{in } \Omega_0, \\ \mathbf{v}_s &=& \nabla N_s(g) & \text{in } Q^0_T, \\ \mathbf{v} &=& \mathbf{v}_e + \mathbf{v}_s & \text{in } Q^0_T, \\ qq_s &=& \pi(\mathbf{f}) - N_s(g_t) & \text{in } Q^0_T. \end{array} \tag{3.23}$$

We have the following result

Proposition 3.8. *Let g be in $H^{3/2,3/4}_{\#}(\Sigma^{s,0}_T)$, $\mathbf{f}$ in $\mathbf{L}^2_{\#}(Q^0_T)$ and $\mathbf{v}^0$ in $\mathbf{V}^1_{\#}(\Omega_0)$ with the compatibility condition $\mathbf{v}^0 = \mathbf{0}$ on Γ and $\mathbf{v}^0 = g(0)\mathbf{e}_2$ on Γ^s_0. Then, (3.23) admits a unique solution $(\mathbf{v}_e,\mathbf{v}_s,q_e)$ in $X^{e,s}_T = \mathbf{V}^{2,1}_{\#}(Q^0_T)\times L^2(0,T;\mathbf{H}^2_{\#}(\Omega_0))\cap H^{3/4}(0,T;\mathbf{H}^{1/2}_{\#}(\Omega_0))\times L^2(0,T;\mathcal{H}^1_{\#}(\Omega_0))$. We have the estimate*

$$\|(\mathbf{v}_e,\mathbf{v}_s,q_e)\|_{X^{e,s}_T} \le c\Big(\|\mathbf{v}^0\|_{\mathbf{V}^1_{\#}(\Omega)} + \|g\|_{H^{3/2,3/4}_{\#}(\Sigma^{s,0}_T)} + \|\mathbf{f}\|_{\mathbf{L}^2_{\#}(Q^0_T)}\Big).$$

We can now construct the contraction mapping to prove Theorem 3.6.

3.5.3 Contruction of a solution of system (3.15).

In this section, the initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ and the right-hand side $(\mathbf{f}, h)$ are fixed respectively in $X^{0,\varepsilon}_{\mathbf{cc}}$ and Z_T.

We consider the mapping $\mathcal{G}$ defined by

$$\begin{array}{rrcl} \mathcal{G}: & L^2(0,T;\mathcal{H}^1_\#(\Omega_0)) & \longrightarrow & X^{e,s,\kappa}_T = X^{e,s}_T \times E^\kappa_T \\ & \overline{q}_e & \longmapsto & (\mathbf{v}_e, \mathbf{v}_s, q_e, \eta) \text{ solution of system (3.24)} \end{array}$$

$$\begin{array}{rcll} \mathbf{v}_{e,t} - \operatorname{div}(\mathbf{v}_e, q_e) & = & P\mathbf{f} & \text{in } Q^0_T, \\ \mathbf{v}_e & = & -\gamma_\tau \nabla N_s(\eta_t) & \text{on } \Sigma^0_T, \\ \mathbf{v}_e(0) & = & P\mathbf{v}^0 & \text{in } \Omega_0, \\ \mathbf{v}_s & = & \nabla N_s(\eta_t) & \text{in } Q^0_T, \\ (I + \gamma^s_\# N_s)\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} & = & \gamma^s_\# \overline{q}_e + \widetilde{h} & \text{on } \Sigma^{s,0}_T, \\ (\eta(0), \eta_t(0)) & = & (\eta^{1,0}, \eta^{2,0}) & \\ q & = & q_e + q_s & \text{in } Q^0_T, \\ \mathbf{v} & = & \mathbf{v}_e + \mathbf{v}_s & \text{in } Q^0_T, \\ q_s & = & \pi(\mathbf{f}) - N_s(\eta_{tt}) & \text{in } Q^0_T. \end{array} \tag{3.24}$$

We have the following result.

Proposition 3.9. *The mapping $\mathcal{G}$ is well-defined from $L^2(0,T;\mathcal{H}^1_\#(\Omega_0))$ into $X^{e,s,\kappa}_T$. Moreover, we have the estimate, for $\theta = \frac{1}{2-\kappa} - \frac{1}{2} > 0$:*

$$\|\mathcal{G}(\overline{q}_e)\|_{X^{e,s,\kappa}_T} \leq C\Big(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^{0,\varepsilon}} + \|(\mathbf{f}, h)\|_{Z_T} + T^\theta \|\overline{q}_e\|_{L^2(0,T;\mathcal{H}^1_\#(\Omega_0))}\Big). \tag{3.25}$$

Furthermore, for two pressures $\overline{q}_{e,1}$ and $\overline{q}_{e,2}$ in $L^2(0,T;\mathcal{H}^1_\#(\Omega_0))$, the term $\mathcal{G}(\overline{p}_{e,1}) - \mathcal{G}(\overline{p}_{e,2}) = (\mathbf{v}_{e,1} - \mathbf{v}_{e,2}, \mathbf{v}_{s,1} - \mathbf{v}_{s,2}, q_{e,1} - q_{e,2}, \eta_1 - \eta_2)$ is the solution of the system corresponding with $\mathcal{G}(\overline{q}_{e,1} - \overline{q}_{e,2})$ in (3.24) with zero for initial data and right-hand sides. Moreover, $\mathcal{G}(\overline{q}_{e,1}) - \mathcal{G}(\overline{q}_{e,2})$ satisfies the estimate

$$\|\mathcal{G}(\overline{q}_{e,1}) - \mathcal{G}(\overline{q}_{e,2})\|_{X^{e,s,\kappa}_T} \leq cT^\theta \|\overline{q}_{e,1} - \overline{q}_{e,2}\|_{L^2(0,T;\mathcal{H}^1_\#(\Omega_0))}.$$

From the mapping $\mathcal{G}$, we define another mapping $\mathcal{F}$ from $L^2(0,T;\mathcal{H}^1(\Omega))$ into itself defined by $\mathcal{F} = \mathcal{P} \circ \mathcal{G}$ where $\mathcal{P}$ is the projection from $X^{e,s,\kappa}_T$ into $L^2(0,T;\mathcal{H}^1_\#(\Omega_0))$ defined by $\mathcal{P}(\mathbf{v}_e, \mathbf{v}_s, q_e, \eta) = q_e$.

Proposition 3.10. *$\mathcal{F}$ is well-defined from $L^2(0,T;\mathcal{H}^1_\#(\Omega_0))$ into itself and, for any $R > 0$, there exists a time $T^* > 0$ such that $\mathcal{F}$ is a contraction in*

$$\mathcal{B}_{L^2(0,T^*;\mathcal{H}^1_\#(\Omega_0))}(R) = \Big\{r_e \in L^2(0,T^*;\mathcal{H}^1_\#(\Omega_0)) \text{ such that } \|r_e\|_{L^2(0,T^*;\mathcal{H}^1_\#(\Omega_0))} \leq R\Big\}.$$

The proofs of Propositions 3.9 and 3.10 can be easily adapted from the results of the previous sections by following the proofs of Propositions 2.10 and 2.11.

We now can prove Theorem 3.6. By the Banach fixed point Theorem, the previous proposition is equivalent to the existence of a unique solution $(\mathbf{v}_e, \mathbf{v}_s, q_e, \eta)$ of system (3.24) in $X^{e,s,\kappa}_{T^*}$. Using the same method but beginning the procedure with $\overline{q}_e = q_e$ on $(0, T^*)$ and $\overline{q}_e = 0$ on $(T^*, 2T^*)$ where q_e is the previous solution on $(0, T^*)$, the same estimates occur, then the Banach fixed point Theorem can be apply on $(0, 2T^*)$. It gives a solution which extends on $(0, 2T^*)$ the previous one found on $(0, T^*)$. Finally, repeting the same idea enough times, we prove the existence of a solution $(\mathbf{v}_e, \mathbf{v}_s, q_e, \eta)$ of system (3.24) in $X^{e,s,\kappa}_T$. Then, we use the following proposition to get a better regularity for η (and then for $\mathbf{v}_s$):

Proposition 3.11. *Let $(\eta^{1,0}, \eta^{2,0})$ be in $H^{2+\varepsilon}_\#(\Gamma^s_0) \times H^{1+\varepsilon}_\#(\Gamma^s_0)$ and let $\overline{q}_e$ be in $L^2(0,T;\mathcal{H}^1_\#(\Omega_0))$ and $\widetilde{h}$ in $L^2(0,T;H^{1/2}_\#(\Gamma^s_0))$. Then, equation (3.21) admits a unique solution η in*

$$\mathcal{E}^\varepsilon_T = H^1(0,T;H^{2+\varepsilon}_\#(\Gamma^s_0)) \cap H^2(0,T;H^\varepsilon_\#(\Gamma^s_0))$$

with the estimate:

$$\|\eta\|_{\mathcal{E}^\varepsilon_T} \leq C\Big(\|(\eta_0, \eta_1)\|^2_{H^{2+\varepsilon}_\#(\Gamma^s_0) \times H^{1+\varepsilon}_\#(\Gamma^s_0)} + \|\overline{q}_e\|^2_{L^2(0,T;\mathcal{H}^1_\#(\Omega_0))} + \|\widetilde{h}\|^2_{L^2(0,T;H^{1/2}_\#(\Gamma^s_0))}\Big).$$

Proof. First, thanks to a classical result (see Theorem 2.12 in Chapter 2), if $Y^0 = (\eta^{1,0}, \eta^{2,0})^{\mathbf{tr}}$ belongs to $[D(\mathcal{A}_{\beta,\gamma}), H_s]_{1/2}$, f belongs to $L^2(\Sigma_T^{s,0})$, we know that equation (3.22) admits a unique solution $Y = (\eta, \eta_t)^{\mathbf{tr}}$ in the space

$$L^2(0,T; D(\mathcal{A}_{\beta,\gamma})) \cap H^1(0,T; H_s) \cap \mathcal{C}([0,T]; [D(\mathcal{A}_{\beta,\gamma}), H_s]^{1/2}).$$

Thanks to the calculations above $D(\mathcal{A}_{\beta,\gamma}) = H^2_\#(\Gamma_0^s) \times H^2_\#(\Gamma_0^s)$ on $H_s = H^2_\#(\Gamma_0^s) \times L^2_{\#,0}(\Gamma_0^s)$. Thus, the solution $Y = (\eta, \eta_t)^{\mathbf{tr}}$ of (3.22) belongs to

$$L^2(0,T; H^2_\#(\Gamma_0^s) \times H^2_\#(\Gamma_0^s)) \cap H^1(0,T; H^2_\#(\Gamma_0^s) \times L^2_{\#,0}(\Gamma_0^s)) \cap \mathcal{C}^0([0,T]; H^2_\#(\Gamma_0^s) \times H^1_\#(\Gamma_0^s)),$$

that is, equation (3.21) admits a solution η in the space

$$H^1(0,T; H^2_\#(\Gamma_0^s)) \cap H^2(0,T; L^2_{\#,0}(\Gamma_0^s)).$$

We consider now a right-hand side $\gamma^s_\# \overline{q}_e + \widetilde{h}$ in $L^2(0,T; H^1_\#(\Gamma_0^s))$ and we set $\mu = \eta_x$. Now, we want to apply Theorem 2.12 to the system satisfies by μ, where η is the solution of the previous system. We get formally that μ satisfies equation

$$\begin{aligned} (I + \gamma^s_\# N_s)\mu_{tt} - \beta\mu_{xx} - \gamma\mu_{txx} &= \left[\gamma^s_\# \overline{q}_e + \widetilde{h}\right]_x \\ (\mu(0), \mu_t(0)) &= (\eta^0_x, \eta^1_x) \end{aligned} \tag{3.26}$$

Then, for $(\eta_x^{1,0}, \eta_x^{2,0})$ in $H^2_\#(\Gamma_0^s) \times H^1_\#(\Gamma_0^s)$, that is $(\eta^{1,0}, \eta^{2,0})$ in $H^3_\#(\Gamma_0^s) \times H^2_\#(\Gamma_0^s)$, we get

$$\mu \in H^1(0,T; H^2_\#(\Gamma_0^s)) \cap H^2(0,T; L^2_{\#,0}(\Gamma_0^s)),$$

that is

$$\eta \in H^1(0,T; H^3_\#(\Gamma_0^s)) \cap H^2(0,T; H^1_\#(\Gamma_0^s)).$$

Finally, by interpolation, for $(\eta^{1,0}, \eta^{2,0})$ in $[H^3_\#(\Gamma_0^s) \times H^2_\#(\Gamma_0^s), H^2_\#(\Gamma_0^s) \times H^1_\#(\Gamma_0^s)]_{1-\varepsilon}$, we get, provided that $\gamma^s_\# \overline{q}_e + h$ belongs to $L^2(0,T; [H^1_\#(\Gamma_0^s), L^2_{\#,0}(\Gamma_0^s)]_{1-\varepsilon})$ (that is for $0 < \varepsilon \le 1/2$),

$$\eta \in H^1(0,T; [H^3_\#(\Gamma_0^s), H^2_\#(\Gamma_0^s)]_{1-\varepsilon}) \cap H^2(0,T; [H^1_\#(\Gamma_0^s), L^2_{\#,0}(\Gamma_0^s)]_{1-\varepsilon}).$$

The previous result means that for $(\eta^{1,0}, \eta^{2,0})$ in $[H^3_\#(\Gamma_0^s) \times H^2_\#(\Gamma_0^s), H^2_\#(\Gamma_0^s) \times H^1_\#(\Gamma_0^s)]_{1-\varepsilon} = H^{2+\varepsilon}_\#(\Gamma_0^s) \times H^{1+\varepsilon}_\#(\Gamma_0^s)$, for $0 < \varepsilon \le 1/2$ and for $\gamma^s_\# q + h$ in $L^2(0,T; H^{1/2}_\#(\Gamma_0^s))$, we get $\eta \in H^1(0,T; H^{2+\varepsilon}_\#(\Gamma_0^s)) \cap H^2(0,T; H^\varepsilon_\#(\Gamma_0^s))$. □

3.6 Proof of Theorems 3.2 and 3.3.

We now have all the tools to prove the main results of this chapter. We first use a fixed point procedure to prove existence ans uniqueness of system (3.6) in the fixed cylinder Q_T^0. Then, using Definition 3.4, we prove the existence and uniqueness of system (3.2)–(3.4).

3.6.1 In the cylindrical domain $Q_T^0 = (0,T) \times \Omega_0$.

We use a second fixed point procedure. First, we have to estimate $(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], h[\mathbf{u}, \eta])$ (defined in (3.7)) in terms of $(\mathbf{u}, p, \eta)$ in X_T^ε. Namely, we have the following result:

Proposition 3.12. *Let* $(\mathbf{u}, p, \eta)$ *be in* X_T^ε*, defined in* (3.17)*, then* $(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], H[\mathbf{u}, \eta])$*, obtained from* $(\mathbf{u}, p, \eta)$ *in* (3.7)*, belongs to*

$$W_T = \mathbf{L}^2_\#(Q_T^0) \times G_T \times L^2(0,T; H^{1/2}_\#(\Gamma_0^s)).$$

Furthermore, there exists $\delta > 0$ such that

$$\|(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], H[\mathbf{u}, \eta])\|_{W_T} \leq c_2 T^\delta (1 + \|(\mathbf{u}, p, \eta)\|_{X_T^\varepsilon})\|(\mathbf{u}, p, \eta)\|_{X_T^\varepsilon}^2. \tag{3.27}$$

Let $(\mathbf{u}_1, p_1, \eta_1)$ and $(\mathbf{u}_2, p_2, \eta_2)$ be two triplets in X_T^ε such that for $i = 1, 2$, $\|(\mathbf{u}_i, p_i, \eta_i)\|_{X_T^\varepsilon} \leq R$ for some $R > 0$, we get

$$\|(\mathbf{F}_1, \mathbf{w}_1, H_1) - (\mathbf{F}_2, \mathbf{w}_2, H_2)\|_{W_T} \leq C(1 + R)RT^\delta \|(\mathbf{u}_1, p_1, \eta_1) - (\mathbf{u}_2, p_2, \eta_2)\|_{X_T^\varepsilon}$$

with the notations $(\mathbf{F}_i, \mathbf{w}_i, H_i) = (\mathbf{F}[\mathbf{u}_i, p_i, \eta_i], \mathbf{w}[\mathbf{u}_i, \eta_i], H[\mathbf{u}_i, \eta_i])$.

Proof. The proof relies on the Lemmas 2.15 and 2.16 in Chapter 2. More precisely, the smoothness of η gives the good estimates of the different products. Indeed, η in $\mathcal{E}_T^\varepsilon$ gives by interpolation

$$\eta \in H^{1+\delta}\left(0, T; H_\#^{2(1-\delta)+\varepsilon}(\Gamma_0^s)\right), \quad 0 < \delta < 1.$$

Then,

$$\eta \in \mathcal{C}([0, T]; H_\#^2(\Gamma_0^s)), \quad \eta_x \in L^\infty(\Sigma_T^{s,0}), \quad \eta_{xx} \in H^1(0, T; H_\#^\varepsilon(\Gamma_0^s)) \quad \text{and} \quad \eta_{tx} \in L^\infty(0, T; L_\#^2(\Gamma_0^s)).$$

The main difference in this part is the space of the divergence term G_T. As already mentionned, we have to prove that $\mathbf{w}[\mathbf{u}, \eta]$ satisfies

$$\operatorname{div} \mathbf{w}[\mathbf{u}, \eta] \in L^2(0, T; H_\#^1(\Omega_0)), \qquad \mathbf{w}_t[\mathbf{u}, \eta] \in \mathbf{L}_\#^2(Q_T^0).$$

The worst term to estimate in all the different calculations is $\eta_{xx} u_{1,z}$. From Proposition B.1 in [12], we have

$$\|\eta_{xx} u_{1,z}\|_{L_\#^2(\Omega_0)} \leq C\|\eta_{xx}\|_{H_\#^\varepsilon(\Gamma_0^s)} \|u_{1,z}\|_{H_\#^1(\Omega_0)}. \tag{3.28}$$

Then, because η_{xx} belongs to $H^1(0, T; H_\#^\varepsilon(\Gamma_0^s))$ and $H^1(0, T) \hookrightarrow L^\infty(0, T)$, we have

$$\|\eta_{xx}\|_{L^\infty(0,T;H_\#^\varepsilon(\Gamma_0^s))} \leq CT^\delta \|\eta\|_{H^1(0,T;H_\#^\varepsilon(\Gamma_0^s))}, \qquad \text{for } \delta > 0.$$

This gives

$$\begin{aligned} \|\eta_{xx} u_{1,z}\|_{L_\#^2(Q_T^0)} &\leq CT^\delta \|\eta\|_{H^1(0,T;H_\#^\varepsilon(\Gamma_0^s))} \|u_{1,z}\|_{L^2(0,T;H_\#^1(\Omega_0))} \\ &\leq CT^\delta \|(\mathbf{u}, p, \eta)\|_{X_T^\varepsilon}^2. \end{aligned}$$

The other terms can be estimates using the classic Sobolev embeddings. □

With this proposition, we follow exactly the proof of the fixed point procedure in section 2.6. Namely, we now state Proposition 3.13 corresponding to Proposition 2.17 in Chapter 2.

Proposition 3.13. *For a given triplet $(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})$ in X_T^ε, system (3.6) with right-hand sides $(\overline{\mathbf{F}}, \overline{\mathbf{w}}, \overline{H}) = (\mathbf{F}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], \mathbf{w}[\overline{\mathbf{u}}, \overline{\eta}], H[\overline{\mathbf{u}}, \overline{\eta}])$ and initial data $(\mathbf{u}^0, \eta^0, \eta^1)$ in $X^{0,\varepsilon}$ satisifying (3.14) admits a unique solution $(\mathbf{u}, p, \eta)$ in X_T^ε with the estimate*

$$\|(\mathbf{u}, p, \eta)\|_{X_T^\varepsilon} \leq c_1(\|(\mathbf{u}^0, \eta^0, \eta^1)\|_{X^{0,\varepsilon}} + c_2 T^\delta (1 + \|(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})\|_{X_T^\varepsilon})\|(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})\|_{X_T^\varepsilon}^2) \tag{3.29}$$

where $\delta > 0$ is defined in Proposition 3.12. In other terms, we can construct a mapping

$$\begin{array}{rccc} \mathcal{X}_T : & X_T^\varepsilon & \longrightarrow & X_T^\varepsilon \\ & (\overline{\mathbf{u}}, \overline{p}, \overline{\eta}) & \longmapsto & \mathcal{X}_T(\overline{\mathbf{u}}, \overline{p}, \overline{\eta}) = (\mathbf{u}, p, \eta) \text{ is a solution of the system (3.6)} \\ & & & \text{with } (\mathbf{F}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], \mathbf{w}[\overline{\mathbf{u}}, \overline{\eta}], H[\overline{\mathbf{u}}, \overline{\eta}]) \text{ for right-hand sides.} \end{array} \tag{3.30}$$

which satisfies

$$\begin{aligned} &\|\mathcal{X}_T(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})\|_{X_T^\varepsilon} \\ \leq\ & c_1\Big(\|(\mathbf{u}^0, \eta^0, \eta^1)\|_{X^{0,\varepsilon}} + c_2 T^\delta (1 + \|(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})\|_{X_T^\varepsilon})\|(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})\|_{X_T^\varepsilon}^2\Big). \end{aligned} \tag{3.31}$$

Proof. The proof relies directly on Theorem 3.6 and Proposition 3.12. Indeed, from Proposition 3.12, we know that for a triplet $(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})$ in X_T^ε, the triplet $(\mathbf{F}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], \mathbf{w}[\overline{\mathbf{u}}, \overline{\eta}], H[\overline{\mathbf{u}}, \overline{\eta}])$ belongs to W_T and in particular that $\mathbf{w}[\overline{\mathbf{u}}, \overline{\eta}]$ belongs to G_T. Thus first, the lifting $(\mathbf{z}[\overline{\mathbf{u}}, \overline{\eta}], \pi[\overline{\mathbf{u}}, \overline{\eta}])$ belongs to $\mathbf{H}^{2,1}_\#(Q_T^0) \times L^2(0,T;\mathcal{H}^1_\#(\Omega_0))$ and that $(\mathbf{z}[\mathbf{u}^0, \eta^{1,0}]) = \mathbf{z}[\mathbf{u}, \eta](0)$ is well-defined thanks to the embedding $\mathbf{H}^{2,1}_\#(Q_T^0) \hookrightarrow \mathcal{C}([0,T]; \mathbf{H}^1_\#(\Omega_0))$ and satisfies the estimates

$$\|\mathbf{z}[\overline{\mathbf{u}}, \overline{\eta}]\|_{\mathbf{H}^{2,1}_\#(Q_T^0)} + \|\pi[\overline{\mathbf{u}}, \overline{\eta}])\|_{L^2(0,T;\mathcal{H}^1_\#(\Omega_0))} \leq C\|\mathbf{w}[\overline{\mathbf{u}}, \overline{\eta}]\|_{G_T}.$$

and

$$\|\mathbf{z}[\mathbf{u}^0, \eta^{1,0}]\|_{\mathbf{H}^1_\#(\Omega_0)} \leq C\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^{0,\varepsilon}}.$$

Second, the couple $(\mathbf{f}[\overline{\mathbf{u}}, \overline{\eta}], h[\overline{\mathbf{u}}, \overline{\eta}])$ defined in (3.10) and (3.11) belongs to Z_T and the initial data $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ in $X^{0,\varepsilon}$ satisfying (3.14) gives that $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$, with $\mathbf{v}^0$ defined in (3.12), belongs to $X^{0,\varepsilon}_{\mathbf{cc}}$. We now apply Theorem 3.6 to obtain a unique solution $(\mathbf{v}, q, \eta)$ of system (3.9) with $(\mathbf{f}[\overline{\mathbf{u}}, \overline{\eta}], h[\overline{\mathbf{u}}, \overline{\eta}])$ for right-hand side and $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ for initial data. Then, the correspondance between systems (3.9) and (3.6) gives that system (3.6) admits a unique solution $(\mathbf{u}, p, \eta)$ in X_T^ε satsifying the estimate

$$\|(\mathbf{u}, p, \eta)\|_{X_T^\varepsilon} \leq C_1 \left(\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^{0,\varepsilon}} + \|(\mathbf{F}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], \mathbf{w}[\overline{\mathbf{u}}, \overline{\eta}], H[\overline{\mathbf{u}}, \overline{\eta}])\|_{W_T}\right).$$

Now, using the estimate in Proposition 3.12, we get the estimate of the proposition. □

Then, we can write Theorems 3.2 and 3.3 in the fixed domain, acting in (3.31) either on the time of existence or on the smallness of the initial data.

Proposition 3.14. *Let* $(\mathbf{u}^0, \eta^0, \eta^1)$ *be in* $X^{0,\varepsilon}$ *satisifying* (3.14).

(i) There exists a time $T_0 > 0$ *such that system* (3.6) *admits a unique local strong solution* $(\mathbf{u}, p, \eta)$ *in* $X_{T_0}^\varepsilon$.

(ii) There exists r *small enough such that, under condition* $\|(\mathbf{u}^0, \eta^0, \eta^1)\|_{X^{0,\varepsilon}} \leq r$*, system* (3.6) *admits a unique global strong solution* $(\mathbf{u}, p, \eta)$ *in* X_T^ε.

3.6.2 In the domain Q_T^η.

The regularity of the solution $(\mathbf{u}, p, \eta)$ of system (3.6) in both cases of Proposition 3.14 gives that the change of variables

$$\begin{array}{rccc} \phi^{-1}_{\eta(t)}: & \Omega_0 & \longrightarrow & \Omega_{\eta(t)} \\ & (x,z) & \longmapsto & (x,y) = (x, (1+\eta(t,x))z) \end{array}$$

is a $\mathcal{C}^1-$diffeomorphism from Ω_0 into $\Omega_{\eta(t)}$ because η is smooth and satisfies condition (3.1). Indeed, by a Sobolev embedding, we have

$$\|\eta\|_{L^\infty(\Sigma_T^{s,0})} \leq c\|\eta\|_{\mathcal{E}_T^\varepsilon}$$

and up to a change of T_0 in the first case of Proposition 3.14 or a change to r in the second case, we can always prescribe $\|\eta\|_{\mathcal{E}_T^\varepsilon} \leq \frac{1-\delta_0}{c}$ and thus

$$\|\eta\|_{L^\infty(\Sigma_T^{s,0})} < 1 - \delta_0,$$

that is η satisfying assumption (3.1).

All these results set us in the case of Definition 3.4.

Chapitre 4

Existence et unicité de solution pour un système couplant les équations de Navier-Stokes et une équation des plaques/des ondes amorties en trois dimensions

In this chapter, we consider the two different kinds of coupled fluid-structure model introduced in the Chapters 2 and 3. More precisely, we study the three dimensional cases of the previous systems. That is the Navier-Stokes equations coupled first with a plate equation and second with a strongly damped wave equation. These two last equations correspond with the two different cases $\alpha > 0$ and $\alpha = 0$ in the plate equation, see (4.3).

In these cases, the strategy of the proof is to use a Banach fixed point precedure. In the first case, the estimates of the nonlinear terms are easily done thanks to the high regularity of the solution of the plate equation (see Chapter 2 and section 4.1). In the second case, we consider the periodic setting (in the first two space variables) due to the low regularity of the solution of the damped wave equation, see Chapter 3 and section 4.2 for details.

4.1 Navier-Stokes equations and plate equation.

Let ω_0 be a bounded open subset of $\mathbb{R}^2$ with smooth boundary $\partial\omega_0$. We define the cylindrical domain Ω_0 by $\Omega_0 = \omega_0 \times (0,1)$. In our problem, the domain $\Gamma_0^s = \omega_0 \times \{1\}$ is the reference state of the plate. It corresponds with the configuration at rest of the plate. This plate is characterized by the vertical displacement η from the reference state. It depends on the time t and on the position (x,y) in ω_0. A priori, the function η is defined from $\mathbb{R}^+ \times \omega_0$ into $(-1,+\infty)$. At time t, the displacement of the plate define the domain $\Gamma^s_{\eta(t)}$ by

$$\Gamma^s_{\eta(t)} = \Big\{(x,y,z) \in \mathbb{R}^3 \text{ s.t. } (x,y) \in \omega_0 \text{ and } z = 1 + \eta(t,x,y)\Big\}.$$

Then, at time t, the fluid occupies the domain $\Omega_{\eta(t)}$ is defined by a subgraph. That is

$$\Omega_{\eta(t)} = \Big\{(x,y,z) \in \mathbb{R}^3 \text{ s.t. } (x,y) \in \omega_0 \text{ and } 0 < z < 1 + \eta(t,x,y)\Big\}.$$

The displacement η has to satisfy the following assumption

$$\exists \delta_0 > 0 \text{ such that } \forall t \geq 0\ \forall (x,y) \in \omega_0 \quad 1 + \eta(t,x,y) \geq \delta_0 > 0 \tag{4.1}$$

to ensure that, for every time t, there is no contact between the boundaries of $\Omega_{\eta(t)}$.

The fixed part of the boundary of $\Omega_{\eta(t)}$ is denoted Γ. It consists in two parts: the lateral part denoted Γ_l and the bottom part Γ_b corresponding with $z = 0$, see Figure 4.1. Namely,

$$\Gamma_l = \partial\omega_0 \times (0,1), \quad \Gamma_b = \omega_0 \times \{0\} \qquad \text{and} \qquad \Gamma = \Gamma_l \cup \Gamma_b.$$

Then, the unit normal to $\partial\Omega_{\eta(t)}$ outward $\Omega_{\eta(t)}$ has the three following expressions depending on the position on the boundary:

- on $\Gamma^s_{\eta(t)}$,

$$\mathbf{n}(t) = \frac{1}{\sqrt{1+\eta_x^2(t)+\eta_y^2(t)}} \begin{pmatrix} -\eta_x(t) \\ -\eta_y(t) \\ 1 \end{pmatrix},$$

- on Γ_l, $\mathbf{n}(t) = (\mathbf{n}_0, 0)$ where $\mathbf{n}_0$ is the unit normal to $\partial\omega_0$ outward ω_0 in $\mathbb{R}^2$,
- on Γ_b, $\mathbf{n}(t) = -\mathbf{e}_3$.

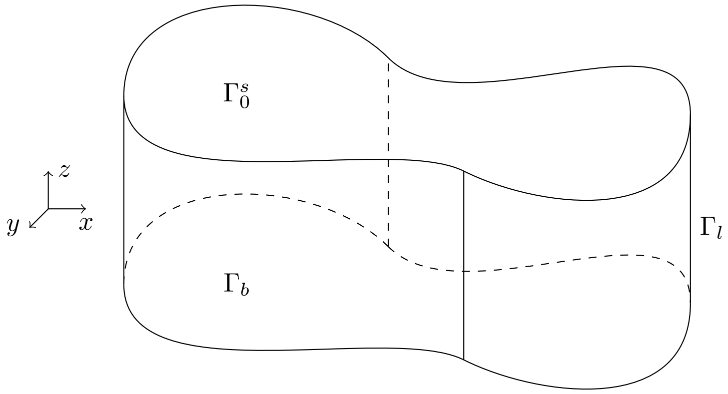

Figure 4.1: The domain Ω_0 for $\alpha > 0$.

Let $T > 0$, we introduce some notations.

$$Q_T^\eta = \bigcup_{t\in(0,T)} \{t\}\times\Omega_{\eta(t)}, \quad \Sigma_T^{s,\eta} = \bigcup_{t\in(0,T)} \{t\}\times\Gamma^s_{\eta(t)}, \quad \Sigma_T^0 = (0,T)\times\Gamma_0, \quad \Gamma_0 = \partial\Omega_0 = \Gamma_0^s \cup \Gamma,$$
$$Q_T^0 = (0,T)\times\Omega_0, \quad \Sigma_T^{s,0} = (0,T)\times\Gamma_0^s, \quad \sigma_T^{s,0} = (0,T)\times\partial\Gamma_0^s, \quad \Sigma_T = (0,T)\times\Gamma.$$

The equations of the system are the three dimensional Navier-Stokes equations in the variables $(\mathbf{u}, p)$ respectively the velocity and the pressure of the fluid

$$\begin{array}{rcll} \mathbf{u}_t - \operatorname{div}\sigma(\mathbf{u},p) + (\mathbf{u}\cdot\nabla)\mathbf{u} &=& \mathbf{0} & \text{in } Q_T^\eta, \\ \operatorname{div}\mathbf{u} &=& 0 & \text{in } Q_T^\eta, \\ \mathbf{u} &=& \eta_t\mathbf{e}_3 & \text{on } \Sigma_T^{s,\eta}, \\ \mathbf{u} &=& \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}(0) &=& \mathbf{u}^0 & \text{in } \Omega_{\eta^{1,0}} \end{array} \tag{4.2}$$

and the two dimensional plate equation:

$$\begin{array}{rcll} \eta_{tt} + \alpha\Delta_s^2\eta - \beta\Delta_s\eta - \gamma\Delta_s\eta_t &=& -\sigma(\mathbf{u},p)\sqrt{1+\eta_x^2+\eta_y^2}\,\mathbf{n}\cdot\mathbf{e}_3 & \text{on } \Sigma_T^{s,0}, \\ \eta &=& 0 & \text{on } \sigma_T^{s,0}, \\ \dfrac{\partial\eta}{\partial\mathbf{n}_s} &=& 0 & \text{on } \sigma_T^{s,0}, \\ (\eta(0),\eta_t(0)) &=& (\eta^{1,0},\eta^{2,0}) & \text{in } \Gamma_0^s. \end{array} \tag{4.3}$$

In these equations, $\sigma(\mathbf{u},p)$ is the Cauchy stress tensor defined by $\sigma(\mathbf{u},p) = \nu\Big(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{tr}}\Big) - p\mathbf{I}$. In equation (4.3), the vector $\mathbf{n}_s$ is the unit normal vector to $\partial\Gamma_0^s$ outward Γ_0^s, that is $\mathbf{n}_s = (\mathbf{n}_0, 0)$.

Note that the right-hand side of (4.3) is taken in the variables (x,y,z) on the boundary $\Sigma_T^{s,\eta}$, that is

$$\mathbf{u} = \mathbf{u}(t,x,y,1+\eta(t,x,y)), \quad p = p(t,x,y,1+\eta(t,x,y)) \quad \text{with } t\in(0,T) \text{ and } (x,y)\in\omega_0.$$

In (4.3), the symbols Δ_s and Δ_s^2 represent respectively the Laplace operator and the bilaplace operator on Γ_0^s defined respectively by

$$D(\Delta_s) = H^2_{(0)}(\Gamma_0^s) \quad \text{and} \quad \Delta_s \mu = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mu \quad \text{for all } \mu \in D(\Delta_s).$$

$$D(\Delta_s^2) = H^4_{(0)}(\Gamma_0^s) \quad \text{and} \quad \Delta_s^2 \mu = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2\mu = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)\mu \quad \text{for all } \mu \in D(\Delta_s^2).$$

In the previous definitions we use the notations:

$$L_0^2(\Gamma_0^s) = \left\{\mu \in L^2(\Gamma_0^s) \text{ s.t. } \int_{\Gamma_0^s} \mu = 0\right\}$$

and

$$H^\sigma_{(0)}(\Gamma_0^s) = \begin{cases} H^\sigma(\Gamma_0^s) \cap L_0^2(\Gamma_0^s) & \text{if } \ 0 \le \sigma \le 1 \\ \left\{\mu \in H^\sigma(\Gamma_0^s) \cap L_0^2(\Gamma_0^s) \quad \text{s.t. } \mu = 0 \text{ on } \partial\Gamma_0^s\right\} & \text{if } \ 1 < \sigma \le 2, \\ \left\{\mu \in H^\sigma(\Gamma_0^s) \cap L_0^2(\Gamma_0^s) \quad \text{s.t. } \mu, \dfrac{\partial \mu}{\partial \mathbf{n}_s} = 0 \text{ on } \partial\Gamma_0^s\right\} & \text{if } \ 2 < \sigma. \end{cases}$$

It means that the plate equation (4.3) is in fact projected on the space $L_0^2(\Gamma_0^s)$. Another way to understand it is to introduce the Lagrange multiplier associated to the constraint of zero mean value. We introduce the projection M_s from $L^2(\Gamma_0^s)$ to $L_0^2(\Gamma_0^s)$ defined by

$$M_s\mu = \mu - \frac{1}{|\Gamma_0^s|}\int_{\Gamma_0^s} \mu \qquad \text{for all } \mu \in L^2(\Gamma_0^s),$$

and the trace operator γ_s associated to M_s defined by

$$\gamma_s q = M_s(q_{|\Gamma_0^s}) = q_{|\Gamma_0^s} - \frac{1}{|\Gamma_0^s|}\int_{\Gamma_0^s} q_{|\Gamma_0^s} \quad \text{for all } q \in H^\sigma(\Omega_0) \text{ with } \sigma > \frac{1}{2}.$$

We remark that for μ in $H^3_{(0)}(\Gamma_0^s)$, then

$$\int_{\Gamma_0^s} \Delta_s \mu = -\int_{\Gamma_0^s} \nabla\mu \cdot \nabla 1 + \int_{\partial\Gamma_0^s} \frac{\partial \mu}{\partial \mathbf{n}_s} 1 = 0,$$

thus $\Delta_s\eta$ already belongs to $L_0^2(\Gamma_0^s)$ for η in $H^3_{(0)}(\Gamma_0^s)$. But the terms $\Delta_s^2\eta$ and $\Delta_s\eta_t$ does not belong *a priori* to $L_0^2(\Gamma_0^s)$ in the left-hand side of the plate equation (4.3). We have to rewrite it as follows:

$$\begin{array}{rcll} \eta_{tt} + \alpha M_s \Delta_s^2 \eta - \beta \Delta_s \eta - \gamma M_s \Delta_s \eta_t & = & -\gamma_s\left[\sigma(\mathbf{u}, p)\sqrt{1 + \eta_x^2 + \eta_y^2}\mathbf{n} \cdot \mathbf{e}_3\right] & \text{on } \Sigma_T^{s,0}, \\ \eta & = & 0 & \text{on } \sigma_T^{s,0}, \\ \dfrac{\partial \eta}{\partial \mathbf{n}_s} & = & 0 & \text{on } \sigma_T^{s,0}, \\ (\eta(0), \eta_t(0)) & = & (\eta^{1,0}, \eta^{2,0}) & \text{in } \Gamma_0^s. \end{array} \tag{4.4}$$

4.1.1 Main results.

This part of the chapter is devoted to the proof of Theorems 4.1 and 4.2 which extend to the three dimensional case the results of Chapter 2.

Theorem 4.1. *Let* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0}) \in \mathbf{V}^1(\Omega_{\eta^{1,0}}) \times H^3_{(0)}(\Gamma_0^s) \times H^1_{(0)}(\Gamma_0^s)$. *There exists* $R > 0$ *such that for any initial data satisfying*

$$\|\mathbf{u}^0\|^2_{\mathbf{V}^1(\Omega_{\eta^{1,0}})} + \|\eta^{1,0}\|^2_{H^3_{(0)}(\Gamma_0^s)} + \|\eta^{2,0}\|^2_{H^1_{(0)}(\Gamma_0^s)} \le R^2$$

and the compatibility condition

$$\mathbf{u}^0 = \mathbf{0} \quad \textit{in } \Gamma \qquad \textit{and} \qquad \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_3 \quad \textit{on } \Gamma^s_{\eta^{1,0}}, \tag{4.5}$$

system (4.2)–(4.4) *has a unique global strong solution* $(\mathbf{u}, p, \eta)$ *in*

$$\mathbf{V}^{2,1}(Q^\eta_T) \times L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1\left(\Omega_{\eta(t)}\right)\right) \times H^{4,2}_{(0)}(\Sigma^{s,0}_T).$$

Theorem 4.2. *Let* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0}) \in \mathbf{V}^1(\Omega_{\eta^{1,0}}) \times H^3_{(0)}(\Gamma^s_0) \times H^1_{(0)}(\Gamma^s_0)$ *satisfying the compatibility condition* (4.5)*. There exists a time* $T_0 > 0$ *such that system* (4.2)–(4.4) *has a unique strong solution* $(\mathbf{u}, p, \eta) \in \mathbf{V}^{2,1}(Q^\eta_{T_0}) \times L^2\left(\bigcup_{t\in(0,T_0)} \{t\} \times \mathcal{H}^1\left(\Omega_{\eta(t)}\right)\right) \times H^{4,2}_{(0)}(\Sigma^{s,0}_{T_0})$.

The proof of this results follows exactly the one in Chapter 2. Let us state the different steps of the proof. First, by a change of variables, we state an equivalent problem in a fixed cylindrical domain $Q^0_T = (0,T) \times \Omega_0$ in section 4.1.2. Second, we prove the existence of global in time (on the fixed time interval $[0,T]$) strong solutions for the linearized system with nonhomogeneous right-hand sides. The proof of existence of the linearized system does not depend on the dimension, thus this part is exactly the same as in section 2.5 in Chapter 2. Third, by a fixed point precedure, we prove either existence of global in time strong solutions for small initial data or local in time strong solutions for any initial data. These results are both due to the fixed point procedure. Indeed, we prove the contraction by acting either on the initial data or on the length of the time interval.

4.1.2 Change of variables.

As introduced in [4], the change of variables is very simple due to the special form of the domain. Namely, we have

$$\begin{array}{rccl} \phi_{\eta(t)} : & \Omega_{\eta(t)} & \longrightarrow & \Omega_0 \\ & (x,y,z) & \longmapsto & (x,y,z_0) \quad \text{where} \quad z_0 = \dfrac{z}{1+\eta(t,x,y)}. \end{array}$$

Then we can calculate the derivatives of $f(x,y,z)$ using the derivatives of $\hat{f}(x,y,z_0)$:

$$f_t = \hat{f}_t - z_0\frac{\eta_t}{1+\eta}\hat{f}_{z_0}, \qquad f_x = \hat{f}_x - z_0\frac{\eta_x}{1+\eta}\hat{f}_{z_0}, \qquad f_y = \hat{f}_y - z_0\frac{\eta_y}{1+\eta}\hat{f}_{z_0}$$

$$f_z = \frac{1}{1+\eta}\hat{f}_{z_0}, \qquad f_{xx} = \hat{f}_{xx} - 2z_0\frac{\eta_x}{1+\eta}\hat{f}_{xz_0} + \left(z_0\frac{\eta_x}{1+\eta}\right)^2\hat{f}_{z_0z_0} - z_0\frac{(1+\eta)\,\eta_{xx} - \eta_x^2}{(1+\eta)^2}\hat{f}_{z_0},$$

$$f_{yy} = \hat{f}_{yy} - 2z_0\frac{\eta_y}{1+\eta}\hat{f}_{yz_0} + \left(z_0\frac{\eta_y}{1+\eta}\right)^2\hat{f}_{z_0z_0} - z_0\frac{(1+\eta)\,\eta_{yy} - \eta_y^2}{(1+\eta)^2}\hat{f}_{z_0}, \qquad f_{zz} = \frac{1}{(1+\eta)^2}\hat{f}_{z_0z_0}.$$

With this formulas, we can now state the Navier-Stokes equations in the cylindrical domain Q^0_T. The method is to multiply the equation by $1+\eta$ and to put all the nonlinear terms in the right-hand side. Let us write the different terms first:

$$\begin{aligned}
\mathbf{u}_t &= \hat{\mathbf{u}}_t - \frac{z_0\eta_t}{1+\eta}\hat{\mathbf{u}}_{z_0}, \\
\Delta\mathbf{u} &= \hat{\mathbf{u}}_{xx} + \hat{\mathbf{u}}_{yy} + \frac{1}{(1+\eta)^2}\hat{\mathbf{u}}_{z_0z_0} - \frac{2z_0}{1+\eta}\left[\eta_x\hat{\mathbf{u}}_{xz_0} + \eta_y\hat{\mathbf{u}}_{yz_0}\right] + \frac{z_0^2}{(1+\eta)^2}\left[\eta_x^2 + \eta_y^2\right]\hat{\mathbf{u}}_{z_0z_0} \\
&\quad - \frac{z_0}{1+\eta}\left[(1+\eta)(\eta_{xx} + \eta_{yy}) - (\eta_x^2 + \eta_y^2)\right]\hat{\mathbf{u}}_{z_0}, \\
(\mathbf{u}\cdot\nabla)\mathbf{u} &= \hat{u}_1\left[\hat{\mathbf{u}}_x - \frac{z_0\eta_x}{1+\eta}\hat{\mathbf{u}}_{z_0}\right] + \hat{u}_2\left[\hat{\mathbf{u}}_y - \frac{z_0\eta_y}{1+\eta}\hat{\mathbf{u}}_{z_0}\right] + \hat{u}_3\left[\frac{1}{1+\eta}\hat{\mathbf{u}}_{z_0}\right] \\
\nabla p &= \left[\hat{p}_x - \frac{z_0}{1+\eta}\hat{p}_{z_0\eta_x}\right]\mathbf{e}_1 + \left[\hat{p}_y - \frac{z_0\eta_y}{1+\eta}\hat{p}_{z_0}\right]\mathbf{e}_2 + \left[\frac{1}{1+\eta}\hat{p}_{z_0}\right]\mathbf{e}_3.
\end{aligned}$$

Then, equation $\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{0}$ becomes

$$\begin{aligned}\hat{\mathbf{u}}_t - \nu\hat{\Delta}\hat{\mathbf{u}} + \hat{\nabla}\hat{p} \;=\;& -\eta\hat{\mathbf{u}}_t + z_0\eta_t\hat{\mathbf{u}}_{z_0} + \nu\eta(\hat{\mathbf{u}}_{xx} + \hat{\mathbf{u}}_{yy}) - \frac{\nu\eta}{1+\eta}\hat{\mathbf{u}}_{z_0z_0} - 2\nu z_0(\eta_x\hat{\mathbf{u}}_{xz_0} + \eta_y\hat{\mathbf{u}}_{yz_0})\\ &+\frac{\nu z_0^2}{1+\eta}(\eta_x^2+\eta_y^2)\hat{\mathbf{u}}_{z_0z_0} - \nu z_0\Big[(1+\eta)(\eta_{xx}+\eta_{yy}) - (\eta_x^2+\eta_y^2)\Big]\hat{\mathbf{u}}_{z_0}\\ &-(1+\eta)\hat{u}_1\hat{\mathbf{u}}_x - (1+\eta)\hat{u}_2\hat{\mathbf{u}}_y + \Big[z_0(\eta_x\hat{u}_1 + \eta_y\hat{u}_2) - \hat{u}_3\Big]\hat{\mathbf{u}}_{z_0}\\ &+\Big[z_0\eta_x\hat{p}_{z_0} - \eta\hat{p}_x\Big]\mathbf{e}_1 + \Big[z_0\eta_y\hat{p}_{z_0} - \eta\hat{p}_y\Big]\mathbf{e}_2.\end{aligned}$$

In the same way, equation div $\mathbf{u} = 0$ becomes

$$\begin{aligned}\hat{\text{div}}\;\hat{\mathbf{u}} &= -\eta(\hat{u}_{1,x} + \hat{u}_{2,y}) + z_0(\eta_x\hat{u}_{1,z_0} + \eta_y\hat{u}_{2,z_0})\\ &= \hat{\text{div}}\;\hat{\mathbf{w}}[\hat{\mathbf{u}},\eta],\end{aligned}$$

with

$$\hat{\mathbf{w}}[\hat{\mathbf{u}},\eta] = -\eta\hat{u}_1\mathbf{e}_1 - \eta\hat{u}_2\mathbf{e}_2 + z_0(\eta_x\hat{u}_1 + \eta_y\hat{u}_2)\mathbf{e}_3. \tag{4.6}$$

Finally, the right-hand side of the plate equation becomes

$$\begin{aligned}-\sigma(\mathbf{u},p)\sqrt{1+\eta_x^2+\eta_y^2}\,\mathbf{n}\cdot\mathbf{e}_3 &= p - 2\nu u_{3,z} + \nu\eta_x\Big[u_{1,z} + u_{3,x}\Big] + \nu\eta_y\Big[u_{2,z} + u_{3,y}\Big]\\ &= \hat{p} - 2\nu\hat{u}_{3,z} + \frac{\nu}{1+\eta}\Big[2\eta - (\eta_x^2+\eta_y^2)\Big]\hat{u}_{3,z_0} + \frac{\nu\eta_x}{1+\eta}\hat{u}_{1,z_0} + \frac{\nu\eta_y}{1+\eta}\hat{u}_{2,z_0}\\ &\quad+\nu\Big[\eta_x\hat{u}_{3,x} + \eta_y\hat{u}_{3,y}\Big].\end{aligned}$$

Let us drop out the notation $\hat{\cdot}$. The Navier-Stokes equations in the cylindrical domain Q_T^0 are

$$\begin{aligned}\mathbf{u}_t - \text{div}\;\sigma(\mathbf{u},p) &= \mathbf{F}[\mathbf{u},p,\eta] && \text{in } Q_T^0,\\ \text{div}\;\mathbf{u} &= \text{div}\;\mathbf{w}[\mathbf{u},\eta] && \text{in } Q_T^0,\\ \mathbf{u} &= \eta_t\mathbf{e}_3 && \text{on } \Sigma_T^{s,0},\\ \mathbf{u} &= \mathbf{0} && \text{on } \Sigma_T,\\ \mathbf{u}(0) &= \mathbf{u}^0 && \text{in } \Omega_0\end{aligned} \tag{4.7}$$

where the right-hand side $\mathbf{F}[\mathbf{u},p,\eta]$ is

$$\begin{aligned}\mathbf{F}[\mathbf{u},p,\eta] \;=\;& -\eta\mathbf{u}_t + z_0\eta_t\mathbf{u}_{z_0} + \nu\eta(\mathbf{u}_{xx} + \mathbf{u}_{yy}) - \frac{\nu\eta}{1+\eta}\mathbf{u}_{z_0z_0} - 2\nu z_0(\eta_x\mathbf{u}_{xz_0} + \eta_y\mathbf{u}_{yz_0})\\ &+\frac{\nu z_0^2}{1+\eta}(\eta_x^2+\eta_y^2)\mathbf{u}_{z_0z_0} - \nu z_0\Big[(1+\eta)(\eta_{xx}+\eta_{yy}) - (\eta_x^2+\eta_y^2)\Big]\mathbf{u}_{z_0}\\ &-(1+\eta)u_1\mathbf{u}_x - (1+\eta)u_2\mathbf{u}_y + \Big[z_0(\eta_x u_1 + \eta_y u_2) - u_3\Big]\mathbf{u}_{z_0}\\ &+\Big[z_0\eta_x p_{z_0} - \eta p_x\Big]\mathbf{e}_1 + \Big[z_0\eta_y p_{z_0} - \eta p_y\Big]\mathbf{e}_2.\end{aligned} \tag{4.8}$$

The plate equation becomes

$$\begin{aligned}\eta_{tt} + \alpha M_s\Delta_s^2\eta - \beta\Delta_s\eta - \gamma M_s\Delta_s\eta_t &= \gamma_s p - 2\nu\gamma_s u_{3,z} + \gamma_s H[\mathbf{u},\eta] && \text{on } \Sigma_T^{s,0},\\ \eta &= 0 && \text{on } \sigma_T^{s,0},\\ \frac{\partial\eta}{\partial\mathbf{n}_s} &= 0 && \text{on } \sigma_T^{s,0},\\ (\eta(0),\eta_t(0)) &= (\eta^{1,0},\eta^{2,0}) && \text{in } \Gamma_0^s\end{aligned} \tag{4.9}$$

with

$$H[\mathbf{u},\eta] = \frac{\nu}{1+\eta}\Big[2\eta - (\eta_x^2+\eta_y^2)\Big]\hat{u}_{3,z_0} + \frac{\nu\eta_x}{1+\eta}\hat{u}_{1,z_0} + \frac{\nu\eta_y}{1+\eta}\hat{u}_{2,z_0} + \nu\Big[\eta_x\hat{u}_{3,x} + \eta_y\hat{u}_{3,y}\Big]. \tag{4.10}$$

To say that system (4.7)–(4.9) is equivalent to system (4.2)–(4.4), we have to state the following definition:

Definition 4.3. $(\mathbf{u}, p, \eta)$ *in* $\mathbf{H}^{2,1}(Q_T^\eta) \times L^2(\bigcup_{t\in(0,T)}\{t\} \times \mathcal{H}^1(\Omega_{\eta(t)})) \times H^{4,2}_{(0)}(\Sigma_T^{s,0})$ *is solution of* (4.2)–(4.3) *when the following conditions are satisfied:*

(i) $(\hat{\mathbf{u}}, \hat{p}, \eta)$ *obtained for the change of variables* $\hat{\mathbf{u}}(x, y, z_0) = \mathbf{u}(x, y, z)$, $\hat{p}(x, y, z_0) = p(x, y, z)$ *with* $z_0 = \frac{z}{1+\eta(t,x,y)}$ *is a solution of* (4.7)–(4.9),

(ii) for any time t *in* $(0, T)$*, the previous change of variables is a* $\mathcal{C}^1$*-diffeomorphism from* $\Omega_{\eta(t)}$ *into* Ω_0*,*

(iii) η *satisfies condition* (4.1).

We set $\mathbf{u} = \mathbf{v} + \mathbf{w}[\mathbf{u}, \eta]$. Noticing that div $\mathbf{v} = 0$, then $(\mathbf{v}, p, \eta)$ is solution of the system:

$$\begin{array}{rcll} \mathbf{v}_t - \text{div } \sigma(\mathbf{v}, p) &=& \mathbf{f}[\mathbf{u}, p, \eta] & \text{in } Q_T^0, \\ \text{div } \mathbf{v} &=& 0 & \text{in } Q_T^0, \\ \mathbf{v} &=& \eta_t \mathbf{e}_3 - \mathbf{w}[\mathbf{u}, \eta] & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v} &=& -\mathbf{w}[\mathbf{u}, \eta] & \text{on } \Sigma_T, \\ \eta_{tt} + \alpha M_s \Delta_s^2 \eta - \beta \Delta_s \eta - \gamma M_s \Delta_s \eta_t &=& \gamma_s p - 2\nu\gamma_s v_{3,z_0} + h[\mathbf{u}, \eta] & \text{on } \Sigma_T^{s,0}, \\ (\mathbf{v}(0), \eta(0), \eta_t(0)) &=& (\mathbf{u}^0 - \mathbf{w}[\mathbf{u}, \eta](0), \eta^{1,0}, \eta^{2,0}) & \end{array} \tag{4.11}$$

with

$$\begin{array}{rcl} \mathbf{f}[\mathbf{u}, p, \eta] &=& \mathbf{F}[\mathbf{u}, p, \eta] - \mathbf{w}_t[\mathbf{u}, \eta] + \nu \Delta \mathbf{w}[\mathbf{u}, \eta], \\ h[\mathbf{u}, \eta] &=& \gamma_s H[\mathbf{u}, \eta] - 2\nu\gamma_s w_{3,z_0}[\mathbf{u}, \eta]. \end{array} \tag{4.12}$$

The expression of $\mathbf{w}[\mathbf{u}, \eta]$ only depends on u_1, u_2 and η. Then, on Γ_0^s, $\mathbf{w}[\mathbf{u}, \eta] = 0$ due to $u_1 = 0$ and $u_2 = 0$. Furthermore, in the plate equation $(4.11)_5$, the term $-2\nu\gamma_s v_{3,z}$ vanishes. Indeed, for $\mathbf{v}$ in $\mathbf{V}^{2,1}(Q_T^0)$, the conditions div $\mathbf{v} = v_{1,x} + v_{2,y} + v_{3,z_0} = 0$ in Q_T^0 and $v_1 = 0$, $v_2 = 0$ on $\Sigma_T^{s,0}$ give together that $v_{1,x|\Sigma_T^{s,0}} = 0$, $v_{2,y|\Sigma_T^{s,0}} = 0$ and then $v_{3,z_0|\Sigma_T^{s,0}} = 0$.

Finally, system (4.11) is equivalent to the following one:

$$\begin{array}{rcll} \mathbf{v}_t - \text{div } \sigma(\mathbf{v}, p) &=& \mathbf{f}[\mathbf{u}, p, \eta] & \text{in } Q_T^0, \\ \text{div } \mathbf{v} &=& 0 & \text{in } Q_T^0, \\ \mathbf{v} &=& \eta_t \mathbf{e}_3 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v} &=& \mathbf{0} & \text{on } \Sigma_T, \\ \eta_{tt} + \alpha M_s \Delta_s^2 \eta - \beta \Delta_s \eta - \gamma M_s \Delta_s \eta_t &=& \gamma_s p + h[\mathbf{u}, \eta] & \text{on } \Sigma_T^{s,0}, \\ (\mathbf{v}(0), \eta(0), \eta_t(0)) &=& (\mathbf{v}^0, \eta^{1,0}, \eta^{2,0}) & \end{array} \tag{4.13}$$

where

$$\mathbf{v}^0 = \mathbf{u}^0 - \mathbf{w}[\mathbf{u}, \eta](0) = \mathbf{u}^0 + \eta^{1,0} u_1^0 \mathbf{e}_1 + \eta^{1,0} u_2^0 \mathbf{e}_2 - z_0(\eta_x^0 u_1^0 + \eta_y^0 u_2^0)\mathbf{e}_3. \tag{4.14}$$

The compatiblity conditions on the initial data to obtain continuity at time $t = 0$ are the following

$$\text{div } \mathbf{v}^0 = 0 \quad \text{in } \Omega_0, \qquad \mathbf{v}^0 = \eta^{2,0}\mathbf{e}_3 \quad \text{on } \Gamma_0^s \quad \text{and} \quad \mathbf{v}^0 = \mathbf{0} \text{ on } \Gamma. \tag{4.15}$$

That is, for the original initial data $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$:

$$\begin{array}{c} \text{div } \left(\mathbf{u}^0 + \eta^{1,0} u_1^0 \mathbf{e}_1 + \eta^{1,0} u_2^0 \mathbf{e}_2 - z_0(\eta_x^0 u_1^0 + \eta_y^0 u_2^0)\mathbf{e}_3\right) = 0 \quad \text{in } \Omega_0, \\ \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_3 \quad \text{on } \Gamma_0^s \qquad \text{and} \qquad \mathbf{u}^0 = \mathbf{0} \quad \text{on } \Gamma. \end{array} \tag{4.16}$$

4.1.3 Study of the linearized system.

We introduce the following system in the cylindrical domain Q_T^0:

$$\begin{array}{rcll} \mathbf{v}_t - \text{div } \sigma(\mathbf{v}, p) &=& \mathbf{f} & \text{in } Q_T^0, \\ \text{div } \mathbf{v} &=& 0 & \text{in } Q_T^0, \\ \mathbf{v} &=& \eta_t \mathbf{e}_3 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v} &=& \mathbf{0} & \text{on } \Sigma_T, \\ \eta_{tt} + \alpha M_s \Delta_s^2 \eta - \beta \Delta_s \eta - \gamma M_s \Delta_s \eta_t &=& \gamma_s p + h & \text{on } \Sigma_T^{s,0}, \\ (\mathbf{v}(0), \eta(0), \eta_t(0)) &=& (\mathbf{v}^0, \eta^{1,0}, \eta^{2,0}). & \end{array} \tag{4.17}$$

In (4.17), the right-hand sides $(\mathbf{f}, h)$ belong to the space

$$Z_T = \mathbf{L}^2(Q_T^0) \times L^2(0,T; H^{1/2}_{(0)}(\Gamma^s_0)).$$

The initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ is in $X^0_{\mathbf{cc}}$ where

$$X^0 = \mathbf{H}^1(\Omega_0) \times H^3_{(0)}(\Gamma^s_0) \times H^1_{(0)}(\Gamma^s_0)$$

and

$$X^0_{\mathbf{cc}} = \Big\{ (\mathbf{z}^0, \mu^{1,0}, \mu^{2,0}) \in X^0 \text{ s.t. } (\mathbf{z}^0, \mu^{1,0}, \mu^{2,0}) \text{ satisfies (4.15)} \Big\}.$$

We now can state the main result of this section:

Theorem 4.4. *Let $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ in $X^0_{\mathbf{cc}}$ and $(\mathbf{f}, h)$ in Z_T. Then, system* (4.17) *admits a unique solution $(\mathbf{v}, p, \eta)$ in the space*

$$X_T = \Big\{ (\mathbf{z}, q, \mu) \in \mathbf{H}^{2,1}(Q_T^0) \times L^2(0,T; H^1(\Omega_0)) \times H^{4,2}_{(0)}(\Sigma_T^{s,0}) \text{ s.t. } \mathbf{z} = \mathbf{0} \text{ on } \Sigma_T \text{ and } \mathbf{z} = \mu_t \mathbf{e}_3 \text{ on } \Sigma_T^{s,0} \Big\}.$$

Furthermore, the solution $(\mathbf{v}, p, \eta)$ satisfies the estimate

$$\|(\mathbf{u}, p, \eta)\|_{X_T} \leq C_1 \left(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} + \|(\mathbf{f}, h)\|_{Z_T} \right).$$

We do not detail the proof of this result. It can be easily adapted from section 2.5 in Chapter 2. It relies on the decomposition of the Stokes system in two parts *via* the Leray operator P from $\mathbf{L}^2(\Omega_0)$ into $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$ where

$$\mathbf{V}^0_{\mathbf{n}}(\Omega_0) = \Big\{ \mathbf{z} \in \mathbf{L}^2(\Omega_0) \text{ s.t. } \operatorname{div} \mathbf{z} = 0 \text{ in } \Omega_0 \text{ and } \mathbf{z} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_0 \Big\}.$$

The splitting of the Stokes equation makes appear equations in the variables $\mathbf{v}_e = P\mathbf{v}$ and $\mathbf{v}_s = (I - P)\mathbf{v}$ and their associated pressure terms p_e and p_s respectively. After some calculations, we get the following system

$$\begin{array}{rcll}
\mathbf{v}_{e,t} - \nu \Delta \mathbf{v}_e + \nabla p_e & = & P\mathbf{f} & \text{in } Q_T^0, \\
\mathbf{v}_e & = & -\gamma_\tau \mathbf{v}_s & \text{on } \Sigma_T^0, \\
\mathbf{v}_s & = & \nabla N_s(\eta_t) & \text{in } Q_T^0, \\
(I + \gamma_s N_s)\eta_{tt} + \alpha M_s \Delta_s^2 \eta - \beta \Delta_s \eta_{xx} - \gamma M_s \Delta_s \eta_t & = & \gamma_s p_e + \tilde{h} & \text{on } \Sigma_T^{s,0}, \\
(\mathbf{v}_e(0), \eta(0), \eta_t(0)) & = & (P\mathbf{v}^0, \eta^{1,0}, \eta^{2,0}) & \\
p & = & p_e + p_s & \text{in } Q_T^0, \\
\mathbf{v} & = & \mathbf{v}_e + \mathbf{v}_s & \text{in } Q_T^0, \\
p_s & = & \pi(\mathbf{f}) - N_s(\eta_{tt}) & \text{in } Q_T^0.
\end{array} \tag{4.18}$$

One can find the proof of the equivalence between systems (4.18) and (4.17) for the two dimensional case in Chapter 2.

The next part of this section is devoted to the fixed point procedure proving the existence of solution in the fixed domain.

4.1.4 Proof of Theorems 4.1 and 4.2.

In this section, we first prove Theorems 4.1 and 4.2 in the fixed cylindrical domain. Then, we check that the change of variables satisfies the conditions of Definition 4.3.

First, we estimate the right-hand sides $(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], H[\mathbf{u}, \eta])$ in terms of $(\mathbf{u}, p, \eta)$ in the following proposition.

Proposition 4.5. *Let $(\mathbf{u}, p, \eta)$ be in X_T, then the triplet $(\mathbf{F}[\mathbf{u}, \mathbf{w}, \eta], \mathbf{w}[\mathbf{u}, \eta], H[\mathbf{u}, \eta])$ belongs to*

$$W_T = \Big\{ (\mathbf{G}, \mathbf{z}, G) \in \mathbf{L}^2(Q_T^0) \times \mathbf{H}^{2,1}(Q_T^0) \times L^2(0,T; H^1(\Omega_0)) \text{ s.t. } \mathbf{z} = \mathbf{0} \text{ on } \Sigma_T^0 \Big\}$$

endowed with the norm

$$\|(\mathbf{G},\mathbf{z},G)\|_{W_T} = \Big(\|\mathbf{G}\|^2_{\mathbf{L}^2(Q_T^0)} + \|\mathbf{z}\|^2_{\mathbf{H}^{2,1}(Q_T^0)} + \|G\|^2_{L^2(0,T;H^1(\Omega_0))}\Big)^{1/2}.$$

Futhermore, we have the estimate

$$\|(\mathbf{F}[\mathbf{u},\mathbf{w},\eta],\mathbf{w}[\mathbf{u},\eta],H[\mathbf{u},\eta])\|_{W_T} \le C_2T^\delta(1+\|(\mathbf{u},p,\eta)\|_{X_T})\|(\mathbf{u},p,\eta)\|^2_{X_T}. \tag{4.19}$$

Let $(\mathbf{u}_1,p_1,\eta_1)$ *and* $(\mathbf{u}_2,p_2,\eta_2)$ *be two triplets in* X_T *such that for* $i=1,2$, $\|(\mathbf{u}_i,p_i,\eta_i)\|_{X_T} \le R$ *for some* $R>0$, *we get*

$$\|(\mathbf{F}_1,\mathbf{w}_1,H_1)-(\mathbf{F}_2,\mathbf{w}_2,H_2)\|_{W_T} \le C(1+R)RT^\delta\|(\mathbf{u}_1,p_1,\eta_1)-(\mathbf{u}_2,p_2,\eta_2)\|_{X_T}$$

with the notations $(\mathbf{F}_i,\mathbf{w}_i,H_i)=(\mathbf{F}[\mathbf{u}_i,p_i,\eta_i],\mathbf{w}[\mathbf{u}_i,\eta_i],H[\mathbf{u}_i,\eta_i])$.

