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Contributions to polynomial interpolation in one and several variables

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Abstract

This thesis deals with polynomial interpolation of functions in one and several variables. We shall be mostly concerned with Lagrange interpolation but one of our work deals with Kergin and Hakopian interpolants. We denote by $\mathbb{K}$ the field that may be either $\mathbb{R}$ or $\mathbb{C}$, and $\mathscr{P}_d(\mathbb{K}^N)$ the vector space of all polynomials of N variables of degree at most d. The set $A \subset \mathbb{K}^N$ is said to be an unisolvent set of degree d if it is not included in the zero set of a polynomial of degree not greater than d. For every function f defined on A, there exists a unique $\mathbf{L}[A; f] \in \mathscr{P}_d(\mathbb{K}^N)$ such that $\mathbf{L}[A; f] = f$ on A, which is called the Lagrange interpolation polynomial of a function f at A. Kergin and Hakopian interpolants are natural multivariate generalizations of univariate Lagrange interpolation. The construction of these interpolation polynomials requires the use of points with which one obtains a number of natural mean value linear forms which provide the interpolation conditions.

The quality of approximation furnished by interpolation polynomials much depends on the choice of the interpolation points. In turn, the quality of the interpolation points is best measured by the growth of the norm of the linear linear operator that associates to a continuous function its interpolation polynomial. This norm is called the Lebesgue constant. Most of this thesis is dedicated to the study of such constant. We provide for instances the first general examples of multivariate points having a Lebesgue constant that grows like a polynomial. This is an important advance in the field. This thesis contains five chapters that correspond to the following five papers.

I. J.-P. Calvi and Phung V. M., **On the Lebesgue constant of Leja sequences for the disk and its applications to multivariate interpolation** (Journal of Approximation Theory, 163(5):608-622, 2011). We estimate the growth of the Lebesgue constant of any Leja sequence for the unit disk. The main application is the construction of new multivariate interpolation points in a polydisk (and in the Cartesian product of many plane compact sets) whose Lebesgue constant grows (at most) like a polynomial.

II. J.-P. Calvi and Phung V. M., **Lagrange interpolation at real projections of Leja sequences for the unit disk** (Proc. Amer. Math. Soc. (accepted)). We show that the Lebesgue constants of the real projection of Leja sequences for the unit disk grow like a polynomial. The main application is the first construction of explicit multivariate interpolation points in $[-1, 1]^N$ whose Lebesgue constants also grow like a polynomial.

III. Phung V. M., **On the convergence of Kergin and Hakopian interpolants at Leja sequences for the disk** (Preprint 2011). We prove that Kergin interpolation polynomials and Hakopian interpolation polynomials at the points of a Leja sequence for the unit disk D of a sufficiently smooth function f in a neighbourhood of D converge uniformly to f on D. Moreover, when $f \in C^\infty(D)$, all the derivatives of the interpolation polynomials converge uniformly to the corresponding derivatives of f.

IV. Phung V. M, **On the limit points of pseudo Leja sequences** (DRNA, vol. 4, p. 1-7, 2011). We prove the existence of pseudo Leja sequences with large sets of limit points for many plane compact sets.

V. J.-P. Calvi and Phung V. M., **On the continuity of multivariate Lagrange interpolation at Chung-Yao lattices** (Preprint 2010). We give a natural geometric condition of Chung-Yao lattices (of fixed degree) in which Lagrange interpolation polynomials of sufficiently smooth functions converge to Taylor polynomial.

Résumé

Cette thèse traite de l'interpolation polynomiale des fonctions d'une ou plusieurs variables. Nous nous intéresserons principalement à l'interpolation de Lagrange mais un de nos travaux concerne les interpolations de Kergin et d'Hakopian. Nous dénotons par $\mathbb{K}$ le corps de base qui sera toujours $\mathbb{R}$ ou $\mathbb{C}$, $\mathscr{P}_d(\mathbb{K}^N)$ l'espace des polynômes de N variables et de degré au plus d à coefficients dans K. Un ensemble $A \subset \mathbb{K}^N$ contenant autant de points que la dimension de $\mathscr{P}_d(\mathbb{K}^N)$ est dit unisolvent s'il n'est pas contenu dans l'ensemble des zéros d'un polynôme de degré $\leq d$. Pour toute fonction f définie sur A, il existe un unique $\mathbf{L}[A; f] \in \mathscr{P}_d(\mathbb{K}^N)$ tel que $\mathbf{L}[A; f] = f$ sur A, appelé le polynôme d'interpolation de Lagrange de f en A. Les polynômes d'interpolation de Kergin et d'Hakopian sont deux généralisations naturelles en plusieurs variables de l'interpolation de Lagrange à une variable. La construction de ces polynômes nécessite le choix de points à partir desquels on construit certaines formes linéaires qui sont des moyennes intégrales et qui fournissent les conditions d'interpolation.

La qualité des approximations fournies par les polynômes d'interpolation dépend pour une large mesure du choix des points d'interpolation. Cette qualité est mesurée par la croissance de la norme de l'opérateur linéaire qui à toute fonction continue associe son polynôme d'interpolation. Cette norme est appelée la constante de Lebesgue (associée au compact et aux points d'interpolation considérés). La majeure partie de cette thèse est consacrée à l'étude de cette constante. Nous donnons par exemples le premier exemple général explicite de familles de points possédant une constante de Lebesgue qui croit comme un polynôme. C'est une avancée significative dans ce domaine de recherche. Cette thèse est constituée de cinq chapitres qui correspondent aux articles suivants.

I. J.-P. Calvi and Phung V. M., **On the Lebesgue constant of Leja sequences for the disk and its applications to multivariate interpolation** (Journal of Approximation Theory, 163(5):608-622, 2011). Nous estimons la croissance de la constante de Lebesgue d'une suite de Leja quelconque pour le disque unité. L'application principale est la construction de nouveaux points d'interpolation en plusieurs variables dans un polydisque (ou le produit cartésien de compact plan assez généraux) dont la constante de Lebesgue ne croît pas plus vite qu'un polynôme.

II. J.-P. Calvi and Phung V. M., **Lagrange interpolation at real projections of Leja sequences for the unit disk** (Proc. Amer. Math. Soc. (accepted)). Nous montrons que la constante de Lebesgue des projections sur l'axe réel des suite de Leja du disque unité dans le plan complexe croît comme un polynôme. L'application principale est la première construction de points explicite dans $[-1, 1]^N$ dont la constante de Lebesgue croît aussi comme un polynôme.

III. Phung V. M., **On the convergence of Kergin and Hakopian interpolants at Leja sequences for the disk** (Preprint 2011). Nous montrons que les suites de polynômes d'interpolation de Kergin et d'Hakopian construits aux points d'une suite de Leja dans le disque unité (points regardés comme des éléments de $\mathbb{R}^2$) convergent uniformément vers

la fonction interpolée dès lors que celle-ci est suffisamment différentiable sur un voisinage de D. En outre, lorsque $f \in C^{\infty}(D)$, toutes les dérivées des polynômes d'interpolation convergent uniformément vers les dérivées correspondantes de la fonction interpolée.

IV. Phung V. M, **On the limit points of pseudo Leja sequences** (DRNA, vol. 4, p. 1-7, 2011). Répondant à une question de Białas-Cież et Calvi, nous établissons l'existence de pseudo suites de Leja possédant un large ensemble de points limites et ceci pour une classe générale de compact plans.

V. J.-P. Calvi and Phung V. M., **On the continuity of multivariate Lagrange interpolation at Chung-Yao lattices** (Preprint 2010). Nous donnons une condition géométrique naturelle sur une suite de tableaux de points de Chung-Yao (de degré fixé) assurant que les polynômes d'interpolation de Lagrange en ces points de toute fonction suffisamment régulière convergent vers le polynôme de Taylor.

Contents

CHAPTER 1

On the Lebesgue constant of Leja sequences for the unit disk and its applications to multivariate interpolation

Abstract. We estimate the growth of the Lebesgue constant of any Leja sequence for the unit disk. The main application is the construction of new multivariate interpolation points in a polydisk (and in the Cartesian product of many plane compact sets) whose Lebesgue constant grows (at most) like a polynomial.

1.1. Introduction

Let A be a set of $d+1$ pairwise distinct points in a compact set K in the complex plane. Given a function f defined on A, the unique polynomial of degree at most d which coincides with f on A is the *Lagrange interpolation polynomial* of f and is denoted by $\mathbf{L}[A;f]$. We have

$$\mathbf{L}[A;f] = \sum_{a\in A} f(a)\,\ell(A,a;\cdot), \tag{1.1}$$

where $\ell(A,a;\cdot)$ is the fundamental Lagrange interpolation polynomial (*FLIP*) corresponding to a, that is, the unique polynomial of degree at most d such that $\ell(A,a;a)=1$ and $\ell(A,a;\cdot)=0$ on $A\setminus\{a\}$,

$$\ell(A,a;z) = \prod_{b\in A, b\neq a}\frac{z-b}{a-b} = \frac{w_A(z)}{w_A'(a)(z-a)} \quad \text{where} \quad w_A(z)=\prod_{a\in A}(z-a), \quad z\in\mathbb{C}. \tag{1.2}$$

The *Lebesgue constant* $\Delta(A)$ is the norm of the continuous operator $\mathbf{L}[A] : f\in C(K)\mapsto \mathbf{L}[A;f]\in C(K)$. As is well-known, $\Delta(A)$ is the uniform norm on K of the Lebesgue function $\delta(A,\cdot) := \sum_{a\in A}|\ell(A,a;\cdot)|$, that is,

$$\Delta(A) = \Big\| \sum_{a\in A} \big|\ell(A,a;\cdot)\big| \Big\|_K. \tag{1.3}$$

In the multivariate case ($A\subset\mathbb{C}^N$), Lagrange interpolation polynomials and Lebesgue constants are defined in a similar way, see Section 1.6, with the (fundamental) differences that not every set A (with a good cardinality) can be taken as an interpolation set and, even when interpolation is possible, in general, the FLIPs no longer have a simple expression – a fact which makes theoretical studies of multivariate Lebesgue constants difficult.

The Lebesgue constant is a basic object of interpolation theory because it controls the stability of interpolation at A as well as the approximation capabilities of the interpolation polynomial via the Lebesgue inequality

$$\|f-\mathbf{L}[A;f]\|_K \le (1+\Delta(A))\mathrm{dist}_K(f,\mathscr{P}_d),$$

where d is the degree of interpolation, $\mathscr{P}_d$ denotes the space of polynomials of degree at most d and $\mathrm{dist}_K(f,\mathscr{P}_d)$ the uniform distance on K between f and $\mathscr{P}_d$. A large part of classical Lagrange interpolation theory is devoted to the study of Lebesgue constants of natural interpolation points, such as, for instance, the roots of standard orthogonal polynomials. Unlike the classical cases, which, it seems, always deal with arrays of points (when we go from degree $d-1$ to degree d, we take $d+1$ new points), in this note we exhibit *sequences* $(e_k : k \in \mathbb{N})$ in the unit disk $D := \{|z| \leq 1\}$ such that $\Delta(\{e_0,\dots,e_k\})$ grows at most like $k \ln k$. These sequences are Leja sequences for the unit disk. They are defined by a simple extremal metric property (Section 1.2). Using classical works of Alper, we may then construct sequences whose Lebesgue constant grows polynomially not only for a disk but also for a large class of plane compact sets (Section 1.5). The main application (and motivation) of our study is the construction of explicit multivariate interpolation sets (in the Cartesian products on many plane compact sets) having a Lebesgue constant that grows polynomially (Section 1.6). Very few such examples are currently available. We mention the beautiful Padua points recently discovered in the square of $\mathbb{R}^2$ and which have a Lebesgue constant that grows like $\ln^2 d$ where d is the degree, but their construction seems to be hardly generalizable to the higher dimensional cases, see [**10**]. The other striking properties of our multivariate interpolation points is that they are nested in the sense that the points used for the degree $d-1$ are still used for the degree d.

Let us finally point out that we may define Leja sequences for every non empty compact subset of the plane and, in a recent interesting paper, Taylor and Totik [**42**] showed that the Lebesgue constant of Leja sequences for many plane compact sets has a subexponential growth. These sequences took their name from Franciszek Leja which used them in a classical paper on the approximation of exterior conformal mappings [**30**] but they were first considered by Albert Edrei in 1939, see [**21**].

1.2. Leja sequences

1.2.1. Definition and structure. A k-tuple $E_k = (e_0,\dots,e_{k-1}) \in D^k$ with $e_0 = 1$ is a *k-Leja section* for the unit disk D if, for $j = 1,\dots,k-1$, the $(j+1)$-st entry e_j maximizes the product of the distances to the j previous points, that is

$$\prod_{m=0}^{j-1} |e_j - e_m| = \max_{z\in D} \prod_{m=0}^{j-1} |z - e_m|, \quad j = 1,\dots,k-1.$$

The maximum principle implies that all the e_i's actually lie on the unit circle ∂D. A sequence $E = (e_k : k \in \mathbb{N})$ for which $E_k := (e_0,\dots,e_{k-1})$ is a k-Leja section for every $k \in \mathbb{N}$ is called a *Leja sequence* for D.

The first purpose of this note is to estimate the Lebesgue constant $\Delta(E_k)$. As recalled in (1.3), it is given by

$$\Delta(E_k) = \Big\| \sum_{j=0}^{k-1} \big|\ell(E_k, e_j; \cdot)\big| \Big\|_D = \Big\| \sum_{j=0}^{k-1} \big|\ell(E_k, e_j; \cdot)\big| \Big\|_{\partial D}. \tag{1.4}$$

The second equality is perhaps not obvious. It follows for instance from the maximum principle applied to the Lebesgue function $\delta(E_k, \cdot)$ which is subharmonic on $\mathbb{C}$.

It is not difficult to describe the structure of a Leja sequence for D. The following theorem is proved in [**4**]. If A is the r-tuple $(a_0,\dots,a_{r-1})$ and B is the s-tuple $(b_0,\dots,b_{s-1})$ we denote by (A,B) the $r+s$-tuple $(a_0,\dots,a_{r-1},b_0,\dots,b_{s-1})$.

THEOREM 1.1. *The underlying set of a 2^n-Leja section for D is formed of the 2^n-th roots of unity. If $E_{2^{n+1}}$ is a 2^{n+1}-Leja section then there exist a 2^n-root ρ of -1 and a 2^n-Leja section U_{2^n} such that $E_{2^{n+1}} = (E_{2^n}, \rho U_{2^n})$.*

Repeated applications of this theorem shows that if E_k is a k-Leja section with $k = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_s}$, $n_1 > n_2 > \cdots > n_s \geq 0$, then the underlying set of E_k is formed of the union of images under certain rotations of the complete sets of roots of unity of order 2^{n_j}, $j = 1, \ldots, s$. Also, if $2^n + 1 \leq k \leq 2^{n+1}$, we have $E_k = (E_{2^n}, \rho U_{k-2^n})$.

The structure of a Leja sequence suggests that the binary expansion of k plays a role in the behavior of $\Delta(E_k)$. We also may expect to use a classical result of Gronwall [**27, 13**] showing that the Lebesgue constant for complete sets of roots of unity grows like the logarithm of the degree. Indeed, since E_{2^n} is a complete set of roots of unity of degree 2^n, Gronwall theorem ensures that $\Delta(E_{2^n}) = O(n)$, $n \to \infty$, see below.

Most of our results strongly rely on Theorem 1.1 and are obtained by successive reductions of the lengths of the Leja sections considered. This will be generally indicated by the use of a laconic expression like 'by continuing in this way'. Further specific consequences of Theorem 1.1 are given in Subsection 1.2.3.

1.2.2. An example. Given a sequence of complex numbers $\eta = (\eta_s : s \in \mathbb{N})$ such that $\eta_s^{2^s} = -1$, we may define a Leja sequence $E = E(\eta)$ such that $E_{2^{n+1}} = (E_{2^n}, \eta_n E_{2^n})$, $n \in \mathbb{N}$. Such a Leja sequence will be said to be *simple*. There are $2^{n(n+1)/2}$ simple 2^{n+1}-Leja sections. The elements of a simple Leja sequence are readily expressed in terms of η.

LEMMA 1.2. *If $E(\eta) = (a_n : n \in \mathbb{N})$ and $k = 2^n + \sum_{j=0}^{n-1} \varepsilon_j 2^j$, $\varepsilon_j \in \{0, 1\}$, then*

$$a_k = \eta_n \eta_{n-1}^{\varepsilon_{n-1}} \eta_{n-2}^{\varepsilon_{n-2}} \cdots \eta_1^{\varepsilon_1} \eta_0^{\varepsilon_0}, \quad k \geq 1. \tag{1.5}$$

PROOF. Since

$$E_{2^{n+1}} = (E_{2^n}, \eta_n E_{2^n}) = (a_0, \ldots, a_{2^n-1}, \eta_n a_0, \eta_n a_1, \ldots \eta_n a_{2^n-1}),$$

we have $a_k = \eta_n a_{k-2^n}$. Now if i is the biggest index for which $\varepsilon_i = 1$ then $k - 2^n = 2^i + \sum_{j=0}^{i-1} \varepsilon_j 2^j$ and the same reasoning as above gives $a_{k-2^n} = \eta_i a_{k-2^n-2^i}$ so that $a_k = \eta_n \eta_i a_{k-2^n-2^i}$. Continuing in this way, we obtain equation (1.5). □

1.2.3. Consequences of Theorem 1.1. The first lemma uses the structure theorem to compute the polynomial w_{E_k} (as defined in (1.2)) for a Leja section E_k and the second one computes the sup norm of this polynomial on D.

LEMMA 1.3. *Let E be a Leja sequence for D. If $k = 2^{n_1} + \cdots + 2^{n_s}$ with $n_1 > n_2 > \cdots > n_s \geq 0$, then, for every $z \in \mathbb{C}$, we have*

$$\prod_{m=0}^{k-1}(z - e_m) = c\left[z^{2^{n_1}} - 1\right] \cdot \left[(z\rho_1^{-1})^{2^{n_2}} - 1\right] \cdot \left[(z\rho_1^{-1}\rho_2^{-1})^{2^{n_3}} - 1\right] \cdots \left[(z\rho_1^{-1} \ldots \rho_{s-1}^{-1})^{2^{n_s}} - 1\right], \tag{1.6}$$

where $|c| = 1$ and $\rho_j^{2^{n_j}} = -1$ for every $1 \leq j \leq s-1$.

PROOF. Theorem 1.1 tells us that $E_k = (E_{2^{n_1}}, \rho_1 U_{k-2^{n_1}})$ with $\rho_1^{2^{n_1}} = -1$ and $U_{k-2^{n_1}}$ a $(k - 2^{n_1})$-Leja section, say $U_{k-2^{n_1}} = (u_0 = 1, u_1, \ldots, u_{k-2^{n_1}-1})$. Since the 2^{n_1} first elements of E form a complete set of roots of unity of degree 2^{n_1}, we have

$$\prod_{m=0}^{k-1}(z-e_m) = \prod_{m=0}^{2^{n_1}-1}(z-e_m)\cdot\prod_{m=2^{n_1}}^{k-1}(z-e_m) \tag{1.7}$$

$$= [z^{2^{n_1}}-1]\cdot\prod_{m=0}^{k-2^{n_1}-1}(z-\rho_1 u_m) \tag{1.8}$$

$$= \rho_1^{k-2^{n_1}}\cdot[z^{2^{n_1}}-1]\cdot\prod_{m=0}^{k-2^{n_1}-1}(\rho_1^{-1}z-u_m). \tag{1.9}$$

Now, since $U_{k-2^{n_1}}$ is itself a Leja section and $k-2^{n_1}=2^{n_2}+\cdots+2^{n_s}$ we may factorize the third factor in the same fashion and continuing in this way we arrive at the required expression with $c=\rho_1^{k-2^{n_1}}\rho_2^{k-2^{n_1}-2^{n_2}}\cdots\rho_{s-1}^{2^{n_s}}$. □

LEMMA 1.4. *Under the same assumptions as in Lemma 1.3, we have*

$$\prod_{m=0}^{k-1}|e_k-e_m| = 2^s. \tag{1.10}$$

PROOF. Equation (1.9) in the proof of the previous lemma yields

$$\prod_{m=0}^{k-1}|e_k-e_m| = |e_k^{2^{n_1}}-1|\cdot\prod_{m=0}^{k-2^{n_1}-1}|\rho_1^{-1}e_k-u_m|.$$

However, as above, in view of Theorem 1.1, $e_k=\rho_1 u_{k-2^{n_1}}$. Hence, since $u_{k-2^{n_1}}$ is a 2^{n_1}-st root of unity and $\rho_1^{2^{n_1}}=-1$, we have

$$\prod_{m=0}^{k-1}|e_k-e_m| = 2\cdot\prod_{m=0}^{k-2^{n_1}-1}|u_{k-2^{n_1}}-u_m|.$$

Likewise, using the fact that $U_{k-2^{n_1}}$ is a Leja section and $k-2^{n_1}=2^{n_2}+\cdots+2^{n_s}$, we may apply the same idea to $\prod_{m=0}^{k-2^{n_1}-1}|u_{k-2^{n_1}}-u_m|$ to obtain

$$\prod_{m=0}^{k-2^{n_1}-1}|u_{k-2^{n_1}}-u_m| = 2\prod_{m=0}^{k-2^{n_1}-2^{n_2}-1}|v_{k-2^{n_1}-2^{n_2}}-v_m|,$$

where the v_m are the points of a certain Leja section. Continuing in this way we arrive to (1.10). □

We now give another consequence of Theorem 1.1 regarding the form of the FLIPs for Leja points.

LEMMA 1.5. *Let $2^n+1\le k\le 2^{n+1}-1$ and let $E_k=(E_{2^n},\rho U_{k-2^n})$ be a k-Leja section for D.*

(1) *If $0\le j\le 2^n-1$ then*

$$\ell(E_k,e_j;z)=\ell(E_{2^n},e_j;z)\cdot\prod_{m=2^n}^{k-1}(z-e_m)/(e_j-e_m),\quad z\in\mathbb{C}.$$

(2) *If $2^n\le j\le k-1$ then*

$$\ell(E_k,e_j;z)=\ell(U_{k-2^n},u_{j-2^n};\rho^{-1}z)\cdot(1-z^{2^n})/2,\quad z\in\mathbb{C}.$$

PROOF. We easily check that the polynomials on the right hand sides are polynomials of degree $k-1$ that vanish at e_s for $s \neq j$ but take the value 1 at e_j. In the second case, we need again to use that, when $2^n \leq j \leq k-1 (\leq 2^{n+1}-2)$, $e_j^{2^n} = -1$ or, equivalently, $\rho^{2^n} = -1$. □

1.3. The estimates on the Lebesgue constants

1.3.1. Upper bound. Here is the key estimate from which the more general statements presented in the last two sections are derived.

THEOREM 1.6. *Let $2^n + 1 \leq k \leq 2^{n+1} - 1$. If $E_k = (E_{2^n}\,,\ \rho U_{k-2^n})$ is a k-Leja section for D then*

$$\Delta(E_k) \leq 2^n \Delta(E_{2^n}) + \Delta(U_{k-2^n}). \tag{1.11}$$

The proof of this result is given in Section 1.4. Our result on the asymptotic behavior of the Lebesgue constant of a Leja sequence is an immediate consequence of Theorem 1.6.

COROLLARY 1.7. *Let E be a Leja sequence for D. As $k \to \infty$, $\Delta(E_k) = O(k \ln k)$.*

The constant involved in the notation O does not depend on E.

PROOF. First recall that since E_{2^n} is a complete set of 2^n-roots of unity, the theorem of Gronwall cited above gives $\Delta(E_{2^n}) = O(\ln(2^n)) = O(n)$. Hence, in view of inequality (1.11), we have

$$\Delta(E_k) = O(n2^n) + \Delta(U_{k-2^n}), \quad 2^n + 1 \leq k \leq 2^{n+1} - 1.$$

Since U_{k-2^n} itself is a Leja section, we may bound its Lebesgue constant in the same fashion. Continuing in this way, if $k = 2^n + \sum_{j=0}^{n-1} \varepsilon_j 2^j$, $\varepsilon_j \in \{0,1\}$, we arrive at

$$\Delta(E_k) = O\left(n2^n + \sum_{j=0}^{n-1} j 2^j \varepsilon_j\right) = O(k \ln k). \tag{1.12}$$ □

1.3.2. Lower bound. As shown by the next result, the Lebesgue constant of any Leja sequence cannot grow slower than k. We conjecture that $\Delta(E_k) \leq k$ for every k.

THEOREM 1.8. *For every Leja sequence E and every $n \in \mathbb{N}^\star$ we have $\Delta(E_{2^n-1}) = 2^n - 1$.*

PROOF. We know that E_{2^n-1} is formed of the 2^n-th roots of unity with only one missing. The property is therefore a consequence of Theorem 1.9 below. □

THEOREM 1.9. *Let $a_k = \exp(2ik\pi/n)$, $R = \{a_k\,:\, k = 0, \ldots, n-1\}$ and $R^j = R \setminus \{a_j\}$ with $0 \leq j \leq n-1$. Then we have $\Delta(R^j) = n-1$.*

LEMMA 1.10. *Let a be a n-th root of unity, $n \geq 3$, $a \neq 1$. We have*

$$\left|\frac{z^n-1}{z-1}\right| \cdot |a-1| \cdot \left[\frac{1}{|z-a|} + \frac{1}{|z-\overline{a}|}\right] \leq 2n, \quad |z| = 1. \tag{1.13}$$

PROOF. We call $F(z,a)$ the left hand side of (1.13). Since F is invariant by conjugation, we may assume that $\arg(z) \in [0,\pi]$ and $\arg(a) \in]0,\pi]$. Setting $z = \exp(i\theta)$ and $a = \exp(i\phi)$, a simple calculation shows

$$\frac{F(z,a)}{|a-1||z^n-1|} = \frac{1}{4}\left[\left|\frac{1}{\sin(\theta/2)\sin((\theta-\phi)/2)}\right| + \left|\frac{1}{\sin(\theta/2)\sin((\theta+\phi)/2)}\right|\right].$$

(A) We assume $0 < \phi \leq \theta \leq \pi$. In that case, $\sin(\theta/2)$, $\sin((\theta-\phi)/2)$ and $\sin((\theta+\phi)/2)$

are nonnegative, hence

$$\frac{F(z,a)}{|a-1||z^n-1|} = \frac{1}{4}\left[\frac{1}{\sin(\theta/2)\sin((\theta-\phi)/2)} + \frac{1}{\sin(\theta/2)\sin((\theta+\phi)/2)}\right].$$

Returning to z and a, using $|1-a| = |1-\overline{a}|$ and an easily checked expansion, we find

(1.14)

$$F(z,a) = |a-1|\left|a\frac{z^n-1}{(z-1)(z-a)} + \frac{z^n-1}{(z-1)(z-\overline{a})}\right| = \left|\frac{(a-1)(z^n-1)}{(z-1)(z-a)} + \frac{(1-\overline{a})(z^n-1)}{(z-1)(z-\overline{a})}\right|$$
$$= \left|\sum_{k=0}^{n-2} z^k(a^{n-k-1} - \overline{a}^{n-k-1})\right| \le \sum_{k=0}^{n-2} |2\sin((k+1)\phi)| \le 2(n-1) < 2n.$$

(B) We now assume $0 \le \theta \le \phi \le \pi$. In that case, $\sin(\theta/2)$, $\sin((\theta+\phi)/2)$ and $-\sin((\theta-\phi)/2)$ are nonnegative. Thus

$$\frac{F(z,a)}{|a-1||z^n-1|} = \frac{1}{4}\left[\frac{-1}{\sin(\theta/2)\sin((\theta-\phi)/2)} + \frac{1}{\sin(\theta/2)\sin((\theta+\phi)/2)}\right],$$

and working as in the previous case, we get

$$\begin{aligned}
F(z,a) &= |a-1|\left|-a\frac{z^n-1}{(z-1)(z-a)} + \frac{z^n-1}{(z-1)(z-\overline{a})}\right| \\
&= \left|\frac{(1-a)(z^n-1)}{(z-1)(z-a)} + \frac{(1-\overline{a})(z^n-1)}{(z-1)(z-\overline{a})}\right| \\
&= \left|\sum_{k=0}^{n-2} z^k\Big(2 - a^{n-k-1} - \overline{a}^{n-k-1}\Big)\right| = \left|\sum_{k=0}^{n-2} z^k\Big(2 - 2\cos\big((k+1)\phi\big)\Big)\right| \\
&\le \sum_{k=0}^{n-2}\Big(2 - 2\cos\big((k+1)\phi\big)\Big) = \sum_{k=1}^{n-1}\Big(2 - 2\cos(k\phi)\Big) = 2(n-1) - 2\sum_{k=1}^{n-1}\cos(k\phi) = 2n.
\end{aligned}$$

□

PROOF OF THEOREM 1.9. Since Lebesgue constants are invariant under rotation, we may assume that $j = 0$ so that the missing point a_j equals 1.

We first prove $\Delta(R^0) \ge n-1$. In view of (1.2), with $w_{R^0}(z) = w(z) = (z^n-1)/(z-1)$, the FLIPs for R^0 are given by

$$\ell(R^0, a; z) = \frac{w(z)}{w'(a)(z-a)} = \frac{(a-1)}{na^{n-1}}\,\frac{z^n-1}{(z-1)(z-a)}.$$

We have $w(1) = n$ and it follows that

$$\Delta(R^0) \ge \sum_{a^n=1, a\neq 1} |\ell(R^0, a; 1)| = \sum_{a^n=1, a\neq 1} 1 = n-1.$$

This shows that $\Delta(R^0) \ge n-1$.

To prove the converse, we first assume that n is odd so that the interpolation points can be written as

$$R^0 = \{1\} \cup \bigcup_{a\in B}\{a, \overline{a}\},$$

with $\sharp B = (n-1)/2$. Then

(1.15)

$$\sum_{a^n=1, a\neq 1} |\ell(R^0, a; z)| = \sum_{a\in B} \left(|\ell(R^0, a; z)| + |\ell(R^0, \overline{a}; z)|\right)$$

$$= \frac{1}{n}\sum_{a\in B}\left|\frac{z^n-1}{z-1}\right|\cdot|a-1|\cdot\left[\frac{1}{|z-a|}+\frac{1}{|z-\overline{a}|}\right] \le \frac{n-1}{2}\frac{2n}{n} = n-1, \quad |z|=1, \tag{1.16}$$

where we used the estimate given in Lemma 1.10. This shows $\Delta(R^0) \le n-1$. (Recall that, according to (1.4), it suffices to bound the Lebesgue function on the unit circle.) When n is even, the proof is similar with the sole difference that $a=-1$ must be treated separately. □

1.4. Proof of Theorem 1.6

The starting point is equation (1.4) which gives

$$\Delta(E_k) \le \Big\|\sum_{j=0}^{2^n-1}\big|\ell(E_k, e_j; \cdot)\big|\Big\|_D + \Big\|\sum_{j=2^n}^{k-1}\big|\ell(E_k, e_j; \cdot)\big|\Big\|_D, \quad 2^n+1 \le k \le 2^{n+1}-1. \tag{1.17}$$

The estimate of the first sum is the difficult part. Our bound is based on the following lemma.

LEMMA 1.11. *Let $E_k = (e_0, \dots, e_{k-1})$ be a k-Leja section with $0 < k \le 2^n - 1$. If a is a 2^n-root of -1 then $\prod_{m=0}^{k-1}|e_k - e_m| \le 2^n \prod_{m=0}^{k-1}|a - e_m|$.*

To prove this lemma we need the following classical inequalities that we state as a lemma.

