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*Optimal Capital Structure with Endogenous Bankruptcy:
Payouts, Tax Benefits Asymmetry and Volatility Risk*

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INTRODUCTION

The dissertation deals with modeling credit risk through a structural model approach. The thesis consists of three papers ([3], [4], [1]) in which we build on the capital structure of a firm proposed by Leland and we study different extensions of his seminal paper [17] with the purpose of obtaining results more in line with historical norms and empirical evidence, studying in details all mathematical aspects.

Modeling credit risk aims at developing and applying option pricing techniques in order to study corporate liabilities and analyzing which is the market perception about the credit quality of a firm. The theory is focused on the capital structure of a firm subject to default risk. In order to study the default process, two different approaches exist in credit risk literature: structural and reduced form models.

For each firm the decision about its own corporate capital structure is a very complex choice since it is affected by a large number of economic and financial factors. Structural models of credit risk represent an analytical framework in which the capital structure of a firm is analyzed in terms of derivatives contracts. This idea has been proposed at first in Merton's work [23] considering Black and Scholes [6] option pricing theory to model the debt issued by a firm. Structural models consider the dynamic of firm's activities value as a determinant of the default time, providing a link between the credit quality of a firm and its financial and economic conditions. The main idea is that default is strictly related to the evolution of firm's activities value. The structural approach considers the value of the firm as the state variable and assume its dynamic being described by a stochastic process in order to determine the time of default. From an economic point of view default is determined by the inability of the firm to cover its debt obligations. From a mathematical point of view default is triggered by firm's value crossing a specified level. Depending on the model, this threshold is assumed to be exogenous or endogenously determined, providing different economic and mathematical implications. Default is endogenously related to firm's parameters and derived within the model following a structural approach, while it is completely exogenous in reduced form models: this is the reason why structural models are also defined *firm value models* while reduced form models are named *intensity models*.

While a direct link between the evolution of firm's value and the time of default is assumed in structural models, the reduced form approach does not consider an explicit relation between these two factors, specifying a default process governing bankruptcy and thus providing different insights from both economic and mathematical point of view. Reduced form approach treats default as the first jump of an exogenously given jump process (independent from firm's value), usually a Poisson-like process, from no-default to default and the probability of a jump in a given time interval is governed by the *default intensity* (or *hazard rate*) (e.g. see [14], [8]): default arrives as a "surprise", it is a totally inaccessible hitting time. Assuming a continuous stochastic process describing firm's assets

value and complete information about asset value and default barrier, structural models treat the bankruptcy time as a predictable stopping time.

Among firm value models, the first structural model is [23]. Merton assumes that the capital structure of the firm is composed by equity and a single liability with promised final payoff B . The ability of the firm to cover its obligations depends on the total value of its assets V . Debt can be seen as a claim on V : it is a zero-coupon bond with maturity T and face value B . As suggested in [6], by issuing debt, equity holders are selling the firm's assets to bond holders and keeping a call option to buy back the assets. Equivalently we can see the same problem as: equity holders own firm's assets and buy a put option from bond holders. In such a framework equity can be represented as a European call option with underlying asset represented by firm's value V and strike price equal to face value of debt B . This model allows for default only at maturity: at time T the firm will default if its value V is lower than the barrier B . The economic reason is that in such a case the firm is not able to pay its obligations to bond holders since its assets are below the value of outstanding debt. The criticism towards this approach relies on the fact that default can occur only at maturity, which is not quite realistic.

A natural extension are the so called *first passage models* allowing for possible default before the maturity of debt. This approach is firstly proposed in [5], where default is triggered by the *first time* a certain threshold is reached by firm value. First passage models define default as the first time the firm's assets value crosses a lower barrier, letting default occur at any time. The default barrier can be exogenously fixed or endogenously determined: when it is an exogenous level, as in [5], [21], it works as a covenant protecting bond holders, otherwise it is endogenously determined as a consequence of equity holders maximizing behavior as in [17]. In such a case the the firm is liquidated immediately after the default triggering event.

Our research focus on structural models of credit risk with endogenous default in the spirit of Leland's model [17]¹. As starting point the author claims: *"The value of corporate debt and capital structure are interlinked variables. Debt values (and therefore yield spreads) cannot be determined without knowing the firm's capital structure, which affects the potential for default and bankruptcy. But capital structure cannot be optimized without knowing the effect of leverage on debt value"*.