Proof. The proof follows the one in Chapter 2, that is Lemmas 2.15 and 2.16 and Proposition 2.14. But, in the three dimension case, we have to be very careful. Indeed, for $(\mathbf{u},p,\eta)$ in X_T, it is not obvious that $\mathbf{w}[\mathbf{u},\eta]$ belongs to $\mathbf{H}^{2,1}(Q_T^0)$. Following Proposition B.2 in [12], we obtain (with the notations of this chapter)

$$\|\mathbf{w}[\mathbf{u},\eta]\|_{\mathbf{H}^{2,1}(Q_T^0)} \le C\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T^0)}\|\eta_x\|_{H^{3,3/2}(\Sigma_T^{s,0})}$$

because (with the notations of [12]) $\lambda=2$, $\mu=0$, $\omega=1$, $n=3$ and $d=2$ satisfy

$$3=\lambda+\mu+\omega \ge (n+d)/2 = 5/2.$$

But, this estimate does not make appear a power T^δ in the right-hand side and so, it is not good enough for us.

The estimates are proved tediously by checking that every derivatives (until the second order) of $\mathbf{w}[\mathbf{u},\eta]$ has a meaning in $\mathbf{L}^2(Q_T^0)$. That is, we calculate $\mathbf{w}_x$, $\mathbf{w}_y$, $\mathbf{w}_{z_0}$, $\mathbf{w}_{xx}$, $\mathbf{w}_{yy}$, $\mathbf{w}_{z_0z_0}$, $\mathbf{w}_{xy}$, $\mathbf{w}_{xz_0}$, $\mathbf{w}_{yz_0}$ and $\mathbf{w}_t$. The worst terms to estimate are $\|\eta_{tx}u_1\|_{L^2(Q_T^0)}$ and $\|\eta_{xxx}u_1\|_{L^2(Q_T^0)}$. In both cases, because η belongs to $H^{4,2}_{(0)}(\Sigma_T^{s,0})$, we get that η_{tx} and η_{xxx} belong to $H^{1,1/2}_{(0)}(\Sigma_T^{s,0}) = H^{1/2}(0,T;L^2_0(\Gamma_0^s)) \cap L^2(0,T;H^1_{(0)}(\Gamma_0^s))$.

Then, we use the interpolate spaces for the velocity $\mathbf{u}$. More precisely, $\mathbf{u}$ belongs to $\mathbf{H}^{2,1}(Q_T^0)$ and thus to $H^\theta(0,T;\mathbf{H}^{2(1-\theta)}(\Omega_0))$ for $0\le\theta\le1$. That is, for $\theta=1/2+\kappa$ for κ small enough but non zero (we will see in the calculation below that $0<\kappa<1/4$), we get $\mathbf{u}$ in $H^{1/2+\kappa}(0,T;\mathbf{H}^{1-2\kappa}(\Omega_0))$.

From Proposition B.1 in [12], we get, for any time t and for $0<\kappa<1/4$, that $\eta_{tx}(t)u_1(t)$ belongs to $L^2(\Omega_0)$ and satisfies the estimate

$$\|\eta_{tx}(t)u_1(t)\|_{L^2(\Omega_0)} \le C\|\eta_{tx}\|_{H^1_{(0)}(\Gamma_0^s)}\|u_1(t)\|_{H^{1-2\kappa}(\Omega_0)}. \tag{4.20}$$

Indeed, (with the notations of [12]) for $\lambda=0$, $\mu=1-2\kappa$, $\omega=1$ the inequality $\lambda+\mu+\omega\ge n/2=3/2$ is satsifed only if $\kappa<1/4$ (the limit case $\kappa=1/4$ does not work, see details in [12]).

For $0<\widetilde{\kappa}<\kappa$, we have the embeddings $H^{1/2+\kappa}(0,T)\hookrightarrow H^{1/2+\widetilde{\kappa}}(0,T)\hookrightarrow \mathcal{C}([0,T])$ with the estimate

$$\|f\|_{L^\infty(0,T)} \le C\|f\|_{H^{1/2+\widetilde{\kappa}}(0,T)} \le CT^{\widetilde{\delta}}\|f\|_{H^{1/2+\kappa}(0,T)} \text{ for all } f\in H^{1/2+\kappa}(0,T) \text{ with } \widetilde{\delta}=\frac{1/2+\widetilde{\kappa}}{1/2+\kappa}.$$

Then, from (4.20), taking the $L^2(0,T)-$norm on both sides and using the $L^\infty(0,T)-$norm for the velocity and the $L^2(0,T)-$norm for the displacement on the right-hand side, we get

$$\begin{aligned}\|\eta_{tx}u_1\|_{L^2(Q_T^0)} &\le C\|\eta_{tx}\|_{L^2(0,T;H^1_{(0)}(\Gamma_0^s))}\|u_1(t)\|_{L^\infty(0,T;H^{1-2\kappa}(\Omega_0))}\\ &\le CT^{\widetilde{\delta}}\|\eta\|_{H^{4,2}_{(0)}(\Sigma_T^{s,0})}\|\mathbf{u}\|_{\mathbf{H}^{2,1}(Q_T^0)}.\end{aligned}$$

All the differents terms containing a derivative of η can be estimated as above.

We now consider the terms $u_1\mathbf{u}_x$, $u_2\mathbf{u}_y$ and $u_3\mathbf{u}_{z_0}$ in $\mathbf{F}[\mathbf{u},\eta]$. We follow Lemma 2.16. From Theorem B.3, for u_1 in $H^{2,1}(Q_T^0)$ and $\mathbf{u}_x$ in $\mathbf{H}^{1,1/2}(Q_T^0)$, we get that $u_1\mathbf{u}_x$ belongs to $\mathbf{H}^{1/2-\rho,1/4-\rho/4}(Q_T^0)$ for

$0 < \rho < 1/2$. Then, by a classic embedding formula, we know $H^{1/4-\rho/2}(0,T) \hookrightarrow L^\xi(0,T)$ where $\frac{1}{\xi} = \frac{1}{2} - \left(\frac{1}{4} - \frac{\rho}{2}\right)$ (see [1]), that is $\xi = \frac{4}{1+2\rho}$. Furthermore, we have $L^\xi(0,T) \hookrightarrow L^2(0,T)$ (because $2 < \xi$) with the estimate

$$\|f\|_{L^2(0,T)} \leq CT^{\frac{1}{2}-\frac{1}{\xi}} \|f\|_{L^\xi(0,T)} \text{ for all } f \in L^\xi(0,T).$$

But $\frac{1}{2} - \frac{1}{\xi} = \frac{1}{2} - \left(\frac{1}{2} - \left(\frac{1}{4} - \frac{\rho}{2}\right)\right) = \frac{1}{4} - \frac{\rho}{2}$ which is nonnegative thanks to $0 < \rho < 1/2$ in the application of Proposition B.1. Finally, going back to the estimate of $u_1 \mathbf{u}_x$, we get

$$\begin{aligned} \|u_1 \mathbf{u}_x\|_{\mathbf{L}^2(Q_T^0)} &\leq CT^{\frac{1}{2}-\frac{\rho}{2}} \|u_1 \mathbf{u}_x\|_{H^{\frac{1}{2}-\frac{\rho}{2}}(0,T;\mathbf{L}^2(\Omega_0))} \\ &\leq cT^{\frac{1}{2}-\frac{\rho}{2}} \|u_1\|_{H^{2,1}(Q_T^0)} \|\mathbf{u}_x\|_{\mathbf{H}^{1,1/2}(Q_T^0)} \\ &\leq CT^{\frac{1}{2}-\frac{\rho}{2}} \|\mathbf{u}\|^2_{\mathbf{H}^{2,1}(Q_T^0)}. \end{aligned}$$

The second part of the proposition comes directly from the first one and the fact that the functions $\mathbf{F}[\mathbf{u}, p, \eta]$, $\mathbf{w}[\mathbf{u}, \eta]$ and $H[\mathbf{u}, \eta]$ are, by construction, at least quadratic in the variables $(\mathbf{u}, p, \eta)$. □

Proposition 4.6. *Let $(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})$ be in X_T, then system (4.7)–(4.9) with initial data $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ in X^0 satisfying (4.16) and right-hand side $(\mathbf{F}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], \mathbf{w}[\overline{\mathbf{u}}, \overline{\eta}], H[\overline{\mathbf{u}}, \overline{\eta}])$ admits a unique solution $(\mathbf{u}, p, \eta)$ in X_T with the estimate*

$$\|(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})\|_{X_T} \leq C_1 \Big(\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} + C_2 T^\delta (1 + \|(\mathbf{u}, p, \eta)\|_{X_T}) \|(\mathbf{u}, p, \eta)\|^2_{X_T} \Big)$$

where δ is a stricly positive constant. That is we have construct a mapping

$$\begin{aligned} \mathcal{X}_T : \quad & X_T \longrightarrow X_T \\ & (\overline{\mathbf{u}}, \overline{p}, \overline{\eta}) \longmapsto \begin{array}{l} (\mathbf{u}, p, \eta) = \mathcal{X}_T(\overline{\mathbf{u}}, \overline{p}, \overline{\eta}) \text{ is the solution of system } (4.7) - (4.9) \\ \qquad \text{with } (\mathbf{F}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], \mathbf{w}[\overline{\mathbf{u}}, \overline{\eta}], H[\overline{\mathbf{u}}, \overline{\eta}]) \text{ for right-hand side} \end{array} \end{aligned}$$

which satisfies the estimate

$$\|\mathcal{X}_T(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})\|_{X_T} \leq C_1 \Big(\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} + C_2 T^\delta (1 + \|(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})\|_{X_T}) \|(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})\|^2_{X_T} \Big).$$

Proof. The proof of this proposition can be adapted from the proof of Proposition 2.17 in Chapter 2 using Theorem 4.4 and Proposition 4.5. Indeed, for $(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})$ in X_T, we get by Proposition 4.5 that $(\mathbf{F}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], \mathbf{w}[\overline{\mathbf{u}}, \overline{\eta}], H[\overline{\mathbf{u}}, \overline{\eta}])$ belongs to W_T and thus that $(\mathbf{f}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], h[\overline{\mathbf{u}}, \overline{\eta}])$ (defined in (4.12)) belongs to Z_T. From $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ in X^0 satisfying (4.16), we get that $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ where $\mathbf{v}^0$ is defined in (4.14) belongs to $X^0_{\mathbf{cc}}$. Thus Theorem 4.4 gives the existence of a solution $(\mathbf{v}, p, \eta)$ to system (4.13) with $(\mathbf{f}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], h[\overline{\mathbf{u}}, \overline{\eta}])$ for right-hand side and $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ for initial data satisfying

$$\|(\mathbf{v}, p, \eta)\|_{X_T} \leq C_1 \left(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} + \|(\mathbf{f}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], h[\overline{\mathbf{u}}, \overline{\eta}])\|_{Z_T} \right).$$

The correspondence between systems (4.13) and (4.7)–(4.9) gives the existence of a solution $(\mathbf{u}, p, \eta)$ in X_T satisfying the estimate

$$\|(\mathbf{u}, p, \eta)\|_{X_T} \leq C_1 \left(\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} + \|(\mathbf{F}[\overline{\mathbf{u}}, \overline{p}, \overline{\eta}], \mathbf{w}[\overline{\mathbf{u}}, \overline{\eta}], H[\overline{\mathbf{u}}, \overline{\eta}])\|_{W_T} \right)$$

thanks to $(\mathbf{u}, p, \eta) = (\mathbf{v} + \mathbf{w}[\mathbf{u}, \eta], p, \eta)$. Estimate (4.19) and the previous inequality give the expected result. □

We can conclude the proofs of Theorem 4.1 and 4.2 in the fixed domain by the following proposition:

Proposition 4.7. *Let $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ be in X^0 satisfying (4.16). Then,*

(i) there exists a time $T_0 > 0$ such that system (4.7)–(4.9) admits a unique local strong solution $(\mathbf{u}, p, \eta)$ in X_{T_0}.

(ii) there exists r small enough such that, under the condition $\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0} \leq r$, system (4.7)–(4.9) admits a unique global strong solution $(\mathbf{u}, p, \eta)$ in X_T.

Proof. This proposition is exactly Proposition 2.18 in Chapter 2. □

Now, thanks to η in $H^{4,2}_{(0)}(\Sigma^{s,0}_T)$ or $H^{4,2}_{(0)}(\Sigma^{s,0}_{T_0})$, we can prove that, either for r or for T_0 small enough (depending on the case we consider), η satisfies assumption (4.1). Together with

$$\phi^{-1}_{\eta(t)} : \begin{array}{rcl} \Omega_0 & \longrightarrow & \Omega_{\eta(t)} \\ (x,y,z_0) & \longmapsto & (x,y,z) \end{array}$$

is a $\mathcal{C}^1-$ diffeomorphism for any time t, that finishes the proof.

4.2 Navier-Stokes equations and strongly damped wave equation.

In this section, we consider the corresponding periodic setting of the previous system (4.2)–(4.3) with $\alpha = 0$ in the plate equation. That is, from now on, for L_1, $L_2 > 0$, we define $\omega_0 = \mathbb{R}/L_1 \times \mathbb{R}/L_2$. Then, we define the fixed domain $\Omega_0 = \omega_0 \times (0,1)$. The boundary Γ_0 of the domain Ω_0 is split into two parts, the reference state of the membrane $\Gamma^s_0 = \omega_0 \times \{1\}$ and the bottom part $\Gamma = \omega_0 \times \{0\}$, see Figure 4.2. For a displacement η, we define in the same way as in section 4.1 the moving part of the boundary:

$$\Gamma^s_{\eta(t)} = \Big\{(x,y,z) \in \mathbb{R}^3 \text{ s.t. } (x,y) \in \omega_0 \quad \text{and} \quad z = 1 + \eta(t,x,y)\Big\}.$$

Then, the domain occupied by the fluid at time $t \geq 0$ is

$$\Omega_{\eta(t)} = \Big\{(x,y,z) \in \mathbb{R}^3 \text{ s.t. } (x,y) \in \omega_0 \quad \text{and} \quad 0 < z < 1 + \eta(t,x,y)\Big\}.$$

the displacement η has to satisfy the condition (4.1) too. With this change of notations, we can keep the definitions, for $T > 0$,

$$\begin{array}{lll} Q^\eta_T = \displaystyle\bigcup_{t\in(0,T)} \{t\} \times \Omega_{\eta(t)}, & \Sigma^{s,\eta}_T = \displaystyle\bigcup_{t\in(0,T)} \{t\} \times \Gamma^s_{\eta(t)}, & \Sigma^0_T = (0,T) \times \Gamma_0, \\ Q^0_T = (0,T) \times \Omega_0, & \Sigma^{s,0}_T = (0,T) \times \Gamma^s_0, & \Sigma_T = (0,T) \times \Gamma. \end{array}$$

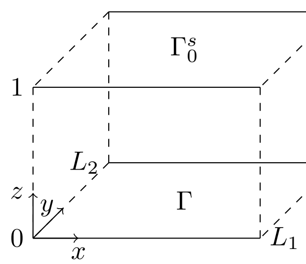

Figure 4.2: The domain Ω_0 in three dimensions in the periodic setting ($\alpha = 0$).

Then, the system in this setting becomes

$$\begin{array}{rcll} \mathbf{u}_t - \text{div } \sigma(\mathbf{u},p) + (\mathbf{u}\cdot\nabla)\mathbf{u} & = & \mathbf{0} & \text{in } Q^\eta_T, \\ \text{div } \mathbf{u} & = & 0 & \text{in } Q^\eta_T, \\ \mathbf{u} & = & \eta_t \mathbf{e}_3 & \text{on } \Sigma^{s,\eta}_T, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \eta_{tt} - \beta\Delta_s\eta - \gamma\Delta_s\eta_t & = & -\gamma^s_\#\Big[\sigma(\mathbf{u},p)\sqrt{1+\eta_x^2+\eta_y^2}\mathbf{n}\cdot\mathbf{e}_3\Big] & \text{on } \Sigma^{s,0}_T, \\ (\mathbf{u}(0),\eta(0),\eta_t(0)) & = & (\mathbf{u}^0,\eta^{1,0},\eta^{2,0}) & \end{array} \tag{4.21}$$

Here, $\gamma^s_\#$ is the trace function from $H^\sigma_\#(\Omega_0)$ into $H^{\sigma-1/2}_\#(\Gamma^s_0)$ for $\sigma > 1/2$ (see Chapter 3 for details). The operator Δ_s is the Laplace operator on Γ^s_0 defined by $D(\Delta_s) = H^2_\#(\Gamma^s_0)$ on $L^2_{\#,0}(\Gamma^s_0)$ by $\Delta_s\mu = \frac{\partial^2\mu}{\partial x^2} + \frac{\partial^2\mu}{\partial y^2}$ for all μ in $D(\Delta_s)$.

Note that, the spaces $L^2_{\#,0}(\Gamma_0^s)$ and $H^\sigma_\#(\Gamma_0^s)$ have been defined in Chapter 3 in the one dimensional periodic case. Here they are the generalization to the two dimensions periodic space (in the variables x and y). Thus, $L^2_{\#,0}$ is the space of all periodic (in the variables x and y) functions in $L^2(\Gamma_0^s)$ of zero mean value on Γ_0^s and the spaces $H^\sigma_\#(\Gamma_0^s)$ are defined as $H^\sigma(\Gamma_0^s) \cap L^2_{\#,0}(\Gamma_0^s)$.

4.2.1 Main results.

The aim of this section is to prove the same alternative for system (4.21) than for system (4.2)–(4.3), that is either existence and uniqueness of local in time strong solution for any initial data or existence and uniqueness of global in time on $[0,T]$ ($T>0$ fixed) strong solution for small initial data. The main difference is the regularity of initial data and the low regularity of the membrane displacement contrary to the high regularity of the displacement of the plate. Indeed, in Theorems 4.8 and 4.9, we only get η in $H^1(0,T;H^2_\#(\Gamma_0^s)) \cap H^2(0,T;L^2_{\#,0}(\Gamma_0^s))$ instead of η in $H^{4,2}_{(0)}(\Sigma_T^{s,0})$ in Theorems 4.1 and 4.2. We can now state this results

Theorem 4.8. *Let $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0}) \in \mathbf{V}^1_\#(\Omega_{\eta^{1,0}}) \times H^{5/2}_\#(\Gamma_0^s) \times H^{3/2}_\#(\Gamma_0^s)$. There exists $R>0$ such that for any initial data satisfying*

$$\|\mathbf{u}^0\|^2_{\mathbf{V}^1_\#(\Omega_{\eta^{1,0}})} + \|\eta^{1,0}\|^2_{H^{5/2}_\#(\Gamma_0^s)} + \|\eta^{2,0}\|^2_{H^{3/2}_\#(\Gamma_0^s)} \leq R^2$$

and the compatibility condition

$$\mathbf{u}^0 = \mathbf{0} \quad \text{on } \Gamma \qquad \text{and} \qquad \mathbf{u}^0 = \eta^{2,0}\mathbf{e}_3 \quad \text{in } \Gamma^s_{\eta^{1,0}}, \tag{4.22}$$

system (4.21) has a unique global strong solution $(\mathbf{u}, p, \eta)$ in

$$\mathbf{V}^{2,1}_\#(Q_T^\eta) \times L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathcal{H}^1_\#\left(\Omega_{\eta(t)}\right)\right) \times \mathcal{E}_T$$

where

$$\mathcal{E}_T = H^1\left(0,T;H^{5/2}_\#\left(\Gamma_0^s\right)\right) \cap H^2\left(0,T;H^{1/2}_\#\left(\Gamma_0^s\right)\right). \tag{4.23}$$

Theorem 4.9. *Let $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0}) \in \mathbf{V}^1_\#(\Omega_{\eta^{1,0}}) \times H^{5/2}_\#(\Gamma_0^s) \times H^{3/2}_\#(\Gamma_0^s)$ satisfying the compatibility condition (4.22). There exists a time $T_0>0$ such that system (4.21) has a unique strong solution $(\mathbf{u}, p, \eta) \in \mathbf{V}^{2,1}_\#(Q^\eta_{T_0}) \times L^2(\bigcup_{t\in(0,T_0)}\{t\} \times \mathcal{H}^1_\#(\Omega_{\eta(t)})) \times \mathcal{E}_{T_0}$.*

The proof of Theorems can be found in the previous section and in Chapter 3 where Theorems 3.2 and 3.3 are the equivalent results in the two dimensional case. Namely, we use the previous change of variables to set the problem in the fixed cylinder Q_T^0. Then, we prove existence and uniqueness of solution for the linearized system. Next, using a fixed point procedure, we prove the existence and uniqueness of the nonlinear system in the fixed domain. Finally, the regularity of the change of variables give the solution in the moving domain (in the sense of Definition 4.3).

4.2.2 Change of variables.

Using the same change of variables as in section 4.1.2, we obtain the following system

$$\begin{array}{rcll}
\mathbf{u}_t - \operatorname{div}\sigma(\mathbf{u},p) &=& \mathbf{F}[\mathbf{u},p,\eta] & \text{in } Q_T^0, \\
\operatorname{div}\mathbf{u} &=& \operatorname{div}\mathbf{w}[\mathbf{u},\eta] & \text{in } Q_T^0, \\
\mathbf{u} &=& \eta_t\mathbf{e}_3 & \text{on } \Sigma_T^{s,0}, \\
\mathbf{u} &=& \mathbf{0} & \text{on } \Sigma_T, \\
\mathbf{u}(0) &=& \mathbf{u}^0 & \text{in } \Omega_0
\end{array} \tag{4.24}$$

and

$$\begin{array}{rcll}
\eta_{tt} - \beta\Delta_s\eta - \gamma\Delta_s\eta_t &=& \gamma^s_\# p - 2\nu\gamma^s_\# u_{3,z} + \gamma^s_\# H[\mathbf{u},\eta] & \text{on } \Sigma_T^{s,0}, \\
(\eta(0), \eta_t(0)) &=& (\eta^{1,0}, \eta^{2,0}) & \text{in } \Gamma_0^s
\end{array} \tag{4.25}$$

where $\mathbf{F}[\mathbf{u},p,\eta]$, $\mathbf{w}[\mathbf{u},\eta]$ and $H[\mathbf{u},\eta]$ are defined respectively in (4.8), (4.6) and (4.10). Then, we condider $\mathbf{z}[\mathbf{u},\eta]=\widetilde{L}\mathbf{w}[\mathbf{u},\eta]$ and $\pi[\mathbf{u},\eta]=\widetilde{L}_p\mathbf{w}[\mathbf{u},\eta]$ defined by

$$-\nu\Delta\mathbf{z}[\mathbf{u},\eta]+\nabla\pi[\mathbf{u},\eta]=\mathbf{0}\quad\text{and}\quad \text{div }\mathbf{z}[\mathbf{u},\eta]=\text{div }\mathbf{w}[\mathbf{u},\eta]\quad\text{in }\Omega_0\quad\text{and}\quad\mathbf{z}[\mathbf{u},\eta]=\mathbf{g}\quad\text{on }\Gamma_0. \tag{4.26}$$

Thanks to Proposition 3.5 in Chapter 3 which state a result in [24], we know that for $\mathbf{w}[\mathbf{u},\eta]$ in $G_T^\#$ defined by

$$G_T^\#=\Big\{\mathbf{k}\in\mathbf{L}^2_\#(Q_T^0)\text{ s.t. div }\mathbf{k}\in L^2(0,T;\mathbf{H}^1_\#(\Omega_0)),\ \mathbf{k}_t\in\mathbf{L}^2_\#(Q_T^0)\text{ and }\mathbf{k}=0\text{ on }\Sigma_T^0\Big\},$$

we get that (4.26) has a unique solution $(\mathbf{z}[\mathbf{u},\eta],\pi[\mathbf{u},\eta])$ in $\mathbf{H}^{2,1}_\#(Q_T^0)\cap L^2(0,T;\mathcal{H}^1_\#(\Omega_0))$ satisfying the estimate

$$\|(\mathbf{z}[\mathbf{u},\eta],\pi[\mathbf{u},\eta])\|_{\mathbf{H}^{2,1}_\#(Q_T^0)\cap L^2(0,T;\mathcal{H}^1_\#(\Omega_0))}\le C\|\mathbf{w}[\mathbf{u}\eta]\|_{G_T^\#},$$

where the norm on $G_T^\#$ is

$$\|\mathbf{k}\|_{G_T^\#}=\Big(\|\mathbf{k}\|^2_{\mathbf{L}^2_\#(Q_T^0)}+\|\text{div }\mathbf{k}\|^2_{L^2(0,T;H^1_\#(\Omega_0))}+\|\mathbf{k}_t\|^2_{\mathbf{L}^2_\#(Q_T^0)}\Big)^{1/2}\qquad\text{for all }\mathbf{k}\in G_T^\#.$$

See section 3.4 in Chapter 3 for details.

Thus, we look for solution $(\mathbf{u},p,\eta)$ of system (4.24)–(4.25) under the form $\mathbf{u}=\mathbf{v}+\mathbf{z}[\mathbf{u},\eta]$, $p=q+\pi[\mathbf{u},\eta]$ where $\mathbf{z}[\mathbf{u},\eta]$ and $\pi[\mathbf{u},\eta]$ is the solution of (4.26). Then, $(\mathbf{v},q,\eta)$ is solution of the following system

$$\begin{array}{rcll}
\mathbf{v}_t-\text{div }\sigma(\mathbf{v},q) &=& \mathbf{f}[\mathbf{u},p,\eta] & \text{in } Q_T^0,\\
\text{div }\mathbf{v} &=& 0 & \text{in } Q_T^0,\\
\mathbf{v} &=& \eta_t\mathbf{e}_3 & \text{on } \Sigma_T^{s,0},\\
\mathbf{v} &=& \mathbf{0} & \text{on } \Sigma_T,\\
\mathbf{v}(0) &=& \mathbf{v}^0 & \text{in } \Omega_0,\\
\eta_{tt}-\beta\Delta_s\eta-\gamma\Delta_s\eta_t &=& \gamma^s_\# q+h[\mathbf{u},\eta] & \text{on } \Sigma_T^{s,0},\\
(\eta(0),\eta_t(0)) &=& (\eta^{1,0},\eta^{2,0}) & \text{in } \Gamma_0^s
\end{array}\tag{4.27}$$

where

$$\mathbf{f}[\mathbf{u},p,\eta]=\mathbf{F}[\mathbf{u},p,\eta]-\mathbf{z}_t[\mathbf{u},\eta],$$

$$h[\mathbf{u},p,\eta]=\gamma_s H[\mathbf{u},p,\eta]-2\nu\left(\mathbf{z}[\mathbf{u},\eta]\right)_{2,z_0}+\gamma_s\psi[\mathbf{u},\eta].$$

and

$$\mathbf{v}^0=\mathbf{u}^0-\mathbf{z}[\mathbf{u}^0,\eta^{1,0}].$$

The compatiblity condition at time $t=0$ becomes in the variables $(\mathbf{v}^0,\eta^{1,0},\eta^{2,0})$

$$\text{div }\mathbf{v}^0=0\quad\text{in }\Omega_0,\quad\mathbf{v}^0=\mathbf{0}\quad\text{on }\Gamma\quad\text{and}\quad\mathbf{v}^0=\eta^{2,0}\mathbf{e}_3\quad\text{on }\Gamma_0^s \tag{4.28}$$

and in the variables $(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})$:

$$\text{div }(\mathbf{u}^0-\mathbf{z}[\mathbf{u}^0,\eta^{1,0}])=0\quad\text{in }\Omega_0,\quad\mathbf{u}^0=\mathbf{0}\quad\text{on }\Gamma\quad\text{and}\quad\mathbf{u}^0=\eta^{2,0}\mathbf{e}_3\quad\text{on }\Gamma_0^s. \tag{4.29}$$

4.2.3 Study of a linear auxiliary system.

In this section, we consider the following linear system:

$$\begin{array}{rcll}
\mathbf{v}_t-\text{div }\sigma(\mathbf{v},q) &=& \mathbf{f} & \text{in } Q_T^0,\\
\text{div }\mathbf{v} &=& 0 & \text{in } Q_T^0,\\
\mathbf{v} &=& \eta_t\mathbf{e}_3 & \text{on } \Sigma_T^{s,0},\\
\mathbf{v} &=& \mathbf{0} & \text{on } \Sigma_T,\\
\mathbf{v}(0) &=& \mathbf{v}^0 & \text{in } \Omega_0,\\
\eta_{tt}-\beta M_s\Delta_s\eta-\gamma M_s\Delta_s\eta_t &=& \gamma^s_\# q+h & \text{on } \Sigma_T^{s,0},\\
(\eta(0),\eta_t(0)) &=& (\eta^{1,0},\eta^{2,0}) & \text{in } \Gamma_0^s.
\end{array}\tag{4.30}$$

The initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ belongs to $X^{0,\#}_{\mathbf{cc}}$ defined from

$$X^{0,\#} = \mathbf{V}^1_\#(\Omega_0) \times H^{5/2}_\#(\Gamma^s_0) \times H^{3/2}_\#(\Gamma^s_0)$$

by

$$X^{0,\#}_{\mathbf{cc}} = \Big\{(\mathbf{z}^0, \mu^{1,0}, \mu^{2,0}) \in X^{0,\#} \text{ s.t. } (\mathbf{z}^0, \mu^{1,0}, \mu^{2,0}) \text{ satisfies (4.28)}\Big\}.$$

The right-hand side $(\mathbf{f}, h)$ belongs to $Z^\#_T = \mathbf{L}^2_\#(Q^0_T) \times L^2(0,T; H^{1/2}_\#(\Gamma^s_0))$. We have the following result:

Theorem 4.10. *Let $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ be in $X^{0,\#}_{\mathbf{cc}}$ and $(\mathbf{f}, h)$ be in $Z^\#_T$. Then, system (4.30) admits a unique solution $(\mathbf{v}, q, \eta)$ in $X^\#_T = \mathbf{V}^{2,1}_\#(Q^0_T) \times L^2(0,T; \mathcal{H}^1_\#(\Omega_0)) \times \mathcal{E}_T$ (the space $\mathcal{E}_T$ is defined in (4.23)). Furthermore, we have the estimate*

$$\|(\mathbf{v}, q, \eta)\|_{X^\#_T} \le C_1\Big(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^{0,\#}} + \|(\mathbf{f}, h)\|_{Z^\#_T}\Big).$$

Proof. The proof is exactly the same as in the two dimensional case (see section 3.5 in Chapter 3). First, the fixed point procedure in the space $X^{e,s,\kappa}_T$ (for $0 < \kappa < 1$) is the same. Then, to obtain a better regularity of η, we use Theorem 2.12 in Chapter 2 (see page 32). Finally, the proof by interpolation of the best regularity for η works in the same way but we have to be careful. In this setting, there are two periodic variables. Thus, we have to consider η_x the solution of

$$\begin{array}{rcll} \mu_{tt} - \beta\Delta_s\mu - \gamma\Delta_s\mu_t &=& \big[\gamma^s_\# q + h\big]_x & \text{on } \Sigma^{s,0}_T, \\ (\mu(0), \mu_t(0)) &=& (\eta^{1,0}_x, \eta^{2,0}_x) & \text{in } \Gamma^s_0. \end{array}$$

and η_y the solution of

$$\begin{array}{rcll} \mu_{tt} - \beta\Delta_s\mu - \gamma\Delta_s\mu_t &=& \big[\gamma^s_\# q + h\big]_y & \text{on } \Sigma^{s,0}_T, \\ (\mu(0), \mu_t(0)) &=& (\eta^{1,0}_y, \eta^{2,0}_y) & \text{in } \Gamma^s_0. \end{array}$$

Thus, for $\big[\gamma^s_\# q + h\big]$ in $L^2(0,T; H^1_\#(\Gamma^s_0))$ and $(\eta^{1,0}, \eta^{2,0})$ in $H^3_\#(\Gamma^s_0) \times H^2_\#(\Gamma^s_0)$, μ is in $H^1(0,T; H^2_\#(\Gamma^s_0)) \cap H^2(0,T; L^2_{\#,0}(\Gamma^s_0))$, that is η belongs to

$$H^1(0,T; H^3_\#(\Gamma^s_0)) \cap H^2(0,T; H^1_\#(\Gamma^s_0)).$$

The interpolation, with the limit case $\varepsilon = 1/2$ (because both h and $\gamma^s_\# q$ belong to $L^2(0,T; H^{1/2}_\#(\Gamma^s_0))$), gives that for initial data $(\eta^{1,0}, \eta^{2,0})$ in $H^{5/2}_\#(\Gamma^s_0) \times H^{3/2}_\#(\Gamma^s_0)$, the solution η to equation

$$\begin{array}{rcll} \eta_{tt} - \beta\Delta_s\eta - \gamma\Delta_s\eta_t &=& \big[\gamma^s_\# q + h\big] & \text{on } \Sigma^{s,0}_T, \\ (\eta(0), \eta_t(0)) &=& (\eta^{1,0}, \eta^{2,0}) & \text{in } \Gamma^s_0 \end{array}$$

belongs to $\mathcal{E}_T$. □

It is important here to stress that it is the point of the proof where we need the periodic setting. Indeed, thanks to this setting, we can obtain by interpolation a better regularity for η from the fact that h and $\gamma^s_\# q$ belong to $L^2(0,T; H^{1/2}_\#(\Gamma^s_0))$ (and not only $L^2(\Sigma^{s,0}_T)$) and from the corresponding regularity of the initial data. The case of the Dirichlet homogeneous condition should have been a problem in the next section due to the non compatibility of the boundary condition with the derivatives of η in that case.

4.2.4 Proof of Theorems 4.8 and 4.9.

We begin by proving this results in the fixed domain. It relies on the same fixed point procedure as 3.6. We introduce the space

$$W^\#_T = \mathbf{L}^2_\#(Q^0_T) \times G^\#_T \times L^2(0,T; H^1_\#(\Omega_0))$$

endowed with the norm

$$\|(\mathbf{G}, \mathbf{r}, G)\|_{W^\#_T} = \Big(\|\mathbf{G}\|^2_{\mathbf{L}^2_\#(Q^0_T)} + \|\mathbf{r}\|^2_{G^\#_T} + \|G\|^2_{L^2(0,T; H^1_\#(\Omega_0))}\Big)^{1/2}.$$

Now, we can estimate the nonlinearities in system (4.24)–(4.25) in terms of $(\mathbf{u}, p, \eta)$ in $X^\#_T$.

Proposition 4.11. *Let* $(\mathbf{u}, p, \eta)$ *be in* $X_T^{\#}$, *then* $(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], H[\mathbf{u}, \eta])$, *obtained from* $(\mathbf{u}, p, \eta)$ *in* (4.8), (4.6) *and* (4.10), *belongs to* $W_T^{\#}$. *Furthermore, there exists* $\delta > 0$ *such that*

$$\|(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], H[\mathbf{u}, \eta])\|_{W_T^{\#}} \leq c_2 T^{\delta}(1 + \|(\mathbf{u}, p, \eta)\|_{X_T^{\varepsilon}})\|(\mathbf{u}, p, \eta)\|^2_{X_T^{\#}}.$$

Let $(\mathbf{u}_1, p_1, \eta_1)$ *and* $(\mathbf{u}_2, p_2, \eta_2)$ *be two triplets in* $X_T^{\#}$ *such that for* $i = 1, 2$, $\|(\mathbf{u}_i, p_i, \eta_i)\|_{X_T^{\#}} \leq R$ *for some* $R > 0$, *we get*

$$\|(\mathbf{F}_1, \mathbf{w}_1, H_1) - (\mathbf{F}_2, \mathbf{w}_2, H_2)\|_{W_T^{\#}} \leq C(1 + R)RT^{\delta}\|(\mathbf{u}_1, p_1, \eta_1) - (\mathbf{u}_2, p_2, \eta_2)\|_{X_T^{\#}}$$

with the notations $(\mathbf{F}_i, \mathbf{w}_i, H_i) = (\mathbf{F}[\mathbf{u}_i, p_i, \eta_i], \mathbf{w}[\mathbf{u}_i, \eta_i], H[\mathbf{u}_i, \eta_i])$.

Proof. This proposition can be proved using the Sobolev embeddings and the nonlinear estimates in the Appendix B in [12], especially Proposition B.1 and Theorem B.3.

The difference between this proposition and Proposition 4.5 comes from the low regularity of the displacement of the beam η. Indeed, for η in $\mathcal{E}_T$, we only get

$$\eta_{xx} \in H^1(0, T; H^{1/2}_{\#}(\Gamma_0^s)) \cap H^{5/4}(0, T; L^2_{\#,0}(\Gamma_0^s))$$

and $\eta_{tx} in H^{3/2,3/4}_{\#}(\Sigma_T^{s,0})$. But the worst terms to estimate comes from the divergence term. Indeed, $\mathbf{w}[\mathbf{u}, \eta] = -\eta u_1 \mathbf{e}_1 - \eta u_2 \mathbf{e}_2 + z_0\,(\eta_x u_1 + \eta_y u_2)\,\mathbf{e}_3$ gives the terms $-z_0 \eta_{tx} u_1 \mathbf{e}_3$ and $-z_0 \eta_{ty} u_2 \mathbf{e}_3$ to estimate in $L^2(Q_T^0)$ ($\mathbf{w}_t$ has to be in $\mathbf{L}^2(Q_T^0)$). Because $H^{3/2,3/4}_{\#}(\Sigma_T^{s,0}) \hookrightarrow H^{1,1/2}_{\#}(\Sigma_T^{s,0})$, we can use the same tedious method as in the proof Proposition 4.5.

The second «worst» estimate to get is $\|\eta_{xx} u_{1,z_0}\|_{L^2(Q_T^0)}$. Indeed, η is only in $H^1(0, T; H^{1/2}_{\#}(\Gamma_0^s))$. But Proposition B.1 in [12], gives for any time t, that $\eta_{xx}(t) u_{1,z_0}(t)$ belongs to $L^2(\Omega_0)$ thanks to the calculation (with the notations of [12])

$$\lambda = 0,\ \mu = \frac{1}{2},\ \omega = 1,\ n = 3 \text{ and } 3/2 = \lambda + \mu + \omega \geq (n)/2 = 3/2.$$

It is the limit case which is possible here because both $\mu > 0$ and $\omega > 0$. Thus,

$$\|\eta_{xx}(t) u_{1,z_0}(t)\|_{L^2(\Omega_0)} \leq C\|\eta_{xx}(t)\|_{H^{1/2}_{\#}(\Gamma_0^s)}\|u_{1,z_0}\|_{H^1_{\#}(\Omega_0)}. \tag{4.31}$$

We introudce the embeddings $H^1(0, T) \hookrightarrow H^{1/2+\kappa}(0, T) \hookrightarrow \mathcal{C}([0, T])$ (for $0 < \kappa < \frac{1}{2}$) with the estimate

$$\|f\|_{L^\infty(0,T)} \leq C\|f\|_{H^{1/2+\kappa}(0,T)} \leq CT^{\frac{1}{4}-\frac{\kappa}{2}}\|f\|_{H^1(0,T)} \text{ for all } f \in H^1(0, T).$$

Thus, taking the $L^2(0, T)-$norm in (4.31) and using the $L^2(0, T)-$norm for the velocity u_{1,z_0} and the $L^\infty(0, T)-$norm for the displacement on the right-hand side, we get

$$\begin{aligned} \|\eta_{tx} u_1\|_{L^2(Q_T^0)} &\leq C\|\eta_{tx}\|_{L^\infty(0,T;H^{1/2}_{\#}(\Gamma_0^s))}\|u_1(t)\|_{L^2(0,T;H^1_{\#}(\Omega_0))} \\ &\leq CT^{\frac{1}{4}-\frac{\kappa}{2}}\|\eta\|_{\mathcal{E}_T}\|\mathbf{u}\|_{\mathbf{H}^{2,1}_{\#}(Q_T^0)}. \end{aligned}$$

The other terms containing a derivative of η are estimated in the same way. The estimate of the products $u_1 \mathbf{u}_x$, $u_2 \mathbf{u}_y$ and $u_3 \mathbf{u}_{z_0}$ are the same as in the proof of Proposition 4.5. □

Proposition 4.12. *Let* $(\mathbf{u}, p, \eta)$ *be in* $X_T^{\#}$, *then system* (4.24)–(4.25) *with initial data* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *in* $X^{0,\#}$ *satisfying* (4.29) *and right-hand side* $(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], H[\mathbf{u}, \eta])$ *admits a unique solution* $(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})$ *in* $X_T^{\#}$ *with the estimate*

$$\|(\overline{\mathbf{u}}, \overline{p}, \overline{\eta})\|_{X_T^{\#}} \leq C_1\Big(\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^{0,\#}} + C_2 T^{\delta}(1 + \|(\mathbf{u}, p, \eta)\|_{X_T^{\#}})\|(\mathbf{u}, p, \eta)\|^2_{X_T^{\#}}\Big)$$

where δ *is a stricly positive constant. That is we have construct a mapping*

$$\begin{array}{rccl} \mathcal{X}_T^{\#}: & X_T^{\#} & \longrightarrow & X_T^{\#} \\ & (\mathbf{u}, p, \eta) & \longmapsto & (\overline{\mathbf{u}}, \overline{p}, \overline{\eta}) = \mathcal{X}_T(\mathbf{u}, p, \eta) \text{ is the solution of system } (4.24) - (4.25) \\ & & & \text{with } (\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{w}[\mathbf{u}, \eta], H[\mathbf{u}, \eta]) \text{ for right-hand side} \end{array}$$

which satisfies the estimate

$$\|\mathcal{X}_T(\mathbf{u},p,\eta)\|_{X_T^\#} \leq C_1\Big(\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{X^{0,\#}} + C_2T^\delta(1+\|(\mathbf{u},p,\eta)\|_{X_T^\#})\|(\mathbf{u},p,\eta)\|^2_{X_T^\#}\Big).$$

The proof of this proposition relies on Theorem 4.10 and Proposition 4.11. One can find the idea in the proof of Proposition 3.13.

We can conclude this section showing existence of solutions in the fixed domain:

Proposition 4.13. *Let* $(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})$ *be in* $X^{0,\#}$ *satisfying* (4.29)*. Then,*

(i) there exists a time $T_0>0$ *such that system* (4.24)–(4.25) *admits a unique local strong solution* $(\mathbf{u},p,\eta)$ *in* $X_{T_0}^\#$.

(ii) there exists r *small enough such that, under the condition* $\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{X^{0,\#}} \leq r$*, system* (4.24)–(4.25) *admits a unique global strong solution* $(\mathbf{u},p,\eta)$ *in* $X_T^\#$.

Proof. The proof is clear thanks to the previous Proposition. We have to act on the size of the initial data or on the size of the time interval to get the alternative. □

To conclude, thanks to η in $\mathcal{E}_T$ or $\mathcal{E}_{T_0}$, we can prove that either for r or for T_0 small enough (depending on the case we consider in Proposition 4.13), η satisfies condition (4.1). Together with

$$\begin{array}{rccc} \phi_{\eta(t)}^{-1}: & \Omega_0 & \longrightarrow & \Omega_{\eta(t)} \\ & (x,y,z_0) & \longmapsto & (x,y,z) \end{array}$$

is a $\mathcal{C}^1-$ diffeomorphism for any time t, that finishes the proof.

Partie II

Contrôlabilité et stabilisabilité

Chapitre 5

Contrôlabilité à zéro d'un système couplé fluide-structure

5.1 Introduction.

Controllability for fluid-structure systems has been studied recently. In a series of papers, J.P. Raymond and M. Vanninathan prove null controllability for different kinds of linear coupled systems modeling, with an increasing difficulty, fluid-structure interaction in 2D. The fluid is modeled respectively by the Helmholtz equation [25], the Heat equation [27, 26] and the Stokes equation [28].

In [9], A. Doubova and E. Fernandez-Cara consider a 1D interaction problem of a particle in a fluid modeled by the Burgers equation. They prove null controllability for the linearized model and then local null controllability for the nonlinear system.

Very recently, M. Boulakia, A. Oxel in [6] and O. Imanuvilov, T. Takahashi in [14] prove independently local exact controllability for a system modeling a rigid body moving in a viscous incompressible fluid described by the Navier-Stokes equations in 2D with a control acting in a fixed subset of the fluid domain.

In this paper, we are interested in the null controllability of a system coupling the Navier-Stokes equations and an ordinary differential equation (see equations (5.7)–(5.6)). More precisely, we prove that for any time $T > 0$ and any initial data small enough, we can find a control acting in a subdomain of the fluid part such that the solution of our system vanishes at time T (see Theorem 5.3).

The systems in [9, 6, 14] deal with nonlinear fluid equations. The strategy of the different proofs is quite the same. First, a change of variables sets the problem in a fixed domain. Then, the different authors prove that the obtained linearized system is null controllable with some control. Finally, a fixed point procedure gives the local null controllability.

The way used to prove the controllability of the linear [25, 27, 26, 28] or the linearized [9, 6, 14] systems is based on the duality between the controllability of a system and the existence of an observability inequality for the adjoint system. Such an observability inequality relies in fact on a Carleman estimate. The proofs of Carleman estimates are really tricky and not straightforward.

5.1.1 The system.

We consider a viscous incompressible fluid in a two dimensional domain. The boundary of the domain is split into two parts. One part is fixed, the other one is a moving beam. At rest, the beam is in its reference state $\Gamma_0^s = (0, L) \times \{1\}$, where $L > 0$ is the characteristic length of the beam. The domain of the fluid at rest is denoted Ω_0. Then its boundary Γ_0 is the union of two curves Γ_0^s and Γ. We suppose that the boundary Γ_0 is smooth, that is at least $\mathcal{C}^4$.

The displacement of the beam is given by a function η depending on the time t and on the position x in the reference state Γ_0^s. Then, *a priori*, the function η is from $(0, +\infty) \times (0, L)$ in $\mathbb{R}$. For any $t \geq 0$, the moving boundary given by the displacement η is

$$\Gamma_{\eta(t)}^s = \Big\{(x, y) \in \mathbb{R}^2 \text{ s.t. } x \in (0, L) \text{ and } y = 1 + \eta(t, x)\Big\}.$$

Then, the fluid at time t occupies a domain noted $\Omega_{\eta(t)}$ which has for boundary $\partial\Omega_{\eta(t)} = \Gamma \bigcup \Gamma_{\eta(t)}^s$.

We have the following assumption on the displacement

$$\exists \varepsilon > 0 \text{ such that } \forall t \in [0,T] \quad \forall x \in (0,L) \quad 1+\eta(t,x) \geq \varepsilon > 0 \tag{5.1}$$

to ensure that, for every time t, $\Omega_{\eta(t)}$ is a connected domain.

Let us introduce some notations. We fix a time $T > 0$, then

$$\begin{array}{lll} Q_T^0 = (0,T)\times\Omega_0, & Q_T^\eta = \displaystyle\bigcup_{t\in(0,T)} \{t\}\times\Omega_{\eta(t)}, & \Sigma_T = (0,T)\times\Gamma, \\ \Sigma_T^{s,0} = (0,T)\times\Gamma_0^s, & \Sigma_T^{s,\eta} = \displaystyle\bigcup_{t\in(0,T)} \{t\}\times\Gamma_{\eta(t)}^s, & \Sigma_T^0 = (0,T)\times\Gamma_0. \end{array}$$

Following the model in [21, 4, 16], the velocity $\mathbf{u}$ and the pressure p of the fluid in the domain Q_T^η are described by the Navier-Stokes equations

$$\begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \text{div }\sigma(\mathbf{u},p) & = & \mathbf{0} & \text{in } Q_T^\eta, \\ \text{div }\mathbf{u} & = & 0 & \text{in } Q_T^\eta, \\ \mathbf{u} & = & \eta_t\mathbf{e}_2 & \text{on } \Sigma_T^{s,\eta}, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}(0) & = & \mathbf{u}^0 & \text{in } \Omega_{\eta^{1,0}}. \end{array} \tag{5.2}$$

In the first equation, the term $\sigma(\mathbf{u},p)$ is the Cauchy stress tensor defined by

$$\sigma(\mathbf{u},p) = -p\mathbf{I} + \nu\Big(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\Big).$$

The coefficient $\nu > 0$ is the viscosity of the fluid. Finally, $\mathbf{e}_1$ and $\mathbf{e}_2$ are the two vectors of $\mathbb{R}^2$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Remark 5.1. *Due to the incompressibility condition of the fluid, solutions* $(\mathbf{u},p)$ *of system* (5.2) *and the Dirchlet boundary condition* $\eta_t\mathbf{e}_2$ *satisfy, for any time* t,

$$\int_{\Omega_{\eta(t)}} \mathit{div}\,\mathbf{u}(t) = \int_{\partial\Omega_{\eta(t)}} \mathbf{u}(t)\cdot\mathbf{n}(t) = \int_{\Gamma_0^s} \eta_t(t) = 0.$$

The vector $\mathbf{n}(t)$ *is the unit normal to* $\partial\Omega_{\eta(t)}$ *outward* $\Omega_{\eta(t)}$. *It is fixed on* Γ *and is given on* $\Gamma_{\eta(t)}^s$ *by*

$$\mathbf{n}(t) = \frac{1}{\sqrt{1+\eta_x^2(t)}}\Big(-\eta_x(t)\mathbf{e}_1 + \mathbf{e}_2\Big).$$

Thus, we will consider functions η *in*

$$L_0^2(\Gamma_0^s) = \left\{\mu \in L^2(\Gamma_0^s) \text{ s.t. } \int_{\Gamma_0^s} \mu = 0\right\}.$$

We assume that the displacement of the beam is a Galerkin approximation of the Euler-Bernoulli beam model. Thus, the function η is of the form

$$\eta(t,x) = \sum_{k=1}^N q_k(t)\zeta_k(x), \text{ for } x \in (0,L) \text{ and } t \geq 0 \tag{5.3}$$

where N is a fixed integer greater than 1. The familly $(\zeta_k)_{k=1,\dots,N}$ is a Hilbertian basis of $L_0^2(\Gamma_0^s)$ (see Remark 5.1). For each $k \geq 1$, ζ_k belongs to $\mathcal{C}^\infty(\Gamma_0^s;\mathbb{R})$ and satisfies

$$\zeta(x) = 0, \quad \zeta_x(x) = 0 \qquad \text{for} \quad x = 0, L.$$

The unknown $q(t)$ is a $N \times 1$ vector,

$$q(t) = \begin{pmatrix} q_1(t) \\ \vdots \\ q_N(t) \end{pmatrix},$$

which satisfies the following ordinary differential equation:

$$\begin{array}{rcl} q''(t) + Aq(t) & = & \Pi_N\Big[-\sigma(\mathbf{u},p)\Big(-\eta_x\mathbf{e}_1 + \mathbf{e}_2\Big)\cdot\mathbf{e}_2\Big] \\ & = & \left(\displaystyle\int_{\Gamma_0^s} -\sigma(\mathbf{u},p)\Big(-\eta_x\mathbf{e}_1 + \mathbf{e}_2\Big)\cdot\zeta_k\mathbf{e}_2\right)^T_{k=1,\ldots,N} \\ \big(q(0), q'(0)\big) & = & \big(q^{1,0}, q^{2,0}\big). \end{array} \tag{5.4}$$

In this equation, A is the symmetric positive matrix defined by

$$A = \left(\int_{\Gamma_0^s}\Big(\alpha\zeta_{k,xx}\zeta_{l,xx} + \beta\zeta_{k,x}\zeta_{l,x}\Big)\right)_{k,l=1,\ldots,N},$$

Π_N is the projection from $L_0^2(\Gamma_0^s)$ to $\mathbb{R}^N$. Then, Π_N satisfies, for every f in $L_0^2(\Gamma_0^s)$,

$$\Pi_N(f) = \begin{pmatrix} \displaystyle\int_{\Gamma_0^s}\zeta_1 f \\ \vdots \\ \displaystyle\int_{\Gamma_0^s}\zeta_N f \end{pmatrix}.$$

Introducing M the $\mathbb{R}^{2\times N}$ matrix,

$$M = \Big(\zeta_1\mathbf{e}_2, \ldots, \zeta_N\mathbf{e}_2\Big) = \begin{pmatrix} 0 & \ldots & 0 \\ \zeta_1 & \ldots & \zeta_N \end{pmatrix},$$

we have a quite simplier notation for the right-hand side of (5.4):

$$q''(t) + Aq(t) \quad = \quad -\int_{\Gamma_0^s} M^T\sigma(\mathbf{u},p)\Big(-\eta_x\mathbf{e}_1 + \mathbf{e}_2\Big).$$

The displacement we consider can be seen as a Galerkin approximation of the one in [4, 23, 16]. Indeed, let us introduce the following partial differential equation, called beam equation:

$$\begin{array}{rcll} \eta_{tt} + \alpha M_s\eta_{xxxx} - \beta\eta_{xx} & = & -\gamma_s\Big[\sigma(\mathbf{u},p)(-\eta_x\mathbf{e}_1 + \mathbf{e}_2)\cdot\mathbf{e}_2\Big] & \text{on } \Sigma_T^{s,0}, \\ \eta & = & 0 & \text{on } (0,T)\times\{0,L\}, \\ \eta_x & = & 0 & \text{on } (0,T)\times\{0,L\}, \\ (\eta(0), \eta_t(0)) & = & (\eta^{1,0}, \eta^{2,0}) & \text{in } \Gamma_0^s. \end{array} \tag{5.5}$$

The coefficients $\alpha > 0$ and $\beta \geq 0$ are respectively the rigidity and the stretching of the beam. The operator M_s is the projection from $L^2(\Gamma_0^s)$ onto $L_0^2(\Gamma_0^s)$ defined by

$$M_s\mu = \mu - \frac{1}{|\Gamma_0^s|}\int_{\Gamma_0^s}\mu, \qquad \forall\mu \in L^2(\Gamma_0^s).$$

We use the trace γ_s defined by

$$\gamma_s p = M_s(p_{|\Gamma_0^s}) = p_{|\Gamma_0^s} - \frac{1}{|\Gamma_0^s|}\int_{\Gamma_0^s} p_{|\Gamma_0^s} \qquad \forall p \in H^\sigma(\Omega_0) \text{ with } \sigma > 1/2.$$

Let us define the operator $(\mathcal{A}_{\alpha,\beta}, D(\mathcal{A}_{\alpha,\beta}))$ on $L^2_0(\Gamma^s_0)$ by

$$\begin{array}{rcl} D(\mathcal{A}_{\alpha,\beta}) & = & \Big\{\mu \in H^4(\Gamma^s_0) \cap L^2_0(\Gamma^s_0) \text{ s.t. } \mu(x) = \mu_x(x) = 0 \text{ for } x = 0, L\Big\}, \\ \mathcal{A}_{\alpha,\beta}\mu & = & \alpha M_s \mu_{xxxx} - \beta\mu_{xx} \text{ for all } \mu \in D(\mathcal{A}_{\alpha,\beta}). \end{array}$$

We can easily see that $(\mathcal{A}_{\alpha,\beta}, D(\mathcal{A}_{\alpha,\beta}))$ is a symmetric positive operator. We denote $\Big\{(\lambda_k, \zeta_k)\Big\}_{k\geq 1}$ its pairs of eigenvalues-eigenfunctions satisfying first $\zeta_k \in D(\mathcal{A}_{\alpha,\beta})$ for all $k \geq 1$ and second

$$\begin{array}{rcll} \mathcal{A}_{\alpha,\beta}\zeta_k & = & \lambda_k\zeta_k & \text{for all } k \geq 1, \\ \Big(\zeta_k, \zeta_l\Big)_{L^2(\Gamma^s_0)} & = & 0 & \text{for } k, l \geq 1 \text{ s.t. } k \neq l, \\ \Big(\zeta_k, \zeta_l\Big)_{H^2(\Gamma^s_0)} & = & \delta_{kl} & \text{for all } k, l \geq 1. \end{array}$$

Then, the family $(\zeta_k)_{k\geq 1}$ constitutes a Hilbertian basis of $L^2_0(\Gamma^s_0)$. Furthermore, each ζ_k for $k \geq 1$ belongs to $\mathcal{C}^\infty(\Gamma^s_0; \mathbb{R})$ as sums of exponential functions.

With a direct calculation, we can verify that the right-hand side of the beam equation (5.4) is

$$\sigma(\mathbf{u}, p)\Big(-\eta_x\mathbf{e}_1 + \mathbf{e}_2\Big) \cdot \mathbf{e}_2 = p - 2\nu u_{2,y} - \nu\eta_x\Big(u_{1,y} + u_{2,x}\Big).$$

Using the projection Π_N, it becomes

$$\Pi_N\Big[p - 2\nu u_{2,y}\Big] - \nu\Pi_N\Big[\eta_x\Big(u_{1,y} + u_{2,x}\Big)\Big].$$

The first term is linear in the variables $(\mathbf{u}, p, q)$ whereas the second is quadratic in the same variables. Then the finite dimensional beam equation is

$$\begin{array}{rcl} q'' + Aq & = & \Pi_N\Big[p - 2\nu u_{2,y}\Big] - \nu\Pi_N\Big[\eta_x\Big(u_{1,y} + u_{2,x}\Big)\Big], \\ (q(0), q'(0)) & = & (q^{1,0}, q^{2,0}). \end{array} \tag{5.6}$$

We set a control $\mathbf{c}$ in a subset ω of the fluid domain. In assumption (5.1), we can take ε such that the set ω will never touch the boundary $\Gamma^s_{\eta(t)}$. For that, let us suppose that

$$\sup_{(x,y)\in\omega} y \leq \varepsilon.$$

This is a physical issue because the domain ω is supposed to be in the fluid part of the domain and the control force cannot be out of the domain.

Denoting $Z(x)$ the $1 \times N$ vector $Z(x) = \Big(\zeta_1(x), \ldots, \zeta_N(x)\Big)$, we have equivalently

$$\eta(t, x) = Z(x)q(t), \text{ for } x \in (0, L) \text{ and } t \geq 0.$$

The equality of the velocities on the boundary becomes $\mathbf{u} = \eta_t\mathbf{e}_2 = Zq'\mathbf{e}_2$. Then, the equations of the fluid part are:

$$\begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\sigma(\mathbf{u}, p) & = & \mathbf{c}\chi_\omega & \text{in } Q^\eta_T, \\ \operatorname{div}\mathbf{u} & = & 0 & \text{in } Q^\eta_T, \\ \mathbf{u} & = & Zq'\mathbf{e}_2 & \text{on } \Sigma^{s,\eta}_T, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}(0) & = & \mathbf{u}^0 & \text{in } \Omega_{\eta^{1,0}}. \end{array} \tag{5.7}$$

The function χ_ω above is the indicator function of the domain ω.

5.1.2 Functional setting.

In the fixed domain Ω_0, we define the classic Hilbert space in two dimensions $\mathbf{L}^2(\Omega_0) = L^2(\Omega_0;\mathbb{R}^2)$ and in the same way the Sobolev spaces $\mathbf{H}^s(\Omega_0) = H^s(\Omega_0;\mathbb{R}^2)$. We denote

$$\mathbf{V}^\sigma(\Omega_0) = \Big\{\mathbf{u} \in \mathbf{H}^\sigma(\Omega_0)\ ;\ \text{div}\ \mathbf{u} = 0 \text{ in } \Omega_0\Big\}.$$

Then we define

$$\begin{aligned}\mathbf{H}^{\sigma,\tau}(Q_T^0) &= L^2(0,T;\mathbf{H}^\sigma(\Omega_0)) \cap H^\tau(0,T;\mathbf{L}^2(\Omega_0)),\\ \mathbf{V}^{\sigma,\tau}(Q_T^0) &= L^2(0,T;\mathbf{V}^\sigma(\Omega_0)) \cap H^\tau(0,T;\mathbf{V}^0(\Omega_0)).\end{aligned}$$

We need a definition of Sobolev spaces in the time dependent domain $\Omega_{\eta(t)}$:

Definition 5.2. *We say that* $\mathbf{u}$ *belongs to* $H^\tau(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{H}^\sigma(\Omega_{\eta(t)}))$ *(respectively to* $H^\tau(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^\sigma(\Omega_{\eta(t)}))$*) if*
- *for almost every* t *in* $(0,T)$*,* $\mathbf{u}(t)$ *is in* $\mathbf{H}^\sigma(\Omega_{\eta(t)})$ *(resp. in* $\mathbf{V}^\sigma(\Omega_{\eta(t)})$*),*
- $t \mapsto \|\mathbf{u}(t)\|_{\mathbf{H}^\sigma(\Omega_{\eta(t)})}$ *(resp.* $t \mapsto \|\mathbf{u}(t)\|_{\mathbf{V}^\sigma(\Omega_{\eta(t)})}$*) is in* $H^\tau(0,T;\mathbb{R})$*.*

We finally define

$$\begin{aligned}\mathbf{H}^{\sigma,\tau}\left(Q_T^\eta\right) &= L^2\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{H}^\sigma\left(\Omega_{\eta(t)}\right)\right)\bigcap H^\tau\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{L}^2\left(\Omega_{\eta(t)}\right)\right),\\ \mathbf{V}^{\sigma,\tau}\left(Q_T^\eta\right) &= L^2\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^\sigma\left(\Omega_{\eta(t)}\right)\right)\bigcap H^\tau\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^0\left(\Omega_{\eta(t)}\right)\right).\end{aligned}$$

The pressure term p is defined in the Navier-Stokes equations up to a constant: only the derivatives of p appears in (5.7). Then, we define the space $\mathcal{H}^\sigma(\Omega_0)$ by

$$\mathcal{H}^\sigma(\Omega_0) = \left\{p \in H^\sigma(\Omega_0) \text{ such that } \int_{\Omega_0} p = 0\right\}.$$

We will look for p in $L^2\left(\bigcup_{t\in(0,T)}\{t\}\times\mathcal{H}^1\left(\Omega_{\eta(t)}\right)\right)$ (see Definition 5.2).

5.1.3 Main result.

The aim of this paper is to prove the following result of null controlability of the system (5.7)–(5.6):

Theorem 5.3. *Let* $T > 0$*. Let* $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ *be in* $\mathbf{V}^1(\Omega_{\eta^{1,0}}) \times \mathbb{R}^N \times \mathbb{R}^N$ *satisfying the compatibility condition* $\mathbf{u}^0 = Zq^{2,0}\mathbf{e}_2$ *on* $\Gamma^s_{\eta^{1,0}}$ *and* $\mathbf{u}^0 = \mathbf{0}$ *on* Γ*. Then there exists* $r > 0$ *such that if*

$$\|\mathbf{u}^0\|_{\mathbf{V}^1(\Omega_{\eta^{1,0}})} + |q^{1,0}|_{\mathbb{R}^N} + |q^{2,0}|_{\mathbb{R}^N} < r,$$

then the system (5.7)–(5.6) *is null controllable at time* T *in* $(\mathbf{u}, q, q')$*. That means exactly there exists* $\mathbf{c} \in L^2(0,T;\mathbf{L}^2(\omega))$ *such that*

$$\mathbf{u}(T) = \mathbf{0}, \quad q(T) = 0 \quad \textit{and} \quad q'(T) = 0.$$

Like the other results of controllability of nonlinear coupled systems already mentioned in the introduction, the first step of the proof is to use a suitable change of variables to set the system in a fixed domain without changing the domain ω of the control. This change of variables and the equivalent system are introduced in the Section 5.1.4. Then, in section 5.2, we prove the null controllability for the linearized system with nonhomogeneous right-hand sides using a duality method and a Carleman estimate. The proof of the Carleman estimate is postponed to section 5.4. Section 5.3 is devoted to the proof of Theorem 5.3. It relies on a fixed point procedure.

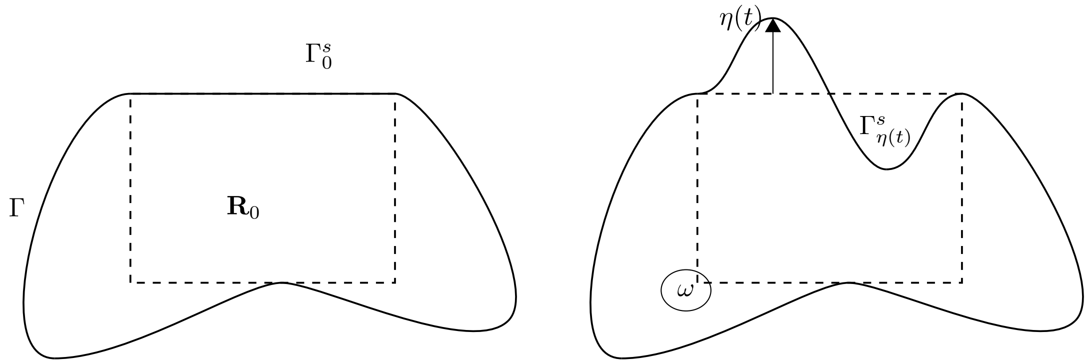

Figure 5.1: The domains Ω_0 (on the left), $\Omega_{\eta(t)}$(on the right) and $\mathbf{R}_0$.

5.1.4 The system in a fixed domain.

We suppose that the rectangle $\mathbf{R}_0 = (0, L) \times (0, 1)$ is included in the domain Ω_0, see Figure 5.1.