LEMMA 1.12. (1) *If $0 \le \alpha \le \pi/2$ then $\sin\alpha \ge 2\alpha/\pi$.*
(2) *If $m \in \mathbb{N}^\star$ and $\alpha \in \mathbb{R}$ then $2^m|\sin\alpha| \ge 2^m|\sin\alpha\cos\alpha| \ge |\sin 2^m\alpha|$.*

PROOF. The second inequality follows at once from repeated applications of

$$|\sin\alpha| \ge |\sin\alpha\cos\alpha| = |\sin 2\alpha|/2.$$

□

PROOF OF LEMMA 1.11. We assume

$$k = 2^{n_1} + \cdots + 2^{n_s} \quad \text{with} \quad n-1 \ge n_1 > \cdots > n_s \ge 0, \tag{1.18}$$

and use the same notation as in Lemma 1.3. In particular $\rho_j^{2^{n_j}} = -1$ so that for some $t_j \in \mathbb{N}$,

$$\theta_j := \arg(\rho_j^{-1}) = (2t_j+1)\pi/2^{n_j}, \quad 1 \le j \le s-1. \tag{1.19}$$

Equation (1.6) yields

$$\prod_{m=0}^{k-1}|a-e_m| = 2^s\prod_{j=0}^{s-1}|\sin 2^{n_{j+1}-1}(\theta_0+\cdots+\theta_j)|,$$

where $\arg a = \theta_0 = (2t_0+1)\pi/2^n$. Thus, in view of (1.10), the lemma will be proved if we show

$$\prod_{j=0}^{s-1}\left|\sin 2^{n_{j+1}-1}(\theta_0+\cdots+\theta_j)\right| \ge 1/2^n. \tag{1.20}$$

We first treat the case $s=1$, that is, $k=2^{n_1}$. Here we just need to prove

$$\left|\sin 2^{n_1-1}\theta_0\right| \geq 1/2^n.$$

Since

$$2^{n_1-1}\theta_0 = \pi/2^{n-n_1+1} + 2t_0\pi/2^{n-n_1+1},$$

we have

$$\left|\sin\left(2^{n_1-1}\theta_0\right)\right| \geq \sin\left(\pi/2^{n-n_1+1}\right) \geq (2/\pi)\pi/2^{n-n_1+1} \geq 1/2^n,$$

where we use Lemma 1.12 (1).

We now assume $s \geq 2$ in (1.18). We first look at the factor corresponding to $j=s-1$ in (1.20). Applying Lemma 1.12 (2) with $m = n_{s-1}-n_s$ and $\alpha = 2^{n_s-1}(\theta_0+\cdots+\theta_{s-1})$, we obtain

$$\left|\sin 2^{n_s-1}(\theta_0+\cdots+\theta_{s-1})\right| \geq 2^{n_s-n_{s-1}}\left|\sin 2^{n_{s-1}-1}(\theta_0+\cdots+\theta_{s-1})\right|. \tag{1.21}$$

But, in view of (1.19),

$$2^{n_{s-1}-1}\theta_{s-1} = 2^{n_{s-1}-1}(2t_{s-1}+1)\pi/2^{n_{s-1}} = \pi/2 + t_{s-1}\pi$$

which gives

$$\left|\sin 2^{n_{s-1}-1}(\theta_0+\cdots+\theta_{s-1})\right| = \left|\cos 2^{n_{s-1}-1}(\theta_0+\cdots+\theta_{s-2})\right|. \tag{1.22}$$

Hence, (1.21) becomes

$$\left|\sin 2^{n_s-1}(\theta_0+\cdots+\theta_{s-1})\right| \geq 2^{n_s-n_{s-1}}\left|\cos 2^{n_{s-1}-1}(\theta_0+\cdots+\theta_{s-2})\right|. \tag{1.23}$$

We now concentrate on the last two factors of (1.20), those corresponding to $j=s-1$ and $j=s-2$. Thanks to (1.23), we have

$$\begin{aligned}\prod_{j=s-2}^{s-1}&\left|\sin 2^{n_{j+1}-1}(\theta_0+\cdots+\theta_j)\right| \\ &\geq 2^{n_s-n_{s-1}}\left|\sin 2^{n_{s-1}-1}(\theta_0+\cdots+\theta_{s-2})\cdot\cos 2^{n_{s-1}-1}(\theta_0+\cdots+\theta_{s-2})\right|.\end{aligned} \tag{1.24}$$

Another use of Lemma 1.12 (2) with $m = n_{s-2}-n_{s-1}$ yields

$$\begin{aligned}&\left|\sin 2^{n_{s-1}-1}(\theta_0+\cdots+\theta_{s-2})\cdot\cos 2^{n_{s-1}-1}(\theta_0+\cdots+\theta_{s-2})\right| \\ &\qquad\geq 2^{n_{s-1}-n_{s-2}}\left|\sin 2^{n_{s-2}-1}(\theta_0+\cdots+\theta_{s-2})\right|\end{aligned} \tag{1.25}$$

and, again, since

$$2^{n_{s-2}-1}\theta_{s-2} = 2^{n_{s-2}-1}(2t_{s-2}+1)\pi/2^{n_{s-2}} = \pi/2 + t_{s-2}\pi,$$

the absolute value of the sine on the right hand side of (1.25) actually equals $|\cos 2^{n_{s-2}-1}(\theta_0+\cdots+\theta_{s-3})|$. Thus, at this point, taking into account (1.24) and (1.25), we have

$$\begin{aligned}\prod_{j=s-3}^{s-1}&\left|\sin 2^{n_{j+1}-1}(\theta_0+\cdots+\theta_j)\right| \\ &\geq 2^{n_s-n_{s-2}}\left|\sin 2^{n_{s-2}-1}(\theta_0+\cdots+\theta_{s-3})\cdot\cos 2^{n_{s-2}-1}(\theta_0+\cdots+\theta_{s-3})\right|.\end{aligned} \tag{1.26}$$

Continuing in this fashion, we finally arrive at

$$\prod_{j=0}^{s-1}\left|\sin 2^{n_{j+1}-1}(\theta_0+\cdots+\theta_j)\right| \geq 2^{n_s-n_1}\left|\sin 2^{n_1-1}\theta_0\cdot\cos 2^{n_1-1}\theta_0\right| = 2^{n_s-n_1-1}|\sin 2^{n_1}\theta_0|.$$

Now, working as in the case $s=1$ above, we obtain

$$2^{n_s-n_1-1}|\sin 2^{n_1}\theta_0| \geq 2^{n_s-n_1-1}(2/\pi)\pi/2^{n-n_1} = 2^{n_s-n} \geq 2^{-n}.$$

This completes the proof of the lemma. □

CONCLUSION OF THE PROOF OF THEOREM 1.6. We assume that $E_{2^{n+1}} = (E_{2^n}\,,\, \rho U_{2^n})$ and $2^n+1 \leq k \leq 2^{n+1}-1$. As indicated above, starting from (1.17), we want to estimate

$$\Big\| \sum_{j=0}^{2^n-1} |\ell(E_k, e_j; \cdot)| \Big\|_D \quad \text{and} \quad \Big\| \sum_{j=2^n}^{k-1} |\ell(E_k, e_j; \cdot)| \Big\|_D.$$

(A) Thanks to Lemma 1.5 (2), we have

$$\Big\| \sum_{j=2^n}^{k-1} \big|\ell(E_k, e_j; \cdot)\big| \Big\|_D \leq \Big\| \frac{|1-z^{2^n}|}{2} \Big\|_D \Big\| \sum_{j=2^n}^{k-1} \big|\ell(U_{k-2^n}, u_{j-2^n}; \rho^{-1}\cdot)\big| \Big\|_D = \Delta(U_{k-2^n}).$$

(B) On the other hand, in view of Lemma 1.5 (1), for every z in D,

$$\begin{aligned} \sum_{j=0}^{2^n-1} |\ell(E_k, e_j; z)| &= \sum_{j=0}^{2^n-1} \Big\{ \big|\ell(E_{2^n}, e_j; z)\big| \prod_{m=2^n}^{k-1} |z-e_m|/|e_j-e_m| \Big\} \\ &\leq \Delta(E_{2^n}) \max_{j=0,\ldots,2^n-1} \prod_{m=2^n}^{k-1} |z-e_m|/|e_j-e_m|. \end{aligned}$$

Hence, to prove the theorem, it suffices to show that

$$\Big\| \prod_{m=2^n}^{k-1} \frac{|z-e_m|}{|e_j-e_m|} \Big\|_D \leq 2^n, \quad 0 \leq j \leq 2^n-1.$$

To see this, we observe that for $0 \leq j \leq 2^n-1$ and $z \in D$, we have

$$\begin{aligned} \prod_{m=2^n}^{k-1} |z-e_m| &= \prod_{m=0}^{k-1-2^n} |\rho^{-1}z-u_m| \leq \prod_{m=0}^{k-1-2^n} |u_{k-2^n}-u_m| \\ &\leq 2^n \prod_{m=0}^{k-1-2^n} |\rho^{-1}e_j-u_m| = 2^n \prod_{m=2^n}^{k-1} |e_j-e_m|, \end{aligned} \tag{1.27}$$

where the equalities come from the relation $E_{2^{n+1}} = (E_{2^n}\,,\, \rho U_{2^n})$; the first inequality follows from the fact that U_{2^n} is a Leja section and the second inequality is given by Lemma 1.11. The use of this lemma is permitted since $(\rho^{-1}e_j)^{2^n} = -1$. Indeed $\rho^{2^n} = -1$ and, since $0 \leq j \leq 2^n-1$, $e_j^{2^n} = 1$. □

1.5. Alper-smooth jordan curves

Using classical works of Alper [**1, 2**] as in [**4**], we now show that if K is a compact set whose boundary is a Alper-smooth Jordan curve (see below) and ϕ denotes the exterior conformal mapping from $\overline{\mathbb{C}} \setminus D$ onto $\overline{\mathbb{C}} \setminus K$ then the image under ϕ of every Leja sequence for the disk (which lies in ∂K) still has a Lebesgue constant (with respect to K) that grows (at most) like a polynomial.

Let Γ be a smooth Jordan curve. The angle between the tangent at $\Gamma(s)$ and the positive real axis is denoted by $\theta(s)$ where s is the arc-length parameter. Following a terminology used by Kövari and Pommerenke [**29**], we say that Γ is *Alper-smooth* if the modulus of continuity ω of θ satisfies

$$\int_0^h \frac{\omega(x)}{x} |\ln x| dx < \infty.$$

Twice continuously differentiable Jordan curves are Alper-smooth.

An important property of the exterior conformal mapping ϕ is the following (see [**1**, §1 and §2] or [**2**, eqn (3) p. 45]). There exist positive constants $M_1 < M_2$ such that

$$0 < M_1 \le \left| \frac{\phi(z) - \phi(w)}{z - w} \right| \le M_2 < \infty, \quad z, w \in \partial D, \quad z \ne w. \tag{1.28}$$

THEOREM 1.13. *Assume that K is a compact set whose boundary is an Alper-smooth Jordan curve. We denote by ϕ the conformal mapping of the exterior to the unit disc onto the exterior of K. If $E = (e_k : k \in \mathbb{N})$ is a Leja sequence for D then the Lebesgue constant $\Delta(\phi(E_k))$ grows at most like a polynomial in k as $k \to \infty$. Here $\phi(E_k) := (\phi(e_0), \ldots, \phi(e_{k-1}))$.*

We need the following lemma.

LEMMA 1.14. *Under the same assumptions as in the theorem, for any w on the unit circle, $w \ne e_i$, $i = 0, \ldots, k-1$, we have*

$$C^k(K) \frac{1}{c_k} \le \prod_{l=0}^{k-1} \frac{|\phi(w) - \phi(e_l)|}{|w - e_l|} \le C^k(K)\, c_k, \tag{1.29}$$

where $C(K)$ is the logarithmic capacity of K,

$$c_k = \exp\left(A \sum_{j=0}^{s} \varepsilon_j \right), \quad k = \sum_{j=0}^{s} \varepsilon_j 2^j, \quad \varepsilon_j \in \{0, 1\},$$

and A is a positive constant depending only on K.

PROOF. The proof can be found in [**4**, Lemma 3]. It is an adaptation of a method due to Alper. □

PROOF OF THEOREM 1.13. As in (1.4), we have

$$\Delta(\phi(E_k)) = \left\| \sum_{j=0}^{k-1} \left| \ell(\phi(E_k), \phi(e_j); \cdot) \right| \right\|_{\partial K}.$$

Thus, since $\partial K = \phi(\partial D)$, we just need to consider terms of the form $|\ell(\phi(E_k), \phi(e_j); \phi(w))|$ with $|w| = 1$.

Now, since, for $w \ne e_l$, $l = 0, \ldots, k-1$,

$$\prod_{l=0, l\ne m}^{k-1} \frac{|\phi(w) - \phi(e_l)|}{|w - e_l|} = \prod_{l=0}^{k-1} \frac{|\phi(w) - \phi(e_l)|}{|w - e_l|} \times \frac{|w - e_m|}{|\phi(w) - \phi(e_m)|}, \quad 0 \le m \le k-1,$$

equations (1.28) and (1.29) give

$$\frac{C^k(K)}{M_2 c_k} \le \prod_{l=0, l\ne m}^{k-1} \frac{|\phi(w) - \phi(e_l)|}{|w - e_l|} \le \frac{C^k(K) c_k}{M_1}.$$

By continuity, this inequality remains true for $w = e_m$. Next, dividing the estimates for w and for e_m, we obtain for $w \ne e_l$, $l \ne m$,

$$\left| \ell\big(\phi(E_k), \phi(e_m), \phi(w)\big) \right| \le \frac{M_2}{M_1} c_k^2 \cdot \left| \ell\big(E_k, e_m, w\big) \right|.$$

Again by continuity, the above inequality holds for every w on the unit circle. Now applying the inequality for every point e_m we get

$$\Delta(\phi(E_k)) \le \frac{M_2}{M_1} c_k^2 \Delta(E_k),$$

from which the conclusion readily follows since, in view of Lemma 1.14, $c_k = O(k^{A/\ln(2)})$. □

As shown by the proof $\Delta(\phi(E_k))$ grows (at most) like $\Delta(E_k)$ apart from the factor $c_k^2 = O(k^{2A/\ln(2)})$. A precise definition of A is given in [**4**, proof of Lemma 3, eqn. (44)]. The way this number depends on the geometry of K does not seem to be simple. It would be interesting to have estimates on A using as little information as possible on the Jordan curve that defines K.

1.6. Multivariate interpolation sets

1.6.1. Intertwining of block unisolvent arrays. The dimension of the space $\mathscr{P}_n(\mathbb{C}^N)$ of complex polynomials of (total) degree at most n in N complex variables is $\binom{n+N}{n}$. A finite set A formed of $\binom{n+N}{n}$ distinct points is said to be *unisolvent* of degree n if Lagrange interpolation at the points of A by polynomials of degree at most n is well defined. The condition is satisfied if and only if A is not included in an algebraic hypersurface of degree $\leq n$. In that case, the Lagrange interpolation polynomial of a function f, still denoted by $\mathbf{L}[A;f]$, is given by (1.1) but the FLIPs $\ell(A,a;\cdot)$ no longer have a simple expression. If K is a compact subset in $\mathbb{C}^N$ containing A, the Lebesgue constant $\Delta(A)$ or $\Delta(A\,|\,K)$ is still defined as the operator norm on $C(K)$ of the interpolation operator and is given by the multivariate form of (1.3). For basic definitions and facts on multivariate Lagrange interpolation from the complex analysis point of view, the reader may consult [**6**].

It is useful to label the elements of a unisolvent set with multi-indexes. The length $\sum_{i=1}^N \alpha_i$ of a N-index $\alpha = (\alpha_1,\ldots,\alpha_N)$ is denoted by $|\alpha|$. The indexes are ordered according to the graded lexicographic order $\prec$. Recall that $\alpha \prec \beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and the leftmost non zero entry of $\alpha - \beta$ is negative.

We say that a $\binom{n+N}{n}$-tuple $A = (x_\alpha = (x_{\alpha_1},\ldots,x_{\alpha_N}) : |\alpha| \leq n)$ is *block-unisolvent* of degree n if for every i, $i \in \{0,1,\ldots,n\}$, (the underlying set of) the i-th block $\mathscr{B}_i(A) := (x_\alpha : |\alpha| \leq i)$ is unisolvent of degree i. Note that when A is a tuple of unidimensional interpolation points, $\mathscr{B}_i(A)$ is simply formed of the first $i+1$ entries of A.

Given two *block-unisolvent* families of degree n,

$$A = (x_{\alpha^1} = (x_{\alpha^1_1},\ldots,x_{\alpha^1_{N_1}}) : |\alpha^1| \leq n) \text{ in } \mathbb{C}^{N_1} \text{ and } B = (y_{\alpha^2} = (y_{\alpha^2_1},\ldots,y_{\alpha^2_{N_2}}) : |\alpha^2| \leq n) \text{ in } \mathbb{C}^{N_2},$$

the *intertwining* of A and B is

$$A \oplus B = \big((x_{\alpha^1}, y_{\alpha^2}) : \big|(\alpha^1,\alpha^2)\big| \leq n\big).$$

It is known [**14**, Theorem 3.1], that $A \oplus B$ is itself block-unisolvent of degree n with blocks

$$\mathscr{B}_i(A\oplus B) = \mathscr{B}_i(A) \oplus \mathscr{B}_i(B), \quad 0 \leq i \leq n. \tag{1.30}$$

Note that, by repeating the process, we may construct the intertwining of any finite number of block-unisolvent families of same degree. In particular, iterations of (1.30) give

$$\mathscr{B}_i(A_1 \oplus A_2 \oplus \cdots \oplus A_N) = \mathscr{B}_i(A_1) \oplus \mathscr{B}_i(A_2) \oplus \cdots \oplus \mathscr{B}_i(A_N). \tag{1.31}$$

1.6.2. The Lebesgue constant of the intertwining of two block-unisolvent family. Our main tool is the following result.

THEOREM 1.15. *Let K be a compact set in $\mathbb{C}^{N_1+N_2}$ containing $A \oplus B$. We denote by K_1 (resp., K_2) the projection of K on $\mathbb{C}^{N_1}$ (resp., $\mathbb{C}^{N_2}$). We have*

$$\Delta(A\oplus B|K) \leq 4\binom{n+N_1+N_2}{n} \sum_{i+j\leq n} \Delta\left(\mathscr{B}_i(A)|K_1\right)\cdot \Delta\left(\mathscr{B}_j(B)|K_2\right).$$

PROOF. See [**14**, Theorem 4.4]. □

In the notation for the Lebesgue constant, we indicate the compact set (containing the interpolation points) with respect to which the Lebesgue constant is computed. In order to estimate the Lebesgue constant of $A \oplus B$, we must therefore use a bound for all the blocks of A and B. The above theorem certainly greatly overestimates the Lebesgue constant but it is sufficient to prove our main application in the following subsection.

1.6.3. Intertwining of Leja sections and related families. For $i = 1, \dots, N$ we let $E^{(i)} = (e_n^{(i)} \,:\, n \in \mathbb{N})$ denotes a Leja sequence for D and K_i a plane compact set whose boundary is an Alper-smooth Jordan curve (with conformal exterior mapping $\phi_i : \overline{\mathbb{C}} \setminus D \to \overline{\mathbb{C}} \setminus K_i$). For every $n \in \mathbb{N}$, we define a family $\mathbf{P}_{N,n}$ as

$$\mathbf{P}_{N,n} = \phi_1(E_{n+1}^{(1)}) \oplus \cdots \oplus \phi_N(E_{n+1}^{(N)}).$$

The $\binom{n+N}{n}$ points of $\mathbf{P}_{N,n}$ lie in $K := K_1 \times K_2 \times \cdots \times K_N \subset \mathbb{C}^N$ and are given by the relation

$$\mathbf{P}_{N,n} = \left(\mathbf{p}_\alpha = \left(\phi_1(e_{\alpha_1}^{(1)}), \dots, \phi_N(e_{\alpha_N}^{(N)}) \right) \,:\, |\alpha| \le n \right).$$

The family $\mathbf{P}_{N,n}$ is block-unisolvent of degree n in $\mathbb{C}^N$. It is obtained by induction via the relations $\mathbf{P}_{1,n} = \phi_1(E_{n+1}^{(1)})$ and

$$\mathbf{P}_{d+1,n} = \mathbf{P}_{d,n} \oplus \phi_{d+1}\left(E_{n+1}^{(d+1)}\right), \quad 1 \le d \le N-1. \tag{1.32}$$

We obtain the following theorem as an consequence of Theorems 1.13 and 1.15. The proof relies on the fact that $\mathbf{P}_{N,n}$ is a sub-family of $\mathbf{P}_{N,n+1}$.

THEOREM 1.16. *The Lebesgue constant $\Delta(\mathbf{P}_{N,n})$ grows at most like a polynomial in n as $n \to \infty$.*

Here the Lebesgue constant is computed with respect to K, the Cartesian product of the K_i's.

PROOF. The proof is by induction on N.

The case $N = 1$ is given by Theorem 1.13. We assume that the estimate holds true up to N and prove it for $N+1$. In view of (1.32) and Theorem 1.15, we have

$$\Delta(\mathbf{P}_{N+1,n}) \le 4\binom{n+N+1}{n} \sum_{i+j \le n} \Delta(\mathscr{B}_i(\mathbf{P}_{N,n})) \cdot \Delta\left(\mathscr{B}_j(\phi(E_{n+1}^{(N+1)}))\right), \tag{1.33}$$

where we use the previous notation for the blocks of both factors. Now, the important point is that

$$\mathscr{B}_i(\mathbf{P}_{N,n}) = \mathbf{P}_{N,i} \quad \text{and} \quad \mathscr{B}_j(\phi(E_{n+1}^{(N+1)})) = \phi(E_{j+1}^{(N+1)}),$$

where we use (1.31) for the first equality. Equation (1.33) thus becomes

$$\Delta(\mathbf{P}_{N+1,n}) \le 4\binom{n+N+1}{n} \sum_{i+j \le n} \Delta(\mathbf{P}_{N,i})) \cdot \Delta\left(\phi(E_{j+1}^{(N+1)})\right). \tag{1.34}$$

Now, in view of Theorem 1.13, for some constant C_{N+1} we have $\Delta\left(\phi(E_{j+1}^{(N+1)})\right) = O(j^{C_{N+1}})$ and the claim now readily follows from (1.34) and the induction hypothesis. □

In the case of an intertwining of Leja sequences, inequality (1.34) together with Corollary 1.7 yields the (almost certainly pessimistic) bound

$$\Delta\left(E_{n+1}^{(1)} \oplus E_{n+1}^{(2)} \oplus \cdots \oplus E_{n+1}^{(N)}\right) = O\left(n^{(N^2+7N-6)/2}(\ln n)^N\right), \quad n \to \infty.$$

The proof also shows that the intertwining of sequences having a Lebesgue constant growing sub-exponentially also has a Lebesgue constant that grows sub-exponentially. Thus starting from Leja sequences for compact K_i of the kind considered in [**42**], we obtain sets of interpolation points whose Lebesgue constant grows at most sub-exponentially.

1.6.4. Application to the construction of weakly admissible meshes. It will be shown in a forthcoming paper that the projections on the real axis of Leja sequences for D still have a Lebesgue constant that grows polynomially. Here, we conclude with a few words on the connection with a topic of recent interest. Let Ω be a compact set in $\mathbb{C}^N$ and for $n \in \mathbb{N}$ a finite subset A_n of Ω. We say that $(A_n : n \in \mathbb{N})$ is a weakly admissible mesh for Ω if the following two conditions are satisfied.

(1) The cardinality of A_n grows sub-exponentially (i.e. $(\sharp A_n)^{1/n} \to 1$ as $n \to \infty$).
(2) There exists a sequence M_n growing sub-exponentially such that

$$\|p\|_\Omega \leq M_n \|p\|_{A_n}, \quad p \in \mathscr{P}_n(\mathbb{C}^N).$$

Weakly admissible meshes are good evaluation points for approximation by discrete least squares polynomials [**15**]. They also contains good points for Lagrange interpolation that, in principles, can be numerically retrieved [**12**]. However, for computational reasons, it is desirable to have both M_n and the cardinality of the A_n as small as possible, see [**12**]. Both conditions however compete with each other and such meshes are not easy to produce. An acceptable compromise is obtained with meshes having both the cardinality of A_n and the constant M_n growing at most polynomially. Now, it is readily seen that (new) examples of such meshes are given by finite unions of images of sets of the form $\mathbf{P}_{N,n}$ under affine mappings (for the union of the corresponding compact sets).

CHAPTER 2

Lagrange interpolation at real projections of Leja sequences for the unit disk

Abstract. We show that the Lebesgue constants of the real projection of Leja sequences for the unit disk grow like a polynomial. The main application is the first construction of explicit multivariate interpolation points in $[-1,1]^N$ whose Lebesgue constants also grow like a polynomial.

2.1. Introduction

We pursue the work initiated in Chapter 1 that aims to construct *explicit* and (or) *easily computable* sets of efficient points for multivariate Lagrange interpolation by using the process of intertwining (see below) certain univariate sequences of points. Here, the efficiency of the interpolation points is measured by the growth of their Lebesgue constants (the norms of the interpolation operator). Namely, we look for sets of points $\mathbf{P}_n \subset \mathbb{R}^N$ — for interpolation by polynomials of total degree at most n — for which the Lebesgue constants $\Delta(\mathbf{P}_n)$ grows at most like a polynomial in n. We say that such points are good interpolation points in the sense that if $\Delta(\mathbf{P}_n) = O(n^\alpha)$ with $\alpha \in \mathbb{N}^\star$ then, in view of a classical result of Jackson, the Lagrange interpolation polynomials at $\mathbf{P}_n$ of any function with $\alpha+1$ continuous (total) derivatives converge uniformly. This is detailed in this chapter. In our former work in Chapter 1, multivariate interpolation points with good Lebesgue constants were constructed on the Cartesian product of many plane compact subsets bounded by sufficiently regular Jordan curves (including, of course, polydiscs) starting from Leja points for the unit disc (see below). Yet, from a practical point of view, especially if we have in mind applications to numerical analysis, the real case is more interesting. It is the purpose of this paper to exhibit explicit interpolation points in $[-1,1]^N$ with Lebesgue constants growing at most like a polynomial. As far as we know, this is the first general construction of such points. This will be done by suitably modifying the methods employed in Chapter 1. Actually, the unidimensional points will be taken as the projections on the real axis of the points of a Leja sequence. We shall first show how to describe (and compute) these points and, in particular, prove that they are Chebyshev-Lobatto points (of increasing degree) arranged in a certain manner. We shall then study their Lebesgue constants to prove that they grow at most like $n^3 \log n$ where n is the degree of interpolation. The passage to the multivariate case is identical to that shown in Chapter 1 and will not be detailed.

Notation. We refer to Chapter 1 for basic definitions on Lagrange interpolation theory. Let us just indicate that, given a finite set A, we write $w_A := \prod_{a\in A}(\cdot - a)$. The fundamental Lagrange interpolation polynomial (FLIP) for $a \in A$ is denoted by $\ell(A,a;\cdot)$. We have

$$\ell(A,a;\cdot) = \prod_{b\in A, b\neq a} (\cdot - b)/(a-b) = w_A/\big(w_A'(a)(\cdot - a)\big).$$

The Lagrange interpolation polynomial of f is $\mathbf{L}[A;f]=\sum_{a\in A} f(a)\ell(A,a;\cdot)$ and the norm of $\mathbf{L}[A;\cdot]$ as an operator on $C(K)$ (where K is a compact subset containing A) is the Lebesgue constant $\Delta(A)=\Delta(A,K)$. It is known that $\Delta(A)=\max_{z\in K}\sum_{a\in A}|\ell(A,a;z)|$.

We shall denote by M, M', etc constants independent of the relevant parameters. Occurrences of the same letter in different places do not necessarily refer to the same constant.

2.2. Leja sequences and their projections on the real axis

2.2.1. Leja sequences for the unit disk. We briefly recall the definition and the structure of a Leja sequence for the unit disk $D=\{|z|\le 1\}\subset\mathbb{C}$. A k-tuple $E_k=(e_0,\dots,e_{k-1})\in D^k$ with $e_0=1$ is a *k-Leja section* for D if, for $j=1,\dots,k-1$, the $(j+1)$-st entry e_j maximizes the product of the distances to the j previous points, that is

$$\prod_{m=0}^{j-1}\left|e_j-e_m\right|=\max_{z\in D}\prod_{m=0}^{j-1}|z-e_m|,\quad j=1,\dots,k-1.$$

The maximum principle implies that the points e_i actually lie on the unit circle ∂D. A sequence $E=(e_k : k\in\mathbb{N})$ for which $E_k:=(e_0,\dots,e_{k-1})$ is a k-Leja section for every $k\in\mathbb{N}^\star$ is called a *Leja sequence* (for D). Of course, the points of a Leja sequence are pairwise distinct.

The structure of a Leja sequence is studied in [**4**] where one can find the following result.

THEOREM 2.1 (Białas-Cież and Calvi). *A Leja sequence is characterized by the following two properties.*

1. *The underlying set of a 2^n-Leja section for D is formed of the 2^n-th roots of unity.*
2. *If $E_{2^{n+1}}$ is a 2^{n+1}-Leja section then there exist a 2^n-root ρ of -1 and a 2^n-Leja section $E^{(1)}_{2^n}$ such that $E_{2^{n+1}}=(E_{2^n},\rho E^{(1)}_{2^n})$.*

Here, the second assertion means the following. The element of index s in $E_{2^{n+1}}$ is

1. the element of index s in E_{2^n} when $0\le s\le 2^n-1$,
2. ρ times the element of index $(s-2^n)$ in $E^{(1)}_{2^n}$ when $2^n\le s\le 2^{n+1}-1$.

Repeated applications of Theorem 2.1 show that if $d=2^{n_0}+2^{n_1}+\cdots+2^{n_r}$ with $n_0>n_1>\cdots>n_r\ge 0$ then

$$E_d=(E_{2^{n_0}},\rho_0E^{(1)}_{d-2^{n_0}})=(E_{2^{n_0}},\rho_0E^{(1)}_{2^{n_1}},\rho_1\rho_0E^{(2)}_{d-2^{n_0}-2^{n_1}}) \tag{2.1}$$

$$=\cdots=(E_{2^{n_0}},\rho_0E^{(1)}_{2^{n_1}},\rho_1\rho_0E^{(2)}_{2^{n_2}},\dots,\rho_{r-1}\cdots\rho_1\rho_0E^{(r)}_{2^{n_r}}), \tag{2.2}$$

where each $E^{(j)}_{2^{n_j}}$ consists of a complete set of 2^{n_j}-roots of unity, arranged in a certain order (actually, a 2^{n_j}-Leja section), and ρ_j is a 2^{n_j}-th root of -1.

2.2.2. Projections of Leja sequences. We are interested in polynomial interpolation at the projections on the real axis of the points of a Leja sequence. We eliminate repeated values and this somewhat complicates the description of the resulting sequence. We use $\Re(\cdot)$ to denote the real part of a complex number (or sequence).

DEFINITION 2.2. A sequence X (in $[-1,1]$) is said to be a $\Re$-Leja sequence if there exists a Leja sequence $E=(e_k : k\in\mathbb{N})$ such that X is obtained by eliminating repetitions in $\Re(e_k : k\in\mathbb{N})$. Here, we mean that the entry $\Re(e_j)$ is eliminated whenever there exists $i<j$ such that $\Re(e_j)=\Re(e_i)$. We write $X=X(E)$.

In particular $X(E)$ is a subsequence of $\Re(e_k : k \in \mathbb{N})$. Since for every $n \in \mathbb{N}$, the underlying set of a 2^{n+1}-Leja section is a complete set of 2^{n+1}-st roots of unity (Theorem 2.1), the corresponding real parts form the set $\mathscr{L}_{2^n}$ of Chebyshev-Lobatto (or Gauss-Lobatto) points of degree 2^n,

$$\mathscr{L}_{2^n} = \{\cos(j\pi/2^n) : j = 0, \dots, 2^n\}.$$

These points are the extremal points of the usual Chebyshev polynomial (of degree 2^n) and are sometimes referred to as the “Chebyshev extremal points”.

For future reference, we state this observation as a lemma.

LEMMA 2.3. *Let X be a $\Re$-Leja sequence. For every $n \in \mathbb{N}$, the underlying set of $X_{2^n+1} = (x_0, \dots, x_{2^n})$ is the set of Chebyshev-Lobatto points $\mathscr{L}_{2^n}$.*

Theorem 2.4 below gives two descriptions of $\Re$-Leja sequences. The first one is particularly adapted to the computations of $\Re$-Leja sequences when one is given Leja sequences. Examples of easily computable (and explicit) Leja sequences can be found in Lemma 1.2. In Figure 1 (I), we show the first 16 points of a Leja sequence E and the first 9 points of the corresponding $\Re$-Leja sequence $X(E)$. (The Leja sequence we use is given by the rule $E_2 = (1, -1)$ and $E_{2^{n+1}} = (E_{2^n}, \exp(i\pi/2^n)E_{2^n})$.)