In the classical Leland [17] framework firm's assets value evolves as a geometric Brownian motion and an infinite time horizon is considered. The firm realizes its capital from both debt and equity. Moreover, the firm has only one perpetual debt outstanding, which pays a constant coupon stream C per instant of time and this determines tax benefits proportional to coupon payments. This assumption of perpetual debt can be justified, as Leland suggests, thinking about two alternative scenarios: a debt with very long maturity

¹[17] has been awarded with the first Stephen A. Ross Prize in Financial Economics: *"...the prize committee chose this paper because of the substantial influence it has given on research about capital structure and corporate debt valuation..."*, see [16].

(in this case the return of principal has no value) or a debt which is continuously rolled over at a fixed interest rate (as in [19]). Assuming an infinite time horizon is a reasonable first approximation for long term corporate debt and enables to have an analytic framework where all corporate securities depending on the underlying variable (firm value) are time independent, thus obtaining closed form solutions. Bankruptcy is triggered endogenously by the inability of the firm to raise sufficient capital to meet its current obligations. On the failure time T , agents which hold debt claims will get the residual value of the firm (because of bankruptcy costs), and those who hold equity will get nothing (meaning the strict priority rule holds). The riskiness of the firm is assumed to be constant: given limited liability of equity, as [15] suggest, equity holders may have incentives to increase the riskiness of the firm, while the opposite happens for debt holders, since a higher volatility decreases debt value (asset substitution problem, see also [18]).

While a huge theoretical literature on risky corporate debt pricing exists, less attention has been paid on empirical tests of these models. The main empirical results in credit risk literature emphasize a poor job of structural models in predicting credit spreads for short maturities. To simplify the discussion two main motivations of structural models failure in predicting bond spreads could be: i) failure in predicting the credit exposure; ii) influence of other non credit related variables on corporate debt spreads. Moreover, structural models usually overestimates leverage ratios.

We depart from Leland's work generalizing the model in different directions with the aim at obtaining results more in line with empirical evidence, further providing detailed mathematical proofs of all results. We extend it by introducing payouts in [2, 3], then, keeping this assumption, in [4] we consider an even more realistic framework under an asymmetric corporate tax schedule. As Leland suggests in [20], a possible way to follow in order to improve empirical predictions of structural models is to modify some critical hypothesis, mainly in the direction of introducing jumps and/or removing the assumption of constant volatility in the underlying firm's assets value stochastic evolution. This latter case is what motivates [1].

In [2], [3] (**Paper 1**) we extend Leland model [17] to the case where the firm has net cash outflows resulting from payments to bondholders or stockholders, for instance if dividends are paid to equity holders, and we study its effect on all financial variables and on the choice of optimal capital structure. The interest in this problem is posed in [17] section VI-B, nevertheless the resulting optimal capital structure is not analyzed in detail. Our aim is twofold: from one hand we complete the study of corporate debt and optimal leverage in the presence of payouts in all analytical aspects, from the other hand we study numerically the effects of this variation on the capital structure. Moreover in [3] we conduct a quantitative analysis on the effects of payouts on the probability of default, on expected time to default and on agency costs from both an economic and mathematical point of view. We follow Leland [17] by considering only cash outflows which are proportional to firm's assets value but our analysis differs from Leland's one since we solve the optimal control problem as an optimal stopping problem (see also [7] for a similar

approach) and not with ordinary differential equations. We find that the increase of the payout rate parameter δ affects not only the level of endogenous bankruptcy, but modifies the magnitude of a change on the endogenous failure level as a consequence of an increase in risk free rate, corporate tax rate, riskiness of the firm and coupon payments. Further the introduction of payouts allows to obtain lower optimal leverage ratios and higher yield spreads, compared to Leland's [17] results, thus making these empirical predictions more in line with historical norms. Our analysis suggests that adding payouts has an actual influence on all financial variables: for an arbitrary coupon level C , a positive payout rate δ increases equity value and decreases both debt and total value of the firm, making bankruptcy more likely. In line with [19] suggestion, in [3] we show that the probability of default is quite dependent on the drift of the process describing firm's activities value, thus on payouts. Analyzing cumulative probability of going bankruptcy over a period longer than 10 years suggests that introducing δ makes debt riskier, strongly increasing the likelihood of default. Studying the influence of payouts on the *asset substitution problem* we find that this problem still exists and its magnitude is increased by payouts, making higher potential agency costs arising from the model.