The change of variables is

$$\begin{array}{rcccl} \theta_t : & \Omega_{\eta(t)} & \longrightarrow & \Omega_0 & \\ & (x,y) & \longmapsto & (x,z) & \text{with} \left\{ \begin{array}{ll} z = \varepsilon + (1-\varepsilon)\dfrac{y-\varepsilon}{1-\varepsilon+\eta(t,x)} & \text{if } 0 \le x \le L \text{ and } \varepsilon \le y < 1+\eta(t,x) \\ z = y & \text{otherwise.} \end{array} \right. \end{array}$$

Setting $\hat{f}(x,z) = f(x,y)$, we can calculate the derivatives of $f(x,y)$ using the derivatives of $\hat{f}(x,z)$ in $(0, L) \times (\varepsilon, 1)$:

$$\left\{ \begin{array}{rcl} f_t &=& \hat{f}_t - (z-\varepsilon)\dfrac{\eta_t}{1-\varepsilon+\eta}\hat{f}_z, \\ f_x &=& \hat{f}_x - (z-\varepsilon)\dfrac{\eta_x}{1-\varepsilon+\eta}\hat{f}_z, \\ f_y &=& \dfrac{1-\varepsilon}{1-\varepsilon+\eta}\hat{f}_z, \\ f_{xx} &=& \hat{f}_{xx} - 2(z-\varepsilon)\dfrac{\eta_x}{1-\varepsilon+\eta}\hat{f}_{xz} + \left((z-\varepsilon)\dfrac{\eta_x}{1-\varepsilon+\eta}\right)^2 \hat{f}_{zz} - (z-\varepsilon)\dfrac{(1-\varepsilon+\eta)\eta_{xx} - \eta_x^2}{(1-\varepsilon+\eta)^2}\hat{f}_z, \\ f_{yy} &=& \dfrac{(1-\varepsilon)^2}{(1-\varepsilon+\eta)^2}\hat{f}_{zz}. \end{array} \right.$$

Now, we state the system satisfied by $\hat{\mathbf{u}}(x,z) = \mathbf{u}(x,y)$ and $\hat{p}(x,z) = p(x,y)$:

$$\begin{array}{rcll} \hat{\mathbf{u}}_t - \text{div } \sigma(\hat{\mathbf{u}}, \hat{p}) &=& \hat{\mathbf{c}}\chi_\omega + \mathbf{F}[\hat{\mathbf{u}}, \hat{p}, \eta] & \text{in } Q_T^0, \\ \text{div } \hat{\mathbf{u}} &=& \text{div } \mathbf{w}[\hat{\mathbf{u}}, \eta] & \text{in } Q_T^0, \\ \hat{\mathbf{u}} &=& Zq'\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \hat{\mathbf{u}} &=& \mathbf{0} & \text{on } \Sigma_T, \\ \hat{\mathbf{u}}(0) &=& \hat{\mathbf{u}}^0 & \text{in } \Omega_0 \end{array}$$

with $\mathbf{F}[\hat{\mathbf{u}},\hat{p},\eta]=-(\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}}=-(\mathbf{u}\cdot\nabla)\mathbf{u}$, $\hat{\mathbf{c}}=\mathbf{c}$ and $\mathbf{w}[\hat{\mathbf{u}},\eta]=\mathbf{0}$ for $(x,z)\in\Omega\setminus(0,L)\times(\varepsilon,1)$. For (x,z) in $(0,L)\times(\varepsilon,1)$, we have:

$$\begin{aligned}\mathbf{F}(t,x,z) &= \frac{1}{1-\varepsilon}\Bigg(-\eta\hat{\mathbf{u}}_t+\left[(z-\varepsilon)\eta_t+\nu(z-\varepsilon)\left(\frac{2\eta_x^2}{1-\varepsilon+\eta}-\eta_{xx}\right)\right]\hat{\mathbf{u}}_z\\ &\quad+\nu\left\{-2(z-\varepsilon)\eta_x\hat{\mathbf{u}}_{xz}+\eta\hat{\mathbf{u}}_{xx}+\frac{(z-\varepsilon)^2\eta_x^2-\eta(1-\varepsilon)}{1-\varepsilon+\eta}\hat{\mathbf{u}}_{zz}\right\}\\ &\quad+((z-\varepsilon)\eta_x\hat{p}_z-\eta\hat{p}_x)\mathbf{e}_1-(1-\varepsilon+\eta)\hat{u}_1\hat{\mathbf{u}}_x+((z-\varepsilon)\eta_x\hat{u}_1-(1-\varepsilon)\hat{u}_2)\hat{\mathbf{u}}_z\Bigg)\end{aligned}$$

and

$$\mathbf{w}(t,x) = \frac{1}{1-\varepsilon}\left(-\eta\hat{u}_1\mathbf{e}_1+(z-\varepsilon)\eta_x\hat{u}_1\mathbf{e}_2\right). \tag{5.8}$$

The change of variables gives us a new formula for the right-hand side of (5.6):

$$\Pi_N\Big[\hat{p}-2\nu\hat{u}_{2,z}\Big]+h[\hat{\mathbf{u}},\eta]$$

where

$$h[\hat{\mathbf{u}},\eta]=\nu\Pi_N\left(\frac{\eta_x}{1+\eta}\hat{u}_{1,z}+\eta_x\hat{u}_{2,x}-\frac{\eta_x^2-2\eta}{1+\eta}\hat{u}_{2,z}\right). \tag{5.9}$$

With the identification (5.3), we can use the notation $h[\hat{\mathbf{u}},q]=h[\hat{\mathbf{u}},\eta]$ and the same for $\mathbf{F}[\hat{\mathbf{u}},\hat{p},q]$ and $\mathbf{w}[\hat{\mathbf{u}},q]$. To simplify the notation, we drop out the symbol $\hat{\cdot}$ and we get the following system:

$$\begin{array}{rcll}\mathbf{u}_t-\operatorname{div}\sigma(\mathbf{u},p) &=& \mathbf{c}\chi_\omega+\mathbf{F}[\mathbf{u},p,q] & \text{in } Q_T^0,\\ \operatorname{div}\mathbf{u} &=& \operatorname{div}\mathbf{w}[\mathbf{u},q] & \text{in } Q_T^0,\\ \mathbf{u} &=& Zq'\mathbf{e}_2 & \text{on } \Sigma_T^{s,0},\\ \mathbf{u} &=& \mathbf{0} & \text{on } \Sigma_T,\\ q''+Aq &=& \Pi_N\Big[p-2\nu u_{2,z}\Big]+h[\mathbf{u},q] & \text{in } (0,T),\\ \big(\mathbf{u}(0),q(0),q'(0)\big) &=& \big(\mathbf{u}^0,q^{1,0},q^{2,0}\big). & \end{array} \tag{5.10}$$

A way to solve the system (5.10) is to find a equivalent problem with divergence free (see [4, 16]). Due to the expression of the nonhonmogeneous divergence term div $\mathbf{w}$, we look for a solution $\mathbf{u}$ of (5.10) under the form $\mathbf{u}=\mathbf{v}+\mathbf{w}$. The new system in the variables $(\mathbf{v},p,q)$ is

$$\begin{array}{rcll}\mathbf{v}_t-\operatorname{div}\sigma(\mathbf{v},p) &=& \mathbf{c}\chi_\omega+\overline{\mathbf{F}}[\mathbf{u},p,q] & \text{in } Q_T^0,\\ \operatorname{div}\mathbf{v} &=& 0 & \text{in } Q_T^0,\\ \mathbf{v} &=& Zq'\mathbf{e}_2 & \text{on } \Sigma_T^{s,0},\\ \mathbf{v} &=& \mathbf{0} & \text{on } \Sigma_T,\\ q''+Aq &=& \Pi_N p+\overline{h}[\mathbf{u},q] & \text{in } (0,T),\\ \big(\mathbf{v}(0),q(0),q'(0)\big) &=& \big(\mathbf{u}^0-\mathbf{w}(0),q^{1,0},q^{2,0}\big). & \end{array} \tag{5.11}$$

Indeed, the formula of $\mathbf{w}[\mathbf{u},q]$ gives us directly that $\mathbf{w}(0)=\frac{1}{1-\varepsilon}\left(-\eta^{1,0}u_1^0\mathbf{e}_1+(z-\varepsilon)\eta_x^{1,0}u_1^0\mathbf{e}_2\right)$ only depends on $(\mathbf{u}^0,q^{1,0},q^{2,0})$ and that $\mathbf{w}[\mathbf{u},q]_{|\Gamma}=\mathbf{0}$ for $(\mathbf{u},p,q)$ solution of the system (5.10). Furthermore, the term $\Pi_N\Big[-2\nu v_{2,z}\Big]$ does not appear in the right-hand side of $(5.11)_5$ because if $\mathbf{v}$ in $\mathbf{H}^{2,1}(Q_T^0)$ is solution of (5.11) then div $\mathbf{v}=0$ and $v_1=0$ on Γ_0, which together give that $v_{2,z}=0$ on Γ_0^s.

In system (5.11), $\overline{\mathbf{F}}$ and $\overline{h}$ are defined by

$$\overline{\mathbf{F}}[\mathbf{u},p,q]=\mathbf{F}[\mathbf{u},p,q]+\nu\Delta\mathbf{w}[\mathbf{u},q]-\mathbf{w}[\mathbf{u},q]_t,\qquad \overline{h}[\mathbf{u},q]=h[\mathbf{u},q]-2\nu\Pi_N\Big[w_{2,z}[\mathbf{u},q]\Big] \tag{5.12}$$

with

$$\mathbf{v}=v_1\mathbf{e}_1+v_2\mathbf{e}_2\quad\text{and}\quad\mathbf{w}[\mathbf{u},q]=w_1[\mathbf{u},q]\mathbf{e}_1+w_2[\mathbf{u},q]\mathbf{e}_2.$$

From now on, we denote

$$\mathbf{v}^0=\mathbf{u}^0-\mathbf{w}[\mathbf{u},q](0). \tag{5.13}$$

On the other hand, we have to add a compatibility condition at time $t = 0$ for $(\mathbf{v}^0, q^{1,0}, q^{2,0})$:

$$\text{div } (\mathbf{v}^0) = 0 \quad \text{in } \Omega_0, \qquad \mathbf{v}^0 = Zq^{2,0}\mathbf{e}_2 \quad \text{on } \Gamma_0^s \quad \text{and} \quad \mathbf{v}^0 = \mathbf{0} \quad \text{on } \Gamma. \tag{5.14}$$

For $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ the compatibility conditions are

$$\begin{array}{c} \text{div } \Big(\mathbf{u}^0 + \frac{1}{1-\varepsilon}\left(Zq^{1,0}u_1^0\mathbf{e}_1 - (z-\varepsilon)Z_x q^{1,0}u_1^0\mathbf{e}_2\right)\Big) = 0 \quad \text{in } \Omega_0, \\ \mathbf{u}^0 = Zq^{2,0}\mathbf{e}_2 \quad \text{on } \Gamma_0^s \quad \text{and} \quad \mathbf{u}^0 = \mathbf{0} \quad \text{on } \Gamma. \end{array} \tag{5.15}$$

5.2 Null controllability of the linearized system with nonhomogeneous right-hand sides.

Fixing intial data $(\mathbf{v}^0, q^{1,0}, q^{2,0})$ and right-hand sides $(\overline{\mathbf{F}}, \overline{h})$, our goal in this section is to prove the null controllability of system (5.16).

$$\begin{array}{rcll} \mathbf{v}_t - \text{div } \sigma(\mathbf{v}, p) & = & \mathbf{c}\chi_\omega + \overline{\mathbf{F}} & \text{in } Q_T^0, \\ \text{div } \mathbf{v} & = & 0 & \text{in } Q_T^0, \\ \mathbf{v} & = & Zq'\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{v} & = & \mathbf{0} & \text{on } \Sigma_T, \\ q'' + Aq & = & \Pi_N p + \overline{h} & \text{in } (0,T), \\ \big(\mathbf{v}(0), q(0), q'(0)\big) & = & \big(\mathbf{v}^0, q^{1,0}, q^{2,0}\big). & \end{array} \tag{5.16}$$

This section is split into three parts. First, in section 5.2.1, we introduce an auxiliary linear system and we state a result of controllability for this system under some assumptions. In section 5.2.2, we set system (5.16) in the abstract general setting of the previous section. Then, in the last section, we prove the controllability of system (5.16).

5.2.1 An auxiliary result.

This part is adapted from [14]. We consider the following abstract linear system:

$$\begin{array}{rcl} z'(t) & = & Az(t) + Bu(t) + Jf(t) \\ z(0) & = & z^0. \end{array} \tag{5.17}$$

Here, U, H, F are Hilbert spaces and A is an unbounded linear operator generator of an analytic semigroup on H denoted $(\mathrm{e}^{tA})_{t\geq 0}$. B and J are two linear continuous operators respectively from U into H and from F into H, z^0 is an element of H.

Let us introduce weight functions ρ_i $(i = 1, 2, 3)$ defined by

$$\rho_i : [0,T] \to \mathbb{R} \text{ continuous functions satisfying } \rho_i(T) = 0, \quad \rho_i(t) > 0 \; \forall t \in [0,T). \tag{5.18}$$

Then, we define three time-dependent weighted function spaces $\mathfrak{F}$, $\mathfrak{Z}$ and $\mathfrak{U}$ by

$$\begin{array}{rcl} \mathfrak{F} & = & \left\{ f \in L^2(0,T;F) \text{ s.t. } \rho_1^{-1} f \in L^2(0,T;F) \right\}, \\ \mathfrak{Z} & = & \left\{ z \in L^2(0,T,H) \text{ s.t. } \rho_2^{-1} z \in L^2(0,T;H) \right\}, \\ \mathfrak{U} & = & \left\{ u \in L^2(0,T,U) \text{ s.t. } \rho_3^{-1} u \in L^2(0,T;U) \right\}. \end{array}$$

In this general abstract setting, we prove the following lemma:

Lemma 5.4. *We have the equivalence between*
(i) For any ψ in $L^2(0,T;H)$, the solution ϕ of

$$\begin{array}{rcl} -\phi'(t) & = & A^*\phi(t) + \psi(t) \\ \phi(T) & = & 0 \end{array} \tag{5.19}$$

satisfies the estimate

$$\|\phi(0)\|_H^2 + \int_0^T \rho_1^2(t)\|J^*\phi(t)\|_F^2 \leq C\left(\int_0^T \rho_2^2(t)\|\psi(t)\|_H^2 + \int_0^T \rho_3^2(t)\|B^*\phi(t)\|_U^2\right). \tag{5.20}$$

(ii) For any (z^0, f) in $H \times \mathfrak{F}$, there exists u in $\mathfrak{U}$ such that the solution z of (5.17) belongs to $\mathfrak{Z}$.

Proof. Remember that the general form of solution for system (5.17) can be written *via* the Duhamel formula

$$z(t) = \mathrm{e}^{tA} z^0 + \int_0^t \mathrm{e}^{(t-s)A} Bu(s)\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A} Jf(s)\mathrm{d}s$$

which can also be written

$$z(t) - \int_0^t \mathrm{e}^{(t-s)A} Bu(s)\mathrm{d}s = \mathrm{e}^{tA} z^0 + \int_0^t \mathrm{e}^{(t-s)A} Jf(s)\mathrm{d}s.$$

We introduce two operators L_T and M_T as follows

$$\begin{array}{rccl} L_T: & H \times \mathfrak{F} & \longrightarrow & L^2(0,T;H) \\ & (z^0, f) & \longmapsto & \left(t \mapsto \mathrm{e}^{tA} z^0 + \displaystyle\int_0^t \mathrm{e}^{(t-s)A} Jf(s)\mathrm{d}s\right) \end{array}$$

and

$$\begin{array}{rccl} M_T: & \mathfrak{Z} \times \mathfrak{U} & \longrightarrow & L^2(0,T;H) \\ & (z, u) & \longmapsto & \left(t \mapsto z(t) - \displaystyle\int_0^t \mathrm{e}^{(t-s)A} Bu(s)\mathrm{d}s\right). \end{array}$$

Then, condition (ii) of the Lemma is equivalent to

$$\mathrm{Range}\, L_T \subset \mathrm{Range}\, M_T.$$

This last inclusion is equivalent to the existence of a constant $C > 0$ such that

$$\|L_T^* \psi\|_{H \times \mathfrak{F}'} \leq C \|M_T^* \psi\|_{\mathfrak{Z}' \times \mathfrak{U}'} \qquad \text{for all } \psi \in L^2(0,T;H). \tag{5.21}$$

The spaces $\mathfrak{F}'$, $\mathfrak{Z}'$ and $\mathfrak{U}'$ are the dual spaces of $\mathfrak{F}$, $\mathfrak{Z}$ and $\mathfrak{U}$:

$$\begin{array}{rcl} \mathfrak{F}' & = & \left\{f \in L^2(0,T,F) \text{ s.t. } \rho_1 f \in L^2(0,T;F)\right\}, \\ \mathfrak{Z}' & = & \left\{z \in L^2(0,T,H) \text{ s.t. } \rho_2 z \in L^2(0,T;H)\right\}, \\ \mathfrak{U}' & = & \left\{u \in L^2(0,T,U) \text{ s.t. } \rho_3 u \in L^2(0,T;U)\right\} \end{array}$$

with the identifications $H \equiv H'$, $F' \equiv F$ and $U \equiv U'$.

By a simple calculation, we get, for ϕ solution of (5.19),

$$\begin{array}{rccl} L_T^*: & L^2(0,T;H) & \longrightarrow & H \times \mathfrak{F}', \\ & \psi & \longmapsto & (\phi(0), J^*\phi) \end{array} \qquad \begin{array}{rccl} M_T^*: & L^2(0,T;H) & \longrightarrow & \mathfrak{Z}' \times \mathfrak{U}' \\ & \psi & \longmapsto & (\psi, B^*\phi). \end{array}$$

Then, (5.21) becomes

$$\|\phi(0)\|_H^2 + \int_0^T \rho_1^2(t)\|J^*\phi(t)\|_F^2 \leq C\left(\int_0^T \rho_2^2(t)\|\psi(t)\|_H^2 + \int_0^T \rho_3^2(t)\|B^*\phi(t)\|_U^2\right),$$

which is exactly (5.20). □

Then, we have the following stronger result:

Theorem 5.5. *Under the hypothesis of Lemma 5.4, assume that (i) holds. Then we can define a linear bounded operator U_T from $H \times \mathfrak{F}$ into $\mathfrak{U}$ by*

$$\begin{array}{rccl} U_T: & H \times \mathfrak{F} & \longrightarrow & \mathfrak{U} \\ & (z^0, f) & \longmapsto & u_{(z^0, f)}, \end{array}$$

such that the solution z of system (5.17) corresponding with the control $u_{(z^0,f)}$ belongs to $\mathfrak{Z}$.

Moreover, if z^0 belongs to $D((-A)^{1/2})$ and if there exists ρ_0 in $\mathcal{C}^2([0,T];\mathbb{R})$ such that

$$\begin{aligned} &\rho_0(t) \geq 0 \quad \forall t \in (0,T) \quad \text{and} \quad \rho_0(t) = 0 \Longleftrightarrow t = T, \\ &\frac{\rho_i}{\rho_0} \in L^\infty(0,T) \text{ for } i = 1,2,3, \quad \frac{\rho_0'\rho_j}{\rho_0^2} \in L^\infty(0,T) \text{ for } j = 1 \text{ or } j = 2, \end{aligned} \tag{5.22}$$

then, z satisfies

$$\frac{z}{\rho_0} \in L^2(0,T;D(-A)) \cap H^1(0,T;H) \cap \mathcal{C}([0,T];D((-A)^{1/2})),$$

with the estimate

$$\left\| \frac{z}{\rho_0} \right\|_{L^2(0,T;D(-A))\cap H^1(0,T;H)\cap\mathcal{C}([0,T];D((-A)^{1/2}))} \leq C\Big(\|z^0\|_{D((-A)^{1/2})} + \|f\|_{\mathfrak{F}} \Big).$$

Proof. We begin by proving the existence of the bounded linear operator U_T. Assuming condition (i) in Lemma 5.4, we know that there exists for any initial data z^0 in H and right-hand side f in $\mathfrak{F}$ at least a function u in $\mathfrak{U}$ such that z belongs to $\mathfrak{Z}$. Now, we consider the following functional

$$\mathcal{J}(z,u) = \frac{1}{2}\|z\|_{\mathfrak{F}}^2 + \frac{1}{2}\|u\|_{\mathfrak{U}}^2.$$

Then, we can find among all the previous control u, the one minimizing this functional, with the corresponding z. Thanks to the observability inequality, a direct calculation gives that this control $\overline{u}$ satisfies the estimate

$$\|\overline{u}\|_{\mathfrak{U}} \leq C\Big(\|z^0\|_H + \|f\|_{\mathfrak{F}} \Big).$$

Denoting $\overline{u} = U_T(z^0, f)$, then U_T is a linear operator from $H \times \mathfrak{F}$ into $\mathfrak{U}$. Furthermore, it is bounded thanks to the previous inequality.

The second part relies on the following classical proposition:

Proposition 5.6. *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{X}$ into $\mathbb{X}$ where $\mathbb{X}$ is a Hilbert space and $\mathcal{A}$ an operator generator of an analytic semigroup on $D(\mathcal{A})$ with a compact resolvent in $\mathbb{X}$. If $\mathcal{Y}^0$ belongs to $D((-\mathcal{A})^{1/2})$ and $\mathcal{B}$ belongs to $L^2(0,T;\mathbb{X})$, then equation*

$$\begin{aligned} \mathcal{Y}'(t) &= \mathcal{A}\mathcal{Y}(t) + \mathcal{B}(t) \\ \mathcal{Y}(0) &= \mathcal{Y}^0 \end{aligned}$$

admits a unique solution $\mathcal{Y}$ in $L^2(0,T;D(\mathcal{A})) \cap H^1(0,T;\mathbb{X}) \cap \mathcal{C}([0,T];D((-\mathcal{A})^{1/2}))$. Furthermore, we get the estimate

$$\|\mathcal{Y}\|_{L^2(0,T;D(\mathcal{A}))\cap H^1(0,T;\mathbb{X})\cap\mathcal{C}([0,T];D((-\mathcal{A})^{1/2}))} \leq C\Big(\|\mathcal{Y}^0\|_{D((-\mathcal{A})^{1/2})} + \|\mathcal{B}\|_{L^2(0,T;\mathbb{X})} \Big).$$

Because $u_{(z^0,f)}$, f and z^0 belongs respectively to $L^2(0,T;H)$, $L^2(0,T;F)$ and $D((-A)^{1/2})$, we can apply the previous proposition and we get that that the solution z of (5.17) belongs to $L^2(0,T;D(-A)) \cap H^1(0,T;H) \cap \mathcal{C}([0,T];D((-A)^{1/2}))$. Futhermore, dividing equation (5.17) by ρ_0, we obtain

$$\begin{aligned} \left(\frac{z}{\rho_0}\right)' &= A\left(\frac{z}{\rho_0}\right) + \frac{f}{\rho_0} - \frac{\rho_0'}{\rho_0^2} z, \\ \left(\frac{z}{\rho_0}\right)(0) &= \frac{z^0}{\rho_0(0)}. \end{aligned} \tag{5.23}$$

Then, we get that $\left(\frac{z}{\rho_0}\right)'$ belongs to $L^2(0,T;H)$ provided that $\frac{z}{\rho_0}$ belongs to $L^2(0,T;D(-A))$. From the previous lemma, we have $\frac{z}{\rho_2}$ in $L^2(0,T;H)$; second, from the choice of the function ρ_0, we have

$$-\frac{\rho_0'}{\rho_0^2} z = -\frac{\rho_0'\rho_2}{\rho_0^2}\frac{z}{\rho_2}$$

which belongs to $L^2(0,T;H)$. Then, applying Proposition 5.6 to system (5.23), we get that

$$\frac{z}{\rho_0} \in L^2(0,T;D(-A)) \cap H^1(0,T;H) \cap \mathcal{C}([0,T];D((-A)^{1/2}))$$

with the estimate

$$\left\| \frac{z}{\rho_0} \right\|_{L^2(0,T;D(-A))\cap H^1(0,T;H)\cap\mathcal{C}([0,T];D((-A)^{1/2}))} \leq C\Big(\|z^0\|_{D((-A)^{1/2})} + \|f\|_{\mathfrak{F}} \Big).$$

□

5.2.2 Equivalent system.

In this section, we fix the initial data $(\mathbf{v}^0, q^{1,0}, q^{2,0})$ in $X^0_{\mathbf{cc}}$ defined by

$$X^0 = \mathbf{H}^1(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N$$

and

$$X^0_{\mathbf{cc}} = \Big\{ (\mathbf{z}^0, k^0, k^1) \in X^0 \text{ such that } (\mathbf{z}^0, k^0, k^1) \text{ verifies (5.14)} \Big\}.$$

The space X^0 is equipped with the norm

$$\|(\mathbf{z}^0, k^0, k^1)\|_{X^0} = \Big(\|\mathbf{z}^0\|^2_{\mathbf{H}^1(\Omega_0)} + |A^{1/2}k^0|^2_{\mathbb{R}^N} + |k^1|^2_{\mathbb{R}^N} \Big)^{1/2}.$$

The right-hand side $(\overline{\mathbf{F}}, \overline{h})$ belongs to the time-dependent weighted function space $\overline{\mathcal{W}}_T$ (see below). Let us define

$$\mathbb{V} = \mathbf{V}^0(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N \tag{5.24}$$

equipped with the norm

$$\Big\| (\mathbf{v}, q, r) \Big\|^2_{\mathbb{V}} = \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega_0)} + |A^{1/2}q|^2_{\mathbb{R}^N} + |r|^2_{\mathbb{R}^N} \quad \text{ for all } (\mathbf{v}, q, r) \in \mathbb{V}.$$

At this point, we introduce time dependent weighted functions ρ_i (for $i = 0, 1, 2, 3$) satisfying the conditions of the previous section, that is ρ_i (for $i = 1, 2, 3$) satisfy (5.18) and ρ_0 is as in Theorem 5.5, that is ρ_0 is $\mathcal{C}^2([0,T];\mathbb{R})$ and satisfies (5.22). More details about these weighted functions are given in Theorem 5.13 and in section 5.4 (in particular, such functions exist).

We introduce the spaces

$$\begin{aligned}
\overline{\mathcal{W}}_T &= \Big\{ (\mathbf{G}, g) \in L^2(0,T;\mathbf{L}^2(\Omega_0) \times \mathbb{R}^N) \text{ s.t. } \rho_1^{-1}(\mathbf{G}, g) \text{ belongs to } L^2(0,T;\mathbf{L}^2(\Omega_0) \times \mathbb{R}^N) \Big\}, \\
\mathcal{Z}_T &= \Big\{ (\mathbf{z}, r) \text{ s.t. } (\mathbf{z}, r, r') \text{ and } \rho_2^{-1}(\mathbf{z}, r, r') \text{ are in } L^2(0,T;\mathbb{V}) \Big\}, \\
\mathcal{U}_T &= \Big\{ \mathbf{d} \in L^2(0,T;\mathbf{L}^2(\omega)) \text{ s.t. } \rho_3^{-1}\mathbf{d} \text{ is in } L^2(0,T;\mathbf{L}^2(\omega)) \Big\}.
\end{aligned}$$

These spaces are equipped with the norms

$$\begin{aligned}
\|(\mathbf{G}, g)\|_{\overline{\mathcal{W}}_T} &= \int_0^T \rho_1^{-2}(t) \Big[\|\mathbf{G}(t)\|^2_{\mathbf{L}^2(\Omega_0)} + |g(t)|^2_{\mathbb{R}^N} \Big] \mathrm{d}t \quad && \text{for all } (\mathbf{G}, g) \in \overline{\mathcal{W}}_T, \\
\|(\mathbf{z}, r)\|_{\mathcal{Z}_T} &= \int_0^T \rho_2^{-2}(t) \|(\mathbf{z}, r, r')(t)\|^2_{\mathbb{V}} \mathrm{d}t && \text{for all } (\mathbf{z}, r) \in \mathcal{Z}_T, \\
\|\mathbf{d}\|_{\mathcal{U}_T} &= \int_0^T \rho_3^{-2}(t) \|\mathbf{d}(t)\|^2_{\mathbf{L}^2(\omega)} \mathrm{d}t && \text{for all } \mathbf{d} \in \mathcal{U}_T.
\end{aligned}$$

We now write system (5.16) as a first order in time linear partial differential equation. Let us introduce the so-called Leray projection P from $\mathbf{L}^2(\Omega_0)$ in $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$ where

$$\mathbf{V}^0_{\mathbf{n}}(\Omega_0) = \Big\{\mathbf{u} \in \mathbf{L}^2(\Omega_0) \text{ such that } \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma_0 \text{ and div } \mathbf{u} = 0 \text{ in } \Omega_0\Big\}.$$

We split system (5.16) using the equality $\mathbf{v} = P\mathbf{v} + (I-P)\mathbf{v}$. Let us denote $\mathbf{v}_e = P\mathbf{v}$ and $\mathbf{v}_s = (I-P)\mathbf{v}$. Each part of the velocity field $\mathbf{v}$ is associated with a corresponding pressure term p_e and p_s. We have the following proposition:

Proposition 5.7. *System* (5.16) *can be splitted into two systems. One, system* (5.25)*, is an evolutionary system in the variables* $(\mathbf{v}_e, q_1, q_2)$ *(where* $q_1 = q$ *and* $q_2 = q'$*) and the other, system* (5.26)*, is a stationary system giving* $(\mathbf{v}_s, p_e, p_s)$ *as functions of* $(\mathbf{v}_e, q_1, q_2)$*. That is system* (5.16) *is equivalent to* (5.25)–(5.26) *(see the notation below):*

$$\begin{array}{rcl}\begin{pmatrix}\mathbf{v}_e\\ q_1\\ q_2\end{pmatrix}' & = & K_s\begin{pmatrix} A_0 & 0 & (-A_0)PD_s\\ 0 & 0 & I_N\\ \nu\Pi_N\mathcal{N}(\Delta(\cdot)\cdot\mathbf{n}) & -A & 0\end{pmatrix}\begin{pmatrix}\mathbf{v}_e\\ q_1\\ q_2\end{pmatrix}\\ & & +K_s\left[\begin{pmatrix}P\overline{\mathbf{F}}\\ 0\\ \Pi_N\pi(\overline{\mathbf{F}}) + \overline{h}\end{pmatrix} + \begin{pmatrix}P(\mathbf{c}\chi_\omega)\\ 0\\ \Pi_N\pi_0(\mathbf{c}\chi_\omega)\end{pmatrix}\right],\\ \begin{pmatrix}\mathbf{v}_e\\ q_1\\ q_2\end{pmatrix}(0) & = & \begin{pmatrix}P\mathbf{v}^0\\ q^{1,0}\\ q^{2,0}\end{pmatrix}\end{array} \tag{5.25}$$

and secondly

$$\begin{array}{rcll}\mathbf{v}_s & = & \nabla\mathcal{N}_s(Zq_2) & \textit{in } Q_T^0,\\ p_e & = & \mathcal{N}(\Delta\mathbf{v}_e\cdot\mathbf{n}) & \textit{in } Q_T^0,\\ p_s & = & \pi(\overline{\mathbf{F}}) + \pi_0(\mathbf{c}\chi_\omega) - \mathcal{N}_s(Zq_2') & \textit{in } Q_T^0,\\ p & = & p_e + p_s & \textit{in } Q_T^0,\\ \mathbf{v} & = & \mathbf{v}_e + \mathbf{v}_s & \textit{in } Q_T^0.\end{array} \tag{5.26}$$

Furthermore, system (5.25) *is exactly under the form of system* (5.17)*.*

Proof. We use a method due to Raymond (see [22]). In particular, we adapt here the decomposition of a similar system made in [23]. We write it in this paper for sake of completness. From the Stokes system

$$\begin{array}{rcll}\mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p & = & \mathbf{c}\chi_\omega + \overline{\mathbf{F}} & \text{in } Q_T^0,\\ \text{div } \mathbf{v} & = & 0 & \text{in } Q_T^0,\\ \mathbf{v} & = & Zq'\mathbf{e}_2 & \text{on } \Sigma_T^{s,0},\\ \mathbf{v} & = & \mathbf{0} & \text{on } \Sigma_T,\\ \mathbf{v}(0) & = & \mathbf{v}^0 & \text{in } \Omega_0,\end{array}$$

we get the following equivalent system

$$\begin{array}{rcll}\mathbf{v}_{e,t} - \nu\Delta\mathbf{v}_e + \nabla p_e & = & P(\mathbf{c}\chi_\omega) + P\overline{\mathbf{F}} & \text{in } Q_T^0,\\ \mathbf{v}_e & = & -\gamma_\tau\mathbf{v}_s & \text{on } \Sigma_T^0,\\ \mathbf{v}_e(0) & = & P\mathbf{v}^0 & \text{in } \Omega_0,\\ \mathbf{v}_s & = & \nabla\mathcal{N}_s(Zq') & \text{in } Q_T^0,\\ p_s & = & \pi(\overline{\mathbf{F}}) + \pi_0(\mathbf{c}\chi_\omega) - \mathcal{N}_s(Zq'') & \text{in } Q_T^0,\\ \mathbf{v} & = & \mathbf{v}_e + \mathbf{v}_s & \text{in } Q_T^0,\\ p & = & p_s + p_e & \text{in } Q_T^0.\end{array} \tag{5.27}$$

In (5.27), we denote $\mathcal{N}_s(\cdot) = \mathcal{N}(\cdot\chi_{\Gamma_0^s})$ where $\mathcal{N}$ the operator from $H^\sigma(\Gamma_0)$ to $H^{\sigma+3/2}(\Omega_0)$ (for $\sigma \geq -1/2$) defined by $r = \mathcal{N}(j)$ for j in $H^\sigma(\Gamma_0)$ if and only if

$$\Delta r = 0 \quad \text{in } \Omega_0, \qquad \frac{\partial r}{\partial \mathbf{n}} = j \quad \text{on } \Gamma_0.$$

and π and π_0 are operators from $\mathbf{L}^2(\Omega_0)$ into $H^1(\Omega_0)$ defined by

$$\left\{\begin{array}{rcll} \Delta\pi(\overline{\mathbf{F}}) & = & \mathrm{div}\,\overline{\mathbf{F}} & \text{in } \Omega_0, \\ \dfrac{\partial\pi(\overline{\mathbf{F}})}{\partial\mathbf{n}} & = & \overline{\mathbf{F}}\cdot\mathbf{n} & \text{on } \Gamma_0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcll} \Delta\pi_0(\mathbf{c}\chi_\omega) & = & \mathrm{div}\,(\mathbf{c}\chi_\omega) & \text{in } \Omega_0, \\ \dfrac{\partial\pi_0(\mathbf{c}\chi_\omega)}{\partial\mathbf{n}} & = & 0 & \text{on } \Gamma_0. \end{array}\right. \tag{5.28}$$

We have an explicit formula for π and π_0:

$$\pi(\overline{\mathbf{F}}) = -(-\Delta_D)^{-1}(\mathrm{div}\,\overline{\mathbf{F}}) + \mathcal{N}((\overline{\mathbf{F}} + \nabla(-\Delta_D)^{-1}(\mathrm{div}\,\overline{\mathbf{F}}))\cdot\mathbf{n}),$$

$$\pi_0(\mathbf{c}\chi_\omega) = -(-\Delta_D)^{-1}(\mathrm{div}\,(\mathbf{c}\chi_\omega)) + \mathcal{N}((\nabla(-\Delta_D)^{-1}(\mathrm{div}\,(\mathbf{c}\chi_\omega)))\cdot\mathbf{n}),$$

where $\pi_1 = -(-\Delta_D)^{-1}(g)$ if and only if $\pi_1 \in H_0^1(\Omega_0)$ and $\Delta\pi_1 = g$ in Ω_0 for any $g \in H^{-1}(\Omega_0)$.

From the first equation in (5.27), we get that p_e satisfies for any time t in $(0,T)$:

$$\Delta p_e(t) = 0 \quad \text{in } \Omega_0, \qquad \frac{\partial p_e(t)}{\partial\mathbf{n}} = \nu\Delta\mathbf{v}_e(t) \quad \text{on } \Gamma_0,$$

that is $p_e = \nu\mathcal{N}(\Delta\mathbf{v}_e\cdot\mathbf{n})$.

In conclusion, $p = p_s + p_e$ is equal to

$$p = \pi(\overline{\mathbf{F}}) + \pi_0(\mathbf{c}\chi_\omega) - \mathcal{N}_s(Zq'') + \nu\mathcal{N}(\Delta\mathbf{v}_e\cdot\mathbf{n}) \quad \text{in } \Omega_0.$$

Then the beam equation becomes

$$(I_N + \Pi_N\mathcal{N}_s(Z\,(\cdot)))q'' + Aq = \nu\Pi_N\mathcal{N}(\Delta\mathbf{v}_e\cdot\mathbf{n}) + \Pi_N\pi(\overline{\mathbf{F}}) + \Pi_N\pi_0(\mathbf{c}\chi_\omega) + \overline{h}.$$

System (5.16) is equivalent to system

$$\begin{array}{rcll} \mathbf{v}_{e,t} - \Delta\mathbf{v}_e + \nabla p_e & = & P(\mathbf{c}\chi_\omega) + P\overline{\mathbf{F}} & \text{in } Q_T^0, \\ \mathbf{v}_e & = & -\gamma_\tau\mathbf{v}_s & \text{on } \Sigma_T^0, \\ \mathbf{v}_e(0) & = & P\mathbf{v}^0 & \text{in } \Omega_0, \\ \mathbf{v}_s & = & \nabla\mathcal{N}_s(Zq') & \text{in } Q_T^0, \\ p_s & = & \pi(\overline{\mathbf{F}}) - \mathcal{N}_s(Zq'') & \text{in } Q_T^0, \\ (I_N + \Pi_N\mathcal{N}_s(Z(\cdot)))q'' + Aq & = & \nu\Pi_N\mathcal{N}(\Delta\mathbf{v}_e\cdot\mathbf{n}) + \Pi_N\pi(\overline{\mathbf{F}}) + \Pi_N\pi_0(\mathbf{c}\chi_\omega) + \overline{h} & \text{in } (0,T), \\ \big(q(0), q'(0)\big) & = & \big(q^{1,0}, q^{2,0}\big), & \\ \mathbf{v} & = & \mathbf{v}_e + \mathbf{v}_s & \text{in } Q_T^0, \\ p & = & p_s + p_e & \text{in } Q_T^0. \end{array} \tag{5.29}$$

From this system, we can obtain an evolution equation. Indeed, $(\mathbf{v}_e, q, q')$ is uncoupled to $(\mathbf{v}_s, p_e, p_s)$. Then, we have first, with obvious notation $q = q_1$ and $q' = q_2$:

$$\begin{array}{rcl} \begin{pmatrix} \mathbf{v}_e \\ q_1 \\ q_2 \end{pmatrix}' & = & K_s\begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I_N \\ \nu\Pi_N\mathcal{N}(\Delta(\cdot)\cdot\mathbf{n}) & -A & 0 \end{pmatrix}\begin{pmatrix} \mathbf{v}_e \\ q_1 \\ q_2 \end{pmatrix} \\ & & +K_s\left[\begin{pmatrix} P\overline{\mathbf{F}} \\ 0 \\ \Pi_N\pi(\overline{\mathbf{F}}) + \overline{h} \end{pmatrix} + \begin{pmatrix} P(\mathbf{c}\chi_\omega) \\ 0 \\ \Pi_N\pi_0(\mathbf{c}\chi_\omega) \end{pmatrix}\right], \\ \begin{pmatrix} \mathbf{v}_e \\ q_1 \\ q_2 \end{pmatrix}(0) & = & \begin{pmatrix} P\mathbf{v}^0 \\ q^{1,0} \\ q^{2,0} \end{pmatrix} \end{array} \tag{5.30}$$

and secondly

$$\begin{array}{rcll} \mathbf{v}_s & = & \nabla\mathcal{N}_s(Zq_2) & \text{in } Q_T^0, \\ p_e & = & \mathcal{N}(\Delta\mathbf{v}_e\cdot\mathbf{n}) & \text{in } Q_T^0, \\ p_s & = & \pi(\overline{\mathbf{F}}) + \pi_0(\mathbf{c}\chi_\omega) - \mathcal{N}_s(Zq_2') & \text{in } Q_T^0, \\ p & = & p_e + p_s & \text{in } Q_T^0, \\ \mathbf{v} & = & \mathbf{v}_e + \mathbf{v}_s & \text{in } Q_T^0, \end{array} \tag{5.31}$$

where K_s an isomorphism from $\mathbf{V}^0_{\mathbf{n}}(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N$ into itself defined by

$$K_s = \begin{pmatrix} \mathbf{Id} & 0 & 0 \\ 0 & I_N & 0 \\ 0 & 0 & (I_N + \Pi_N \mathcal{N}_s(Z(\cdot)))^{-1} \end{pmatrix}, \tag{5.32}$$

A_0 is the Stokes operator defined by $D(A_0) = \mathbf{V}^2(\Omega_0) \cap \mathbf{V}^1_0(\Omega_0)$ in $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$ and $A_0 \mathbf{z}_e = \nu P \Delta \mathbf{z}_e$, for all $\mathbf{z}_e$ in $D(A_0)$. The operator D_s is a lifting of the nonhomogeneous Dirichlet condition $\mathbf{v} = Zq_2\mathbf{e}_2$ on Γ^s_0 defined from $\mathbb{R}^N$ into $\mathbf{V}^2(\Omega_0)$ for r in $\mathbb{R}^N$ by $\mathbf{z} = D_s r$ if and only if there exists a function ρ in $\mathcal{H}^1(\Omega_0)$ such that

$$\begin{aligned} -\nu\Delta\mathbf{z} + \nabla\rho &= \mathbf{0} && \text{in } \Omega_0, \\ \operatorname{div} \mathbf{z} &= 0 && \text{in } \Omega_0, \\ \mathbf{z} &= Zr\mathbf{e}_2 && \text{on } \Gamma^s_0, \\ \mathbf{z} &= \mathbf{0} && \text{on } \Gamma. \end{aligned}$$

We finally get that system (5.25)–(5.26) is equivalent to system (5.29), that is system (5.25)–(5.26) is equivalent to system (5.16).

We now can identify notations from (5.25) with those from the previous section. The Hilbert spaces H, U and F are now respectively

$$\mathbb{V}_{\mathbf{n}} = \mathbf{V}^0_{\mathbf{n}}(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N, \qquad \mathbf{L}^2(\omega) \quad \text{and} \quad \mathbf{L}^2(\Omega_0) \times \mathbb{R}^N.$$

The operator A in (5.17) is remplaced by

$$\mathcal{A} = K_s \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I_N \\ \nu \Pi_N \mathcal{N}(\Delta(\cdot)\cdot\mathbf{n}) & -A & 0 \end{pmatrix}$$

which is defined from

$$D(\mathcal{A}) = \Big\{ (\mathbf{z}_e, q_1, q_2) \in \mathbf{V}^2(\Omega_0) \cap \mathbf{V}^0_{\mathbf{n}}(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N \text{ s.t. } \mathbf{z}_e = -\gamma_\tau \nabla \mathcal{N}_s(Zq_2) \text{ on } \Gamma_0 \Big\}$$

in $\mathbb{V}_{\mathbf{n}}$. We have

$$\begin{aligned} B : \quad \mathbf{L}^2(\omega) &\longrightarrow \mathbb{V}_{\mathbf{n}} \\ \mathbf{c}\chi_\omega &\longmapsto \begin{pmatrix} P(\mathbf{c}\chi_\omega) \\ 0 \\ (I_N + \Pi_N \mathcal{N}_s(Z(\cdot)))^{-1} \Pi_N \pi_0(\mathbf{c}\chi_\omega) \end{pmatrix} \end{aligned}, \qquad J\begin{pmatrix} \overline{\mathbf{F}} \\ \overline{h} \end{pmatrix} = J_1 \overline{\mathbf{F}} + J_2 \overline{h}$$

with

$$\begin{aligned} J_1 : \quad \mathbf{L}^2(\Omega_0) &\longrightarrow \mathbb{V}_{\mathbf{n}} \\ \overline{\mathbf{F}} &\longmapsto K_s \begin{pmatrix} P\overline{\mathbf{F}} \\ 0 \\ \Pi_N \pi_0(\overline{\mathbf{F}}) \end{pmatrix} \end{aligned}, \qquad \begin{aligned} J_2 : \quad \mathbb{R}^N &\longrightarrow \mathbb{V}_{\mathbf{n}} \\ \overline{h} &\longmapsto K_s \begin{pmatrix} \mathbf{0} \\ 0 \\ \overline{h} \end{pmatrix} \end{aligned}.$$

This gives, with $f = (\overline{\mathbf{F}}, \overline{h})$ in $\mathbf{L}^2(\Omega_0) \times \mathbb{R}^N$,

$$Jf = \begin{pmatrix} P\overline{\mathbf{F}} \\ 0 \\ (I_N + \Pi_N \mathcal{N}_s(Z(\cdot)))^{-1} \Big[\Pi_N \pi(\overline{\mathbf{F}}) + \overline{h} \Big] \end{pmatrix}.$$

Finally,

$$z = \begin{pmatrix} \mathbf{v}_e \\ q_1 \\ q_2 \end{pmatrix} \qquad \text{and} \qquad z^0 = \begin{pmatrix} P\mathbf{v}^0 \\ q^{1,0} \\ q^{2,0} \end{pmatrix}.$$

□

5.2.3 Null Controllability of system (5.16).

We can now state the main result of this section:

Theorem 5.8. *Let* $(\mathbf{v}^0, q^{1,0}, q^{2,0})$ *be in* $X^0_{\mathbf{cc}}$. *There exists a linear bounded operator* $\overline{U}_T$ *from* $\mathbb{V}\times\overline{\mathcal{W}}_T$ *into* $L^2(0,T;\mathbf{L}^2(\omega))$ *such that for all* $(\overline{\mathbf{F}},\overline{h})$ *in* $\overline{\mathcal{W}}_T$ *the solution of system* (5.16) *associated with the function* $\mathbf{c}=\overline{U}_T\Big((\mathbf{v}^0, q^{1,0}, q^{2,0}),(\overline{\mathbf{F}},\overline{h})\Big)$ *in the right-hand side belongs to* $\mathcal{X}_T$ *defined by*

$$\mathcal{X}_T=\big\{(\mathbf{x},\pi,r)\in X_T\,;\rho_0^{-1}(\mathbf{x},\pi,r)\in X_T\big\}\ \text{equipped with the norm}\ \|(\mathbf{x},\pi,r)\|_{\mathcal{X}_T}=\big\|\rho_0^{-1}(\mathbf{x},\pi,r)\big\|_{X_T}$$

where $X_T=\mathbf{H}^{2,1}(Q^0_T)\times L^2(0,T;\mathcal{H}^1(\Omega_0))\times H^2(0,T;\mathbb{R}^N)$. *Furthermore, we have the estimate:*

$$\|(\mathbf{v},p,q)\|_{\mathcal{X}_T}\leq C\Big(\|(\mathbf{v}^0,q^{1,0},q^{2,0})\|_{X^0}+\|(\overline{\mathbf{F}},\overline{h})\|_{\overline{\mathcal{W}}_T}\Big).$$

That is, system (5.16) *is null controllable at time* $T>0$*:*

$$\mathbf{v}(T)=\mathbf{0}\ \text{in}\ \Omega_0,\quad q(T)=0\quad \text{and}\quad q'(T)=0.$$

The proof of this proposition relies on the two previous sections. First, thanks to section 5.2.2, system (5.16) is equivalent to system (5.25)–(5.26). Then, we can apply results of section 5.2.1 to system (5.25). Finally, this results and an observability inequality finish the proof.

First, we want to write Lemma 5.4 for system (5.25). Thus, we have to calculate the adjoint operators $\mathcal{A}^*$, B^* and J^*.

Lemma 5.9. *We define the bilinear form* ϕ *on* $\mathbb{V}_\mathbf{n}$ *by*

$$\phi\Big((\mathbf{v}_e,q_1,q_2),(\mathbf{y}_e,k_1,k_2)\Big)=(\mathbf{v}_e,\mathbf{y}_e)_{\mathbf{L}^2(\Omega_0)}+(A^{1/2}q_1,A^{1/2}k_1)_{\mathbb{R}^N}+(q_2,(I_n+\Pi_N\mathcal{N}_s(Z(\cdot)))k_2)_{\mathbb{R}^N},$$

for $(\mathbf{v}_e,q_1,q_2)$ *and* $(\mathbf{y}_e,k_1,k_2)$ *in* $\mathbb{V}_\mathbf{n}$. *Then,* ϕ *is a scalar product on* $\mathbb{V}_\mathbf{n}$. *We still denote* $\mathbb{V}_\mathbf{n}$ *the space* $\mathbb{V}_\mathbf{n}$ *endowed with this scalar product. In the following, we set*

$$\langle\cdot,\cdot\rangle_{\mathbb{V}_\mathbf{n}}=\phi(\cdot,\cdot).$$

Proof. We have to prove that the operator $\Pi_N\mathcal{N}_s(Z\cdot):\mathbb{R}^N\to\mathbb{R}^N$ is symmetric and positive. Let us take q_2 and k_2 in $\mathbb{R}^N$, we calculate

$$(q_2,\Pi_N\mathcal{N}_s(Zk_2))_{\mathbb{R}^N}.$$

By definition, the function $a=\mathcal{N}_s(Zk_2)$ belongs to $H^2(\Omega_0)$ and satisfies $\left\{\begin{array}{rcll}\Delta a&=&0&\text{in }\Omega_0,\\ \dfrac{\partial a}{\partial\mathbf{n}}&=&Zk_2\chi_{\Gamma^s_0}&\text{on }\Gamma_0.\end{array}\right.$

In the same way, we denote $b=\mathcal{N}_s(Zq_2)$. First, with the previous notation

$$\begin{array}{rcl}(q_2,\Pi_N\mathcal{N}_s(Zk_2))_{\mathbb{R}^N}&=&(Zq_2,a)_{L^2(\Gamma^s_0)}\\ &=&\left(\dfrac{\partial b}{\partial\mathbf{n}},a\right)_{L^2(\Gamma^s_0)}.\end{array}$$

Second, an integration by parts gives

$$(\Delta b,a)_{L^2(\Omega_0)}=-(\nabla b,\nabla a)_{\mathbf{L}^2(\Omega_0)}+\left(\frac{\partial b}{\partial\mathbf{n}},a\right)_{L^2(\Gamma^s_0)}$$

or

$$(b,\Delta a)_{L^2(\Omega_0)}=-(\nabla b,\nabla a)_{\mathbf{L}^2(\Omega_0)}+\left(b,\frac{\partial a}{\partial\mathbf{n}}\right)_{L^2(\Gamma^s_0)}.$$

That is, because $\Delta a=0$ and $\Delta b=0$ in Ω_0,

$$\left(\frac{\partial b}{\partial\mathbf{n}},a\right)_{L^2(\Gamma^s_0)}=(\nabla b,\nabla a)_{\mathbf{L}^2(\Omega_0)},\qquad \left(b,\frac{\partial a}{\partial\mathbf{n}}\right)_{L^2(\Gamma^s_0)}=(\nabla b,\nabla a)_{\mathbf{L}^2(\Omega_0)}.$$

Putting all the calculations together, we get

$$\begin{array}{rcl}(q_2, \Pi_N \mathcal{N}_s(Zk_2))_{\mathbb{R}^N} & = & (\nabla b, \nabla a)_{\mathbf{L}^2(\Omega_0)} \\ & = & (\Pi_N \mathcal{N}_s(Zq_2), k_2)_{\mathbb{R}^N}.\end{array}$$

To prove the positivity, we calculate $(q_2, \Pi_N \mathcal{N}_s(Zq_2))_{\mathbb{R}^N}$ for q_2 in $\mathbb{R}^N$. With the previous equality, we obtain

$$(q_2, \Pi_N \mathcal{N}_s(Zq_2))_{\mathbb{R}^N} = \|\nabla b\|^2_{\mathbf{L}^2(\Omega_0)},$$

which concludes the proof. □

Proposition 5.10. - *The operator $\mathcal{A}$ is a generator of an analytic semigroup on $\mathbb{V}_{\mathbf{n}}$. Furthermore, it has a compact resolvent. The adjoint operator $\mathcal{A}^*$ is given by $D(\mathcal{A}^*) = D(\mathcal{A})$ and*

$$\mathcal{A}^* = \begin{pmatrix} \mathbf{Id} & 0 & 0 \\ 0 & I_N & 0 \\ 0 & 0 & (I_N + \Pi_N \mathcal{N}_s(Z(\cdot)))^{-1} \end{pmatrix} \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & -I_N \\ \nu \Pi_N \mathcal{N}(\Delta(\cdot)\cdot \mathbf{n}) & A & 0 \end{pmatrix}.$$

- *The operator B^* is defined from $\mathbb{V}_{\mathbf{n}}$ into $\mathbf{L}^2(\omega)$ by*

$$B^* \begin{pmatrix} \mathbf{y}_e \\ r_1 \\ r_2 \end{pmatrix} = (\mathbf{y}_e + \nabla \mathcal{N}_s(Zr_2))\chi_\omega.$$

The operator J^* is defined from $\mathbb{V}_{\mathbf{n}}$ into $\mathbf{L}^2(\Omega_0) \times \mathbb{R}^N$ by

$$J^* \begin{pmatrix} \mathbf{y}_e \\ r_1 \\ r_2 \end{pmatrix} = \Big((\mathbf{y}_e + \nabla \mathcal{N}_s(Zr_2)), r_2\Big).$$

Proof. The first point of the proof can be easily adapted from [23, Section 3.] and is left to the reader.

We now prove the second point. Let $\mathbf{d}$ be in $\mathbf{L}^2(\omega)$ and $(\mathbf{y}_e, r_1, r_2)$ be in $\mathbb{V}_{\mathbf{n}}$, then, by definition of B,

$$\left\langle B\mathbf{d}, \begin{pmatrix} \mathbf{y}_e \\ r_1 \\ r_2 \end{pmatrix} \right\rangle_{\mathbb{V}_{\mathbf{n}}} = (P\mathbf{d}, \mathbf{y}_e)_{\mathbf{V}^0_{\mathbf{n}}(\Omega_0)} + (\Pi_N \pi_0(\mathbf{d}), r_2)_{\mathbb{R}^N}.$$

By an integration by parts, we have

$$(\pi_0(\mathbf{d}), \Delta q)_{L^2(\Omega_0)} = -(\nabla \pi_0(\mathbf{d}), \nabla q)_{\mathbf{L}^2(\Omega_0)} + \left(\pi_0(\mathbf{d}), \frac{\partial q}{\partial \mathbf{n}}\right)_{L^2(\partial\Omega_0)}. \tag{5.33}$$

Denoting $q = \mathcal{N}_s(Zr_2)$, from equation (5.33), we obtain

$$(\Pi_N \pi_0(\mathbf{d}), r_2)_{\mathbb{R}^N} = (\pi_0(\mathbf{d}), Zr_2)_{L^2(\Omega_0)} = \left(\pi_0(\mathbf{d}), \frac{\partial q}{\partial \mathbf{n}}\right)_{L^2(\partial\Omega_0)} = (\nabla \pi_0(\mathbf{d}), \nabla q)_{\mathbf{L}^2(\Omega_0)}.$$

Then, setting $\mathbf{y} = \mathbf{y}_e + \nabla q$, we see that $\mathbf{y}$ is an element of $\mathbf{V}^0(\Omega)$ satisfying

$$\mathbf{y}\cdot\mathbf{n} = Zr_2 \text{ on } \Gamma^s_0, \qquad \mathbf{y}\cdot\mathbf{n} = 0 \text{ on } \Gamma^s_0$$

and furthermore, thanks to the definition of $\pi_0(\mathbf{d})$ (see (5.28)), we have $\mathbf{d} = P(\mathbf{d}) + \nabla\pi_0(\mathbf{d})$. Thus,

$$\begin{array}{rcl}(\mathbf{y}, \mathbf{d})_{\mathbf{L}^2(\Omega_0)} & = & (\mathbf{y}_e + \nabla q, P(\mathbf{d}) + \nabla \pi_0(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)} \\ & = & (\mathbf{y}_e, P(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)} + (\nabla q, \nabla \pi_0(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)} + (\mathbf{y}_e, \nabla \pi_0(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)} + (\nabla q, P(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)}.\end{array}$$

To conclude, we see that $\mathbf{y}_e$ and $P(\mathbf{d})$ belong to $\mathbf{V}^0_{\mathbf{n}}(\Omega_0)$ whereas ∇q and $\nabla \pi_0(\mathbf{d})$ belongs to $(\mathbf{V}^0_{\mathbf{n}}(\Omega_0))^\perp$. Then,

$$(\mathbf{y}, \mathbf{d})_{\mathbf{L}^2(\Omega_0)} = (\mathbf{y}_e, P(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)} + (\nabla q, \nabla \pi_0(\mathbf{d}))_{\mathbf{L}^2(\Omega_0)}.$$

Finally, putting all this calculations together, we get

$$\left\langle B(\mathbf{d}), \begin{pmatrix} \mathbf{y}_e \\ r_1 \\ r_2 \end{pmatrix} \right\rangle_{\mathbb{V}_{\mathbf{n}}} = (\mathbf{d}, \mathbf{y})_{\mathbf{L}^2(\Omega_0)} = (\mathbf{d}, \mathbf{y})_{\mathbf{L}^2(\omega)}.$$

That is B^*, the adjoint operator of B, is defined from $\mathbb{V}_{\mathbf{n}}$ into $\mathbf{L}^2(\omega)$ by

$$B^* \begin{pmatrix} \mathbf{y}_e \\ r_1 \\ r_2 \end{pmatrix} = (\mathbf{y}_e + \nabla \mathcal{N}_s(Zr_2))\chi_\omega.$$

We directly deduce J^* from the calculations above. □

Then, we have the following proposition:

Proposition 5.11. *The two following statements are equivalent:*
(i) For all $(\mathbf{a}_e, b, c)$ in $L^2(0,T;\mathbb{V}_{\mathbf{n}})$, the solution $(\mathbf{y}_e, k_1, k_2)$ of equation

$$\begin{aligned} -\begin{pmatrix} \mathbf{y}_e \\ k_1 \\ k_2 \end{pmatrix}'(t) &= \mathcal{A}^* \begin{pmatrix} \mathbf{y}_e \\ k_1 \\ k_2 \end{pmatrix}(t) + \begin{pmatrix} \mathbf{a}_e \\ b \\ c \end{pmatrix}(t), \\ \begin{pmatrix} \mathbf{y}_e(T) \\ k_1(T) \\ k_2(T) \end{pmatrix} &= \begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix} \end{aligned} \tag{5.34}$$

satisfies the inequality

$$\begin{aligned} &\|(\mathbf{y}_e(0), k_1(0), k_2(0))\|^2_{\mathbb{V}_{\mathbf{n}}} + \int_0^T \rho_1^2(t) \Big[\|\mathbf{y}_e + \nabla \mathcal{N}_s(Zk_2)\|^2_{\mathbf{L}^2(\Omega_0)} + |k_2|^2_{\mathbb{R}^N} \Big] \\ \leq\ & C \left(\int_0^T \rho_2^2(t) \, \|(\mathbf{a}_e(t), b(t), c(t))\|^2_{\mathbb{V}_{\mathbf{n}}} + \int_0^T \rho_3^2(t) \, \|\mathbf{y}_e + \nabla \mathcal{N}_s(Zk_2)\|^2_{\mathbf{L}^2(\omega)} \right). \end{aligned}$$

(ii) For all $\Big((P\mathbf{v}^0, q^{1,0}, q^{2,0}), (\overline{\mathbf{F}}, \overline{h})\Big)$ in $\mathbb{V}_{\mathbf{n}} \times \overline{\mathcal{W}}_T$, there exists a control $\mathbf{c}$ in $\mathcal{U}_T$ such that the solution $(\mathbf{v}_e, q_1, q_2)$ of (5.25) belongs to $\mathcal{Z}_T^e$ with

$$\mathcal{Z}_T^e = \Big\{ (\mathbf{x}_e, r_1, r_2) \in L^2(0,T;\mathbb{V}_{\mathbf{n}}) \ s.t.\ \rho_2^{-1}(\mathbf{x}_e, r_1, r_2) \in L^2(0,T;\mathbb{V}_{\mathbf{n}}) \Big\}.$$

Using the same idea as in section 5.2.2, we get that there exists a pressure term π such that $(\mathbf{y}, \pi, k_1, k_2)$ defined from $(\mathbf{y}_e, k_1, k_2)$ solution of (5.34) by $\mathbf{y} = \mathbf{y}_e + \nabla \mathcal{N}_s(Zk_2)$ is solution of the system

$$\begin{aligned} -\mathbf{y}_t - \text{div}\ \sigma(\mathbf{y}, \pi) &= \mathbf{a} && \text{in } Q_T^0, \\ \text{div}\ \mathbf{y} &= 0 && \text{in } Q_T^0, \\ \mathbf{y} &= 0 && \text{on } \Sigma_T, \\ \mathbf{y} &= Zk_2\mathbf{e}_2 && \text{on } \Sigma_T^{s,0}, \\ k_1' &= k_2 - b && \text{in } (0,T), \\ k_2' + Ak_1 &= -\Pi_N \pi - c && \text{in } (0,T), \\ \big(\mathbf{y}(T), k_1(T), k_2(T)\big) &= (\mathbf{0}, 0, 0) \end{aligned} \tag{5.35}$$

with $\mathbf{a} = \mathbf{a}_e + \nabla \mathcal{N}_s(Zc)$. System (5.35) is exactly the adjoint of system (5.16). Furthermore, with the notation $\mathbf{y} = \mathbf{y}_e + \nabla \mathcal{N}_s(Zk_2)$ for $(\mathbf{y}_e, k_1, k_2)$ in $\mathbb{V}_{\mathbf{n}}$, we have first that $(\mathbf{y}, k_1, k_2)$ belongs to $\mathbb{V}$ and second that

$$\begin{aligned} \Big\|(\mathbf{y}_e, k_1, k_2)\Big\|^2_{\mathbb{V}_{\mathbf{n}}} &= \|\mathbf{y}_e\|^2_{\mathbf{L}^2(\Omega_0)} + |A^{1/2}k_1|^2_{\mathbb{R}^N} + (k_2, (I_n + \Pi_N \mathcal{N}_s(Z\cdot))k_2)_{\mathbb{R}^N} \\ &= \|\mathbf{y}\|^2_{\mathbf{L}^2(\Omega_0)} + |A^{1/2}k_1|^2_{\mathbb{R}^N} + |k_2|^2_{\mathbb{R}^N} \\ &= \Big\|(\mathbf{y}, k_1, k_2)\Big\|^2_{\mathbb{V}} \end{aligned}$$

(see this calculation in the proof of Lemma 5.9 above).

Finally, Proposition 5.11 can be written in term of system (5.16) and its adjoint (5.35) as follows:

Proposition 5.12. *The two following statements are equivalent:*

(i) For all $(\mathbf{a}, b, c)$ *in* $L^2(0,T;\mathbb{V})$*, the solution* $(\mathbf{y}, \pi, k_1, k_2)$ *of system* (5.35) *satisfies the inequality:*

$$\begin{aligned} & \left\| \Big(\mathbf{y}(0), k_1(0), k_2(0)\Big) \right\|_{\mathbb{V}}^2 + \int_0^T \rho_1^2 \Big[\|\mathbf{y}\|_{\mathbf{L}^2(\Omega_0)}^2 + |k_2|_{\mathbb{R}^N}^2 \Big] \\ \leq \ & C \left(\int_0^T \rho_2^2(t) \|(\mathbf{a}(t), b(t), c(t))\|_{\mathbb{V}}^2 \, dt + \int_0^T \rho_3^2(t) \|\mathbf{y}(t)\|_{\mathbf{L}^2(\omega)}^2 \right). \end{aligned}$$

(ii) For all $(\mathbf{v}^0, q^{1,0}, q^{2,0})$ *in* $\mathbb{V}$ *and all* $(\overline{\mathbf{F}}, \overline{h})$ *in* $\overline{\mathcal{W}}_T$*, there exists* $\mathbf{c}$ *in* $\mathcal{U}_T$ *such that the solution* $(\mathbf{v}, p, q)$ *of system* (5.16) *satisfies* $(\mathbf{v}, q) \in \mathcal{Z}_T$*.*

We set here the result on the observability inequality.

Theorem 5.13. *We introduce the weight functions* $(\rho_i)_{i=0,1,2,3}$

$$\begin{array}{rcl} \rho_0(t) & = & e^{-\frac{3s}{4}\delta^*(t)}, \\ \rho_1(t) & = & (s\lambda)^{3/2}(\sigma^*(t))^{3/2} e^{-s\delta^*(t)}, \\ \rho_2(t) & = & \lambda^{5/2} s^{15/4} (\sigma^*(t))^{15/4} e^{-s\delta^*(t)}, \\ \rho_3(t) & = & \rho_2(t). \end{array} \tag{5.36}$$

where σ^* *and* δ^* *are given at the end of section 5.4. Then, there exists* $C > 0$ *such that all the smooth solutions* $(\mathbf{y}, \pi, k_1, k_2)$ *of system* (5.35) *with any right-hand side* $(\mathbf{a}, b, c)$ *in* $L^2(0,T;\mathbb{V})$ *satisfy the inequality*

$$\begin{aligned} & \left\| \Big(\mathbf{y}(0), k_1(0), k_2(0)\Big) \right\|_{\mathbb{V}}^2 + \int_0^T \rho_1^2 \Big[\|\mathbf{y}\|_{\mathbf{L}^2(\Omega_0)}^2 + |k_2|_{\mathbb{R}^N}^2 \Big] \\ \leq \ & C \left(\int_0^T \rho_2^2(t) \|(\mathbf{a}(t), b(t), c(t))\|_{\mathbb{V}}^2 \, dt + \int_0^T \rho_3^2(t) \|\mathbf{y}(t)\|_{\mathbf{L}^2(\omega)}^2 \right) \end{aligned}$$

for s *and* λ *large enough* ($s \geq \hat{s}$ *and* $\lambda \geq \hat{\lambda}$)*.*

The proof is postponed to section 5.4 and relies on a Carleman inequality. Now, we are able to prove the main result of section 5.2.3.