The concatenation of tuples is denoted by $\wedge$,

$$(x_1, \dots, x_m) \wedge (y_1, \dots, y_n) := (x_1, \dots, x_m, y_1, \dots, y_n).$$

For every sequence of complex numbers $S = (s_k : k \in \mathbb{N})$ we define $S(j : k) := (s_j, s_{j+1}, \dots, s_k)$. As before, $S_k := S(0 : k-1)$.

THEOREM 2.4. *A sequence $X = (x_k : k \in \mathbb{N})$ is a $\Re$-Leja sequence if and only if there exists a Leja sequence $E = (e_k : k \in \mathbb{N})$ such that*

$$X = (1, -1) \wedge \bigwedge_{j=1}^{\infty} \Re\Big(E(2^j : 2^j + 2^{j-1} - 1)\Big). \tag{2.3}$$

Equivalently, $x_k = \Re(e_{\phi(k)})$, $k \in \mathbb{N}$, with $\phi(0) = 0$, $\phi(1) = 1$ and

$$\phi(k) = \begin{cases} \frac{3k}{2} - 1 & k = 2^n \\ 2^{\lfloor \log_2(k) \rfloor} + k - 1 & k \neq 2^n \end{cases}, \quad k \geq 2, \tag{2.4}$$

where $\lfloor \cdot \rfloor$ is used for the ordinary floor function.

PROOF. Let $X = X(E)$ with $E = (e_s : s \in \mathbb{N})$. We prove that if $2^j \leq k < 2^j + 2^{j-1}$ then $\Re(e_k)$ does not appear in $\Re(E_k)$ and therefore provides a new point for X. To do that, since e_k itself does not belong to E_k, it suffices to check that $\overline{e_k}$ is not a point of E_k, equivalently $e_k \neq \overline{e_s}$, $0 \leq s \leq k-1$. If $s < 2^j$ then $\overline{e_s}$ is a 2^j-th root of unity whereas e_k is not. On the other hand, if $2^j \leq s \leq k-1$ then, in view of Theorem 2.1, we have

$$E_{2^{j+1}} = (E_{2^j}\, , \rho E^{(1)}_{2^j}) = (E_{2^j}\, , \rho E^{(1)}_{2^{j-1}}, \rho\rho' E^{(2)}_{2^{j-1}}),$$

hence e_k and e_s appear in the second tuple so that $e_k = \rho a$ and $e_s = \rho b$ where ρ is a 2^j-th root of -1 and both a and b are 2^{j-1}-st roots of unity. The relation $e_k = \overline{e_s}$ yields $\rho/\overline{\rho} = \overline{b}/a$. The argument of the first number is of the form $2(2l+1)\pi/2^j$ and the argument of the second one is $2t\pi/2^{j-1}$ (with $l, t \in \mathbb{Z}$). Equality is therefore impossible.

Now, in view of Lemma 2.3, from $E_{2^{j+1}}$ we obtain $2^j + 1$ points for X, namely the points in $\mathscr{L}_{2^j}$ arranged in a certain way. Yet, the $2^j + 1$ first points of X are already given by E_{2^j} ($2^{j-1} + 1$ points) together with, according to the first part of this proof, the 2^{j-1}

points $\Re(e_k)$, $2^j \leq k < 2^j + 2^{j-1}$. This implies that if $2^j + 2^{j-1} \leq k < 2^{j+1}$ then $\Re(e_k)$ is not a new point for X. This achieves the proof of (2.3).

To prove (2.4) we observe that, in view of (2.3), we have

$$\Re(e_{2^k+i}) = x_{2^{k-1}+i+1}, \quad 0 \leq i \leq 2^{k-1} - 1.$$

Hence $\phi(2^{k-1}+i+1) = 2^k + i$ and the expression for ϕ easily follows. □

COROLLARY 2.5 (to the proof). *If $X = X(E)$ then $X(2^n + 1 : 2^{n+1}) = \Re(E(2^{n+1} : 2^{n+1} + 2^n - 1))$.*

Decompositions (3.31) and (2.3) are fundamental to this work. In particular, the binary expansion of k will be used in the study of the tuple $X(0 : k)$.

Note that, of course, the decomposition would be different if we projected the Leja points on another segment, say on $[-e^{i\theta}, e^{i\theta}]$. The distribution of the projected points in general depends on arithmetic properties of θ. We shall not discuss the general case in this paper.

Finally, let us point out that our $\Re$-Leja sequences are not Leja sequences for the interval. It can easily be shown that they are pseudo Leja sequences. A pseudo Leja sequence for a plane compact set K is a sequence (a_n) in K that satisfies an inequality of the form

$$\max_{z\in K} \prod_{i=0}^{d} |z - a_i| \leq M_{d+1} \prod_{i=0}^{d} |a_{d+1} - a_i|, \quad d \geq 0,$$

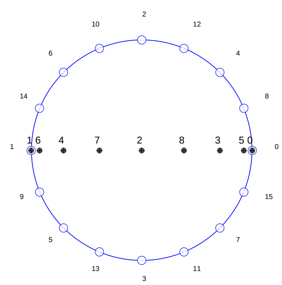
where M_d is a sequence of sub-exponential growth ($M_d^{1/d} \to 1$ as $d \to \infty$). We refer to [**4**] for results on pseudo Leja sequences. There is no known expression for Leja points for the interval. For that matter, such an expression is very unlikely to exist. Note however that by a result of Taylor and Totik [**42**], the Lebesgue constants of a Leja sequence for the interval grow at most sub-exponentially. A referee also pointed out that, in a recent work, A. Goncharov [**25**] arranged ordinary Chebyshev points for $[-1, 1]$ into a certain sequence and showed that the Lebesgue constants for this sequence grow sub-exponentially. They are actually shown to be $O(k^{\log k/\log 3} k^8)$ as the degree k tends to ∞. Goncharov also shows that the Lebesgue constants for this set of points cannot grow polynomially. These sequences could be used to construct further multivariate interpolation points with Lebesgue constants that grow sub-exponentially. This is not sufficient to derive approximation results for differentiable functions yet implies optimal approximation properties for analytic functions, see [**4**, Subsection 2.3].

2.3. Lebesgue constants of $\Re$-Leja sequences

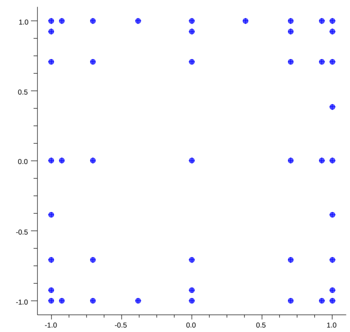
2.3.1. The upper bound and its consequences. Recall that given a set A of $n+1$ interpolation points in $[-1, 1]$ and $f \in C^s([-1, 1])$, the Lebesgue inequality together with the Jackson theorem [**39**, Theorem 1.5] yield the well known estimate

$$\max_{[-1,1]} \|f - \mathbf{L}[A; f]\| \leq M(1 + \Delta(A))\, \omega(f^{(s)}, 1/n)/n^s$$

where $\omega(f^{(s)}, \cdot)$ denotes the modulus of continuity of $f^{(s)}$ and M does not depend either on A or n. The following theorem implies in particular that interpolation polynomials at the points of any $\Re$-Leja sequence converge uniformly on $[-1, 1]$ to the interpolated function as soon as it belongs to $C^4([-1, 1])$. A weaker consequence is that the discrete measure $\mu_d := \frac{1}{d+1} \sum_{i=0}^{d} [x_i]$ associated to the $\Re$-Leja sequence $X = (x_s : s \in \mathbb{N})$ weakly converges



(I) First 9 points of a $\Re$-Leja sequence.



(II) 45 interpolation points obtained as the intertwining of the points in (I) with themselves.

FIGURE 1. Examples of points from a $\Re$-Leja sequence and their intertwining.

to the 'arcsin' distribution on $[-1,1]$ which is the equilibrium measure of the interval. Here $[x_i]$ denotes the Dirac measure at x_i.

THEOREM 2.6. *Let X be a $\Re$-Leja sequence. The Lebesgue constant $\Delta(X_k)$ for the interpolation points $x_0,\ldots,x_{k-1}$ satisfies the following estimate*

$$\Delta(X_k) = O(k^3 \log k), \quad k \to \infty.$$

The construction of good multivariate interpolation points is derived as follows. We start from N $\Re$-Leja sequences $X^{(j)} = (x_k^{(j)} : k \in \mathbb{N})$, $j = 1,\ldots,N$. These N sequences need not be distinct. We define $\mathbf{P}_k \subset [-1,1]^N$ as

$$\mathbf{P}_k = \Big\{ x_\alpha = (x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N}) : \sum_{j=1}^{N} \alpha_j \le k \Big\}.$$

It is known [**14**] that this is a valid set for interpolation by N-variable polynomials of degree at most k. Actually, $\mathbf{P}_k$ is the underlying set of the *intertwining* of the univariate tuples $X^{(j)}(0:k)$, $j = 1,\ldots,N$. We refer to Section 1.6 for details on this definition and to [**14**] for a general discussion of the intertwining process. Let us just emphasize that, in order to provide good points, the method requires to use sequences of interpolation points (we add one point when we go from degree k to $k+1$) rather than arrays (the classical case : all the points change when we change degree). The reason for this is explained in Section 1.6.

In Figure 1 (II), we show the points of a set $\mathbf{P}_8$ constructed with the first 9 points of the $\Re$-Leja sequence in (I).

THEOREM 2.7. *The Lebesgue constants $\Delta(\mathbf{P}_k)$ grow at most like a polynomial in k as $k \to \infty$.*

PROOF. The proof of Theorem 1.16 works as well in this case. □

Just as in the univariate case, the multivariate versions of the Lebesgue inequality and the Jackson theorem together with Theorem 2.7 imply that Lagrange interpolants of sufficiently smooth functions converge uniformly. Examining the terms in the proof of

Theorem 1.16, we find

$$\Delta(\mathbf{P}_k) = O\Big(k^{(N^2+11N-6)/2}\log^N k\Big), \quad k \to \infty,$$

which gives a more precise idea of the required level of smoothness. This bound however is certainly pessimistic.

In the rest of the paper, unless otherwise stated, the Lebesgues constants are computed with respect to the interval $[-1,1]$.

2.3.2. Outline of the proof of Theorem 2.6. We take advantage of the structure of the points of a $\Re$-Leja sequence. The first step is a simple algebraic observation. We use the notation recalled at the end of the introduction.

LEMMA 2.8. *Let $N = N_0 \cup \cdots \cup N_{s-1}$ where the N_i form a partition of the finite set $N \subset \mathbb{R}$. We have*

$$\ell(N,a;\cdot) = \frac{w_{N\setminus N_i}}{w_{N\setminus N_i}(a)}\ell(N_i,a;\cdot), \quad a \in N_i, \quad i = 0,\ldots,s-1. \tag{2.5}$$

Consequently,

$$\Delta(N) \leq \sum_{i=0}^{s-1} \max_{x\in K,\, a\in N_i} \left|\frac{w_{N\setminus N_i}(x)}{w_{N\setminus N_i}(a)}\right| \Delta(N_i), \tag{2.6}$$

where the Lebesgue constants are computed with respect to a compact set K containing N.

PROOF. We readily check that the polynomial on the right hand side of (2.5) satisfies the defining properties of $\ell(N,a;\cdot)$. The estimate for the Lebesgue constant $\Delta(N)$ follows from the definition together with the formula for the FLIPs. □

Given a $\Re$-Leja sequence $X = X(E)$, to estimate $\Delta(X_k)$, $2^n+1 < k \leq 2^{n+1}$, we shall first apply the lemma with a partition of X_k into two subsets, namely $A = X(0:2^n) = \mathscr{L}_{2^n}$ and $B = X(2^n+1 : k-1)$. Here and below, when there is no risk of misunderstanding, we confuse a tuple with its underlying set. In other words, when we write $T = Y$ with T a tuple and Y a set, we mean that Y is formed of the entries of T. Our choice for A and B, of course, is motivated by the fact that the Chebyshev-Lobatto points are excellent interpolation points for which there is a large amount of information available, see below. With this choice, Lemma 2.8 gives

$$\Delta(X_k) \leq \Delta(A) \max_{x\in[-1,1],\, a\in A}\left|\frac{w_B(x)}{w_B(a)}\right| + \Delta(B) \max_{x\in[-1,1],\, b\in B}\left|\frac{w_A(x)}{w_A(b)}\right|. \tag{2.7}$$

The factors depending on the first subset A in (2.7) will be easily estimated. The more difficult part will be to estimate w_B and $\Delta(B)$. To do that, we shall use a partition of B and still have recourse to Lemma 2.8 in its most general form. We point out however that our method is unlikely, it seems, to give the best estimates. Intuitively, we think that sharp estimates cannot be obtained by separating the interpolation points into two or more groups.

It is not difficult to see that the Lebesgue constants $\Delta(X_k)$ cannot grow slower than k. This is explained below in Subsection 2.3.4.

2.3.3. Interpolation at Chebyshev-Lobatto points. We collect a few results on Chebyshev-Lobatto points. First, since they are the extremal points of the Chebyshev polynomials, we have

$$w_{\mathscr{L}_d}(x) = (x^2-1)d^{-1}T_d'(x), \quad x \in \mathbb{R},$$

where T_d denotes the *monic* Chebyshev polynomial of degree d. From $2^{d-1}T_d(\cos\theta) = \cos d\theta$ we readily find

$$w_{\mathscr{L}_d}(\cos\theta) = -2^{1-d}\sin\theta\sin d\theta, \quad \theta \in \mathbb{R}. \tag{2.8}$$

A classical result of Ehlich and Zeller [**22, 13**] ensures that

$$\Delta(\mathscr{L}_d) = O(\log d), \quad d \to \infty. \tag{2.9}$$

LEMMA 2.9. *Let $X = X(E)$ be a $\Re$-Leja sequence. If $2^n+1 < k \le 2^{n+1}$, $A = X(0:2^n)$, $B = X(2^n+1:k-1)$ and $K = [-1,1]$ then*

$$\max_{x\in K, b\in B} \frac{|w_A(x)|}{|w_A(b)|} \le 1/\sin(\pi/2^{n+1}).$$

PROOF. If $x = \cos t$ and $b = \cos\beta$ then, since $A = \mathscr{L}_{2^n}$, in view of (2.8), we have

$$|w_A(x)/w_A(b)| = |\sin t\sin(2^n t)|/|\sin\beta\sin(2^n\beta)| \le 1/|\sin\beta\sin(2^n\beta)|.$$

Hence, it suffices to check that

$$|\sin\beta\sin(2^n\beta)| \ge \sin(\pi/2^{n+1}). \tag{2.10}$$

Since $b = x_j$ with $2^n+1 \le j \le k-1 < 2^{n+1}$, equation (2.4) gives

$$x_j = \Re(e_{2^n+j-1}) = \Re(e_{2^{n+1}+u-1}), \quad u = j-2^n,\ 1 \le u \le 2^n-1.$$

Theorem 2.1 says that $e_{2^{n+1}+u-1} = \rho z$ where ρ is a 2^{n+1}-st root of -1 and z is a 2^n-th root of 1. This means that the angle β that gives $b = x_j$ is of the form

$$\beta = (2\tau+1)\pi/2^{n+1} + 2\tau'\pi/2^n \quad \text{with } \tau,\tau' \in \mathbb{Z}.$$

It follows that $|\sin 2^n\beta| = 1$ and $|\sin\beta| \ge \sin(\pi/2^{n+1})$. This gives inequality (2.10) and concludes the proof of the lemma. □

2.3.4. A lower bound. We now show that when we remove one point from $\mathscr{L}_d$, the Lebesgue constants grow significantly faster. Write $a_i = \cos(i\pi/d)$. Suppose that $A_j := \mathscr{L}_d \setminus \{a_j\}$ with $1 \le j \le d-1$ (so that we remove a point different from 1 and -1). We compute the values of the FLIPs for A_j at the missing point a_j. From $w_{\mathscr{L}_d}(\cdot) = w_{A_j}(\cdot)(\cdot - a_j)$ we get

$$w_{A_j}(a_j) = w'_{\mathscr{L}_d}(a_j) \quad \text{and} \quad w'_{\mathscr{L}_d}(a_i) = w'_{A_j}(a_i)(a_i-a_j), \quad i \ne j.$$

Hence,

$$\ell(A_j,a_i;a_j) = \frac{w_{A_j}(a_j)}{(a_j-a_i)w'_{A_j}(a_i)} = -\frac{w'_{\mathscr{L}_d}(a_j)}{w'_{\mathscr{L}_d}(a_i)}.$$

Yet, as easily follows from (2.8),

$$w'_{\mathscr{L}_d}(a_i) = \pm 2^{1-d}d, \quad i = 1,\dots,d-1.$$

Hence $|\ell(A_j,a_i;a_j)| = 1$ for $d-2$ values of i, namely for $i = 1,\dots,d-1$, $i \ne j$. Consequently,

$$\Delta(A_j) \ge \sum_{i=1,i\ne j}^{d-1} |\ell(A_j,a_i;a_j)| = d-2. \tag{2.11}$$

Here is the consequence about our $\mathfrak{R}$-leja sequences. If X is a $\mathfrak{R}$-Leja sequence, then $X(0:2^n-1)$ is formed of all the Chebyshev-Lobatto points of degree 2^n with only one missing and this missing point is different from 1 and -1 (as soon as $n\geq 2$) which are the first two points of the sequence. Hence, according to (2.11), $\Delta(X(0:2^n-1))\geq 2^n-2$ which shows that the Lebesgue constants $\Delta(X_k)$ cannot grow slower than k.

2.3.5. Interpolation at modified Chebyshev points. We introduce other sets of interpolation points that will naturally come into play when dealing with $B=X(2^n+1:k-1)$. When $\cos\beta$ is not an extremal point of T_d, that is $\cos\beta\notin\mathscr{L}_d$, then the equation $T_d(x)=T_d(\cos\beta)$ has d roots in $[-1,1]$. The set of these roots — which we call *modified Chebyshev points* — will be denoted by $\mathscr{T}_d^{(\beta)}$. Since the change of variable $x=\cos t$ transforms the equation to $\cos dt=\cos d\beta$, we have

$$\mathscr{T}_d^{(\beta)}=\{\cos\beta_j:\ \beta_j:=\beta+2j\pi/d,\ j=0,\dots,d-1\}.$$

The following result is probably known but we are unable to provide references.

LEMMA 2.10. *We have $\Delta(\mathscr{T}_d^{(\beta)})=O(\log d/|\sin d\beta|)$ as $d\to\infty$ and the constant involved in O does not depend on β. Equivalently, there exists M such that,*

$$\Delta(\mathscr{T}_d^{(\beta)})\leq M\log(d+1)/|\sin d\beta|,\quad d\geq 1.$$

PROOF. First, since

$$w_{\mathscr{T}_d^{(\beta)}}(x)=T_d(x)-T_d(\cos\beta),$$

(see the introduction for the notation $w_{\mathscr{T}_d^{(\beta)}}$) we have

$$\ell(\mathscr{T}_d^{(\beta)},\cos\beta_j;x)=\frac{T_d(x)-T_d(\cos\beta)}{T_d'(\cos\beta_j)\,(x-\cos\beta_j)}=\frac{\sin\beta_j(\cos dt-\cos d\beta)}{d\sin d\beta(\cos t-\cos\beta_j)},\quad x=\cos t.$$

A use of the sum-to-product identity for cosines now yields

$$\ell(\mathscr{T}_d^{(\beta)},\cos\beta_j;x)=\frac{\sin\beta_j}{d\sin d\beta}\,\frac{\sin(d(t+\beta)/2)\sin(d(t-\beta)/2)}{\sin((t+\beta_j)/2)\sin((t-\beta_j)/2)}.\tag{2.12}$$

Yet, we also have

$$\sin\beta_j=\sin((t+\beta_j)/2)\cos((t-\beta_j)/2)-\sin((t-\beta_j)/2)\cos((t+\beta_j)/2).$$

Using this in (2.12), we obtain after simplification,

$$\ell(\mathscr{T}_d^{(\beta)},\cos\beta_j;x)=\frac{1}{d\sin d\beta}\left\{\frac{\cos((t-\beta_j)/2)\sin(d(t+\beta)/2)\sin(d(t-\beta)/2)}{\sin((t-\beta_j)/2)}\right.$$
$$\left.-\ \frac{\cos((t+\beta_j)/2)\sin(d(t+\beta)/2)\sin(d(t-\beta)/2)}{\sin((t+\beta_j)/2)}\right\}.\tag{2.13}$$

It follows that

$$|\ell(\mathscr{T}_d^{(\beta)},\cos\beta_j;x)|\leq\frac{1}{d|\sin d\beta|}\left\{\left|\frac{\sin(d(t-\beta)/2)}{\sin((t-\beta_j)/2)}\right|+\left|\frac{\sin(d(t+\beta)/2)}{\sin((t+\beta_j)/2)}\right|\right\},\quad x=\cos t.$$

Hence,

$$\Delta(\mathscr{T}_d^{(\beta)})\leq\frac{1}{|\sin d\beta|}\left\{\max_{t\in\mathbb{R}}F(t-\beta)+\max_{t\in\mathbb{R}}F(t+\beta)\right\}=\frac{2}{|\sin d\beta|}\max_{t\in\mathbb{R}}F(t),$$

where

$$F(t)=\frac{1}{d}\sum_{j=0}^{d-1}\left|\frac{\sin(dt/2)}{\sin((t-2j\pi/d)/2)}\right|.$$

But $\max_{t\in\mathbb{R}} F(t)$ is exactly the Lebesgue constant for the d-th roots of unity (on the unit circle) which is known to be $O(\log d)$, see [**27**]. □

2.4. Proof of Theorem 2.6

2.4.1. Further reduction. We use Lemma 2.9 and the classical estimate (2.9) of Ehlich and Zeller in (2.7) to obtain the following lemma.

LEMMA 2.11. *Let X be a $\mathfrak{R}$-Leja sequence and let $2^n+1<k\leq 2^{n+1}$. If $A=X(0:2^n)$ and $B=X(2^n+1:k-1)$ then*

$$\Delta(X_k)\leq M\log 2^n \max_{x\in[-1,1],a\in A}\frac{|w_B(x)|}{|w_B(a)|}+\frac{\Delta(B)}{\sin(\pi/2^{n+1})}, \tag{2.14}$$

where M does not depend on k.

The remaining required information is collected in the following two theorems.

THEOREM 2.12. *Let X be a $\mathfrak{R}$-Leja sequence and let $2^n+1<k\leq 2^{n+1}$. If $A=X(0:2^n)$ and $B=X(2^n+1:k-1)$ then*

$$\max_{x\in[-1,1],a\in A}\frac{|w_B(x)|}{|w_B(a)|}\leq 2^{2n+2}.$$

THEOREM 2.13. *Let X be a $\mathfrak{R}$-Leja sequence and let $2^n+1<k\leq 2^{n+1}$. If $B=X(2^n+1:k-1)$ then*

$$\Delta(B)\leq M' 2^{2n}\log 2^n,$$

where the constant M' does not depend on k.

END OF PROOF OF THEOREM 2.6. When $k-1$ is a power of 2, the points of X_k form a complete set of Chebyshev-Lobatto points and the bound is implied by Ehlich and Zeller's estimate (2.9). We assume $2^n+1<k\leq 2^{n+1}$. Using Theorems 2.12 and 2.13 in (2.14), we obtain

$$\Delta(X_k)\leq M2^{2n+2}\log 2^n + M'\frac{2^{2n}\log 2^n}{\sin(\pi/2^{n+1})}. \tag{2.15}$$

Since $n=\lfloor\log_2(k)\rfloor$ (or $\lfloor\log_2(k)\rfloor-1$ in the case $k=2^{n+1}$) and $1/\sin(\pi/2^{n+1})=O(2^n)$, this readily gives the existence of a constant M'' (independent of k) such that

$$\Delta(X_k)\leq M''k^3\log k,\quad \text{for } k \text{ large enough.}$$

Observe that the highest power (that is, k^3) comes from the second term in (2.15). □

2.4.2. A trigonometric inequality. The proofs of the two remaining steps rest on an elementary inequality that we present in this subsection. As in Section 1.4, the key observation is

$$|\sin\alpha|\geq|\sin 2^n\alpha|/2^n,\quad n\in\mathbb{N},\quad \alpha\in\mathbb{R}. \tag{2.16}$$

LEMMA 2.14. *Let $r\geq 1$ and let $n_0>n_1>\cdots>n_r\geq 0$ be a finite decreasing sequence of natural numbers. If $2^{n_j}\varphi_j=\pi\ [2\pi]$ (i.e. $2^{n_j}\varphi_j=\pi \bmod 2\pi$), $j=0,\dots,r-1$, then*

$$\prod_{j=0}^{r-1}|\sin 2^{n_{j+1}-1}(\varphi-\varphi_0-\cdots-\varphi_j)|\geq(1/2^{n_0-n_r})|\cos 2^{n_0-1}\varphi|,\quad \varphi\in\mathbb{R}. \tag{2.17}$$

PROOF. The proof is by induction. To treat the case $r = 1$, we prove that

$$|\sin 2^{n_1-1}(\varphi - \varphi_0)| \geq (1/2^{n_0-n_1})|\cos 2^{n_0-1}\varphi|.$$

Using (2.16) with $\alpha = 2^{n_1-1}(\varphi - \varphi_0)$ and $n = n_0 - n_1$ we obtain

$$|\sin 2^{n_1-1}(\varphi - \varphi_0)| \geq (1/2^{n_0-n_1})|\sin 2^{n_0-1}(\varphi - \varphi_0)|.$$

But, since $2^{n_0}\varphi_0 = \pi\ [2\pi]$, $|\sin 2^{n_0-1}(\varphi - \varphi_0)| = |\cos 2^{n_0-1}\varphi|$ and the claim follows.

We now assume that the inequality is true for $r = k$ and prove it for $r = k + 1$. The induction hypothesis applied to $\varphi - \varphi_0$ instead of φ yields

(2.18)
$$\prod_{j=1}^{k} |\sin 2^{n_{j+1}-1}\big((\varphi - \varphi_0) - \varphi_1 - \cdots - \varphi_j\big)| \geq (1/2^{n_1-n_{k+1}})|\cos 2^{n_1-1}(\varphi - \varphi_0)|, \quad \varphi \in \mathbb{R}.$$

Multiplying by the term corresponding to $j = 0$, we obtain

$$(2.19)\quad \prod_{j=0}^{k} |\sin 2^{n_{j+1}-1}\big((\varphi - \varphi_0) - \varphi_1 - \cdots - \varphi_j\big)| \geq \frac{1}{2^{n_1-n_{k+1}}}|\sin 2^{n_1-1}(\varphi - \varphi_0)\cos 2^{n_1-1}(\varphi - \varphi_0)| = \frac{1}{2^{n_1-n_{k+1}+1}}|\sin 2^{n_1}(\varphi - \varphi_0)|, \quad \varphi \in \mathbb{R}.$$

Another use of (2.16) with $n = n_0 - n_1 - 1$ shows that

$$\frac{1}{2^{n_1-n_{k+1}+1}}|\sin 2^{n_1}(\varphi - \varphi_0)| \geq \frac{1}{2^{n_0-n_{k+1}}}|\sin 2^{n_0-1}(\varphi - \varphi_0)|, \quad \varphi \in \mathbb{R}.$$

The sine on the right hand side is shown to be $|\cos 2^{n_0-1}\varphi|$ as in the case $r = 1$. □

2.4.3. Proof of Theorem 2.12. Let $X = X(E)$ and $B = X(2^n + 1 : k - 1)$ with $2^n + 1 < k \leq 2^{n+1}$. We write

$$(2.20)\quad k - 1 = 2^n + 2^{n_1} + \cdots + 2^{n_r} \quad \text{with } n - 1 \geq n_1 > \cdots > n_r \geq 0,$$

and, to simplify the notation,

$$(2.21)\quad n_0 = n + 1,$$

$$(2.22)\quad d_i = 2^{n_0} + \cdots + 2^{n_i}, \quad i = 0, \dots, r.$$

Then, in view of Corollary 2.5, we have

$$B = X(2^n + 1 : 2^n + (d_1 - d_0)) \wedge \bigwedge_{i=1}^{r-1} X\big(2^n + (d_i - d_0) + 1 : 2^n + (d_{i+1} - d_0)\big) = \bigwedge_{i=0}^{r-1} \Re\big(E(d_i : d_{i+1} - 1)\big).$$

Now using the structure properties of a Leja sequence, see Theorem 2.1 and (3.31), we see that the points of $X(2^n + 1, k - 1)$ are certain modified Chebyshev points (see Subsection 2.3.5). Indeed, for $i \in \{0, \dots, r - 1\}$,

$$(2.23)\quad E(d_i : d_{i+1} - 1) = \rho_i \cdots \rho_1 \rho_0 E_{2^{n_{i+1}}}^{(i+1)}, \quad \text{with } \rho_j^{2^{n_j}} = -1,$$

$$(2.24)\quad \Re(E(d_i : d_{i+1} - 1)) = \mathscr{T}_{2^{n_{i+1}}}^{(\beta_0 + \cdots + \beta_i)}, \quad \text{with } \beta_j = \arg \rho_j = (2t_j + 1)\pi/2^{n_j},\ t_j \in \mathbb{Z}.$$

This implies the following relation for the polynomial w_B,

$$w_B(x) = \prod_{i=0}^{r-1} \Big\{ T_{2^{n_{i+1}}}(x) - T_{2^{n_{i+1}}}(\cos(\beta_0 + \cdots + \beta_i)) \Big\}.$$

It follows that for $a = \cos\varphi \in A = X(0:2^n)$

$$\max_{x\in[-1,1]} \frac{|w_B(x)|}{|w_B(a)|} = \max_{t\in\mathbb{R}} \prod_{i=0}^{r-1} \frac{|\cos(2^{n_{i+1}}t) - \cos(2^{n_{i+1}}(\beta_0 + \cdots + \beta_i))|}{|\cos(2^{n_{i+1}}\varphi) - \cos(2^{n_{i+1}}(\beta_0 + \cdots + \beta_i))|} \tag{2.25}$$

$$\leq 2^r \prod_{i=0}^{r-1} 1/\big|\cos(2^{n_{i+1}}\varphi) - \cos(2^{n_{i+1}}(\beta_0 + \cdots + \beta_i))\big|. \tag{2.26}$$

Now, a use of the sum-to-product formula for cosines together with two applications of Lemma 3.38 (first with $\varphi_i = \beta_i$, then with $\varphi_i = -\beta_i$) enable us to bound the denominator in (2.26) and arrive at

$$\max_{x\in[-1,1]} \frac{|w_B(x)|}{|w_B(a)|} \leq \frac{2^{2(n_0-n_r)}}{\cos^2(2^{n_0-1}\varphi)}.$$

It remains to recall that $n_0 = n+1$ so that $2^{2(n_0-n_r)} \leq 2^{2n+2}$ and observe that, since $A = \mathscr{L}_{2^n}$, $2^{n_0-1}\varphi = 2^n\varphi = 0\,[\pi]$ so that $\cos^2(2^{n_0-1}\varphi) = 1$. This concludes the proof of Theorem 2.12.

2.4.4. Proof of Theorem 2.13. We still use the fact that, for $X = X(E)$ and $2^n + 1 < k \leq 2^{n+1}$, we have $B = \wedge_{i=0}^{r-1} B_i$, where the underlying set of B_i is $\mathscr{T}_{2^{n_{i+1}}}^{(\beta_0+\cdots+\beta_i)}$ with $\beta_j = (2t_j+1)\pi/2^{n_j}$, $t_j \in \mathbb{Z}$.