Proof of Theorem 5.8. Thanks to Theorem 5.13, condition *(i)* of Proposition 5.12 is satisfied. Then, we can apply Theorem 5.5 to system (5.25). That is, there exists a bounded linear operator U_T^e from $\mathbb{V}_{\mathbf{n}} \times \overline{\mathcal{W}}_T$ into $\mathcal{U}_T$ such that the solution $(\mathbf{v}_e, q_1, q_2)$ of system (5.25) associated with $\mathbf{c} = U_T^e\Big((P\mathbf{v}^0, q^{1,0}, q^{2,0}), (\overline{\mathbf{F}}, \overline{h})\Big)$ belongs to $\mathcal{Z}_T^e$. Using (5.26), we get that $\mathbf{v}_s$ belongs to

$$\mathcal{Z}_T^s = \Big\{ \mathbf{x}_s \in L^2(0,T;\mathbf{L}^2(\Omega_0)) \text{ s.t. } \rho_2^{-1}\mathbf{x}_s \in L^2(0,T;\mathbf{L}^2(\Omega_0)) \Big\}.$$

This gives together that $(\mathbf{v}, q_1, q_2) \in \mathcal{Z}_T$. Then, denoting E_T the linear bounded operator from $\mathbb{V} \times \overline{\mathcal{W}}_T$ into $\mathbb{V}_{\mathbf{n}} \times \overline{\mathcal{W}}_T$ defined by

$$E_T\Big((\mathbf{v}^0, q^{1,0}, q^{2,0}), (\overline{\mathbf{F}}, \overline{h})\Big) = \Big((P\mathbf{v}^0, q^{1,0}, q^{2,0}), (\overline{\mathbf{F}}, \overline{h})\Big),$$

we get that $\overline{U}_T = U_T^e \circ E_T$ is the linear bounded operator of the proposition.

Furthermore, for $(\mathbf{v}^0, q^{1,0}, q^{2,0})$ in $X^0_{\mathbf{cc}}$, we get that $(P\mathbf{v}^0, q^{1,0}, q^{2,0})$ belongs to $D((-\mathcal{A})^{1/2}) = \mathbf{V}^1_{\mathbf{n}}(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N$. Applying now the second point of Theorem 5.5 to system (5.25), we get that $\rho_0^{-1}(\mathbf{v}_e, q_1, q_2)$ belongs to

$$\begin{aligned} & L^2(0,T;D(-\mathcal{A})) \cap H^1(0,T;\mathbb{V}_{\mathbf{n}}) \cap \mathcal{C}([0,T];D((-\mathcal{A})^{1/2})) \\ = \ & \mathbf{V}^{2,1}(Q_T^0) \times H^1(0,T;\mathbb{R}^N) \times H^1(0,T;\mathbb{R}^N) \cap \mathcal{C}([0,T];\mathbf{V}^1_{\mathbf{n}}(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N). \end{aligned}$$

Then, using (5.26), we get that $\rho_0^{-1}(\mathbf{v}_s, p_e, p_s)$ belongs to

$$\Big(\mathbf{H}^{2,1}(Q_T^0) \cap \mathcal{C}([0,T];\mathbf{H}^1(\Omega_0))\Big) \times \Big[L^2(0,T;H^1(\Omega_0))\Big]^2.$$

Finally, $\mathbf{v} = \mathbf{v}_e + \mathbf{v}_s$, $p = p_s + p_e$ and q satisfy

$$\begin{array}{rcl} \mathbf{v},\ \rho_0^{-1}\mathbf{v} & \in & \mathbf{H}^{2,1}(Q_T^0) \cap \mathcal{C}([0,T];\mathbf{H}^1(\Omega_0)), \\ p,\ \rho_0^{-1}p & \in & L^2(0,T;H^1(\Omega_0)), \\ q,\ q',\ \rho_0^{-1}q,\ \rho_0^{-1}q' & \in & H^1(0,T;\mathbb{R}^N). \end{array}$$

That is, thanks to the embedding $H^1(0,T;\mathbb{R}^N) \hookrightarrow \mathcal{C}([0,T];\mathbb{R}^N)$ and the definition of ρ_0 (especially, $\rho_0(T) = 0$), that we have the null controllability of system (5.16):

$$\mathbf{v}(T) = \mathbf{0}, \quad \text{in } \Omega_0 \qquad \text{and} \qquad q(T) = q'(T) = 0.$$

□

5.3 Proof of Theorem 5.3.

In this section, we prove Theorem 5.3. First, we use the previous section to prove the theorem in the cylinder $(0,T) \times \Omega_0$. Then, we will derive Theorem 5.3 from this result using the change of variables introduced in section 5.1.4.

5.3.1 In the cylinder $(0,T) \times \Omega_0$.

First, we begin by proving the null controllability of system

$$\begin{array}{rcll} \mathbf{u}_t - \text{div}\ \sigma(\mathbf{u},p) & = & \mathbf{c}\chi_\omega + \mathbf{F} & \text{in } Q_T^0, \\ \text{div}\ \mathbf{u} & = & \text{div}\ \mathbf{w} & \text{in } Q_T^0, \\ \mathbf{u} & = & Zq'\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Sigma_T, \\ q'' + Aq & = & \Pi_N\Big[p - 2\nu u_{2,z}\Big] + h & \text{in } (0,T), \\ \big(\mathbf{u}(0), q(0), q'(0)\big) & = & (\mathbf{u}^0, q^{1,0}, q^{2,0}). & \end{array} \tag{5.37}$$

Because $\mathbf{u}^0$ is not divergence free (see (5.15)), we do not have $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ in the space $\mathbb{V}$. Thus, we introduce another Hilbert space

$$\mathbb{L} = \mathbf{L}^2(\Omega_0) \times \mathbb{R}^N \times \mathbb{R}^N.$$

In system (5.37), the right-hand side $(\mathbf{F}, \mathbf{w}, h)$ belongs to

$$\mathcal{W}_T \;=\; \Big\{(\mathbf{G}, \mathbf{z}, g) \in W_T \text{ s.t. } \rho_1^{-1}(\mathbf{G}, (-\Delta)\mathbf{z}, \mathbf{z}', g) \text{ belongs to } L^2(0,T;[\mathbf{L}^2(\Omega_0)]^3 \times \mathbb{R}^N)\Big\}$$

equipped with the norm

$$\|(\mathbf{G},\mathbf{z},g)\|_{\mathcal{W}_T} \;=\; \int_0^T \rho_1^{-2}(t)\Big[\|(\mathbf{G}(t),(-\Delta)\mathbf{z}(t),\mathbf{z}'(t))\|^2_{[\mathbf{L}^2(\Omega_0)]^3} + |g(t)|^2_{\mathbb{R}^N}\Big]\mathrm{d}t \quad \text{for all } (\mathbf{G},\mathbf{z},g) \in \mathcal{W}_T,$$

where

$$W_T = \Big\{(\mathbf{G},\mathbf{z},g) \in \mathbf{L}^2(Q_T^0) \times \mathbf{H}^{2,1}(Q_T^0) \times L^2(0,T;\mathbb{R}^N) \text{ such that } \mathbf{z} = \mathbf{0} \text{ on } \Gamma_0\Big\}.$$

Remark 5.14. *Conditions* $\frac{\rho_0'\rho_j}{\rho_0^2} \in L^\infty(0,T)$ *and* $\frac{\rho_i}{\rho_0} \in L^\infty(0,T)$ *in* (5.22) *for* $j = 1$ *or* $j = 2$ *give respectively the equivalence between*

$$\frac{\Delta\mathbf{w}}{\rho_1}, \frac{\mathbf{w}'}{\rho_1} \in \mathbf{L}^2(Q_T^0) \quad \textit{and} \quad \frac{\mathbf{w}}{\rho_0} \in \mathbf{H}^{2,1}(Q_T^0)$$

and

$$\frac{\Delta\mathbf{v}}{\rho_2}, \frac{\mathbf{v}'}{\rho_2} \in \mathbf{L}^2(Q_T^0) \quad \textit{and} \quad \frac{\mathbf{v}}{\rho_0} \in \mathbf{H}^{2,1}(Q_T^0).$$

Then, we have the following result:

Proposition 5.15. *Let* $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ *be in* X^0 *satisfying* (5.15)*. There exists a linear bounded operator* U_T *from* $\mathbb{L}\times\mathcal{W}_T$ *into* $L^2(0,T;\mathbf{L}^2(\omega))$ *such that for all* $(\mathbf{F}, \mathbf{w}, h)$ *in* $\mathcal{W}_T$ *the solution of system* (5.37) *associated with the function* $\mathbf{c} = U_T\Big((\mathbf{u}^0, q^{1,0}, q^{2,0}), (\mathbf{F}, \mathbf{w}, h)\Big)$ *in the right-hand side belongs to* $\mathcal{X}_T$*. Furthermore, there exists a constant* $C_1 > 0$ *such that*

$$\|(\mathbf{u}, p, q)\|_{\mathcal{X}_T} \leq C_1\Big(\|(\mathbf{u}^0, q^{1,0}, q^{2,0})\|_{X^0} + \|(\mathbf{F}, \mathbf{w}, h)\|_{\mathcal{W}_T}\Big). \tag{5.38}$$

That is, system (5.37) *is null controllable at time* $T > 0$

$$\mathbf{u}(T) = \mathbf{0} \text{ in } \Omega_0, \quad q(T) = 0 \quad \text{and} \quad q'(T) = 0.$$

Proof. Let us define the operator K_T by

$$\begin{array}{cccc} K_T : & \mathbb{L}\times\mathcal{W}_T & \longrightarrow & \mathbb{V}\times\overline{\mathcal{W}}_T \\ & \Big((\mathbf{u}^0, q^{1,0}, q^{2,0}), (\mathbf{F}, \mathbf{w}, h)\Big) & \longmapsto & \Big((\mathbf{v}^0, q^{1,0}, q^{2,0}), (\overline{\mathbf{F}}, \overline{h})\Big) \end{array}$$

where $\mathbf{v}^0$ is defined by (see (5.13))

$$\mathbf{v}^0 = \mathbf{u}^0 - \mathbf{w}(0)$$

and $(\overline{\mathbf{F}}, \overline{h})$ are defined from $(\mathbf{F}, \mathbf{w}, h)$ as follow (see (5.12))

$$\overline{\mathbf{F}} = \mathbf{F} + \nu\Delta\mathbf{w} - \mathbf{w}_t, \qquad \overline{h} = h - 2\nu\Pi_N\Big[w_{2,z}\Big].$$

The operator K_T is clearly linear. Moreover it is bounded

$$\begin{aligned} \Big\|K_T\Big((\mathbf{u}^0, q^{1,0}, q^{2,0}), (\mathbf{F}, \mathbf{w}, h)\Big)\Big\|_{\mathbb{V}\times\overline{\mathcal{W}}_T} &\leq C\Big(\|(\mathbf{u}^0, q^{1,0}, q^{2,0})\|_{\mathbb{L}} + \|\mathbf{w}(0)\|_{\mathbf{L}^2(\Omega_0)} + \|(\overline{\mathbf{F}}, \overline{h})\|_{\overline{\mathcal{W}}_T}\Big) \\ &\leq C\Big\|\Big((\mathbf{u}^0, q^{1,0}, q^{2,0}), (\mathbf{F}, \mathbf{w}, h)\Big)\Big\|_{\mathbb{L}\times\mathcal{W}_T}. \end{aligned}$$

Indeed, $\mathbf{w}$ belongs to $\mathbf{H}^{2,1}(Q_T^0) \hookrightarrow \mathcal{C}([0,T];\mathbf{H}^1(\Omega_0))$, then $\|\mathbf{w}(0)\|_{\mathbf{L}^2(\Omega_0)} \leq C\|(\mathbf{F}, \mathbf{w}, h)\|_{\mathcal{W}_T}$.

Then, thanks to the existence of a bounded operator $\overline{U}_T$ from $\mathbb{V}\times\overline{\mathcal{W}}_T$ into $L^2(0,T;\mathbf{L}^2(\omega))$ used in Theorem 5.8, we get by composition a linear bounded operator U_T defined from $\mathbb{L}\times\mathcal{W}_T$ into $L^2(0,T;\mathbf{L}^2(\omega))$.

The fact that the solution $(\mathbf{u}, p, q)$ of (5.37) associated to $\mathbf{c} = U_T\Big((\mathbf{u}^0, q^{1,0}, q^{2,0}), (\mathbf{F}, \mathbf{w}, h)\Big)$ belongs to $\mathcal{X}_T$ comes exactly from Theorem 5.8 and $\mathbf{u} = \mathbf{v}+\mathbf{w}$. Indeed, by construction, the solution $(\mathbf{v}, p, q)$ of (5.16) corresponding with $(\mathbf{v}^0, q^{1,0}, q^{2,0})$ and $(\overline{\mathbf{F}}, \overline{h})$—both obtained from $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ and $(\mathbf{F}, \mathbf{w}, h)$—and associated to $\mathbf{c} = U_T\Big((\mathbf{u}^0, q^{1,0}, q^{2,0}), (\mathbf{F}, \mathbf{w}, h)\Big) = \overline{U}_T\Big((\mathbf{v}^0, q^{1,0}, q^{2,0}), (\overline{\mathbf{F}}, \overline{h})\Big)$ belongs to $\mathcal{X}_T$. Moreover, as $\mathbf{w}$ and $\frac{\mathbf{w}}{\rho_0}$ belongs to $\mathbf{H}^{2,1}(Q_T^0)$ (see Remark 5.14 and the definition of ρ_0 in (5.22)), we have first $(\mathbf{u}, p, q) = (\mathbf{v} + \mathbf{w}, p, q)$ belongs to $\mathcal{X}_T$ with the expected estimate and second, thanks to $\mathbf{w}(T) = \mathbf{0}$, that

$$\mathbf{u}(T) = \mathbf{0} \text{ in } \Omega_0 \qquad \text{and} \qquad q(T) = q'(T) = 0.$$

□

From now on, the initial data $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ is fixed in X^0 and satisfies (5.15). The time $T > 0$ is fixed too. We want to prove the controllability of the system written in the fixed domain (5.10). We use a fixed point procedure based on the result for the linearized system (5.37).

Lemma 5.16. *Let* $(\mathbf{u}, p, q)$ *be the solution in* $\mathcal{X}_T$ *of the system* (5.37) *for the intial data* $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ *in* X^0 *satisfying* (5.15) *and right-hand sides* $(\mathbf{F}, \mathbf{w}, h)$ *in* $\mathcal{W}_T$*, then* $(\overline{\mathbf{F}}, \overline{\mathbf{w}}, \overline{h}) = (\mathbf{F}[\mathbf{u}, p, q], \mathbf{w}[\mathbf{u}, q], h[\mathbf{u}, p, q])$ *defined by* (5.8) *and* (5.9) *belongs to* $\mathcal{W}_T$ *and there exists a constant* C_2 *such that*

$$\|(\overline{\mathbf{F}}, \overline{\mathbf{w}}, \overline{h})\|_{\mathcal{W}_T} \leq C_2(1 + \|(\mathbf{u}, p, q)\|_{\mathcal{X}_T})\|(\mathbf{u}, p, q)\|_{\mathcal{X}_T}^2. \tag{5.39}$$

Furthermore, let $(\mathbf{u}_i, p_i, q_i)$ $(i = 1, 2)$ *be solutions in* $\mathcal{X}_T$ *of system* (5.37) *with the same initial data* $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ *in* X^0 *satisfying* (5.15) *and repectively right-hand sides* $(\mathbf{F}_i, \mathbf{w}_i, h_i)$ $(i = 1, 2)$ *in* $\mathcal{W}_T$*. If* $(\mathbf{u}_i, p_i, q_i)$ $(i = 1, 2)$ *satisfies for some* $R > 0$,

$$\|(\mathbf{u}_i, p_i, q_i)\|_{\mathcal{X}_T} \leq R,$$

then, we have the estimate

$$\|(\overline{\mathbf{F}}_1, \overline{\mathbf{w}}_1, \overline{h}_2) - (\overline{\mathbf{F}}_2, \overline{\mathbf{w}}_2, \overline{h}_2)\|_{\mathcal{W}_T} \leq C_2(1 + R)R\|(\mathbf{u}_1, p_1, q_1) - (\mathbf{u}_2, p_2, q_2)\|_{\mathcal{X}_T} \tag{5.40}$$

where $(\overline{\mathbf{F}}_i, \overline{\mathbf{w}}_i, \overline{h}_i) = (\overline{\mathbf{F}}[\mathbf{u}_i, p_i, q_i], \overline{\mathbf{w}}[\mathbf{u}_i, q_i], \overline{h}[\mathbf{u}_i, q_i])$ $(i = 1, 2)$.

Proof. First, ρ_0 and ρ_2 defined in (5.36) satisfy $\frac{\rho_0^2}{\rho_2} \in L^\infty(0, T; \mathbb{R})$. Then, with this, the proof is a consequence of the definition of the right-hand sides $\overline{\mathbf{F}}$, $\overline{\mathbf{w}}$ in (5.8) and $\overline{h}$ in (5.9).The estimate of the $\mathcal{W}_T$-norm of $(\overline{\mathbf{F}}, \overline{\mathbf{w}}, \overline{h})$ is tedious but straightforward from Proposition 6.1 in [16]. □

Proposition 5.17. *Let* $(\overline{\mathbf{u}}, \overline{p}, \overline{q})$ *in* $\mathcal{X}_T$ *be a solution of the control problem of system* (5.37) *associated with* $(\mathbf{u}^0, q^{1,0}, q^{2,0})$, $(\mathbf{F}, \mathbf{w}, h)$ *in* $\mathcal{W}_T$ *and the control* $\mathbf{c} = U_T\Big((\mathbf{u}^0, q^{1,0}, q^{2,0}), (\mathbf{F}, \mathbf{w}, h)\Big)$ *in* $L^2(0, T; \mathbf{L}^2(\omega))$ *(see Proposition 5.15). Then, system*

$$\begin{array}{rcll} \mathbf{u}_t - div\ \sigma(\mathbf{u}, p) & = & \mathbf{c}\chi_\omega + \mathbf{F}[\overline{\mathbf{u}}, \overline{p}, \overline{q}] & in\ Q_T^0, \\ div\ \mathbf{u} & = & div\ \mathbf{w}[\overline{\mathbf{u}}, \overline{q}] & in\ Q_T^0, \\ \mathbf{u} & = & Zq'\mathbf{e}_2 & on\ \Sigma_T^{s,0}, \\ \mathbf{u} & = & \mathbf{0} & on\ \Sigma_T, \\ q'' + Aq & = & \Pi_N p + h[\overline{\mathbf{u}}, \overline{q}] & in\ (0, T), \\ (\mathbf{u}(0), q(0), q'(0)) & = & (\mathbf{u}^0, q^{1,0}, q^{2,0}). & \end{array} \tag{5.41}$$

is null controllable at time T*, that is there exists a control*

$$\mathbf{c} = U_T\Big((\mathbf{u}^0, q^{1,0}, q^{2,0}), (\mathbf{F}[\overline{\mathbf{u}}, \overline{p}, \overline{q}], \mathbf{w}[\overline{\mathbf{u}}, \overline{q}], h[\overline{\mathbf{u}}, \overline{q}])\Big)$$

in $L^2(0, T; \mathbf{L}^2(\omega))$ *such that the solution* $(\mathbf{u}, p, q)$ *of system* (5.41) *corresponding with* $\mathbf{c}$ *belongs to* $\mathcal{X}_T$ *and satisfies*

$$\mathbf{u}(T) = \mathbf{0} \text{ in } \Omega_0, \qquad q(T) = 0, \qquad q'(T) = 0.$$

Furthermore, the triplet $(\mathbf{u}, p, q)$ *satisfies the estimate*

$$\|(\mathbf{u}, p, q)\|^2_{\mathcal{X}_T} \leq C_1\Big(\|(\mathbf{u}^0, q^{1,0}, q^{2,0})\|^2_{X^0} + C_2(1 + \|(\overline{\mathbf{u}}, \overline{p}, \overline{q})\|_{\mathcal{X}_T})\|(\overline{\mathbf{u}}, \overline{p}, \overline{q})\|^2_{\mathcal{X}_T}\Big).$$

In other terms, we can contruct a mapping

$$\begin{array}{rrcl} \mathcal{C}_T : & \mathcal{X}_T & \longrightarrow & \mathcal{X}_T \\ & (\overline{\mathbf{u}}, \overline{p}, \overline{q}) & \longmapsto & \mathcal{C}_T(\overline{\mathbf{u}}, \overline{p}, \overline{q}) = (\mathbf{u}, p, q) \text{ is the solution of the control problem for system (5.41)} \end{array}$$

which satisfies the estimate

$$\|\mathcal{C}_T(\overline{\mathbf{u}}, \overline{p}, \overline{q})\|^2_{\mathcal{X}_T} \leq C_1\Big(\|(\mathbf{u}^0, q^{1,0}, q^{2,0})\|^2_{X^0} + C_2(1 + \|(\overline{\mathbf{u}}, \overline{p}, \overline{q})\|_{\mathcal{X}_T})\|(\overline{\mathbf{u}}, \overline{p}, \overline{q})\|^2_{\mathcal{X}_T}\Big). \tag{5.42}$$

Proof. The proof relies on Proposition 5.15 and estimate (5.39) in the previous lemma. The constants C_1 and C_2 are defined respectively in (5.38) and (5.39). □

We now are able to state the main result of this section:

Proposition 5.18. *Let* $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ *be in* X^0 *satisfying* (5.15)*. Then, there exists* r *small enough such that, under condition*

$$\|(\mathbf{u}^0, q^{1,0}, q^{2,0})\|_{X^0} \leq r,$$

system (5.10) *is null controllable at time* $T > 0$*, that is there exists a control* $\mathbf{c}$ *in* $L^2(0, T; \mathbf{L}^2(\omega))$ *such that system* (5.10) *associated with this control* $\mathbf{c}$ *admits a solution* $(\mathbf{u}, p, q)$ *in* $\mathcal{X}_T$ *satisfying*

$$\mathbf{u}(T) = \mathbf{0} \text{ in } \Omega_0, \qquad q(T) = 0, \qquad q'(T) = 0.$$

Proof. For $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ in X^0 as above, we denote $r = \|(\mathbf{u}^0, q^{1,0}, q^{2,0})\|_{X^0}$ and $R = 2C_1 r$ (with C_1 defined in (5.38)). We choose r such that $C_2 r(1 + 2C_1 r) = 1$ (with C_2 defined in (5.39)), that is

$$r = \frac{1}{2C_1^2 C_2} \frac{1}{\sqrt{1 + \frac{2}{C_1 C_2}}}.$$

Then, we define a ball of the space $\mathcal{X}_T$ of radius R as follows:

$$\mathcal{X}_T^R = \Big\{ (\mathbf{z}, \rho, r) \in \mathcal{X}_T \text{ s.t. } \|(\mathbf{z}, \rho, r)\|_{\mathcal{X}_T} \leq R \Big\}.$$

Then, $\mathcal{C}_T$ is a contraction mapping in $\mathcal{X}_T^R$. Indeed, for two triplets $(\mathbf{u}_i, p_i, q_i)$ in $\mathcal{X}_T$, by definition of $\mathcal{C}_T$, we get first that $\mathcal{C}_T(\mathbf{u}_i, p_i, q_i)$ $(i = 1, 2)$ is solution of the control problem of system (5.10) corresponding with initial data $(\mathbf{u}^0, q^{1,0}, q^{2,0})$, right-hand sides $(\mathbf{F}[\mathbf{u}_i, p_i, q_i], \mathbf{w}[\mathbf{u}_i, q_i], h[\mathbf{u}_i, q_i])$ and the control $\mathbf{c}_i = U_T\Big((\mathbf{u}^0, q^{1,0}, q^{2,0}), (\mathbf{F}[\mathbf{u}_i, p_i, q_i], \mathbf{w}[\mathbf{u}_i, q_i], h[\mathbf{u}_i, q_i])\Big)$. This means that $\mathcal{C}_T(\mathbf{u}_i, p_i, q_i)$ $(i = 1, 2)$ satifies

$$\|\mathcal{C}_T(\mathbf{u}_i, p_i, q_i)\|_{\mathcal{X}_T} \leq \frac{R}{2} + \frac{R}{2} = R.$$

Furthermore, the difference $\mathcal{C}_T(\mathbf{u}_1, p_1, q_1) - \mathcal{C}_T(\mathbf{u}_2, p_2, q_2)$ satisifies by linearity system (5.10) with $(\mathbf{0}, 0, 0)$ for initial data and $(\mathbf{F}_1, \mathbf{w}_1, h_2) - (\mathbf{F}_2, \mathbf{w}_2, h_2)$ for right-hand sides. Then, *via* the estimates (5.38) in Proposition 5.15 and (5.40) in Lemma 5.16 and the choice of r, we have

$$\|\mathcal{C}_T(\mathbf{u}_1, p_1, q_1) - \mathcal{C}_T(\mathbf{u}_2, p_2, q_2)\|_{\mathcal{X}_T} \leq \frac{1}{2} \|(\mathbf{u}_1, p_1, q_1) - (\mathbf{u}_2, p_2, q_2)\|_{\mathcal{X}_T}.$$

For r chosen as above, $\mathcal{C}_T$ is a contraction mapping from $\mathcal{X}_T^R$ into itself. Then, using the Picard-Banach fixed point theorem, this mapping admits a fixed point $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{q})$ in $\mathcal{X}_T$ solution of the control problem (5.10) corresponding with initial data $(\mathbf{u}^0, q^{1,0}, q^{2,0})$ in $X^0_{\mathbf{cc}}$, right-hand sides $(\mathbf{F}[\tilde{\mathbf{u}}, \tilde{p}, \tilde{q}], \mathbf{w}[\tilde{\mathbf{u}}, \tilde{q}], h[\tilde{\mathbf{u}}, \tilde{q}])$ and the control $\mathbf{c} = U_T\Big((\mathbf{u}^0, q^{1,0}, q^{2,0}), (\mathbf{F}[\tilde{\mathbf{u}}, \tilde{p}, \tilde{q}], \mathbf{w}[\tilde{\mathbf{u}}, \tilde{q}], h[\tilde{\mathbf{u}}, \tilde{q}])\Big)$. That is exactly $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{q})$ is a solution of (5.10) in X_T and satisfies:

$$\tilde{\mathbf{u}}(T) = \mathbf{0} \text{ in } \Omega_0, \qquad \tilde{q}(T) = 0 \quad \text{and} \quad \tilde{q}'(T) = 0.$$

□

5.3.2 In the moving domain.

In this section, we have to check the conditions on the change of variables. That is we have to prove that the change of variables

$$\begin{array}{rccc} \phi_t : & \Omega_0 & \longrightarrow & \Omega_{\eta(t)} \\ & (x, z) & \longmapsto & (x, y) \end{array}$$

is well-defined as a $\mathcal{C}^1$–diffeomorphism from Ω_0 into $\Omega_{\eta(t)}$ for every t in $[0, T]$ and that condition (5.1) is checked. The regularity of q and of the functions ζ_k $(k = 1, \ldots, N)$ gives together with the formula of change of variables in section 5.1.4 that ϕ_t is a $\mathcal{C}^1$–diffeomorphism. We just need to check assumption (5.1). Since $\eta(t, x) = Zq$, η would satisfy the hypothesis (5.1) if we have an estimate on q like

$$\|q\|_{L^\infty(0,T;\mathbb{R}^N)} \leq \frac{1 - \varepsilon}{3\|Z\|_{L^\infty(0,L)}}.$$

Indeed, the maximum of the function η in $\Sigma_T^{s,0}$ can be roughly bounded by

$$\|\eta\|_{L^\infty(\Sigma_T^{s,0})} \leq \|Z\|_{L^\infty(\Gamma_0^s)} \|q\|_{L^\infty(0,T)}.$$

Then $1 + \eta(t, x) \geq \varepsilon$ for $(t, x) \in \Sigma_T^{s,0}$ if $\|\eta\|_{L^\infty(\Sigma_T^{s,0})} \leq 1 - \varepsilon$. Because of the following estimate

$$\|q\|_{L^\infty(0,T;\mathbb{R}^N)} \leq C \left\| \frac{q'}{\rho_0} \right\|_{H^1(0,T;\mathbb{R}^N)} \leq C(\|(\mathbf{u}^0, q^{1,0}, q^{2,0})\|_{X^0} + \|(\mathbf{F}, \mathbf{w}, h)\|_{\mathcal{W}_T}),$$

if both the conditions $\|(\mathbf{u}^0, q^{1,0}, q^{2,0})\|_{X^0} \leq r$ and $(\mathbf{F}, \mathbf{w}, h) \in \mathcal{W}_T$ such that $\|(\mathbf{F}, \mathbf{w}, h)\|_{\mathcal{W}_T} \leq r$ are satisfied then

$$\|q\|_{L^\infty(0,T;\mathbb{R}^N)} \leq 2Cr \leq \frac{2(1-\varepsilon)}{3\|Z\|_{L^\infty(0,L)}} \leq \frac{1-\varepsilon}{\|Z\|_{L^\infty(0,L)}}$$

for r small enough and the hypothesis (5.1) is checked. That is, up to the change of parameter r_1 defined by

$$r_1 = \min\left(r, \frac{1}{C}\frac{1-\varepsilon}{3\|Z\|_{L^\infty(0,L)}}\right),$$

instead of r in the previous section, we have the result of Theorem 5.3 and in the same time the assumption 5.1 is checked.

To conclude, we can remark that the control $\mathbf{c}$ stated in Theorem 5.3 is exactely the one obtained by the fixed point procedure in section 5.3.1. Indeed, the change of variables does not change the subdomain ω where the control acts. In other words, we have, with obvious notations, $\phi_t(\mathbf{c}) = \mathbf{c}$.

5.4 Proof of Theorem 5.13.

Our goal is to prove an observability inequality for the system

$$\begin{array}{rcll} -\mathbf{y}_t - \operatorname{div}\sigma(\mathbf{y},\pi) &=& \mathbf{a} & \text{in } Q_T^0, \\ \operatorname{div}\mathbf{y} &=& 0 & \text{in } Q_T^0, \\ \mathbf{y} &=& Zk_2\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{y} &=& \mathbf{0} & \text{on } \Sigma_T, \\ k_1' &=& k_2 - b & \text{in } (0,T), \\ k_2' &=& -Ak_1 - \Pi_N\pi - c & \text{in } (0,T), \\ (\mathbf{y}(T), k_1(T), k_2(T)) &=& (\mathbf{0}, 0, 0). & \end{array} \tag{5.43}$$

The observability inequality we want to prove comes from a Carleman estimate for the coupled system. In the following, we first prove this Carleman estimate (see sections 5.4.1 to 5.4.4). Then, in section 5.4.5, we derive the expected result from the previous estimates.

The proof of this Carleman estimate relies mainly on two papers. Indeed, numerous steps of the proof can be found in [28] with details in [27]. On the other hand, the integral of the pressure term (see sections 5.4.2 and 5.4.3) is estimated *via* the method of Fernandez-Cara, Guerrero, Imanuvilov and Puel in [10] itself using [15]. This step is different from the one in [28] where the authors consider a fictitious control acting on the divergence term (see [28, section 10]). This is the main different between the proof in [28] and ours.

Roughly speaking, the results of this section can be found in the literature cited above. Some obvious parts — as integrations by parts — or some details are mentionned but not clearly proved, the interested reader can adapt them from either [28] or [10].

Let ϕ be a $\mathcal{C}^2(\overline{\Omega_0})$ function satisfying
- $\phi(x) > 0$, for all $x \in \overline{\Omega_0}$,
- $|\nabla\phi(x)| > 0$ for all $x \in \overline{\Omega_0 \setminus \omega_0}$,
- $\phi(x) = C$ for all $x \in \Gamma$,
- $\partial_{\mathbf{n}}\phi(x) \leq 0$ for all $x \in \Gamma_0$,
- $\partial_{\mathbf{n}}\phi(x) = -1$, $\Delta\phi(x) = \mathbf{0}$ for all $x \in \Gamma_0^s$.

We define for a large parameter $\lambda \geq 1$, the functions

$$\begin{array}{rcll} \xi(x,t) &=& \dfrac{\mathrm{e}^{\lambda(\phi + m\|\phi\|_\infty)}}{t^k(T-t)^k}, & m > 1 \\ \kappa(x) &=& \mathrm{e}^{\lambda m K_1} - \mathrm{e}^{\lambda(\phi(x)+m\|\phi\|_\infty)}, & \forall x \in \overline{\Omega_0}, \end{array}$$

where $K_1 > 0$ is a constant such that $K_1 \geq 2\|\phi\|_\infty$. We set next $\varphi_\lambda(x,t) = \frac{\kappa(x)}{t^k(T-t)^k}$ and $\rho(x,t) = \mathrm{e}^{\varphi_\lambda(x,t)}$ where k is a constant number such that $k \geq 2$. The number k will be fixed to 4 in section 5.4.3, following [10, 28, 14].

Let us define $\mathbf{z}(x,t) = \rho^{-s}(x,t)\mathbf{y}(x,t)$. System (5.43) written in the variables $(\mathbf{z}, \pi, k_1, k_2)$ is

$$\begin{array}{rcll} M_1\mathbf{z} + M_2\mathbf{z} & = & \mathbf{f}_s & \text{in } Q_T^0, \\ \text{div } \mathbf{z} & = & -s\nabla\varphi_\lambda \cdot \mathbf{z} & \text{in } Q_T^0, \\ \mathbf{z} & = & \rho^{-s} Z k_2 \mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{z} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{z}(0) = \mathbf{z}(T) & = & \mathbf{0} & \text{in } \Omega_0, \\ k_1' & = & k_2 - b & \text{in } (0,T), \\ k_2' + Ak_1 & = & \Pi_N \pi - c & \text{in } (0,T), \\ k_1(0) = k_1(T) & = & 0 & \\ k_2(0) = k_2(T) & = & 0 & \end{array} \tag{5.44}$$

with

$$\begin{array}{rcl} M_1\mathbf{z} & = & \mathbf{z}' - 2s\nu\nabla\varphi_\lambda \cdot \nabla\mathbf{z} \qquad \text{and} \qquad M_2\mathbf{z} = s\varphi_\lambda'\mathbf{z} - \nu\Delta\mathbf{z} - s^2\nu|\nabla\varphi_\lambda|^2\mathbf{z}, \\ \mathbf{f}_s & = & \rho^{-s}\mathbf{a} - \rho^{-s}\nabla\pi + s\nu(\Delta\varphi_\lambda)\mathbf{z}. \end{array} \tag{5.45}$$

Indeed, we have calculated

$$\rho^{-s}\Big(\partial_t - \nu\Delta\Big)\rho^s\mathbf{z} = -\rho^{-s}\nabla\pi.$$

We first have

$$\begin{array}{rcl} \partial_t(\rho^s\mathbf{z}) & = & \rho^s(s\partial_t\varphi_\lambda\mathbf{z} + \mathbf{z}'), \\ \partial_{x_i}(\rho^s\mathbf{z}) & = & \rho^s(s\partial_{x_i}\varphi_\lambda\mathbf{z} + \partial_{x_i}\mathbf{z}), \\ \partial_{x_ix_i}(\rho^s\mathbf{z}) & = & \rho^s\Big(s\partial_{x_i}\varphi_\lambda(s\partial_{x_i}\varphi_\lambda\mathbf{z} + \partial_{x_i}\mathbf{z}) + s\partial_{x_ix_i}\varphi_\lambda\mathbf{z} + s\partial_{x_i}\varphi_\lambda\partial_{x_i}\mathbf{z} + \partial_{x_ix_i}\mathbf{z}\Big) \\ & = & \rho^s\Big(s^2(\partial_{x_i}\varphi_\lambda)^2\mathbf{z} + s\partial_{x_ix_i}\varphi_\lambda\mathbf{z} + 2s\partial_{x_i}\varphi_\lambda\partial_{x_i}\mathbf{z} + \partial_{x_ix_i}\mathbf{z}\Big). \end{array}$$

Then,

$$\rho^{-s}\Delta(\rho^s\mathbf{z}) = s^2|\nabla\varphi_\lambda|^2\mathbf{z} + s\Delta\varphi_\lambda\mathbf{z} + 2s\nabla\mathbf{z}\nabla\varphi_\lambda + \Delta\mathbf{z}.$$

In the next section, we will only consider the Stokes system. We will set a Carleman inequality on this system. Then, in the next sections, we will estimate some terms in the right-hand side of this inequality thanks partly to the beam equation.

5.4.1 Carleman Estimate for the Stokes system.

After a change of time variable $t \leftrightarrow T - t$, we want to have a Carleman estimate on the Stokes part of the previous system, namely on

$$\begin{array}{rcll} \mathbf{y}_t - \text{div } \sigma(\mathbf{y}, \pi) & = & \mathbf{a} & \text{in } Q_T^0, \\ \text{div } \mathbf{y} & = & 0 & \text{in } Q_T^0, \\ \mathbf{y} & = & g\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{y} & = & \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{y}(0) & = & \mathbf{y}^0 & \text{in } \Omega_0. \end{array}$$

Then, we have, with the notations of (5.45),

$$\|M_1\mathbf{z}\|^2_{\mathbf{L}^2(Q_T^0)} + \|M_2\mathbf{z}\|^2_{\mathbf{L}^2(Q_T^0)} + 2(M_1\mathbf{z}, M_2\mathbf{z})_{\mathbf{L}^2(Q_T^0)} = \|\mathbf{f}_s\|^2_{\mathbf{L}^2(Q_T^0)}. \tag{5.46}$$

Thus, we have to estimate from below $2(M_1\mathbf{z}, M_2\mathbf{z})_{\mathbf{L}^2(Q_T^0)}$. We rewrite this term as follows

$$2(M_1\mathbf{z}, M_2\mathbf{z})_{\mathbf{L}^2(Q_T^0)} = I_1 + I_2 + I_3$$

where

$$\begin{array}{rcl} I_1 & = & 2\displaystyle\int_{Q_T^0} (s\varphi_\lambda'\mathbf{z} - \nu\Delta\mathbf{z} - s^2\nu|\nabla\varphi_\lambda|^2\mathbf{z}) \cdot \mathbf{z}', \\ I_2 & = & 4s\nu^2\displaystyle\int_{Q_T^0} (\nabla\mathbf{z}\nabla\varphi_\lambda) \cdot \Delta\mathbf{z}, \\ I_3 & = & 4s\nu\displaystyle\int_{Q_T^0} (s^2\nu|\nabla\varphi_\lambda|^2\mathbf{z} - s\varphi_\lambda'\mathbf{z}) \cdot (\nabla\mathbf{z}\nabla\varphi_\lambda). \end{array}$$

After some calculations (see [27, 28]), we get

$$\begin{array}{rclcl} I_{11} &=& 2\displaystyle\int_{Q_T^0} s\varphi_\lambda'\mathbf{z}\cdot\mathbf{z}' &=& -s\displaystyle\int_{Q_T^0}\varphi_\lambda''|\mathbf{z}|^2,\\ I_{12} &=& -2\nu\displaystyle\int_{Q_T^0}\Delta\mathbf{z}\cdot\mathbf{z}' &=& -2\nu\displaystyle\int_{\Sigma_T^0}\nabla\mathbf{z}\mathbf{n}\cdot\mathbf{z}',\\ I_{13} &=& -2s^2\nu\displaystyle\int_{Q_T^0}|\nabla\varphi_\lambda|^2\mathbf{z}\cdot\mathbf{z}' &=& s^2\nu\displaystyle\int_{Q_T^0}\nabla\varphi_\lambda\cdot\nabla\varphi_\lambda'|\mathbf{z}|^2. \end{array}$$

But $\nabla\mathbf{z}\mathbf{n}\cdot\mathbf{n} = z_{2,y}$ on Γ_0^s and $\mathbf{z}=0$ on Γ. Then, thanks to $z_{1,x}+z_{2,y} = -s\nabla\varphi_\lambda\cdot\mathbf{z}$, we get that

$$\begin{array}{rcll} \nabla\mathbf{z}\mathbf{n}\cdot\mathbf{n} &=& 0 & \text{on } \Gamma,\\ \nabla\mathbf{z}\mathbf{n}\cdot\mathbf{n} &=& -s\lambda\xi\rho^{-s}g & \text{on } \Gamma_0^s. \end{array}$$

Thus,

$$I_1 = \int_{Q_T^0}\Big(2s^2\nu\nabla\varphi_\lambda\cdot\nabla\varphi_\lambda' - s\varphi_\lambda''\Big)|\mathbf{z}|^2 + 2s\lambda\nu\int_{\Sigma_T^{s,0}}\xi\big(\rho^{-s}g\big)'\big(\rho^{-s}g\big).$$

An integration by parts gives that

$$2s\lambda\nu\int_{\Sigma_T^{s,0}}\xi\big(\rho^{-s}g\big)'\big(\rho^{-s}g\big) \;=\; -s\lambda\nu\int_{\Sigma_T^{s,0}}\xi'\rho^{-2s}|g|^2.$$

Then,

$$I_1 = \int_{Q_T^0}\Big(2s^2\nu\nabla\varphi_\lambda\cdot\nabla\varphi_\lambda' - s\varphi_\lambda''\Big)|\mathbf{z}|^2 - s\lambda\nu\int_{\Sigma_T^{s,0}}\xi'\rho^{-2s}|g|^2.$$

In the same way, we have

$$I_2 = 2s\nu^2\int_{\Sigma_T^0}\partial_{\mathbf{n}}\varphi_\lambda|\partial_{\mathbf{n}}\mathbf{z}|^2 - 4s\nu^2\int_{Q_T^0}\partial_{i,j}^2\varphi_\lambda\partial_i\mathbf{z}\cdot\partial_j\mathbf{z} + 2s\nu^2\int_{Q_T^0}\Delta\varphi_\lambda|\nabla\mathbf{z}|^2.$$

and

$$\begin{array}{rcl} I_3 &=& -2s^3\nu^2\displaystyle\int_{Q_T^0}|\nabla\varphi_\lambda|^2\Delta\varphi_\lambda|\mathbf{z}|^2 - 4s^3\nu^2\displaystyle\int_{Q_T^0}\partial_{i,j}^2\varphi_\lambda\partial_i\varphi_\lambda\partial_j\varphi_\lambda|\mathbf{z}|^2 + 2s^2\nu\displaystyle\int_{Q_T^0}\nabla\varphi_\lambda'\cdot\nabla\varphi_\lambda|\mathbf{z}|^2\\ && +2s^2\nu\displaystyle\int_{Q_T^0}\varphi_\lambda'\Delta\varphi_\lambda|\mathbf{z}|^2 + 2s^3\nu^2\displaystyle\int_{\Sigma_T^0}(\partial_{\mathbf{n}}\varphi_\lambda)^3|\mathbf{z}|^2 - 2s^2\nu\displaystyle\int_{\Sigma_T^0}\varphi_\lambda'\partial_{\mathbf{n}}\varphi_\lambda|\mathbf{z}|^2. \end{array}$$

Remembering that $\mathbf{z}=\mathbf{0}$ on Γ and $\mathbf{z}=g\mathbf{e}_2$ on Γ_0^s, we get that

$$\begin{array}{rcl} I_3 &=& -2s^3\nu^2\displaystyle\int_{Q_T^0}|\nabla\varphi_\lambda|^2\Delta\varphi_\lambda|\mathbf{z}|^2 - 4s^3\nu^2\displaystyle\int_{Q_T^0}\partial_{i,j}^2\varphi_\lambda\partial_i\varphi_\lambda\partial_j\varphi_\lambda|\mathbf{z}|^2 + 2s^2\nu\displaystyle\int_{Q_T^0}\nabla\varphi_\lambda'\cdot\nabla\varphi_\lambda|\mathbf{z}|^2\\ && +2s^2\nu\displaystyle\int_{Q_T^0}\varphi_\lambda'\Delta\varphi_\lambda|\mathbf{z}|^2 + 2s^3\nu^2\displaystyle\int_{\Sigma_T^{s,0}}(\partial_{\mathbf{n}}\varphi_\lambda)^3\rho^{-2s}|g|^2 - 2s^2\nu\displaystyle\int_{\Sigma_T^{s,0}}\varphi_\lambda'\partial_{\mathbf{n}}\varphi_\lambda\rho^{-2s}|g|^2. \end{array}$$

Reordering the different terms, we get the following expression of the cross product.

$$2(M_1\mathbf{z}, M_2\mathbf{z})_{\mathbf{L}^2(Q_T^0)} = J_1 + J_2 + J_3 + \tilde{J}_4 + J_5 + 2J_6$$

where

$$J_1 = -4s^3\nu^2\int_{Q_T^0}\partial_{i,j}^2\varphi_\lambda\partial_i\varphi_\lambda\partial_j\varphi_\lambda|\mathbf{z}|^2,\quad J_2 = 2s\nu^2\int_{\Sigma_T^0}\partial_{\mathbf{n}}\varphi_\lambda|\partial_{\mathbf{n}}\mathbf{z}|^2,$$

$$J_3 = 2s^2\nu\int_{Q_T^0}\varphi_\lambda'\Delta\varphi_\lambda|\mathbf{z}|^2 - s\int_{Q_T^0}\varphi_\lambda''|\mathbf{z}|^2 + 4s^2\nu\int_{Q_T^0}\nabla\varphi_\lambda'\cdot\nabla\varphi_\lambda|\mathbf{z}|^2,$$

$$\tilde{J}_4 = -s\lambda\nu\int_{\Sigma_T^{s,0}}\xi'\rho^{-2s}|g|^2 + 2s^3\nu^2\int_{\Sigma_T^{s,0}}(\partial_{\mathbf{n}}\varphi_\lambda)^3\rho^{-2s}|g|^2 - 2s^2\nu\int_{\Sigma_T^{s,0}}\varphi_\lambda'\partial_{\mathbf{n}}\varphi_\lambda\rho^{-2s}|g|^2,$$

$$J_5 = -4s\nu^2\int_{Q_T^0}\partial_{i,j}^2\varphi_\lambda\partial_i\mathbf{z}\cdot\partial_j\mathbf{z},\quad J_6 = \int_{Q_T^0}\Big(s\nu^2\Delta\varphi_\lambda|\nabla\mathbf{z}|^2 - s^3\nu^2\Delta\varphi_\lambda|\nabla\varphi_\lambda|^2|\mathbf{z}|^2\Big).$$

Remark 5.19. *We use the same notation as in [27]. We only put the symbol $\tilde{}$ for $\tilde{J}_4$ because this term is slightly different than the term J_4 in [27].*

The next step is to obtain estimates on J_1,J_2, J_3 and J_5.

$$\begin{array}{rcl} J_1+J_3 & \geq & \dfrac{1}{2}C_1 s^3\lambda^4 \displaystyle\int_{Q_T^0} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}|\mathbf{z}|^2 - Cs^3\lambda^4 \int_{\omega_0\times(0,T)} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}|\mathbf{z}|^2, \\ J_2 & \geq & 2s\lambda \displaystyle\int_{\Sigma_T^{s,0}} \frac{e^{\lambda(\phi+m\|\phi\|_\infty)}}{t^k(T-t)^k}|\partial_{\mathbf{n}}\mathbf{z}|^2, \\ J_5 & \geq & \dfrac{1}{2}\|M_2\mathbf{z}\|^2_{\mathbf{L}^2(Q_T^0)} - Cs^2\lambda^2 \displaystyle\int_{Q_T^0} \frac{e^{2\lambda(\phi+m\|\phi\|_\infty)}}{t^{2k}(T-t)^{2k}}|\mathbf{z}|^2 - Cs^2\lambda \int_{Q_T^0} \frac{e^{\lambda m K_1}}{t^{2k+1}(T-t)^{2k+1}}|\mathbf{z}|^2 \\ & & -Cs^3\lambda^3 \displaystyle\int_{Q_T^0} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}|\nabla\phi|^2|\mathbf{z}|^2 + Cs^2\lambda^2 \int_{\Sigma_T^{s,0}} \frac{e^{2\lambda(\phi+m\|\phi\|_\infty)}}{t^{2k}(T-t)^{2k}}\rho^{-2s}|g|^2. \end{array}$$

We now calculate an estimate from below of J_6 like in [27, 28]. By integrations by parts, we get

$$J_6 = \frac{s\nu^2}{2}\int_{Q_T^0} \Delta^2\varphi_\lambda|\mathbf{z}|^2 - s^2\nu^2\int_{Q_T^0} \varphi'_\lambda\Delta\varphi_\lambda|\mathbf{z}|^2 + s\nu^2\int_{Q_T^0} \Delta\varphi_\lambda(\mathbf{f}_s - M_1\mathbf{z} + s\Delta\varphi_\lambda\mathbf{z})\cdot\mathbf{z} + S_1 + S_2$$

with $S_1 = -s^2\nu^2 \int_{\Sigma_T^{s,0}} \rho^{-2s}|g|^2\partial_{\mathbf{n}}\varphi_\lambda$ and $S_2 = -\frac{s\nu^2}{2}\int_{\Sigma_T^{s,0}} \partial_{\mathbf{n}}(\Delta\varphi_\lambda)\rho^{-2s}|g|^2$. Then,

$$\begin{array}{rcl} J_6 & \geq & -Cs\lambda^4 \displaystyle\int_{Q_T^0} \frac{e^{\lambda(\phi+m\|\phi\|_\infty)}}{t^k(T-t)^k}|\mathbf{z}|^2 - Cs^2\lambda^2 \int_{Q_T^0} \frac{e^{\lambda m K_1}}{t^{2k+1}(T-t)^{2k+1}}|\mathbf{z}|^2 - Cs^2\lambda^4 \int_{Q_T^0} \frac{e^{2\lambda(\phi+m\|\phi\|_\infty)}}{t^{2k}(T-t)^{2k}}|\mathbf{z}|^2 \\ & & -\dfrac{1}{4}\displaystyle\int_{Q_T^0} |\mathbf{f}_s|^2 - \frac{1}{4}\int_{Q_T^0} |M_1\mathbf{z}|^2 + S_1 + S_2. \end{array}$$

Then, putting all the above estimates in (5.46), we get

$$\begin{array}{rl} & \|M_1\mathbf{z}\|^2_{\mathbf{L}^2(Q_T^0)} + \|M_2\mathbf{z}\|^2_{\mathbf{L}^2(Q_T^0)} + s^3\lambda^4 \displaystyle\int_{Q_T^0} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}|\mathbf{z}|^2 \\ & +s^2\lambda^2 \displaystyle\int_{\Sigma_T^{s,0}} \frac{e^{2\lambda(\phi+m\|\phi\|_\infty)}}{t^{2k}(T-t)^{2k}}\rho^{-2s}|g|^2 + S_1 + S_2 + \tilde{J}_4 \\ \leq & C\Big[\|\mathbf{f}_s\|^2_{\mathbf{L}^2(Q_T^0)} + s^3\lambda^4 \displaystyle\int_{\omega_0\times(0,T)} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}|\mathbf{z}|^2\Big]. \end{array}$$

We have to calculate now the boundary terms. If we write $\tilde{J}_4 = T_4 + \tilde{T}_5 + T_6$, with

$$\begin{array}{rcl} T_4 & = & 2s^3\nu^2 \displaystyle\int_{\Sigma_T^{s,0}} (\partial_{\mathbf{n}}\varphi_\lambda)^3|\mathbf{z}|^2 = 2s^3\lambda^3\nu^2 \int_{\Sigma_T^{s,0}} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}\rho^{-2s}|g|^2, \\ \tilde{T}_5 & = & -s\lambda\nu \displaystyle\int_{\Sigma_T^{s,0}} \xi'\rho^{-2s}|g|^2, \\ T_6 & = & -2s^2 \displaystyle\int_{\Sigma_T^{s,0}} \varphi'_\lambda\partial_{\mathbf{n}}\varphi_\lambda\rho^{-2s}|g|^2. \end{array}$$

We get that $|\tilde{T}_5| \leq \frac{1}{8}T_4$, $|T_6| \leq \frac{1}{8}T_4$, $|S_1| \leq \frac{1}{8}T_4$ and $|S_2| \leq \frac{1}{8}T_4$ for λ and s large enough (s depending of λ). Then

$$\begin{array}{rl} & \|M_1\mathbf{z}\|^2_{\mathbf{L}^2(Q_T^0)} + \|M_2\mathbf{z}\|^2_{\mathbf{L}^2(Q_T^0)} + s^3\lambda^4 \displaystyle\int_{Q_T^0} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}|\mathbf{z}|^2 \\ & +s^2\lambda^2 \displaystyle\int_{\Sigma_T^{s,0}} \frac{e^{2\lambda(\phi+m\|\phi\|_\infty)}}{t^{2k}(T-t)^{2k}}\rho^{-2s}|g|^2 + \frac{1}{4}s^3\lambda^3 \int_{\Sigma_T^{s,0}} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}\rho^{-2s}|g|^2 \\ \leq & C\Big[\|\mathbf{f}_s\|^2_{\mathbf{L}^2(Q_T^0)} + s^3\lambda^4 \displaystyle\int_{\omega_0\times(0,T)} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}|\mathbf{z}|^2 + s^2\lambda^2 \int_{\Sigma_T^{s,0}} \rho^{-2s}\frac{e^{\lambda(\phi+m\|\phi\|_\infty)}}{t^k(T-t)^k}|g|^2\Big]. \end{array}$$

Then, $s^2\lambda^2 \int_{\Sigma_T^{s,0}} \rho^{-2s} \frac{e^{\lambda(\phi+m\|\phi\|_\infty)}}{t^k(T-t)^k}|g|^2$ can be absorbed in the left-hand side for s and λ large enough thanks to $\frac{1}{4}s^3\lambda^3 \int_{\Sigma_T^{s,0}} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}\rho^{-2s}|g|^2$. Namely we get

$$\begin{aligned} &\|M_1\mathbf{z}\|^2_{\mathbf{L}^2(Q_T^0)} + \|M_2\mathbf{z}\|^2_{\mathbf{L}^2(Q_T^0)} + s^3\lambda^4 \int_{Q_T^0} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}|\mathbf{z}|^2 + s^3\lambda^3 \int_{\Sigma_T^{s,0}} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}\rho^{-2s}|g|^2 \\ \leq\ & C\Big[\|\mathbf{f}_s\|^2_{\mathbf{L}^2(Q_T^0)} + s^3\lambda^4 \int_{\omega_0\times(0,T)} \frac{e^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}|\mathbf{z}|^2\Big]. \end{aligned}$$

Let us introduce some notations,

$$\phi_* = \min_{x\in\overline{\Omega_0}} \phi(x)$$

and then

$$\begin{aligned} \varphi_\lambda^*(t) &= \max_{x\in\overline{\Omega_0}} \varphi_\lambda(x,t) = \frac{e^{\lambda m K_1} - e^{\lambda(\phi_*+m\|\phi\|_\infty)}}{t^k(T-t)^k}, \\ \hat{\varphi_\lambda}(t) &= \min_{x\in\overline{\Omega_0}} \varphi_\lambda(x,t) = \frac{e^{\lambda m K_1} - e^{\lambda(m+1)\|\phi\|_\infty}}{t^k(T-t)^k}, \\ \xi^*(t) &= \min_{x\in\overline{\Omega_0}} \xi(x,t) = \frac{e^{\lambda(\phi_*+m\|\phi\|_\infty)}}{t^k(T-t)^k}, \\ \hat{\xi}(t) &= \max_{x\in\overline{\Omega_0}} \xi(x,t) = \frac{e^{\lambda(1+m)\|\phi\|_\infty}}{t^k(T-t)^k}. \end{aligned}$$

And obviously $\rho^*(t) = \exp(\varphi_\lambda^*(t))$ and $\hat{\rho}(t) = \exp(\hat{\varphi_\lambda}(t))$.

We act exactely like in [28] to improve the previous inequality adding terms depending on $\nabla\mathbf{z}$, $\Delta\mathbf{z}$ or $\mathbf{z}'$ in its left-hand side, namely we have

$$\begin{aligned} & s^{-1}\int_{Q_T^0} \xi^{-1}\Big(|\mathbf{z}'|^2 + |\Delta\mathbf{z}|^2\Big) + \int_{Q_T^0} |M_1\mathbf{z}|^2 + \int_{Q_T^0} |M_2\mathbf{z}|^2 + \int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2 \\ & + s\lambda^2 \int_{Q_T^0} \xi|\nabla\mathbf{z}|^2 + s^3\lambda^4 \int_{Q_T^0} \xi^3|\mathbf{z}|^2 + s^3\lambda^3 \int_{\Sigma_T^{s,0}} \xi^3\rho^{-2s}|Zk_2|^2 \\ \leq\ & C\left[\int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2 + s^3\lambda^4 \int_{\omega_1\times(0,T)} \xi^3|\mathbf{z}|^2\right]. \end{aligned}$$

We need to improve the previous estimate by adding terms of the beam equation in the left-hand side. Thanks to the following estimate

$$\int_0^T \rho_\Gamma^{-2s}(|k_2'|^2_{\mathbb{R}^N} + |A^{1/2}k_1|^2_{\mathbb{R}^N}) \leq C\left\{\int_0^T \rho_\Gamma^{-2s}\,|\Pi_N\pi|^2_{\mathbb{R}^N} + \int_0^T \rho_\Gamma^{-2s}(|A^{1/2}k_1|^2_{\mathbb{R}^N} + |c|^2_{\mathbb{R}^N})\right\},$$

we can sum up all the previous results in the following theorem

Theorem 5.20. *For λ large enough, there is $s_0(\lambda) > 0$ such that for all $s \geq s_0(\lambda)$ and for all the solutions $(\mathbf{z}, k_1, k_2)$ of (5.44), we have*

$$\begin{aligned} & s^{-1}\int_{Q_T^0} \xi^{-1}\Big(|\mathbf{z}'|^2 + |\Delta\mathbf{z}|^2\Big) + \int_{Q_T^0} |M_1\mathbf{z}|^2 + \int_{Q_T^0} |M_2\mathbf{z}|^2 + \int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2 \\ & + s\lambda^2 \int_{Q_T^0} \xi|\nabla\mathbf{z}|^2 + s^3\lambda^4 \int_{Q_T^0} \xi^3|\mathbf{z}|^2 + s^3\lambda^3 \int_{\Sigma_T^{s,0}} \xi^3\rho^{-2s}|Zk_2|^2 + \int_0^T \rho_\Gamma^{-2s}\Big(|k_2'|^2_{\mathbb{R}^N} + |A^{1/2}k_1|^2_{\mathbb{R}^N}\Big) \\ \leq\ & C\left[\int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2 + s^3\lambda^4 \int_{\omega_1\times(0,T)} \xi^3|\mathbf{z}|^2 + \int_0^T \rho_\Gamma^{-2s}(|A^{1/2}k_1|^2_{\mathbb{R}^N} + |c|^2_{\mathbb{R}^N}) + \int_0^T \rho_\Gamma^{-2s}\,|\Pi_N\pi|^2_{\mathbb{R}^N}\right] \end{aligned} \tag{5.47}$$

where $\omega_0 \subset\subset \omega_1 \subset\subset \Omega_0$.

We have to estimate from above the terms of the right-hand side depending of the pressure π and of the displacement of the beam k_1.

5.4.2 First treatment of the pressure term integral.

We need to get rid of the term $\int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2$ in the righ-hand side of the previous inequality. Here, we follow the idea of [15, 10]. We cannot follow the idea of [28] (that is using a fictitious second control on the divergence term) because we cannot obtain enough time regularity for the fictitious control (because of the the right-hand sides) to lift these condition at the end of the proof (see section 10. in [28] for details).

First of all, we have for ω_2 such that $\omega_1 \subset\subset \omega_2 \subset\subset \omega$ the following inequality:

$$\int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2 \leq C\left[s^{1/2}\int_0^T (\rho^*)^{-2s}(\xi^*)^{1/2}\|\pi\|^2_{H^{1/2}(\Omega_0)} + s^2\lambda^2\int_{\omega_2\times(0,T)} \rho^{-2s}\xi^2|\pi|^2\right].$$

Second, to estimate the trace of the pressure π, we need to consider the system in the unknowns $(\mathbf{y}^*, \pi^*, k_1^*, k_2^*) = (\theta^*\mathbf{y}, \theta^*\pi, \theta^*k_1, \theta^*k_2)$ where $\theta^* = s^{1/4}(\rho^*)^{-s}(\xi^*)^{1/4}$:

$$\begin{array}{rcll}
-\mathbf{y}^{*\prime} - \operatorname{div}\,\sigma(\mathbf{y}^*, \pi^*) & = & \theta^{*\prime}\mathbf{y} + \theta^*\mathbf{a} & \text{in } Q_T^0,\\
\operatorname{div}\,\mathbf{y}^* & = & 0 & \text{in } Q_T^0,\\
\mathbf{y}^* & = & \mathbf{0} & \text{on } \Sigma_T,\\
\mathbf{y}^* & = & Zk_2^*\mathbf{e}_2 & \text{on } \Sigma_T^{s,0},\\
k_1^{*\prime} & = & k_2^* + \theta^{*\prime}k_1 - \theta^*b & \text{in } (0,T),\\
k_2^{*\prime} + Ak_1^* & = & -\Pi_N\pi^* + \theta^{*\prime}k_2 - \theta^*c & \text{in } (0,T),\\
(\mathbf{y}^*(T), k_1^*(T), k_2^*(T)) & = & (\mathbf{0}, 0, 0). &
\end{array}$$

Then, we have

$$\|\pi^*\|_{L^2(0,T;H^1(\Omega_0))} \leq C\Big(\|\theta^{*\prime}\mathbf{y}\|_{\mathbf{L}^2(Q_T^0)} + \|\theta^*\mathbf{a}\|_{\mathbf{L}^2(Q_T^0)} + \|k_2^*\|_{H^{3/4}(0,T;\mathbb{R}^N)}\Big).$$

But, we can estimate the $H^{3/4}(0,T;\mathbb{R}^N)$-norm thanks to an interpolation inequality (see [17]). Indeed, we have easily

$$H^{3/4}(0,T;\mathbb{R}^N) = \Big[H^1(0,T;\mathbb{R}^N), L^2(0,T;\mathbb{R}^N)\Big]_{1/4}$$

and then, the estimate:

$$\|\cdot\|_{H^{3/4}(0,T;\mathbb{R}^N)} \leq C\|\cdot\|^{1/4}_{L^2(0,T;\mathbb{R}^N)}\|\cdot\|^{3/4}_{H^1(0,T;\mathbb{R}^N)}.$$

Finally, we just have to estimate the $H^1(0,T;\mathbb{R}^N)$-norm of k_2^*. If π^* belongs to $L^2(\Sigma_T^{s,0})$ and $\theta^{*\prime}A^{1/2}k_1$, $\theta^{*\prime}k_2$ belong to $L^2(0,T;\mathbb{R}^N)$, then (k_1^*, k_2^*) belongs to $H^1(0,T;\mathbb{R}^N)\times H^1(0,T;\mathbb{R}^N)$ with the estimate:

$$\|(k_1^*, k_2^*)\|_{M_T} \leq C\Big(\|\theta^{*\prime}k_1\|_{L^2(0,T;\mathbb{R}^N)} + \|\theta^{*\prime}k_2\|_{L^2(0,T;\mathbb{R}^N)} + \|\theta^*A^{1/2}b\|_{L^2(0,T;\mathbb{R}^N)} + \|\theta^*c\|_{L^2(0,T;\mathbb{R}^N)} + \|\pi^*\|_{L^2(\Sigma_T^{s,0})}\Big).$$

Then, k_2^* belongs to $H^1(0,T;\mathbb{R}^N)$ and

$$\begin{array}{rcl}
\|k_2^*\|_{H^{3/4}(0,T;\mathbb{R}^N)} & \leq & C\|k_2^*\|^{1/4}_{L^2(0,T;\mathbb{R}^N)}\Big(\|\theta^{*\prime}A^{1/2}k_1\|^{3/4}_{L^2(0,T;\mathbb{R}^N)} + \|\theta^{*\prime}k_2\|^{3/4}_{L^2(0,T;\mathbb{R}^N)}\\
& & +\|\pi^*\|^{3/4}_{L^2(\Sigma_T^{s,0})} + \|\theta^*A^{1/2}b\|_{L^2(0,T;\mathbb{R}^N)} + \|\theta^*c\|_{L^2(0,T;\mathbb{R}^N)}\Big).
\end{array}$$

For $\varepsilon > 0$, we use a Young inequality to get

$$\begin{array}{rcl}
\|\pi^*\|_{L^2(0,T;H^1(\Omega_0))} & \leq & C\Big(\dfrac{1}{\varepsilon^3}\|k_2^*\|_{L^2(0,T;\mathbb{R}^N)} + \varepsilon\|\theta^{*\prime}A^{1/2}k_1\|_{L^2(0,T;\mathbb{R}^N)} + \varepsilon\|\theta^{*\prime}k_2\|_{L^2(0,T;\mathbb{R}^N)}\\
& & +\varepsilon\|\pi^*\|_{L^2(\Sigma_T^{s,0})} + \|\theta^{*\prime}\mathbf{y}\|_{\mathbf{L}^2(Q_T^0)} + \|\theta^*\mathbf{a}\|_{\mathbf{L}^2(Q_T^0)} + \|\theta^*A^{1/2}b\|_{L^2(0,T;\mathbb{R}^N)} + \|\theta^*c\|_{L^2(0,T;\mathbb{R}^N)}\Big).
\end{array}$$

Finally, we have

$$\begin{array}{rcl}
\|\pi^*\|_{L^2(0,T;H^1(\Omega_0))} & \leq & C\Big(\|\theta^*k_2\|_{L^2(0,T;\mathbb{R}^N)} + \varepsilon\|\theta^{*\prime}A^{1/2}k_1\|_{L^2(0,T;\mathbb{R}^N)} + \varepsilon\|\theta^{*\prime}k_2\|_{L^2(0,T;\mathbb{R}^N)}\\
& & +\|\theta^*\mathbf{a}\|_{\mathbf{L}^2(Q_T^0)} + \|\theta^*A^{1/2}b\|_{L^2(0,T;\mathbb{R}^N)} + \|\theta^*c\|_{L^2(0,T;\mathbb{R}^N)}\Big).
\end{array}$$

Remember that $\theta^*(t) = (\xi^*)^{1/4} \mathrm{e}^{-s\varphi^*_\lambda(t)}$. We have

$$|\theta^{*\prime}(t)| \leq CsT(\xi^*(t))^{3/2}\mathrm{e}^{-s\varphi^*_\lambda(t)} \quad \text{and} \quad |\theta^{*\prime}(t)|^2 \leq Cs^2T^2(\xi^*(t))^3\mathrm{e}^{-2s\varphi^*_\lambda(t)}$$

We finally get

$$\begin{aligned} s^{1/2}\int_0^T (\rho^*)^{-2s}(\xi^*)^{1/2}\|\pi\|^2_{H^{1/2}(\Omega_0)} \quad \leq \quad & C\Bigg[s^{5/2}\int_{Q^0_T} \xi^3|\mathbf{z}|^2 + s^{5/2}\int_0^T \xi^{*3}(\rho^*)^{-2s}|k_2|^2_{\mathbb{R}^N} \\ & + s^{5/2}\varepsilon\int_0^T (\xi^*(t))^3\mathrm{e}^{-2s\varphi^*_\lambda(t)}\Big(|k_2(t)|^2_{\mathbb{R}^N} + |A^{1/2}k_1(t)|^2_{\mathbb{R}^N}\Big)\mathrm{d}t \\ & + s^{1/2}\int_0^T (\xi^*(t))^{1/2}\mathrm{e}^{-2s\varphi^*_\lambda(t)}\|(\mathbf{a}(t), b(t), c(t))\|^2_{\mathbb{V}}\mathrm{d}t\Bigg]. \end{aligned}$$

The term $s^{5/2}\int_{Q^0_T} \xi^3|\mathbf{z}|^2$ and the two terms $s^{5/2}\int_0^T \xi^{*3}\rho^{*\,-2s}|k_2|^2_{\mathbb{R}^N}$ in the righ-hand side can be absorbed respectively by $s^3\lambda^4\int_{Q^0_T} \dfrac{\mathrm{e}^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}|\mathbf{z}|^2$ and $s^3\lambda^3\int_{\Sigma^{s,0}_T} \dfrac{\mathrm{e}^{3\lambda(\phi+m\|\phi\|_\infty)}}{t^{3k}(T-t)^{3k}}\rho^{-2s}|Zk_2|^2$. We get the following inequality:

$$\begin{aligned} & s^{-1}\int_{Q^0_T} \xi^{-1}\Big(|\mathbf{z}'|^2 + |\Delta\mathbf{z}|^2\Big) + \int_{Q^0_T} |M_1\mathbf{z}|^2 + \int_{Q^0_T} |M_2\mathbf{z}|^2 + s\lambda^2\int_{Q^0_T} \xi|\nabla\mathbf{z}|^2 \\ & + s^3\lambda^4\int_{Q^0_T} \xi^3|\mathbf{z}|^2 + s^3\lambda^3\int_{\Sigma^{s,0}_T} \xi^3\rho^{-2s}|k_2|^2_{\mathbb{R}^N} + \int_{Q^0_T} \rho^{-2s}|\nabla\pi|^2 \\ \leq \quad & C\Bigg[s^3\lambda^4\int_{(0,T)\times\omega_1} \xi^3|\mathbf{z}|^2 + s^{5/2}\varepsilon\int_0^T (\xi^*(t))^3\mathrm{e}^{-2s\varphi^*_\lambda(t)}|A^{1/2}k_1(t)|^2_{\mathbb{R}^N}\mathrm{d}t \\ & + s^2\lambda^2\int_{(0,T)\times\omega_2} \rho^{-2s}\xi^2|\pi|^2 + s^{1/2}\int_0^T (\xi^*)^{1/2}\mathrm{e}^{-2s\varphi^*_\lambda(t)}\|(\mathbf{a}(t), b(t), c(t))\|^2_{\mathbb{V}}\mathrm{d}t\Bigg]. \end{aligned}$$

We need now to estimate the integrals $s^2\lambda^2\int_{(0,T)\times\omega_2} \rho^{-2s}\xi^2|\pi|^2$ and $\int_0^T (\xi^*(t))^3\mathrm{e}^{-2s\varphi^*_\lambda(t)}|A^{1/2}k_1(t)|^2_{\mathbb{R}^N}$.

The pressure π is determined up to an additive constant, we fix it by adding the following condition

$$\int_{\omega_2} \pi(x,t)\mathrm{d}x = 0 \quad \text{for a.e. } t \in (0,T),$$

then we can apply the Poincaré-Wirtinger inequality

$$\int_{(0,T)\times\omega_1} s^2\lambda^2\mathrm{e}^{2\lambda\eta}\mathrm{e}^{-2s\varphi_\lambda}|\pi|^2 \leq C\int_{(0,T)\times\omega_1} s^2\lambda^2\hat{\xi}^2\mathrm{e}^{-2s\hat{\varphi}_\lambda}|\nabla\pi|^2$$

to obtain for λ and s large enough

$$\int_{(0,T)\times\omega_1} (s^2\lambda^2\mathrm{e}^{2\lambda\eta}|\pi|^2 + |\nabla\pi|^2)\mathrm{e}^{-2s\varphi_\lambda} \leq C\int_{(0,T)\times\omega_1} s^2\lambda^2\hat{\xi}^2\mathrm{e}^{-2s\hat{\varphi}_\lambda}|\nabla\pi|^2.$$

We use the fact that $\nabla\pi = \mathbf{y}' + \nu\Delta\mathbf{y} + \mathbf{a}$ to finally get

$$\begin{aligned} I(s,\lambda;\xi) \quad \leq \quad & C\Bigg(s^3\lambda^4\int_{\omega_2\times(0,T)} \xi^3|\mathbf{z}|^2 + s^{5/2}\int_0^T (\xi^*)^3\mathrm{e}^{-2s\varphi^*_\lambda}(|A^{1/2}k_1|^2_{\mathbb{R}^N} + |k_2|^2_{\mathbb{R}^N}) \\ & + \int_0^T \rho_2^2(t)\|(\mathbf{a}(t), b(t), c(t))\|^2_{\mathbb{V}} + \int_{(0,T)\times\omega_2} s^2\lambda^2\hat{\xi}^2\mathrm{e}^{-2s\hat{\varphi}_\lambda}(|\mathbf{a}|^2 + |\Delta\mathbf{y}|^2 + |\mathbf{y}'|^2)\Bigg) \end{aligned}$$

where $I(s,\lambda;\xi)$ is the left-hand side of inequality (5.47), namely

$$\begin{aligned} I(s,\lambda;\xi) &= s^{-1}\int_{Q_T^0}\xi^{-1}\Big(|\mathbf{z}'|^2+|\Delta\mathbf{z}|^2\Big)+\int_{Q_T^0}|M_1\mathbf{z}|^2+\int_{Q_T^0}|M_2\mathbf{z}|^2+\int_{Q_T^0}\rho^{-2s}|\nabla\pi|^2 \\ &\quad +s\lambda^2\int_{Q_T^0}\xi|\nabla\mathbf{z}|^2+s^3\lambda^4\int_{Q_T^0}\xi^3|\mathbf{z}|^2+s^3\lambda^3\int_{\Sigma_T^{s,0}}\xi^3\rho^{-2s}|Zk_2|^2+\int_0^T\rho_\Gamma^{-2s}\Big(|k_2'|^2_{\mathbb{R}^N}+|A^{1/2}k_1|^2_{\mathbb{R}^N}\Big). \end{aligned}$$

5.4.3 Estimates of the local integrals of $\Delta\mathbf{y}$ and $\mathbf{y}'$.

The next steps are to estimate the two local integrals in the right-hand side of the previous inequality. From now on, we fix $k=4$ as in [10, 28, 14]. As usual, we act as in [10, 14], we note $\hat\theta(t)=s\lambda\hat{\xi}\mathrm{e}^{-s\hat\varphi_\lambda}$. Then we have

$$\int_{(0,T)\times\omega_2}|\hat\theta(t)|^2|\Delta\mathbf{y}|^2\le\int_{(0,T)\times\omega_3}|\hat\theta'(t)|^2|\mathbf{y}|^2+\int_{(0,T)\times\omega_3}|\hat\theta(t)|^2(|\mathbf{a}|^2+|\mathbf{y}|^2)$$

for ω_3 such that $\omega_2\subset\subset\omega_3\subset\subset\omega$.

The second local integral estimate is more complicated to obtain. We have to fix another time dependent function θ defined by

$$\theta(t)=(s\xi^*)^a\mathrm{e}^{-s\varphi^*_\lambda},\ \text{ where } a \text{ is a parameter.}$$

Then, we want to find an estimate on $\int_{\omega_2\times(0,T)}|\hat\theta(t)|^2|\mathbf{y}'|^2$. Thus, we look for an estimate on $(\mathbf{z},\psi,l_1,l_2)=\theta(\mathbf{y},\pi,k_1,k_2)$, solution of the following system

$$\begin{aligned} -\mathbf{z}'-\mathrm{div}\,\sigma(\mathbf{z},\psi) &= \theta\mathbf{a}-\theta'\mathbf{y} && \text{in } Q_T^0,\\ \mathrm{div}\,\mathbf{z} &= 0 && \text{in } Q_T^0,\\ \mathbf{z} &= \mathbf{0} && \text{on } \Sigma_T,\\ \mathbf{z} &= Zl_2\mathbf{e}_2 && \text{on } \Sigma_T^{s,0},\\ l_1{}' &= l_2-\theta b+\theta' k_1 && \text{in } (0,T),\\ l_2{}'+Al_1 &= -\Pi_N\psi-\theta c+\theta' k_2 && \text{in } (0,T),\\ (\mathbf{z}(T),l_1(T),l_2(T)) &= (\mathbf{0},0,0). \end{aligned}$$

We split the system into two parts, looking for $\mathbf{z}=\mathbf{z}^1+\mathbf{z}^2$ where $(\mathbf{z}^1,\psi^1,l_1^1,l_2^1)$ and $(\mathbf{z}^2,\psi^2,l_1^2,l_2^2)$ are respectively solutions of

$$\begin{aligned} -\mathbf{z}^{1\prime}-\mathrm{div}\,\sigma(\mathbf{z}^1,\psi^1) &= \theta\mathbf{a} && \text{in } Q_T^0,\\ \mathrm{div}\,\mathbf{z}^1 &= 0 && \text{in } Q_T^0,\\ \mathbf{z}^1 &= \mathbf{0} && \text{on } \Sigma_T,\\ \mathbf{z}^1 &= Zl_2^1 && \text{on } \Sigma_T^{s,0},\\ l_1^{1\prime} &= l_2^1-\theta b && \text{in } (0,T),\\ l_2^{1\prime}+Al_1^1 &= -\Pi_N\psi^1-\theta c && \text{in } (0,T),\\ (\mathbf{z}^1(T),l_1^1(T),l_2^1(T)) &= (\mathbf{0},0,0). \end{aligned}$$

and

$$\begin{aligned} -\mathbf{z}^{2\prime}-\mathrm{div}\,\sigma(\mathbf{z}^2,\psi^2) &= -\theta'\mathbf{y} && \text{in } Q_T^0,\\ \mathrm{div}\,\mathbf{z}^2 &= 0 && \text{in } Q_T^0,\\ \mathbf{z}^2 &= \mathbf{0} && \text{on } \Sigma_T,\\ \mathbf{z}^2 &= Zl_2^2 && \text{on } \Sigma_T^{s,0},\\ l_1^{2\prime} &= l_2^2+\theta' k_1 && \text{in } (0,T),\\ l_2^{2\prime}+Al_2^1 &= -\Pi_N\psi^2+\theta' k_2 && \text{in } (0,T),\\ (\mathbf{z}^2(T),l_1^2(T),l_2^2(T)) &= (\mathbf{0},0,0). \end{aligned}$$

Thanks to Theorem 5.8 for instance, we have the uniqueness of the system and thus

$$\theta\mathbf{y}=\mathbf{z}^1+\mathbf{z}^2,\quad \theta\psi=\psi^1+\psi^2,\quad \theta k_1=l_1^1+l_1^2\ \text{ and }\ \theta k_2=l_2^1+l_2^2,$$

and in the same time that

$$\|\mathbf{z}^1\|_{\mathbf{H}^{2,1}(Q_T^0)} + \|l_1^1\|_{H^1(0,T;\mathbb{R}^N)} + \|l_2^1\|_{H^1(0,T;\mathbb{R}^N)} \leq C\left(\int_0^T |\theta|^2\|(\mathbf{f},g,h)\|_{\mathbb{V}}^2\right)^{1/2}.$$

The same work as in [14, page 423 and next] gives us, with $a = \frac{15}{4}$ and for s, λ large enough

$$\begin{aligned}\left\|\hat{\theta}\mathbf{y}'\right\|^2_{\mathbf{L}^2(\omega_2\times(0,T))} \leq\ & C\left(\int_{(0,T)\times\omega_2} \lambda^2 s^{9/2}(\xi^*)^{9/2}\mathrm{e}^{-2s\varphi_\lambda^*}|\mathbf{y}|^2 + \lambda^5\|\theta\mathbf{y}\|^2_{\mathbf{L}^2(\omega_2\times(0,T))}\right.\\ & +\lambda^5\int_0^T |\theta|^2\|(\mathbf{a},b,c)\|^2_{\mathbb{V}} + \int_0^T \lambda^{-1}s^{3/2}(\xi^*)^{3/2}\mathrm{e}^{-2s\varphi_\lambda^*}\|(\mathbf{y},k_1,k_2)\|^2_{\mathbb{V}}\\ & \left.+\int_0^T \lambda^{-1}s^{-1}\hat{\xi}^{-1}\mathrm{e}^{-2s\hat{\varphi}_\lambda}\|(\mathbf{y}',k_1',k_2')\|^2_{\mathbb{V}}\right).\end{aligned}$$

Combining all the previous estimates, we get that

$$\begin{aligned}I(s,\lambda;\xi) \leq\ & C\left(\int_{(0,T)\times\omega_2} \lambda^5(s\xi^*)^{15/2}\mathrm{e}^{-2s\varphi_\lambda^*}|\mathbf{y}|^2 + \int_{(0,T)\times\omega_2} \lambda^5(s\xi^*)^{15/2}\mathrm{e}^{-2s\varphi_\lambda^*}\|(\mathbf{a},b,c)\|^2_{\mathbb{V}}\right.\\ & +\int_0^T \lambda^{-1}s^{3/2}(\xi^*)^{3/2}\mathrm{e}^{-2s\varphi_\lambda^*}\|(\mathbf{y},k_1,k_2)\|^2_{\mathbb{V}} + \int_0^T \lambda^{-1}s^{-1}\hat{\xi}^{-1}\mathrm{e}^{-2s\hat{\varphi}_\lambda}\|(\mathbf{y}',k_1',k_2')\|^2_{\mathbb{V}}\\ & \left.+s^{5/2}\int_0^T (\xi^*)^3\mathrm{e}^{-2s\varphi_\lambda^*}\left(|A^{1/2}k_1|^2_{\mathbb{R}^N} + |k_2|^2_{\mathbb{R}^N}\right)\right).\end{aligned} \tag{5.48}$$

The terms in the second line and the one depending on k_2 in the last line of the right-hand side of (5.48) can be absorbed in the left-hand side because of the factor λ^{-1} and estimates on the derivatives of $(\mathbf{y}, k_1, k_2)$ in Theorem 5.8.

Remember that $\mathbf{y} = \mathrm{e}^{s\varphi_\lambda}\mathbf{z}$, we can rewrite inequality $I(s,\lambda,\xi)$ in terms of $\mathbf{y}$ thanks to

$$\begin{aligned}\mathbf{y}' &= \mathrm{e}^{s\varphi_\lambda}\left(\mathbf{z}' + s\varphi_\lambda'\mathbf{z}\right),\\ \nabla\mathbf{y} &= \mathrm{e}^{s\varphi_\lambda}\left(\nabla\mathbf{z} + s\mathbf{z}\nabla\varphi_\lambda\right)\\ \Delta\mathbf{y} &= \mathrm{e}^{s\varphi_\lambda}\left(\Delta\mathbf{z} + 2s\nabla\varphi_\lambda\nabla\mathbf{z} + (s^2|\nabla\varphi_\lambda|^2 + s\Delta\varphi_\lambda)\mathbf{z}\right)\end{aligned}$$

as follow

$$\begin{aligned}I(s,\lambda;\xi) =\ & s^{-1}\int_{Q_T^0} \xi^{-1}\rho^{-2s}\left(|\mathbf{y}'|^2 + |\Delta\mathbf{y}|^2\right) + \int_{Q_T^0} \rho^{-2s}|\nabla\pi|^2 + s\lambda^2\int_{Q_T^0} \xi\rho^{-2s}|\nabla\mathbf{y}|^2\\ & +s^3\lambda^4\int_{Q_T^0} \xi^3\rho^{-2s}|\mathbf{y}|^2 + s^3\lambda^3\int_{\Sigma_T^{s,0}} \xi^3\rho_\Gamma^{-2s}|Zk_2|^2 + \int_0^T \rho_\Gamma^{-2s}\left(|k_2'|^2_{\mathbb{R}^N} + |A^{1/2}k_1|^2_{\mathbb{R}^N}\right).\end{aligned} \tag{5.49}$$

Finaly, we can sum up all the previous results in the following proposition:

Proposition 5.21. *For λ large enough, there is $s_0(\lambda) > 0$ such that for all $s \geq s_0(\lambda)$ and for all the solutions $(\mathbf{z}, k_1, k_2)$ of* (5.44)*, we have*

$$\begin{aligned}I(s,\lambda;\xi) \leq\ & C\left(\int_{(0,T)\times\omega_2} \lambda^5(s\xi^*)^{15/2}e^{-2s\varphi_\lambda^*}|\mathbf{y}|^2 + \int_0^T \lambda^5(s\xi^*)^{15/2}e^{-2s\varphi_\lambda^*}\|(\mathbf{a},b,c)\|^2_{\mathbb{V}}\right.\\ & \left.+s^{5/2}\int_0^T (\xi^*)^3 e^{-2s\varphi_\lambda^*}|A^{1/2}k_1|^2_{\mathbb{R}^N}\right)\end{aligned} \tag{5.50}$$

where $I(s,\lambda,\xi)$ has been redefined in (5.49).

5.4.4 Treatment of the integral of k_1.

The last step is to put the term $s^{5/2}\int_0^T(\xi^*)^3 e^{-2s\varphi_\lambda^*}|A^{1/2}k_1|^2_{\mathbb{R}^N}$ in the left-hand side of the previous inequality. For that, we follow the proof in [28, section 7]. Here we strongly use the finite dimensional beam equation. First, we set

$$d(t) = (\xi^*(t))^{3/2} e^{-s\varphi_\lambda^*(t)}.$$

Thus,

$$d'(t) = e^{-s\varphi_\lambda^*(t)}|\xi^*(t)|^{3/2}\frac{k}{t^{k+1}(T-t)^{k+1}}D(t)$$

with

$$D(t) = -se^{\lambda(\eta_*+m\|\eta\|_\infty)}(-kT+2kt) - \frac{3}{2}kt^k(T-t)^{k+1} + \frac{3}{2}kt^{k+1}(T-t)^k$$

The roots of d' are exactely the roots of the polynomial D. We will note them $T_1,\dots,T_n$ with

$$0 = T_0 < T_1 < \dots < T_n < T_{n+1} = T.$$

Then, d is monotone on the intervals (T_i, T_{i+1}) for $i = 0,\dots,n$. We introduce the space $\mathcal{E}$ of the solution of (5.44) obtained by varying the right-hand sides $(\mathbf{f}_s, b, c)$. Then, we introduce a subspace of $\mathcal{E}$ denoted $\mathcal{E}_{inf}$ and defined by

$$\mathcal{E}_{inf} = \{(\mathbf{z},\psi,k_1,k_2)\in\mathcal{E}\ ;\ k_1(T_i) = 0 \text{ for } i = 1,\dots,n\}.$$

This space $\mathcal{E}_{inf}$ is of infinite dimension with a codimension less than $N\times n$ where N represents the finite dimension of the beam displacement in (5.3) and n is defined just above. We will treat the case $\dim\mathcal{E}_{inf} = Nn$, the proof for the others cases is quite the same. Under this statement, for any $i = 0,\dots,n$, there exists N quadruplets $(\mathbf{z}^{i,j},\psi^{i,j},k_1^{i,j},k_2^{i,j})$ (for $j = 1,\dots,N$) in $\mathcal{E}$ such that

$$k_1^{i,j}(T_i) = (\delta_{j,l})_{l=1,\dots,N}.$$

Let $\mathcal{E}_0$ be the space

$$\mathcal{E}_0 = \operatorname{span}\left\{k_1^{i,j}\ 1\le i\le n,\ 1\le j\le N\right\}$$

and $\mathcal{E}_{fin}$ be the subspace of $\mathcal{E}$ spanned by $(\mathbf{z}^{i,j},\psi^{i,j},k_1^{i,j},k_2^{i,j})_{1\le i\le n, 1\le j\le N}$. Thus

$$\mathcal{E} = \mathcal{E}_{inf}\oplus\mathcal{E}_{fin}.$$

Then, we define $\Pi : \mathcal{E}\to\mathcal{E}_0$ by

$$\Pi(\mathbf{z},\psi,k_1,k_2) = \sum_{i=1}^n\sum_{j=1}^N\Big(k_1(T_i)\cdot(\delta_{j,l})_{l=1,\dots,n}(\mathbf{z}^{i,j},\pi^{i,j},k_1^{i,j},k_2^{i,j})\Big).$$

Then, we have the following lemma

Lemma 5.22. *If $(\mathbf{z},\psi,k_1,k_2)$ is in $\mathcal{E}_{inf}$, then*

$$\int_0^T d(t)^2|A^{1/2}k_1|^2_{\mathbb{R}^N} \le C\left(\int_0^T(\xi^*(t))^3 e^{-2s\varphi_\lambda^*(t)}(|k_2|^2_{\mathbb{R}^N}+|b|^2_{\mathbb{R}^N})\right).$$

Proof. The proof can be found in [28]. We have adapted it with $|A^{1/2}k_1'|^2_{\mathbb{R}^N}\le C(|k_2|^2_{\mathbb{R}^N}+|b|^2_{\mathbb{R}^N})$. □

We can have the following estimate seeing that $k_1 = k_1 - \Pi(\mathbf{z},\psi,k_1,k_2) + \Pi(\mathbf{z},\psi,k_1,k_2)$ and that $k_1 - \Pi(\mathbf{z},\psi,k_1,k_2)\in\mathcal{E}_{inf}$:

$$I(s,\lambda;\xi)\le C(K(s,\lambda;\xi)+J(s,\lambda;\xi))$$

with

$$K(s,\lambda;\xi) = \int_{(0,T)\times\omega_2}\lambda^5(s\xi^*)^{15/2}e^{-2s\varphi_\lambda^*}|\mathbf{y}|^2 + \int_0^T\lambda^5(s\xi^*)^{15/2}e^{-2s\varphi_\lambda^*}\|(\mathbf{a},b,c)\|^2_{\mathbb{V}}$$

and

$$J(s,\lambda;\xi) = s^{5/2}\int_0^T (\xi^*(t))^3 \mathrm{e}^{-2s\varphi^*_\lambda(t)}\Big(|\Pi(\mathbf{z},\psi,k_1,k_2)|^2_{\mathbb{R}^N} + |\Pi(\mathbf{z},\psi,k_1,k_2)'|^2_{\mathbb{R}^N}\Big).$$

Finally, thanks to a compactness argument like in [28, section 7], we have the last Carleman estimate :

$$I(s,\lambda;\xi) \le C\left(\int_{(0,T)\times\omega_2} \lambda^5 (s\xi^*)^{15/2}\mathrm{e}^{-2s\varphi^*_\lambda}|\mathbf{y}|^2 + \int_0^T \lambda^5 (s\xi^*)^{15/2}\mathrm{e}^{-2s\varphi^*_\lambda}\|(\mathbf{a},b,c)\|^2_{\mathbb{V}}\right)$$

because the terms in $|k_2|^2$ in the right-hand side can be absorbed by the one in $I(s,\lambda;\xi)$ for λ and s large enough.

5.4.5 From the Carleman estimate to the observability inequality.

We introduce here a piecewise continuous function l defined in $[0,T]$ by

$$l(t) = \begin{cases} T^2/4 & \text{if} \quad t\in[0,T/2], \\ t(T-t) & \text{if} \quad t\in[T/2,T]. \end{cases}$$

which gives us two new weight functions $\delta(x,t) = \frac{\kappa(x)}{l^k(t)}$ and $\sigma(x,t) = \frac{\mathrm{e}^{\lambda(\phi(x)+m\|\phi\|_\infty)}}{l^k(t)}$.

We use here the energy estimates for the system

$$\begin{array}{rcll} -\mathbf{y}_t - \mathrm{div}\,\sigma(\mathbf{y},\pi) &=& \mathbf{a} & \text{in } Q_T^0, \\ \mathrm{div}\,\mathbf{y} &=& 0 & \text{in } Q_T^0, \\ \mathbf{y} &=& Zk_2\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{y} &=& \mathbf{0} & \text{on } \Sigma_T, \\ k_1' &=& k_2 - b & \text{in } (0,T), \\ k_2' + Ak_1 &=& -\Pi_N\pi - c & \text{in } (0,T), \\ (\mathbf{y}(T),k_1(T),k_2(T)) &=& (\mathbf{0},0,0). & \end{array}$$

The energy equality is

$$\begin{array}{rl} & -\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{y}\|^2_{\mathbf{L}^2(\Omega_0)} + |k_2|^2_{\mathbb{R}^N} + |A^{1/2}k_1|^2_{\mathbb{R}^N}\Big) + \nu\|\nabla\mathbf{y}\|^2_{\mathbf{L}^2(\Omega_0)} \\ = & (\mathbf{a},\mathbf{y})_{L^2(\Omega_0)} + (k_2,b)_{\mathbb{R}^N} + (A^{1/2}k_1,A^{1/2}b)_{\mathbb{R}^N}. \end{array}$$

Then, integrating from t to T (with t in $(0,T)$) and taking the supremum, we get the classic energy estimate

$$\begin{array}{rl} & \|\mathbf{y}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_0))} + \|A^{1/2}k_1\|^2_{L^\infty(0,T;\mathbb{R}^N)} + \|k_2\|^2_{L^\infty(0,T;\mathbb{R}^N)} + \nu\|\nabla\mathbf{y}\|^2_{L^2(Q_T^0)} \\ \le & C\Big(\|\mathbf{a}\|^2_{\mathbf{L}^2(Q_T^0)} + \|A^{1/2}b\|^2_{L^2(0,T;\mathbb{R}^N)} + \|c\|^2_{L^2(0,T;\mathbb{R}^N)}\Big). \end{array} \tag{5.51}$$

That is, using the notation of the space $\mathbb{V}$ defined in (5.24), we have

$$\|(\mathbf{y},k_1,k_2)\|^2_{L^\infty(0,T;\mathbb{V})} + \nu\|\nabla\mathbf{y}\|^2_{L^2(Q_T^0)} \le C\Big(\|(\mathbf{a},b,c)\|^2_{\mathbf{L}^2(0,T;\mathbb{V})}\Big).$$

We introduce a weight function θ in $\mathcal{C}^1([0,T];\mathbb{R})$ satisfying

$$\theta\equiv 1 \text{ in } [0,T/2], \quad \theta\equiv 0 \text{ in } [3T/4,T] \quad \text{and } |\theta'|\le 1/T.$$

Let us now consider the system satisfied by $(\theta\mathbf{y},\theta\pi,k_1,k_2) = (\mathbf{y}^*,\pi^*,k_1,k_2)$:

$$\begin{array}{rcll} -\mathbf{y}^*_t - \mathrm{div}\,\sigma(\mathbf{y}^*,\pi^*) &=& \theta\mathbf{a} - \theta'\mathbf{y} & \text{in } Q_T^0, \\ \mathrm{div}\,\mathbf{y}^* &=& 0 & \text{in } Q_T^0, \\ \mathbf{y}^* &=& \theta Zk_2\mathbf{e}_2 & \text{on } \Sigma_T^{s,0}, \\ \mathbf{y}^* &=& \mathbf{0} & \text{on } \Sigma_T, \\ \theta k_1' &=& \theta k_2 - \theta b & \text{in } (0,T), \\ \theta k_2' + \theta Ak_1 &=& -\Pi_N\pi^* - \theta c & \text{in } (0,T), \\ (\mathbf{y}(T),k_1(T),k_2(T)) &=& (\mathbf{0},0,0). & \end{array}$$

By some integrations by parts, we get the energy identity of the previous system:

$$\begin{aligned}
&\left\|\Big(\mathbf{y},k_1,k_2\Big)\right\|^2_{L^2(0,T/2;\mathbb{V})}+\left\|\Big(\mathbf{y},k_1,k_2\Big)\right\|^2_{L^\infty(0,T/2;\mathbb{V})}+\nu\|\nabla\mathbf{y}\|^2_{L^2(0,T/2;\mathbf{L}^2(\Omega_0))}\\
\leq\ & C\left[\left\|\Big(\mathbf{a},b,c\Big)\right\|^2_{L^2(0,3T/4;\mathbb{V})}+\frac{1}{T}\left\|\Big(\mathbf{y},k_1,k_2\Big)\right\|^2_{L^2(T/2,3T/4;\mathbb{V})}\right].
\end{aligned}\tag{5.52}$$

Then, by the first energy estimate (5.51), we can estimate the last term on the right-hand side of (5.52) as above and get the classic estimate

$$\left\|\Big(\mathbf{y},k_1,k_2\Big)\right\|^2_{L^2(0,T/2;\mathbb{V})}+\left\|\Big(\mathbf{y},k_1,k_2\Big)\right\|^2_{L^\infty(0,T/2;\mathbb{V})}+\nu\|\nabla\mathbf{y}\|^2_{L^2(0,T/2;\mathbf{L}^2(\Omega_0))}\leq C\left\|\Big(\mathbf{a},b,c\Big)\right\|^2_{L^2(0,3T/4;\mathbb{V})}.\tag{5.53}$$

Because the weigths δ and σ are constant in time on $[0,T/2]$ and the weights in s and λ are bigger in the right-hand side than in the left-hand side, this gives in particular,

$$\begin{aligned}
&\left\|\Big(\mathbf{y}(0),k_1(0),k_2(0)\Big)\right\|^2_{\mathbb{V}}+s^3\lambda^4\int_0^{T/2}\int_{\Omega_0}\mathrm{e}^{-2s\delta}\sigma^3|\mathbf{y}|^2\\
&+s\lambda^2\int_0^{T/2}\int_{\Omega_0}\mathrm{e}^{-2s\delta}\sigma|\nabla\mathbf{y}|^2+s^3\lambda^3\int_0^{T/2}\mathrm{e}^{-2s\delta}\sigma^3|k_2|^2_{\mathbb{R}^N}\\
\leq\ & C\left[\int_0^{T/2}\lambda^5(s\sigma^*)^{15/2}\mathrm{e}^{-2s\delta^*}\|(\mathbf{a},b,c)\|^2_{\mathbb{V}}\right].
\end{aligned}\tag{5.54}$$

On the other hand, the Carleman estimate (5.50) in Proposition 5.21 gives, because $\delta=\varphi_\lambda$ and $\xi=\sigma$ for t in $[T/2,T]$, the same result:

$$\begin{aligned}
&s\lambda^2\int_{T/2}^T\int_{\Omega_0}\sigma|\nabla\mathbf{y}|^2\mathrm{e}^{-2s\delta}+s^3\lambda^4\int_{T/2}^T\int_{\Omega_0}\sigma^3|\mathbf{y}|^2\mathrm{e}^{-2s\delta}+s^3\lambda^3\int_{T/2}^T\int_{\Gamma_0^s}\sigma^3\mathrm{e}^{-2s\delta}|k_2|^2_{\mathbb{R}^N}\\
\leq\ & C\left(\int_{T/2}^T\int_{\omega_2}\lambda^5(s\sigma^*)^{15/2}\mathrm{e}^{-2s\delta^*}|\mathbf{y}|^2+\int_{T/2}^T\lambda^5(s\sigma^*)^{15/2}\mathrm{e}^{-2s\delta^*}\|(\mathbf{a},b,c)\|^2_{\mathbb{V}}\right).
\end{aligned}\tag{5.55}$$

Finally, adding inequalies (5.54) and (5.55), we get the expected observability inequality

$$\begin{aligned}
&\left\|\Big(\mathbf{y}(0),k_1(0),k_2(0)\Big)\right\|^2_{\mathbb{V}}+s^3\lambda^3\int_0^T\sigma^{*3}(t)\mathrm{e}^{-2s\delta^*(t)}\Big(\|\mathbf{y}(t)\|^2_{\mathbf{L}^2(\Omega_0)}+|k_2|^2_{\mathbb{R}^N}\Big)\\
\leq\ & C\left(\int_0^T\lambda^5(s\sigma^*(t))^{15/2}\mathrm{e}^{-2s\delta^*(t)}\|(\mathbf{a}(t),b(t),c(t))\|^2_{\mathbb{V}}+\int_0^T\lambda^5(s\sigma^*(t))^{15/2}\mathrm{e}^{-2s\delta^*(t)}\|\mathbf{y}\|^2_{\mathbf{L}^2(\omega_2)}\right).
\end{aligned}$$

Chapitre 6

Stabilisation d'un système couplé fluide - structure.

6.1 Introduction.

In this chapter, we consider a fluid flow in a two dimensional periodic channel. The boundary of the channel is split into two parts (the upper and the lower part, see Figure 6.1). Each part is a mobile structure which is modeled by a beam. Our goal is to prove the stabilization (for any decay rate) of this system with two controls acting on the upper part of the boundary (see section 6.3 for details).

The boundary conditions at the interface between the fluid and the beams are the one studied in [13]. This conditions allow us to prove the approximate controllability of the linearized system around the zero stationary solution using the explicit expression of the eigenfunctions of the problem (see sections 6.5.1 and 6.5.2). Then, by classic results of controllability, the stabilizability of this system follows. We are able to prove the stabilization for the nonlinear system thanks to a fixed point method.

Let us denote $\Omega_0 = \mathbb{R}/2\pi \times (-1, 1)$ the reference domain. In the following, we set κ for $\pm$ and Γ_0^κ the two parts of the boundary of Ω_0, that is $\Gamma_0^+ = \mathbb{R}/2\pi \times \{+1\}$ and $\Gamma_0^- = \mathbb{R}/2\pi \times \{-1\}$. We now introduce two periodic (in the x-variable) functions η^κ from $(0, +\infty) \times (0, 2\pi)$ into *a priori* $\mathbb{R}$. These functions model the displacement (with respect to the reference state Γ_0^κ) of the boundary of the domain of the fluid denoted, at time t, $\Omega_{\eta(t)}$ (with $\eta = (\eta^+, \eta^-)$) . Thus, the boundaries $\Gamma^\kappa_{\eta^\kappa(t)}$ are defined by

$$\Gamma^\kappa_{\eta^\kappa(t)} = \Big\{(x, y) \in \mathbb{R}/2\pi \times \mathbb{R} \text{ s.t. } y = \kappa 1 + \eta^\kappa(t, x)\Big\}$$

and the domain $\Omega_{\eta(t)}$ by

$$\Omega_{\eta(t)} = \Big\{(x, y) \in \mathbb{R}/2\pi \times \mathbb{R} \text{ s.t. } -1 + \eta^-(t, x) < y < 1 + \eta^+(t, x)\Big\}.$$

The deplacements η^+ and η^- have to satisfy the assumption

$$\text{There exists } \delta_0 > 0 \text{ such that for all } t \geq 0 \text{ and } x \in (0, 2\pi) \qquad 2 + \eta^+(t, x) - \eta^-(t, x) \geq \delta_0 > 0. \tag{6.1}$$

Under assumption (6.1), the domain $\Omega_{\eta(t)}$ is connected at every time t. We denote for a time $0 < T \leq +\infty$ the different cylindrical domains

$$Q_T^0 = (0, T) \times \Omega_0, \qquad \Sigma_T^{\kappa,0} = (0, T) \times \Gamma_0^\kappa, \qquad Q_T^\eta = \bigcup_{t\in(0,T)} \{t\} \times \Omega_{\eta(t)}, \qquad \Sigma_T^{\kappa,\eta^\kappa} = \bigcup_{t\in(0,T)} \{t\} \times \Gamma^\kappa_{\eta^\kappa(t)}.$$

The system is

$$\begin{array}{rcll}
\mathbf{u}_t - \text{div } \sigma(\mathbf{u}, p) + (\mathbf{u} \cdot \nabla)\mathbf{u} & = & \mathbf{0} & \text{in } Q^\eta_\infty, \\
\text{div } \mathbf{u} & = & 0 & \text{in } Q^\eta_\infty, \\
\mathbf{u} \cdot \widetilde{\mathbf{n}}^\kappa & = & \kappa \eta^\kappa_t & \text{on } \Sigma^{\kappa,\eta^\kappa}_\infty, \\
\sigma(\mathbf{u}, p)\mathbf{n}^\kappa \cdot \mathbf{t}^\kappa & = & 0 & \text{on } \Sigma^{\kappa,\eta^\kappa}_\infty, \\
\eta^\kappa_{tt} + \alpha \eta^\kappa_{xxxx} - \beta \eta^\kappa_{xx} - \gamma \eta^\kappa_{txx} & = & -\kappa \sigma(\mathbf{u}, p)\mathbf{n}^\kappa \cdot \mathbf{n}^\kappa & \text{on } \Sigma^{\kappa,0}_\infty, \\
(\mathbf{u}(0), \eta(0), \eta_t(0)) & = & (\mathbf{u}^0, \eta^{1,0}, \eta^{2,0}) &
\end{array} \tag{6.2}$$

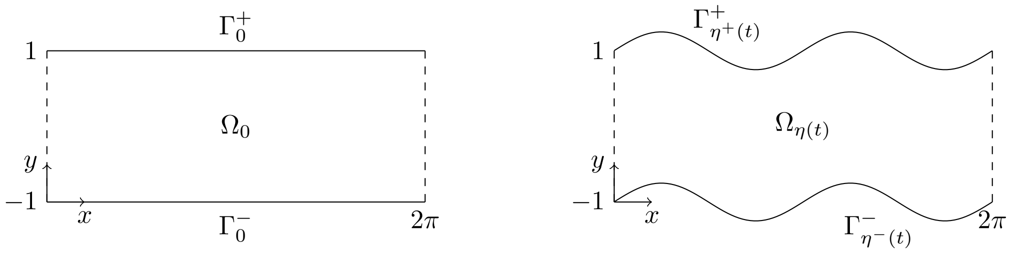

Figure 6.1: The domains Ω_0 (left) and $\Omega_{\eta(t)}$ (right).

In these equations, $\sigma(\mathbf{u},p)$ is the Cauchy stress tensor of the fluid given by $\sigma(\mathbf{u},p) = \nu\mathbf{S}(\mathbf{u}) - p\mathbf{I}_2 = \nu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{tr}}\right) - p\mathbf{I}_2$ with $\mathbf{I}_2$ the 2×2 identity matrix. The vectors $\mathbf{n}^\kappa$ and $\mathbf{t}^\kappa$ are respectively the outward normal and tangent vectors of the boundary $\Gamma^\kappa_{\eta^\kappa(t)}$, they are given, for any time t, by

$$\mathbf{n}^\kappa(t) \quad = \quad \frac{\kappa}{\sqrt{1+(\eta^\kappa_x(t))^2}}\begin{pmatrix} -\eta^\kappa_x(t) \\ 1 \end{pmatrix}, \qquad \mathbf{t}^\kappa(t) \quad = \quad \frac{1}{\sqrt{1+(\eta^\kappa_x(t))^2}}\begin{pmatrix} 1 \\ \eta^\kappa_x(t) \end{pmatrix}.$$

The vector $\widetilde{\mathbf{n}}^\kappa$ is obtained from $\mathbf{n}^\kappa$ by $\widetilde{\mathbf{n}}^\kappa = \kappa\big(-\eta^\kappa_x\mathbf{e}_1 + \mathbf{e}_2\big) = \sqrt{1+(\eta^\kappa_x)^2}\mathbf{n}^\kappa$.

6.2 Functional setting.

As we consider the periodic setting, we will strongly use the Fourier decomposition in the periodic variable. Namely, we write $L^2_\#(\Omega_0;\mathbb{R})$ the space of all functions f in $L^2_{\text{loc}}(\mathbb{R}\times(-1,1);\mathbb{R})$ which are 2π-periodic in the x-variable. Such a function f can be easily characterized by

$$f(x,y) = \sum_{k\in\mathbb{Z}} f_k(y)\mathrm{e}^{ikx}, \text{ for all } (x,y)\in\Omega_0, \quad f_k = \overline{f_{-k}}, \quad f_0\in\mathbb{R} \text{ and } \qquad \sum_{k\in\mathbb{Z}}\int_{-1}^1 |f_k(y)|^2\mathrm{d}y < +\infty.$$

The space $L^2_\#(\Omega_0;\mathbb{R})$ is endowed with the classic L^2-norm:

$$\|f\|^2_{L^2_\#(\Omega_0;\mathbb{R})} = \sum_{k\in\mathbb{Z}}\|f_k\|^2_{L^2(-1,1;\mathbb{C})} = \sum_{k\in\mathbb{Z}}\int_{-1}^1 |f_k(y)|^2\mathrm{d}y.$$

We denote $\mathbf{L}^2_\#(\Omega_0;\mathbb{R}) = [L^2_\#(\Omega_0;\mathbb{R})]^2$. In the same way as above, a vector $\mathbf{u}$ in $\mathbf{L}^2_\#(\Omega_0;\mathbb{R})$ is characterized by

$$\mathbf{u}(x,y) = \sum_{k\in\mathbb{Z}}\mathbf{u}_k(y)\mathrm{e}^{ikx}, \text{ for all } (x,y)\in\Omega_0, \quad \mathbf{u}_k = (u^1_k,u^2_k), \quad \mathbf{u}_k = \overline{\mathbf{u}_{-k}}, \quad \mathbf{u}_0\in\mathbb{R}^2$$
$$\text{and} \qquad \|\mathbf{u}\|^2_{\mathbf{L}^2_\#(\Omega_0;\mathbb{R})} = \sum_{k\in\mathbb{Z}}\|\mathbf{u}_k\|^2_{\mathbf{L}^2(-1,1;\mathbb{C})} < +\infty.$$

We can define Sobolev spaces for more regular functions in the Fourier setting as follows:

$$H^1_\#(\Omega_0;\mathbb{R}) \quad = \quad \Bigg\{ f = \sum_{k\in\mathbb{Z}} f_k\mathrm{e}^{ik\cdot} \quad f_k = \overline{f_{-k}}, \quad f_0\in\mathbb{R}$$
$$\text{and} \quad \|f_0\|^2_{H^1(-1,1;\mathbb{R})} + \sum_{k\in\mathbb{Z}}\Big(k^2\|f_k\|^2_{L^2(-1,1;\mathbb{C})} + \|f_{k,y}\|^2_{L^2(-1,1;\mathbb{C})}\Big) < +\infty\Bigg\}.$$

For any f in $H^1_\#(\Omega_0;\mathbb{R})$, we defined the norm on the Sobolev space $H^1_\#(\Omega_0;\mathbb{R})$ by

$$\|f\|_{H^1_\#(\Omega_0;\mathbb{R})} = \left(\|f_0\|^2_{H^1(-1,1;\mathbb{R})} + \sum_{k\in\mathbb{Z}\setminus\{0\}}\Big(k^2\|f_k\|^2_{L^2(-1,1;\mathbb{C})} + \|f_{k,y}\|^2_{L^2(-1,1;\mathbb{C})}\Big)\right)^{1/2}.$$

More generally, we have for any $n \in \mathbb{N} \setminus \{0\}$ the space $H^n_\#(\Omega_0;\mathbb{R})$ as follows

$$H^n_\#(\Omega_0;\mathbb{R}) = \left\{ f = \sum_{k\in\mathbb{Z}} f_k \mathrm{e}^{ik\cdot} \text{ s.t. } \|f_0\|^2_{H^n(-1,1;\mathbb{R})} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \|f_k\|^2_{H^n_{\#,k}(-1,1;\mathbb{C})} < +\infty \right\}$$

where

$$\|f_k\|_{H^n_{\#,k}(-1,1;\mathbb{C})} = \left(\sum_{p=0}^{n} \begin{pmatrix} n \\ p \end{pmatrix} k^{2(n-p)} \left\| \frac{\partial^p f_k}{\partial y^p} \right\|^2_{L^2(-1,1;\mathbb{C})} \right)^{1/2}.$$

Then, we can define the space $H^\sigma_\#(\Omega_0;\mathbb{R})$ for $\sigma > 0$ by interpolation and for $\sigma < 0$ by duality. In the same way, we denote the spaces $\mathbf{H}^\sigma_\#(\Omega_0;\mathbb{R}) = [H^\sigma_\#(\Omega_0;\mathbb{R})]^2$.

The space of function for the Stokes system is

$$\mathbf{V}^0_\#(\Omega_0;\mathbb{R}) = \left\{ \mathbf{u} = (u^1,u^2) \in \mathbf{L}^2_\#(\Omega_0;\mathbb{R}) \text{ s.t. } u^1_x + u^2_y = 0 \text{ in } \Omega_0 \right\}.$$

This Hilbert space is endowed with the classic norm in $\mathbf{L}^2_\#(\Omega_0;\mathbb{R})$. Note that the vector functions in $\mathbf{V}^0_\#(\Omega_0;\mathbb{R})$ can be characterized by

$$\begin{aligned} \mathbf{V}^0_\#(\Omega_0;\mathbb{R}) = & \Big\{ \mathbf{u} = (u^1,u^2) \in \mathbf{L}^2_\#(\Omega_0;\mathbb{R}) \quad i.e. \quad u^j = \sum_{k\in\mathbb{Z}} u^j_k \mathrm{e}^{ik\cdot} \text{ s.t. } u^j_k = \overline{u^j_{-k}}, \quad u^j_0 \in \mathbb{R} \quad \text{for } j=1,2, \\ & \text{s.t. } \mathrm{div}_k \mathbf{u}_k = iku^1_k + u^2_{k,y} = 0 \text{ in } \Omega_0 \text{ for all } k \in \mathbb{Z} \Big\}. \end{aligned}$$

Thus for any $k \in \mathbb{Z}$, the vector $\mathbf{u}_k$ belongs to $\mathbf{L}^2(1,1;\mathbb{C}) = [L^2(-1,1;\mathbb{C})]^2$ and satisfies $\mathrm{div}_k\mathbf{u}_k = 0$ in $(-1,1)$, which is exactly the divergence in the Fourier setting (because the derivation by the first component becomes in the Fourier setting the multiplication by ik). We define the space

$$\mathbf{V}^0_{\#,k}(-1,1;\mathbb{C}) = \Big\{ \mathbf{z}_k = (z^1_k, z^2_k) \in \mathbf{L}^2(-1,1;\mathbb{C}) \text{ s.t. } \mathrm{div}_k \mathbf{z}_k(y) = 0 \quad \text{for a.e. } y \in (-1,1) \Big\}$$

and in the same way,

$$\begin{aligned} \mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C}) = & \Big\{ \mathbf{z}_k = (z^1_k, z^2_k) \in \mathbf{L}^2(-1,1;\mathbb{C}) \text{ s.t. } \mathrm{div}_k \mathbf{z}_k(y) = 0 \text{ for a.e. } y \in (-1,1) \\ & \text{and } z^2_k = 0 \text{ for } y = \kappa 1 \Big\}, \end{aligned}$$

$$\begin{aligned} \mathbf{V}^1_{\#,k,0}(-1,1;\mathbb{C}) = & \Big\{ \mathbf{z}_k = (z^1_k, z^2_k) \in \mathbf{H}^1(-1,1;\mathbb{C}) \text{ s.t. } \mathrm{div}_k \mathbf{z}_k(y) = 0 \text{ for a.e. } y \in (-1,1) \\ & \text{and } \mathbf{z}_k = 0 \text{ for } y = \kappa 1 \Big\}. \end{aligned}$$

We define for a function f_k in $H^1_{\#,k}(-1,1;\mathbb{C})$, the gradient $\nabla_k f_k$ by

$$\nabla_k f = \begin{pmatrix} ikf_k \\ f_{k,y} \end{pmatrix}.$$

In the same way, we define the Laplace operator for functions $\mathbf{z}_k$ in $\mathbf{V}^2_{\#,k}(-1,1;\mathbb{C}) \cap \mathbf{V}^1_{\#,k,0}(-1,1;\mathbb{C})$ by

$$\Delta_k \mathbf{z}_k = \begin{pmatrix} z^1_{k,yy} - k^2 z^1_k \\ z^2_{k,yy} - k^2 z^2_k \end{pmatrix}.$$

Due to the incompressible condition in the Stokes system and an integration by parts, we have the following identity for $(\mathbf{u}, p, \eta)$ the solution of (6.2) at any time t:

$$\int_{\Omega_{\eta(t)}} \mathrm{div}\ \mathbf{u}(t) = \int_{\partial\Omega_{\eta(t)}} \mathbf{u}(t)\cdot\mathbf{n}(t) = \sum_\kappa \int_{\Gamma^\kappa_{\eta^\kappa(t)}} \mathbf{u}(t)\cdot\mathbf{n}^\kappa = \sum_\kappa \int_{\Gamma^\kappa_0} \mathbf{u}\cdot\widetilde{\mathbf{n}}^\kappa(t) = \sum_\kappa \int_{\Gamma^\kappa_0} \kappa\eta^\kappa_t(t) = 0.$$

Thus, we take the displacement functions η^κ in the space of the periodic functions in the x-variable with zero mean value on Γ_0^κ (as for the second component of the velocity $\mathbf{u}$), that is η^κ belongs to $L^2_{\#,0}(\Gamma_0^\kappa;\mathbb{R})$ characterized in the Fourier setting by

$$L^2_{\#,0}(\Gamma_0^\kappa;\mathbb{R}) = \left\{ \mu^\kappa = \sum_{k\in\mathbb{Z}} \mu_k^\kappa \mathrm{e}^{ik\cdot} \quad \text{with} \quad \mu_k^\kappa = \overline{\mu_{-k}^\kappa} \quad \text{and} \quad \mu_0^\kappa = 0 \quad \text{s.t.} \quad \sum_{k\in\mathbb{Z}\setminus\{0\}} |\mu_k^\kappa|^2 < +\infty \right\}.$$

We can define more regular Sobolev spaces for the displacement function. For instance, we will endow the space $H^2_\#(\Gamma_0^\kappa;\mathbb{R})$ with the norm induced by the beam operator $A_{\alpha,\beta}$ defined from $D(A_{\alpha,\beta}) = H^4_\#(\Gamma_0^\kappa)$ into $L^2_{\#,0}(\Gamma_0^\kappa;\mathbb{R})$ by $A_{\alpha,\beta}\mu^\kappa = \alpha\mu^\kappa_{xxxx} - \beta\mu^\kappa_{xx}$, for all $\mu^\kappa \in D(A_{\alpha,\beta})$(see [23] for details). That is

$$H^2_\#(\Gamma_0^\kappa;\mathbb{R}) = \left\{ \mu^\kappa = \sum_{k\in\mathbb{Z}\setminus\{0\}} \mu_k^\kappa \mathrm{e}^{ik\cdot} \text{ with } \mu_k^\kappa = \overline{\mu_{-k}^\kappa} \text{ and } \mu_0^\kappa = 0 \text{ s.t. } \sum_{k\in\mathbb{Z}\setminus\{0\}} (\alpha k^4 + \beta k^2)|\mu_k^\kappa|^2 < +\infty \right\}.$$

To keep in mind that the family $(\mu_k^\kappa)_{k\in\mathbb{Z}\setminus\{0\}}$ can be taken such that $\mu^\kappa = \sum_{k\in\mathbb{Z}\setminus\{0\}} \mu_k^\kappa \mathrm{e}^{ik\cdot}$ belongs to a Sobolev space $H^\sigma_\#(\Gamma_0^\kappa;\mathbb{R})$, we introduce on $\mathbb{C}$ some «Sobolev spaces» corresponding for any $k \in \mathbb{Z}\setminus\{0\}$ to $H^\sigma_\#(\Gamma_0^\kappa;\mathbb{R})$. Let us denote $H^\sigma_{\#,k}(\mathbb{C})$ the space $\mathbb{C}$ endowed with the (equivalent) scalar product

$$\langle \mu_k^\kappa, \nu_k^\kappa \rangle_{H^\sigma_{\#,k}(\mathbb{C})} = (\alpha k^4 + \beta k^2)^{\sigma/2} \langle \mu_k^\kappa, \nu_k^\kappa \rangle \qquad \text{for any } \mu_k^\kappa, \nu_k^\kappa \in H^\sigma_{\#,k}(\mathbb{C}).$$

Thus, for any μ^κ, ν^κ in $H^\sigma_\#(\Gamma_0^\kappa;\mathbb{R})$, the classic scalar product $(\mu^\kappa,\nu^\kappa)_{H^\sigma_\#(\Gamma_0^\kappa;\mathbb{R})}$ becomes

$$(\mu^\kappa,\nu^\kappa)_{H^\sigma_\#(\Gamma_0^\kappa;\mathbb{R})} = \sum_{k\in\mathbb{Z}} \langle \mu_k^\kappa, \nu_k^\kappa \rangle_{H^\sigma_{\#,k}(\mathbb{C})} = \sum_{k\in\mathbb{Z}\setminus\{0\}} (\alpha k^4 + \beta k^2)^{\sigma/2} \langle \mu_k^\kappa, \nu_k^\kappa \rangle .$$

where $\mu^\kappa = \sum_{k\in\mathbb{Z}\setminus\{0\}} \mu_k^\kappa \mathrm{e}^{ik\cdot}$, $\nu^\kappa = \sum_{k\in\mathbb{Z}\setminus\{0\}} \nu_k^\kappa \mathrm{e}^{ik\cdot}$ are the Fourier series of μ^κ, ν^κ. By abuse, we denote $L^2_{\#,k}(\mathbb{C}) = H^0_{\#,k}(\mathbb{C})$ the set $\mathbb{C}$ endowed with the classic scalar product $\langle \mu,\nu\rangle = \mu\overline{\nu}$ for all μ,ν in $\mathbb{C}$.

For simplicity, we denote $\eta \in H^\sigma_\#(\Gamma_0;\mathbb{R})$ instead of $(\eta^+,\eta^-) \in H^\sigma_\#(\Gamma_0^+;\mathbb{R}) \times H^\sigma_\#(\Gamma_0^-;\mathbb{R})$ where $\eta = (\eta^+,\eta^-)$ and $\Gamma_0 = \Gamma_0^+ \times \Gamma_0^-$.

The pressure term p in the first Navier-Stokes equation or in the beam equations is defined up to an additive constant. Thus, to get the uniqueness of the triplet $(\mathbf{u},p,\eta)$ solution of (6.4), we define the space for the pressure

$$\mathcal{H}^\sigma_\#(\Omega_0;\mathbb{R}) = \left\{ q \in H^\sigma_\#(\Omega_0;\mathbb{R}) \text{ s.t. } \int_{\Omega_0} q = 0 \right\}.$$

We introduce the space time Sobolev spaces for the velocity, for $0 < T \le \infty$,

$$\begin{aligned} \mathbf{H}^{\sigma,\tau}_\#(Q_T^0) &= L^2(0,T;\mathbf{H}^\sigma_\#(\Omega_0)) \cap H^\sigma(0,T;\mathbf{L}^2_\#(\Omega_0)) \\ \mathbf{V}^{\sigma,\tau}_\#(Q_T^0) &= L^2(0,T;\mathbf{V}^\sigma_\#(\Omega_0)) \cap H^\sigma(0,T;\mathbf{V}^0_\#(\Omega_0)) \end{aligned}$$

and the equivalent in the cylinder Q_T^η:

$$\begin{aligned} \mathbf{H}^{\sigma,\tau}_\#(Q_T^\eta) &= L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathbf{H}^\sigma_\#\left(\Omega_{\eta(t)}\right) \right) \cap H^\tau \left(\bigcup_{t\in(0,T)} \{t\} \times \mathbf{L}^2_\#\left(\Omega_{\eta(t)}\right) \right) \\ \mathbf{V}^{\sigma,\tau}_\#(Q_T^\eta) &= L^2\left(\bigcup_{t\in(0,T)} \{t\} \times \mathbf{V}^\sigma_\#\left(\Omega_{\eta(t)}\right) \right) \cap H^\tau \left(\bigcup_{t\in(0,T)} \{t\} \times \mathbf{V}^0_\#\left(\Omega_{\eta(t)}\right) \right). \end{aligned}$$

Here, we use the following definition.

Definition 6.1. *We say that the function* $\mathbf{u}$ *belongs to* $H^\tau\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{H}^\sigma_\#\left(\Omega_{\eta(t)}\right)\right)$ *(respectively to* $H^\tau\left(\bigcup_{t\in(0,T)}\{t\}\times\mathbf{V}^\sigma_\#\left(\Omega_{\eta(t)}\right)\right)$*) if*
- *for almost every* t *in* $(0,T)$*, the function* $\mathbf{u}(t,\cdot)$ *belongs to* $\mathbf{H}^\sigma_\#(\Omega_0)$ *(resp. to* $\mathbf{V}^\sigma_\#(\Omega_0)$*),*
- *the function* $t\mapsto\|\mathbf{u}(t)\|_{\mathbf{H}^\sigma_\#(\Omega_0)}$ *(resp.* $t\mapsto\|\mathbf{u}(t)\|_{\mathbf{V}^\sigma_\#(\Omega_0)}$*) belongs to* $H^\tau(0,T)$*.*

We set the spaces for the displacement $\eta=(\eta^+,\eta^-)$, for $0<T\leq\infty$, as follows:

$$\begin{aligned}H^{\sigma,\tau}_\#(\Sigma^0_T) &= L^2(0,T;H^\sigma_\#(\Gamma_0))\cap H^\sigma(0,T;L^2_{\#,0}(\Gamma_0))\\ &= H^{\sigma,\tau}_\#(\Sigma^{+,0}_T)\times H^{\sigma,\tau}_\#(\Sigma^{-,0}_T).\end{aligned}$$

Using the Fourier decomposition, we introduce, in the same way, the space time Sobolev spaces for the Fourier coefficient of $(\mathbf{u},p,\eta)$ (in the fixed domain Q^0_∞). That is, for $\mathbf{u}=\sum_{k\in\mathbb{Z}}\mathbf{u}_k\mathrm{e}^{ik\cdot}$, we introduce the spaces for $\mathbf{u}_k$ corresponding with $\mathbf{H}^{\sigma,\tau}_\#(Q^0_\infty)$ and $\mathbf{V}^{\sigma,\tau}_\#(Q^0_\infty)$:

$$\begin{aligned}\mathbf{H}^{\sigma,\tau}_{\#,k}((0,T)\times(-1,1);\mathbb{C}) &= L^2(0,T;\mathbf{H}^\sigma_{\#,k}(-1,1;\mathbb{C}))\cup H^\tau(0,T;\mathbf{L}^2_{\#,k}(-1,1;\mathbb{C})),\\ \mathbf{V}^{\sigma,\tau}_{\#,k}((0,T)\times(-1,1);\mathbb{C}) &= L^2(0,T;\mathbf{V}^\sigma_{\#,k}(-1,1;\mathbb{C}))\cup H^\tau(0,T;\mathbf{V}^0_{\#,k}(-1,1;\mathbb{C})).\end{aligned}$$

The same idea for the displacements give the space time function spaces (for $k\neq0$), for $0<T\leq\infty$,

$$H^{\sigma,\tau}_{\#,k}((0,T)\times\mathbb{C}^2)=L^2(0,T;H^\sigma_{\#,k}(\mathbb{C}^2))\cap H^\tau(0,T;L^2_{\#,k}(\Gamma_0)).$$

Finally, the coefficients p_k of the pressure term p will belong to $L^2(0,T;H^\sigma_{\#,k}(-1,1;\mathbb{C}))$ for $k\neq0$ and p_0 will be zero (see the space $\mathcal{H}^\sigma_\#(\Omega_0)$).

6.3 Main result.

A simple stationary solution of system (6.2) is given by $(\mathbf{u},p,\eta)=(\mathbf{0},0,0,0)$. The aim of this paper is to prove the stabilization localy around $(\mathbf{0},0,0,0)$ with any decay rate $\omega>0$ with two controls f^+_0 and f^+ acting on each boundary equation of the upper part. The first control acts as a discontinuity of the tangential component of the normal Cauchy tensor stress (see equation $(6.4)_5$ below). The second control, namely f^+, acts as a force in the right-hand side of the upper beam equation on the whole boundary (see $(6.4)_7$).

More precisely, we prove the following result.

Theorem 6.2. *For any decay rate* $\omega>0$*, there exists a constant* $r_0>0$ *and a increasing function* R *from* $\mathbb{R}^+$ *into itself such that if* r *belongs to* $(0,r_0)$ *and* $(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})$ *is in* $\mathbf{V}^1_\#(\Omega_{\eta^{1,0}})\times H^3_\#(\Gamma_0)\times H^1_\#(\Gamma_0)$ *satisfying the compatibility condition (note that* $\mathbf{u}^0=(u^{0,1},u^{0,2})$*)*

$$-\eta^{1,0,+}_x u^{0,1}+u^{0,2} = \eta^{2,0,+}\quad on\ \Gamma^+_{\eta^{1,0,+}},\qquad -\eta^{1,0,-}_x u^{0,1}+u^{0,2} = \eta^{2,0,-}\quad on\ \Gamma^-_{\eta^{1,0,-}} \tag{6.3}$$

and

$$\|(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})\|_{\mathbf{V}^1_\#(\Omega_{\eta^{1,0}})\times H^3_\#(\Gamma_0)\times H^1_\#(\Gamma_0)}\leq R(r),$$

then system (6.4)

$$\begin{array}{rcll}\mathbf{u}_t-div\,\sigma(\mathbf{u},p)+(\mathbf{u}\cdot\nabla)\mathbf{u} &=& \mathbf{0} & in\ Q^\eta_\infty,\\ div\,\mathbf{u} &=& 0 & in\ Q^\eta_\infty,\\ \mathbf{u}\cdot\widetilde{\mathbf{n}}^+ &=& \eta^+_t & on\ \Sigma^{+,\eta^+}_\infty,\\ \mathbf{u}\cdot\widetilde{\mathbf{n}}^- &=& -\eta^-_t & on\ \Sigma^{-,\eta^-}_\infty,\\ \mathbf{S}(\mathbf{u})\mathbf{n}^+\cdot\mathbf{t}^+ &=& f^+_0 & on\ \Sigma^{+,\eta^+}_\infty,\\ \mathbf{S}(\mathbf{u})\mathbf{n}^-\cdot\mathbf{t}^- &=& 0 & on\ \Sigma^{-,\eta^-}_\infty,\\ \eta^+_{tt}+\alpha\eta^+_{xxxx}-\beta\eta^+_{xx}-\gamma\eta^+_{txx} &=& -\sigma(\mathbf{u},p)\mathbf{n}^+\cdot\mathbf{n}^++f^+ & on\ \Sigma^{+,0}_\infty,\\ \eta^-_{tt}+\alpha\eta^-_{xxxx}-\beta\eta^-_{xx}-\gamma\eta^-_{txx} &=& \sigma(\mathbf{u},p)\mathbf{n}^-\cdot\mathbf{n}^- & on\ \Sigma^{-,0}_\infty,\\ (\mathbf{u}(0),\eta(0),\eta_t(0)) &=& (\mathbf{u}^0,\eta^{1,0},\eta^{2,0}) & \end{array} \tag{6.4}$$

with the feedback laws

$$f_0^+ = -\mathcal{B}_0^* \Pi_0 \left[\left(\mathbf{u}\circ\phi_\eta^{-1}\right)_0^1 \right] \qquad \text{and} \qquad f^+ = -\sum_{|k|\le M_\omega; k\neq 0} \Pi_k^{3,+} \left[\left(\mathbf{P}_k \left[\mathbf{u}\circ\phi_\eta^{-1}\right]_k, \eta_k, \eta_{k,t} \right) \right]$$

admits a unique solution $(\mathbf{u}, p, \eta)$ *such that*

$$\left\| e^{\omega\cdot}\mathbf{u}\circ\phi_\eta^{-1} \right\|_{\mathbf{H}^{2,1}_{\#}(Q^0_\infty)} + \left\| e^{\omega\cdot} p\circ\phi_\eta^{-1} \right\|_{L^2(0,+\infty;\mathcal{H}^1_{\#}(\Omega_0))} \| e^{\omega\cdot}\eta \|_{H^{4,2}_{\#}(\Sigma^{s,0}_\infty)} + \| e^{\omega\cdot}\eta_t \|_{H^{2,1}_{\#}(\Sigma^{s,0}_\infty)} \le r,$$

where $\phi_{\eta(t)}$ *is defined in the next section as the change of variables from* $\Omega_{\eta(t)}$ *into* Ω_0.