Using first Lemma 2.8 (with $N_i = B_i$) and then Lemma 2.10 to bound $\Delta(B_j)$, we obtain

$$\Delta(B) \leq M \sum_{j=0}^{r-1} \max_{x\in[-1,1],\, a\in B_j} \frac{|w_{B\setminus B_j}(x)|}{|w_{B\setminus B_j}(a)|} \, \frac{\log(2^{n_{j+1}}+1)}{\big|\sin(2^{n_{j+1}}(\beta_0+\cdots+\beta_j))\big|}. \tag{2.27}$$

Now, just as in (2.26) (we just remove one factor), for $a = \cos\theta_j \in B_j$, we have

$$\max_{x\in[-1,1]} \frac{|w_{B\setminus B_j}(x)|}{|w_{B\setminus B_j}(a)|} = \max_{t\in\mathbb{R}} \prod_{i=0, i\neq j}^{r-1} \frac{|\cos(2^{n_{i+1}}t) - \cos(2^{n_{i+1}}(\beta_0 + \cdots + \beta_i))|}{|\cos(2^{n_{i+1}}\theta_j) - \cos(2^{n_{i+1}}(\beta_0 + \cdots + \beta_i))|} \tag{2.28}$$

$$\leq 2^{r-1} \prod_{i=0, i\neq j}^{r-1} 1/|\cos(2^{n_{i+1}}\theta_j) - \cos(2^{n_{i+1}}(\beta_0 + \cdots + \beta_i))|. \tag{2.29}$$

Again the sum-to-product formula for cosines transforms the right-hand side in a product of sines and it follows that the j-term in the right-hand side of (2.27) is bounded by the maximum when $a = \cos\theta_j$ runs over B_j of

$$\log(2^{n_{j+1}}+1) \prod_{i=0, i\neq j}^{r-1} \left|\sin^{-1}\left(2^{n_{i+1}-1}(\theta_j - \beta_0 - \cdots - \beta_i)\right)\right| \times \prod_{i=0}^{r-1} \left|\sin^{-1}\left(2^{n_{i+1}-1}(\theta_j + \beta_0 + \cdots + \beta_i)\right)\right|. \tag{2.30}$$

Here we used the fact that

$$|\sin\left(2^{n_{j+1}-1}(\theta_j + \beta_0 + \cdots + \beta_j)\right)| = |\sin(2^{n_{j+1}}(\beta_0 + \cdots + \beta_j))| \tag{2.31}$$

which enabled us to insert the isolated sine in (2.27) into the second product of (2.30). To prove (2.31), we observe that, since $a = \cos\theta_j \in B_j$, we have

$$2^{n_{j+1}}\theta_j = 2^{n_{j+1}}(\beta_0 + \cdots + \beta_j)\ [2\pi]. \tag{2.32}$$

We now estimate independently both products in (2.30). The same bound is valid for every $a \in B_j$ and therefore provides an upper bound for the maximum over B_j as required.

I) We start with the first product. In view of (2.32), since $n_{i+1} > n_{j+1}$ whenever $i < j$, we have

$$2^{n_{i+1}}(\theta_j - \beta_0 - \cdots - \beta_i) = 2^{n_{i+1}}(\beta_{i+1} + \cdots + \beta_j)\ [2\pi], \quad 0 \le i < j.$$

On the other hand, since $2^{n_s}\beta_s = \pi\ [2\pi]$, we also have

$$2^{n_{i+1}}(\beta_{i+1} + \cdots + \beta_j) = \pi\ [2\pi], \quad 0 \le i < j.$$

Thus the absolute value of the first j sines equals 1 and we just need to estimate

$$\prod_{i=j+1}^{r-1} |\sin 2^{n_{i+1}-1}(\theta_j - \beta_0 - \cdots - \beta_i)|. \tag{2.33}$$

To do that, we apply Lemma 2.14 with $\varphi = \theta_j - \beta_0 - \cdots - \beta_j$. We obtain the lower bound $(1/2^{n_{j+1}-n_r})|\cos 2^{n_{j+1}-1}(\theta_j - \beta_0 - \cdots - \beta_j)|$. Yet in view of (2.32) this cosine equals ± 1 and we obtain

$$\prod_{i=0, i\neq j}^{r-1} \left|\sin^{-1}\left(2^{n_{i+1}-1}(\theta_j - \beta_0 - \cdots - \beta_i)\right)\right| \le 2^{n_{j+1}-n_r}. \tag{2.34}$$

Note that, in the case $j = r-1$, the whole product equals 1 which obviously implies the inequality. The inequality is likewise satisfied in the case $r = 1$ (for which the product is empty).

II) We now turn to the second product in (2.30). We obtain an upper bound on using again Lemma 2.14 as in (2.19). Indeed, we have

$$\prod_{i=0}^{r-1} \left|\sin\left(2^{n_{i+1}-1}(\theta_j + \beta_0 + \cdots + \beta_i)\right)\right| \\ = |\sin(2^{n_1-1}(\theta_j + \beta_0))| \prod_{i=1}^{r-1} \left|\sin\left(2^{n_{i+1}-1}(\theta_j + \beta_0 + \cdots + \beta_i)\right)\right|.$$

Applying Lemma 2.14 with $\varphi = \theta_j + \beta_0$ to the second factor on the left hand side, we obtain

$$\prod_{i=1}^{r-1} \left|\sin\left(2^{n_{i+1}-1}(\theta_j + \beta_0 + \cdots + \beta_i)\right)\right| \ge \frac{1}{2^{n_1-n_r}} |\cos(2^{n_1-1}(\theta_j + \beta_0))|.$$

It follows that

$$\prod_{i=0}^{r-1} \left|\sin\left(2^{n_{i+1}-1}(\theta_j + \beta_0 + \cdots + \beta_i)\right)\right| \\ \ge \frac{1}{2^{n_1-n_r+1}} \left|2\sin(2^{n_1-1}(\theta_j + \beta_0))\cos(2^{n_1-1}(\theta_j + \beta_0)\right| \\ = \frac{1}{2^{n_1-n_r+1}} |\sin(2^{n_1}(\theta_j + \beta_0)|.$$

Thus we get the following upper bound for the second product in (2.30),

$$2^{n_1-n_r+1}/|\sin 2^{n_1}(\theta_j + \beta_0)|.$$

However, since for every s, $\beta_s = (2t_s+1)\pi/2^{n_s}$ and $\theta_j = \beta_0 + \cdots + \beta_j + 2q_j\pi/2^{n_{j+1}}$ with $t_s, q_j \in \mathbb{Z}$, we have

$$2^{n_1}(\theta_j + \beta_0) = 2^{n_1}(2\beta_0 + \beta_1 + \cdots + \beta_j + 2q_j\pi/2^{n_{j+1}})$$
$$= 2^{n_1+1}\beta_0 + p\pi = \frac{(2t_0+1)\pi}{2^{n_0-n_1-1}} + p\pi \quad \text{with } p \in \mathbb{Z}.$$

Note that since $n_0 = n+1$ and $n > n_1$ we have $n_0 - n_1 - 1 > 0$. This shows that $|\sin 2^{n_1}(\theta_j + \beta_0)| \geq \sin(\pi/2^{n_0-n_1-1}) \geq 2/2^{n_0-n_1-1} = 1/2^{n_0-n_1-2}$. We have therefore proved

$$\prod_{i=0}^{r-1} \left|\sin^{-1}\left(2^{n_{i+1}-1}(\theta_j + \beta_0 + \cdots + \beta_i)\right)\right| \leq 2^{n_1-n_r+1} \cdot 2^{n_0-n_1-2} = 2^{n_0-n_r-1} \leq 2^n. \tag{2.35}$$

III) It remains to insert (2.34) and (2.35) in (2.27) with the aid of (2.30). Indeed, we obtain

$$\Delta(B) \leq M \sum_{j=0}^{r-1} 2^{n_{j+1}-n_r} \cdot 2^n \cdot \log(2^{n_{j+1}} + 1) \tag{2.36}$$

$$\leq M2^n \sum_{j=0}^{r-1} 2^{n_{j+1}} \log(2^{n_{j+1}} + 1) \leq M2^n \log(2^n + 1) \sum_{j=0}^{r-1} 2^{n_{j+1}} = O(2^{2n} \log 2^n). \tag{2.37}$$

This achieves the proof of Theorem 2.13.

CHAPTER 3

On the convergence of Kergin and Hakopian interpolants at Leja sequences for the disk

Abstract. We prove that Kergin interpolation polynomials and Hakopian interpolation polynomials at the points of a Leja sequence for the unit disk D of a sufficiently smooth function f in a neighbourhood of D converge uniformly to f on D. Moreover, when $f \in C^\infty(D)$, all the derivatives of the interpolation polynomials converge uniformly to the corresponding derivatives of f.

3.1. Introduction

Kergin and Hakopian interpolants were introduced independently about thirty years ago as natural multivariate generalizations of univariate Lagrange interpolation. The construction of these interpolation polynomials requires the use of points, usually called nodes, with which one obtains a number of natural mean value linear forms which provide the interpolation conditions. Kergin interpolation polynomials also interpolate in the usual sense, that is, the interpolation polynomial and the interpolated function coincide on the set of nodes but this condition no longer characterizes it. The general definition is recalled below. Approximation properties of Kergin and Hakopian interpolation polynomials have been deeply investigated, see e.g., [**3, 5, 8, 9, 23**]. Elegant results were in particular obtained in the two-dimensional case when the nodes forms a complete set of roots of unity (viewed as a subset of $\mathbb{R}^2$). Thus, in [**31**], Liang established a formula for Hakopian interpolation at the roots of unity in $\mathbb{R}^2$, and later together with Lü [**33**] estimated the remainder and proved that Hakopian interpolation polynomials at the roots of unity of a function of class C^2 in a neighbourhood of the closed unit disk $D \subset \mathbb{R}^2$ converge uniformly to the function on D. Thanks to Liang's formula, further authors investigated (weighted) mean convergence of Hakopian interpolation (see [**33, 32**]). On other hand, in 1997, using a beautiful formula for Kergin interpolation at nodes in general position in $\mathbb{R}^2$, Bos and Calvi [**11**] independently established a similar convergence result for Kergin interpolation. If C_n denotes the set of n-th roots of unity, $\mathscr{H}[C_n;\cdot]$ (resp. $\mathscr{K}[C_n;\cdot]$) the Hakopian (resp. Kergin) projector, the results can be stated as

$$\mathscr{K}[C_n;f] \to f \text{ and } \mathscr{H}[C_n;f] \to f, \quad \text{uniformly on } D, \text{ for every } f \in C^2(D).$$

In the above results, going from n to $n+1$, we need to change all the nodes and it seems natural to look for similar results in which C_n would be replaced by a set E_n such that $E_n \subset E_{n+1}$, which comes to find *sequences* of nodes rather than sequences of arrays of nodes. It is the purpose of this note to exhibit such sequences. They enable us to obtain beautiful series expansions of the form

$$f(x) = \sum_{d=0}^{\infty} \mu\left(e_0,\ldots,e_d, \mathrm{D}^d f(\cdot)(x-e_0,\ldots,x-e_{d-1})\right)$$

for $f \in C^\infty(D)$, where the $\mu(e_0, \dots, e_d, \cdot)$ are certain mean value linear forms (whose definition will be specified below) and $\mathrm{D}^d f(a)$ denotes the d-th total derivative of f. The sequences that we shall use are Leja sequences for D and the results of the present paper have been made possible by recent progresses on the study of Leja sequences (and associated constants) contained in [**4, 18, 16**]. To treat the case of Hakopian interpolation, we shall prove a formula for Hakopian interpolation at nodes in general position in $\mathbb{R}^2$ which reduces to Liang's formula when the nodes form a complete set of roots of unity and which is of independent interest. The proof of our convergence results requires a somewhat higher level of smoothness than in the case of interpolation at the roots of unity. The question whether we can weaken the smoothness of the interpolated function is still unanswered. Moreover, when the interpolated function is in the class C^∞, we show that all the derivatives of the interpolation polynomials converge uniformly to the corresponding derivatives of the interpolated function.

Notations. The scalar product of $x = (x^1, \dots, x^N)$ and $y = (y^1, \dots, y^N)$ in $\mathbb{R}^N$ is defined by $\langle x, y\rangle := \sum_{j=1}^N x^j y^j$, and the corresponding norm of x is $\|x\| = \sqrt{\langle x, x\rangle}$. Let K be a compact set in $\mathbb{R}^N$. For each continuous function f on K we set $\|f\|_K = \sup\{|f(x)| : x \in K\}$. The space of k-times continuously differentiable functions on a neighbourhood of K is denoted by $C^k(K)$. For $f \in C^k(K)$, $k \geq 1$, we set

$$\mathrm{D}^\alpha f = \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} \cdots (\partial x^N)^{\alpha_N}}, \quad \alpha = (\alpha_1, \dots, \alpha_N),\ |\alpha| = \alpha_1 + \cdots + \alpha_N \leq k,$$

$$\mathrm{D}_y f(x) = \mathrm{D}f(x)(y) = \sum_{j=1}^N \frac{\partial f}{\partial x^j}(x) y^j, \quad x \in K,\ y = (y^1, \dots, y^N).$$

The euclidean norm of the linear form $\mathrm{D}f(x)$ is denoted by $\|\mathrm{D}f(x)\|$. We have

$$\|\mathrm{D}f(x)\| = \Big[\sum_{j=1}^N \big(\frac{\partial f}{\partial x^j}(x)\big)^2\Big]^{\frac{1}{2}}.$$

We also denote by $\mathscr{P}_d(\mathbb{R}^N)$ the space of polynomials of N variables and degree at most d.

3.2. The definition of Kergin and Hakopian interpolants

It is convenient to recall some definitions and properties of interpolation polynomials in their full generality. In particular, we shall introduce Kergin and Hakopian interpolation polynomials as particular cases of a more general procedure.

Given a convex subset $\Omega \subset \mathbb{R}^N$ and a tuple A of $d+1$ not necessarily distinct points in Ω, $A = (a_0, a_1, \dots, a_d) \in \Omega^{d+1}$, the simplex functional $\int_{[a_0, \dots, a_d]}$ is defined on the space of continuous functions $C(\Omega)$ by the relation

$$\int_{[a_0,\dots,a_d]} f := \int_{\Delta_d} f\Big(a_0 + \sum_{j=1}^d t_j (a_j - a_0)\Big) \mathrm{d}t, \quad f \in C(\Omega),\ d \geq 1, \tag{3.1}$$

where $\mathrm{d}t = \mathrm{d}t_1 \cdots \mathrm{d}t_d$ stands for the ordinary Lebesgue measure on the standard simplex $\Delta_d = \{(t_1, t_2, \dots, t_d) \in [0, 1]^d,\ \sum_{j=1}^d t_j \leq 1\}$. In the case $d = 0$ we set $\int_{[a_0]} f = f(a_0)$.

The following theorem leads us to the definition of mean-value interpolation. Its proof can be found in [**26**] or [**23**].

THEOREM 3.1. *Let Ω be an open convex subset of $\mathbb{R}^N$, $A = (a_0,\dots,a_d)$ be a tuple in Ω and let $k \in \{0,\dots,d\}$. For every function $f \in C^{d-k}(\Omega)$, the set of all $(d-k)$-times continuously differentiable functions on Ω, there exists a unique polynomial P on $\mathbb{R}^N$ of degree at most $d-k$ such that*

$$\int_{[a_0,\dots,a_{j+k}]} \mathrm{D}^\alpha (f-P) = 0, \quad |\alpha| = j, \quad j = 0,\dots,d-k. \tag{3.2}$$

DEFINITION 3.2. The polynomial P in (3.2) is called the *k-th mean-value interpolation polynomial* of f at A and is denoted by $\mathscr{L}^{(k)}[A;f]$ or $\mathscr{L}^{(k)}[a_0,\dots,a_d;f]$.

There is an explicit but rather complicated formula for mean-value interpolation, see [**26**, Theorem 1] or [**23**, Theorem 4.3]. Here we summarize a few basic properties of mean-value interpolation.

(1) The polynomial $\mathscr{L}^{(k)}[A;f]$ does not depend on the ordering of the points in A,
(2) The operator $\mathscr{L}^{(k)}[A] : f \in C^{d-k}(\Omega) \longmapsto \mathscr{L}^{(k)}[A;f] \in \mathscr{P}_{d-k}(\mathbb{R}^N)$ is a continuous linear projector (when $C^{d-k}(\Omega)$ is equipped with its standard topology),
(3) For each $f \in C^{d-k}(\Omega)$, the map $A \in \Omega^{d+1} \longmapsto \mathscr{L}^{(k)}[A;f] \in \mathscr{P}_{d-k}(\mathbb{R}^N)$ is continuous,
(4) For any affine mapping $\Psi : \mathbb{R}^N \to \mathbb{R}^M$ and any suitably defined function f, we have $\mathscr{L}^{(k)}[A; f\circ\Psi] = \mathscr{L}^{(k)}[\Psi(A); f]\circ\Psi$
(5) The polynomial $\mathscr{L}^{(0)}[A;f]$ interpolates f at the a_j's and becomes Taylor polynomial of f at a of order d when $a_0 = \cdots = a_d = a$.

The most interesting mean-value interpolation polynomials are Kergin interpolants which correspond to the case $k = 0$,

$$\mathscr{K}[a_0,\dots,a_d;f] = \mathscr{L}^{(0)}[a_0,\dots,a_d;f], \tag{3.3}$$

and Hakopian interpolants which correspond to the case $k = N-1$ and $d \ge N-1$,

$$\mathscr{H}[a_0,\dots,a_d;f] = \mathscr{L}^{(N-1)}[a_0,\dots,a_d;f]. \tag{3.4}$$

When the a_j's are in general position in $\mathbb{R}^N$- that is, every subset of $N+1$ points of A defines an affine basis of $\mathbb{R}^N$- then Kergin operator extends to functions of class C^{N-1}, see [**11**, p. 206-207]. On the other hand, under the same condition on the points, Hakopian interpolation is characterized by the following relation. For $P \in \mathscr{P}_{d-N+1}(\mathbb{R}^N)$,

$$P = \mathscr{H}[a_0,\dots,a_d;f] \iff \int_{[a_{i_1},\dots,a_{i_N}]} (f-P) = 0, \quad 0 \le i_1 < \cdots < i_N \le d. \tag{3.5}$$

Hence, in that case, derivatives are no longer involved and the Hakopian operator extends to continuous functions.

3.3. Error formulas for Hakopian and Kergin interpolants in $\mathbb{R}^2$

We now restrict ourselves to the two-dimensional case.

3.3.1. For $x = (x^1, x^2) \in \mathbb{R}^2$, we denote by $x^\perp := (-x^2, x^1)$, the image of x under the rotation of center the origin and angle $\pi/2$. As usual, to $x = (x^1, x^2)$, we associate the complex number $x^1 + ix^2$ with $i = \sqrt{-1}$ which we still denote by x. With this notation we have $x^{\perp} = ix$. Assume that the points a_i are in general position (so that no three of them are aligned). We consider an one-variable polynomial of degree $d-1$ defined by

$$h_{st}(w) = \prod_{m=0, m\neq s}^{d-1} \big(w - \langle (a_s - a_t)^{\perp}, a_m \rangle\big),\ w \in \mathbb{R},\ s \neq t,\ 0 \leq s,t \leq d-1. \tag{3.6}$$

The polynomial h_{st} appears in the formulas for Kergin and Hakopian interpolation polynomials and plays an important role in our arguments. It is worth pointing out that h_{st} is a multiple of the polynomial q_{ts} used in relation (2.18) in [**11**] where basic properties of q_{ts} are established. To make our exposition self-contained we state and prove a few properties of h_{st}.

LEMMA 3.3. *Let $d \geq 3$ and let $A = (a_0, a_1, \ldots, a_{d-1})$ be a d-tuple of points in general position in $\mathbb{R}^2$. Then*

1. $h_{st}\big(\langle (a_s - a_t)^{\perp}, a_v \rangle\big) = 0, \quad 0 \leq v \leq d-1$;
2. $h'_{st}\big(\langle (a_s - a_t)^{\perp}, a_s \rangle\big) = \prod_{m=0, m\neq s,t}^{d-1} \langle (a_s - a_t)^{\perp}, (a_s - a_m) \rangle$;
3. *If u, v and s are pairwise distinct and $\langle (a_s - a_t)^{\perp}, (a_u - a_v) \rangle = 0$, then* $h'_{st}\big(\langle (a_s - a_t)^{\perp}, a_u \rangle\big) = 0$.

PROOF. Observe that $h_{st}\big(\langle (a_s - a_t)^{\perp}, a_v \rangle\big) = \prod_{m=0, m\neq s}^{d-1} \langle (a_s - a_t)^{\perp}, (a_v - a_m) \rangle$. The product in the right hand side has the vanishing factor $\langle (a_s - a_t)^{\perp}, a_v - a_v \rangle$ when $v \neq s$ and the vanishing factor $\langle (a_s - a_t)^{\perp}, a_s - a_t \rangle$ when $v = s$. Thus $h_{st}(\langle (a_s - a_t)^{\perp}, a_v \rangle) = 0$. For the proof of assertion (2), we just compute the derivative of h_{st}, that is

$$h'_{st}(w) = \sum_{m=0, m\neq s}^{d-1} \prod_{j=0, j\neq m,s}^{d-1} \big(w - \langle (a_s - a_t)^{\perp}, a_j \rangle\big). \tag{3.7}$$

It is easy to see that the vanishing factor $\langle (a_s - a_t)^{\perp}, (a_s - a_t) \rangle$ is contained in the product $\prod_{j=0, j\neq m,s}^{d-1} \langle (a_s - a_t)^{\perp}, (a_s - a_j) \rangle$ whenever $m \neq t$. Hence, in view of (3.7), we have

$$h'_{st}\big(\langle (a_s - a_t)^{\perp}, a_s \rangle\big) = \prod_{j=0, j\neq t,s}^{d-1} \langle (a_s - a_t)^{\perp}, (a_s - a_j) \rangle. \tag{3.8}$$

For the last assertion it is enough to verify that $\prod_{j=0, j\neq m,s}^{d-1} \langle (a_s - a_t)^{\perp}, (a_u - a_j) \rangle = 0$ for every $m \neq s$. To prove this we only notice that the product in the left hand side contains the vanishing factor $\langle (a_s - a_t)^{\perp}, (a_u - a_u) \rangle$ if $m = v$ and the vanishing factor $\langle (a_s - a_t)^{\perp}, (a_u - a_v) \rangle$ if $m \neq v$. □

Theorem 3.5 below gives a formula for Hakopian interpolation polynomial in $\mathbb{R}^2$. It is similar to that of Kergin interpolation polynomial found by Bos and Calvi. Here, we denote by $cv(A)$ the convex hull of the set A.

THEOREM 3.4 (Bos and Calvi). *Let $A = (a_0, a_1, \ldots, a_{d-1})$ be a tuple of d points in general position in the plane. Then the (extended) Kergin operator $\mathscr{K}[A]$ is continuously defined on $C^1(cv(A))$ by the formula*

$$\mathscr{K}[A; f] = \sum_{j=0}^{d-1} f(a_j) P_j + \sum_{0 \leq s < t \leq d-1} P_{st} \int_{[a_s, a_t]} D_{(a_t - a_s)^{\perp}} f,$$

where P_j is the real part of the $(j+1)$-st fundamental Lagrange polynomial corresponding to the complex nodes $a_0,\dots,a_{d-1}$, that is

$$P_j(x^1,x^2)=\Re\Big(\prod_{m=0,m\neq j}^{d-1}\frac{(x^1+ix^2)-a_m}{a_j-a_m}\Big), \tag{3.9}$$

and

$$P_{st}(x)=\frac{h_{st}\big(\langle (a_s-a_t)^{\perp},x\rangle\big)}{\|a_s-a_t\|^2 h'_{st}\big(\langle (a_s-a_t)^{\perp},a_s\rangle\big)}. \tag{3.10}$$

THEOREM 3.5. *Let $A=(a_0,a_1,\dots,a_{d-1})$ be a tuple of d points in general position in the plane. Then the (extended) Hakopian operator $\mathscr{H}[A]$ is continuously defined on $C(cv(A))$ by the formula*

$$\mathscr{H}[A;f]=\sum_{0\le s<t\le d-1} Q_{st}\int_{[a_s,a_t]} f, \tag{3.11}$$

where

$$Q_{st}(x)=\frac{h'_{st}\big(\langle (a_s-a_t)^{\perp},x\rangle\big)}{h'_{st}\big(\langle (a_s-a_t)^{\perp},a_s\rangle\big)}. \tag{3.12}$$

PROOF. Of course, when $d=2$ then $Q_{01}=1$ and (3.11) is trivial. Now, suppose that $d\ge 3$. Since, for all $0\le s<t\le d-1$, Q_{st} is a polynomial of degree at most $d-2$, the polynomial defined in the right hand side of (3.11) belongs to $\mathscr{P}_{d-2}(\mathbb{R}^2)$. Let us call H this polynomial. Thanks to (3.5), we have to show that

$$\int_{[a_u,a_v]} H=\int_{[a_u,a_v]} f,\quad \text{for all } 0\le u<v\le d-1. \tag{3.13}$$

In view of (3.11), it suffices to verify that

$$\int_{[a_u,a_v]} Q_{st}=\delta_{us}\delta_{vt},\quad \text{for all } 0\le s<t\le d-1,\ 0\le u<v\le d-1, \tag{3.14}$$

where δ is the Kronecker symbol. Looking at (3.12), we have

$$\begin{aligned}\int_{[a_u,a_v]} Q_{st} &= \int_0^1 Q_{st}(a_u+w(a_v-a_u))\mathrm{d}w \\ &= \frac{1}{h'_{st}\big(\langle (a_s-a_t)^{\perp},a_s\rangle\big)}\int_0^1 h'_{st}\big(\langle (a_s-a_t)^{\perp},a_u\rangle + w\langle (a_s-a_t)^{\perp},(a_v-a_u)\rangle\big)\mathrm{d}w.\end{aligned} \tag{3.15}$$

To deal with the last integral we examine three cases.
First, we assume that $(s,t)=(u,v)$. Then, since $\langle (a_s-a_t)^{\perp},(a_t-a_s)\rangle=0$, relation (3.15) gives

$$\int_{[a_s,a_t]} Q_{st}=1.$$

The second case occurs when $(s,t)\neq(u,v)$ and $\langle (a_s-a_t)^{\perp},(a_v-a_u)\rangle=0$. Then $s\neq u$ and $s\neq v$. Indeed, if, for exemple, $s=u$, then the relation $\langle (a_s-a_t)^{\perp},(a_v-a_s)\rangle=0$ implies that a_s, a_t and a_v are collinear, contrary to the hypothesis. Now, the integral term in (3.15) reduces to

$$\int_{[a_u,a_v]} Q_{st} = \frac{h'_{st}\big(\langle (a_s-a_t)^\perp, a_u\rangle\big)}{h'_{st}\big(\langle (a_s-a_t)^\perp, a_s\rangle\big)} = 0,$$

where we use Lemma 3.3(3) in the second equality. The last case is when $(s,t) \neq (u,v)$ and $\langle (a_s-a_t)^\perp, (a_v-a_u)\rangle \neq 0$ then, calculating the intergral (3.15), we have

$$\int_{[a_u,a_v]} Q_{st} = \frac{h_{st}\big(\langle (a_s-a_t)^\perp, a_v\rangle\big) - h_{st}\big(\langle (a_s-a_t)^\perp, a_u\rangle\big)}{h'_{st}\big(\langle (a_s-a_t)^\perp, a_s\rangle\big)\langle (a_s-a_t)^\perp, (a_v-a_u)\rangle}. \tag{3.16}$$

Now, Lemma 3.3(1) follows that the right hand side of (3.16) vanishes and the proof is complete. □

3.3.2. We use the above formulas to establish multivariate analogues of the classical Lebesgue inequality for Lagrange interpolation. For the proof, we refer to [**11**, Theorem 1.1] and [**33**, Theorem 5].

LEMMA 3.6. *Let $A = (a_0, a_1, \dots, a_{d-1})$ be a tuple of d points in general position in the plane, and let $K \subset \mathbb{R}^2$ be a convex compact set containing A. Then for every $f \in C^1(K)$ and every $Q_{d-1} \in \mathscr{P}_{d-1}(\mathbb{R}^2)$ we have*

(3.17)

$$\|f - \mathscr{K}[A;f]\|_K \leq \big(1 + \sum_{j=0}^{d-1} \|P_j\|_K\big)\|f - Q_{d-1}\|_K + \operatorname{diam}(K) \sum_{0\leq s<t\leq d-1} \|P_{st}\|_K \|\mathrm{D}f - \mathrm{D}Q_{d-1}\|_K,$$

where $\operatorname{diam}(K)$ *is the diameter of K, the polynomials P_j and P_{st} are defined in (3.9) and (3.10) respectively.*

LEMMA 3.7. *Let $A = (a_0, a_1, \dots, a_{d-1})$ be a tuple of d points in general position in the plane, and let $K \subset \mathbb{R}^2$ be a convex compact set containing A. Then for every $f \in C(K)$ and every $Q_{d-2} \in \mathscr{P}_{d-2}(\mathbb{R}^2)$ we have*

$$\|f - \mathscr{H}[A;f]\|_K \leq \big(1 + \sum_{0\leq s<t\leq d-1} \|Q_{st}\|_K\big)\|f - Q_{d-2}\|_K, \tag{3.18}$$

where the polynomials Q_{st} are defined in (3.12).

3.4. Kergin and Hakopian interpolants at Leja sequences for the disk

DEFINITION 3.8. Let D be the closed unit disk in the complex plane and $E = (e_n : n \in \mathbb{N})$ be a sequence of points in D. One says that E is a Leja sequence for D if the following property hold true,

$$\Big|\prod_{j=0}^{d-1}(e_d - e_j)\Big| = \max_{z\in D}\Big|\prod_{j=0}^{d-1}(z - e_j)\Big|, \quad \text{for all } d \geq 1.$$

A d-tuple $E_d = (e_0, e_1, \dots, e_{d-1})$ is called a d-Leja section. In this paper we only consider Leja sequences whose first entry is equal to 1. It is not diffcult to describe the structure of Leja sequences for D. The following theorem is proved in [**4**].

THEOREM 3.9 (Białas-Cież and Calvi). *The structure of a Leja sequence $E = (e_n : n \in \mathbb{N})$ for the unit disk D with $e_0 = 1$ is given by the following rules.*

(1) *The underlying set of the 2^n-Leja section E_{2^n} consists of the 2^n-th roots of unity*

(2) *The* 2^{n+1}*-Leja section is* $(E_{2^n}, \rho E^{(1)}_{2^n})$*, where* ρ *is a* 2^n*-th roots of -1 and* $E^{(1)}_{2^n}$ *is the* 2^n*-Leja section of a Leja sequence* $E^{(1)} = (e^{(1)}_n : n \in \mathbb{N})$ *for the unit disk with* $e^{(1)}_0 = 1$.

Next, we use Lebesgue-type inequalities for Kergin and Hakopian interpolants along with the method of Bos and Calvi to prove the following convergence results.

THEOREM 3.10. *Let* $\mathscr{K}[E_d; f]$ *denote the Kergin interpolation polynomial of* f *with respect to the Leja section* $E_d = (e_0, \ldots, e_{d-1})$ *of a Leja sequence* $E = (e_n : n \in \mathbb{N})$ *for* D.

(1) *If* $f \in C^4(D)$*, then* $\mathscr{K}[E_d; f]$ *converges uniformly to* f *on* D *as* $d \to \infty$*;*
(2) *If* $f \in C^\infty(D)$*, then* $\mathrm{D}^\beta\left(\mathscr{K}[E_d; f]\right)$ *converges uniformly to* $\mathrm{D}^\beta f$ *on* D *as* $d \to \infty$*, for every two-dimensional index* β.

COROLLARY 3.11. *For every* $f \in C^\infty(D)$*, the series*

$$\sum_{d=0}^{\infty} \int_{[e_0,\ldots,e_d]} \mathrm{D}^d f(\cdot, x-e_0, \ldots, x-e_{d-1})$$

converges to f *uniformly on* D*. Moreover, the convergence extends to all derivatives.*

PROOF. In view of Newton's formula for Kergin interpolation (see [**36**, Theorem 2]), the $(d+1)$-st partial sum of the series is exactly $\mathscr{K}[E_d; f]$. □

THEOREM 3.12. *Let* $\mathscr{H}[E_d; f]$ *denote the Hakopian interpolation polynomial of a function* f *with respect to the Leja section* $E_d = (e_0, \ldots, e_{d-1})$ *of a Leja sequence* $E = (e_n : n \in \mathbb{N})$ *for* D.

(1) *If* $f \in C^5(D)$*, then* $\mathscr{H}[E_d; f]$ *converges uniformly to* f *on* D *as* $d \to \infty$*;*
(2) *If* $f \in C^\infty(D)$*, then* $\mathrm{D}^\beta\left(\mathscr{H}[E_d; f]\right)$ *converges uniformly to* $\mathrm{D}^\beta f$ *on* D *as* $d \to \infty$*, for every two-dimensional index* β.