Moreover, the feedback laws Π_k *(for k in* $\mathbb{Z}$ *such that* $|k| \le M_\omega$ *and* $k \neq 0$*) and* Π_0 *are obtained as unique solutions of finite dimensional algebraic Riccati equations (see sections 6.5.3 and 6.6.2, especially equations* (6.45) *and* (6.54)*).*

Remark 6.3. *The term* $\left(\mathbf{u}\circ\phi_\eta^{-1}\right)_0^1$ *above means the coefficient of mean value (coefficient number zero) in the Fourier series of the first component of vector* $\left(\mathbf{u}\circ\phi_\eta^{-1}\right)$*. The operator* $\mathbf{P}_k$ *is the Leray projector from* $\mathbf{L}^2(-1,1;\mathbb{C})$ *onto* $\mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})$.

These kind of systems can model blood flows in large vessels (see [21] and references therein). The periodic boundary condition has obviously no physical meaning but leads to interesting mathematic challenges.

The feedback stabilization of the linearized Navier-Stokes equations around a Poiseuille profile in a channel has been studied by different authors, see for instance [2, 3, 29]. In [23], the author proves the feedback stabilization of a coupled system (dealing with the Navier-Stokes equations and a beam equation) for small initial data. The system in [23] is slightly different from the one in this paper. Indeed, the boundary conditions at the interface with the beam are different. In [23], the author considers the boundary condition introduced in [21], namely $\mathbf{u} = \eta_t \mathbf{e}_2$, whereas the boundary conditions in (6.2) are the ones studied in [13].

With this choice of boundary condition and a Fourier decomposition in the periodic variable, the problem of unique continuation for any Fourier modus crucially relies on the explicit formula of the eigenfunctions obtained by solving a linear ordinary differential equation with constant (but depending on the eigenvalue) coefficients (see section 6.5.1 and in particular Proposition 6.8 for details).

This result may be interesting due to the finite dimensional characterization of the controls either for the number of the Fourier coefficients for f^+ or for the finite dimensional algebraic Riccati equations giving the different feedback laws. Both have an interest in numerical simulation of the problem.

The idea to prove this type of result is first to write an equivalent system in a fixed domain thanks to a change of variables. Then, we consider the linearized system around the solution $(\mathbf{0}, 0, 0, 0)$. For this linear system, we prove the stabilization with standard argument. Finally, we use a fixed point method to conclude.

6.4 Change of variables.

Thanks to the specific form of the domain $\Omega_{\eta(t)}$, we can construct a change of variables only depending on $\eta(t) = (\eta^+(t), \eta^-(t))$ as follows:

$$\begin{array}{rccl} \phi_{\eta(t)}: & \Omega_{\eta(t)} & \longrightarrow & \Omega_0 \\ & (x,y) & \longmapsto & (x,z) = \left(x, \dfrac{2y - (\eta^+(t,x) + \eta^-(t,x))}{2 + \eta^+(t,x) - \eta^-(t,x)} \right). \end{array}$$

Then, setting $\hat{f}(x,z) = f(x,y)$, we have the formula:

$$\hat{f}(x,z) = f(x, (1 + d(t,x))z + m(t,x)), \qquad f(x,y) = \hat{f}\left(x, \frac{y - m(t,x)}{1 + d(t,x)} \right)$$

with
$$m(t,x) = \frac{\eta^+(t,x)+\eta^-(t,x)}{2} \qquad \text{and} \qquad d(t,x) = \frac{\eta^+(t,x)-\eta^-(t,x)}{2}.$$
The formula of the different derivatives are
$$f_t = \hat{f}_t - \frac{m_t + zd_t}{1+d}\hat{f}_z, \qquad f_x = \hat{f}_x - \frac{m_x+zd_x}{1+d}\hat{f}_z, \qquad f_y = \frac{\hat{f}_z}{1+d}, \qquad f_{yy} = \frac{\hat{f}_{zz}}{(1+d)^2},$$
$$f_{xx} = \hat{f}_{xx} - \frac{2(m_x+zd_x)}{1+d}\hat{f}_{xz} + \left(\frac{m_x+zd_x}{1+d}\right)^2 \hat{f}_{zz} - \frac{(m_{xx}+zd_{xx})(1+d) - d_x(m_x+zd_x)}{(1+d)^2}\hat{f}_z.$$

Thus,
$$\begin{aligned}
\mathbf{u}_t &= \hat{\mathbf{u}}_t - \frac{m_t+zd_t}{1+d}\hat{\mathbf{u}}_z,\\
\Delta\mathbf{u} &= \hat{\mathbf{u}}_{xx} - \frac{2(m_x+zd_x)}{1+d}\hat{\mathbf{u}}_{xz} + \frac{(m_x+zd_x)^2+1}{(1+d)^2}\hat{\mathbf{u}}_{zz} - \frac{(m_{xx}+zd_{xx})(1+d)-d_x(m_x+zd_x)}{(1+d)^2}\hat{\mathbf{u}}_z,\\
(\mathbf{u}\cdot\nabla)\mathbf{u} &= \hat{u}_1\left(\hat{\mathbf{u}}_x - \frac{m_x+zd_x}{1+d}\hat{\mathbf{u}}_z\right) + \hat{u}_2\frac{\hat{\mathbf{u}}_z}{1+d},\\
\nabla p &= \left(\hat{p}_x - \frac{m_x+zd_x}{1+d}\hat{p}_z\right)\mathbf{e}_1 + \frac{\hat{p}_z}{1+d}\mathbf{e}_2,
\end{aligned}$$

Then, putting all these terms together, mutiplying by $1+d$, we get
$$\begin{aligned}
\hat{\mathbf{u}}_t - \nu\hat{\Delta}\hat{\mathbf{u}} + \hat{\nabla}\hat{p} &= \mathbf{F}[\hat{\mathbf{u}},\hat{p},\eta] && \text{in } Q^0_\infty,\\
\hat{\text{div}}\ \hat{\mathbf{u}} &= g[\hat{\mathbf{u}},\eta] && \text{in } Q^0_\infty,\\
\hat{u}_2 &= \eta^\kappa_t + j^\kappa[\hat{\mathbf{u}},\eta] && \text{on } \Sigma^{\kappa,0}_\infty,\\
\hat{u}_{1,y} + \hat{u}_{2,x} &= l^\kappa[\hat{\mathbf{u}},\eta] + f_0^+\chi_{\Gamma_0^+} && \text{on } \Sigma^{\kappa,0}_\infty,\\
\eta^\kappa_{tt} + \alpha\eta^\kappa_{xxxx} - \beta\eta^\kappa_{xx} - \gamma\eta^\kappa_{txx} &= \kappa[\hat{p} - 2\nu\hat{u}_{2,y}] + H^\kappa[\hat{\mathbf{u}},\eta] + f^+\chi_{\Gamma_0^+} && \text{on } \Sigma^{\kappa,0}_\infty,\\
(\hat{\mathbf{u}}(0),\eta(0),\eta_t(0)) &= (\hat{\mathbf{u}}^0,\eta^{1,0},\eta^{2,0})
\end{aligned}$$

where
$$\begin{aligned}
\mathbf{F}[\hat{\mathbf{u}},\hat{p},\eta] &= -d\hat{\mathbf{u}}_t + \left((m_t+zd_t) - \nu(m_{xx}+zd_{xx}) + \nu\frac{d_x(m_x+zd_x)}{1+d}\right)\hat{\mathbf{u}}_z + \nu\hat{\mathbf{u}}_{xx}\\
&\quad -2\nu(m_x+zd_x)\hat{\mathbf{u}}_{xz} + \nu\frac{(m_x+zd_x)^2-d}{1+d}\hat{\mathbf{u}}_{zz} - (1+d)\hat{u}_1\hat{\mathbf{u}}_x - \hat{u}_2\hat{\mathbf{u}}_z\\
&\quad +(m_x+zd_x)\hat{u}_1\hat{\mathbf{u}}_z - (d\hat{p}_x - (m_x+zd_x)\hat{p}_z)\mathbf{e}_1,\\
g[\hat{\mathbf{u}},\eta] &= -d\hat{u}_{1,x} + (m_x+zd_x)\hat{u}_{1,z}, \qquad \mathbf{g}[\hat{\mathbf{u}},\eta] = -d\hat{u}_1\mathbf{e}_1 + (m_x+zd_x)\hat{u}_1\mathbf{e}_2,\\
j^\kappa[\hat{\mathbf{u}},\eta] &= \eta^\kappa_x\hat{u}_1,\\
l^\kappa[\hat{\mathbf{u}},\eta] &= 2\eta^\kappa_x(1+d)\hat{u}_{1,x} - 2(\eta^\kappa_x)^2\hat{u}_{1,z} - (d\hat{u}_{2,x} + \eta^\kappa_x\hat{u}_{2,z})(1-(\eta^\kappa_x)^2) - 2\eta^\kappa_x\hat{u}_{2,z} + (\eta^\kappa_x)^2(\hat{u}_{1,z}+\hat{u}_{2,x}),\\
H^\kappa[\hat{\mathbf{u}},\eta] &= -\frac{2\nu}{1+(\eta^\kappa_x)^2}\left((\eta^\kappa_x)^2\hat{u}_{1,x} - \eta^\kappa_x\hat{u}_{2,x}\right) + \frac{2\nu\eta^\kappa_x}{1+d}\hat{u}_{1,z} + \frac{2\nu d}{1+d}\hat{u}_{2,z}.
\end{aligned} \tag{6.5}$$

From now on, we drop the notation $\hat{\cdot}$ and we consider the linearized system around $(\mathbf{0},0,0,0)$, that is, we look at the system satisfied by $(\mathbf{u},p,\eta,\eta_t)$:
$$\begin{aligned}
\mathbf{u}_t - \text{div}\ \sigma(\mathbf{u},p) &= \mathbf{F}[\mathbf{u},p,\eta] && \text{in } Q^0_\infty,\\
\text{div}\ \mathbf{u} &= g[\mathbf{u},\eta] = \text{div}\ \mathbf{g}[\mathbf{u},\eta] && \text{in } Q^0_\infty,\\
u_2 &= \eta^\kappa_t + j^\kappa[\mathbf{u},\eta] && \text{on } \Sigma^{\kappa,0}_\infty,\\
u_{1,y} + u_{2,x} &= l^\kappa[\mathbf{u},\eta] + f_0^+\chi_{\Gamma_0^+} && \text{on } \Sigma^{\kappa,0}_\infty,\\
\eta^\kappa_{tt} + \alpha\eta^\kappa_{xxxx} - \beta\eta^\kappa_{xx} - \gamma\eta^\kappa_{txx} &= \kappa[p - 2\nu u_{2,y}] + H^\kappa[\mathbf{u},\eta] + f^+\chi_{\Gamma_0^+} && \text{on } \Sigma^{\kappa,0}_\infty,\\
(\mathbf{u}(0),\eta(0),\eta_t(0)) &= (\mathbf{u}^0,\eta^{1,0},\eta^{2,0}).
\end{aligned} \tag{6.6}$$

System (6.6) is equivalent to system (6.4) in the sens of the following Definition.

Definition 6.4. *The triplet* $(\mathbf{u},p,\eta)$ *in* $\mathbf{H}^{2,1}_\#(Q^\eta_\infty) \times L^2\left(\bigcup_{t\in(0,\infty)}\{t\}\times\mathcal{H}^1_\#\left(\Omega_{\eta(t)}\right)\right) \times H^{4,2}_\#\left(\Sigma^0_\infty\right)$ *is solution of system* (6.4) *when the following conditions are satisfied:*

(i) the triplet $(\hat{\mathbf{u}}, \hat{p}, \eta)$ *obtained by the change of variables* $\hat{\mathbf{u}}(x,z) = \mathbf{u}(x,y)$ *and* $\hat{p}(x,z) = p(x,y)$ *with* $(x,z) = \phi_\eta(x,y)$ *is a solution of* (6.6),

(ii) for any time t *in* $(0,\infty)$, *the previous change of variables* $\phi_{\eta(t)}$ *is a* $\mathcal{C}^1$*-diffeomorphism from* $\Omega_{\eta(t)}$ *into* Ω_0,

(iii) η *satisfies condition* (6.1).

We introduce a decomposition of the velocity $\mathbf{u}$ into $\mathbf{u} = \mathbf{v} + \mathbf{L}[\mathbf{u},\eta]$ and of the pressure $p = q + L_p[\mathbf{u},\eta]$, where $\mathbf{L}[\mathbf{u},\eta] = \mathbf{g}[\mathbf{u},\eta] + \underline{\mathbf{w}}[\mathbf{u},\eta]$ and $L_p[\mathbf{u},\eta] = \underline{\pi}[\mathbf{u},\eta]$ with $\mathbf{g}[\mathbf{u},\eta]$ is the vector field in the nonhomogeneous divergence condition and $(\underline{\mathbf{w}}[\mathbf{u},\eta], \underline{\pi}[\mathbf{u},\eta])$ is the solution of the nonhomogeneous boundary instationary Stokes problem

$$\begin{array}{rcll} \mathbf{w}_t[\mathbf{u},\eta] - \text{div } \sigma(\mathbf{w}[\mathbf{u},\eta], \pi[\mathbf{u},\eta]) &=& \mathbf{0} & \text{in } Q_\infty^\eta, \\ \text{div } \mathbf{w}[\mathbf{u},\eta] &=& 0 & \text{in } Q_\infty^\eta, \\ w_2[\mathbf{u},\eta] &=& 0 & \text{on } \Sigma_\infty^{\kappa,0}, \\ \mathbf{S}(\mathbf{w}[\mathbf{u},\eta])\mathbf{e}_2 \cdot \mathbf{e}_1 &=& l^\kappa[\mathbf{u},\eta] - \mathbf{S}(\mathbf{g}[\mathbf{u},\eta])\mathbf{e}_2 \cdot \mathbf{e}_1 & \text{on } \Sigma_\infty^{\kappa,0}, \\ \mathbf{w}[\mathbf{u},\eta](0) &=& 0 & \text{in } \Omega_0. \end{array}$$

We will see in section 6.8 that the right-hand sides are smooth enough to get the well-definedness of these liftings. Then, we look for solution of system (6.6) under the form $(\mathbf{v}, q, \eta) = (\mathbf{u} - \mathbf{L}[\mathbf{u},\eta], p - L_p[\mathbf{u},\eta], \eta)$. The system satisfied by $(\mathbf{v}, q, \eta)$ is the following

$$\begin{array}{rcll} \mathbf{v}_t - \text{div } \sigma(\mathbf{v}, q) &=& \mathbf{f}[\mathbf{u}, q, \eta] & \text{in } Q_\infty^0, \\ \text{div } \mathbf{v} &=& 0 & \text{in } Q_\infty^0, \\ v_2 &=& \eta_t^\kappa & \text{on } \Sigma_\infty^{\kappa,0}, \\ \mathbf{S}(\mathbf{v})\mathbf{e}_2 \cdot \mathbf{e}_1 &=& f_0^+ \chi_{\Gamma_0^+} & \text{on } \Sigma_\infty^{\kappa,0}, \\ \eta_{tt}^\kappa + \alpha\eta_{xxxx}^\kappa - \beta\eta_{xx}^\kappa - \gamma\eta_{txx}^\kappa &=& \kappa[q - 2\nu v_{2,y}] + h^\kappa[\mathbf{u},\eta] + f^+\chi_{\Gamma_0^+} & \text{on } \Sigma_\infty^{\kappa,0}, \\ (\mathbf{v}(0), \eta(0), \eta_t(0)) &=& (\mathbf{v}^0, \eta^{1,0}, \eta^{2,0}). \end{array} \tag{6.7}$$

where

$$\begin{array}{rcl} \mathbf{f}[\mathbf{u},p,\eta] &=& \mathbf{F}[\mathbf{u},p,\eta] - \mathbf{g}_t[\mathbf{u},\eta] + \nu\Delta\mathbf{g}[\mathbf{u},\eta], \\ h^\kappa[\mathbf{u},\eta] &=& H^\kappa[\mathbf{u},\eta] - 2\nu g_{2,y}[\mathbf{u},\eta](\kappa 1) + \kappa[\pi[\mathbf{u},\eta] - 2\nu w_{2,y}[\mathbf{u},\eta]](\kappa 1), \end{array} \tag{6.8}$$

and

$$\mathbf{v}^0 = \mathbf{u}^0 - \mathbf{g}[\mathbf{u},\eta](0). \tag{6.9}$$

We add compatibility conditions for the initial data to get continuity at time $t = 0$ to the solution of system (6.11)

$$\text{div } \mathbf{v}^0 = 0 \quad \text{in } \Omega_0, \qquad v_2^0 = \eta^{2,0,\kappa} \text{ on } \Gamma_0^\kappa.$$

This can be written in the variables $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ as follows:

$$\text{div } (\mathbf{u}^0 - \mathbf{g}[\mathbf{u},\eta](0)) = 0 \text{ in } \Omega_0, \qquad u_2^0 - g_2[\mathbf{u},\eta](0) = \eta^{2,0,\kappa} \text{ on } \Gamma_0^\kappa. \tag{6.10}$$

In the following, we begin with studying the corresponding linear system to system (6.7). It is obtained by taking all the right-hand sides $(\mathbf{f}[\mathbf{u},p,\eta], h[\mathbf{u},\eta])$ independent of $(\mathbf{u},p,\eta)$. That is, we consider first the following system

$$\begin{array}{rcll} \mathbf{v}_t - \text{div } \sigma(\mathbf{v}, q) &=& \mathbf{f} & \text{in } Q_\infty^0, \\ \text{div } \mathbf{v} &=& 0 & \text{in } Q_\infty^0, \\ v_2 &=& \eta_t^\kappa & \text{on } \Sigma_\infty^0, \\ \mathbf{S}(\mathbf{v})\mathbf{e}_2 \cdot \mathbf{e}_1 &=& f_0^+ \chi_{\Gamma_0^+} & \text{on } \Sigma_\infty^0, \\ \eta_{tt}^\kappa + \alpha\eta_{xxxx}^\kappa - \beta\eta_{xx}^\kappa - \gamma\eta_{txx}^\kappa &=& \kappa[q - 2\nu v_{2,y}] + h^\kappa + f^+\chi_{\Gamma_0^+} & \text{on } \Sigma_\infty^0, \\ (\mathbf{v}(0), \eta(0), \eta_t(0)) &=& (\mathbf{v}^0, \eta^{1,0}, \eta^{2,0}) \end{array} \tag{6.11}$$

with fixed right-hand sides $(\mathbf{f}, h)$ and initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ in certain spaces.

Our first goal is to prove the stabilization for any decay rate $\omega > 0$ of this system with two controls f_0^+ and f^+ (respectively in $L^2(0,+\infty;\mathbb{R})$ and $L^2(\Sigma_{+\infty}^{+,0})$) obtained by several feedback laws. The method we apply here is to split system (6.11) into an infinite number of simplier systemsusing the Fourier decomposition of (6.11). Then, putting the results for all these systems together, we are able to prove the stabilization of system (6.11). Finally, we use the linear feedback laws to stabilize the nonlinear system (6.7) locally around the stationary solution $(\mathbf{0},0,0,0)$.

Using the periodicity in the first variable and following [2] for instance, we decompose the previous system in the Fourier setting. That is, for now on, all the functions will be considered as Fourier series as follows

$$\begin{array}{rclrcl} v^1(t,x,y) &=& \displaystyle\sum_{k\in\mathbb{Z}} v_k^1(t,y)\mathrm{e}^{ikx}, & v^2(t,x,y) &=& \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} v_k^2(t,y)\mathrm{e}^{ikx}, \\ \eta^{1,\kappa}(t,x) &=& \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} \eta_k^{1,\kappa}(t)\mathrm{e}^{ikx}, & \eta^{2,\kappa}(t,x) &=& \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} \eta_k^{2,\kappa}(t)\mathrm{e}^{ikx}, \\ q(t,x,y) &=& \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} q_k(t,y)\mathrm{e}^{ikx}, & f^+(t,x) &=& \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} f_k^+(t)\mathrm{e}^{ikx}. \end{array}$$

This gives directly the two systems (6.12) for $k \in \mathbb{Z}\setminus\{0\}$ and (6.13) for $k = 0$

$$\begin{array}{rcll} \mathbf{v}_{k,t} - \nu\Delta_k\mathbf{v}_k + \nabla_k q_k &=& \mathbf{f}_k & \text{in } (0,\infty)\times(-1,1), \\ ikv_k^1 + v_{k,y}^2 &=& 0 & \text{in } (0,\infty)\times(-1,1), \\ v_k^2 &=& \eta_k^{2,\kappa} & \text{on } (0,\infty)\times\{\kappa 1\}, \\ v_{k,y}^1 + ikv_k^2 &=& 0 & \text{on } (0,\infty)\times\{\kappa 1\}, \\ \eta_{k,t}^{1,\kappa} &=& \eta_k^{2,\kappa}, & \text{on } (0,\infty), \\ \eta_{k,t}^{2,\kappa} + (\alpha k^4 + \beta k^2)\eta_k^{1,\kappa} + \gamma k^2\eta_k^{2,\kappa} &=& \kappa(q_k - 2\nu v_{k,y}^2)(\kappa 1) + h_k^\kappa + f_k^+\chi_+, & \text{on } (0,\infty), \\ (\mathbf{v}_k(0), \eta_k^1(0), \eta_k^2(0)) &=& (\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0}). & \end{array} \tag{6.12}$$

and

$$\begin{array}{rcll} v_{0,t}^1 - \nu\Delta v_0^1 &=& \mathbf{f}_0 & \text{in } (0,\infty)\times(-1,1), \\ v_{0,y}^1 &=& f_0^+\chi_+ & \text{on } (0,\infty)\times\{\kappa 1\}, \\ v_0^1(0) &=& v_0^{0,1} & \text{in } (-1,1). \end{array} \tag{6.13}$$

In the two following sections, we prove the stabilization (for any decay rate $\omega > 0$) of either system (6.12) with a control f_k^+ acting as a force in the right-hand side of the upper beam equation or the stabilization of equation (6.13) with a control f_0^+ acting on the upper Neuman boundary condition.

The method we follow here is adapted from a paper of J.P. Raymond [23] where he considers a slightly different system. Furthermore, the method he uses is applied in this paper for each system (6.12) and (6.13) to get specific controls. In section 6.7, we collect these results to obtain the stabilization for any decay rate of the solution of system (6.11). In the next section, by a fixed point method, we are able to prove the stabilization for any decay rate of the solution of the complete system (6.4) locally around the stationary state $(\mathbf{0},0,0,0)$.

6.5 Stabilization of (6.12).

We begin by considering the homogeneous system

$$\begin{array}{rcll} \mathbf{v}_{k,t} - \nu\Delta_k\mathbf{v}_k + \nabla_k q_k &=& \mathbf{0} & \text{in } (0,\infty)\times(-1,1), \\ ikv_k^1 + v_{k,y}^2 &=& 0 & \text{in } (0,\infty)\times(-1,1), \\ v_k^2 &=& \eta_k^{2,\kappa} & \text{on } (0,\infty)\times\{\kappa 1\}, \\ v_{k,y}^1 + ikv_k^2 &=& 0 & \text{on } (0,\infty)\times\{\kappa 1\}, \\ \eta_{k,t}^{1,\kappa} &=& \eta_k^{2,\kappa}, & \text{on } (0,\infty), \\ \eta_{k,t}^{2,\kappa} + (\alpha k^4 + \beta k^2)\eta_k^{1,\kappa} + \gamma k^2\eta_k^{2,\kappa} &=& \kappa(q_k - 2\nu v_{k,y}^2)(\kappa 1) + f_k^+\chi_{\Gamma_0^+}, & \text{on } (0,\infty), \\ (\mathbf{v}_k(0), \eta_k^1(0), \eta_k^2(0)) &=& (\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0}). & \end{array} \tag{6.14}$$

All the results obtained for (6.14) will be generalized to system (6.12) in the end of this section (see Proposition 6.16). First, using the Leray projection $\mathbf{P}_k$, we can rewrite following [23] system (6.14) as two systems, one evolutionnary system in the variables $(\mathbf{P}_k\mathbf{v}_k, \eta_k^1, \eta_k^2)$ and a stationary system in the variable $(\mathbf{I}_k - \mathbf{P}_k)\mathbf{v}_k$. We introduce $\mathbb{H}_k = \mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C}) \times H^2_{\#,k}(\mathbb{C}^2) \times L^2_{\#,k}(\mathbb{C}^2)$. This space is endowed with the norm $\|\cdot\|_{\mathbb{H}_k}$ defined, for all $(\mathbf{P}_k\mathbf{z}_k, \mu_k^1, \mu_k^2) \in \mathbb{H}_k$ by

$$\|(\mathbf{P}_k\mathbf{z}_k, \mu_k^1, \mu_k^2)\|_{\mathbb{H}_k} = \left(\|\mathbf{P}_k\mathbf{z}_k\|^2_{\mathbf{L}^2(-1,1;\mathbb{C})} + \|\mu_k^1\|^2_{H^2_{\#,k}(\mathbb{C}^2)} + \left\langle \mu_k^2, (\mathbf{I}_2 + \mathbf{N}_k)\mu_k^2\right\rangle_{\mathbb{C}^2}\right)^{1/2}$$

where the matrix $\mathbf{N}_k$ is defined in (6.20) and $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$ is the standard scalar product on $\mathbb{C}^2$.

Proposition 6.5. *System* (6.14) *can be written in the variables* $(\mathbf{P}_k\mathbf{v}_k, (\mathbf{I}_k - \mathbf{P}_k)\mathbf{v}_k, \eta_k^1, \eta_k^2)$ *where* $\eta_k^1 = \eta_k$ *and* $\eta_k^2 = \eta_{k,t}$ *as follows*

$$\begin{array}{rcl} \begin{pmatrix} \mathbf{P}_k\mathbf{v}_k \\ \eta_k^1 \\ \eta_k^2 \end{pmatrix}' & = & \mathcal{A}_k \begin{pmatrix} \mathbf{P}_k\mathbf{v}_k \\ \eta_k^1 \\ \eta_k^2 \end{pmatrix} + \mathcal{B}_k f_k^+, \qquad \begin{pmatrix} \mathbf{P}_k\mathbf{v}_k \\ \eta_k^1 \\ \eta_k^2 \end{pmatrix}(0) = \begin{pmatrix} \mathbf{P}_k\mathbf{v}_k^0 \\ \eta_k^{1,0} \\ \eta_k^{2,0} \end{pmatrix}, \\ (\mathbf{I}_k - \mathbf{P}_k)\mathbf{v}_k & = & \nabla_k N_k\left(\eta_k^2\right), \\ q_k & = & -N_k(\eta_{k,t}^2 + 4\nu k^2\eta_k^2), \end{array} \tag{6.15}$$

where

$$\mathcal{A}_k = \left(\begin{array}{c|cc} \mathbf{I}_2 & 0 & 0 \\ \hline 0 & \mathbf{I}_2 & 0 \\ 0 & 0 & (\mathbf{I}_2 + \mathbf{N}_k)^{-1} \end{array}\right)\left(\begin{array}{c|cc} \mathbf{A}_k & \mathbf{0} & (-\mathbf{A}_k)\mathbf{P}_k\mathbf{D}_k \\ \hline 0 & 0 & \mathbf{I}_2 \\ \mathbf{S}_k & \mathbf{M}_k^1 & \mathbf{M}_k^2 \end{array}\right) \quad \text{and} \quad \mathcal{B}_k f_k^+ = \left(\begin{array}{c} 0 \\ \hline 0 \\ (\mathbf{I}_2 + \mathbf{N}_k)^{-1}\begin{pmatrix} f_k^+ \\ 0 \end{pmatrix} \end{array}\right).$$

The operator $\mathcal{A}_k$ *is an operator with domain* $D(\mathcal{A}_k)$ *in* $\mathbb{H}_k$ *defined by*

$$D(\mathcal{A}_k) = \left\{ \left(\mathbf{P}_k\mathbf{z}_k, \mu_k^1, \mu_k^2\right) \in \mathbf{V}^2_{\#,k,\mathbf{n}}(-1,1;\mathbb{C}) \times H^4_{\#,k}(\mathbb{C}^2) \times H^2_{\#,k}(\mathbb{C}^2) \ \text{s.t.}\ \mathbf{P}_k\mathbf{z}_k - \mathbf{P}_k\mathbf{D}_k\mu_k^2 \in D(\mathbf{A}_k)\right\}$$

whereas the operator $\mathcal{B}_k$ *is a bounded linear operator from* $\mathbb{C}$ *in* $\mathbb{H}_k$. *The operators* $\mathbf{D}_k$, $\mathbf{N}_k$ *and* $\mathbf{S}_k$ *are defined respectively in* (6.16) *and in* (6.20). *The operator* $\mathbf{A}_k$ *is the Stokes operator defined in* (6.18). *The matrices* $\mathbf{M}_k^1$ *and* $\mathbf{M}_k^2$ *are defined in* (6.22).

Proof. We begin by splitting the Stokes system written in the variable $\mathbf{v}_k = (v_k^1, v_k^2)$ into two parts, using the Leray projection $\mathbf{P}_k$ from $\mathbf{L}^2(-1,1;\mathbb{C})$ into $\mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})$. We introduce a lifting of the nonhomogenous Dirichlet boundary condition for the velocity (v_k^1, v_k^2). Namely, we denote $\mathbf{w}_k = \mathbf{D}_k\eta_k^2$ the function solution of the following system

$$\begin{array}{rcll} -\nu\Delta_k\mathbf{w}_k + \nabla_k\rho_k & = & 0 & \text{in } (-1,1), \\ ikw_k^1 + w_{k,y}^2 & = & 0 & \text{in } (-1,1), \\ w_k^2 & = & \eta_k^{2,\kappa} & \text{on } \{\kappa 1\}, \\ w_{k,y}^1 + ikw_k^2 & = & 0 & \text{on } \{\kappa 1\}, \end{array} \tag{6.16}$$

Then, the velocity and the pressure can be search under the form $\mathbf{v}_k = \widetilde{\mathbf{v}}_k + \mathbf{w}_k$ and $q_k = \widetilde{q}_k + \rho_k$. Taking the divergence and the normal trace of the first equation, we obtain the equation satisfied by ρ_k:

$$\rho_{k,yy} - k^2\rho_k = 0 \quad \text{in } (-1,1), \qquad \rho_{k,y} = -2\nu k^2\eta_k^{2,\kappa} \text{ on } \kappa 1.$$

We introduce the two operators denoted N_k^+ and N_k^-. They are defined by $m_k^+ = N_k^+ g^+$ and $m_k^- = N_k^- g^-$ (for g_k^+ and g_k^- in $\mathbb{C}$) iff

$$m_{k,yy}^+(y) - k^2 m_k^+(y) = 0 \quad \text{for } y \in (-1,1), \qquad m_{k,y}^+(y) = g^+ \text{ for } y = 1, \qquad m_{k,y}^+(y) = 0 \text{ for } y = -1$$

and

$$m^-_{k,yy}(y) - k^2 m^-_k(y) = 0 \quad \text{for } y \in (-1,1), \qquad m^-_{k,y}(y) = 0 \text{ for } y = 1, \qquad m^-_{k,y}(y) = g^-_k \text{ for } y = -1.$$

Thus, introducing the operator N_k from $\mathbb{C}^2$ into $H^1_{\#,k}(-1,1;\mathbb{C})$ defined for $g = (g^+, g^-)$ in $\mathbb{C}^2$ by $m_k = N_k g$ iff $m_k = N^+_k g^+ + N^-_k g^-$, we have $\rho_k = -2\nu k^2 N_k \eta^{2,\kappa}_k$.

We can write the system in the variable $(\widetilde{\mathbf{v}}_k, \eta^1_k, \eta^2_k)$:

$$\begin{array}{rcll}
\widetilde{\mathbf{v}}_{k,t} - \nu\Delta_k \widetilde{\mathbf{v}}_k + \nabla_k q_k & = & -\mathbf{w}_{k,t} & \text{in } (0,\infty)\times(-1,1),\\
ik\widetilde{v}^1_k + \widetilde{v}^2_{k,y} & = & 0 & \text{in } (0,\infty)\times(-1,1),\\
\widetilde{v}^2_k & = & 0 & \text{on } (0,\infty)\times\{\kappa 1\},\\
\widetilde{v}^1_{k,y} + ik\widetilde{v}^2_k & = & 0 & \text{on } (0,\infty)\times\{\kappa 1\},\\
\eta^{1,\kappa}_{k,t} & = & \eta^{2,\kappa}_k & \text{on } (0,\infty),\\
\eta^{2,\kappa}_{k,t} + (\alpha k^4 + \beta k^2)\eta^{1,\kappa}_k + \gamma k^2 \eta^{2,\kappa}_k & = & \kappa(q_k - 2\nu \widetilde{v}^2_{k,y}) + \kappa(\rho_k - 2\nu w^2_{k,y}), & \text{on } (0,\infty),\\
(\widetilde{\mathbf{v}}_k(0), \eta^1_k(0), \eta^2_k(0)) & = & (\widetilde{\mathbf{v}}^0_k, \eta^{1,0}_k, \eta^{2,0}_k) &
\end{array} \tag{6.17}$$

We see, thanks to equation $(6.17)_2$ and $(6.17)_3$, that $\widetilde{\mathbf{v}}_k$ already belongs to $\mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})$. We introduce the Stokes operator $\mathbf{A}_k$ with domain

$$D(\mathbf{A}_k) = \Big\{ \mathbf{z}_k \in \mathbf{V}^2_{\#,k}(-1,1;\mathbb{C}) \text{ s.t. } z^2_k(y) = 0 \text{ and } z^1_{k,y}(y) = 0 \text{ for } y = \kappa 1 \Big\}$$

in $\mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})$ defined by

$$\mathbf{A}_k \mathbf{z}_k = \nu\Delta_k \mathbf{z}_k = \begin{pmatrix} \nu z^1_{k,yy} - \nu k^2 z^1_k \\ \nu z^2_{k,yy} - \nu k^2 z^2_k \end{pmatrix} \text{ for all } \mathbf{z}_k \in D(\mathbf{A}_k). \tag{6.18}$$

On the other hand, the right-hand side becomes $-\mathbf{P}_k \mathbf{w}_{k,t}$ in $\mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})$. Thus, the Stokes equation becomes

$$\widetilde{\mathbf{v}}_{k,t}(t) = \mathbf{A}_k \widetilde{\mathbf{v}}_k(t) - \mathbf{P}_k \mathbf{w}_{k,t}(t), \qquad \widetilde{\mathbf{v}}_k(0) = \mathbf{v}^0_k.$$

Using an integration by parts from the Duhamel formula, we get

$$\begin{aligned}
\widetilde{\mathbf{v}}_k(t) &= e^{t\mathbf{A}_k}\widetilde{\mathbf{v}}^0_k + \int_0^t e^{(t-\tau)\mathbf{A}_k}[-\mathbf{P}_k \mathbf{w}_{k,t}](\tau)\mathrm{d}\tau,\\
&= e^{t\mathbf{A}_k}\widetilde{\mathbf{v}}^0_k - \int_0^t e^{(t-\tau)\mathbf{A}_k}(-\mathbf{A}_k)[-\mathbf{P}_k \mathbf{w}_k](\tau)\mathrm{d}\tau + \Big[e^{(t-\tau)\mathbf{A}_k}[-\mathbf{P}_k \mathbf{w}_k](\tau) \Big]_{\tau=0}^{\tau=t}
\end{aligned}$$

which gives

$$\widetilde{\mathbf{v}}_k(t) + \mathbf{P}_k \mathbf{w}_k(t) = e^{t\mathbf{A}_k}[\widetilde{\mathbf{v}}^0_k + \mathbf{P}_k \mathbf{w}_k(0)] + \int_0^t e^{(t-s)\mathbf{A}_k}(-\mathbf{A}_k)\mathbf{P}_k \mathbf{w}_k(\tau)\mathrm{d}\tau.$$

That is, $\mathbf{P}_k \mathbf{v}_k = \widetilde{\mathbf{v}}_k + \mathbf{P}_k \mathbf{w}_k$ is solution of

$$\mathbf{P}_k \mathbf{v}_{k,t}(t) = \mathbf{A}_k \mathbf{P}_k \mathbf{v}_k(t) + (-\mathbf{A}_k)\mathbf{P}_k \mathbf{w}_k(t), \qquad \mathbf{P}_k \mathbf{v}_k(0) = \mathbf{P}_k \mathbf{v}^0_k. \tag{6.19}$$

In the equation above, $\mathbf{A}_k$ has to be understood as the extended operator with domain $\mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})$ in $(D(\mathbf{A}^*_k))'$ (still denoted $\mathbf{A}_k$) obtained by transposition from the operator $\mathbf{A}_k$.

The right-hand side of the beam equation has to be written in terms of $(\mathbf{P}_k \mathbf{v}_k, \eta^1_k, \eta^2_k)$ too. A simple calculation gives that $q_k = -N_k(\eta^2_{k,t} + 2\nu k^2 \eta^2_k)$. The term $-2\nu v^2_{k,y}$ can be split into

$$-2\nu v^2_{k,y} = -2\nu[(\mathbf{P}_k \mathbf{v}_k)_{2,y}] - 2\nu[(\mathbf{I}_k - \mathbf{P}_k)\mathbf{v}_k]_{2,y}.$$

But $(\mathbf{I}_k - \mathbf{P}_k)\mathbf{v}_k = \nabla_k \phi_k$ with ϕ_k solution of

$$\phi_{k,yy}(y) - k^2\phi_k(y) = 0 \quad \text{for all } y \in (-1,1), \qquad \phi_{k,y}(y) = \eta^{2,\kappa}_k \text{ for } y = \kappa 1.$$

Thus, $[(\mathbf{I}_k - \mathbf{P}_k)\mathbf{v}_k]_{2,y} = \phi_{k,yy} = k^2\phi_k = k^2 N_k \eta_k^2$ which gives finally

$$q_k - 2\nu v_{k,y}^2 = -2\nu[(\mathbf{P}_k\mathbf{v}_k)_{2,y}] - N_k(\eta_{k,t}^2 + 4\nu k^2 \eta_k^2)$$

or using the definition of N_k:

$$q_k - 2\nu v_{k,y}^2 = -2\nu[(\mathbf{P}_k\mathbf{v}_k)_{2,y}] - N_k^+(\eta_{k,t}^{2,+} + 4\nu k^2 \eta_k^{2,+}) - N_k^-(\eta_{k,t}^{2,-} + 4\nu k^2 \eta_k^{2,-}).$$

Finally, we denote γ_κ the evaluation in $y = \kappa 1$ of a smooth enough function. Then, the right-hand sides of the beam equation are

$$\begin{array}{rcl} \gamma_+(q_k - 2\nu v_{k,y}^2) & = & -2\nu\gamma_+[(\mathbf{P}_k\mathbf{v}_k)_{2,y}] - \gamma_+ N_k^+(\eta_{k,t}^{2,+} + 4\nu k^2 \eta_k^{2,+}) - \gamma_+ N_k^-(\eta_{k,t}^{2,-} + 4\nu k^2 \eta_k^{2,-}), \\ -\gamma_-(q_k - 2\nu u_{k,y}^2) & = & +2\nu\gamma_-[(\mathbf{P}_k\mathbf{v}_k)_{2,y}] + \gamma_- N_k^+(\eta_{k,t}^{2,+} + 4\nu k^2 \eta_k^{2,+}) + \gamma_- N_k^-(\eta_{k,t}^{2,-} + 4\nu k^2 \eta_k^{2,-}). \end{array}$$

We sum up all these calculations by defining two new operators $\mathbf{N}_k$ from $\mathbb{C}^2$ into $\mathbb{C}^2$ and $\mathbf{S}_k$ from $\mathbf{V}^2_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})$ into $\mathbb{C}^2 \times \mathbb{C}^2$ by

$$\mathbf{N}_k = \begin{pmatrix} \gamma_+ N_k^+ & \gamma_+ N_k^- \\ -\gamma_- N_k^+ & -\gamma_- N_k^- \end{pmatrix}, \qquad \mathbf{S}_k(\mathbf{P}_k\mathbf{v}_k) = \left(\begin{array}{c} 0 \\ 0 \\ \hline -2\nu\gamma_+[(\mathbf{P}_k\mathbf{v}_k)_{2,y}] \\ +2\nu\gamma_-[(\mathbf{P}_k\mathbf{v}_k)_{2,y}] \end{array} \right). \tag{6.20}$$

We now can write the beam equations as two first order in time equations as follows

$$\left(\begin{array}{c|c} \mathbf{I}_2 & 0 \\ \hline 0 & \mathbf{I}_2 + \mathbf{N}_k \end{array} \right) \left(\begin{array}{c} \eta_k^1 \\ \hline \eta_k^2 \end{array} \right)' = \left(\begin{array}{c|c} 0 & \mathbf{I}_2 \\ \hline \mathbf{M}_k^1 & \mathbf{M}_k^2 \end{array} \right) \left(\begin{array}{c} \eta_k^1 \\ \hline \eta_k^2 \end{array} \right) + \mathbf{S}_k(\mathbf{P}_k\mathbf{v}_k) \tag{6.21}$$

with

$$\mathbf{M}_k^1 = -(\alpha k^4 + \beta k^2)\mathbf{I}_2 \qquad \text{and} \qquad \mathbf{M}_k^2 = -\gamma k^2 \mathbf{I}_2 - 4\nu k^2 \mathbf{N}_k. \tag{6.22}$$

Finally, equations (6.19) and (6.21) gives system (6.15).

□

To construct the operator $\mathcal{A}_k$ from equation (6.21), we used the following lemma:

Lemma 6.6. *The matrix $\mathbf{N}_k$ is symmetric positive and thus $\mathbf{I}_2 + \mathbf{N}_k$ is an element of $\mathcal{GL}_2(\mathbb{C})$.*

Proof. Let $a_k = (a_k^+, a_k^-)$ and $b_k = (b_k^-, b_k^+)$ be two elements of $\mathbb{C}^2$ and r_k, s_k two functions defined by $r_k = N_k a_k$ and $s_k = N_k b_k$. Then, first

$$\int_{-1}^{1} (r_{k,yy}(y) - k^2 r_k(y))\overline{r_k(y)}\mathrm{d}y = -\int_{-1}^{1} \left(|r_{k,y}(y)|^2 + k^2 |r_k(y)|^2 \right) \mathrm{d}y + \left[r_{k,y}(y)\overline{r_k(y)} \right]_{y=-1}^{y=1}.$$

By definition of N_k, we have $r_{k,yy} - k^2 r_k = 0$ in $(-1,1)$ and $r_{k,y}(\kappa 1) = a_k^\kappa$, then

$$0 \le \int_{-1}^{1} |\nabla_k r_k(y)|^2_{\mathbb{C}^2} = a_k^+ \overline{\gamma_+ N_k a_k} - a_k^- \overline{\gamma_- N_k a_k} = \left\langle a_k, \begin{pmatrix} \gamma_+ N_k a_k \\ -\gamma_- N_k a_k \end{pmatrix} \right\rangle_{\mathbb{C}^2} = \langle a_k, \mathbf{N}_k a_k \rangle_{\mathbb{C}^2}.$$

Thus, $\langle a_k, \mathbf{N}_k a_k \rangle_{\mathbb{C}^2} \ge 0$. Second,

$$\int_{-1}^{1} (r_{k,yy}(y) - k^2 r_k(y))\overline{s_k(y)}\mathrm{d}y = \int_{-1}^{1} r_k(y)\overline{(s_{k,yy}(y) - k^2 s_k(y))}\mathrm{d}y + \left[r_k(y)\overline{s_{k,y}(y)} - r_{k,y}(y)\overline{s_k(y)} \right]_{y=-1}^{y=1},$$

which gives directly

$$a_k^+ \overline{\gamma_+ N_k b_k} - a_k^- \overline{\gamma_- N_k b_k} = \gamma_+ N_k a_k \overline{b_k^+} - \gamma_- N_k a_k \overline{b_k^-}.$$

That is the symmetry:

$$\langle a_k, \mathbf{N}_k b_k \rangle_{\mathbb{C}^2} = \langle \mathbf{N}_k a_k, b_k \rangle_{\mathbb{C}^2}.$$

□

6.5.1 Study of the operator $\mathcal{A}_k$.

We have some property for the operator $\mathcal{A}_k$ above.

Proposition 6.7. *The operator $\mathcal{A}_k$ generates an analytic semigroup on $\mathbb{H}_k$. For each λ in the resolvent set $\rho(\mathcal{A}_k)$ of $\mathcal{A}_k$, the operator $(\lambda\mathcal{I}_k - \mathcal{A}_k)^{-1}$ is compact. Morevover, we have, for $\omega > 0$*

$$\sigma(\mathcal{A}_k) \subset \Big\{\delta \in \mathbb{C} \ s.t. \ \mathfrak{Re}(\delta) \leq -\omega\Big\}, \qquad \forall |k| \geq M_\omega$$

where $M_\omega = \max\left\{\sqrt{\frac{\omega}{\nu}}, \sqrt{\frac{\omega}{\gamma}}\right\}$.

Proof. Step 1. Analytic semigroup. The idea is to adapt the proof in [23]. First, we write the operator $\mathcal{A}_k$ as a sum of two operators $\mathcal{A}_k^0$ and $\mathcal{A}_k^1$ and second we prove that the first one generates an analytic semigroup on $\mathbb{H}_k$ and the second one is $\mathcal{A}_k^0$-bounded with relative bound zero.

We define

$$\mathcal{A}_k^0 = \left(\begin{array}{c|cc} \mathbf{A}_k & 0 & (-\mathbf{A}_k)\mathbf{P}_k\mathbf{D}_k \\ \hline 0 & 0 & \mathbf{I}_2 \\ 0 & \mathbf{M}_k^1 & \mathbf{M}_k^2 \end{array}\right) \qquad \text{and} \qquad \mathcal{A}_k^1 = \left(\begin{array}{c|cc} \mathbf{A}_k & 0 & (-\mathbf{A}_k)\mathbf{P}_k\mathbf{D}_k \\ \hline 0 & 0 & \mathbf{I}_2 \\ (\mathbf{I}_2+\mathbf{N}_k)^{-1}\mathbf{S}_k & \mathbf{K}_k\mathbf{M}_k^1 & \mathbf{K}_k\mathbf{M}_k^2 \end{array}\right)$$

where $\mathbf{K}_k = (\mathbf{I}_2 + \mathbf{N}_k)^{-1} - \mathbf{I}_2$. Then, we can adpat the proof in [23], in particular Theorems 3.6 and 3.7.
Step 2. Compact resolvent. We consider the following stationary problem:

$$\begin{array}{rcll} \lambda\mathbf{v}_k - \nu\Delta_k\mathbf{v}_k + \nabla_k q_k &=& \mathbf{a}_k & y \in (-1,1), \\ ikv_k^1 + v_{k,y}^2 &=& 0 & y \in (-1,1), \\ v_k^2 &=& \eta_k^{2,\kappa} & y = \kappa 1, \\ v_{k,y}^1 + ikv_k^2 &=& 0 & y = \kappa 1, \\ \lambda\eta_k^{1,\kappa} &=& \eta_k^{2,\kappa} + \mu_k^{1,\kappa} & y = \kappa 1, \\ \lambda\eta_k^{2,\kappa} + (\alpha k^4 + \beta k^2)\eta_k^{1,\kappa} + \gamma k^2\eta_k^{2,\kappa} &=& \kappa(q_k - 2\nu v_{k,y}^2) + \mu_k^{2,\kappa} & y = \kappa 1. \end{array}$$

for $\lambda > 0$ large enough and $(\mathbf{a}_k, \mu_k^1, \mu_k^2)$ in $\mathbb{H}_k$. This system is equivalent to

$$\begin{array}{rcll} \lambda\mathbf{v}_k - \nu\Delta_k\mathbf{v}_k + \nabla_k q_k &=& \mathbf{a}_k & y \in (-1,1), \\ ikv_k^1 + v_{k,y}^2 &=& 0 & y \in (-1,1), \\ v_k^2 &=& \lambda\eta_k^{1,\kappa} - \mu_k^{1,\kappa} & y = \kappa 1, \\ v_{k,y}^1 + ikv_k^2 &=& 0 & y = \kappa 1, \\ \lambda\eta_k^{1,\kappa} &=& \eta_k^{2,\kappa} + \mu_k^{1,\kappa} & y = \kappa 1, \\ \Big(\lambda^2 + \gamma k^2\lambda + \alpha k^4 + \beta k^2\Big)\eta_k^{1,\kappa} &=& \kappa(q_k - 2\nu v_{k,y}^2) + (\lambda + \gamma k^2)\mu_k^{1,\kappa} + \mu_k^{2,\kappa} & y = \kappa 1. \end{array} \tag{6.23}$$

We adapt Section 3.4 in [23] here. For $\lambda > 0$ large enough, the coefficient $\lambda^2 + \gamma k^2\lambda + (\alpha k^4 + \beta k^2)$ is invertible. Then, we can rewrite system (6.23) as follows:

$$\begin{array}{rcll} \lambda\mathbf{v}_k - \nu\Delta_k\mathbf{v}_k + \nabla_k q_k &=& \mathbf{a}_k & y \in (-1,1), \\ ikv_k^1 + v_{k,y}^2 &=& 0 & y \in (-1,1), \\ v_k^2 &=& \lambda\dfrac{\kappa(q_k - 2\nu v_{k,y}^2) + (\lambda + \gamma k^2)\mu_k^{1,\kappa} + \mu_k^{2,\kappa}}{\lambda^2 + \gamma k^2\lambda + \alpha k^4 + \beta k^2} - \mu_k^{1,\kappa} & y = \kappa 1, \\ v_{k,y}^1 + ikv_k^2 &=& 0 & y = \kappa 1, \\ \lambda\eta_k^{1,\kappa} &=& \eta_k^{2,\kappa} + \mu_k^{1,\kappa} & y = \kappa 1, \\ \Big(\lambda^2 + \gamma k^2\lambda + \alpha k^4 + \beta k^2\Big)\eta_k^{1,\kappa} &=& \kappa(q_k - 2\nu v_{k,y}^2) + (\lambda + \gamma k^2)\mu_k^{1,\kappa} + \mu_k^{2,\kappa} & y = \kappa 1. \end{array}$$

From now on, we only consider the Stokes system

$$\begin{array}{rcll} \lambda \mathbf{v}_k - \nu\Delta_k \mathbf{v}_k + \nabla_k q_k &=& \mathbf{a}_k & y\in(-1,1),\\ ikv_k^1 + v_{k,y}^2 &=& 0 & y\in(-1,1),\\ v_k^2 &=& \lambda\dfrac{\kappa(q_k - 2\nu v_{k,y}^2)}{\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2} + \mu_k^\kappa & y=\kappa 1,\\ v_{k,y}^1 + ikv_k^2 &=& 0 & y=\kappa 1 \end{array} \tag{6.24}$$

where

$$\mu_k^\kappa = \lambda\frac{(\lambda+\gamma k^2)\mu_k^{1,\kappa}+\mu_k^{2,\kappa}}{\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2} - \mu_k^{1,\kappa}. \tag{6.25}$$

Let $\mathbf{w}_k$ be in the Hilbert space

$$\mathbf{E}_k = \Big\{\mathbf{z}_k = (z_k^1, z_k^2)\in \mathbf{V}^1_{\#,k}(-1,1;\mathbb{C}) \text{ s.t. } z_{k,y}^1(y) + ikz_k^2(y) = 0 \text{ for } y=\kappa 1\Big\}$$

endowed with the norm

$$\|\mathbf{z}_k\|_{\mathbf{E}_k} = \left(\|\mathbf{z}_k\|^2_{\mathbf{V}^1_{\#,k}(-1,1;\mathbb{C})} + \frac{\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2}{\lambda}\Big(|z_k^2(1)|^2 + |z_k^2(-1)|^2\Big)\right)^{1/2}.$$

We multiply scalarly the first equation in (6.24) by $\mathbf{w}_k$ to obtain:

$$\begin{array}{l} \displaystyle\int_{-1}^1 \Big(\lambda\mathbf{v}_k\cdot\mathbf{w}_k + \nu\nabla_k\mathbf{v}_k : \nabla_k\mathbf{w}_k\Big)\mathrm{d}y + \Big[(q_k(y) - 2\nu v_{k,y}^2(y))\overline{w_k^2(y)}\Big]_{y=-1}^{y=1}\\ \qquad\qquad +2\nu\mathfrak{Re}\Big[v_{k,y}^2(y)\overline{w_k^2(y)}\Big]_{y=-1}^{y=1} \quad = \displaystyle\int_{-1}^1 \mathbf{a}_k\cdot\mathbf{w}_k\mathrm{d}y. \end{array}$$

In the previous equality the term $\nabla_k\mathbf{v}_k : \nabla_k\mathbf{w}_k$ means:

$$\begin{array}{rcl} \nabla_k\mathbf{v}_k : \nabla_k\mathbf{w}_k &=& \begin{pmatrix} ikv_k^1 & v_{k,y}^1\\ ikv_k^2 & v_{k,y}^2\end{pmatrix} : \begin{pmatrix} \overline{ikw_k^1} & \overline{w_{k,y}^1}\\ \overline{ikw_k^2} & \overline{w_{k,y}^2}\end{pmatrix}\\ &=& \displaystyle\sum_{j=1,2} (ikv_k^j, ikw_k^j)_{L^2(-1,1;\mathbb{C})} + (v_{k,y}^j, w_{k,y}^j)_{L^2(-1,1;\mathbb{C})}\\ &=& \displaystyle\sum_{j=1,2} k^2(v_k^j, w_k^j)_{L^2(-1,1;\mathbb{C})} + (v_{k,y}^j, w_{k,y}^j)_{L^2(-1,1;\mathbb{C})}. \end{array}$$

The boundary terms, namely $(q_k(y) - 2\nu v_{k,y}^2(y))\overline{w_k^2(y)}$ for $y=\kappa 1$, can be remplaced by the value of $(q_k(y) - 2\nu v_{k,y}^2(y))$ on the boundary. Indeed, remember that $v_k^2 = \lambda\frac{\kappa(q_k-2\nu v_{k,y}^2)}{\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2} + \mu_k^\kappa$ for $y=\kappa 1$, then we have

$$\begin{array}{rl} & \displaystyle\int_{-1}^1 \Big(\lambda\mathbf{v}_k\cdot\mathbf{w}_k + \nu\nabla_k\mathbf{v}_k : \nabla_k\mathbf{w}_k\Big)\mathrm{d}y + 2\nu\mathfrak{Re}\Big[v_{k,y}^2(y)\overline{w_k^2(y)}\Big]_{y=-1}^{y=1}\\ & +\dfrac{\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2}{\lambda}\Big(v_k^2(1)\overline{w_k^2(1)} + v_k^2(-1)\overline{w_k^2(-1)}\Big)\\ = & \displaystyle\int_{-1}^1 \mathbf{a}_k\cdot\mathbf{w}_k\mathrm{d}y + \frac{\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2}{\lambda}\Big(\mu_k^+\overline{w_k^2(1)} + \mu_k^-\overline{w_k^2(-1)}\Big). \end{array} \tag{6.26}$$

Now, we set

$$\begin{array}{rcl} a_k(\mathbf{v}_k,\mathbf{w}_k) &=& \mathfrak{Re}\displaystyle\int_{-1}^1 \Big(\lambda\mathbf{v}_k\cdot\mathbf{w}_k + \nu\nabla_k\mathbf{v}_k : \nabla_k\mathbf{w}_k\Big) + 2\nu\mathfrak{Re}\Big[v_{k,y}^2(y)\overline{w_k^2(y)}\Big]_{y=-1}^{y=1}\\ && +\dfrac{\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2}{\lambda}\Big(v_k^2(1)\overline{w_k^2(1)} + v_k^2(-1)\overline{w_k^2(-1)}\Big) \end{array}$$

and

$$l_k(\mathbf{w}_k) = \mathfrak{Re}\int_{-1}^1 \mathbf{a}_k\cdot\mathbf{w}_k + \frac{\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2}{\lambda}\Big(\mu_k^+\overline{w_k^2(1)} + \mu_k^-\overline{w_k^2(-1)}\Big).$$

Taking the real part in (6.26), we get that system (6.24) is equivalent to

$$\begin{array}{rcll} a_k(\mathbf{v}_k,\mathbf{w}_k) &=& l_k(\mathbf{w}_k) & \text{for all } \mathbf{w}_k\in\mathbf{E}_k,\\ \lambda q_k(\kappa 1) &=& 2\nu\lambda v^2_{k,y}(\kappa 1)+\kappa(\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2)(v^2_k(\kappa 1)-\mu^\kappa_k) & \text{for } y=\kappa 1 \end{array}$$

(remember that μ^κ_k is defined from $(\mu^{1,\kappa}_k,\mu^{2,\kappa}_k)$ in (6.25)). Then, using the Lax-Milgram Theorem, we prove that the variational problem

$$\text{Find } \mathbf{v}_k\in\mathbf{E}_k \text{ s.t. } a_k(\mathbf{v}_k,\mathbf{w}_k)=l_k(\mathbf{w}_k) \quad \text{for all } \mathbf{w}_k\in\mathbf{E}_k$$

has a unique solution. This solution satisfies the estimate

$$\|\mathbf{v}_k\|_{\mathbf{E}_k}\le C\left(\|\mathbf{a}_k\|_{\mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})}+(\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2)\Big(|\mu^+_k|+|\mu^-_k|\Big)\right).$$

Remember that

$$\mu^\kappa_k=\lambda\frac{(\lambda+\gamma k^2)\mu^{1,\kappa}_k+\mu^{2,\kappa}_k}{\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2}-\mu^{1,\kappa}_k,$$

then,

$$(\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2)\Big(|\mu^+_k|+|\mu^-_k|\Big) \quad\le\quad C\left((\alpha k^4+\beta k^2)\Big(|\mu^{1,+}_k|+|\mu^{1,-}_k|\Big)+\Big(|\mu^{2,+}_k|+|\mu^{2,-}_k|\Big)\right).$$

Thus,

$$\|\mathbf{v}_k\|_{\mathbf{E}_k}\le C\Big(\|\mathbf{a}_k\|_{\mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})}+\|\mu^1_k\|_{H^2_{\#,k}(\mathbb{C}^2)}+\|\mu^2_k\|_{L^2_{\#,k}(\mathbb{C}^2)}\Big).$$

Finally, a regularity argument gives that the weak solution $\mathbf{v}_k$ in $\mathbf{E}_k$ belongs in fact to $\mathbf{V}^2_{\#,k}(-1,1;\mathbb{C})\cap\mathbf{E}_k$. Thus, from the equations satisfied by η^1_k and η^2_k, we get the estimate

$$\|\mathbf{v}_k\|_{\mathbf{V}^2_{\#,k}(-1,1;\mathbb{C})}+\|\eta^1_k\|_{H^4_{\#,k}(\mathbb{C}^2)}+\|\eta^2_k\|_{H^2_{\#,k}(\mathbb{C}^2)}\le C\Big(\|\mathbf{a}_k\|_{\mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})}+\|\mu^1_k\|_{H^2_{\#,k}(\mathbb{C}^2)}+\|\mu^2_k\|_{L^2_{\#,k}(\mathbb{C}^2)}\Big),$$

that is, the resolvent of $\mathcal{A}_k$ is compact in $\mathbf{V}^0_{\#,k,\mathbf{n}}(-1,1;\mathbb{C})\times\mathbb{C}^2\times\mathbb{C}^2$.

Step 3. Estimate of the eigenvalues. Let us consider the following eigenvalue/eigenfunction problem:

$$\begin{array}{rcll} \lambda\mathbf{v}_k-\nu\Delta_k\mathbf{v}_k+\nabla_k q_k &=& 0 & \text{for } y\in(-1,1),\\ ikv^1_k+v^2_{k,y} &=& 0 & \text{for } y\in(-1,1),\\ v^2_k &=& \eta^{2,\kappa}_k & \text{for } y=\kappa 1,\\ v^1_{k,y}+ikv^2_k &=& 0 & \text{for } y=\kappa 1,\\ \lambda\eta^{1,\kappa}_k &=& \eta^{2,\kappa}_k & \\ \lambda\eta^{2,\kappa}_k+(\alpha k^4+\beta k^2)\eta^{1,\kappa}_k+\gamma k^2\eta^{2,\kappa}_k &=& \kappa(q_k-2\nu v^2_{k,y})(\kappa 1). & \end{array} \tag{6.27}$$

The same calculation as in *Step 2.* gives, with $v^2_k(\kappa 1)=\lambda\eta^{1,\kappa}_k$,

$$\begin{array}{c} \displaystyle\int_{-1}^1\Big(\lambda\mathbf{v}_k\cdot\mathbf{v}_k+\nu\nabla_k\mathbf{v}_k:\nabla_k\mathbf{v}_k\Big)\mathrm{d}y+2\nu\mathfrak{Re}\Big[v^2_{k,y}(y)\overline{v^2_k(y)}\Big]_{y=-1}^{y=1}\\ \displaystyle+\frac{\lambda^2+\gamma k^2\lambda+\alpha k^4+\beta k^2}{\lambda}\Big(|\eta^{2,+}_k|^2+|\eta^{2,-}_k|^2\Big) \quad=\quad 0. \end{array}$$

Thus, taking the real part and using a Young inequality, we get that

$$\begin{array}{c} \Big(\mathfrak{Re}(\lambda)+\nu k^2\Big)\Big(\|v^1_k\|^2+\|v^2_k\|^2\Big)+\nu\Big(\|v^1_{k,y}\|^2+\|v^2_{k,y}\|^2\Big)\\ \displaystyle+\left[(\mathfrak{Re}(\lambda)+\gamma k^2)+\frac{\mathfrak{Re}(\lambda)(\alpha k^4+\beta k^2)}{|\lambda|^2}\right](|\eta^{2,-}_k|^2+|\eta^{2,+}_k|^2) \quad\le\quad 0. \end{array}$$

That is, because the left-hand side is negative, we have either $\mathfrak{Re}(\lambda)\le-\nu k^2$ or $\mathfrak{Re}(\lambda)\le-\gamma k^2$. Finally, if k satisfies the two inequalities $\nu k^2\ge\omega$ and $\gamma k^2\ge\omega$, then

$$\mathfrak{Re}(\lambda)\le-\omega.$$

□

Let us state a result on the eigenvalues of the operator $\mathcal{A}_k$:

Proposition 6.8. *Every eigenvalue of the operator $\mathcal{A}_k$ is simple.*

Proof. We are going to calculate explicitly the form of the eigenfunction. More preciesely, from the system (6.27) in the variables $(\mathbf{v}_k, \eta_k^1, \eta_k^2)$ we can obtain an equivalent system written only in the variable v_k^2. Indeed, taking the curl of the Stokes equation (that derivating the first component by y, multiplying the second one by ik and substracting one to the other), we get

$$\begin{array}{rcll} \lambda v_{k,y}^1 - \nu(v_{k,yyy}^1 - k^2 v_{k,y}^1) + ikq_{k,y} &=& 0 & \text{for } y \in (-1,1) \\ \lambda ikv_k^2 - ik\nu(v_{k,yy}^2 - k^2 v_k^2) + ikq_{k,y} &=& 0 & \text{for } y \in (-1,1) \end{array}$$

and thus

$$(\lambda + \nu k^2)(v_{k,y}^1 - ikv_k^2) - \nu(v_{k,yyy}^1 - ikv_{k,yy}^2) = 0 \text{ for } y \in (-1,1).$$

Using now the divergence free condition $ikv_k^1 + v_{k,y}^2 = 0$ in $(-1,1)$, we can replace v_k^1 by $-(ik)^{-1}v_{k,y}^2$, this gives

$$\left(v_{k,yy}^2 - k^2 v_k^2\right)_{yy} - \left(k^2 + \frac{\lambda}{\nu}\right)\left(v_{k,yy}^2 - k^2 v_k^2\right) = 0. \tag{6.28}$$

The boundary conditions $v_k^2(\kappa 1) = \eta_k^{2,\kappa}$ and $v_{k,y}^1(\kappa 1) + ikv_k^2(\kappa 1) = 0$ can be replace by

$$\begin{array}{rcl} (\lambda^2 + \gamma k^2 \lambda + \alpha k^4 + \beta k^2) v_k^2(\kappa 1) &=& \kappa\lambda\left(\dfrac{\nu}{k^2} v_{k,yyy}^2(\kappa 1) - \dfrac{(\lambda + 3\nu k^2)}{k^2} v_{k,y}^2(\kappa 1)\right), \\ v_{k,yy}^2(\kappa 1) + k^2 v_k^2(\kappa 1) &=& 0. \end{array}$$

These boundary conditions written in the variable v_k^2 come from the divergence free condition, the value of the pressure term q_k obtained in the first equation and from the beam equations.

Thus v_k^2 satisfies a fourth order ordinary differential equation with constant coefficients (but depending on the eigenvalue λ). This gives the form of the solution v_k^2:

$$v_k^2(y) = a\cosh(ky) + b\sinh(ky) + c\cosh(\psi_k y) + d\sinh(\psi_k y), \qquad \text{with } \psi_k = \left(k^2 + \frac{\lambda}{\nu}\right)^{1/2},$$

and the different values of the derivatives of v_k^2:

$$\begin{array}{rcl} v_{k,y}^2(y) &=& ak\sinh(ky) + bk\cosh(ky) + c\psi_k\sinh(\psi_k y) + d\psi_k\cosh(\psi_k y), \\ v_{k,yy}^2(y) &=& ak^2\cosh(ky) + bk^2\sinh(ky) + c\psi_k^2\cosh(\psi_k y) + d\psi_k^2\sinh(\psi_k y), \\ v_{k,yyy}^2(y) &=& ak^3\sinh(ky) + bk^3\cosh(ky) + c\psi_k^3\sinh(\psi_k y) + d\psi_k^3\cosh(\psi_k y). \end{array}$$

Putting all the common terms together in the first boundary condition, we get:

$$\begin{array}{rl} & \left[(\lambda^2 + \gamma k^2\lambda + \alpha k^4)\cosh(k) - \lambda\kappa\left(\dfrac{\nu}{k^2}k^3\sinh(k\kappa) - \dfrac{(\lambda + 3\nu k^2)}{k^2}k\sinh(k\kappa)\right)\right] a \\ + & \left[(\lambda^2 + \gamma k^2\lambda + \alpha k^4)\sinh(k\kappa) - \lambda\kappa\left(\dfrac{\nu}{k^2}k^3\cosh(k) - \dfrac{(\lambda + 3\nu k^2)}{k^2}k\cosh(k)\right)\right] b \\ + & \left[(\lambda^2 + \gamma k^2\lambda + \alpha k^4)\cosh(\psi_k) - \lambda\kappa\left(\dfrac{\nu}{k^2}\psi_k^3\sinh(\psi_k\kappa) - \dfrac{(\lambda + 3\nu k^2)}{k^2}\psi_k\sinh(\psi_k\kappa)\right)\right] c \\ + & \left[(\lambda^2 + \gamma k^2\lambda + \alpha k^4)\sinh(\psi_k\kappa) - \lambda\kappa\left(\dfrac{\nu}{k^2}\psi_k^3\cosh(\psi_k) - \dfrac{(\lambda + 3\nu k^2)}{k^2}\psi_k\cosh(\psi_k)\right)\right] d \\ = & 0. \end{array}$$

Then, substracting or adding the two boundary conditions (obtained in $y = -1$ or $y = 1$), we have:

$$\begin{array}{rl} & \left[(\lambda^2 + \gamma k^2\lambda + \alpha k^4)\cosh(k) - \lambda\left(\dfrac{\nu}{k^2}k^3\sinh(k) - \dfrac{(\lambda + 3\nu k^2)}{k^2}k\sinh(k)\right)\right] a \\ + & \left[(\lambda^2 + \gamma k^2\lambda + \alpha k^4)\cosh(\psi_k) - \lambda\left(\dfrac{\nu}{k^2}\psi_k^3\sinh(\psi_k) - \dfrac{(\lambda + 3\nu k^2)}{k^2}\psi_k\sinh(\psi_k)\right)\right] c \\ = & 0. \end{array} \tag{6.29}$$

and

$$\begin{aligned} &+\left[(\lambda^2+\gamma k^2\lambda+\alpha k^4)\sinh(k)-\lambda\left(\frac{\nu}{k^2}k^3\cosh(k)-\frac{(\lambda+3\nu k^2)}{k^2}k\cosh(k)\right)\right]b \\ &+\left[(\lambda^2+\gamma k^2\lambda+\alpha k^4)\sinh(\psi_k)-\lambda\left(\frac{\nu}{k^2}\psi_k^3\cosh(\psi_k)-\frac{(\lambda+3\nu k^2)}{k^2}\psi_k\cosh(\psi_k)\right)\right]d \\ =\ &0. \end{aligned} \tag{6.30}$$

The other boundary conditions are, for $y=\kappa 1$,

$$v_{k,yy}^2+k^2v_k^2=0.$$

From these ones, we get

$$2ak^2\cosh(k)+2bk^2\sinh(k\kappa)+c(\psi_k^2+k^2)\cosh(\psi_k)+d(\psi_k^2+k^2)\sinh(\psi_k\kappa)=0,$$

that is again two relations between respectively a and c and b and d:

$$2ak^2\cosh(k)+c(\psi_k^2+k^2)\cosh(\psi_k)=0 \tag{6.31}$$

and

$$2bk^2\sinh(k)+d(\psi_k^2+k^2)\sinh(\psi_k)=0. \tag{6.32}$$

Thus, the eigenvalue λ has to satisfy either system (6.29)–(6.31) or system (6.30)–(6.32).

Let us begin with the couple (a,c). We consider the system (6.29)–(6.31). This system has a non trivial solution if and only if the determinant of the matrix is zero. Namely, if we have the condition:

$$\begin{aligned} &2k^2\cosh(k)\left[(\lambda^2+\gamma k^2\lambda+\alpha k^4)\cosh(\psi_k)+\lambda\times 2\nu\psi_k\sinh(\psi_k)\right] \\ &-(\psi_k^2+k^2)\cosh(\psi_k)\left[(\lambda^2+\gamma k^2\lambda+\alpha k^4)\cosh(k)+\lambda\frac{(\lambda+2\nu k^2)}{k^2}k\sinh(k)\right]a \\ =\ &0. \end{aligned}$$

That is,

$$e^{2\psi_k}=\frac{4\nu k^2\lambda\psi_k\cosh(k)+\left((\lambda^2+\gamma k^2\lambda+\alpha k^4)\cosh(k)+\frac{\nu^2(\psi_k^4-k^4)}{k^2}k\sinh(k)\right)}{4\nu k^2\lambda\psi_k\cosh(k)-\left((\lambda^2+\gamma k^2\lambda+\alpha k^4)\cosh(k)+\frac{\nu^2(\psi_k^4-k^4)}{k^2}k\sinh(k)\right)} \tag{6.33}$$

The corresponding eigenfunction is

$$v_k^2(y)=c\Big[-(\psi_k^2+k^2)\cosh(\psi_k)\cosh(ky)+k^2\cosh(k)\cosh(\psi_k y)\Big].$$

Then, at the boundary $y=\kappa 1$, we have

$$v_k^2(\kappa 1)=\eta_k^{2,\kappa}=-c\psi_k^2\cosh(\psi_k)\cosh(k)$$

which is different of 0 if both $\psi_k^2\neq 0$ and $\cosh(\psi_k)\neq 0$. The first case implies that $\lambda=-\nu k^2$ and then (after some calculations) equation (6.28) has no nonzero solutions. We have to consider the case $\cosh(\psi_k)=0$. It is impossible too thanks to equation (6.33). Indeed, we rewrite it and obtain

$$e^{2\psi_k}=-1+\frac{8\nu^2k^2(\psi_k^2-k^2)\psi_k}{4\nu^2k^2(\psi_k^2-k^2)\psi_k-\left((\lambda^2+\gamma k^2\lambda+\alpha k^4)+\frac{\nu^2(\psi_k^4-k^4)}{k^2}k\tanh(k)\right)}.$$

The second term in the right-hand side is 0 if and only if $\psi_k=0$ which is not (see above) or if $\psi_k^2=k^2$ which is not too (same idea).

We can make the exact same work for the case (b,d). In this case, we obtain the following equation for λ (remember that $\psi_k=\left(k^2+\frac{\lambda}{\nu}\right)^{1/2}$):

$$e^{2\psi_k}=1-\frac{8\nu^2k^2(\psi_k^2-k^2)\psi_k}{4\nu^2k^2(\psi_k^2-k^2)\psi_k-\left((\lambda^2+\gamma k^2\lambda+\alpha k^4)+\frac{\nu^2(\psi_k^4-k^4)}{k^2}k\coth(k)\right)}$$

and the corresponding eigenvalue is

$$v_k^2(y) = d\Big[- (\psi_k^2 + k^2)\sinh(\psi_k)\sinh(ky) + k^2\sinh(k)\sinh(\psi_k y)\Big]$$

which satisfies at the boundary

$$v_k^2(\kappa 1) = \eta_k^{2,\kappa} = \kappa d \psi_k^2 \sinh(k)\sinh(\psi_k).$$

Thus, the eigenfunction v_k^2 associated to an eigenvalue λ satisfies $v_k^2(\kappa 1) \neq 0$. Let us assume that there exists (at least) two different eigenfunctions $v_k^{2,1}$ and $v_k^{2,2}$ corresponding to the same eigenvalue $\lambda \in \mathbb{C}$. Then, because $v_k^{2,1}(\kappa 1) \neq 0$ and $v_k^{2,2}(\kappa 1) \neq 0$, we can define the constant $\theta = \frac{v_k^{2,1}(1)}{v_k^{2,2}(1)}$ and a new function $v_k^2(y) = v_k^{2,1}(y) - \theta v_k^{2,2}(y)$ for every y in $(-1,1)$. By linearity v_k^2 is a solution to the same eigenvalue problem as $v_k^{2,1}$ and $v_k^{2,2}$ and satisfies $v_k^2(1) = 0$ by construction. Then $v_k^2(y) = 0$ for every y in $(-1,1)$. The two eigenfunctions are colinear. The eigenvalue λ is simple. □

6.5.2 Approximate controllability and stabilization of system (6.14).

We begin by proving the stabilization of the system with a control in $L^2(0,\infty;\mathbb{C})$. The stabilization for any decay rate will be obtained by a shift of the operator. Let us introduce some notations. We define the space of initial data $X^0_{k,\mathbf{cc}}$ by

$$X^0_{k,\mathbf{cc}} = \Big\{ (\mathbf{z}_k^0, \mu_k^{1,0}, \mu_k^{2,0}) \text{ in } X_k^0 \quad \text{s.t.} \quad z_k^{0,2}(\kappa 1) = \mu_k^{2,0,\kappa} \Big\}$$

where

$$X_k^0 = \mathbf{L}^2(-1,1;\mathbb{C}) \times H^2_{\#,k}(\mathbb{C}^2) \times L^2_{\#,k}(\mathbb{C}^2).$$

Proposition 6.9. *For any initial data $(\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0})$ in $X^0_{k,\mathbf{cc}}$ and any time $T > 0$, system (6.14) is approximately controllable in time T by controls f_k^+ in $L^2(0,T;L^2_{\#,k}(\mathbb{C}))$.*

Proof. The idea is to prove that for $(\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0}) = (\mathbf{0}, 0_{\mathbb{C}^2}, 0_{\mathbb{C}^2})$ (by linearity), the set

$$R(T) = \big\{ (\mathbf{v}_k(T), \eta_k^1(T), \eta_k^2(T)) \text{ where } (\mathbf{v}_k, q_k, \eta_k^1, \eta_k^2) \text{ is solution of (6.34) with } f_k^+ \text{ in } L^2(0,T;L^2_{\#,k}(\mathbb{C})) \big\}$$

$$\begin{array}{rcll}
\mathbf{v}_{k,t} - \nu\Delta_k \mathbf{v}_k + \nabla_k q_k & = & \mathbf{0} & \text{in } (0,T)\times(-1,1),\\
ikv_k^1 + v_{k,y}^2 & = & 0 & \text{in } (0,T)\times(-1,1),\\
v_k^2 & = & \eta_k^{2,\kappa} & \text{on } (0,T)\times\{\kappa 1\},\\
v_{k,y}^1 + ikv_k^2 & = & 0 & \text{on } (0,T)\times\{\kappa 1\},\\
\eta_{k,t}^{1,\kappa} & = & \eta_k^{2,\kappa}, & \text{on } (0,T),\\
\eta_{k,t}^{2,\kappa} + (\alpha k^4 + \beta k^2)\eta_k^{1,\kappa} + \gamma k^2 \eta_k^{2,\kappa} & = & \kappa(q_k - 2\nu v_{k,y}^2)(\kappa 1) + f_k^+ \chi_{\Gamma_0^+}, & \text{on } (0,T),\\
(\mathbf{v}_k(0), \eta_k^1(0), \eta_k^2(0)) & = & (\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0}). &
\end{array} \tag{6.34}$$

is dense in $X^0_{k,\mathbf{cc}}$. That is, taking $(\mathbf{z}_k, \mu_k^1, \mu_k^2)$ in $R(T)^\perp$, we have to prove that $(\mathbf{z}_k, \mu_k^1, \mu_k^2) = (\mathbf{0}, 0_{\mathbb{C}^2}, 0_{\mathbb{C}^2})$.

Let us introduce the adjoint system of system (6.14):

$$\begin{array}{rcll}
-\mathbf{z}_{k,t} - \nu\Delta_k \mathbf{z}_k + \nabla_k \pi_k & = & \mathbf{0} & \text{in } (0,T)\times(-1,1),\\
ikz_k^1 + z_{k,y}^2 & = & 0 & \text{in } (0,T)\times(-1,1),\\
z_k^2 & = & \mu_k^{2,\kappa} & \text{on } (0,T)\times\{\kappa 1\},\\
z_{k,y}^1 + ikz_k^2 & = & 0 & \text{on } (0,T)\times\{\kappa 1\},\\
-\mu_{k,t}^{1,\kappa} & = & -\mu_k^{2,\kappa}, & \text{on } (0,T),\\
-\mu_{k,t}^{2,\kappa} - (\alpha k^4 + \beta k^2)\mu_k^{1,\kappa} + \gamma k^2 \mu_k^{2,\kappa} & = & \kappa(\pi_k - 2\nu z_{k,y}^2)(\kappa 1), & \text{on } (0,T),\\
(\mathbf{z}_k(T), \mu_k^1(T), \mu_k^2(T)) & = & (\mathbf{z}_k^T, \mu_k^{1,T}, \mu_k^{2,T}). &
\end{array}$$

This system gives us from the inner product of equation $(6.14)_1$ by $\mathbf{z}_k$ (after some integrations by parts), the following equality.

$$\int_{-1}^{1} \mathbf{v}_k\overline{\mathbf{z}_k} + \sum_{\kappa}(\alpha k^4+\beta k^2)\eta_k^{1,\kappa}(T)\overline{\mu_k^{1,T,\kappa}} + \eta_k^{2,\kappa}(T)\overline{\mu_k^{2,T,\kappa}} = \int_0^T f_k^+(t)\overline{\mu_k^{2,T,+}(t)}\mathrm{d}t.$$

Thus, with $(\mathbf{v}(T),\eta_k^1(T),\eta_k^2(T))$ in $R(T)$ and $(\mathbf{z}_k^T,\mu_k^{1,T},\mu_k^{2,T})$ in $R(T)^\perp$, we get that

$$\int_0^T f_k^+(t)\overline{\mu_k^{2,T,+}(t)}\mathrm{d}t = 0 \text{ for all } f_k^+ \text{ in } L^2(0,T;L^2_{\#,k}(\mathbb{C})).$$

This gives $\mu_k^{2,T,+}=0$. The proof is now reduced to the following unique continuation problem:

Does $(\mathbf{z}_k,\mu_k^1,\mu_k^2)$ the solution to the following eigenvalue problem

$$\begin{array}{rcll}
-\lambda\mathbf{z}_k-\nu\Delta_k\mathbf{z}_k+\nabla_k\pi_k &=& \mathbf{0} & \text{in } (-1,1),\\
ikz_k^1+z_{k,y}^2 &=& 0 & \text{in } (-1,1),\\
z_k^2 &=& \mu_k^{2,\kappa} & \text{on } \{\kappa 1\},\\
z_{k,y}^1+ikz_k^2 &=& 0 & \text{on } \{\kappa 1\},\\
-\lambda\mu_k^{1,\kappa} &=& -\mu_k^{2,\kappa}, & \\
-\lambda\mu_k^{2,\kappa}-(\alpha k^4+\beta k^2)\mu_k^{1,\kappa}+\gamma k^2\mu_k^{2,\kappa} &=& \kappa(\pi_k-2\nu z_{k,y}^2)(\kappa 1) &
\end{array} \tag{6.35}$$

with the extra condition $\mu_k^{2,+}=0$ satisfy $(\mathbf{z}_k,\mu_k^1,\mu_k^2)=(\mathbf{0},0_{\mathbb{C}^2},0_{\mathbb{C}^2})$?

The answer comes from the study of the operator $\mathcal{A}_k$ in the previous section, especially Proposition 6.8. Indeed, system (6.35) is the same as the one studied in the proof of Proposition 6.8 where we see that the solution $(\mathbf{z}_k,\mu_k^1,\mu_k^2)$ of this eigenvalue problem satisfies $\mu_k^{2,+}\neq 0$. Thus, the unique continuation property holds true and so does the approximate controllability. □

Now, let us consider the shifted system corresponding with (6.14). We obtain this system by introducing new variables $(\widetilde{\mathbf{v}}_k,\widetilde{\eta}_k^1,\widetilde{\eta}_k^2)=\mathrm{e}^{\omega\cdot}(\mathbf{v}_k,\eta_k^1,\eta^2)$ where $\omega>0$ is the prescribe decay rate. Then, for $(\mathbf{v}_k,\eta_k^1,\eta^2)$ solution of system (6.14) with initial data $(\mathbf{v}_k^0,\eta_k^{1,0},\eta_k^{2,0})$ in $X^0_{k,\mathbf{cc}}$, the system satisfied by $(\widetilde{\mathbf{v}}_k,\widetilde{\eta}_k^1,\widetilde{\eta}_k^2)$ is

$$\begin{array}{rcll}
\widetilde{\mathbf{v}}_{k,t}-\omega\widetilde{\mathbf{v}}_k-\nu\Delta_k\widetilde{\mathbf{v}}_k+\nabla_k\widetilde{q}_k &=& \mathbf{0} & \text{in } (0,\infty)\times(-1,1),\\
ik\widetilde{v}_k^1+\widetilde{v}_{k,y}^2 &=& 0 & \text{in } (0,\infty)\times(-1,1),\\
\widetilde{v}_k^2 &=& \widetilde{\eta}_k^{2,\kappa} & \text{on } (0,\infty)\times\{\kappa 1\},\\
\widetilde{v}_{k,y}^1+ik\widetilde{v}_k^2 &=& 0 & \text{on } (0,\infty)\times\{\kappa 1\},\\
\widetilde{\eta}_{k,t}^{1,\kappa} &=& \eta_k^{2,\kappa}+\omega\widetilde{\eta}_k^{1,\kappa}, & \text{on } (0,\infty),\\
\widetilde{\eta}_{k,t}^{2,\kappa}-\omega\widetilde{\eta}_k^{2,\kappa}+(\alpha k^4+\beta k^2)\widetilde{\eta}_k^{1,\kappa}+\gamma k^2\widetilde{\eta}_k^{2,\kappa} &=& \kappa(\widetilde{q}_k-2\nu\widetilde{v}_{k,y}^2)(\kappa 1)+\widetilde{f}_k^+\chi_{\Gamma_0^+}, & \text{on } (0,\infty),\\
(\widetilde{\mathbf{v}}_k(0),\widetilde{\eta}_k^1(0),\widetilde{\eta}_k^2(0)) &=& (\mathbf{v}_k^0,\eta_k^{1,0},\eta_k^{2,0}) &
\end{array} \tag{6.36}$$

where $\widetilde{f}_k^+=\mathrm{e}^{\omega\cdot}f_k^+$.

From the correspondence between (6.14) and (6.15), we get the directly the correspondence between system (6.36) and system (6.37)

$$\begin{array}{rcl}
\begin{pmatrix}\mathbf{P}_k\widetilde{\mathbf{v}}_k\\ \widetilde{\eta}_k^1\\ \widetilde{\eta}_k^2\end{pmatrix}' &=& \mathcal{A}_{k,\omega}\begin{pmatrix}\mathbf{P}_k\widetilde{\mathbf{v}}_k\\ \widetilde{\eta}_k^1\\ \widetilde{\eta}_k^2\end{pmatrix}+\mathcal{B}_k\widetilde{f}_k^+, \qquad \begin{pmatrix}\mathbf{P}_k\widetilde{\mathbf{v}}_k\\ \widetilde{\eta}_k^1\\ \widetilde{\eta}_k^2\end{pmatrix}(0)=\begin{pmatrix}\mathbf{P}_k\mathbf{v}_k^0\\ \eta_k^{1,0}\\ \eta_k^{2,0}\end{pmatrix},\\
(\mathbf{I}_k-\mathbf{P}_k)\widetilde{\mathbf{v}}_k &=& \nabla_k N_k\left(\widetilde{\eta}_k^2\right),\\
\widetilde{q}_k &=& -N_k(\widetilde{\eta}_{k,t}^2-\omega\widetilde{\eta}_k^2+4\nu k^2\widetilde{\eta}_k^2),
\end{array} \tag{6.37}$$

where $\mathcal{A}_{k,\omega}$ is the operator defined by $D(\mathcal{A}_{k,\omega})=D(\mathcal{A}_k)$ and

$$\mathcal{A}_{k,\omega}=\mathcal{A}_k+\omega\left(\begin{array}{c|cc}\mathbf{I}_k & \mathbf{0} & \mathbf{0}\\ \hline 0 & \mathbf{I}_2 & 0\\ 0 & 0 & (\mathbf{I}_2+\mathbf{N}_k)^{-1}\end{array}\right).$$

Thus, we get the following result.

Theorem 6.10. *For all $\omega > 0$ and all $(\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0})$ in $X^0_{k,\mathbf{cc}}$, there exists a control f_k^+ in $L^2(0,\infty;\mathbb{C})$ such that the solution $(\widetilde{\mathbf{v}}, \widetilde{q}_k, \widetilde{\eta}_k^1, \widetilde{\eta}_k^2)$ of system (6.36) satisfies*

$$\|(\widetilde{\mathbf{v}}_k, \widetilde{\eta}_k^1, \widetilde{\eta}_k^2)\|_{L^2(0,\infty;X_k^0)} < \infty.$$

Proof. The idea is to use the semigroup approach for system

$$r_k'(t) = \mathcal{A}_k r_k(t) + \mathcal{B}_k f_k^+(t), \qquad r_k(0) = r_k^0 \tag{6.38}$$

where $r_k = (\mathbf{P}_k \mathbf{v}_k, \eta_k^1, \eta_k^2)^{\mathbf{tr}}$ and $r_k^0 = (\mathbf{P}_k \mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0})$, that is the first equation in (6.15). We follow the proof of Theorem in [23].

Let us consider $\omega > 0$ in the resolvent set of $\mathcal{A}_k$. From Propositions 6.7 and 6.8, we know that the spectrum of $-\mathcal{A}_k$ is constituted of pointwise and simple eigenvalues. We can number and denote them as follows:

$$\ldots \leq \mathfrak{Re}(\lambda_k^{n_k+1}) < -\omega < \mathfrak{Re}(\lambda_k^{n_k}) \leq \ldots \leq \mathfrak{Re}(\lambda_k^1).$$

We denote $\phi_k^p = (\mathbf{P}_k \mathbf{v}_k^p, \eta_k^{1,p}, \eta^{2,p})^{\mathbf{tr}}$ the eigenfunction of $\mathcal{A}_k$ corresponding with λ_k^p, for $p = 1, \ldots, \infty$. Then, we introduce $\mathbb{H}_k^u$ (respectively $\mathbb{H}_k^s$) the unstable (respectively stable) eigenspace of the operator $\mathcal{A}_{k,\omega}$. That is, $\mathbb{H}_k^u$ is constituted of all the eigenfunctions corresponding with the eigenvalues λ of $\mathcal{A}_k$ satisfying $\mathfrak{Re}(\lambda) \geq -\omega$ and $\mathbb{H}_k^s$ is constituted with all the other eigenfunctions (corresponding with the eigenvalues λ satsfying $\mathfrak{Re}(\lambda) < -\omega$). That is,

$$\mathbb{H}_k^u = \mathrm{Vect}\Big\{\phi_k^p, p = 1, \ldots, n_k\Big\} \quad \text{and} \quad \mathbb{H}_k^s = \mathrm{Vect}\Big\{\phi_k^p, p = n_k+1, \ldots, \infty\Big\}.$$

Note that the sum is direct $\mathbb{H}_k = \mathbb{H}_k^s \oplus \mathbb{H}_k^u$. We denote finally by P_k^u the orthogonal projection from $\mathbb{H}_k$ onto the unstable space $\mathbb{H}_k^u$. Applying the projections P_k^u or $(I - P_k^u)$ to system (6.38), we obtain two systems, one in the variable $r_k^u = P_k^u r_k$ on $\mathbb{H}_k^u$ and this other in the variable $r_k^s = (I - P_k^u) r_k$ on $\mathbb{H}_k^s$:

$$r_{k,t}^u = \mathcal{A}_k^u r_k^u + P_k^u \mathcal{B}_k f_k^+, \qquad r_k^u(0) = P_k^u r_k^0, \tag{6.39}$$

and

$$r_{k,t}^s = \mathcal{A}_k^s r_k^s + (I - P_k^u)\mathcal{B}_k f_k^+, \qquad r_k^s(0) = (I - P_k^u) r_k^0 \tag{6.40}$$

where, with obvious notations, $\mathcal{A}_k^u = P_k^u \mathcal{A}_k$ and $\mathcal{A}_k^s = (I - P_k^u)\mathcal{A}_k$.

Now, we follow classic control theory results. First, from Proposition 6.9, we know that there exists a control f_k^+ in $L^2(0,T;\mathbb{C})$ such that system (6.14) is approximatively controllable at time $T > 0$. Thus, system $(6.15)_1$ is approximatively controllable just like the projected system (6.39). Then, system (6.39) is controllable (because it is of finite dimension). Let f_k^+ be a control satisfying $P_k^u(\mathbf{P}_k \mathbf{v}_k, \eta_k^1, \eta_k^2)(T) = (\mathbf{0}, 0_{\mathbb{C}^2}, 0_{\mathbb{C}^2})$, then still denoting f_k^+ its extension by 0 on (T, ∞), we use the correspondence between system (6.15) and system (6.37): $r_k = (\mathbf{P}_k \mathbf{v}_k, \eta_k^1, \eta_k^2)$ is solution of (6.15) if and only if $\widetilde{r}_k = (\mathbf{P}_k \widetilde{\mathbf{v}}_k, \widetilde{\eta}_k^1, \widetilde{\eta}_k^2)$ is solution of system

$$\widetilde{r}_{k,t}(t) = \mathcal{A}_{k,\omega}\widetilde{r}_k(t) + \mathcal{B}_k \widetilde{f}_k^+(t), \qquad \widetilde{r}_k(0) = r_k^0. \tag{6.41}$$

Indeed, using the decomposition of $\mathbb{H}_k$ into $\mathbb{H}_k^u \oplus \mathbb{H}_k^s$, we get

$$\widetilde{r}_{k,t}^u = \mathcal{A}_{k,\omega}^u \widetilde{r}_k^u + \mathcal{B}_k^u \widetilde{f}_k^+, \qquad \widetilde{r}_k^u(0) = P_k^u r_k^0 \tag{6.42}$$

and

$$\widetilde{r}_{k,t}^u = \mathcal{A}_{k,\omega}^s \widetilde{r}_k^u + \mathcal{B}_k^s \widetilde{f}_k^+, \qquad \widetilde{r}_k^u(0) = (I - P_k^u) r_k^0 \tag{6.43}$$

with $\mathcal{A}_{k,\omega}^u = P_k^u \mathcal{A}_{k,\omega}$, $\mathcal{A}_{k,\omega}^s = (I - P_k^u)\mathcal{A}_{k,\omega}$, $\mathcal{B}_k^u = P_k^u \mathcal{B}_k$ and $\mathcal{B}_k^s = (I - P_k^u)\mathcal{B}_k$ and $\widetilde{f}_k^+ = \mathrm{e}^{\omega \cdot} f_k^+$. Thus, system (6.42) is stabilizable with a control $\widetilde{f}_k^+$ satisfying

$$\Big|\widetilde{f}_k^+(t)\Big| \leq C_k \mathrm{e}^{-\omega_k} \|P_k^u r_k^0\| \text{ for all } t \geq 0$$

for $\omega_k > 0$. This result comes from the construction of the control above and the fact that the control is zero on (T, ∞). Thus, we have this estimate for a large enough constante $C > 0$ (depending on T, ω and ω_k).

Now, we prove that the complete system (6.37) is stabilizable. Using the fact that $\mathcal{A}^s_{k,\omega}$ is stable on $\mathbb{H}^s_k$, the Duhamel formula and the estimate of the control above, we can prove that equation (6.43) is stabilizable.

To conclude the proof of the stabilization of system (6.37), we note that $(6.37)_2$ gives $(\mathbf{I}_k - \mathbf{P}_k)\widetilde{\mathbf{v}}_k$ in term of $\widetilde{\eta}^2_k$. Thus, the estimate of $\widetilde{\eta}^2_k$ in $\mathbb{C}$ gives the same estimate for $(\mathbf{I}_k - \mathbf{P}_k)\widetilde{\mathbf{v}}_k$ in $\mathbf{L}^2(-1, 1; \mathbb{C})$.

The stabilization of system (6.36) follows. Indeed, the correspondence correspondance between system (6.36) and system (6.37) gives that the solution $(\widetilde{\mathbf{v}}_k, \widetilde{q}_k, \widetilde{\eta}^1_k, \widetilde{\eta}^2_k)$ of (6.36) is linked to the solution $(\mathbf{P}_k\widetilde{\mathbf{v}}_k, \widetilde{\eta}^1_k, \widetilde{\eta}^2_k)$ of (6.37) by $\widetilde{\mathbf{v}}_k = \mathbf{P}_k\widetilde{\mathbf{v}}_k + \nabla_k N_k(\widetilde{\eta}^2_k)$. Thus, for $(\mathbf{v}^0_k, \eta^{1,0}_k, \eta^{2,0}_k)$ in $X^0_{k,\mathbf{cc}}$, we get $(\mathbf{P}_k\mathbf{v}^0_k, \eta^{1,0}_k, \eta^{2,0}_k)$ in $\mathbb{H}_k$. Then, there exists a control $\widetilde{f}^+_k$ in $L^2(0, \infty; \mathbb{C})$ such that the solution $(\mathbf{P}_k\widetilde{\mathbf{v}}_k, \widetilde{\eta}^1_k, \widetilde{\eta}^2_k)$ of $(6.37)_1$ with initial data $(\mathbf{P}_k\mathbf{v}^0_k, \eta^{1,0}_k, \eta^{2,0}_k)$ in $\mathbb{H}_k$ and f^+_k as right-hand side satisfies

$$\|(\mathbf{P}_k\widetilde{\mathbf{v}}_k, \widetilde{\eta}^1_k, \widetilde{\eta}^2_k)\|_{L^2(0,\infty;\mathbb{H}_k)} \leq C\|(\mathbf{P}_k\mathbf{v}^0_k, \eta^{1,0}_k, \eta^{2,0}_k)\|_{\mathbb{H}_k}, \qquad \text{for all } t \geq 0.$$

A little calculation made in the proof of Lemma 6.6 gives that $\langle \mathbf{N}_k\widetilde{\eta}^2_k, \widetilde{\eta}^2_k\rangle_{\mathbb{C}^2} = \|\nabla_k N_k\widetilde{\eta}^2_k\|^2_{\mathbf{L}^2(-1,1;\mathbb{C})}$. Furthermore,

$$\|\widetilde{\mathbf{v}}_k\|^2_{\mathbf{L}^2(-1,1;\mathbb{C})} = \|\mathbf{P}_k\widetilde{\mathbf{v}}_k\|^2_{\mathbf{L}^2(-1,1;\mathbb{C})} + \|(\mathbf{I}_k - \mathbf{P}_k)\widetilde{\mathbf{v}}_k\|^2_{\mathbf{L}^2(-1,1;\mathbb{C})} = \|\mathbf{P}_k\widetilde{\mathbf{v}}_k\|^2_{\mathbf{L}^2(-1,1;\mathbb{C})} + \|\nabla_k N_k\widetilde{\eta}^2_k\|^2_{\mathbf{L}^2(-1,1;\mathbb{C})}.$$

Thus, $\|(\mathbf{P}_k\widetilde{\mathbf{v}}_k, \widetilde{\eta}^1_k, \widetilde{\eta}^2_k)\|_{\mathbb{H}_k} = \|(\widetilde{\mathbf{v}}_k, \widetilde{\eta}^1_k, \widetilde{\eta}^2_k)\|_{X^0_k}$ and finally, the solution $(\widetilde{\mathbf{v}}_k, \widetilde{q}_k, \widetilde{\eta}^1_k, \widetilde{\eta}^2_k)$ of system (6.36) satisfies the expected estimate. □

Remark 6.11. *Thanks to the estimate of the eigenvalues for the operator $\mathcal{A}_k$, if k satisfies $|k| \geq M_\omega = \max\left\{\sqrt{\frac{\omega}{\nu}}, \sqrt{\frac{\omega}{\gamma}}\right\}$, then all the eigenvalues of $\mathcal{A}_k$ satisfies $\mathfrak{Re}(\lambda) < -\omega$. Thus, the eigenspace $\mathbb{H}^u_k$ for such value of k is reduced to $\{0\}$ and system (6.36) is already stable. In the following, for such value of k, we will consider the control $f^+_k \equiv 0$.*

6.5.3 Feedback stabilization of system (6.14).

In this section, we follow the previous decomposition to prove the feedback stabilization of each systems (6.14) (for k such that $|k| \leq M_\omega$ and $k \neq 0$). We prove the feedback stabilization with a finite dimension control (each control is one dimensional and stabilize the finite dimensional unstable eigenspace of the operator $\mathcal{A}_{k,\omega}$). Thus the feedback controls will be obtained by solving finite dimensional algebraic Riccati equations, which can be very usefull in applications.

A way to do that is to consider the infinite time horizon control problems $(\mathcal{P}^{k,\infty}_{0,(\mathbf{P}_k\mathbf{v}^0_k,\eta^{1,0}_k,\eta^{2,0}_k)})$ (k in $\mathbb{Z}$ such that $|k| \leq M_\omega$ and $k \neq 0$)

$$\inf\left\{\mathcal{J}_k(\mathbf{P}_k\mathbf{v}_k, \eta^1_k, \eta^2_k; f^+_k) \quad \text{s.t.} \quad (\mathbf{P}_k\mathbf{v}_k, \eta^1_k, \eta^2_k; f^+_k) \text{ satisfies } (6.37)_1 \text{ with } f^+_k \text{ in } L^2(0, \infty; \mathbb{C})\right\}$$

where

$$\mathcal{J}_k(\mathbf{P}_k\mathbf{v}_k, \eta^1_k, \eta^2_k; f^+_k) = \frac{1}{2}\int_0^\infty \|P^u_k(\mathbf{P}_k\mathbf{v}_k(t), \eta^1_k(t), \eta^2_k(t))\|^2_{\mathbb{H}^u_k}\mathrm{d}t + \frac{1}{2}\int_0^\infty |f^+_k(t)|^2\mathrm{d}t.$$

Following [23], we directly consider the equivalent system (6.37) in the variables $(\mathbf{P}_k\mathbf{v}_k, \eta^1_k, \eta^2_k)$ instead of the system (6.36) in the variables $(\mathbf{v}_k, \eta^1_k, \eta^2_k)$. First, we can prove the following result.

Theorem 6.12. *For all $(\mathbf{P}_k\mathbf{v}^0_k, \eta^{1,0}_k, \eta^{2,0}_k)$ in $\mathbb{H}_k$, problem $(\mathcal{P}^{k,\infty}_{0,(\mathbf{P}_k\mathbf{v}^0_k,\eta^{1,0}_k,\eta^{2,0}_k)})$ admits a unique solution $(\mathbf{P}_k\mathbf{v}_{k(\mathbf{P}_k\mathbf{v}^0_k,\eta^{1,0}_k,\eta^{2,0}_k)}, \eta^1_{k(\mathbf{P}_k\mathbf{v}^0_k,\eta^{1,0}_k,\eta^{2,0}_k)}, \eta^2_{k(\mathbf{P}_k\mathbf{v}^0_k,\eta^{1,0}_k,\eta^{2,0}_k)}; f^+_{k(\mathbf{P}_k\mathbf{v}^0_k,\eta^{1,0}_k,\eta^{2,0}_k)})$. There exists Π^u_k in $\mathcal{L}(\mathbb{H}^u_k, (\mathbb{H}^u_k)^*)$ obeying $\Pi^u_k = (\Pi^u_k)^* \geq 0$ such that the optimal cost is given by*

$$\inf(\mathcal{P}^{k,\infty}_{0,(\mathbf{P}_k\mathbf{v}^0_k,\eta^{1,0}_k,\eta^{2,0}_k)}) = \frac{1}{2}\left(P^u_k(\mathbf{P}_k\mathbf{v}^0_k, \eta^{1,0}_k, \eta^{2,0}_k), \Pi^u_k P^u_k(\mathbf{P}_k\mathbf{v}^0_k, \eta^{1,0}_k, \eta^{2,0}_k)\right)_{\mathbb{H}^u_k,(\mathbb{H}^u_k)^*}.$$

Moreover, Π_k^u *is the solution to the finite dimensional algebraic Riccati equation*

$$\Pi_k^u \text{ in } \mathcal{L}(\mathbb{H}_k^u, (\mathbb{H}_k^u)^*), \quad \Pi_k^u = (\Pi_k^u)^* \geq 0, \qquad \Pi_k^u \mathcal{A}_{k,\omega}^u + (\mathcal{A}_{k,\omega}^u)^* \Pi_k^u - \Pi_k^u \mathcal{B}_k^u (\mathcal{B}_k^u)^* \Pi_k^u + I_k^u = 0, \quad (6.44)$$

where

$$\begin{aligned} \mathcal{A}_{k,\omega}^u &= P_k^u \mathcal{A}_{k,\omega} P_k^u \in \mathcal{L}(\mathbb{H}_k^u, \mathbb{H}_k^u), \qquad \left(\mathcal{A}_{k,\omega}^u\right)^* = (P_k^u \mathcal{A}_{k,\omega} P_k^u)^* = (P_k^u)^* \mathcal{A}_{k,\omega}^* (P_k^u)^* \in \mathcal{L}((\mathbb{H}_k^u)^*, (\mathbb{H}_k^u)^*), \\ \mathcal{B}_k^u &= P_k^u \mathcal{B}_k \in \mathcal{L}(\mathbb{C}, \mathbb{H}_k^u), \qquad (\mathcal{B}_k^u)^* = (P_k^u \mathcal{B}_k)^* \in \mathcal{L}((\mathbb{H}_k^u)^*, \mathbb{C}), \\ I_k^u &\in \mathcal{L}(\mathbb{H}_k^u, (\mathbb{H}_k^u)^*) \text{ is the identity.} \end{aligned}$$

Denoting $\Pi_k = (P_k^u)^* \Pi_k^u P_k^u$*, we have* Π_k *in* $\mathcal{L}(\mathbb{H}_k, (\mathbb{H}_k)^*)$ *solution to the following algebraic Riccati equation*

$$\Pi_k \text{ in } \mathcal{L}(\mathbb{H}_k, (\mathbb{H}_k)^*), \quad \Pi_k = (\Pi_k)^* \geq 0, \qquad \Pi_k \mathcal{A}_{k,\omega}^u + (\mathcal{A}_{k,\omega}^u)^* \Pi_k - \Pi_k \mathcal{B}_k (\mathcal{B}_k)^* \Pi_k + (P_k^u)^* P_k^u = 0. \quad (6.45)$$

Proof. This proof is very classical. It can be found in [5]. The reduction of the problem to a control problem on the finite dimensional space $\mathbb{H}_k^u$ comes from the previous section. Hence, because the operator $\mathcal{B}_k$ is bounded from $\mathbb{C}$ into $\mathbb{H}_k$ and P_k^u is the projection from $\mathbb{H}_k$ onto $\mathbb{H}_k^u$, the control operator is bounded and the observation operator in the functional is $i_k^u \in \mathcal{L}(\mathbb{H}_k^u, \mathbb{H})$ such that $I_k^u = (i_k^u)^* i_k^u$. □

The algebraic equation (6.44) is set in the space $\mathbb{H}_k^u$ the finite dimensional unstable eigenspace of the operator $\mathcal{A}_{k,\omega}$. The identity $\Pi_k^u = (\Pi_k^u)^* \geq 0$ in (6.44) has to be understood in the sens of the quadratic linear form:

$$(r_k^u, \Pi_k^u s_k^u)_{\mathbb{H}_k^u, (\mathbb{H}_k^u)^*} = (\Pi_k^u r_k^u, s_k^u)_{(\mathbb{H}_k^u)^*, \mathbb{H}_k^u} \text{ for all } r_k^u, s_k^u \in \mathbb{H}_k^u \quad \text{and} \quad (r_k^u, \Pi_k^u r_k^u)_{\mathbb{H}_k^u, (\mathbb{H}_k^u)^*} \geq 0 \text{ for all } r_k^u \in \mathbb{H}_k^u.$$

Second, Π_k^u satisfies $\Pi_k^u \mathcal{A}_{k,\omega}^u + (\mathcal{A}_{k,\omega}^u)^* \Pi_k^u - \Pi_k^u B_k^u (B_k^u)^* \Pi_k^u + I_k^u = 0$ when

$$\begin{aligned} &\left(\Pi_k^u \mathcal{A}_{k,\omega}^u r_k^u, s_k^u\right)_{(\mathbb{H}_k^u)^*, \mathbb{H}_k^u} + \left(\left(\mathcal{A}_{k,\omega}^u\right)^* \Pi_k^u r_k^u, s_k^u\right)_{(\mathbb{H}_k^u)^*, \mathbb{H}_k^u} \\ &- \left\langle (\mathcal{B}_k^u)^* \Pi_k^u r_k^u, (\mathcal{B}_k^u)^* \Pi_k^u s_k^u \right\rangle_{\mathbb{C}} + (I_k^u r_k^u, s_k^u)_{(\mathbb{H}_k^u)^*, \mathbb{H}_k^u} = 0 \quad \text{for all } r_k^u, s_k^u \in \mathbb{H}_k^u. \end{aligned}$$

The operator $\mathcal{A}_{k,\omega}^*$ is defined by $D(\mathcal{A}_{k,\omega}^*) = D(\mathcal{A}_{k,\omega})$ on $(\mathbb{H}_k)^*$ and

$$\mathcal{A}_{k,\omega}^* = \left(\begin{array}{c|cc} \mathbf{I}_2 & 0 & 0 \\ \hline 0 & \mathbf{I}_2 & 0 \\ 0 & 0 & (\mathbf{I}_2 + \mathbf{N}_k)^{-1} \end{array}\right) \left(\begin{array}{c|cc} \mathbf{A}_k & \mathbf{0} & (-\mathbf{A}_k)\mathbf{P}_k \mathbf{D}_k \\ \hline 0 & 0 & -\mathbf{I}_2 \\ \mathbf{S}_k & -\mathbf{M}_k^1 & \mathbf{M}_k^2 \end{array}\right) \quad \text{and} \quad \mathcal{B}_k^* \begin{pmatrix} \mathbf{P}_k \mathbf{z}_k \\ \mu_k^1 \\ \mu_k^2 \end{pmatrix} = \mu_k^{2,+}.$$

Because Π_k belongs to $\mathcal{L}(\mathbb{H}_k, \mathbb{H}_k)$, we can see it as a matrix of operators

$$\Pi_k = \begin{pmatrix} \Pi_k^{11} & \Pi_k^{12} & \Pi_k^{13} \\ \Pi_k^{21} & \Pi_k^{22} & \Pi_k^{23} \\ \Pi_k^{31} & \Pi_k^{32} & \Pi_k^{33} \end{pmatrix} = \begin{pmatrix} \Pi_k^1 \\ \Pi_k^2 \\ \Pi_k^3 \end{pmatrix}$$

adn thanks to the simple form of $\mathcal{B}_k^*$, we can easily calculate

$$f_k^+ = -\mathcal{B}_k^* \Pi_k \left(\mathbf{P}_k \mathbf{v}_k, \eta_k^1, \eta_k^2\right) = -\begin{pmatrix} 1 & 0 \end{pmatrix} \Pi_k^3 \left(\mathbf{P}_k \mathbf{v}_k, \eta_k^1, \eta_k^2\right) = -\Pi_k^{3,+} \left(\mathbf{P}_k \mathbf{v}_k, \eta_k^1, \eta_k^2\right).$$

We use here the notation $\Pi_k^{3,+}$ to denote the first line of the matrix $\Pi_k^3 = \begin{pmatrix} \Pi_k^{3,+} & \Pi_k^{3,-} \end{pmatrix}^{\mathbf{tr}}$.

Then, the control obtained above by the feedback law still stabilizes system (6.14) because of the correspondence between the two systems. Namely, we have

Proposition 6.13. *For all* $(\mathbf{v}^0, \eta_k^{1,0}, \eta_k^{2,0})$ *in* $X^0_{k,\mathbf{cc}}$*, system*

$$\begin{array}{rcll}
\mathbf{v}_{k,t} - \nu\Delta_k \mathbf{v}_k + \nabla_k q_k &=& \mathbf{0} & \textit{in } (0,\infty)\times(-1,1),\\
ikv_k^1 + v_{k,y}^2 &=& 0 & \textit{in } (0,\infty)\times(-1,1),\\
v_k^2 &=& \eta_k^{2,\kappa} & \textit{on } (0,\infty)\times\{\kappa 1\},\\
v_{k,y}^1 + ikv_k^2 &=& 0 & \textit{on } (0,\infty)\times\{\kappa 1\},\\
\eta_{k,t}^{1,\kappa} &=& \eta_k^{2,\kappa}, & \textit{on } (0,\infty),\\
\eta_{k,t}^{2,\kappa} + (\alpha k^4 + \beta k^2)\eta_k^{1,\kappa} + \gamma k^2 \eta_k^{2,\kappa} &=& \kappa(q_k - 2\nu v_{k,y}^2)(\kappa 1) & \\
 & & -\Pi_k^{3,+}\left(\mathbf{P}_k \mathbf{v}_k, \eta_k^1, \eta_k^2\right)\chi_+ & \textit{on } (0,\infty),\\
(\mathbf{v}_k(0), \eta_k^1(0), \eta_k^2(0)) &=& (\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0}). &
\end{array}$$

obtained from system (6.14) *with the feedback law* $f_k^+ = -\Pi_k^{3,+}\left(\mathbf{P}_k \mathbf{v}_k, \eta_k^1, \eta_k^2\right)$ *admits a unique solution* $(\mathbf{v}_k, q_k, \eta_k^1, \eta_k^2)$ *satisfying*

$$\|e^{\omega\cdot}(\mathbf{v}_k, \eta_k^1, \eta_k^2)\|_{L^2(0,\infty;X_k^0)} \le C\|(\mathbf{v}^0, \eta_k^{1,0}, \eta_k^{2,0})\|_{X_k^0}.$$

6.5.4 Feedback stabilization of system (6.12).

The idea is to follow the three previous sections. From now on, until the end of section, we will consider the control obtained by feedback law (for system (6.36)) $f_k^+ = -\Pi_k^{3,+}\left(\mathbf{P}_k \mathbf{v}_k, \eta_k^1, \eta_k^2\right)$ in the previous section.

Using the same change of unknowns, we write from system (6.12) system (6.47) in the variable $(\widetilde{\mathbf{v}}_k, \widetilde{\eta}_k^1, \widetilde{\eta}_k^2)$. Then, we write the equivalent system (6.48) to system (6.47) in the variables $(\mathbf{P}_k\widetilde{\mathbf{v}}_k, \widetilde{\eta}_k^1, \widetilde{\eta}_k^2)$.

The control f_k^+ above makes the operator $\mathcal{A}_{k,\omega,\Pi_k} = \mathcal{A}_{k,\omega} - \mathcal{B}_k\mathcal{B}_k^*\Pi_k = \mathcal{A}_{k,\omega} - \mathcal{B}_k\Pi_k^{3,+}$ stable on $\mathbb{H}_k$. A classic regularity result allows us to conclude this section.

More precisely, we consider the system

$$\begin{array}{rcll}
\mathbf{v}_{k,t} - \nu\Delta_k \mathbf{v}_k + \nabla_k q_k &=& \mathbf{f}_k & \text{in } (0,\infty)\times(-1,1),\\
ikv_k^1 + v_{k,y}^2 &=& 0 & \text{in } (0,\infty)\times(-1,1),\\
v_k^2 &=& \eta_k^{2,\kappa} & \text{on } (0,\infty)\times\{\kappa 1\},\\
v_{k,y}^1 + ikv_k^2 &=& 0 & \text{on } (0,\infty)\times\{\kappa 1\},\\
\eta_{k,t}^{1,\kappa} &=& \eta_k^{2,\kappa}, & \text{on } (0,\infty),\\
\eta_{k,t}^{2,\kappa} + (\alpha k^4 + \beta k^2)\eta_k^{1,\kappa} + \gamma k^2 \eta_k^{2,\kappa} &=& \kappa(q_k - 2\nu v_{k,y}^2)(\kappa 1) + h_k^\kappa & \\
 & & -\Pi_k^{3,+}(\mathbf{P}_k \mathbf{v}_k, \eta_k^1, \eta_k^2)\chi_+, & \text{on } (0,\infty),\\
(\mathbf{v}_k(0), \eta_k^1(0), \eta_k^2(0)) &=& (\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0}). &
\end{array} \qquad (6.46)$$

Then, we obtain the shifted system in the variables $(\widetilde{\mathbf{v}}_k, \widetilde{q}_k, \widetilde{\eta}_k^1, \widetilde{\eta}_k^2)$, denoting $\widetilde{\mathbf{f}}_k = \mathrm{e}^{\omega\cdot}\mathbf{f}_k$ and $\widetilde{h}_k^\kappa = \mathrm{e}^{\omega\cdot}h_k^\kappa$:

$$\begin{array}{rcll}
\widetilde{\mathbf{v}}_{k,t} - \omega\widetilde{\mathbf{v}}_k - \nu\Delta_k \widetilde{\mathbf{v}}_k + \nabla_k \widetilde{q}_k &=& \widetilde{\mathbf{f}}_k & \text{in } (0,\infty)\times(-1,1),\\
ik\widetilde{v}_k^1 + \widetilde{v}_{k,y}^2 &=& 0 & \text{in } (0,\infty)\times(-1,1),\\
\widetilde{v}_k^2 &=& \widetilde{\eta}_k^{2,\kappa} & \text{on } (0,\infty)\times\{\kappa 1\},\\
\widetilde{v}_{k,y}^1 + ik\widetilde{v}_k^2 &=& 0 & \text{on } (0,\infty)\times\{\kappa 1\},\\
\widetilde{\eta}_{k,t}^{1,\kappa} &=& \eta_k^{2,\kappa} + \omega\widetilde{\eta}_k^{1,\kappa}, & \text{on } (0,\infty),\\
\widetilde{\eta}_{k,t}^{2,\kappa} - \omega\widetilde{\eta}_k^{2,\kappa} + (\alpha k^4 + \beta k^2)\widetilde{\eta}_k^{1,\kappa} + \gamma k^2 \widetilde{\eta}_k^{2,\kappa} &=& \kappa(\widetilde{q}_k - 2\nu \widetilde{v}_{k,y}^2)(\kappa 1) + \widetilde{h}_k^\kappa & \\
 & & -\Pi_k^{3,+}(\mathbf{P}_k \widetilde{\mathbf{v}}_k, \widetilde{\eta}_k^1, \widetilde{\eta}^2)\chi_+ & \text{on } (0,\infty),\\
(\widetilde{\mathbf{v}}_k(0), \widetilde{\eta}_k^1(0), \widetilde{\eta}_k^2(0)) &=& (\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0}) &
\end{array} \qquad (6.47)$$

Now, we prove the following equivalence.

Proposition 6.14. *System* (6.47) *can be written in the variables* $(\mathbf{P}_k\widetilde{\mathbf{v}}_k, (\mathbf{I}_k-\mathbf{P}_k)\widetilde{\mathbf{v}}_k, \widetilde{\eta}_k^1, \widetilde{\eta}_k^2)$ *as follows*

$$\begin{array}{rcl}\begin{pmatrix}\mathbf{P}_k\widetilde{\mathbf{v}}_k\\ \widetilde{\eta}_k^1\\ \widetilde{\eta}_k^2\end{pmatrix}' & = & \mathcal{A}_{k,\omega,\Pi_k}\begin{pmatrix}\mathbf{P}_k\widetilde{\mathbf{v}}_k\\ \widetilde{\eta}_k^1\\ \widetilde{\eta}_k^2\end{pmatrix} + \begin{pmatrix}\mathbf{P}_k\widetilde{\mathbf{f}}_k\\ 0\\ (\mathbf{I}_2+\mathbf{N}_k)^{-1}(\widetilde{h}_k^\kappa+\pi_{\widetilde{\mathbf{f}}_k})\end{pmatrix}, \quad \begin{pmatrix}\mathbf{P}_k\widetilde{\mathbf{v}}_k\\ \widetilde{\eta}_k^1\\ \widetilde{\eta}_k^2\end{pmatrix}(0) = \begin{pmatrix}\mathbf{P}_k\mathbf{v}_k^0\\ \eta_k^{1,0}\\ \eta_k^{2,0}\end{pmatrix},\\ (\mathbf{I}_k-\mathbf{P}_k)\widetilde{\mathbf{v}}_k & = & \nabla_k N_k\widetilde{\eta}_k^2,\\ \widetilde{q}_k & = & \pi_{\widetilde{\mathbf{f}}_k} - N_k(\widetilde{\eta}_{k,t}^2 + 4\nu k^2\widetilde{\eta}_k^2),\end{array} \tag{6.48}$$

where $\pi_{\widetilde{\mathbf{f}}_k}$ *is defined from* $\widetilde{\mathbf{f}}_k$ *in* (6.50).

Proof. Taking the projection with $\mathbf{P}_k$ in the first equation of (6.46), we get directly the equation in the variable $\mathbf{P}_k\widetilde{\mathbf{v}}_k$ as follows:

$$\mathbf{P}_k\widetilde{\mathbf{v}}_k' = \mathbf{A}_k\mathbf{P}_k\widetilde{\mathbf{v}}_k + (-\mathbf{A}_k)\mathbf{P}_k\mathbf{D}_k\widetilde{\eta}_k^2 + \mathbf{P}_k\widetilde{\mathbf{f}}_k$$

but $\mathbf{P}_k\widetilde{\mathbf{f}}_k = \widetilde{\mathbf{f}}_k - \nabla_k\pi_{\widetilde{\mathbf{f}}_k}$ with $\pi_{\widetilde{\mathbf{f}}_k}$ the solution of

$$\Delta_k\pi_{\widetilde{\mathbf{f}}_k} = \mathrm{div}_k\widetilde{\mathbf{f}}_k \quad \text{in } (-1,1) \qquad \text{and} \qquad \pi_{\widetilde{\mathbf{f}}_k,y} = f_k^2 \quad \text{for } y=\kappa 1.$$

Remark that $\widetilde{\mathbf{f}}_k = (f_k^1, f_k^2)$. We decompose $\pi_{\widetilde{\mathbf{f}}_k}$ in two pressure terms, one, namely $\pi^1_{\widetilde{\mathbf{f}}_k}$ solution of the homogeneous Dirichlet Laplace equation

$$\Delta_k\pi^1_{\widetilde{\mathbf{f}}_k} = \mathrm{div}_k\widetilde{\mathbf{f}}_k \quad \text{in } (-1,1) \qquad \text{and} \qquad \pi^1_{\widetilde{\mathbf{f}}_k} \in H_0^1(-1,1;\mathbb{C}), \tag{6.49}$$

the other, namely $\pi^2_{\widetilde{\mathbf{f}}_k}$, solution of the following equation depending on $\pi^1_{\widetilde{\mathbf{f}}_k}$

$$\Delta_k\pi^2_{\widetilde{\mathbf{f}}_k} = 0 \quad \text{in } (-1,1) \qquad \text{and} \qquad \pi^2_{\widetilde{\mathbf{f}}_k,y} = f_k^2 - \pi^1_{\widetilde{\mathbf{f}}_k,y} \text{ for } y=\kappa 1$$

such that $\pi_{\widetilde{\mathbf{f}}_k} = \pi^1_{\widetilde{\mathbf{f}}_k} + \pi^2_{\widetilde{\mathbf{f}}_k}$. Denoting $\pi^1_{\widetilde{\mathbf{f}}_k} = -(-\Delta_{k,D})^{-1}(\mathrm{div}_k\widetilde{\mathbf{f}}_k)$ the solution of (6.49), we get that

$$\pi_{\widetilde{\mathbf{f}}_k} = -(-\Delta_{k,D})^{-1}(\mathrm{div}_k\widetilde{\mathbf{f}}_k) + N_k\left(\left(\widetilde{\mathbf{f}}_k + \nabla_k((-\Delta_{k,D})^{-1}(\mathrm{div}_k\widetilde{\mathbf{f}}_k))(\kappa 1)\right)\cdot\mathbf{e}_2\right). \tag{6.50}$$

This calculation allows us to obtain the pressure term $\widetilde{q}_k$ in the right-hand side of the beam equation. Indeed, $(\mathbf{I}_k-\mathbf{P}_k)\widetilde{\mathbf{v}}_k = \nabla_k\phi_k$ with $\phi_k = N_k\widetilde{\eta}_k^2$ and putting this term in the first equation of (6.12), we get

$$\phi_{k,t} - \nu\Delta_k\phi_k + \widetilde{q}_k^s = \pi_{\widetilde{\mathbf{f}}_k}.$$

Thus, $\widetilde{q}_k^s = \pi_{\widetilde{\mathbf{f}}_k} - \phi_{k,t} = \pi_{\widetilde{\mathbf{f}}_k} - N_k\widetilde{\eta}_{k,t}^2$. Furthermore, the pressure term $\widetilde{q}_k^e$ associated to $\mathbf{P}_k\widetilde{\mathbf{v}}_k$ satisfies the equation

$$\Delta_k\widetilde{q}_k^e = 0 \quad \text{in } (-1,1) \qquad \text{and} \qquad \widetilde{q}_{k,y}^e = -2\nu k^2\widetilde{\eta}_k^{2,\kappa} \quad \text{for } y=\kappa 1,$$

that is $\widetilde{q}_k^e = -2\nu k^2 N_k\widetilde{\eta}_k^2$. Finally, $\widetilde{q}_k = \widetilde{q}_k^s + \widetilde{q}_k^e$ satisfies

$$\widetilde{q}_k = \pi_{\widetilde{\mathbf{f}}_k} - N_k(\widetilde{\eta}_{k,t}^2 + 4\nu k^2\widetilde{\eta}_k^2).$$

Then, following Proposition 6.5, we get the equivalent system (6.48). □

We can now state the main result of this section.

Theorem 6.15. *Let* $(\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0})$ *in* $\mathbf{V}^1_{\#,k}(-1,1;\mathbb{C}) \times H^3_{\#,k}(\mathbb{C}^2) \times H^1_{\#,k}(\mathbb{C}^2)$ *satisfying the compatibility condition* $v_k^{0,2}(\kappa 1) = \eta_k^{2,0,\kappa}$ *and* $(\widetilde{\mathbf{f}}_k, \widetilde{h}_k)$ *in* $L^2(0,\infty;\mathbf{L}^2(-1,1;\mathbb{C})\times\mathbb{C}^2)$*, then system* (6.47) *admits a unique solution* $(\widetilde{\mathbf{v}}_k, \widetilde{q}_k, \widetilde{\eta}_k^1, \widetilde{\eta}_k^2)$ *in* $\mathbf{H}^{2,1}_{\#,k}((0,\infty)\times(-1,1);\mathbb{C}) \times L^2(0,\infty;H^1_{\#,k}(-1,1;\mathbb{C})) \times H^{4,2}_{\#,k}((0,\infty)\times\mathbb{C}^2) \times H^{2,1}_{\#,k}((0,\infty)\times\mathbb{C}^2)$ *which satisfies*

$$\begin{array}{rl} & \|\widetilde{\mathbf{v}}_k\|_{\mathbf{H}^{2,1}_{\#,k}((0,\infty)\times(-1,1);\mathbb{C})} + \|\widetilde{q}_k\|_{L^2(0,\infty;H^1_{\#,k}(-1,1;\mathbb{C}))} + \|\widetilde{\eta}_k^1\|_{H^{4,2}_{\#,k}((0,\infty)\times\mathbb{C}^2)} + \|\widetilde{\eta}_k^2\|_{H^{2,1}_{\#,k}((0,\infty)\times\mathbb{C}^2)}\\ \leq & C\left(\|(\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0})\|_{\mathbf{H}^1_{\#,k}(-1,1;\mathbb{C})\times H^3_{\#,k}(\mathbb{C}^2)\times H^1_{\#,k}(\mathbb{C}^2)} + \|(\widetilde{\mathbf{f}}_k, \widetilde{h}_k)\|_{L^2(0,\infty;\mathbf{L}^2(-1,1;\mathbb{C})\times\mathbb{C}^2)}\right).\end{array}$$

Proof. Thanks to the previous section, we know that the feedback control $\widetilde{f}_k^+ = -\Pi_k^{3,+}(\mathbf{P}_k\widetilde{\mathbf{v}}_k, \widetilde{\eta}_k^1, \widetilde{\eta}_k^2)$ stabilize on $\mathbb{H}_k$ the system $(6.37)_1$. From $(\widetilde{\mathbf{f}}_k, \widetilde{h}_k)$ in $L^2(0,\infty;\mathbf{L}^2(-1,1;\mathbb{C})\times\mathbb{C}^2)$, we get that $(\mathbf{P}_k\widetilde{\mathbf{f}}_k, 0, (\mathbf{I}_2 + \mathbf{N}_k)^{-1}(\widetilde{h}_k + \pi_{\widetilde{\mathbf{f}}_k}))^{\mathbf{tr}}$ belongs to $L^2(0,\infty;\mathbb{H}_k)$. Then, because the operator $\mathcal{A}_{k,\omega,\Pi_k}$ is exponentially stable on $\mathbb{H}_k$, we get that system $(6.48)_1$ admits a unique solution $(\mathbf{P}_k\widetilde{\mathbf{v}}_k, \widetilde{\eta}_k^1, \widetilde{\eta}_k^2)$ in $L^2(0,\infty;\mathbb{H}_k)$. Thanks now to the regularity of the initial data, we get by classical regularity result that this solution belongs to $L^2(0,\infty;D(\mathcal{A}_k)) \cap H^1(0,\infty;\mathbb{H}_k) \cap \mathcal{C}(0,\infty;[D(\mathcal{A}_k),\mathbb{H}_k]_{1/2})$, which concludes the proof. □

We go back to system (6.12) using again the correspondance between systems (6.12) and (6.47).