COROLLARY 3.13. *For every* $f \in C^\infty(D)$*, the series*

$$\sum_{d=1}^{\infty} \sum_{0 \le j_1 < j_2 < \cdots < j_{d-1} \le d-1} \int_{[e_0,\ldots,e_d]} \mathrm{D}^d f(\cdot, x-e_{j_1}, \ldots, x-e_{j_{d-1}}),$$

converges to f *uniformly on* D*. Moreover, the convergence extends to all derivatives.*

PROOF. Looking at the formula for Hakopian interpolation (see [**28, 26**]), the d-th partial sum of the series is exactly $\mathscr{H}[E_d; f]$. □

REMARK 3.14. We denote by F_p the set of functions from $\{0, \ldots, d-1\}$ to $\{1, 2\}$. For $\tau \in F_d$, we set $\alpha(\tau) = (a, b)$ with a (resp. b) the number of times that τ takes on the value 1 (resp. the value 2) and we write $(x-e)^\tau := \prod_{i=0}^{d-1}(x-e_i)_{\tau(i)}$, where $(x-e_i)_{\tau(i)}$ is $x^1 - \Re e_i$ (resp. $x^2 - \Im e_i$) if $\tau(i) = 1$ (resp. $\tau(i) = 2$). In particular, $(x-e)^\tau$ is a polynomial of degree d. We have

$$\mathrm{D}^d f(\cdot, x-e_0, \ldots, x-e_{d-1}) = \sum_{\tau \in F_p} \mathrm{D}^{\alpha(\tau)} f(\cdot)(x-e)^\tau. \tag{3.19}$$

The series expansion in Corollary 3.11 can be rewriten as

$$f(x) = \sum_{d=0}^{\infty} \sum_{\tau \in F_d} \int_{[e_0,\ldots,e_d]} \mathrm{D}^{\alpha(\tau)} f(\cdot)(x-e)^\tau. \tag{3.20}$$

Thus the polynomials $(x-e)^{\tau}$ can be regarded as a generalization of the classical Newton polynomials and the above expansion as a multivariate Newton series expansion. A similar observation could be done with Corollary 3.13.

In the rest of this note we always interpolate at Leja sections E_d. In order to use the Lebesgue-type inequalities given in (3.17) and (3.18), we need a kind of Jackson theorem that we now recall.

For $f \in C^k(D)$, we set

$$\|f\|_k := \sum_{|\beta| \le k} \|\mathrm{D}^{\beta} f\|_D,$$

$$\omega(f;\delta) := \sup\{|f(x)-f(y)| : \|x-y\| \le \delta,\ x,y \in D\},\ \ \omega(f^{(k)};\delta) = \sum_{|\beta|=k} \omega(\mathrm{D}^{\beta} f;\delta),\ \delta > 0.$$

The following theorem, proved in [**37**, pp. 164], is due to Ragozin.

THEOREM 3.15 (Ragozin). *Given f in $C^k(D)$ with $k \ge 0$, there exists polynomials Q_d with $\deg Q_d \le d$ such that*

$$\|f - Q_d\|_D \le M(k) d^{-k} \big[d^{-1} \|f\|_k + \omega(f^{(k)}; 1/d) \big],$$

where $M(k)$ is a positive constant which depends only on k.

In [**11**], Bos and Calvi slightly modified a method of Ragozin and used Theorem 3.15 to prove a simultaneously approximate theorem for C^2 functions on D (see [**11**, Lemma 4.1]). Examining their proof, we see that the proof easily extends to C^k functions. We state the generalized result without proof.

LEMMA 3.16. *Given f in $C^k(D)$ with $k \ge 1$, there exists a sequence of polynomials Q_d with $\deg Q_d \le d$ such that*

$$\lim_{d\to\infty} d^{k-1} \|f - Q_d\|_D = 0 \quad \textit{and} \quad \lim_{d\to\infty} d^{k-1} \|\mathrm{D}f - \mathrm{D}Q_d\|_D = 0.$$

Next, we investigate the growth of Lebesgue-type constants

$$\sum_{j=0}^{d-1} \|P_j\|_D, \qquad \sum_{0 \le s < t \le d-1} \|P_{st}\|_D \quad \text{and} \quad \sum_{0 \le s < t \le d-1} \|Q_{st}\|_D.$$

Looking at the formula for P_j in Theorem 3.4, we see that $\sum_{j=0}^{d-1} \|P_j\|_D$ is dominated by the Lebesgue constant $\Delta(E_d)$ for Lagrange interpolation corresponding d complex nodes e_j, $0 \le j \le d-1$, over D. But Corollary 1.7 tells us that $\Delta(E_d) = O(d \log d)$ as $d \to \infty$. It gives us the estimate

$$\sum_{j=0}^{d-1} \|P_j\|_D = O(d \log d) \text{ as } d \to \infty. \tag{3.21}$$

The estimates for the remaining Lebesgue-type constants are simple consequences of the following two theorems that are proved in the last section. Here, in the formulas for h_{st} and h'_{st}, we take $a_j = e_j$ so that

$$h_{st}(w) = \prod_{m=0, m\neq s}^{d-1} \big(w - \langle (e_s - e_t)^{\perp}, e_m \rangle \big), \quad w \in \mathbb{R}, \tag{3.22}$$

$$h'_{st}\big(\langle (e_s - e_t)^{\perp}, e_s \rangle \big) = \prod_{m=0, m\neq s,t}^{d-1} \langle (e_s - e_t)^{\perp}, (e_s - e_m) \rangle. \tag{3.23}$$

THEOREM 3.17. *We have* $\|P_{st}\|_D \le 2d$ *for all* $0 \le s < t \le d-1$, *where*

$$P_{st}(x) = \frac{h_{st}\big(\langle (e_s - e_t)^\perp, x\rangle\big)}{|e_s - e_t|^2 h'_{st}\big(\langle (e_s - e_t)^\perp, e_s\rangle\big)}, \qquad x \in D.$$

THEOREM 3.18. *We have* $\|Q_{st}\|_D \le 4d^3$ *for all* $0 \le s < t \le d-1$, *where*

$$Q_{st}(x) = \frac{h'_{st}\big(\langle (e_s - e_t)^\perp, x\rangle\big)}{h'_{st}\big(\langle (e_s - e_t)^\perp, e_s\rangle\big)}, \qquad x \in D.$$

Next, we note that D satisfies a Markov inequality, that is

$$\max\left\{\left\|\frac{\partial p}{\partial x^1}\right\|_D, \left\|\frac{\partial p}{\partial x^2}\right\|_D\right\} \le (\deg p)^2 \|p\|_D, \quad p \in \mathscr{P}(\mathbb{R}^2), \tag{3.24}$$

see [**40**]. Repeated application of (3.24) yields estimates for each partial derivatives

$$\|\mathrm{D}^\beta p\|_D \le (\deg p)^{2|\beta|} \|p\|_D, \quad p \in \mathscr{P}(\mathbb{R}^2). \tag{3.25}$$

PROOF OF THEOREM 3.10. Using Lemma 3.16 for $f \in C^k(D)$ we can find a sequence of polynomials $Q_{d-1} \in \mathscr{P}_{d-1}(\mathbb{R}^2)$, $d \in \mathbb{N}^\star$, and a sequence of positive numbers $(\varepsilon_n : n \in \mathbb{N})$ that converges to 0 such that

$$\max\{\|f - Q_{d-1}\|_D, \|\mathrm{D}f - \mathrm{D}Q_{d-1}\|_D\} \le \frac{\varepsilon_d}{d^{k-1}}, \quad d \ge 1. \tag{3.26}$$

But Theorem 3.17 gives $\sum_{0 \le s < t \le d-1} \|P_{st}\|_D \le d^2(d-1)$ and (3.21) gives $1 + \sum_{j=0}^{d-1} \|P_j\|_D = O(d \log d)$. It follows from (3.17) that

$$\|f - \mathscr{K}[E_d; f]\|_D \le \Big(O(d \log d) + 2d^2(d-1)\Big)\frac{\varepsilon_d}{d^{k-1}} \le \frac{M\varepsilon_d}{d^{k-4}}, \tag{3.27}$$

where M is a constant. The right hand side of (3.27) tends to 0 as $d \to \infty$ when $f \in C^4(D)$, i.e., $k = 4$. This follows the first assertion. To prove the second one with the hypothesis that $f \in C^\infty(D)$, we first observe that

$$\|\mathscr{K}[E_{n+1}; f] - \mathscr{K}[E_n; f]\|_D \le \|f - \mathscr{K}[E_{n+1}; f]\|_D + \|f - \mathscr{K}[E_n; f]\|_D \le \frac{M(\varepsilon_{n+1} + \varepsilon_n)}{n^{k-4}}. \tag{3.28}$$

Applying Markov's inequality in (3.25) for $p = \mathscr{K}[E_{n+1}; f] - \mathscr{K}[E_n; f] \in \mathscr{P}_{n+1}(\mathbb{R}^2)$ we obtain

$$\left\|\mathrm{D}^\beta\Big(\mathscr{K}[E_{n+1}; f] - \mathscr{K}[E_n; f]\Big)\right\|_D \le \frac{M(\varepsilon_{n+1} + \varepsilon_n)(n+1)^{2|\beta|}}{n^{k-4}}, \quad \beta \in \mathbb{N}^2. \tag{3.29}$$

Now, we choose $k = k(\beta) = 2|\beta| + 6$. Then the series

$$\sum_{n=1}^{\infty} \frac{M(\varepsilon_{n+1} + \varepsilon_n)(n+1)^{2|\beta|}}{n^{k-4}}$$

converges. This follows the uniform convergence on D of the series

$$\mathrm{D}^\beta\Big(\mathscr{K}[E_1; f]\Big) + \sum_{n=1}^{\infty} \mathrm{D}^\beta\Big(\mathscr{K}[E_{n+1}; f] - \mathscr{K}[E_n; f]\Big).$$

Hence $\mathrm{D}^\beta\Big(\mathscr{K}[E_d; f]\Big)$ converges uniformly on D as $d \to \infty$, for every $\beta \in \mathbb{N}^2$. A classical reasoning show that if $\mathscr{K}[E_d; f]$ converges uniformly to f on D and $\mathrm{D}^\beta\Big(\mathscr{K}[E_d; f]\Big)$

converges uniformly on D for every $\beta \in \mathbb{N}^2$, then

$$\mathrm{D}^\beta\Big(\mathscr{K}[E_d;f]\Big) \to \mathrm{D}^\beta f, \quad \text{uniformly on } D, \quad \text{for every } \beta \in \mathbb{N}^2.$$

□

PROOF OF THEOREM 3.12. Using Theorem 3.15 for $f \in C^k(D)$ we can find a sequence of polynomials $Q_{d-2} \in \mathscr{P}_{d-2}(\mathbb{R}^2)$, $d \geq 2$, and a sequences of positive numbers $(\delta_n : n \in \mathbb{N})$ that converges to 0 such that

$$\|f - Q_{d-2}\|_D \leq \frac{\delta_d}{d^k}, \quad d \geq 2.$$

From Theorem 3.18 we have $\sum_{0\leq s<t\leq d-1}\|Q_{st}\|_D \leq 2d^4(d-1) < 2d^5 - 1$. Hence, in view of Lemma 3.7, we get

$$\|f - \mathscr{H}[E_d;f]\|_D \leq \big(1 + \sum_{0\leq s<t\leq d-1}\|Q_{st}\|_D\big)\|f - Q_{d-2}\|_D \leq \frac{2\delta_d}{d^{k-5}}.$$

This estimate is the same as (3.27). The conclusions of the theorem now follow by repeating the arguments in the proof of Theorem 3.10. □

3.5. Further properties of Leja sequences for the disk

3.5.1. Decomposition of Leja sections. Let $E = (e_n : n \in \mathbb{N})$ be a Leja sequence for D. As observed in Section 2.2, repeated applications of the rule in Theorem 3.9 show that if $d = 2^{n_0} + 2^{n_1} + \cdots + 2^{n_r}$ with $n_0 > n_1 > \cdots > n_r \geq 0$, then

$$E_d = (E_{2^{n_0}}\,,\, \rho_0 E^{(1)}_{d-2^{n_0}}) = (E_{2^{n_0}}\,,\, \rho_0 E^{(1)}_{2^{n_1}}\,,\, \rho_1\rho_0 E^{(2)}_{d-2^{n_0}-2^{n_1}}) \tag{3.30}$$

$$= \cdots = (E_{2^{n_0}}\,,\, \rho_0 E^{(1)}_{2^{n_1}}\,,\, \rho_1\rho_0 E^{(2)}_{2^{n_2}}, \ldots, \rho_{r-1}\ldots\rho_1\rho_0 E^{(r)}_{2^{n_r}}), \tag{3.31}$$

where each $E^{(j)}_{2^{n_j}}$ consists of a complete set of the 2^{n_j}-roots of unity, arranged in a certain order, and ρ_j satisfies $\rho_j^{2^{n_j}} = -1$ for all $0 \leq j \leq r-1$.

For a sequence of complex numbers $Z = (z_k : k \in \mathbb{N})$ we define $Z(j:k) := (z_j, z_{j+1}, \ldots, z_k)$, of course, $Z(0:k-1) = Z_k$. For $d = 2^{n_0} + 2^{n_1} + \cdots + 2^{n_r}$ with $n_0 > n_1 > \cdots > n_r \geq 0$, let us set

$$d_{-1} = 0, \quad d_j = 2^{n_0} + \cdots + 2^{n_j}, \quad \text{for } 0 \leq j \leq r. \tag{3.32}$$

With this notation, in view of (3.31), we have

$$E(d_{-1} : d_0 - 1) = E_{2^{n_0}} \quad \text{and} \quad E(d_j : d_{j+1} - 1) = \rho_j \cdots \rho_0 E^{(j+1)}_{2^{n_{j+1}}},\ 0 \leq j \leq r-1. \tag{3.33}$$

From now on, we always denote $\theta_n := \arg e_n$ for $n \geq 0$ and $\varphi_j := \arg \rho_j$ for $0 \leq j \leq r-1$. Since $\rho_j^{2^{n_j}} = -1$ we may put $\varphi_j = \frac{(2q_j+1)\pi}{2^{n_j}}$, $q_j \in \mathbb{N}$. The following lemma is similar to Lemma 1.3.

LEMMA 3.19. *Let $E = (e_n : n \in \mathbb{N})$ be a Leja sequence for D and $d = 2^{n_0} + 2^{n_1} + \cdots + 2^{n_r}$ with $n_0 > n_1 > \cdots > n_r \geq 0$. Then*

(1) $\prod_{m=d_{-1}}^{d_0-1}|z - e_m| = |z^{2^{n_0}} - 1|$;
(2) $\prod_{m=d_j}^{d_{j+1}-1}|z - e_m| = |(z\rho_0^{-1}\cdots\rho_j^{-1})^{2^{n_{j+1}}} - 1|, \quad 0 \leq j \leq r-1$;
(3) $\prod_{m=d_j, m\neq k}^{d_{j+1}-1}|e_k - e_m| = 2^{n_{j+1}}, \quad d_j \leq k \leq d_{j+1} - 1, \quad -1 \leq j \leq r-1.$

PROOF. Since $E^{(j+1)}_{2^{n_{j+1}}}$ forms a complete set of the $2^{n_{j+1}}$-st roots of unity, the relation in (3.33) gives

$$\prod_{m=d_j}^{d_{j+1}-1}|z-e_m| = \prod_{e\in E^{(j+1)}_{2^{n_{j+1}}}}|z-\rho_j\cdots\rho_0 e| = \prod_{e\in E^{(j+1)}_{2^{n_{j+1}}}}|(\rho_j\cdots\rho_0)^{-1}z-e| = |(z\rho_0^{-1}\cdots\rho_j^{-1})^{2^{n_{j+1}}}-1|.$$

This proves the first two assertions. For the third one, we observe that $e_k=\rho_j\cdots\rho_0 e'$ with $e'\in E^{(j+1)}_{2^{n_{j+1}}}$. It follows that

$$\prod_{m=d_j, m\neq k}^{d_{j+1}-1}|e_k-e_m| = \prod_{e\in E^{(j+1)}_{2^{n_{j+1}}}, e\neq e'}|e'-e| = 2^{n_{j+1}},$$

since the middle term is the modulus of the derivative of $z^{2^{n_{j+1}}}-1$ at e'. This completes the proof. $\square$

3.5.2. Some trigonometric inequalities. Let T_n be the *monic* Chebyshev polynomial of degree n, that is $2^{n-1}T_n(\cos\varphi)=\cos(n\varphi)$. If $\cos(n\beta)\neq\pm1$, then the equation $T_n(x)=T_n(\cos\beta)$ has n distinct roots: $\cos(\beta+2m\pi/n)$, $m=0,\ldots,n-1$. Hence

$$T_n(\cos\varphi)-T_n(\cos\beta)=\prod_{m=0}^{n-1}[\cos\varphi-\cos(\beta+2m\pi/n)]. \tag{3.34}$$

Since both sides of (3.34) are continuous functions of β, relation (3.34) holds true for all $\beta\in\mathbb{R}$. Now, the relation in (3.33) implies that, for $-1\le j\le r-1$,

$$\{\theta_m : d_j\le m\le d_{j+1}-1\}=\{\varphi_0+\cdots+\varphi_j+2\pi k/2^{n_{j+1}}[2\pi] : 0\le k\le 2^{n_{j+1}}-1\}, \tag{3.35}$$

where we write $\alpha=\beta[2\pi]$ if $\alpha=\beta \bmod 2\pi$ and the φ_j's do not appear when $j=-1$. Using (3.34) for $n=2^{n_{j+1}}$ and $\beta=\varphi_0+\cdots+\varphi_j-\psi$ we obtain from (3.35) the following result.

LEMMA 3.20. *Let $E=(e_n : n\in\mathbb{N})$ be a Leja sequence for D, $\theta_n=\arg e_n$, and $d=2^{n_0}+2^{n_1}+\cdots+2^{n_r}$ with $n_0>n_1>\cdots>n_r\ge 0$. Then for all $\psi\in\mathbb{R}$ and $-1\le j\le r-1$ we have*

$$\prod_{m=d_j}^{d_{j+1}-1}[\cos\varphi-\cos(\theta_m-\psi)] = \frac{1}{2^{2^{n_{j+1}}-1}}\Big(\cos 2^{n_{j+1}}\varphi-\cos 2^{n_{j+1}}(\varphi_0+\cdots+\varphi_j-\psi)\Big),$$

where the d_i's are defined in (3.32).

LEMMA 3.21. *If $n\ge1$, $0\le j\le n-1$ and $\sin(\beta+2j\pi/n)\neq0$, then*

$$\prod_{m=0, m\neq j}^{n-1}|\cos\varphi-\cos(\beta+2m\pi/n)|\le\frac{2n}{2^{n-1}|\sin(\beta+2j\pi/n)|},\quad \varphi\in\mathbb{R}.$$

PROOF. In view of (3.34) we have

$$\prod_{m=0, m\neq j}^{n-1}|\cos\varphi-\cos(\beta+2m\pi/n)| = \Big|\frac{\cos(n\varphi)-\cos(n\beta)}{2^{n-1}[\cos\varphi-\cos(\beta+2j\pi/n)]}\Big|. \tag{3.36}$$

Set $\psi=\beta+2j\pi/n$, $\psi_1=(1/2)(\varphi+\psi), \psi_2=(1/2)(\varphi-\psi)$. Then the sum-to-product formula for cosines transforms the right hand side of (3.36) into

$$2^{-n+1}|\sin n\psi_1\sin n\psi_2|/|\sin\psi_1\sin\psi_2|.$$

Since $\sin\psi = \sin\psi_1\cos\psi_2 - \cos\psi_1\sin\psi_2$, we obtain after simplication

$$\begin{aligned}\prod_{m=0,m\neq j}^{n-1}|\cos\varphi - \cos(\beta+2m\pi/n)| &= \frac{|\sin\psi_1\cos\psi_2-\cos\psi_1\sin\psi_2|\cdot|\sin n\psi_1\sin n\psi_2|}{2^{n-1}|\sin\psi||\sin\psi_1\sin\psi_2|}\\ &\leq \frac{1}{2^{n-1}|\sin\psi|}\Big(\frac{|\sin n\psi_2|}{|\sin\psi_2|}+\frac{|\sin n\psi_1|}{|\sin\psi_1|}\Big)\\ &\leq \frac{2n}{2^{n-1}|\sin(\beta+2j\pi/n)|},\end{aligned}$$

where we use the classical inequality $|\sin n\alpha|\leq n|\sin\alpha|$ for $\alpha\in\mathbb{R}$, $n\in\mathbb{N}$ in the third line. □

Now, for $d_j\leq s\leq d_{j+1}-1$, Lemma 3.21 and equation (3.35) imply that

$$\prod_{m=d_j,m\neq s}^{d_{j+1}-1}[\cos\varphi-\cos(\theta_m+\beta)]\leq\frac{2\cdot 2^{n_{j+1}}}{2^{2^{n_{j+1}}-1}|\sin(\theta_s+\beta)|},\quad \varphi\in\mathbb{R}. \tag{3.37}$$

In (3.37), taking $\beta=-\frac{\theta_s+\theta_t}{2}$ with $s<t$, we get the following result.

LEMMA 3.22. *Under the same assumptions of Lemma 3.20, if $d_j\leq s\leq d_{j+1}-1$ with $-1\leq j\leq r-1$ and $s<t\leq d-1$, then*

$$\prod_{m=d_j,m\neq s}^{d_{j+1}-1}[\cos\varphi-\cos(\theta_m-\frac{\theta_s+\theta_t}{2})]\leq\frac{2\cdot 2^{n_{j+1}}}{2^{2^{n_{j+1}}-1}|\sin\frac{\theta_s-\theta_t}{2}|},\quad \varphi\in\mathbb{R}.$$

3.5.3. Further results. The following lemma will be used to get lower estimates for the denominators of P_{st} and Q_{st}.

LEMMA 3.23. *Let $E=(e_n:n\in\mathbb{N})$ be a Leja sequence for D and $d\geq 2$, $d=2^{n_0}+2^{n_1}+\cdots+2^{n_r}$ with $n_0>n_1>\cdots>n_r\geq 0$. Then $\prod_{m=0,m\neq s}^{d-1}|e_s-e_m|\geq 2^r$ for all $0\leq s\leq d-1$.*

The proof of Lemma 3.23 requires a purely trigonometric inequality given in the following lemma. For the proof we refer the reader to Lemma 2.14.

LEMMA 3.24. *Let $r\geq 1$ and let $n_0>n_1>\cdots>n_r\geq 0$ be a decreasing sequence of natural numbers. If $\varphi_j=(2q_j+1)\pi/2^{n_j}$ with $q_j\in\mathbb{Z}$, $j=0,\ldots,r-1$, then*

$$\prod_{j=0}^{r-1}|\sin 2^{n_{j+1}-1}(\varphi-\varphi_0-\cdots-\varphi_j)|\geq(1/2^{n_0-n_r})|\cos 2^{n_0-1}\varphi|,\quad \varphi\in\mathbb{R}. \tag{3.38}$$

PROOF OF LEMMA 3.23. The case $r=0$ is trivial, since $\prod_{m=0,m\neq s}^{2^{n_0}-1}|e_s-e_m|=2^{n_0}$. Thus we may assume that $r\geq 1$. Notice that, since $\arg e_s=\theta_s$ and $\arg\rho_j=\varphi_j=(2q_j+1)\pi/2^{n_j}$, $0\leq j\leq r-1$,

$$|(e_s\rho_0^{-1}\cdots\rho_k^{-1})^{2^{n_{k+1}}}-1|=2|\sin 2^{n_{k+1}-1}(\theta_s-\varphi_0-\cdots-\varphi_k)|,\qquad 0\leq k\leq r-1. \tag{3.39}$$

First, suppose that $r\geq 2$ and $s\geq 2^{n_0}$. Then there exists a unique $0\leq j\leq r-1$ such that $d_j\leq s\leq d_{j+1}-1$, where the d_j's are defined in (3.32). We write

$$\prod_{m=0,m\neq s}^{d-1}|e_s-e_m|=\prod_{m=d_{-1}}^{d_0-1}|e_s-e_m|\cdot\prod_{m=d_j,m\neq s}^{d_{j+1}-1}|e_s-e_m|\cdot\prod_{k=0,k\neq j}^{r-1}\prod_{m=d_k}^{d_{k+1}-1}|e_s-e_m|. \tag{3.40}$$

We will treat three factors in (3.40) independently. The first and the third part of Lemma 3.19 give $\prod_{m=d_{-1}}^{d_0-1} |e_s - e_m| = |e_s^{2^{n_0}} - 1| = 2$, and $\prod_{m=d_j, m\neq s}^{d_{j+1}-1} |e_s - e_m| = 2^{n_{j+1}}$. On the other hand, the second part of Lemma 3.19 along with equation (3.39) yields

$$\prod_{k=0,k\neq j}^{r-1} \prod_{m=d_k}^{d_{k+1}-1} |e_s - e_m| = \prod_{k=0,k\neq j}^{r-1} |(e_s \rho_0^{-1} \cdots \rho_k^{-1})^{2^{n_{k+1}}} - 1| \tag{3.41}$$

$$= 2^{r-1} \prod_{k=0,k\neq j}^{r-1} |\sin 2^{n_{k+1}-1}(\theta_s - \varphi_0 - \cdots - \varphi_k)|. \tag{3.42}$$

Since $d_j \leq s \leq d_{j+1} - 1$, relation (3.35) tells us that $\theta_s = \varphi_0 + \cdots + \varphi_j + 2q\pi/2^{n_{j+1}}[2\pi]$ for some $q \in \mathbb{Z}$. To estimate the product in (3.42) we proceed as in Subsection 2.4.4. For the convenience of reader, we reproduce the proof . For $0 \leq k < j$, we have $\theta_s - \varphi_0 - \cdots - \varphi_k = \varphi_{k+1} + \cdots + \varphi_j + 2q\pi/2^{n_{j+1}}[2\pi]$. Hence the hypotheses on the values of $\varphi_0, \ldots, \varphi_{r-1}$ give

$$2^{n_{k+1}-1}(\theta_s - \varphi_0 - \cdots - \varphi_k) = \big((2q_{k+1} + 1)\pi/2\big)[\pi]. \tag{3.43}$$

It follows that

$$\prod_{k=0}^{j-1} |\sin 2^{n_{k+1}-1}(\theta_s - \varphi_0 - \cdots - \varphi_k)| = 1. \tag{3.44}$$

On the other hand, using Lemma 3.24 for $\varphi = \theta_s - \varphi_0 - \cdots - \varphi_j$, i.e., $\varphi = 2q\pi/2^{n_{j+1}}[2\pi]$, we obtain

$$\prod_{k=j+1}^{r-1} |\sin 2^{n_{k+1}-1}(\theta_s - \varphi_0 - \cdots - \varphi_k)| \geq (1/2^{n_{j+1}-n_r})|\cos 2^{n_{j+1}-1}\varphi| = 1/2^{n_{j+1}-n_r}. \tag{3.45}$$

Combining (3.44) and (3.45) we get

$$\prod_{k=0,k\neq j}^{r-1} |\sin 2^{n_{k+1}-1}(\theta_s - \varphi_0 - \cdots - \varphi_k)| \geq 1/2^{n_{j+1}-n_r} \geq 1/2^{n_{j+1}}. \tag{3.46}$$

Note that when $j = r-1$, then the left hand side of (3.46) is equal to 1 and inequality (3.46) is obviously true. When $j = 0$, then the factor in (3.44) does not appear. Now, using relation (3.46) in (3.42) we get the following estimate

$$\prod_{k=0,k\neq j}^{r-1} \prod_{m=d_k}^{d_{k+1}-1} |e_s - e_m| \geq 2^{r-1} 2^{-n_{j+1}}.$$

In this case, we finally obtain

$$\prod_{m=0,m\neq s}^{d-1} |e_s - e_m| = 2 \cdot 2^{n_{j+1}} \cdot 2^{r-1} \cdot 2^{-n_{j+1}} = 2^r.$$

We now treat the case $r \geq 1$ and $0 \leq s \leq 2^{n_0} - 1$. The proof is the same as above. Indeed, thanks to Lemma 3.19 and (3.39), we can write

$$\prod_{m=0,m\neq s}^{d-1} |e_s - e_m| = \prod_{m=d_{-1},m\neq s}^{d_0-1} |e_s - e_m| \prod_{k=0}^{r-1} \prod_{m=d_k}^{d_{k+1}-1} |e_s - e_m| \tag{3.47}$$

$$= 2^{n_0} 2^r \prod_{k=0}^{r-1} |\sin 2^{n_{k+1}-1}(\theta_s - \varphi_0 - \cdots - \varphi_k)|. \tag{3.48}$$

Now using Lemma 3.24 in (3.48) we get

$$\prod_{m=0,m\neq s}^{d-1}|e_s-e_m|\geq 2^{n_0}2^r2^{-n_0+n_r}|\cos 2^{n_0-1}\theta_s|=2^{r+n_r}\geq 2^r,$$

since $|\cos 2^{n_0-1}\theta_s|=1$ for $0\leq s\leq 2^{n_0}-1$. The last case is when $r=1$ and $2^{n_0}\leq s\leq 2^{n_0}+2^{n_1}-1$. The proof is simple and we omit it. □

REMARK 3.25. Take $d=2^{n_0}+1$ and $s=2^{n_0}$ then $\prod_{m=0}^{2^{n_0}-1}|e_{2^{n_0}}-e_m|=|e_{2^{n_0}}^{2^{n_0}}-1|=|-1-1|=2$. Thus the conclusion of Lemma 3.23 is optimal in some cases.