Proposition 6.16. *Let $(\mathbf{v}_k^0, \eta_k^{1,0}, \eta_k^{2,0})$ be in $\mathbf{V}^1_{\#,k}(-1,1;\mathbb{C}) \times H^3_{\#,k}(\mathbb{C}^2) \times L^2_{\#,k}(\mathbb{C}^2)$ satisfying the condition $v_k^{0,2}(\kappa 1) = \eta_k^{2,0,\kappa}$ and $(\mathbf{f}_k, h_k)$ such that $(\widetilde{\mathbf{f}}_k, \widetilde{h}_k)$ belongs to $L^2(0,\infty;\mathbf{L}^2(-1,1;\mathbb{C})\times\mathbb{C}^2)$. Then, system (6.46) admits a unique solution $(\mathbf{v}_k, q_k, \eta_k^1, \eta_k^2)$ such that $e^{\omega\cdot}(\mathbf{v}_k, q_k, \eta_k^1, \eta_k^2)$ belongs to $\mathbf{H}^{2,1}_{\#,k}((0,\infty)\times(-1,1);\mathbb{C}) \times L^2(0,\infty;H^1_{\#,k}(-1,1;\mathbb{C})) \times H^{4,2}_{\#,k}((0,\infty)\times\mathbb{C}^2) \times H^{2,1}_{\#,k}((0,\infty)\times\mathbb{C}^2)$ and satisfies the estimate*

$$\begin{aligned} & \|e^{\omega\cdot}\mathbf{v}_k\|^2_{\mathbf{H}^{2,1}_{\#,k}((0,\infty)\times(-1,1);\mathbb{C})} + \|e^{\omega\cdot}q_k\|^2_{L^2(0,\infty;H^1_{\#,k}(-1,1;\mathbb{C}))} + \|e^{\omega\cdot}\eta_k^1\|^2_{H^{4,2}_{\#,k}((0,\infty)\times\mathbb{C}^2)} + \|e^{\omega\cdot}\eta_k^2\|^2_{H^{2,1}_{\#,k}((0,\infty)\times\mathbb{C}^2)} \\ \leq\ & C\left(\|(\mathbf{v}^0, \eta_k^1, \eta^2)\|^2_{\mathbf{H}^1_{\#,k}(-1,1;\mathbb{C})\times H^3_{\#,k}(\mathbb{C}^2)\times H^1_{\#,k}(\mathbb{C}^2)} + \|e^{\omega\cdot}(\mathbf{f}_k, h_k)\|^2_{L^2(0,\infty;\mathbf{L}^2(-1,1;\mathbb{C})\times\mathbb{C}^2)}\right). \end{aligned}$$

6.6 Stabilization of equation (6.13).

For $k = 0$, system (6.11) becomes

$$\begin{array}{rcll} v^1_{0,t} - \nu v^1_{0,yy} & = & f_0^1 & \text{in } (0,\infty)\times(-1,1), \\ v^2_{0,t} - \nu v^2_{0,yy} + q_{0,y} & = & f_0^2 & \text{in } (0,\infty)\times(-1,1), \\ v^2_{0,y} & = & 0 & \text{in } (0,\infty)\times(-1,1), \\ v_0^2 & = & 0 & \text{on } (0,\infty)\times\{-1\}, \\ v^1_{0,y} & = & f_0^+\chi_{\Gamma_0^+} & \text{on } (0,\infty)\times\{1\}, \\ p_0 - 2\nu v^2_{0,y} & = & 0 & \text{on } (0,\infty)\times\{\kappa 1\}, \\ \mathbf{v}_0 & = & \mathbf{v}_0^0 & \text{in } (-1,1). \end{array}$$

This leads to $v_0^2 \equiv 0$ in $(-1,1)$ and v_0^1 satisfies the heat equation with Neumann boundary conditions (6.13). We now consider the homogeneous system

$$\begin{array}{rcll} v^1_{0,t} - \nu v^1_{0,yy} & = & 0 & \text{in } (0,\infty)\times(-1,1), \\ v^1_{0,y} & = & f_0^+ & \text{on } (0,\infty)\times\{1\}, \\ v^1_{0,y} & = & 0 & \text{on } (0,\infty)\times\{-1\}. \end{array} \tag{6.51}$$

This can be written in an abstract setting using the operator $\mathcal{A}_0 = \nu\Delta_N$ Laplace operator with homogeneous Neumann boundary condition defined on $L^2(-1,1;\mathbb{R})$ with domain

$$D(\mathcal{A}_0) = \left\{z_0^1 \in H^2(-1,1) \text{ s.t. } z^1_{0,y} = 0 \text{ on } (\kappa 1)\right\}$$

and a lifting of the nonhomogeneous Neumann boundary condition N_0^+ defined by $w_0^1 = N_0^+ g^+$ iff

$$\begin{array}{rcll} \theta_0 w_0^1 - \nu w^1_{0,yy} & = & 0 & \text{in } (-1,1), \\ w^1_{0,y} & = & f_0^+ & \text{for } y = 1, \\ w^1_{0,y} & = & 0 & \text{for } y = -1 \end{array}$$

for a positive constant θ_0 large enough. Then, system (6.51) becomes, with the notations $v_0^1 = \widetilde{v}_0^1 + w_0^1$,

$$\begin{array}{rcll} \widetilde{v}^1_{0,t} - \nu\widetilde{v}^1_{0,yy} & = & -w^1_{0,t} + \theta_0 w_0^1 & \text{in } (0,\infty)\times(-1,1), \\ \widetilde{v}^1_{0,y} & = & 0 & \text{on } (0,\infty)\times\{\kappa 1\}, \\ \widetilde{v}_0^1(0) & = & v_0^{0,1} & \text{in } (-1,1) \end{array}$$

which gives, with the extension by interpolation on $L^2(-1,1;\mathbb{R})$ of the operator $(-\mathcal{A}_0)$ denoted $(-\widetilde{\mathcal{A}_0})$, the following abstract setting

$$\begin{array}{rcll} v^1_{0,t} & = & \mathcal{A}_0 v^1_0 + (\theta_0 + \nu(-\widetilde{\mathcal{A}_0}))N^+_0 f^+_0 & \text{in } (0,\infty)\times(-1,1), \\ v^1_0(0) & = & v^{0,1}_0 & \text{in } (-1,1). \end{array}$$

We state the different results we need on the operator $-\mathcal{A}_0$ in the following proposition:

Proposition 6.17. *The operator $\mathcal{A}_0$ generates an analytic semigroup on $L^2(1,1;\mathbb{R})$. For each λ in the resolvent set $\rho(\mathcal{A}_0)$ of $\mathcal{A}_0$, the operator $(\lambda\mathcal{I}_0 - \mathcal{A}_0)^{-1}$ is compact. Morevover, the different eigenvalues are given by*

$$\lambda^p_0 = -\nu\left(\frac{p\pi}{2}\right)^2, \quad p \in \mathbb{N}$$

associated with the eigenfunctions

$$\begin{array}{rcll} v^{1,0}_0 & = & \dfrac{1}{\sqrt{2}}, & \\ v^{1,p}_0 & = & \cos\left(\dfrac{p\pi}{2}y\right) & \text{if } p = 2p',\ p' = 1,\dots,\infty, \\ v^{1,p}_0 & = & \sin\left(\dfrac{p\pi}{2}y\right) & \text{if } p = (2p'+1),\ p' = 0,\dots,\infty. \end{array} \tag{6.52}$$

Proof. The proof is very classic and is left to the reader. □

6.6.1 Stabilization of system (6.51).

We can prove, in the same way as system (6.14) (for $k \neq 0$) is stabilizable for any initial data in $X^0_{k,\mathbf{cc}}$ and for any decay rate $\omega > 0$, that system (6.51) is stabilizable for any initial data $v^{0,1}_0$ in $X^0_0 = L^2(-1,1;\mathbb{R})$ and $v^{0,2}_0 = 0$ with a control f^+_0 in $L^2(0,\infty;\mathbb{R})$, namely

Proposition 6.18. *Let $\omega > 0$ be a decay rate. Let $v^{0,1}_0$ be in $L^2(-1,1;\mathbb{R})$. Then, there exists a control f^+_0 in $L^2(0,\infty;\mathbb{R})$ such that the solution v^1_0 of equation (6.51) with initial data $v^{0,1}_0$ and f^+_0 as a Neumann boundary condition satisfies the exponential decay:*

$$\|e^{\omega\cdot} v^1_0\|_{L^2((0,\infty)\times(-1,1);\mathbb{R})} \le C\|v^{0,1}_0\|_{L^2(-1,1;\mathbb{R})}, \qquad \text{for all } t \ge 0.$$

Furthermore, the control function f^+_0 satisfies the exponential decay

$$|f^+_0(t)| \le C_0 e^{-\omega_0 t}\|v^{0,1}_0\|_{L^2(-1,1;\mathbb{R})}, \qquad \text{for all } t \ge 0$$

for $\omega_0 > 0$ large enough.

Proof. The same work as above gives directly the result. Indeed, the unique ocntinuation property (to prove the approximative controllability) is reduced to prove that the eigenfunctions of the operator $\mathcal{A}_0$ do not vanish at the boundary $y = 1$ which is clear thanks to the explicit form of this eigenvalues (see (6.52)).

To obtain the exponential decay rate $\omega > 0$, we use a new variable $\widetilde{v}^1_0 = \mathrm{e}^{\omega\cdot} v^1_0$, then the same work as in Theorem 6.10 for system (6.14) can be done here. □

6.6.2 Feedback stabilization of system (6.51).

The feedback law to the stabilization of system (6.51) can be shown by solving the infinite time horizon control problem $(\mathcal{P}^{0,\infty}_{0,v^{0,1}_0})$ (for $k = 0$)

$$\inf\left\{\mathcal{J}_0(v^1_0; f^+_0) \quad \text{s.t.} \quad (v^1_0; f^+_0) \text{ satisfies (6.51) with } f^+_0 \text{ in } L^2(0,\infty;\mathbb{R})\right\}$$

where

$$\mathcal{J}_0(v^1_0; f^+_0) = \frac{1}{2}\int_0^\infty |P^u_0 v^1_0(t)|^2 \mathrm{d}t + \frac{1}{2}\int_0^\infty |f^+_0(t)|^2 \mathrm{d}t.$$

The different notations here are P_0^u is the projection from $L^2(-1,1;\mathbb{R})$ onto the unstable eigenspace of the operator $\mathcal{A}_{0,\omega} = \mathcal{A}_0 + \omega I$, this space will be denoted $\mathbb{H}_0^u$ to be coherent with the previous section. There exists a number $n_0 \geq 0$ such that $\mathbb{H}_0^u = \mathrm{Span}\Big\{\phi_0^p, p = 0, \dots, n_0\Big\}$ where $\{\phi_0^p\}_{p\geq 0}$ is the family of eigenfunctions of $\mathcal{A}_0$ corresponding with the oredered eigenvalues $\{\lambda_0^p\}_{p\geq 0}$ (see section 6.6).

Then, we can prove the following result

Theorem 6.19. *For all $v_0^{0,1}$ in $\mathbb{H}_0$, problem $(\mathcal{P}^{0,\infty}_{0,v_0^{0,1}})$ admits a unique solution $(v^1_{0v_0^{0,1}}; f^+_{0v_0^{0,1}})$. There exists Π_0 in $\mathcal{L}(\mathbb{H}_0, (\mathbb{H}_0)^*)$ obeying $\Pi_0 = \Pi_0^* \geq 0$ such that the optimal cost is given by*

$$\inf(\mathcal{P}^{0,\infty}_{0,v_0^{0,1}}) = \frac{1}{2}\left(v_0^{0,1}, \Pi_0 v_0^{0,1}\right)_{\mathbb{H}_0,(\mathbb{H}_0)^*}.$$

Moreover, Π_0 is obtained as $\Pi_0 = (P_0^u)^*\Pi_0^u P_0^u$ where Π_0^u is the solution to the finite dimensional algebraic Riccati equation

$$\Pi_0^u \text{ in } \mathcal{L}(\mathbb{H}_0^u, (\mathbb{H}_0^u)^*), \quad \Pi_0^u = (\Pi_0^u)^* \geq 0, \qquad \Pi_0^u\mathcal{A}_0^u + (\mathcal{A}_0^u)^*\Pi_0^u - \Pi_0^u B_0^u (B_0^u)^*\Pi_0^u + I_0 = 0, \tag{6.53}$$

where

$$\begin{aligned}
\mathcal{A}_0^u &= P_0^u\mathcal{A}_0 P_0^u \in \mathcal{L}(\mathbb{H}_0^u, \mathbb{H}_0^u), & (\mathcal{A}_0^u)^* &= (P_0^u\mathcal{A}_0P_0^u)^* = (P_0^u)^*\,\mathcal{A}_0^*\,(P_0^u)^* \in \mathcal{L}((\mathbb{H}_0^u)^*, (\mathbb{H}_0^u)^*),\\
\mathcal{B}_0^u &= P_0^u\mathcal{B}_0 \in \mathcal{L}(\mathbb{R}, \mathbb{H}_0^u), & (\mathcal{B}_0^u)^* &= (P_0^u\mathcal{B}_0)^* \in \mathcal{L}((\mathbb{H}_0^u)^*, \mathbb{R}),\\
I_0 &\in \mathcal{L}(\mathbb{H}_0^u, (\mathbb{H}_0^u)^*) \text{ is the identity.}
\end{aligned}$$

From (6.53), we get the algebraic Riccati equation satisfied by Π_0:

$$\Pi_0 \text{ in } \mathcal{L}(\mathbb{H}_0, (\mathbb{H}_0)^*), \quad \Pi_0 = (\Pi_0)^* \geq 0, \qquad \Pi_0\mathcal{A}_0^u + (\mathcal{A}_0^u)^*\Pi_0 - \Pi_0 B_0 B_0^*\Pi_0 + (P_0^u)^*(P_0^u) = 0. \tag{6.54}$$

6.6.3 Feedback stabilization of system (6.13).

System (6.13) can be written in the semigroup setting as follows:

$$v^1_{0,t} = \mathcal{A}_0 v_0^1 - (\theta_0 + \nu\widetilde{(-\mathcal{A}_0)})N_0^+\Pi_0(v_0^1) + f_0^1 \qquad \text{in } (-1,1), \qquad v_0^1(0) = v_0^{0,1} \tag{6.55}$$

where we take the control obtained by the feedback law of the previous section. Thus, the operator is exponential stable (with decay rate ω) on $\mathbb{H}_0$. Thus, we can state the equivalent to Proposition 6.16 to system (6.13):

Theorem 6.20. *Let $v_0^{0,1}$ be in $H^1(-1,1;\mathbb{R})$ and f_0^1 such that $e^{\omega\cdot}f_0^1$ belongs to $L^2((0,\infty)\times(-1,1);\mathbb{R})$, then system (6.55) admits a unique solution v_0^1 such that*

$$\|e^{\omega\cdot}v_0^1\|_{H^{2,1}((0,\infty)\times(-1,1);\mathbb{R})} \leq C\left(\|v_0^{0,1}\|_{H^1(-1,1;\mathbb{R})} + \|e^{\omega\cdot}f_0^1\|_{L^2((0,\infty)\times(-1,1);\mathbb{R}))}\right).$$

6.7 Stabilization of system (6.11).

In this section, we collect all the different results of the previous sections to get the stabilization of the nonhomogeneous system (6.11) in the variables $(\mathbf{v}, p, \eta)$ in the fix domain. More precisely, we consider system (6.11) where the controls f_0^+ and f^+ are the ones obtained thanks to the previous feedback laws, that is

$$f_0^+ = -\mathcal{B}_0^*\Pi_0(v_0^1), \qquad f^+ = -\sum_{|k|\leq M_\omega; k\neq 0} \Pi_k^{3,+}(\mathbf{P}_k\mathbf{v}_k, \eta_k^1, \eta_k^2)\mathrm{e}^{ik\cdot}.$$

We introduce the space of initial data

$$X^0_{\#,\mathbf{cc}} = \left\{(\mathbf{z}^0, \mu^{1,0}, \mu^{2,0}) \text{ in } \mathbf{V}^1_\#(\Omega_0)\times H^3_\#(\Gamma_0)\times H^1_\#(\Gamma_0) \quad \text{satisfying} \quad z^{0,2} = \mu^{2,0,\kappa} \text{ on } \Gamma_0^\kappa\right\}$$

endowed with the norm of $\|\cdot\|_{X^0_\#}$ of the Hilbert space $X^0_\# = \mathbf{V}^1_\#(\Omega_0)\times H^3_\#(\Gamma_0)\times H^1_\#(\Gamma_0)$ defined, for every $(\mathbf{z}^0, \mu^{1,0}, \mu^{2,0})$ in $X^0_\#$, by

$$\|(\mathbf{z}^0, \mu^{1,0}, \mu^{2,0})\|_{X^0_\#} = \left(\|\mathbf{z}^0\|^2_{\mathbf{V}^1_\#(\Omega_0)} + \|\mu^{1,0}\|^2_{H^3_\#(\Gamma_0)} + \|\mu^{2,0}\|^2_{H^1_\#(\Gamma_0)}\right)^{\frac{1}{2}}.$$

Then, we prove the following result:

Theorem 6.21. *Let* $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ *be in* $X^0_{\#,\mathbf{cc}}$ *and* $(\mathbf{f}, h)$ *be such that* $e^{\omega\cdot}(\mathbf{f}, h)$ *is in* $L^2(0,\infty; \mathbf{L}^2_\#(\Omega_0;\mathbb{R}) \times L^2_\#(\Gamma_0;\mathbb{R}))$*, then system*

$$\begin{array}{rcll} \mathbf{v}_t - div\, \sigma(\mathbf{v}, q) & = & \mathbf{f} & in\ Q^0_\infty, \\ div\, \mathbf{v} & = & 0 & in\ Q^0_\infty, \\ v_2 & = & \eta^\kappa_t & on\ \Sigma^0_\infty, \\ \mathbf{S}(\mathbf{v})\mathbf{e}_2 \cdot \mathbf{e}_1 & = & -\mathcal{B}^*_0 \Pi_0(v^1_0)\chi_+ & on\ \Sigma^0_\infty, \\ \eta^\kappa_{tt} + \alpha\eta^\kappa_{xxxx} - \beta\eta^\kappa_{xx} - \gamma\eta^\kappa_{txx} & = & \kappa[q - 2\nu v_{2,y}] + h^\kappa - \left(\displaystyle\sum_{|k| \leq M_\omega; k \neq 0} \Pi^{3,+}_k (\mathbf{P}_k \mathbf{v}_k, \eta^1_k, \eta^2_k) e^{ik\cdot} \right) \chi_+ & on\ \Sigma^0_\infty, \\ (\mathbf{v}(0), \eta(0), \eta_t(0)) & = & (\mathbf{v}^0, \eta^{1,0}, \eta^{2,0}) & \end{array}$$

admits a unique solution $(\mathbf{v}, q, \eta)$ *such that* $(\mathbf{v}, q, \eta, \eta_t)$ *belongs to*

$$X^\infty_{\#,\omega} = \left\{ \left(\mathbf{z}, r, \mu^1, \mu^2\right)\ s.t.\ e^{\omega\cdot}(\mathbf{z}, r, \mu^1, \mu^2)\ belongs\ to\ X^\infty_{\#,0} \right\}$$

where

$$X^\infty_{\#,0} = \mathbf{H}^{2,1}_\#(Q^0_\infty) \times L^2(0,\infty; \mathcal{H}^1_\#(\Omega_0)) \times H^{4,2}_\#(\Sigma^0_\infty) \times H^{2,1}_\#(\Sigma^0_\infty)$$

endowed with the norm $\|\cdot\|_{X^\infty_{\#,0}}$ *defined, for all* $(\mathbf{z}, r, \mu^1, \mu^2)$ *in* $X^\infty_{\#,0}$*, by*

$$\|(\mathbf{z}, r, \mu^1, \mu^2)\|_{X^\infty_{\#,0}} = \left(\|\mathbf{z}\|^2_{\mathbf{H}^{2,1}_\#(Q^0_\infty)} + \|r\|^2_{L^2(0,\infty;\mathcal{H}^1_\#(\Omega_0))} + \|\mu^1\|^2_{H^{4,2}_\#(\Sigma^0_\infty)} + \|\mu^2\|^2_{H^{2,1}_\#(\Sigma^0_\infty)} \right)^{\frac{1}{2}}.$$

Furthermore, the solution $(\mathbf{v}, q, \eta)$ *satisfies the following estimate*

$$\|e^{\omega\cdot}(\mathbf{v}, q, \eta, \eta_t)\|_{X^\infty_{\#,0}} \leq C_1 \left(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0_\#} + \|e^{\omega\cdot}(\mathbf{f}, h)\|_{L^2(0,\infty;\mathbf{L}^2_\#(\Omega_0)\times L^2_{\#,0}(\Gamma_0))} \right).$$

Proof. From the last two sections, we get the existence and uniqueness. The fact that $(\mathbf{v}, p, \eta)$ are real functions comes from the fact that for conjugate initial data, we get conjugate controls f^+_k. Indeed, let us have a look to the system satsifed by $(\overline{\mathbf{v}_{-k}}, \overline{p_{-k}}, \overline{\eta_{-k}})$. From (6.14), we have first (by changing k in $-k$)

$$\begin{array}{rcll} \mathbf{v}_{-k,t} - \nu\Delta_{-k}\mathbf{v}_{-k} + \nabla_{-k}p_{-k} & = & \mathbf{0} & \text{in } (0,\infty)\times(-1,1), \\ i(-k)v^1_{-k} + v^2_{-k,y} & = & 0 & \text{in } (0,\infty)\times(-1,1), \\ v^2_{-k} & = & \eta^{2,\kappa}_{-k} & \text{on } (0,\infty)\times\{\kappa 1\}, \\ v^1_{-k,y} + i(-k)v^2_{-k} & = & 0 & \text{on } (0,\infty)\times\{\kappa 1\}, \\ \eta^{1,\kappa}_{-k,t} & = & \eta^{2,\kappa}_{-k}, & \text{on } (0,\infty) \\ \eta^{2,\kappa}_{-k,t} + (\alpha k^4 + \beta k^2)\eta^{1,\kappa}_{-k} + \gamma k^2 \eta^{2,\kappa}_{-k} & = & \kappa(p_{-k} - 2\nu v^2_{-k,y})(\kappa 1) + f^+_{-k}\chi_{\Gamma^+_0}, & \text{on } (0,\infty) \\ (\mathbf{v}_{-k}(0), \eta^1_{-k}(0), \eta^2_{-k}(0)) & = & (\mathbf{v}^0_{-k}, \eta^{1,0}_{-k}, \eta^{2,0}_{-k}). & \end{array}$$

Then, we take the conjugate system:

$$\begin{array}{rcll} \overline{\mathbf{v}_{-k,t}} - \nu\Delta_k\overline{\mathbf{v}_{-k}} + \nabla_k\overline{p_{-k}} & = & \mathbf{0} & \text{in } (0,\infty)\times(-1,1), \\ (-i)(-k)\overline{v^1_{-k}} + \overline{v^2_{-k,y}} & = & 0 & \text{in } (0,\infty)\times(-1,1), \\ \overline{v^2_{-k}} & = & \overline{\eta^{2,\kappa}_{-k}} & \text{on } (0,\infty)\times\{\kappa 1\}, \\ \overline{v^1_{-k,y}} + (-i)(-k)\overline{v^2_{-k}} & = & 0 & \text{on } (0,\infty)\times\{\kappa 1\}, \\ \overline{\eta^{1,\kappa}_{-k,t}} & = & \overline{\eta^{2,\kappa}_{-k}}, & \text{on } (0,\infty) \\ \overline{\eta^{2,\kappa}_{-k,t}} + (\alpha k^4 + \beta k^2)\overline{\eta^{1,\kappa}_{-k}} + \gamma k^2 \overline{\eta^{2,\kappa}_{-k}} & = & \kappa\overline{(p_{-k} - 2\nu v^2_{-k,y})}(\kappa 1) + \overline{f^+_{-k}}\chi_{\Gamma^+_0}, & \text{on } (0,\infty) \\ (\overline{\mathbf{v}_{-k}(0)}, \overline{\eta^1_{-k}(0)}, \overline{\eta^2_{-k}(0)}) & = & (\overline{\mathbf{v}^0_{-k}}, \overline{\eta^{1,0}_{-k}}, \overline{\eta^{2,0}_{-k}}). & \end{array}$$

That is, after little simplifications, $(\overline{\mathbf{v}_{-k}}, \overline{p_{-k}}, \overline{\eta_{-k}})$ satisfies system (6.14) with the same initial data (because the initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ are real and thus $\mathbf{v}^0_k$ is conjugate to $\mathbf{v}^0_{-k}$, *e.g.*) and a control $\overline{f^+_{-k}}$. Finally, we get that $f^+_k = \overline{f^+_{-k}}$ which gives $(\mathbf{v}_k, p_k, \eta_k) = (\overline{\mathbf{v}_{-k}}, \overline{p_{-k}}, \overline{\eta_{-k}})$ and that the solution $(\mathbf{v}, p, \eta)$ has real values.

□

6.8 Proof of Theorem 6.2.

In the section, we prove the main result of this paper. We rewrite here system (6.4) in the fixed domain where now the controls f_0^+ and f^+ are obtained with the feedback laws $f_0^+ = -\mathcal{B}_0^*\Pi_0(v_0^1)$ and $f^+ = -\Pi(\mathbf{v},\eta,\eta_t) = -\sum_{|k|\le M_\omega;k\neq 0}\Pi_k^{3,+}(\mathbf{P}_k\mathbf{v}_k,\eta_k^1,\eta_k^2)$,

$$\begin{array}{rcll}
\mathbf{u}_t-\nu\Delta\mathbf{u}+\nabla p &=& \mathbf{F}[\mathbf{u},p,\eta] & \text{in } Q_\infty^0,\\
\text{div }\mathbf{u} &=& g[\mathbf{u},\eta] & \text{in } Q_\infty^0,\\
u_2 &=& \eta_t^\kappa + j^\kappa[\mathbf{u},\eta] & \text{on } \Sigma_\infty^{\kappa,0},\\
u_{1,y}+u_{2,x} &=& l^\kappa[\mathbf{u},\eta]-\Pi_0(u_0^1)\chi_{\Gamma_0^+} & \text{on } \Sigma_\infty^{\kappa,0},\\
\eta_{tt}^\kappa+\alpha\eta_{xxxx}^\kappa-\beta\eta_{xx}^\kappa-\gamma\eta_{txx}^\kappa &=& \kappa[p-2\nu u_{2,y}]+H^\kappa[\mathbf{u},\eta]-\Pi(\mathbf{u},\eta,\eta_t)\chi_{\Gamma_0^+} & \text{on } \Sigma_\infty^{\kappa,0},\\
(\mathbf{u}(0),\eta(0),\eta_t(0)) &=& (\mathbf{u}^0,\eta^{1,0},\eta^{2,0}) &
\end{array} \tag{6.56}$$

The shifted system in the variables $(\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2) = e^{\omega\cdot}(\mathbf{u},p,\eta,\eta_t)$ where $(\mathbf{u},p,\eta)$ is solution of system (6.56) is

$$\begin{array}{rcll}
\widetilde{\mathbf{u}}_t-\omega\widetilde{\mathbf{u}}-\nu\Delta\widetilde{\mathbf{u}}+\nabla\widetilde{p} &=& e^{-\omega\cdot}\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2] & \text{in } Q_\infty^0,\\
\text{div }\widetilde{\mathbf{u}} &=& e^{-\omega\cdot}\widetilde{g}[\widetilde{\mathbf{u}},\widetilde{\eta}^1] & \text{in } Q_\infty^0,\\
\widetilde{u}_2 &=& \widetilde{\eta}^{2,\kappa}+e^{-\omega\cdot}\widetilde{j}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1] & \text{on } \Sigma_\infty^{\kappa,0},\\
\widetilde{u}_{1,y}+\widetilde{u}_{2,x} &=& e^{-\omega\cdot}\widetilde{l}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1]-\Pi_0(\widetilde{u}_0^1)\chi_{\Gamma_0^+} & \text{on } \Sigma_\infty^{\kappa,0},\\
\widetilde{\eta}_t^{1,\kappa} &=& \widetilde{\eta}^{2,\kappa}+\omega\widetilde{\eta}^{1,\kappa} & \text{on } \Sigma_\infty^{\kappa,0},\\
\widetilde{\eta}_t^{2,\kappa}-\omega\widetilde{\eta}^{2,\kappa}+\alpha\widetilde{\eta}_{xxxx}^{1,\kappa}-\beta\widetilde{\eta}_{xx}^{1,\kappa}-\gamma\widetilde{\eta}_{xx}^{2,\kappa} &=& \kappa[\widetilde{p}-2\nu\widetilde{u}_{2,y}]+e^{-\omega\cdot}\widetilde{H}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1]-\Pi(\widetilde{\mathbf{u}},\widetilde{\eta}^1,\widetilde{\eta}^2)\chi_{\Gamma_0^+} & \text{on } \Sigma_\infty^{\kappa,0},\\
(\widetilde{\mathbf{u}}(0),\widetilde{\eta}(0),\widetilde{\eta}_t(0)) &=& (\mathbf{u}^0,\eta^{1,0},\eta^{2,0}) &
\end{array} \tag{6.57}$$

where the right-hand sides $(\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2],\widetilde{g}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{j}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{l}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{H}[\widetilde{\mathbf{u}},\widetilde{\eta}^1])$ are calculated from the right-hand sides of (6.56) $(\mathbf{F}[\mathbf{u},p,\eta],g[\mathbf{u},\eta],j[\mathbf{u},\eta],l[\mathbf{u},\eta],H[\mathbf{u},\eta])$ defined in (6.5) as follows:

$$\begin{array}{rcl}
\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2] &=& -\widetilde{d}\,(\widetilde{\mathbf{u}}_t-\omega\widetilde{\mathbf{u}})+\left(\widetilde{\eta}^{2,+}\left(\dfrac{1+z}{2}\right)+\widetilde{\eta}^{2,-}\left(\dfrac{1-z}{2}\right)-\nu(\widetilde{m}_{xx}+z\widetilde{d}_{xx})+\nu\dfrac{\widetilde{d}_x(\widetilde{m}_x+z\widetilde{d}_x)}{e^{\omega\cdot}+\widetilde{d}}\right)\widetilde{\mathbf{u}}_z\\
&& +\nu\widetilde{\mathbf{u}}_{xx}-2\nu(\widetilde{m}_x+z\widetilde{d}_x)\widetilde{\mathbf{u}}_{xz}+\nu\dfrac{(\widetilde{m}_x+z\widetilde{d}_x)^2-e^{\omega\cdot}\widetilde{d}}{e^{\omega\cdot}+\widetilde{d}}\widetilde{\mathbf{u}}_{zz}-(1+e^{-\omega\cdot}\widetilde{d})\widetilde{u}_1\widetilde{\mathbf{u}}_x-\widetilde{u}_2\widetilde{\mathbf{u}}_z\\
&& +e^{-\omega\cdot}(\widetilde{m}_x+z\widetilde{d}_x)\widetilde{u}_1\widetilde{\mathbf{u}}_z-(\widetilde{d}\widetilde{p}_x-(\widetilde{m}_x+z\widetilde{d}_x)\widetilde{p}_z)\mathbf{e}_1,\\
\widetilde{g}[\widetilde{\mathbf{u}},\widetilde{\eta}^1] &=& -\widetilde{d}\widetilde{u}_{1,x}+(\widetilde{m}_x+z\widetilde{d}_x)\widetilde{u}_{1,z},\\
\widetilde{j}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1] &=& \widetilde{\eta}_x^{1,\kappa}\widetilde{u}_1,\\
\widetilde{l}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1] &=& 2\widetilde{\eta}_x^{1,\kappa}(1+e^{-\omega\cdot}\widetilde{d})\widetilde{u}_{1,x}-2e^{-\omega\cdot}(\widetilde{\eta}_x^{1,\kappa})^2\widetilde{u}_{1,z}-(\widetilde{d}\widetilde{u}_{2,x}+\widetilde{\eta}_x^{1,\kappa}\widetilde{u}_{2,z})(1-e^{-2\omega\cdot}(\eta_x^\kappa)^2)\\
&& -2\widetilde{\eta}_x^{1,\kappa}\widetilde{u}_{2,z}+e^{-\omega\cdot}(\widetilde{\eta}_x^{1\kappa})^2(\widetilde{u}_{1,z}+\widetilde{u}_{2,x}),\\
\widetilde{H}^\kappa[\widetilde{\mathbf{u}},\eta] &=& -\dfrac{2\nu}{1+e^{-2\omega\cdot}(\eta_x^\kappa)^2}\left(e^{-\omega\cdot}(\widetilde{\eta}_x^{1,\kappa})^2\widetilde{u}_{1,x}-\widetilde{\eta}_x^{1,\kappa}\widetilde{u}_{2,x}\right)+\dfrac{2\nu\widetilde{\eta}_x^{1,\kappa}}{1+e^{-\omega\cdot}\widetilde{d}}\widetilde{u}_{1,z}+\dfrac{2\nu\widetilde{d}}{1+e^{-\omega\cdot}\widetilde{d}}\widetilde{u}_{2,z}.
\end{array} \tag{6.58}$$

Remember that d and m are defined from $\eta = (\eta^+,\eta^-)$ in section 6.4. Thus, $\widetilde{d}$ and $\widetilde{m}$ are obtained from $\widetilde{\eta}^1$ by

$$\widetilde{d} = \frac{\widetilde{\eta}^{1,+}-\widetilde{\eta}^{1,-}}{2} \qquad \text{and} \qquad \widetilde{m} = \frac{\widetilde{\eta}^{1,+}+\widetilde{\eta}^{1,-}}{2}.$$

First, we now state a technical lemma which gives the estimates of elements in (6.58) in terms of $(\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2)$ in the space of the solution to the linearized system, that is in $X_{\#,0}^\infty$ (defined in Theorem 6.21).

Proposition 6.22. *Let* $(\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2)$ *in* $X_{\#,0}^\infty$*, then* $(\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2],\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{j}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{l}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{H}[\widetilde{\mathbf{u}},\widetilde{\eta}^1])$ *defined in* (6.58) *belongs to*

$$W_\#^\infty = \mathbf{L}_\#^2(Q_\infty^0)\times\mathbf{H}_\#^{2,1}(Q_\infty^0)\times H_\#^{3/2,3/4}(\Sigma_\infty^0)\times H_\#^{1/2,1/4}(\Sigma_\infty^0)\times L_\#^2(\Sigma_\infty^0)$$

endowed with the norm $\|\cdot\|_{W^\infty_\#}$ *defined, for all* $(\mathbf{G},\mathbf{z},a,b,C)$ *in* $W^\infty_\#$*, by*

$$\|(\mathbf{G},\mathbf{z},a,b,C)\|_{W^\infty_\#}=\left(\|\mathbf{G}\|^2_{\mathbf{L}^2_\#(Q^0_\infty)}+\|\mathbf{z}\|^2_{\mathbf{H}^{2,1}_\#(Q^0_\infty)}+\|a\|^2_{H^{3/2,3/4}_\#(\Sigma^0_\infty)}+\|b\|^2_{H^{1/2,1/4}_\#(\Sigma^0_\infty)}+\|C\|^2_{L^2_\#(\Sigma^0_\infty)}\right)^{\frac{1}{2}}.$$

Furthermore, it satisfies the estimate

$$\|(\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2],\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{j}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{l}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{H}[\widetilde{\mathbf{u}},\widetilde{\eta}^1])\|_{W^\infty_\#}\leq C_2\left(1+\left\|(\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2)\right\|_{X^\infty_{\#,0}}\right)\left\|(\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2)\right\|^2_{X^\infty_{\#,0}}.$$

Let now $(\widetilde{\mathbf{u}}^m,\widetilde{p}^m,\widetilde{\eta}^{1,m},\widetilde{\eta}^{2,m})$ in $X^\infty_{\#,0}$, for $m=1,2$, such that $\|(\widetilde{\mathbf{u}}^m,\widetilde{p}^m,\widetilde{\eta}^{1,m},\widetilde{\eta}^{2,m})\|_{X^\infty_{\#,0}}\leq R_0$ (for some constant $R_0>0$), then the elements $(\widetilde{\mathbf{F}}^m,\widetilde{\mathbf{w}}^m,\widetilde{j}^m,\widetilde{l}^m,\widetilde{H}^m)$ satisfy

$$\|(\widetilde{\mathbf{F}}^1,\widetilde{\mathbf{g}}^1,\widetilde{j}^1,\widetilde{l}^1,\widetilde{H}^1)-(\widetilde{\mathbf{F}}^2,\widetilde{\mathbf{g}}^2,\widetilde{j}^2,\widetilde{l}^2,\widetilde{H}^2)\|_{W^\infty_\#}\leq C(1+R_0)R_0\|(\widetilde{\mathbf{u}}^1,\widetilde{p}^1,\widetilde{\eta}^{1,1},\widetilde{\eta}^{2,1})-(\widetilde{\mathbf{u}}^2,\widetilde{p}^2,\widetilde{\eta}^{1,2},\widetilde{\eta}^{2,2})\|_{X^\infty_{\#,0}}$$

with the notations, for $m=1$, 2,

$$(\widetilde{\mathbf{F}}^m,\widetilde{\mathbf{g}}^m,\widetilde{j}^m,\widetilde{l}^m,\widetilde{H}^m)=(\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}}^m,\widetilde{p}^m,\widetilde{\eta}^{1,m},\widetilde{\eta}^{2,m}],\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}}^m,\widetilde{\eta}^{1,m}],\widetilde{j}[\widetilde{\mathbf{u}}^m,\widetilde{\eta}^{1,m}],\widetilde{l}[\widetilde{\mathbf{u}}^m,\widetilde{\eta}^{1,m}],\widetilde{H}[\widetilde{\mathbf{u}}^m,\widetilde{\eta}^{1,m}]).$$

Proof. These estimates can be proved using Theorem B.3 in [12] and following either [16, section 6.] or [23, section 11.]. For instance, let us prove that $\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$ belongs to $\mathbf{H}^{2,1}_\#(Q^0_\infty)$. We know that $\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]=-\widetilde{d}\widetilde{u}_1\mathbf{e}_1+(\widetilde{m}_x+z\widetilde{d}_x)\widetilde{u}_1\mathbf{e}_2$. The less regular term is the second one. Indeed, we know that $\widetilde{\eta}^1$ belongs to $H^{4,2}_\#(\Sigma^0_\infty)$, thus $\widetilde{\eta}_x$ belongs to $H^{3,3/2}_\#(\Sigma^0_\infty)$. From Theorem B.3 in [12], we have $\widetilde{\eta}^\kappa_x\widetilde{u}_1$ satsifying

$$\|\widetilde{\eta}^\kappa_x\widetilde{u}_1\|_{H^{2,1}_\#(Q^0_\infty)}\leq C\|\widetilde{\eta}^\kappa_x\|_{H^{3,3/2}_\#(\Sigma^0_\infty)}\|\widetilde{u}_1\|_{H^{2,1}_\#(Q^0_\infty)}$$

from η_x in $H^{3,3/2}_\#(\Sigma^0_\infty)$ and u_1 in $H^{2,1}_\#(Q^0_\infty)$ if (with the notations of [12])

$$\lambda+\omega+\mu>\frac{n+d}{2}$$

where here $\lambda=2$, $\omega=1$, $\mu=0$, $n=2$ and $d=2$, that is $3>3/2$. Thus, the vector $\mathbf{g}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$ belongs to $\mathbf{H}^{2,1}_\#(Q^0_\infty)$ and satisfies the estimate

$$\|\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]\|_{\mathbf{H}^{2,1}_\#(Q^0_\infty)}\leq C\|\widetilde{\eta}^1\|_{H^{4,2}_\#(\Sigma^0_\infty)}\|\widetilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}_\#(Q^0_\infty)}.$$

□

Before constructing the mapping for the fixed point procedure, we now introduce the lifting we used in section 6.4.

Proposition 6.23. *System* (6.57) *in the variables* $(\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2)$ *is equivalent to system* (6.62) *in the variables* $(\widetilde{\mathbf{v}},\widetilde{q},\widetilde{\eta}^1,\widetilde{\eta}^2)$ *where* $\widetilde{\mathbf{u}}=\widetilde{\mathbf{v}}+\widetilde{\mathbf{L}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$ *and* $\widetilde{p}=\widetilde{q}+\widetilde{L}_p[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$*. The lifting* $\widetilde{\mathbf{L}}$ *is the sum of two terms,* $\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$ *and* $\underline{\widetilde{\mathbf{w}}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$ *defined in* (6.61) *and the lifting* $\widetilde{L}_p$ *is the pressure term* $\underline{\widetilde{\pi}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$ *corresponding with* $\underline{\widetilde{\mathbf{w}}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$ *in* (6.61)*.*

The right-hand sides of system (6.62) *are obtained from the ones in system* (6.57) *thanks to equations* (6.60) *and* (6.63)*.*

Furthermore, for $(\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2],\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],j[\widetilde{\mathbf{u}},\widetilde{\eta}^1],l[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{H}[\widetilde{\mathbf{u}},\widetilde{\eta}^1])$ *in* $W^\infty_\#$*,* $(\widetilde{\mathbf{f}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2],\widetilde{h}[\widetilde{\mathbf{u}},\widetilde{\eta}^1])$ *belongs to* $L^2(0,\infty;\mathbf{L}^2_\#(\Omega_0)\times L^2_\#(\Gamma_0))$ *and for* $(\mathbf{u}^0,\eta^{1,0},\eta^{2,0})$ *in* $X^0_\#$ *satisfying* (6.10)*, then* $(\mathbf{v}^0,\eta^{1,0},\eta^{2,0})$ *with* $\mathbf{v}^0=\mathbf{u}^0-\widetilde{\mathbf{g}}[\mathbf{u}^0,\eta^{1,0}]$*, belongs to* $X^\#_{0,\mathbf{cc}}$*.*

Proof. Because of the special form of the nonlinear term $\widetilde{g}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$, we can look for solution $(\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2)$ under the form $(\widetilde{\mathbf{w}}+\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2)$. Thus, we get rid of the nonhomogeneous divergence condition but it makes appear new terms (dependent on $\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$), namely system (6.57) becomes

$$\begin{array}{rcll}
\widetilde{\mathbf{w}}_t-\omega\widetilde{\mathbf{w}}-\nu\Delta\widetilde{\mathbf{w}}+\nabla\widetilde{p} &=& \mathrm{e}^{-\omega\cdot}\underline{\widetilde{\mathbf{F}}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2] & \text{in } Q^0_\infty,\\
\mathrm{div}\,\widetilde{\mathbf{w}} &=& 0 & \text{in } Q^0_\infty,\\
\widetilde{w}_2 &=& \widetilde{\eta}^{2,\kappa} & \text{on } \Sigma^{\kappa,0}_\infty,\\
\widetilde{w}_{1,y}+\widetilde{w}_{2,x} &=& \mathrm{e}^{-\omega\cdot}\underline{\widetilde{l}}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1]-\Pi_0(\widetilde{w}^1_0)\chi_{\Gamma_0^+} & \text{on } \Sigma^{\kappa,0}_\infty,\\
\widetilde{\eta}^{1,\kappa}_t &=& \widetilde{\eta}^{2,\kappa}+\omega\widetilde{\eta}^{1,\kappa} & \text{on } \Sigma^{\kappa,0}_\infty,\\
\widetilde{\eta}^{2,\kappa}_t-\omega\widetilde{\eta}^{2,\kappa}+\alpha\widetilde{\eta}^{1,\kappa}_{xxxx}-\beta\widetilde{\eta}^{1,\kappa}_{xx}-\gamma\widetilde{\eta}^{2,\kappa}_{xx} &=& \kappa[\widetilde{p}-2\nu\widetilde{w}_{2,y}]+\mathrm{e}^{-\omega\cdot}\underline{\widetilde{H}}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1]-\Pi(\widetilde{\mathbf{w}},\widetilde{\eta}^1,\widetilde{\eta}^2)\chi_{\Gamma_0^+} & \text{on } \Sigma^{\kappa,0}_\infty,\\
(\widetilde{\mathbf{w}}(0),\widetilde{\eta}(0),\widetilde{\eta}_t(0)) &=& (\mathbf{u}^0-\widetilde{\mathbf{g}}[\mathbf{u}^0,\eta^{1,0}],\eta^{1,0},\eta^{2,0})
\end{array} \tag{6.59}$$

where

$$\begin{array}{rcl}
\underline{\widetilde{\mathbf{F}}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2] &=& \widetilde{\mathbf{F}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2]-\big(\widetilde{\mathbf{g}}_t[\widetilde{\mathbf{u}},\widetilde{\eta}^1]-\omega\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]\big)+\nu\Delta\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\\
\underline{\widetilde{l}}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1] &=& \widetilde{l}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1]-\mathbf{S}(\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1])\mathbf{e}_2\cdot\mathbf{e}_1-\Pi_0(\widetilde{g}_1[\widetilde{\mathbf{u}},\widetilde{\eta}^1])\chi_{\Gamma_0^+},\\
\underline{\widetilde{H}}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1] &=& \widetilde{H}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1]-\kappa\mathbf{S}(\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1])\mathbf{e}_2\cdot\mathbf{e}_2-\Pi(\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],0,0)\chi_{\Gamma_0^+}.
\end{array} \tag{6.60}$$

We recover directly the regularity and estimate of the elements $(\underline{\widetilde{\mathbf{F}}}[\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2],\underline{\widetilde{l}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\underline{\widetilde{H}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1])$ from $(\widetilde{\mathbf{F}}[\widetilde{\mathbf{w}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2],\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{l}[\widetilde{\mathbf{w}},\widetilde{\eta}^1],\widetilde{H}[\widetilde{\mathbf{w}},\widetilde{\eta}^1])$ thanks to the regularity of $\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$. Indeed, with $\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$ in $\mathbf{H}^{2,1}_\#(Q^0_\infty)$, we get that

$$-\widetilde{\mathbf{g}}_t[\widetilde{\mathbf{u}},\widetilde{\eta}^1]+\omega\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]+\nu\Delta\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]\in\mathbf{L}^2_\#(Q^0_\infty) \qquad \text{and} \qquad \mathbf{S}(\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1])\in\Big[H^{1/2,1/4}_\#(\Sigma^0_\infty)\Big]^4.$$

Note that the right-hand side $\widetilde{j}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$ disappeared in (6.59) because we exactly have $\widetilde{g}_2(\kappa 1)=\widetilde{j}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1]=\widetilde{\eta}^{1,\kappa}_x\widetilde{u}_1(\kappa 1)$.

We continue the lifting strategy using a result of [12]. We first look for solution $(\underline{\mathbf{w}},\underline{\pi})$ with a right-hand side in the variables $(\mathbf{u},\eta^1)$, then, we use the classic change of unknows $(\underline{\widetilde{\mathbf{w}}},\underline{\widetilde{\pi}})=\mathrm{e}^{\omega\cdot}(\underline{\mathbf{w}},\underline{\pi})$. More precisely, we use the following result.

Proposition 6.24 (Theorem 5.3 in [12] in the case $r=0$ for $k=2$.). *Let $\underline{l}$ be in $H^{1/2,1/4}_\#(\Sigma^0_\infty)$, then system*

$$\begin{array}{rcll}
\mathbf{w}_t-\nu\Delta\mathbf{w}+\nabla\pi &=& \mathbf{0} & \textit{in } Q^0_\infty,\\
\textit{div}\,\mathbf{w} &=& 0 & \textit{in } Q^0_\infty,\\
w_2 &=& 0 & \textit{on } \Sigma^{\kappa,0}_\infty,\\
w_{1,y}+w_{2,x} &=& \underline{l}^\kappa & \textit{on } \Sigma^{\kappa,0}_\infty,\\
\mathbf{w}(0) &=& \mathbf{0} & \textit{in } \Omega_0
\end{array}$$

admits a unique solution $(\mathbf{w},\pi)$ in $\mathbf{H}^{2,1}_\#(Q^0_\infty)\times L^2(0,\infty;\mathcal{H}^1_\#(\Omega_0))$. Furthermore, this solution satisfies $w^1_0(y)=\frac{1}{2\pi}\int_0^{2\pi}w^1(x,y)dx\equiv 0$ in $(-1,1)$ (this point is obvious).

Then, applying the previous proposition to the lifting of $\underline{l}[\mathbf{u},\eta^1]$ (which corresponds to $\underline{\widetilde{l}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$, that is $\underline{l}[\mathbf{u},\eta^1]=\mathrm{e}^{-2\omega\cdot}\underline{\widetilde{l}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1]$), we obtain a unique solution $(\underline{\mathbf{w}}[\mathbf{u},\eta^1],\underline{\pi}[\mathbf{u},\eta^1])$ in $\mathbf{H}^{2,1}_\#(Q^0_\infty)\times L^2(0,\infty;\mathcal{H}^1_\#(\Omega_0))$. Furthermore, the corresponding variables $(\underline{\widetilde{\mathbf{w}}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\underline{\widetilde{\pi}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1])=\mathrm{e}^{\omega\cdot}(\underline{\mathbf{w}}[\mathbf{u},\eta^1],\underline{\pi}[\mathbf{u},\eta^1])$ satisfies the equation

$$\begin{array}{rcll}
\widetilde{\mathbf{w}}_t-\omega\widetilde{\mathbf{w}}-\nu\Delta\widetilde{\mathbf{w}}+\nabla\widetilde{\pi} &=& \mathbf{0} & \text{in } Q^0_\infty,\\
\mathrm{div}\,\widetilde{\mathbf{w}} &=& 0 & \text{in } Q^0_\infty,\\
\widetilde{w}_2 &=& 0 & \text{on } \Sigma^{\kappa,0}_\infty,\\
\widetilde{w}_{1,y}+\widetilde{w}_{2,x} &=& \mathrm{e}^{-\omega\cdot}\underline{\widetilde{l}}^\kappa[\widetilde{\mathbf{u}},\widetilde{\eta}^1] & \text{on } \Sigma^{\kappa,0}_\infty,\\
\widetilde{\mathbf{w}}(0) &=& \mathbf{0} & \text{in } \Omega_0,
\end{array} \tag{6.61}$$

belongs to $\mathbf{H}^{2,1}_\#(Q^0_\infty)\times L^2(0,\infty;\mathcal{H}^1_\#(\Omega_0))$ and $\underline{\widetilde{w}}^1_0[\widetilde{\mathbf{u}},\widetilde{\eta}^1]\equiv 0$ in $(-1,1)$.

We now look for solution of (6.59) under the form $(\widetilde{\mathbf{w}},\widetilde{p},\widetilde{\eta}^1,\widetilde{\eta}^2)=(\widetilde{\mathbf{v}}+\underline{\widetilde{\mathbf{w}}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{q}+\underline{\widetilde{\pi}}[\widetilde{\mathbf{u}},\widetilde{\eta}^1],\widetilde{\eta}^1,\widetilde{\eta}^2)$.

The system satsfied by $(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\eta}^1, \widetilde{\eta}^2)$ is

$$\begin{array}{rcll} \widetilde{\mathbf{v}}_t - \omega\widetilde{\mathbf{v}} - \nu\Delta\widetilde{\mathbf{v}} + \nabla\widetilde{q} & = & \mathrm{e}^{-\omega\cdot}\widetilde{\mathbf{f}}[\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2] & \text{in } Q^0_\infty, \\ \mathrm{div}\ \widetilde{\mathbf{v}} & = & 0 & \text{in } Q^0_\infty, \\ \widetilde{v}_2 & = & \widetilde{\eta}^{2,\kappa} & \text{on } \Sigma^{\kappa,0}_\infty, \\ \widetilde{v}_{1,y} + \widetilde{v}_{2,x} & = & -\Pi_0(\widetilde{v}^1_0)\chi_{\Gamma_0^+} & \text{on } \Sigma^{\kappa,0}_\infty, \\ \widetilde{\eta}^{1,\kappa}_t & = & \widetilde{\eta}^{2,\kappa} + \omega\widetilde{\eta}^{1,\kappa} & \text{on } \Sigma^{\kappa,0}_\infty, \\ \widetilde{\eta}^{2,\kappa}_t - \omega\widetilde{\eta}^{2,\kappa} + \alpha\widetilde{\eta}^{1,\kappa}_{xxxx} - \beta\widetilde{\eta}^{1,\kappa}_{xx} - \gamma\widetilde{\eta}^{2,\kappa}_{xx} & = & \kappa[\widetilde{q} - 2\nu\widetilde{v}_{2,y}] + \mathrm{e}^{-\omega\cdot}\widetilde{h}^\kappa[\widetilde{\mathbf{v}}, \widetilde{\eta}^1] - \Pi(\widetilde{\mathbf{v}}, \widetilde{\eta}^1, \widetilde{\eta}^2)\chi_{\Gamma_0^+} & \text{on } \Sigma^{\kappa,0}_\infty, \\ (\widetilde{\mathbf{v}}(0), \widetilde{\eta}(0), \widetilde{\eta}_t(0)) & = & (\mathbf{u}^0 - \widetilde{\mathbf{g}}[\mathbf{u}^0, \eta^{1,0}], \eta^{1,0}, \eta^{2,0}) & \end{array} \tag{6.62}$$

where

$$\begin{array}{rcl} \widetilde{\mathbf{f}}[\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2] & = & \underline{\widetilde{\mathbf{F}}}[\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2] \\ \widetilde{h}^\kappa[\widetilde{\mathbf{u}}, \widetilde{\eta}^1] & = & \underline{\widetilde{H}}^\kappa[\widetilde{\mathbf{u}}, \widetilde{\eta}^1] - \kappa\mathrm{e}^{\omega\cdot}\left[\underline{\widetilde{\pi}}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1] - 2\nu\underline{\widetilde{w}}_{2,z}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1]\right] - \Pi(\widetilde{\mathbf{g}}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1] + \underline{\widetilde{\mathbf{w}}}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], 0, 0)\chi_{\Gamma_0^+}. \end{array} \tag{6.63}$$

The point $\underline{\widetilde{w}}^1_0[\widetilde{\mathbf{u}}, \widetilde{\eta}^1] \equiv 0$ in $(-1, 1)$ gives that $-\Pi_0(\underline{\widetilde{w}}^1_0[\widetilde{\mathbf{u}}, \widetilde{\eta}^1]) = 0$ and this term vanishes in (6.62). ☐

All this work has been made to set system (6.62) in the setting of Theorem 6.21 from the previous section. Thus, together with Proposition 6.22 gives the following theorem.

Theorem 6.25. *For a quadruplet $(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)$ in $X^\infty_{\#,0}$, the closed loop system (6.57) with right-hand sides $(\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2], \widetilde{g}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{j}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{l}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{H}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1])$ defined in (6.58) and initial data $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ in $X^0_\#$ satisfying (6.10) admits a unique solution $(\widetilde{\mathbf{u}}^\bullet, \widetilde{p}^\bullet, \widetilde{\eta}^{1,\bullet}, \widetilde{\eta}^{2,\bullet})$ in $X^\infty_{\#,0}$ with the estimate*

$$\|(\widetilde{\mathbf{u}}^\bullet, \widetilde{p}^\bullet, \widetilde{\eta}^{1,\bullet}, \widetilde{\eta}^{2,\bullet})\|_{X^\infty_{\#,0}} \leq C_1\left(\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0_\#} + C_2(1 + \|(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)\|_{X^\infty_{\#,0}})\|(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)\|^2_{X^\infty_{\#,0}}\right). \tag{6.64}$$

That is, we have construct a mapping from $X^\infty_{\#,0}$ into itself defined by

$$\begin{array}{rrcl} \mathcal{M}: & X^\infty_{\#,0} & \longrightarrow & X^\infty_{\#,0} \\ & (\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2) & \longmapsto & \mathcal{M}(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2) = (\widetilde{\mathbf{u}}^\bullet, \widetilde{p}^\bullet, \widetilde{\eta}^{1,\bullet}, \widetilde{\eta}^{2,\bullet}) \text{ the solution of the close loop system (6.57)} \\ & & & \text{with } (\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2], \widetilde{g}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{j}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{l}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{H}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1]) \text{ for right-hand sides} \end{array}$$

which satisfies

$$\|\mathcal{M}(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)\|_{X^\infty_{\#,0}} \leq C_1\left(\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0_\#} + C_2(1 + \|(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)\|_{X^\infty_{\#,0}})\|(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)\|^2_{X^\infty_{\#,0}}\right).$$

Furthermore, for two quadruplets $(\widetilde{\mathbf{u}}^m, \widetilde{p}^m, \widetilde{\eta}^{1,m}, \widetilde{\eta}^{2,m})$ for $m = 1, 2$, such that

$$\|(\widetilde{\mathbf{u}}^m, \widetilde{p}^m, \widetilde{\eta}^{1,m}, \widetilde{\eta}^{2,m})\|_{X^\infty_{\#,0}} \leq R_0$$

for some $R_0 > 0$, the difference (by linearity) $\mathcal{M}(\widetilde{\mathbf{u}}^1, \widetilde{p}^1, \widetilde{\eta}^{1,1}, \widetilde{\eta}^{2,1}) - \mathcal{M}(\widetilde{\mathbf{u}}^2, \widetilde{p}^2, \widetilde{\eta}^{1,2}, \widetilde{\eta}^{2,2})$ corresponds with the solution of system (6.57) with $(\mathbf{F}^1, \mathbf{g}^1, j^1, l^1, H^1) - (\mathbf{F}^2, \mathbf{g}^2, j^2, l^2, H^2)$ as right-hand side and $(\mathbf{0}, 0, 0, 0)$ as initial data. We used again the notations, for $m = 1, 2$,

$$(\widetilde{\mathbf{F}}^m, \widetilde{\mathbf{g}}^m, \widetilde{j}^m, \widetilde{l}^m, \widetilde{H}^m) = (\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}}^m, \widetilde{p}^m, \widetilde{\eta}^{1,m}, \widetilde{\eta}^{2,m}], \widetilde{\mathbf{g}}[\widetilde{\mathbf{u}}^m, \widetilde{\eta}^{1,m}], \widetilde{j}[\widetilde{\mathbf{u}}^m, \widetilde{\eta}^{1,m}], \widetilde{l}[\widetilde{\mathbf{u}}^m, \widetilde{\eta}^{1,m}], \widetilde{H}[\widetilde{\mathbf{u}}^m, \widetilde{\eta}^{1,m}]).$$

Thus, $\mathcal{M}(\widetilde{\mathbf{u}}^1, \widetilde{p}^1, \widetilde{\eta}^{1,1}, \widetilde{\eta}^{2,1}) - \mathcal{M}(\widetilde{\mathbf{u}}^2, \widetilde{p}^2, \widetilde{\eta}^{1,2}, \widetilde{\eta}^{2,2})$ satisfies the estimate

$$\|\mathcal{M}(\widetilde{\mathbf{u}}^1, \widetilde{p}^1, \widetilde{\eta}^{1,1}, \widetilde{\eta}^{2,1}) - \mathcal{M}(\widetilde{\mathbf{u}}^2, \widetilde{p}^2, \widetilde{\eta}^{1,2}, \widetilde{\eta}^{2,2})\|_{X^\infty_{\#,0}} \leq C_1C_2(1+R_0)R_0\|(\widetilde{\mathbf{u}}^1, \widetilde{p}^1, \widetilde{\eta}^{1,1}, \widetilde{\eta}^{2,1}) - (\widetilde{\mathbf{u}}^2, \widetilde{p}^2, \widetilde{\eta}^{1,2}, \widetilde{\eta}^{2,2})\|_{X^\infty_{\#,0}}.$$

Proof. For a quadruplet $(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)$ in $X^\infty_{\#,0}$, from Theorem 6.21, system (6.62) corresponding with the right-hand side $(\widetilde{\mathbf{f}}[\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2], \widetilde{h}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1])$ obtained from $(\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2], \widetilde{g}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{j}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{l}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{H}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1])$ defined in (6.58) using (6.60) and (6.63) and with the initial data $(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})$ in $X^0_{\#,\mathbf{cc}}$ defined from $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ in $X^0_\#$ satisfying (6.10) in (6.9) admits a unique solution $(\widetilde{\mathbf{v}}^\bullet, \widetilde{q}^\bullet, \widetilde{\eta}^{1,\bullet}, \widetilde{\eta}^{2,\bullet})$ in $X^\infty_{\#,0}$ satisfying the estimate

$$\|(\widetilde{\mathbf{v}}^\bullet, \widetilde{q}^\bullet, \widetilde{\eta}^{1,\bullet}, \widetilde{\eta}^{2,\bullet})\|_{X^\infty_{\#,0}} \leq C\left(\|(\mathbf{v}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0_\#} + (1 + \|(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)\|_{X^\infty_\#})\|(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)\|^2_{X^\infty_\#}\right).$$

Now, using Proposition 6.23, we get that $(\widetilde{\mathbf{u}}^\bullet, \widetilde{p}^\bullet, \widetilde{\eta}^{1,\bullet}, \widetilde{\eta}^{2,\bullet}) = (\widetilde{\mathbf{v}}^\bullet + \widetilde{\mathbf{L}}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{q}^\bullet + \widetilde{L}_p[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{\eta}^{1,\bullet}, \widetilde{\eta}^{2,\bullet})$ is solution of (6.57) satisfying estimate (6.64). ☐

From the extimate of the mapping $\mathcal{M}$, we deduce the following proposition which state the existence of the shifted closed loop system (6.57).

Proposition 6.26. *Let* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *in* $X^0_\#$ *satisfying the compatibility condition* (6.10). *There exists a constant* $r_0 > 0$ *and a increasing function* R *from* $\mathbb{R}^+$ *into itself such that if* r *belongs to* $(0, r_0)$ *and* $\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0_\#} \le R(r)$*, system* (6.57) *admits a unique solution* $(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)$ *in* $X^\infty_{\#,0}$ *satsfying the estimate*

$$\|(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2)\|_{X^\infty_{\#,0}} \le r.$$

Proof. We want to use the Banach fixed point method. We begin by considering for $r > 0$ the ball of the space $X^\infty_{\#,0}$ of radius r, that is

$$\mathcal{B}_{X^\infty_{\#,0}}(r) = \Big\{(\mathbf{z}, r, \mu^1, \mu^2) \in X^\infty_{\#,0} \quad \text{s.t.} \quad \|(\mathbf{z}, r, \mu^1, \mu^2)\|_{X^\infty_{\#,0}} \le r\Big\}$$

From Theorem 6.25, we know that $\mathcal{M}$ is a well-defined from $\mathcal{B}_{X^\infty_{\#,0}}(r)$ into itself and is a contraction on $\mathcal{B}_{X^\infty_{\#,0}}(r)$ if the two following inequalities are satisfied

$$C_1\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0_\#} + C_1C_2(1+r)r \le r \quad \text{and} \quad C_1C_2(1+r)r < \frac{1}{2}.$$

Let us consider initial data $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ such that $\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0_\#} \le \frac{r}{2C_1}$. Thus, r has to satisfy

$$C_1C_2(1+r)r \le \frac{1}{2}.$$

which is possible for $0 < r < \frac{1}{C_1C_2\sqrt{1+\frac{2}{C_1C_2}}} = r_0$. The function R is $r \mapsto \frac{r}{2C_1}$. For such a constant $r > 0$, the mapping $\mathcal{M}$ is a contraction from $\mathcal{B}_{X^\infty_{\#,0}}(r)$ into itself, that concludes the proof. ☐

Now, from the correspondence between system (6.56) and (6.57), we get that $(\mathbf{u}^\bullet, p^\bullet, \eta^\bullet)$ is solution to system (6.56) with right-hand side $(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{g}[\mathbf{u}, \eta], j[\mathbf{u}, \eta], l[\mathbf{u}, \eta], H[\mathbf{u}, \eta])$ (defined in (6.5)) if and only if

$$(\widetilde{\mathbf{u}}^\bullet, \widetilde{p}^\bullet, \widetilde{\eta}^{1,\bullet}, \widetilde{\eta}^{2,\bullet}) = \mathrm{e}^{\omega\cdot}(\mathbf{u}^\bullet, p^\bullet, \eta^\bullet, \eta^\bullet_t)$$

is solution to system (6.57) with rigth-hand side $(\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2], \widetilde{g}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{j}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{l}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{H}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1])$ (defined in (6.58)).

Note that the right-hand sides $(\mathbf{F}[\mathbf{u}, p, \eta], \mathbf{g}[\mathbf{u}, \eta], j[\mathbf{u}, \eta], l[\mathbf{u}, \eta], H[\mathbf{u}, \eta])$ of (6.56) multiplied by $\mathrm{e}^{\omega\cdot}$ are exactly the right-hand sides $(\widetilde{\mathbf{F}}[\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\eta}^1, \widetilde{\eta}^2], \widetilde{g}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{j}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{l}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1], \widetilde{H}[\widetilde{\mathbf{u}}, \widetilde{\eta}^1])$ of (6.57). Thus, the first ones give that $(\mathbf{f}[\mathbf{u}, p, \eta], h[\mathbf{u}, \eta])$ obtained in (6.8) satisfies $\mathrm{e}^{\omega\cdot}(\mathbf{f}, h)$ belongs to $L^2(0, \infty; \mathbf{L}^2_\#(\Omega_0) \times L^2_\#(\Gamma_0))$.

This allows us to write the equivalent result for system (6.56).

Proposition 6.27. *Let* $(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})$ *in* $X^0_\#$ *satisfying the compatibility condition* (6.10). *There exists a constant* $r_0 > 0$ *and a increasing function* R *from* $\mathbb{R}^+$ *into itself such that if* r *belongs to* $(0, r_0)$ *and* $\|(\mathbf{u}^0, \eta^{1,0}, \eta^{2,0})\|_{X^0_\#} \le R(r)$*, system* (6.56) *admits a unique solution* $(\mathbf{u}, p, \eta)$ *such that* $(\mathbf{u}, p, \eta, \eta_t)$ *belongs to* $X^\infty_{\#,\omega}$ *(this space is defined in Theorem 6.21) and satisties the estimate*

$$\|(\mathbf{u}, p, \eta, \eta_t)\|_{X^\infty_{\#,\omega}} \le r.$$

Up to a restriction on the value of r_0, we can make the assumption (6.1) comes true. Indeed, thanks to the embedding $H^{4,2}_\#(\Sigma^0_\infty) \hookrightarrow L^\infty(\Sigma^0_\infty)$, we can get $\|\eta^\kappa\|_{L^\infty(\Sigma^{\kappa,0}_\infty)} \le 1 - \delta_0$. This, with the regularity of η gives that the change of variables ϕ_η defined in section 6.4 is a $\mathcal{C}^1-$diffeomorphism. Thanks to Definition 6.4, we finally obtain the main result of this paper in the moving cylinder Q^η_∞.

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Titre de la thèse : Quelques résultats d'existence, de contrôlabilité et de stabilisation pour des systèmes couplés fluide-structure
Auteur : Julien Lequeurre
Directeur de thèse : Jean-Pierre Raymond
Date et lieu de soutenance : le 5 décembre 2011 à l'Université Toulouse III - Paul Sabatier

Résumé :

Dans cette thèse nous étudions des systèmes couplés fluide-structure. Ces systèmes peuvent modéliser un écoulement sanguin dans un vaisseau large ou un problème d'aéroélasticité. La vitesse et la pression du fluide sont décrites par les équations de Navier-Stokes incompressibles et le déplacement de la structure frontière est régi par une équation de poutre/plaque/membrane (selon la dimension du modèle et la nature de la structure).
Dans la première partie, nous montrons l'existence de solutions fortes pour de tels systèmes en deux ou trois dimensions, soit pour des conditions initiales petites (existence globale en temps), soit pour des conditions initiales quelconques (existence locale en temps).
Dans une seconde partie, nous étudions d'abord la contrôlabilité à zéro d'un système couplant les équations de Navier-Stokes à une équation de structure correspondant à une approximation de dimension finie des modèles de poutres ou de plaques. Nous étudions ensuite la stabilisation (pour tout taux de décroissance), locale au voisinage de la solution nulle, d'un système couplant les équations de Navier-Stokes à deux équations de poutres, par deux contrôles de dimension finie agissant dans l'équation de la structure et dans la deuxième condition au bord pour la vitesse. Le second contrôle ne dépend que du temps.

Mots clés :Systèmes couplés, équations de Navier-Stokes, équations des poutres/plaques, contrôlabilité, stabilisation, existence et unicité de solutions.

Title: Some results of existence, controllability and stabilization for fluid-structure systems.

Abstract:

In this thesis, we are interested in the study of fluid-structure systems. These systems may model blood flows in large vessels or aeroelasticy problems. The velocity and the pressure of the blood are described by the incompressible Navier-Stokes equations and the displacement of the structure boundary satisfies a beam/plate/membrane equation (it depends on the dimension of the model and of the nature of the structure).
In the fist part, we prove the exitence and uniqueness of strong solutions to the kind of systems in two or three dimensions, either for small initial data (global in time existence) or for any initial data (local in time existence).
In the second part, we study on one hand the null controllability of a system coupling the Navier-Stokes equations with a structure equation corresponding with a finite dimensional approximation of the beam or plate equation. On the other hand, we study the stabilization (for any decay rate) local around the stationary null solution of a system coupling the Navier-Stokes equations with two beam equations with two finite dimension controls acting on the structure equation and in the second boundary condition for the velocity. The second control only depends on time.

Keywords: Coupled systems, Navier-Stokes equations, beam and plate equations, controllability, stabilization, existence and uniqueness of solutions.