3.6. Proof of Theorems 3.17 and 3.18

We continue to use the notation introduced in the previous two sections. Let $E=(e_n:n\in\mathbb{N})$ be a Leja sequence for D and $\theta_n=\arg e_n$, $n\geq 0$. We use the formula for the polynomials h_{st} and h'_{st} given in (3.22) and (3.23). We start with the following simple observations

$$e_u-e_v=2i\sin\frac{\theta_u-\theta_v}{2}e^{\frac{\theta_u+\theta_v}{2}i}\quad\text{and}\quad (e_u-e_v)^{\perp}=-2\sin\frac{\theta_u-\theta_v}{2}e^{\frac{\theta_u+\theta_v}{2}i}\quad\text{for}\quad u\neq v.$$

Set $\alpha_{uv}:=e^{\frac{\theta_u+\theta_v}{2}i}$ and $\beta_{uv}:=-2\sin\frac{\theta_u-\theta_v}{2}$. Then we immediately see that

$$(e_u-e_v)^{\perp}=\alpha_{uv}\beta_{uv}\quad\text{and}\quad |e_u-e_v|=|(e_u-e_v)^{\perp}|=|\beta_{uv}|\quad\text{for}\quad u\neq v. \tag{3.49}$$

LEMMA 3.26. *Let $E=(e_n:n\in\mathbb{N})$ be a Leja sequence for D and $d\geq 2$, $d=2^{n_0}+2^{n_1}+\cdots+2^{n_r}$ with $n_0>n_1>\cdots>n_r\geq 0$. Then*

$$|h'_{st}(\langle(e_s-e_t)^{\perp},e_s\rangle)|\geq\frac{2^{2r}|\beta_{st}|^{d-4}}{2^{d-2}},\quad 0\leq s<t\leq d-1. \tag{3.50}$$

PROOF. Thanks to (3.49) we may write

$$\begin{aligned}|h'_{st}(\langle(e_s-e_t)^{\perp},e_s\rangle)| &= \prod_{m=0,m\neq s,t}^{d-1}|\langle(e_s-e_t)^{\perp},(e_s-e_m)\rangle| \quad (\text{ see (3.23) })\\ &= |\beta_{st}|^{d-2}\prod_{m=0,m\neq s,t}^{d-1}\left|\left\langle\alpha_{st},\frac{e_s-e_m}{-\beta_{sm}}\right\rangle\right|\prod_{m=0,m\neq s,t}^{d-1}|e_s-e_m|\\ &= |\beta_{st}|^{d-3}\prod_{m=0,m\neq s,t}^{d-1}\left|\left\langle\alpha_{st},\frac{e_s-e_m}{-\beta_{sm}}\right\rangle\right|\prod_{m=0,m\neq s}^{d-1}|e_s-e_m|. \qquad (3.51)\end{aligned}$$

Since $\frac{e_s-e_m}{-\beta_{sm}}=ie^{\frac{i(\theta_s+\theta_m)}{2}}=e^{\frac{i(\pi+\theta_s+\theta_m)}{2}}$ and $\alpha_{st}=e^{\frac{i(\theta_s+\theta_t)}{2}}$, we have

$$\left|\left\langle\alpha_{st},\frac{e_s-e_m}{-\beta_{sm}}\right\rangle\right|=\left|\cos(\frac{\pi+\theta_s+\theta_m}{2}-\frac{\theta_s+\theta_t}{2})\right|=\left|\sin\frac{\theta_t-\theta_m}{2}\right|=\frac{|e_t-e_m|}{2}. \tag{3.52}$$

Combining (3.51) and (3.52) we obtain

$$\begin{aligned}|h'_{st}(\langle(e_s-e_t)^{\perp},e_s\rangle)| &= |\beta_{st}|^{d-3}\prod_{m=0,m\neq s,t}^{d-1}\frac{|e_t-e_m|}{2}\prod_{m=0,m\neq s}^{d-1}|e_s-e_m|\\ &= \frac{|\beta_{st}|^{d-4}}{2^{d-2}}\prod_{m=0,m\neq t}^{d-1}|e_t-e_m|\prod_{m=0,m\neq s}^{d-1}|e_s-e_m|\\ &\geq \frac{2^{2r}|\beta_{st}|^{d-4}}{2^{d-2}},\end{aligned}$$

where we use Lemma 3.23 in the third line. □

LEMMA 3.27. *Let $E = (e_n : n \in \mathbb{N})$ be a Leja sequence for D and $d \geq 2$, $d = 2^{n_0} + 2^{n_1} + \cdots + 2^{n_r}$ with $n_0 > n_1 > \cdots > n_r \geq 0$. Then*

$$|h_{st}\big(\langle (e_s - e_t)^{\perp}, x\rangle\big)| \leq \frac{2^{2r+1} d |\beta_{st}|^{d-2}}{2^{d-2}}, \quad 0 \leq s < t \leq d-1, \quad \|x\| \leq 1. \tag{3.53}$$

PROOF. Let us set $\langle \alpha_{st}, x\rangle = \cos\varphi$. Since $(e_s - e_t)^{\perp} = \alpha_{st}\beta_{st}$ and $\langle \alpha_{st}, e_m\rangle = \cos(\theta_m - \frac{\theta_s+\theta_t}{2})$, in view of (3.22), we get

$$\begin{aligned} |h_{st}\big(\langle (e_s - e_t)^{\perp}, x\rangle\big)| &= |\beta_{st}|^{d-1} \prod_{m=0, m\neq s}^{d-1} |\langle \alpha_{st}, (x - e_m)\rangle| \\ &= |\beta_{st}|^{d-1} \prod_{m=0, m\neq s}^{d-1} |\cos\varphi - \cos(\theta_m - \frac{\theta_s+\theta_t}{2})|. \end{aligned} \tag{3.54}$$

There exists a unique $-1 \leq j \leq r-1$ such that $d_j \leq s \leq d_{j+1} - 1$, where the d_i's are defined in (3.32). We can write the trigonometric expression in the right hand side of (3.54) as follows

$$\prod_{k=-1, k\neq j}^{r-1} \prod_{m=d_k}^{d_{k+1}-1} |\cos\varphi - \cos(\theta_m - \frac{\theta_s+\theta_t}{2})| \cdot \prod_{m=d_j, m\neq s}^{d_{j+1}-1} |\cos\varphi - \cos(\theta_m - \frac{\theta_s+\theta_t}{2})|. \tag{3.55}$$

Thanks to Lemmas 3.20 and 3.22, the first factor and the second factor in (3.55) are dominated respectively by

$$\prod_{k=-1, k\neq j}^{r-1} \frac{2}{2^{2^{n_{k+1}}-1}} \quad \text{and} \quad \frac{2 \cdot 2^{n_{j+1}}}{2^{2^{n_{j+1}}-1} |\sin\frac{\theta_s-\theta_t}{2}|}.$$

Combining these estimates with (3.54) we obtain

$$|h_{st}\big(\langle (e_s - e_t)^{\perp}, x\rangle\big)| \leq |\beta_{st}|^{d-1} \cdot \prod_{k=-1}^{r-1} \frac{2}{2^{2^{n_{k+1}}-1}} \cdot \frac{2^{n_{j+1}}}{|\sin\frac{\theta_s-\theta_t}{2}|} \leq \frac{2^{2r+1} d |\beta_{st}|^{d-2}}{2^{d-2}},$$

here we use the facts that $|\beta_{st}| = 2|\sin\frac{\theta_s-\theta_t}{2}|$ and $2^{n_{j+1}} \leq d$. □

LEMMA 3.28. *Let $E = (e_n : n \in \mathbb{N})$ be a Leja sequence for D and $d \geq 2$, $d = 2^{n_0} + 2^{n_1} + \cdots + 2^{n_r}$ with $n_0 > n_1 > \cdots > n_r \geq 0$. Then*

$$|h'_{st}\big(\langle (e_s - e_t)^{\perp}, x\rangle\big)| \leq \frac{2^{2r+1} d^3 |\beta_{st}|^{d-3}}{2^{d-2}}, \quad 0 \leq s < t \leq d-1, \ \|x\| \leq 1. \tag{3.56}$$

PROOF. Let $\ell : \mathbb{R}^2 \to \mathbb{R}$ be the linear form defined by $\ell(x) := \langle (e_s - e_t)^{\perp}, x\rangle$. Since $|(e_s - e_t)^{\perp}| = |\beta_{st}|$, we have $\ell(D) = [-|\beta_{st}|, |\beta_{st}|]$. It follows that

$$\sup_{x\in D} |h_{st}\big(\langle (e_s - e_t)^{\perp}, x\rangle\big)| = \|h_{st}\|_{[-|\beta_{st}|, |\beta_{st}|]} \quad \text{and} \quad \sup_{x\in D} |h'_{st}\big(\langle (e_s - e_t)^{\perp}, x\rangle\big)| = \|h'_{st}\|_{[-|\beta_{st}|, |\beta_{st}|]}.$$

Since $\deg h_{st} = d-1$, classical Markov's inequality gives

$$\|h'_{st}\|_{[-|\beta_{st}|, |\beta_{st}|]} \leq (1/|\beta_{st}|)(d-1)^2 \|h_{st}\|_{[-|\beta_{st}|, |\beta_{st}|]}.$$

Hence Lemma 3.27 yields

$$\sup_{x\in D} |h'_{st}\big(\langle (e_s - e_t)^{\perp}, x\rangle\big)| \leq \frac{(d-1)^2}{|\beta_{st}|} \sup_{x\in D} |h_{st}\big(\langle (e_s - e_t)^{\perp}, x\rangle\big)| \leq \frac{2^{2r+1} d^3 |\beta_{st}|^{d-3}}{2^{d-2}},$$

□

PROOF OF THEOREMS 3.17 AND 3.18. Lemmas 3.26 and 3.27 give an upper bound $\frac{2^{2r+1}d|\beta_{st}|^{d-2}}{2^{d-2}}$ for the numerator of $P_{st}(x)$ and a lower bound $\frac{2^{2r}|\beta_{st}|^{d-2}}{2^{d-2}}$ for the denominator of $P_{st}(x)$. Thus $|P_{st}(x)| \leq 2d$ for all $x \in D$ and $0 \leq s < t \leq d-1$. At the same time, Lemmas 3.26 and 3.28 follow that $|Q_{st}(x)| \leq 2d^3|\beta_{st}| \leq 4d^3$ for all $x \in D$ and $0 \leq s < t \leq d-1$. $\square$

CHAPTER 4

On the limit points of pseudo Leja sequences

Abstract. We prove the existence of pseudo Leja sequences with large sets of limit points for many plane compact sets.

4.1. Introduction

Let K be a non-empty compact subset of the complex plane and let $A = (a_n : n \in \mathbb{N})$ be a sequence of points in K. We write

$$w\big(A, a_d; z\big) := \prod_{j=0}^{d-1}(z - a_j), \quad d \geq 1. \tag{4.1}$$

One says that A is a Leja sequence for K if, for all $d \geq 1$, the $(d+1)$-st entry a_d maximizes the product of the distances to the d previous ones, that is

$$|w\big(A, a_d; a_d\big)| = \max_{z \in K} |w\big(A, a_d; z\big)|. \tag{4.2}$$

By the maximal principle, all points of A (except perhaps a_0) must lie on the outer boundary $\partial_\infty K$ of K. It is known that every non-polar compact set (that has a positive logarithmic capacity) possesses infinitely many Leja sequences but it is in general impossible to compute them. Recently, Białas-Cież and Calvi [**4**] described the structure of Leja sequences for the unit disk in $\mathbb{C}$. Also in [**4**], the authors introduced the concept of pseudo Leja sequence for K, that is a sequence $Z = (z_n : n \in \mathbb{N})$ in K such that the $(d+1)$-st entry z_d satisfies the inequality

$$M_Z(z_d)|w\big(Z, z_d; z_d\big)| \geq \max_{z \in K} |w\big(Z, z_d; z\big)|, \quad d \geq 1, \tag{4.3}$$

where $(M_Z(z_d) : d \in \mathbb{N}^*)$ is a sequence of positive real numbers greater than or equal to 1 of subexponential growth, i.e., $\lim_{d\to\infty}\big(M_Z(z_d)\big)^{\frac{1}{d}} = 1$. The sequence $M_Z(z_d)$ is called the Edrei growth of the pseudo Leja sequence Z. There are some advantages in working with pseudo Leja sequences. First, unlike Leja sequences, pseudo Leja sequences can be easily computed and are therefore suitable for numerical purposes. For details, we refer the reader to [**4**]. Second, from a theorical point of view, pseudo Leja sequences also provide excellent points for polynomial interpolation. We shall explain the second point. Suppose that K is a non-polar, polynomially convex, compact set in $\mathbb{C}$. Białas-Cież and Calvi showed that

$$\lim_{d\to\infty} |\mathrm{VDM}(z_0, \ldots, z_{d-1})|^{\frac{2}{(d-1)d}} = C(K), \tag{4.4}$$

where $\mathrm{VDM}(z_0, \ldots, z_{d-1}) = \prod_{0 \leq j < k \leq d-1}(z_k - z_j)$ and $C(K)$ is the logarithmic capacity of K. This asymptotic behavior enables one to use [**6**, Theorem 1.5] and get the following two properties.

(1) $\lim_{d\to\infty}(1/d)\sum_{j=0}^{d-1}[z_j] = \mu_K$, where $[z_j]$ is the Dirac measure at z_j and μ_K is the equilibrium measure of K.
(2) For every holomorphic function f in a neighborhood of K, the Lagrange interpolation polynomial of f at $z_0,\dots,z_{d-1}$ converges uniformly to f on K as $d\to\infty$.

According to a remark in [**4**], the first property implies that every point in the support of μ_K is a limit point of $Z = (z_n : n\in\mathbb{N})$. Since this support lies in $\partial_\infty K$, the number of points lying in any compact subset G of the interior of K is small in the sense that

$$\lim_{n\to\infty}\frac{1}{d}\sharp\big(G\cap\{z_0,\dots,z_{d-1}\}\big) = 0.$$

There arises a natural problem to decide whether there exists a pseudo Leja sequence for a compact set with a limit point in the interior of the compact set. The aim of this paper is to give an affirmative answer. It is shown that the sets of limit points of pseudo Leja sequences for many plane compact sets constructed below contain neighborhoods of boundaries of these compact sets, and is even equal to the whole set when the compact set is the unit disk.

Notation. The closed disk of center about $a\in\mathbb{C}$ and radius $r>0$ is denoted by $D(a,r)$. For simplicity, we write $D := D(0,1)$. Let $Z = (z_n : n\in\mathbb{N})$ be a sequence of distinct complex numbers. The index of $z\in Z$ is denoted by $s_Z(z)$, which shows the position of z in Z, so that $s_Z(z_j) = j+1$ for $j\ge 0$. For each $d\ge 1$, we define $Z_d := (z_0,\dots,z_{d-1})$. For $T_k = (t_0,\dots,t_{k-1})$, we write $(Z_d,T_k) := (z_0,\dots,z_{d-1},t_0,\dots,t_{k-1})$.

4.2. Pseudo Leja sequences for the unit disk

Given a Leja sequence A for D and a sequence B of distinct points in $\operatorname{int}(D)$, we show how to insert the entries of B into A in order that the resulting sequence is a pseudo Leja sequence for D. We exploit the structure of Leja sequences for D that is given in [**4**, Theorem 5] and that we recall below. Here a d-tuple $A_d := (a_0,\dots,a_{d-1})$ is called a d-Leja section of the sequence A.

THEOREM 4.1 (Białas-Cież and Calvi). *The structure of a Leja sequence $A = (a_n : n\in\mathbb{N})$ for the unit disk D with $a_0 = 1$ is given by the following rules.*

(1) *The underlying set of the 2^n-Leja section A_{2^n} consists of the 2^n-th roots of unity;*
(2) *The 2^{n+1}-Leja section $A_{2^{n+1}}$ is $(A_{2^n},\rho U_{2^n})$, where ρ is a 2^n-th root of -1 and U_{2^n} is the 2^n-Leja section of a Leja sequence $U = (u_n : n\in\mathbb{N})$ for D with $u_0 = 1$.*

THEOREM 4.2. *Let $A = (a_n : n\in\mathbb{N})$ be a Leja sequence for D with $a_0\in\partial D$ and $B = (b_n : n\in\mathbb{N})$ a sequence of distinct points in $\operatorname{int}(D)$. Then there exists a pseudo Leja sequence for D whose underlying set is $A\cup B$.*

PROOF. Without loss of generality, we assume that $a_0 = 1$. We consider three sequences of positive real numbers whose entries are defined by

$$\alpha_j = \operatorname{dist}(B_j,\partial D) = \inf\{|b_k - a| : 0\le k\le j-1, |a| = 1\},\quad j\ge 1. \tag{4.5}$$

$$\beta_j = |w(B,b_j;b_j)| \quad\text{and}\quad \gamma_j = \sup_{z\in D}|w(B,b_j;z)|,\quad j\ge 1, \tag{4.6}$$

where $w(B,b_j;z)$ is defined as (4.1). Take a subsequence $4 < n_0 < n_1 < \cdots < n_k < \cdots$ such that

$$\lim_{j\to\infty}\Big(\frac{2}{\alpha_{j+1}}\Big)^{\frac{j+1}{2^{n_j}}} = \lim_{j\to\infty}\Big(\frac{2\gamma_j}{(1-|b_j|)\beta_j}\Big)^{\frac{1}{2^{n_j}}} = 1. \tag{4.7}$$

Let X be a new sequence obtained by inserting the entries of B into A such that b_j is inserted between $a_{2^{n_j}-1}$ and $a_{2^{n_j}}$ for all $j \geq 0$,

(4.8)

$$a_0, a_1 \ldots, a_{2^{n_0}-1}, b_0, a_{2^{n_0}}, \ldots, a_{2^{n_1}-1}, b_1, a_{2^{n_1}}, \ldots, a_{2^{n_j}-1}, b_j, a_{2^{n_j}}, \ldots, a_{2^{n_{j+1}}-1}, b_{j+1}, a_{2^{n_{j+1}}}, \ldots$$

We will show that X is a pseudo Leja sequence for D. To do this, we construct a discrete function $M_X : X \longmapsto [1,\infty)$ (of subexponential growth of $s_X(x)$) such that

$$M_X(x)|w(X,x;x)| \geq \sup_{z\in D}|w(X,x;z)|, \quad x \in X. \tag{4.9}$$

We shall define differently M_X on A and on B and prove first for $x = a_d$ and then for $x = b_j$.

Assume first that $x = a_d$. Since the first entries of X play no role in the required property that X is a pseudo Leja sequence, we may assume that $2^{n_j} \leq d \leq 2^{n_{j+1}} - 1$ for $j \geq 1$. Looking at the definition of X in (4.8) and relation (4.1), we have

$$w(X,a_d;z) = w(A,a_d;z) \cdot w(B,b_{j+1};z). \tag{4.10}$$

Since the second factor at the right hand side of (4.10) contains $j+1$ factors $z-b_k$, $0 \leq k \leq j$, and $|z-b_k| \leq 2$ for every $z \in D$, the hypothesis that A is a Leja sequence for D implies

$$\sup_{z\in D}|w(X,a_d;z)| \leq \sup_{z\in D}|w(A,a_d;z)| \cdot \sup_{z\in D}|w(B,b_{j+1};z)| \leq 2^{j+1}|w(A,a_d;a_d)|. \tag{4.11}$$

On the other hand, relation (4.5) gives $|a_d - b_k| \geq \alpha_{j+1}$ for all $0 \leq k \leq j$. Thus

$$|w(X,a_d;a_d)| \geq (\alpha_{j+1})^{j+1}|w(A,a_d;a_d)|. \tag{4.12}$$

From (4.11) and (4.12) we obtain

$$M_X(a_d)|w(X,a_d;a_d)| \geq \sup_{z\in D}|w(X,a_d;z)| \quad \text{with} \quad M_X(a_d) = \big(\frac{2}{\alpha_{j+1}}\big)^{j+1}. \tag{4.13}$$

Since the index of a_d in X is $s_X(a_d) = d + j + 2 > 2^{n_j}$, equation (4.7) yields

$$\lim_{d\to\infty}\Big(M_X(a_d)\Big)^{\frac{1}{s_X(a_d)}} = \lim_{j\to\infty}\Big(\frac{2}{\alpha_{j+1}}\Big)^{\frac{j+1}{2^{n_j}}} = 1. \tag{4.14}$$

Second, we work with $x = b_j$ for $j \geq 1$. We have

$$w(X,b_j;z) = w(A,a_{2^{n_j}};z) \cdot w(B,b_j;z) = (z^{2^{n_j}} - 1)w(B,b_j;z). \tag{4.15}$$

Here we use the fact that $w(A,a_{2^{n_j}};z) = z^{2^{n_j}} - 1$, since the set $\{a_0,\ldots,a_{2^{n_j}-1}\}$ forms a complete set of the 2^{n_j}-roots of unity (see Theorem 4.1). It follows that

$$\sup_{z\in D}|w(X,b_j;z)| \leq 2\sup_{z\in D}|w(B,b_j;z)| = 2\gamma_j, \tag{4.16}$$

and

$$|w(X,b_j;b_j)| = |b_j^{2^{n_j}} - 1| \cdot |w(B,b_j;b_j)| \geq (1-|b_j|)\beta_j. \tag{4.17}$$

We get from (4.16) and (4.17) the following estimate

$$M_X(b_j)|w(X,b_j;b_j)| \geq \sup_{z\in D}|w(X,b_j;z)| \quad \text{with} \quad M_X(b_j) = \frac{2\gamma_j}{(1-|b_j|)\beta_j}. \tag{4.18}$$

Since the index of b_j in X is $s_X(b_j) = 2^{n_j} + j + 1 > 2^{n_j}$, equation (4.7) gives

$$\lim_{j\to\infty}\Big(M_X(b_j)\Big)^{\frac{1}{s_X(b_j)}} = \lim_{j\to\infty}\Big(\frac{2\gamma_j}{(1-|b_j|)\beta_j}\Big)^{\frac{1}{2^{n_j}}} = 1. \tag{4.19}$$

This completes the proof of the theorem. □

If we choose the sequence B in Theorem 4.2 to be dense in D, then we have the following corollary.

COROLLARY 4.3. *There exist pseudo Leja sequences for D such that the set of their limit points is D.*

4.3. Pseudo Leja sequences for compact sets in $\mathbb{C}$

This section is devoted to the construction of pseudo Leja sequences for more general compact sets. Our idea is similar in spirit to [**4**, Theorem 6], but original arguments come from the work of Alper [**2**, pp. 48-49] with a weaker assumption on K.

Let K be a compact set in $\mathbb{C}$ such that ∂K is an analytic Jordan curve. Suppose that $\phi(z)$ is a conformal mapping of $\overline{\mathbb{C}}\setminus D$ onto $\overline{\mathbb{C}}\setminus K$. It is known that

$$\phi(z) = cz + c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots \quad \text{with} \quad |c| = C(K). \tag{4.20}$$

Since ∂K is assumed to be an analytic Jordan curve, there exists an analytic and univalent continuation of ϕ to the domain $\overline{\mathbb{C}}\setminus D(0,\rho_0)$ for some $\rho_0 < 1$ (see for instance [**24**, p.12]). If $1 > \rho_1 > \rho_0$, then the function

$$\psi(t,z) = \begin{cases} \frac{\phi(t)-\phi(z)}{t-z} & \text{if } t \neq z \\ \phi'(z) & \text{if } t = z \end{cases}$$

is continuous and does not vanish when $t, z \in D\setminus \operatorname{int}(D(0,\rho_1))$. Thus, there exist $M_1, M_2 > 0$ such that

$$M_1 \le \Big|\frac{\phi(t)-\phi(z)}{t-z}\Big| \le M_2, \quad t,z \in D\setminus \operatorname{int}(D(0,\rho_1)),\ t \neq z. \tag{4.21}$$

LEMMA 4.4. *Under the above assumptions. If $\rho_0 < \rho_1 < 1$ and $e_k = \exp(2k\pi i/d)$ for $0 \le k \le d-1$, then*

$$\frac{C(K)^d}{V} \le \prod_{k=0}^{d-1} \frac{|\phi(t)-\phi(e_k)|}{|t-e_k|} \le VC(K)^d, \quad \rho_1 \le |t| \le 1, \tag{4.22}$$

where V is a positive constant independent of d.

PROOF. The proof is a slight adaptation of the reasoning used in [**4**]. Since $z \mapsto \psi(z,t)$ is a nowhere-vanishing holomorphic function on $\overline{\mathbb{C}}\setminus D$ for all $\rho_1 \le |t| \le 1$, there exists a branch of $\log \psi(z,t)$ that, as a function of z, is holomorphic on $\overline{\mathbb{C}}\setminus D$ and continuous on $\overline{\mathbb{C}}\setminus \operatorname{int}(D)$. For convenience, we set

$$f_t(z) = \log \frac{\phi(t)-\phi(z)}{t-z}, \quad |z| \ge 1,\ \rho_1 \le |t| \le 1. \tag{4.23}$$

The real part of $f_t(z)$, say $\Re f_t(z)$, is harmonic on $\overline{\mathbb{C}}\setminus D$ and continuous on $\overline{\mathbb{C}}\setminus \operatorname{int}(D)$. It is clear that $\Re f_t(z) = \log\big|\frac{\phi(t)-\phi(z)}{t-z}\big|$. By the mean value theorem for harmonic functions we have

$$\frac{1}{2\pi}\int_0^{2\pi} \Re f_t(e^{i\theta})\mathrm{d}\theta = \lim_{z\to\infty} \Re f_t(z) = \log|c| = \log C(K). \tag{4.24}$$

From (4.23) we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta} f_t(e^{i\theta}) = ie^{i\theta}\Big(\frac{\phi'(e^{i\theta})}{\phi(e^{i\theta})-\phi(t)} - \frac{1}{e^{i\theta}-t}\Big), \quad \theta \in [0,2\pi],\ \rho_1 \le |t| \le 1,\ t \neq e^{i\theta}. \tag{4.25}$$

The limit $\lim_{t\to e^{i\theta}} \frac{\mathrm{d}}{\mathrm{d}\theta} f_t(e^{i\theta})$ exists and is denoted by $\frac{\mathrm{d}}{\mathrm{d}\theta} f_{e^{i\theta}}(e^{i\theta})$, since

$$\lim_{t\to e^{i\theta}} \frac{\mathrm{d}}{\mathrm{d}\theta} f_t(e^{i\theta}) = ie^{i\theta} \lim_{t\to e^{i\theta}} \frac{\phi'(e^{i\theta}) - \frac{\phi(e^{i\theta})-\phi(t)}{e^{i\theta}-t}}{(e^{i\theta}-t)\frac{\phi(e^{i\theta})-\phi(t)}{e^{i\theta}-t}} = \frac{ie^{i\theta}\phi''(e^{i\theta})}{2\phi'(e^{i\theta})}. \tag{4.26}$$

Here we use the Taylor expansion of the holomorphic function ϕ at $e^{i\theta}$ to get the second equality in (4.26). Thus, the function $(\theta,t) \mapsto \frac{\mathrm{d}}{\mathrm{d}\theta} f_t(e^{i\theta})$ is continuous on $[0,2\pi]\times\{\rho_1 \leq |t| \leq 1\}$. It follows that there exists $V_0 < \infty$ such that

$$\int_0^{2\pi} \Big|\frac{\mathrm{d}}{\mathrm{d}\theta} f_t(e^{i\theta})\Big| \mathrm{d}\theta < V_0, \quad \rho_1 \leq |t| \leq 1. \tag{4.27}$$

Consequently, $f_t(e^{i\theta})$ is a function of total variation bounded by V_0 for all $\rho_1 \leq |t| \leq 1$. Therefore, so is its real part $\Re f_t(e^{i\theta})$. Using [**4**, Lemma 1] for $\Re f_t$, the formula for $\Re f_t(e^{i\theta})$ and relation (4.24), we obtain

$$\Big|\log C(K) - \frac{1}{d}\sum_{k=1}^{d-1} \log\Big|\frac{\phi(t)-\phi(e_k)}{t-e_k}\Big|\Big| \leq \frac{V_0}{d}, \quad \rho_1 \leq |t| \leq 1. \tag{4.28}$$

This estimate implies the conclusion of the lemma. □

THEOREM 4.5. *Let K be a compact set in $\mathbb{C}$ such that ∂K is an analytic Jordan curve. Then there exists a pseudo Leja sequence for K whose set of limit points contains $\{z \in K : \mathrm{dist}(z,\partial K) \leq r\}$ for some $r > 0$.*

PROOF. Let $\phi(z)$ be a conformal mapping of $\overline{\mathbb{C}}\setminus D$ onto $\overline{\mathbb{C}}\setminus K$. Then ϕ admits an analytic and univalent continuation to the domain $\overline{\mathbb{C}}\setminus D(0,\rho_0)$ for $\rho_0 < 1$. Let $A = (a_n : n \in \mathbb{N})$ be a Leja sequence for D with $a_0 = 1$. Take $\rho_1 \in (\rho_0, 1)$ and a sequence $B = (b_n : n \in \mathbb{N})$ of distinct points in the open annulus $\{\rho_1 < |z| < 1\}$ such that B is dense in the closure of this annulus. Let X be a sequence defined as in (4.8) such that X is a pseudo Leja sequence for D. Since the closure of B is $\{\rho_1 \leq |z| \leq 1\}$, the set of limit points of $\phi(X)$ contains $\phi(\{\rho_1 \leq |z| \leq 1\})$, a compact subset of K containing $\{z \in K : \mathrm{dist}(z,\partial K) \leq r\}$ for some $r > 0$. Thus, it remains to verify that $\phi(X)$ is a pseudo Leja sequence for K. For simplicity, we write

$$\begin{aligned} &\widetilde{A} := \phi(A), \quad \widetilde{B} := \phi(B), \quad \widetilde{X} := \phi(X), \\ &\tilde{z} := \phi(z), \quad \tilde{a}_d := \phi(a_d), \quad \text{and} \quad \tilde{b}_k := \phi(b_k), \quad d \geq 0, \quad k \geq 0. \end{aligned} \tag{4.29}$$

With this notation, the sequence $\widetilde{X}$ is given by

$$\tilde{a}_0, \tilde{a}_1 \ldots, \tilde{a}_{2^{n_0}-1}, \tilde{b}_0, \tilde{a}_{2^{n_0}}, \ldots, \tilde{a}_{2^{n_1}-1}, \tilde{b}_1, \tilde{a}_{2^{n_1}}, \ldots, \tilde{a}_{2^{n_j}-1}, \tilde{b}_j, \tilde{a}_{2^{n_j}}, \ldots, \tilde{a}_{2^{n_{j+1}}-1}, \tilde{b}_{j+1}, \tilde{a}_{2^{n_{j+1}}}, \ldots \tag{4.30}$$

We also consider two cases as in the proof of Theorem 4.2 and use the formula for the product defined in (4.1).

For $2^{n_j} \leq d \leq 2^{n_{j+1}} - 1$ with $j \geq 1$, we have

$$w\big(\widetilde{X}, \tilde{a}_d; \tilde{z}\big) = w\big(\widetilde{A}, \tilde{a}_d; \tilde{z}\big) \cdot w\big(\widetilde{B}, \tilde{b}_{j+1}; \tilde{z}\big). \tag{4.31}$$

Using [**4**, Lemma 3] we get

$$c_d^{-1} C(K)^d |w(A, a_d; z)| \leq \big|w\big(\widetilde{A}, \tilde{a}_d; \tilde{z}\big)\big| \leq c_d C(K)^d |w(A, a_d; z)|, \quad z \in \partial D, \tag{4.32}$$

where $c_d < (d+1)^{C/\log 2}$ and C is a positive constant depending only on K, see [**4**, Subsection 3.2] for the precise definition for c_d. On the other hand, inequality (4.21) gives

$$M_1|z-b_k| \le \left|\tilde{z}-\tilde{b}_k\right| \le M_2|z-b_k|, \quad \rho_1 \le |z| \le 1,\ k \ge 0. \tag{4.33}$$

Hence

$$M_1^{j+1}|w(B,b_{j+1};z)| \le \left|w(\widetilde{B},\tilde{b}_{j+1};\tilde{z})\right| \le M_2^{j+1}|w(B,b_{j+1};z)|, \quad z\in\partial D. \tag{4.34}$$

Multiplying sides by sides of (4.32) and (4.34), and using (4.31), we obtain

$$c_d^{-1}C(K)^dM_1^{j+1}|w(X,a_d;z)| \le |w\big(\widetilde{X},\tilde{a}_d;\tilde{z}\big)| \le c_dC(K)^dM_2^{j+1}|w(X,a_d;z)|, \quad z\in\partial D \tag{4.35}$$

Since $\phi(\partial D)=\partial K$, the maximum principle and the relations in (4.35) (for $z\in\partial D$ and a_d) now give

$$\begin{aligned}
\sup_{t\in K}|w\big(\widetilde{X},\tilde{a}_d;t\big)| &= \sup_{\tilde{z}=\phi(z),|z|=1}|w\big(\widetilde{X},\tilde{a}_d;\tilde{z}\big)| \\
&\le c_dC(K)^dM_2^{j+1}\sup_{|z|=1}|w(X,a_d;z)| \\
&\le c_dC(K)^dM_2^{j+1}M_X(a_d)|w(X,a_d;a_d)| \\
&\le c_d^2\big(\frac{M_2}{M_1}\big)^{j+1}M_X(a_d)|w\big(\widetilde{X},\tilde{a}_d;\tilde{a}_d\big)|.
\end{aligned}$$

Let us set $M_{\widetilde{X}}(\tilde{a}_d)=c_d^2\big(\frac{M_2}{M_1}\big)^{j+1}M_X(a_d)$. Since $s_{\widetilde{X}}(\tilde{a}_d)=s_X(a_d)=d+2+j>2^{n_j}\ge 2^j$, we have

$$\lim_{d\to\infty}c_d^{\frac{2}{s_{\widetilde{X}}(\tilde{a}_d)}}=\lim_{d\to\infty}c_d^{\frac{2}{d}}=1, \quad \lim_{d\to\infty}\left(\frac{M_2}{M_1}\right)^{\frac{j+1}{s_{\widetilde{X}}(\tilde{a}_d)}}=\lim_{j\to\infty}\left(\frac{M_2}{M_1}\right)^{\frac{j+1}{2^j}}=1, \tag{4.36}$$
$$\lim_{d\to\infty}\Big(M_X(a_d)\Big)^{\frac{1}{s_{\widetilde{X}}(\tilde{a}_d)}}=\lim_{d\to\infty}\Big(M_X(a_d)\Big)^{\frac{1}{s_X(a_d)}}=1.$$

Here we use the fact that c_d grows at most like a polynomial in d and the hypothesis that X is a pseudo Leja sequence for D. Hence

$$\lim_{d\to\infty}\Big(M_{\widetilde{X}}(\tilde{a}_d)\Big)^{\frac{1}{s_{\widetilde{X}}(\tilde{a}_d)}}=1. \tag{4.37}$$

We now turn to $\tilde{b}_j$ for $j\ge 1$. We have

$$w\big(\widetilde{X},\tilde{b}_j;\tilde{z}\big)=w\big(\tilde{A},\tilde{a}_{2^{n_j}};\tilde{z}\big)\cdot w\big(\widetilde{B},\tilde{b}_j;\tilde{z}\big). \tag{4.38}$$

Using relation (4.32) for $d=2^{n_j}$ and relation (4.33) to estimate the first and the second factor at the right hand side of (4.38) respectively, we obtain

$$|w\big(\widetilde{X},\tilde{b}_j;\tilde{z}\big)| \le c_{2^{n_j}}C(K)^{2^{n_j}}M_2^j|w\big(X,b_j;z\big)|, \quad z\in\partial D. \tag{4.39}$$

On the other hand, since $\{a_0,\dots,a_{2^{n_j}-1}\}$ is a complete set of the 2^{n_j}-th roots of unity, Lemma 4.4 gives

$$|w\big(\widetilde{A},\tilde{a}_{2^{n_j}};\tilde{b}_j\big)| \ge \frac{C(K)^{2^{n_j}}}{V}|w(A,a_{2^{n_j}};b_j)|. \tag{4.40}$$

By $|\tilde{b}_j-\tilde{b}_k|\ge M_1|b_j-b_k|$ for all $0\le k\le j-1$, relations (4.38) and (4.40) show that

$$|w\big(\widetilde{X},\tilde{b}_j;\tilde{b}_j\big)| \ge \frac{C(K)^{2^{n_j}}}{V}M_1^j|w\big(X,b_j;b_j\big)|. \tag{4.41}$$

Combining (4.39) and (4.41), and using the maximum principle again, we have

$$\begin{aligned}\sup_{t\in K}|w(\widetilde{X},\tilde{b}_j;t)| &= \sup_{\tilde{z}=\phi(z),|z|=1}|w(\widetilde{X},\tilde{b}_j;\tilde{z})|\\ &\le c_{2^{n_j}}C(K)^{2^{n_j}}M_2^j\sup_{|z|=1}|w(X,b_j;z)|\\ &\le c_{2^{n_j}}C(K)^{2^{n_j}}M_2^jM_X(b_j)|w(X,b_j;b_j)|\\ &\le c_{2^{n_j}}V(\frac{M_2}{M_1})^jM_X(b_j)|w(\widetilde{X},\tilde{b}_j;\tilde{b}_j)|.\end{aligned}$$

Set $M_{\widetilde{X}}(\tilde{b}_j)=c_{2^{n_j}}V(\frac{M_2}{M_1})^jM_X(b_j)$. We also have $s_{\widetilde{X}}(\tilde{b}_j)=s_X(b_j)=2^{n_j}+j+1\ge 2^j+j+1$. A passage to the limit similar to (4.36) implies that

$$\lim_{j\to\infty}\left(M_{\widetilde{X}}(\tilde{b}_j)\right)^{\frac{1}{s_{\widetilde{X}}(\tilde{b}_j)}}=1. \tag{4.42}$$

This completes the proof of the theorem. □

CHAPTER 5

On the continuity of multivariate Lagrange interpolation at Chung-Yao lattices

Abstract. We give a natural geometric condition that ensures that sequences of Chung-Yao interpolation polynomials (of fixed degree) of sufficiently differentiable functions converge to a Taylor polynomial.

5.1. Introduction

5.1.1. Stating the problem. When $d+1$ points $a_0,\ldots,a_d$ in $\mathbb{R}$ converge to a limit point a, the corresponding Lagrange interpolation polynomial $\mathbf{L}[a_0,\ldots,a_d;f]$ of a function f at the a_i's tends to the Taylor polynomial of f at a to the order d and this under the sole assumption that f is d times continuously differentiable on a neighborhood of the limit point. This classical result is an easy consequence of Newton's formula for Lagrange interpolation and of the mean value theorem for divided differences. In this paper, we study a multivariate analogue of this problem. We suppose that the points of a multivariate interpolation lattice A of degree d in $\mathbb{R}^N$ converge to a limit point $a\in\mathbb{R}^N$ and ask under what conditions we can assert that the corresponding multivariate Lagrange interpolation polynomials of a function f converge to the Taylor polynomial of f at a to the order d ? The question is answered for a particular but important class of interpolation lattices, the so-called Chung-Yao lattices, see below.

5.1.2. A known criterion. In the multivariate case, a simple clear-cut answer cannot be expected. This perhaps may be regarded as another consequence of the absence of a multivariate mean value equality. We recall a rather general criterion (which actually works for hermitian interpolations) which can be found in [**7**]. Let us mention that the first results which appeared in the literature concerned the case (of practical importance in finite elements theory) for which the lattices are of the form $A^{(t)}=U^{(t)}(A)$ where $U^{(t)}$ is a sequence of linear transformations whose norms tend to 0 and A is a fixed lattice. We refer to [**7**] for details and references to earlier works.

We denote by $\mathscr{P}^d(\mathbb{R}^N)$ the space of polynomials in N real variables of degree at most d, X^α is the monomial function corresponding to the N-index α, that is $X^\alpha(x)=x_1^{\alpha_1}\cdots x_N^{\alpha_N}$ for $x=(x_1,\ldots,x_N)\in\mathbb{R}^N$. The length of α is the degree of X^α, $|\alpha|=\sum_{i=1}^N\alpha_i$. We denote by m_d the dimension of the vector space $\mathscr{P}^d(\mathbb{R}^N)$. We have $m_d=\binom{N+d}{d}$. In the whole paper, $N\geq 2$.

THEOREM 5.1 (Bloom and Calvi). *Let $A^{(s)}$ be a sequence of interpolation lattices of degree d in $\mathbb{R}^N$. If the following condition holds*

$$|\alpha|=d+1\Longrightarrow\lim_{s\to\infty}\mathbf{L}[A^{(s)};X^\alpha]=0, \tag{5.1}$$

then, for every function f of class C^{m_d-1} in a neighborhood of the origin 0, we have

$$\lim_{s\to\infty} \mathbf{L}[A^{(s)}; f] = \mathbf{T}_0^d(f), \tag{5.2}$$

where $\mathbf{L}[A^{(s)}; \cdot]$ (resp. $\mathbf{T}_0^d(\cdot)$) denotes the Lagrange interpolation projector at the points of $A^{(s)}$ (resp. the Taylor projector at 0 of order d).

Unfortunately condition (5.1) is not easy to verify, especially if the degree of interpolation is not small, and it seems difficult to check it on general classes of interpolation lattices. Besides, Theorem 5.1 requires a high order of smoothness. We point out, however, that although whether the level of differentiability required in Theorem 5.1 is optimal is not known (in the case of Lagrange interpolation), examples do exist for which convergence does not hold for function of class C^{d+1} but holds for function of higher smoothness, see [**7**, Example 5.4].

The aim of this paper is to give a natural geometric condition in the case where the interpolation lattices are Chung-Yao lattices. From an algebraic point of view, they can be regarded as the simplest interpolation lattices : every point is situated at the intersection of N hyperplanes chosen among a minimal family and the corresponding Lagrange fundamental polynomials are products of affine forms. The definition and main properties of Chung-Yao lattices are collected in Section 5.2. Our criterion is given and commented in Section 5.3. The proof is quite technical and is postponed to the next section. It relies on a remainder formula due to Carl de Boor.

We need very few facts from general interpolation theory. They are recalled in the following subsection.

5.1.3. Basic facts on interpolation. Let E be a m-dimensional space of functions on $\mathbb{R}^N$ and $A = \{a_1, \ldots, a_m\} \subset \mathbb{R}^N$. We say that A is an interpolation lattice for E if for every function f defined on A there exists a unique $L \in E$ such that $L = f$ on A. Given a basis $\mathbf{f} = (f_1, \ldots, f_m)$ of E, we define the Vandermonde determinant $\text{VDM}(\mathbf{f}; A)$ by

$$\text{VDM}(\mathbf{f}; A) := \det\big(f_i(a_j)\big)_{i,j=1}^m. \tag{5.3}$$

Then A is an interpolation lattice if and only if

$$\text{VDM}(\mathbf{f}; A) \neq 0. \tag{5.4}$$

Of course, the condition is independent from the choice of the basis $\mathbf{f}$. When (5.4) is satisfied, we have

$$L = \sum_{i=1}^m f(a_i)\, \mathbf{l}(A, a_i, \cdot), \tag{5.5}$$

where $\mathbf{l}(A, a_i, \cdot)$ is the unique element of E which vanishes on $A \setminus \{a_i\}$ and takes the value 1 at a_i,

$$\mathbf{l}(A, a_i, x) = \frac{\text{VDM}(\mathbf{f}; \{a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m\})}{\text{VDM}(\mathbf{f}; A)}, \quad 1 \le i \le m, \quad x \in \mathbb{R}^N. \tag{5.6}$$

In the case where $E = \mathscr{P}^d(\mathbb{R}^N)$ we write $L = \mathbf{L}[A; f]$ and call it the Lagrange interpolation of f at A. We say that A is an interpolation lattice of degree d. The only other case that we consider in this paper is $E = \mathscr{H}^d(\mathbb{R}^N)$, the space of homogeneous polynomials of degree d in N variables whose dimension is $\binom{N+d-1}{d}$.

5.2. Chung-Yao lattices

We recall the construction of the lattices and of some objects attached to them. Despite their apparent simplicity, it seems that these configurations were first considered in 1977's Chung and Yao's paper [**19**]. Here, we essentially follow the presentation and notational conventions of de Boor [**20**].

We work in $\mathbb{R}^N$ endowed with its canonical euclidean structure. The corresponding scalar product is denoted by $\langle\cdot,\cdot\rangle$.

A set of N hyperplanes $H = \{\ell_1,\dots,\ell_N\}$ in $\mathbb{R}^N$ is said to be in *general position* if the intersection of the N hyperplanes is a singleton, that is

$$\bigcap_{i=1}^{N} \ell_i = \{\vartheta_H\}.$$

If $\ell_i = \{x \in \mathbb{R}^N \,:\, \langle n_i, x\rangle = c_i\}$, $i = 1,\dots,N$, then H is in general position if and only if $\det(n_1,\dots,n_N) \neq 0$.

DEFINITION 5.2. A collection $\mathbb{H}$ of (at least N) distinct hyperplanes in $\mathbb{R}^N$ is said to be in general position if

(1) Every $H \in \binom{\mathbb{H}}{N}$ — i.e. every subset of N hyperplanes in $\mathbb{H}$ — is in general position (as defined above).
(2) The map

$$H \in \binom{\mathbb{H}}{N} \longmapsto \vartheta_H := \bigcap_{\ell \in H} \ell \in \mathbb{R}^N \tag{5.7}$$

is one-to-one. Here and in the sequel we confuse the singleton $\bigcap_{\ell\in H}\ell$ with its element.

This definition stands at the basis of the following result.

THEOREM 5.3 (Chung and Yao [**19**]). *Let $\mathbb{H}$ be a set of $d \geq N$ hyperplanes in general position in $\mathbb{R}^N$. The lattice*

$$\Theta_{\mathbb{H}} = \left\{ \vartheta_H = \bigcap_{\ell\in H} \ell \,:\, H \in \binom{\mathbb{H}}{N} \right\} \tag{5.8}$$

is an interpolation lattice of degree $d-N$. Moreover, if $\ell \in \mathbb{H}$ is given by $\ell = \{x \in \mathbb{R}^N \,:\, \langle n_\ell, x\rangle = c_\ell\}$ then we have the interpolation formula

$$\mathbf{L}[\Theta_{\mathbb{H}}\,; f](x) = \sum_{H \in \binom{\mathbb{H}}{N}} f(\vartheta_H) \prod_{\ell \notin H} \frac{\langle n_\ell, x\rangle - c_\ell}{\langle n_\ell, \vartheta_H\rangle - c_\ell}. \tag{5.9}$$

The lattice $\Theta_{\mathbb{H}}$ is called a Chung-Yao lattice *(of degree $d-N$) and the interpolation formula is called the* Chung-Yao interpolation formula *corresponding to $\mathbb{H}$. In particular, we have*

$$\mathbf{l}(\Theta_{\mathbb{H}}, \vartheta_H, x) = \prod_{\ell\notin H} \frac{\langle n_\ell, x\rangle - c_\ell}{\langle n_\ell, \vartheta_H\rangle - c_\ell}, \quad H \in \binom{\mathbb{H}}{N}. \tag{5.10}$$

When the set of hyperplanes we use is clear, we write Θ instead of $\Theta_{\mathbb{H}}$. Of course, in (5.9), different equations for the hyperplanes yield a same formula. In the particular case $N = 1$, every set of interpolation nodes may be regarded as a (trivial) Chung-Yao lattice.

As shown by (5.9), interpolation polynomials at Chung-Yao lattices are easy to compute. Some difficulties, however, must be pointed out. In constructing a Chung-Yao lattice, we start from a family of hyperplanes and compute the interpolation points by solving, in principle, m_d linear systems (of order N). Besides, it is a difficult problem, even in the case $N = 2$, to decide how to choose the hyperplanes if a special requirement is made on the location of the interpolation points. For instance, we currently do not know what kind of limiting distribution we can obtain with a growing number of Chung-Yao points. We mention that an interesting Chung-Yao lattice was constructed by Sauer and Xu [**41**] on bidimensional disks.

5.3. Chung-Yao lattices of points converging to the origin

5.3.1. The convergence theorem. From now on, we shall confuse an hyperplane ℓ with the affine form $\ell(x) = \langle \mathbf{n}, x\rangle - c$ which defines it, where $\mathbf{n}$ is normalized so that $\|\mathbf{n}\| = 1$. This abuse of language (each hyperplane has two normalized equations) should not create confusion. Boldfaced $\mathbf{n}$ will be kept for normalized vectors and vectors derived from them.

Supposing that the points of a sequence $\Theta^{(s)}$ of Chung-Yao lattices of same degree converge to the origin (or to any other fixed point), we study under what conditions the corresponding interpolation operator converges to the Taylor projector at the origin. Our main result is summarized in the following theorem.

THEOREM 5.4. *Let $d \geq N$. Let $\Theta^{(s)}$, $s \in \mathbb{N}$, be a sequence of Chung-Yao lattices of degree $d - N$ in $\mathbb{R}^N$. We assume that $\Theta^{(s)}$ is the lattice given by the family of hyperplanes*

$$\mathbb{H}^{(s)} = \{\ell_1^{(s)}, \ldots, \ell_d^{(s)}\}, \quad \text{with } \ell_i^{(s)} = \langle \mathbf{n}_i^{(s)}, \cdot\rangle - c_i^{(s)},\ \|\mathbf{n}_i^{(s)}\| = 1,\ i = 1, \ldots, d. \tag{5.11}$$

Consider the following two conditions.

(C1) *All the points of the lattice tend to* 0 *as $s \to \infty$, that is* $\max\{\|\vartheta\| : \vartheta \in \Theta^{(s)}\} \to 0$ *as $s \to \infty$,*

(C2) *The volumes*

$$\mathrm{vol}\left(\mathbf{n}_{i_1}^{(s)}, \ldots, \mathbf{n}_{i_N}^{(s)}\right), \quad 1 \leq i_1 < i_2 < \cdots < i_N \leq d, \tag{5.12}$$

of the parallelotope spanned by the vectors $\mathbf{n}_{i_1}^{(s)}, \ldots, \mathbf{n}_{i_N}^{(s)}$ are bounded from below, away from 0*, uniformly in s.*

If conditions (C1) and (C2) are satisfied then, for every function f of class C^{d-N+1} on a neighborhood of the origin, we have

$$\lim_{s\to\infty} \mathbf{L}[\Theta^{(s)}; f] = \mathbf{T}_0^{d-N}(f). \tag{5.13}$$

Of course, (5.13) holds in every normed vector space topology of $\mathscr{P}^{d-N}(\mathbb{R}^N)$.

5.3.2. On condition (C2). The condition on the volume of the parallelotopes is equivalent to the following,

$$\liminf_{s\to\infty} \min_{1\leq i_1<\cdots<i_N\leq d} \left|\det\left(\mathbf{n}_{i_1}^{(s)}, \ldots, \mathbf{n}_{i_N}^{(s)}\right)\right| > 0. \tag{5.14}$$

In $\mathbb{R}^2$ we have

$$\mathrm{vol}(\mathbf{n}_i^{(s)}, \mathbf{n}_j^{(s)}) = \sin(\alpha_{ij}^{(s)}), \tag{5.15}$$

where $\alpha_{ij}^{(s)} \in]0,\pi[$ is the line angle between the lines ℓ_i and ℓ_j. Thus $\mathbb{H}$ satisfies condition (C2) if and only of the angles between any two (distinct) lines in $\mathbb{H}^{(s)}$ remain uniformly bounded from below by a positive constant. An example of Chung-Yao lattice of degree 2 in $\mathbb{R}^2$ and the various parameters involved in Theorem 5.4 are shown in Figure 1.

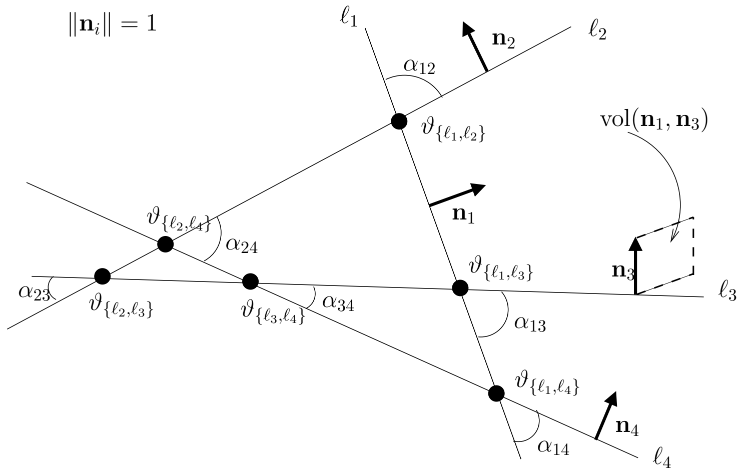


FIGURE 1. A bidimensional Chung-Yao lattice.

Conditions (C1) and (C2) are obviously independent. However, when we know that the second one holds true, the first one is easily checked as shown by the following lemma.

LEMMA 5.5. *If (C2) is satisfied then (C1) is equivalent to*

(C3) $\lim_{s\to\infty} \max_{i=1,\dots,d} |c_i^{(s)}| = 0$ *where* $c_i^{(s)}$ *is defined in (5.11).*

PROOF. We show that (C1) implies (C2). Consider $H^{(s)} \in \binom{\mathbb{H}^{(s)}}{N}$ with $\ell_i^{(s)} \in H^{(s)}$. From $\langle \mathbf{n}_i^{(s)}, \vartheta_{H^{(s)}} \rangle - c_i^{(s)} = 0$, we get

$$|c_i^{(s)}| \le \|\mathbf{n}_i^{(s)}\| \cdot \|\vartheta_{H^{(s)}}\| = \|\vartheta_{H^{(s)}}\| \to 0, \quad s \to \infty.$$

To show the converse, we observe that if $H^{(s)} = \{\ell_{i_1}^{(s)}, \dots, \ell_{i_N}^{(s)}\}$ then the coordinates (x_k) of $\vartheta_{H^{(s)}}$ are solutions of the linear system

$$\sum_{k=1}^{N} \mathbf{n}_{i_j k}^{(s)} x_k = c_j^{(s)}, \quad j = 1, \dots, N,$$

and the claim follows from Cramer's formula in which, thanks to condition (C2), the denominator remains away from 0 whereas the numerator tends to 0. □

5.3.3. Affine transformations of Chung-Yao lattices. Let $\mathscr{L}(x) = L(x) + b$ be an affine transformation (isomorphism) of $\mathbb{R}^N$ (with L its linear part). If $\mathbb{H}$ is in general position so is $\mathscr{L}(\mathbb{H}) := \{\mathscr{L}(\ell_i) : i = 1, \dots, d\}$ and $\mathscr{L}$ induces a one-to-one correspondence between $\binom{\mathbb{H}}{N}$ and $\binom{\mathscr{L}(\mathbb{H})}{N}$. Moreover if $H \in \binom{\mathbb{H}}{N}$ then

$$\vartheta_{\mathscr{L}(H)} = \mathscr{L}(\vartheta_H) \quad \text{and} \quad \Theta_{\mathscr{L}(\mathbb{H})} = \mathscr{L}(\Theta_{\mathbb{H}}).$$

In the following theorem we translate the conditions of Theorem 5.4 when the points of a Chung-Yao lattice are sent to the origin by applying a sequence of affine transformations.

THEOREM 5.6. *Let* $\mathbb{H} = \{\ell_1, \ldots, \ell_d\}$ *be a fixed collection of d hyperplanes in general position in* $\mathbb{R}^N$, $d \geq N$, *with, as above,* $\ell_i = \{x \in \mathbb{R}^N : \langle \mathbf{n}_i, x\rangle - c_i = 0\}$, $\|\mathbf{n}_i\| = 1$. *Let* $\mathscr{L}_s = L_s + b_s$, $s \in \mathbb{N}$, *be a sequence of affine transformations of* $\mathbb{R}^N$. *We set*

$$\mathbb{H}^{(s)} = \mathscr{L}_s(\mathbb{H}), \quad s \in \mathbb{N}. \tag{5.16}$$

We consider the sequence of Chung-Yao lattices $\Theta^{(s)}$ *induced by* $\mathbb{H}^{(s)}$. *The following assertions are equivalent.*

(1) $\Theta^{(s)}$ *satisfies conditions (C1) and (C2).*
(2) *There exists a positive constant* Δ *such that*

$$|\det L_s| \cdot \prod_{j=1}^{N} \|L_s^{-T}(\mathbf{n}_{i_j})\| \leq \Delta, \quad 1 \leq i_1 < \cdots < i_N \leq d, \quad s \in \mathbb{N}, \tag{5.17}$$

and

$$\max_{i=1,\ldots,d} \frac{1}{\|L_s^{-T}(\mathbf{n}_i)\|} \cdot \left| c_i + \langle \mathbf{n}_i, L_s^{-1}(b_s)\rangle \right| \to 0, \quad s \to \infty, \tag{5.18}$$

where L_s^{-T} *denotes the transpose of the inverse of* L_s.

PROOF. It follows from the normalized equation of $\mathscr{L}_s(\ell_i)$ together with Lemma 5.5. Indeed, with $\ell_i(x) = \langle \mathbf{n}_i, x\rangle - c_i$, we have

$$\mathscr{L}(\ell_i) = \{x \in \mathbb{R}^N : \langle \mathbf{n}_i, \mathscr{L}_s^{-1}(x)\rangle - c_i = 0\}.$$

Since for $x \in \mathscr{L}(\ell_i)$,

$$\begin{aligned} 0 = \langle \mathbf{n}_i, L_s^{-1}(x - b_s))\rangle - c_i &= \langle \mathbf{n}_i, L_s^{-1}(x)\rangle - \big(c_i + \langle \mathbf{n}_i, L_s^{-1}(b_s)\rangle\big) \\ &= \langle L_s^{-T}(\mathbf{n}_i), x\rangle - \big(c_i + \langle \mathbf{n}_i, L_s^{-1}(b_s)\rangle\big), \end{aligned} \tag{5.19}$$

a normalized equation of $\mathscr{L}_s(\ell_i)$ is given by

$$\left\langle \frac{L_s^{-T}(\mathbf{n}_i)}{\|L_s^{-T}(\mathbf{n}_i)\|}, x \right\rangle - \frac{1}{\|L_s^{-T}(\mathbf{n}_i)\|} \big\{c_i + \langle \mathbf{n}_i, L_s^{-1}(b_s)\rangle\big\}. \quad \square$$

5.3.4. Examples. In $\mathbb{R}^2$ any interpolation lattice of degree 1 is a Chung-Yao lattice (based on the three distinct lines defined by the interpolation points). Moreover, any such lattice is the image under an affine isomorphism of the lattice $\Theta := \{(0,0), (1,0), (0,1)\}$ constructed with the lines of equations $\ell_1(x_1, x_2) = x_1$, $\ell_2(x_1, x_2) = x_2$ and $\ell_3(x_1, x_2) = x_1 + x_2 - 1$.

Consider the affine transformations $\mathscr{L}_s$ defined by

$$\mathscr{L}_s(x) = \begin{pmatrix} t^2 & 0 \\ 0 & -t^2 u \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t \\ t \end{pmatrix}, \quad x = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \quad t = 1/s, \quad s \in \mathbb{N}^\star,$$

where u is a function of t such that $\lim_{t\to 0} u(t) = 1$, and the lattice

$$\Theta^{(s)} = \mathscr{L}_s(\Theta) = \left\{(t,t), (t^2 + t, t), (t, -t^2 u + t)\right\}, \quad t = 1/s.$$

It is not difficult to see that $\Theta^{(s)}$ satisfies condition (C1) and (C2). For (C2) we use (5.15) and observe that one of the angle is equal to $\pi/2$ while, thanks to the assumption on u, the other two tend to $\pi/4$ as $t \to 0$. Hence, according to Theorem 5.4, the corresponding Lagrange interpolation polynomials at $\Theta^{(s)}$ of any twice continuously differentiable function f on a neighborhood of 0 converge to the Taylor polynomial of f. This example shows that the assumptions of Theorem 5.4, even in the simple case of Theorem 5.6, are weaker that those given in [**7**, Proposition 2.1]. Indeed the assumption

$$\|(t,t)\|^2 \cdot \left|\ell\left(\Theta^{(s)},(t,t),\cdot\right)\right| \to 0, \quad t \to 0,$$

is required in that proposition whereas it clearly does not hold here since, as is easily checked,

$$\ell\left(\Theta^{(s)},(t,t),x\right) = \frac{x_2 - ux_1 + \left(t^2+t\right)u - t}{t^2 u}.$$

We now give an example showing that convergence to the Taylor projector no longer holds, in general, when condition (C2) is not satisfied. We use a computation done in [**7**, Example 1.2.]. We fix $\varepsilon \geq 0$ and define

$$\Theta_{\mathbb{H}^{(s)}} = \left\{(0,0),(t,t^{2+\varepsilon}),(2t,0)\right\} \subset \mathbb{R}^2, \quad t = 1/s, \quad s \in \mathbb{N}^\star.$$

This lattice satisfies (C1) but not (C2) and it is readily checked that

$$\mathbf{L}\left[\Theta_{\mathbb{H}^{(s)}}\,;X^{(2,0)}\right](x) = 2tx_1 - \frac{x_2}{t^\varepsilon},$$

which clearly does not converge to $\mathbf{T}_0^1(X^{(2,0)}) = 0$ as $s = 1/t \to \infty$. The case $\varepsilon = 0$ shows that that the Lagrange polynomials may converge to a limit different from the Taylor polynomial.

5.4. Further properties of Chung-Yao lattices and proof of Theorem 5.4

5.4.1. de Boor's identity. In the following $\mathbb{H}$ always denotes a set of $d \geq N$ hyperplanes in general position in $\mathbb{R}^N$ and $\Theta = \Theta_{\mathbb{H}}$ the corresponding Chung-Yao lattice. We will always assume that

$$\mathbb{H} = \{\ell_1,\ldots,\ell_d\}. \tag{5.20}$$

The elements of $\mathbb{H}$ are ordered according to the indexes. Every subset of $\mathbb{H}$ is endowed with the induced ordering.

If K is a subset of $N-1$ elements in $\mathbb{H}$, that is $K \in \binom{\mathbb{H}}{N-1}$, then $\cap_{\ell\in K}\ell$ is a line in $\mathbb{R}^N$ which contains $d-N+1$ points of Θ. Indeed, it passes through every ϑ_H such that $H \in \binom{\mathbb{H}}{N}$, $K \subset H$. The set of these $d-N+1$ points is denoted by Θ_K,

$$\Theta_K = \Theta \cap \left(\bigcap_{\ell\in K}\ell\right), \quad K \in \binom{\mathbb{H}}{N-1}. \tag{5.21}$$

Assume that $K = \{\ell_{i_1},\ldots,\ell_{i_{N-1}}\}$ with $i_1 < i_2 < \cdots < i_{N-1}$. Since the map

$$v \in \mathbb{R}^N \mapsto \det(v,\mathbf{n}_{i_1},\ldots,\mathbf{n}_{i_{N-1}}) \tag{5.22}$$

is a linear form, there exists a vector, which we denote by $\mathbf{n}_K$, such that

$$\det(v,\mathbf{n}_{i_1},\ldots,\mathbf{n}_{i_{N-1}}) = \langle v,\mathbf{n}_K\rangle, \quad v \in \mathbb{R}^N. \tag{5.23}$$

As defined, the value of $\mathbf{n}_K$ depends on the ordering of the hyperplanes of K. A different ordering may change $\mathbf{n}_K$ in $-\mathbf{n}_K$. It is to avoid further discussion of this detail that we assumed we start with a particular ordering of $\mathbb{H}$ and agreed that every subset of $\mathbb{H}$ is endowed with the induced ordering.

LEMMA 5.7. *If $K = \{\ell_{i_1},\ldots,\ell_{i_{N-1}}\}$ then the direction of the line $\cap_{\ell\in K}\ell$ is given by the (nonzero) vector $\mathbf{n}_K$. We have $\|\mathbf{n}_K\| \leq 1$.*

PROOF. The first claim is a consequence of the equations

$$\langle \mathbf{n}_{i_j}, \mathbf{n}_K \rangle = 0, \quad j = 1, \dots, N-1,$$

which follows readily from (5.23). Next, by Hadamard's inequality, the norm of the linear form (5.22) is smaller than the product of the $\|\mathbf{n}_{i_j}\|$'s which is smaller than one. Hence, in view of (5.23), so is the norm of $\mathbf{n}_K$. □

The vectors $\mathbf{n}_K$ play a fundamental role in our proof of Theorem 5.4.

Note in particular that if $H \in \binom{\mathbb{H}}{N}$ and $\ell \in H$ then we may speak of $\mathbf{n}_{H\setminus\ell}$. From now on, we use $H \setminus \ell_i$ for $H \setminus \{\ell_i\}$.

LEMMA 5.8 (de Boor's identity). *If $H \in \binom{\mathbb{H}}{N}$ then we have*

$$x = \vartheta_H + \sum_{\ell \in H} \frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_{H\setminus\ell})}\, \mathbf{n}_{H\setminus\ell}, \quad x \in \mathbb{R}^N, \tag{5.24}$$

where $\tilde{\ell}$ denotes the linear part of ℓ (thus $\tilde{\ell}(x) = \langle \mathbf{n}, x\rangle$ if $\ell(x) = \langle \mathbf{n}, x\rangle - c$). In particular, for every H, the vectors $\mathbf{n}_{H\setminus\ell}$, $\ell \in H$, form a basis of $\mathbb{R}^N$.

PROOF. See [**20**, p. 37]. □

5.4.2. de Boor's remainder formula. We now recall the definition of multivariate divided differences. Let Ω be an open convex set in $\mathbb{R}^N$, to every set $A = \{a_0, \dots, a_s\} \subset \Omega$ (the points are not necessarily distinct) and $f \in C^s(\Omega)$, we associate a s-linear form on $(\mathbb{R}^N)^s$ defined by

$$(\mathbb{R}^N)^s \ni (v_1, \dots, v_s) \longmapsto [a_0, \dots, a_s | v_1, \dots, v_s] f := \int_{[A]} D_{v_1} \dots D_{v_s} f = \int_{[A]} f^{(s)}(\cdot)(v_1, \dots, v_s), \tag{5.25}$$

where $f^{(s)}$ denotes the s-th total derivative of f,

$$\int_{[A]} g = \int_{\Delta_s} g\left(a_0 + \sum_{i=1}^{s} \xi_i (a_i - a_0)\right) d\xi_1 \dots d\xi_s$$

and Δ_s is the standard simplex $\{\xi = (\xi_1, \dots, \xi_s) : \xi_i \geq 0,\ \sum_{i=1}^{s} \xi_i \leq 1\}$. This symmetric s-linear form is called the *multivariate divided difference* of f at A. Note that, when $f \in C^s(\Omega)$ is fixed, the function

$$\Omega^{s+1} \times (\mathbb{R}^N)^s \ni (a_0, \dots, a_s, v_1, \dots, v_s) \longmapsto [a_0, \dots, a_s \,|\, v_1, \dots, v_s] f$$

is continuous (as a function of its two groups of variables).

We now state a beautiful error formula due to Carl de Boor.

THEOREM 5.9 (de Boor's remainder formula). *Let $\mathbb{H} = \{\ell_1, \dots, \ell_d\}$ be a collection of $d \geq N$ hyperplanes in general position in $\mathbb{R}^N$ and $\Theta = \Theta_{\mathbb{H}}$ the corresponding Chung-Yao lattice. For $K \in \binom{\mathbb{H}}{N-1}$, we define the polynomial P_K of degree $d - N + 1$ by the relation*

$$P_K(x) = \prod_{\ell \in \mathbb{H}\setminus K} \frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_K)}, \tag{5.26}$$

where, as above, $\tilde{\ell}$ is used for the linear part of ℓ.

The error between a function f of class C^{d-N+1} on a convex neighborhood Ω of Θ and the Lagrange interpolation polynomial of f at Θ is given by the following formula.

$$f(x) = \mathbf{L}[\Theta; f](x) + \sum_{K \in \binom{\mathbb{H}}{N-1}} P_K(x) \cdot \Big[\Theta_K, x \,|\, \underbrace{\mathbf{n}_K, \cdots, \mathbf{n}_K}_{d-N+1}\Big] f, \quad x \in \Omega. \tag{5.27}$$

Recall that for $K \in \binom{\mathbb{H}}{N-1}$, Θ_K is the subset formed by the $d-N+1$ points of Θ lying on the line $\cap_{\ell \in K} \ell$, see (5.21).

PROOF. See [**20**, Theorem 3.1.]. □

5.4.3. Some algebraic identities. We now prove two auxiliary lemmas. The first one (Lemma 5.10) shows that the $\mathbf{n}_K$'s, $K \in \binom{\mathbb{H}}{N-1}$, themselves form a certain interpolation lattice. The second one (Lemma 5.12) is a somewhat mysterious representation formula for symmetric multilinear forms.

LEMMA 5.10. *Let $\mathbb{H} = \{\ell_1, \ldots, \ell_d\}$ be a collection of d hyperplanes in general position in $\mathbb{R}^N$ with $d \geq N$. The set*

$$\mathscr{V} := \left\{ \mathbf{n}_K : K \in \binom{\mathbb{H}}{N-1} \right\} \tag{5.28}$$

is an interpolation lattice for the space $\mathscr{H}^{d-N+1}(\mathbb{R}^N)$ of homogeneous polynomials of degree $d-N+1$.

PROOF. It suffices to prove the following two assertions.

(1) The cardinality of $\mathscr{V}$ is equal to the dimension of $\mathscr{H}^{d-N+1}(\mathbb{R}^N)$ which is $\binom{d}{d-N+1} = \binom{d}{N-1}$.
(2) For every $\mathbf{n}_K$ in $\mathscr{V}$ there exists $H_K \in \mathscr{H}^{d-N+1}(\mathbb{R}^N)$ such that $H_K(\mathbf{n}_K) = 1$ but H_K vanishes on $\mathscr{V} \setminus \{\mathbf{n}_K\}$

To verify the first point, we just need to check that if $K, K' \in \binom{\mathbb{H}}{N-1}$ and $K \neq K'$ then $\mathbf{n}_K \neq \mathbf{n}_{K'}$. But, if $K \neq K'$ there exists $\ell \in K \setminus K'$ with $\ell(x) = \langle \mathbf{n}, x \rangle - c$. Assume that $K' = \{\ell_{i_1}, \ldots, \ell_{i_{N-1}}\}$. Since $\ell \cup K'$ is a set of N hyperplanes in general position, we have $\det(\mathbf{n}, \mathbf{n}_{i_1}, \ldots, \mathbf{n}_{i_{N-1}}) \neq 0$ hence, in view of (5.23), $\langle \mathbf{n}, \mathbf{n}_{K'} \rangle \neq 0$. On the other hand, since $\ell \in K$, $\langle \mathbf{n}, \mathbf{n}_K \rangle = 0$. Hence $\mathbf{n}_K \neq \mathbf{n}_{K'}$.

As for the second point, for $K \in \binom{\mathbb{H}}{N-1}$, we set

$$H_K := \tilde{P}_K(x) = \prod_{\ell \in \mathbb{H} \setminus K} \frac{\tilde{\ell}(x)}{\tilde{\ell}(\mathbf{n}_K)}. \tag{5.29}$$

This clearly defines a homogeneous polynomial of degree $d-N+1$ in $\mathbb{R}^N$ satisfying $H_K(\mathbf{n}_K) = 1$. Moreover, if $K' \in \binom{\mathbb{H}}{N-1}$, $K' \neq K$, then we can find ℓ in $(\mathbb{H} \setminus K) \cap K'$. Since $\ell \notin K$, the factor $\tilde{\ell}(\mathbf{n}_{K'})$ appears in $H_K(\mathbf{n}_{K'})$. However, since $\ell \in K'$, $\tilde{\ell}(\mathbf{n}_{K'}) = \langle \mathbf{n}, \mathbf{n}_{K'} \rangle = 0$. Hence $H_K(\mathbf{n}_{K'}) = 0$. □

The interpolation formula corresponding to the interpolation lattice — and using the polynomials $H_K = \tilde{P}_K$ in (5.29) — yields the following identity.

COROLLARY 5.11. *With the assumptions of the lemma, for every symmetric $(d-N+1)$-linear form ϕ on $\mathbb{R}^N$, we have*

$$\phi(v^{d-N+1}) = \sum_{K \in \binom{\mathbb{H}}{N-1}} \tilde{P}_K(v) \cdot \phi(\mathbf{n}_K^{d-N+1}), \quad v \in \mathbb{R}^N, \tag{5.30}$$

where we use $u^{d-N+1} := (u,\dots,u)$ *($d-N+1$ times).*

LEMMA 5.12. *Let* $\mathbb{H} = \{\ell_1,\dots,\ell_d\}$ *be a collection of hyperplanes in* $\mathbb{R}^N$ *in general position with* $d \geq N$. *We set*

$$\mathbb{H}_i = \{\ell_1,\dots,\ell_i\}, \quad 1 \leq i \leq d, \tag{5.31}$$

and

$$P_K^{[i-1]}(x) = \prod_{\ell\in\mathbb{H}_{i-1}\setminus K} \frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_K)}, \quad K \in \binom{\mathbb{H}_{i-1}}{N-1}, \quad N \leq i \leq d+1. \tag{5.32}$$

Then for every symmetric $(d-N+1)$*-linear form* ϕ *on* $\mathbb{R}^N$*, we have*

$$\phi(x^{d-N+1}) = \sum_{i=N}^{d+1} \sum_{K\in\binom{\mathbb{H}_{i-1}}{N-1}} P_K^{[i-1]}(x)\cdot\phi\left(x^{d-i},\ \vartheta_{K\cup\ell_i},\mathbf{n}_K^{i-N}\right), \quad x\in\mathbb{R}^N. \tag{5.33}$$

In the above formula, we agree that when $d-i$ (resp. $i-N$) is not positive then x (resp. $\mathbf{n}_K$) does not appear in $\phi(x^{d-i},\vartheta_{K\cup\ell_i},\mathbf{n}_K^{i-N})$, and when $i=d+1$ then x and $\vartheta_{K\cup\ell_i}$ do not appear. Likewise, if the product in the definition of $P_K^{[i-1]}$ is empty then its value is taken to be 1.

We need the following simple observation.

LEMMA 5.13. *Let* $\mathbb{H} = \{\ell_1,\dots,\ell_d+1\}$ *be a collection of hyperplanes in* $\mathbb{R}^N$ *in general position with* $d\geq N$. *As above, we write* $\mathbb{H}_d = \{\ell_1,\dots,\ell_d\}$. *Let* $K' \in \binom{\mathbb{H}_d}{N-2}$. *If* $K\in\binom{\mathbb{H}_d}{N-1}$ *and* $K' \not\subseteq K$ *then* $\tilde{P}_K^{[d]}(\mathbf{n}_{K'\cup\ell_{d+1}}) = 0$ *where*

$$\tilde{P}_K^{[d]} = \prod_{\ell\in\mathbb{H}_d\setminus K} \frac{\tilde{\ell}(\cdot)}{\tilde{\ell}(\mathbf{n}_K)}.$$

PROOF. Take $\ell_i \in K'\cap(\mathbb{H}_d\setminus K)$. The fact that $\ell_i\in K'$ gives

$$0 = \langle \mathbf{n}_i, \mathbf{n}_{K'\cup\ell_{d+1}}\rangle = \tilde{\ell}_i(\mathbf{n}_{K'\cup\ell_{d+1}})$$

and since $\ell_i\notin K$, $\tilde{\ell}_i(\mathbf{n}_{K'\cup\ell_{d+1}})$ is a factor of $\tilde{P}_K^{[d]}(\mathbf{n}_{K'\cup\ell_{d+1}})$. □

PROOF OF LEMMA 5.12. We prove identity (5.33) by induction on $d\geq N$.

(A) We start with the case $d=N$. In that case (5.33) reduces to

$$\phi(x) = \sum_{K\in\binom{\mathbb{H}_{N-1}}{N-1}} P_K^{[N-1]}(x)\phi(\vartheta_{K\cup\ell_N}) + \sum_{K\in\binom{\mathbb{H}_N}{N-1}} P_K^{[N]}(x)\phi(\mathbf{n}_K) \tag{5.34}$$

$$= \phi(\vartheta_{\mathbb{H}_N}) + \sum_{i=1}^{N} \frac{\ell_i(x)}{\tilde{\ell}_i(\mathbf{n}_{\mathbb{H}_N\setminus\ell_i})}\phi(\mathbf{n}_{\mathbb{H}_N\setminus\ell_i}). \tag{5.35}$$

Since ϕ is a linear form, the claim follows from de Boor's identity (5.24).

(B) We assume that (5.33) holds true for d and prove it for $d+1$. Take ϕ a symmetric $(d+2-N)$-linear form. Fix $y\in\mathbb{R}^N$ and define ϕ_y on $(\mathbb{R}^N)^{d+1-N}$ by $\phi_y(v_1,\dots,v_{d+1-N}) = \phi(v_1,\dots,v_{d+1-N},y)$. Thus ϕ_y is a symmetric $(d+1-N)$-linear form to which we may apply the induction hypothesis to get

$$\phi_y(x^{d-N+1}) = \sum_{i=N}^{d+1} \sum_{K\in\binom{\mathbb{H}_{i-1}}{N-1}} P_K^{[i-1]}(x)\phi_y\left(x^{d-i},\vartheta_{K\cup\ell_i},\mathbf{n}_K^{i-N}\right). \tag{5.36}$$

Putting $y = x$ in the above expression, we obtain

$$\phi(x^{d-N+2}) = \sum_{i=N}^{d+1} \sum_{K \in \binom{\mathbb{H}_{i-1}}{N-1}} P_K^{[i-1]}(x)\phi_x\left(x^{d-i}, \vartheta_{K\cup\ell_i}, \mathbf{n}_K^{i-N}\right). \tag{5.37}$$

On the other hand, we need to prove

$$\phi(x^{d-N+2}) = \sum_{i=N}^{d+2} \sum_{K \in \binom{\mathbb{H}_{i-1}}{N-1}} P_K^{[i-1]}(x)\phi\left(x^{d+1-i}, \vartheta_{K\cup\ell_i}, \mathbf{n}_K^{i-N}\right). \tag{5.38}$$

Expressions (5.37) and (5.38) differ only for $i = d+1$ and $i = d+2$. Thus, to establish (5.38), it suffices to prove that the term corresponding to $d+1$ in (5.37) equals the sum of the terms corresponding to $d+1$ and $d+2$ in (5.38), that is

$$\begin{aligned} &\sum_{K \in \binom{\mathbb{H}_d}{N-1}} P_K^{[d]}(x)\phi\left(\mathbf{n}_K^{d+1-N}, x\right) \\ &\quad = \sum_{K \in \binom{\mathbb{H}_d}{N-1}} P_K^{[d]}(x)\phi\left(\vartheta_{K\cup\ell_{d+1}}, \mathbf{n}_K^{d+1-N}\right) + \sum_{K \in \binom{\mathbb{H}_{d+1}}{N-1}} P_K^{[d+1]}(x)\phi\left(\mathbf{n}_K^{d+2-N}\right). \end{aligned} \tag{5.39}$$

For $K \in \binom{\mathbb{H}_d}{N-1}$ and $x \in \mathbb{R}^N$, using de Boor's identity (5.24) with $H = K \cup \ell_{d+1}$, we may write

$$\begin{aligned} x &= \vartheta_{K\cup\ell_{d+1}} + \sum_{\ell \in K\cup\ell_{d+1}} \frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_{(K\cup\ell_{d+1})\setminus\ell})} \mathbf{n}_{(K\cup\ell_{d+1})\setminus\ell} \\ &= \vartheta_{K\cup\ell_{d+1}} + \frac{\ell_{d+1}(x)}{\tilde{\ell}_{d+1}(\mathbf{n}_K)}\mathbf{n}_K + \sum_{\ell\in K} \frac{\ell(x)}{\tilde{\ell}\left(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}}\right)} \mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}}. \end{aligned}$$

Substituting x with the above expression in the left hand side of (5.39), we arrive to

$$\begin{aligned} \sum_{K \in \binom{\mathbb{H}_d}{N-1}} P_K^{[d]}(x)\cdot\phi\left(\mathbf{n}_K^{d+1-N}, x\right) &= \sum_{K \in \binom{\mathbb{H}_d}{N-1}} P_K^{[d]}(x)\cdot\phi\left(\vartheta_{K\cup\ell_{d+1}}, \mathbf{n}_K^{d+1-N}\right) \\ &+ \sum_{K \in \binom{\mathbb{H}_d}{N-1}} P_K^{[d]}(x)\,\frac{\ell_{d+1}(x)}{\tilde{\ell}_{d+1}(\mathbf{n}_K)}\cdot\phi\left(\mathbf{n}_K^{d+2-N}\right) \\ &+ \sum_{K \in \binom{\mathbb{H}_d}{N-1}} \sum_{\ell\in K} P_K^{[d]}(x)\,\frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}})}\cdot\phi\left(\mathbf{n}_K^{d+1-N}, \mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}}\right). \end{aligned} \tag{5.40}$$

Now, for $K \in \binom{\mathbb{H}_d}{N-1}$, we have

$$P_K^{[d]}(x)\,\frac{\ell_{d+1}(x)}{\tilde{\ell}_{d+1}(\mathbf{n}_K)} = P_K^{[d+1]}(x).$$

Hence, the second term on the right hand side of (5.40) is

$$\sum_{K \in \binom{\mathbb{H}_{d+1}}{N-1},\ \ell_{d+1}\notin K} P_K^{[d+1]}(x)\cdot\phi\left(\mathbf{n}_K^{d+2-N}\right). \tag{5.41}$$

Thus, since $K \in \binom{\mathbb{H}_{d+1}}{N-1}$, $\ell_{d+1} \in K$ means $K = K' \cup \{\ell_{d+1}\}$ with $K' \in \binom{\mathbb{H}_d}{N-2}$, to prove (5.39), it remains to establish

$$\sum_{K \in \binom{\mathbb{H}_d}{N-1}} \sum_{\ell \in K} P_K^{[d]}(x)\, \frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}})} \cdot \phi\left(\mathbf{n}_K^{d+1-N}, \mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}}\right) = \sum_{K' \in \binom{\mathbb{H}_d}{N-2}} P_{K'\cup\ell_{d+1}}^{[d+1]}(x) \cdot \phi\left(\mathbf{n}_{K\cup\ell_{d+1}}^{d+2-N}\right). \tag{5.42}$$

We first concentrate on the term $P_K^{[d]}(x)\frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}})}$ on the left hand side of (5.42). Since $\ell \in K$ we have

$$P_K^{[d]}(x)\frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}})} = \left\{\prod_{h\in\mathbb{H}_d\setminus K}\frac{h(x)}{\tilde{h}(\mathbf{n}_K)}\right\}\cdot\frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}})} \tag{5.43}$$

$$= \left\{\prod_{h\in\mathbb{H}_d\setminus(K\setminus\ell)}\frac{h(x)}{\tilde{h}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}})}\right\}\cdot\left\{\prod_{h\in\mathbb{H}_d\setminus K}\frac{\tilde{h}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}})}{\tilde{h}(\mathbf{n}_K)}\right\} \tag{5.44}$$

$$= P_{(K\setminus\ell)\cup\ell_{d+1}}^{[d+1]}(x)\cdot\tilde{P}_K^{[d]}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}}). \tag{5.45}$$

Using this expression in the left hand side of (5.42), we arrive at

$$\begin{aligned}
&\sum_{K \in \binom{\mathbb{H}_d}{N-1}} \sum_{\ell \in K} P_K^{[d]}(x)\, \frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}})} \cdot \phi\left(\mathbf{n}_K^{d-N+1}, \mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}}\right)\\
&\quad= \sum_{K \in \binom{\mathbb{H}_d}{N-1}} \sum_{\ell \in K} P_{(K\setminus\ell)\cup\ell_{d+1}}^{[d+1]}(x)\, \tilde{P}_K^{[d]}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}}) \cdot \phi\left(\mathbf{n}_K^{d-N+1}, \mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}}\right)\\
&\quad\quad= \sum_{K' \in \binom{\mathbb{H}_d}{N-2}} P_{K'\cup\ell_{d+1}}^{[d+1]}(x) \sum_{K'\subset K \in \binom{\mathbb{H}_d}{N-1}} \tilde{P}_K^{[d]}(\mathbf{n}_{K'\cup\ell_{d+1}}) \cdot \phi\left(\mathbf{n}_K^{d-N+1}, \mathbf{n}_{K'\cup\ell_{d+1}}\right).
\end{aligned} \tag{5.46}$$

Now, for a fixed $K' \in \binom{\mathbb{H}_d}{N-2}$, using Lemma 5.13 for the first equality (we add null terms) and Corollary 5.11 for the second one, we get

$$\sum_{K'\subset K \in \binom{\mathbb{H}_d}{N-1}} \tilde{P}_K^{[d]}(\mathbf{n}_{K'\cup\ell_{d+1}})\phi(\mathbf{n}_K^{d-N+1}, \mathbf{n}_{K'\cup\ell_{d+1}}) = \sum_{K \in \binom{\mathbb{H}_d}{N-1}} \tilde{P}_K^{[d]}(\mathbf{n}_{K'\cup\ell_{d+1}})\phi(\mathbf{n}_K^{d-N+1}, \mathbf{n}_{K'\cup\ell_{d+1}}) = \phi(\mathbf{n}_{K'\cup\ell_{d+1}}^{d-N+2}). \tag{5.47}$$

Using (5.47) in the last term of (5.46), we finally arrive at

$$\sum_{K \in \binom{\mathbb{H}_d}{N-1}} \sum_{\ell \in K} P_K^{[d]}(x)\frac{\ell(x)}{\tilde{\ell}(\mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}})}\phi(\mathbf{n}_K^{d-N+1}, \mathbf{n}_{(K\setminus\ell)\cup\ell_{d+1}}) = \sum_{K' \in \binom{\mathbb{H}_d}{N-2}} P_{K'\cup\ell_{d+1}}^{[d+1]}(x) \cdot \phi\left(\mathbf{n}_{K'\cup\ell_{d+1}}^{d-N+2}\right), \tag{5.48}$$

which is (5.42). This completes the proof of the lemma. □

COROLLARY 5.14. *Let $\mathbb{H}=\{\ell_1,\dots,\ell_d\}$ be a collection of $d\geq N$ hyperplanes in general position in $\mathbb{R}^N$. For every function f of class C^{d-N+1} on a convex neighborhood Ω of the origin in $\mathbb{R}^N$ we have*

$$f(x)-\mathbf{T}_0^{d-N}(f)(x) = \sum_{i=N}^{d+1}\sum_{K\in\binom{\mathbb{H}_{i-1}}{N-1}} P_K^{[i-1]}(x)\cdot \int_{\big[\underbrace{0,\dots,0}_{d-N+1},x\big]} f^{(d-N+1)}(\cdot)\big(x^{d-i},\vartheta_{K\cup\ell_i},\mathbf{n}_K^{i-N}\big),\quad x\in\Omega.$$

PROOF. The remainder formula for Taylor polynomial (as a special case of Kergin interpolation, see e.g. [**36**, Theorem 3]) gives us,

$$f(x)-\mathbf{T}_0^{d-N}(f)(x)=\big[\underbrace{0,\dots,0}_{d-N+1},x\,|\,\underbrace{x,\dots,x}_{d-N+1}\big]f=\int_{[0,\dots,0,x]} f^{(d-N+1)}(\cdot)(x,\dots,x). \tag{5.49}$$

The corollary then follows directly from Lemma 5.12 since, for every $a\in\Omega$, $f^{(d-N+1)}(a)$ is a symmetric $(d-N+1)$-linear form on $\mathbb{R}^N$. ☐

5.4.4. Proof of Theorem 5.4. Let Ω be a neighborhood of the origin on which f is of class C^{d+1}. We may assume that

(i) Ω contains $B(0,R)$, the closed euclidean ball of center the origin and radius R and, in view of condition $(C1)$,

(ii) all the points of $\Theta^{(s)}=\Theta_{\mathbb{H}^{(s)}}$ lie in $B(0,R)$, $s\in\mathbb{N}$.

We set

$$M=\max_{a\in B(0,R)}\|f^{(d-N+1)}(a)\|<\infty, \tag{5.50}$$

where $\|\cdot\|$ here denotes the usual norm of a multilinear form. We use condition (C2) in the form given by (5.14) taking (5.23) into account as follows.

(iii) There exists $\delta>0$ such that

$$|\langle\mathbf{n}_i,\mathbf{n}_K\rangle|\geq\delta,\quad K\in\binom{\mathbb{H}^{(s)}}{N-1},\quad \ell_i\notin K,\quad s\in\mathbb{N}. \tag{5.51}$$

Here we drop the upper indecies s on the vectors and hyperplanes corresponding to $\mathbb{H}^{(s)}$.

(A) We first derive an estimate on the polynomials $P_K=P_K^{[d]}$ defined in (5.26). We claim that

$$|P_K^{[d]}(x)|\leq\left(\frac{2R}{\delta}\right)^{d-N+1},\quad x\in B(0,R),\quad K\in\binom{\mathbb{H}^{(s)}}{N-1},\quad s\in\mathbb{N}. \tag{5.52}$$

Indeed, if $K\in\binom{\mathbb{H}^{(s)}}{N-1}$ and $\ell_i\in\mathbb{H}^{(s)}\setminus K$, since $\vartheta_{K\cup\ell_i}\in\ell_i$, we have

$$|c_i|=|\langle\mathbf{n}_i,\vartheta_{K\cup\ell_i}\rangle|\leq\|\vartheta_{K\cup\ell_i}\|\leq R.$$

Next, using (5.51) and $\|\mathbf{n}_i\|=1$, we have

$$\left|\frac{\ell_i(x)}{\tilde{\ell}_i(\mathbf{n}_K)}\right|\leq\frac{|\langle\mathbf{n}_i,x\rangle|+|c_i|}{|\langle\mathbf{n}_i,\mathbf{n}_K\rangle|}\leq\frac{2R}{\delta},\quad \ell_i\in\mathbb{H}^{(s)}\setminus K, \tag{5.53}$$

which readily implies (5.52).

(B) We now use Theorem 5.9 and Corollary 5.14 to estimate the difference between a Taylor polynomials and a Chung-Yao interpolation polynomial of a same function. To simplify, we omit the index s in the formulas. We have

$$\begin{aligned}\mathbf{L}[\Theta;f](x)-\mathbf{T}_0^{d-N}(f)(x) &= \Big[f(x)-\mathbf{T}_0^{d-N}(f)(x)\Big]-[f(x)-\mathbf{L}[\Theta;f](x)] \\ &= \sum_{K\in\binom{\mathbb{H}_d}{N-1}} P_K^{[d]}(x)\Big([0,\ldots,0,x\,|\,\mathbf{n}_K,\ldots,\mathbf{n}_K]f-[\Theta_K,x\,|\,\mathbf{n}_K,\cdots,\mathbf{n}_K]f\Big) \\ &\quad+\sum_{i=N}^{d}\sum_{K\in\binom{\mathbb{H}_{i-1}}{N-1}} P_K^{[i-1]}(x)\int_{[0,\ldots,0,x]} f^{(d-N+1)}(\cdot)(x^{d-i},\vartheta_{K\cup\ell_i},\mathbf{n}_K^{i-N}),\quad x\in B(0,R).\end{aligned} \tag{5.54}$$

We call $S_1(x)$ and $S_2(x)$ the terms in the above sum and prove that, for every $x\in B(0,R)$, both of them tends to 0 as $s\to\infty$. This will achieve the proof (since simple convergence on a compact set of nonempty interior implies convergence on any normed vector space topology on $\mathscr{P}^{d-N}(\mathbb{R}^N)$).

(C) Since, in view of (5.52), the polynomials $P_K^{[d]}$ are bounded uniformly in s, that $S_1(x)\to 0$ for $x\in B(0,R)$ follows from

$$\Big|[\underbrace{0,\ldots,0}_{d-N+1},x|\underbrace{\mathbf{n}_K,\ldots,\mathbf{n}_K}_{d-N+1}]f-[\Theta_K,x|\mathbf{n}_K,\cdots,\mathbf{n}_K]f\Big|\to 0 \tag{5.55}$$

which is a consequence of the fact that the points of $\Theta=\Theta^{(s)}$ tend to 0 together with the continuity of the divided differences of f as a function of the two groups of its arguments, see Subsection 5.4.2.

(D) As for the term $S_2(x)$, since the right hand side goes to 0 as $s\to\infty$, the conclusion follows from the following estimate.

$$|S_2(x)|\le\frac{M}{(d-N+1)!}R^{d-N}\left(1+\frac{2}{\delta}\right)^{d-1}\|\Theta\|,\quad x\in B(0,R), \tag{5.56}$$

where $\|\Theta\|=\|\Theta^{(s)}\|:=\max\{\|\vartheta\|\,:\,\vartheta\in\Theta\}$. To prove this, we observe that if $N\le i\le d$ and $K\in\binom{\mathbb{H}_{i-1}}{N-1}$, the bound (5.52) (in which $\mathbb{H}$ is replaced by $\mathbb{H}_{i-1}$) gives

$$|P_K^{[i-1]}(x)|\le\left(\frac{2R}{\delta}\right)^{i-N},\quad x\in B(0,R). \tag{5.57}$$

Moreover, for every $a\in B(0,R)$, using $\|\mathbf{n}_K\|\le 1$, we have

$$\Big|f^{(d-N+1)}(a)(x^{d-i},\vartheta_{K\cup\ell_i},\mathbf{n}_K^{i-N})\Big|\le M\|x\|^{d-i}\|\vartheta_{K\cup\ell_i}\|\|\mathbf{n}_K\|^{i-N}\le MR^{d-i}\cdot\|\Theta\|. \tag{5.58}$$

Hence, since $\mathrm{vol}(\Delta_{d-N+1})=1/(d-N+1)!$, for $x\in B(0,R)$ we have

$$\left|\int_{[0,\ldots,0,x]} f^{(d-N+1)}(\cdot)(x^{d-i},\vartheta_{K\cup\ell_i},\mathbf{n}_K^{i-N})\right|\le\frac{M}{(d-N+1)!}R^{d-i}\|\Theta\|. \tag{5.59}$$

Combining the above estimates, we obtain

$$|S_2(x)| \leq \sum_{i=N}^{d} \binom{i-1}{N-1} \left(\frac{2R}{\delta}\right)^{i-N} \frac{M}{(d-N+1)!} R^{d-i} \|\Theta\| \tag{5.60}$$

$$= \frac{M}{(d-N+1)!} \|\Theta\| R^{d-N} \sum_{i=N}^{d} \binom{i-1}{i-N} \left(\frac{2}{\delta}\right)^{i-N} \tag{5.61}$$

$$\leq \frac{M}{(d-N+1)!} \|\Theta\| R^{d-N} \sum_{j=0}^{d-1} \binom{d-1}{j} \left(\frac{2}{\delta}\right)^{j} \tag{5.62}$$

$$= \frac{M}{(d-N+1)!} \|\Theta\| R^{d-N} \left(1+\frac{2}{\delta}\right)^{d-1}. \tag{5.63}$$

This concludes the proof of Theorem 5.4.

5.4.5. An estimate on the error. The proof actually yields some estimate on the error between Chung-Yao interpolation polynomials and the Taylor polynomial at the origin. It is shown in the following corollary.

COROLLARY 5.15. *We assume that the assumptions of Theorem 5.4 are satisfied. If $f \in C^{d-N+2}(\Omega)$ then*

$$\max_{x\in B(0,R)} \|\mathbf{L}[\Theta^{(s)}; f](x) - \mathbf{T}_0^{d-N}(f)(x)\| = O\left(\|\theta^{(s)}\| \cdot \left\{ \max_{a\in B(0,R)} \|f^{(d-N+1)}(a)\| + \max_{a\in B(0,R)} \|f^{(d-N+2)}(a)\| \right\}\right). \tag{5.64}$$

where the constant involved in the symbol O does not depend on f.

PROOF. We turn to the term $S_1(x)$ in the previous proof. For simplicity, we set $m = d-N+1$. Since $f \in C^{m+1}(\Omega)$, for all $K \in \binom{\mathbb{H}_d}{N-1}$, the mean value inequality gives

$$\begin{aligned} &|[0,\ldots,0,x|\mathbf{n}_K,\ldots,\mathbf{n}_K]f - [\Theta_K, x|\mathbf{n}_K,\ldots,\mathbf{n}_K]f| \\ &= \left| \int_{\Delta_m} \left\{ f^{(m)}\Big(x + \sum_{j=1}^{m} (0-x)\xi_j\Big)(\mathbf{n}_K^m) - f^{(m)}\Big(x + \sum_{j=1}^{m} (\theta_{Kj}-x)\xi_j\Big)(\mathbf{n}_K^m) \right\} d\xi \right| \\ &\leq \int_{\Delta_m} \max_{B(0,R)} \|f^{(m+1)}\| \left\| \sum_{j=1}^{m} \theta_{Kj}\xi_j \right\| \|\mathbf{n}_K\|^m d\xi \leq \frac{1}{m!} \max_{B(0,R)} \|f^{(m+1)}\| \|\Theta\|, \end{aligned} \tag{5.65}$$

where $\Theta_K = \{\theta_{Kj} : i = 1,\ldots,m\}$. Using (5.56) and (5.65) in (5.54), we finally get

$$\begin{aligned} &\max_{x\in B(0,R)} \|\mathbf{L}[\Theta, f](x) - \mathbf{T}_0^{d-N}(f)(x)\| \\ &\leq \binom{d}{N-1} \left(\frac{2R}{\delta}\right)^{d-N+1} \frac{1}{(d-N+1)!} \max_{B(0,R)} \|f^{(d-N+2)}\| \|\Theta\| \\ &\quad + \frac{1}{(d-N+1)!} \max_{B(0,R)} \|f^{(d-N+1)}\| R^{d-N} \left(1+\frac{2}{\delta}\right)^{d-1} \|\Theta\| \\ &= \Big(M_1 \max_{B(0,R)} \|f^{(d-N+1)}\| + M_2 \max_{B(0,R)} \|f^{(d-N+2)}\|\Big) \|\Theta\|. \end{aligned} \tag{5.66}$$

□

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Résumé

Cette thèse traite de l'interpolation polynomiale des fonctions d'une ou plusieurs variables. Nous nous intéresserons principalement à l'interpolation de Lagrange mais un de nos travaux concerne les interpolations de Kergin et d'Hakopian. Nous dénotons par K le corps de base qui sera toujours R ou C, $P_d(K^N)$ l'espace des polynômes de N variables et de degré au plus d à coefficients dans K. Un ensemble A dans K^N contenant autant de points que la dimension de $P_d(K^N)$ est dit unisolvent s'il n'est pas contenu dans l'ensemble des zéros d'un polynôme de degré d. Pour toute fonction f définie sur A, il existe un unique L[A;f] dans $P_d(K^N)$ tel que L[A;f]=f sur A, appelé le polynôme d'interpolation de Lagrange de f en A. Les polynômes d'interpolation de Kergin et d'Hakopian sont deux généralisations naturelles en plusieurs variables de l'interpolation de Lagrange à une variable. La construction de ces polynômes nécessite le choix de points à partir desquels on construit certaines formes linéaires qui sont des moyennes intégrales et qui fournissent les conditions d'interpolation.

La qualité des approximations fournies par les polynômes d'interpolation dépend pour une large mesure du choix des points d'interpolation. Cette qualité est mesurée par la croissance de la norme de l'opérateur linéaire qui à toute fonction continue associe son polynôme d'interpolation. Cette norme est appelée la constante de Lebesgue (associée au compact et aux points d'interpolation considérés). La majeure partie de cette thèse est consacrée à l'étude de cette constante. Nous donnons par exemples le premier exemple général explicite de familles de points possédant une constante de Lebesgue qui croit comme un polynôme. C'est une avancée significative dans ce domaine de recherche.