In [4] (**Paper 2**) we keep the introduction of a company's assets payout ratio δ as in [2],[3] and extend the model proposed by [17] in the direction of a switching (even debt dependent) in tax savings. Taxes are a crucial economic variable affecting optimal capital structure, as early recognized by [24] and observed by [25]. While structural models assume constant corporate tax rates, Leland argues that default and leverage decisions might be affected by non constant corporate tax rates, because a loss of tax advantages is possible for low firm values. The empirical analysis of [11] confirms that the corporate tax schedule is asymmetric, in most cases it is convex, and [25] suggests that tax convexity cannot be ignored in corporate financing decisions. We assume the corporate tax schedule based on two different corporate tax rates: the switching from a corporate tax rate to the other is determined by firm's activities value crossing a critical barrier. We consider two different frameworks: at first, the switching barrier is assumed to be a constant exogenous level; secondly, we analyze an even more realistic scenario in which this level depends upon the amount of debt the firm has issued. We obtain an explicit form for the tax benefit claim, which allows us to study monotonicity and convexity of the equity function, to find the endogenous failure level in closed form in case of no-payouts and to prove its existence and uniqueness in the general case with both asymmetric tax scheme and payouts, while literature in this field usually gives only numerical results. Our approach differs from [17] since we solve the optimal control problem as an optimal problem in the set of passage times; the key method is the Laplace transform of the stopping failure time. Our results show that tax asymmetry increases the optimal failure level reduces the optimal leverage ratio, and that this last effect is more pronounced, thus confirming the results in [25]. Nevertheless, as far as the magnitude is concerned, introducing payouts increases this negative effect on both leverage and debt. Optimal leverage ratios are smaller than in case of a flat tax schedule since the potential loss in tax benefits makes debt less attractive

and this effect is more pronounced when we deal with a debt dependent switching barrier (i.e. depending on coupon payments). The joint influence of payouts and corporate tax asymmetry produces a significant impact on corporate financing decisions: it drops down leverage ratios to empirically representative values suggesting a possible way to explain difference in observed leverage across firms facing different tax-code provisions.

In [1] (**Paper 3**) the aim is to analyze the capital structure of a firm under an infinite time horizon removing the assumption of constant volatility and considering default as endogenously triggered. We introduce a process describing the dynamic of the diffusion coefficient driven by a one factor mean-reverting process of Orstein-Uhlenbeck type, negatively correlated with firm's assets value evolution. Differently from a pure Leland framework and even from the more general context with payouts and asymmetric corporate tax rates studied in [2, 3, 4], inside this framework we cannot obtain explicit expressions for all the variables involved in the capital structure by means of the Laplace transform of the stopping failure time. The key point relies in the fact that this transform, which was the key tool used in [2, 3, 4], is not available in closed form under our stochastic volatility framework. Nevertheless debt, equity, bankruptcy costs and tax benefits are claims on the firm's assets, thus we apply ideas and techniques developed in [9] for the pricing of derivatives securities whose underlying asset price's volatility is characterized by means of its time scales fluctuations. This approach has been applied in [10] to price a defaultable zero coupon bond. Here a one-factor stochastic volatility model is considered and single perturbation theory as in [9] is applied in order to find approximate closed form solutions for derivatives involved in our economic problem. Each financial variable is analyzed in terms of its *approximate* value through an asymptotic expansion depending on the volatility mean-reversion speed. Moreover, we study the effects of the stochastic volatility assumption on the endogenous failure level determined by equity holders maximizing behavior. Under our approach, the failure level derived from standard smooth-fit principle is not the solution of the optimal stopping problem, but only represents a lower bound which has to be satisfied due to limited liability of equity. Choosing that failure level would mean a *non-optimal exercise* of the option to default. A *corrected smooth-pasting condition* must be applied in order to find the endogenous failure level solution of the optimal stopping problem. Moreover, we show the convergence of our results to Leland case [17] as the particular case of zero-perturbation. By taking into account the stochastic volatility risk component of the firm's asset dynamic, our aim is to better capture extreme returns behavior which could be a robust way to improve empirical predictions about spreads and leverage. Introducing randomness in volatility allows to deal with a structural model in which the distribution of stock prices returns is not symmetric: in our mind this seems to be the right way for capturing what structural models are not able to explain with a constant diffusion coefficient. The numerical results show that the assumption of stochastic volatility model produces relevant effects on the optimal capital structure in terms of higher credit spreads and lower leverage ratios, if compared with the original Leland case.

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I

Payouts



An Endogenous Bankruptcy Model with Firm's Net Cash Payouts

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Abstract

In this paper a structural model of corporate debt is analyzed following an approach of optimal stopping problem. We extend Leland model [7] introducing a payout δ paid to equity holders and studying its effect on corporate debt and optimal capital structure. Varying the parameter δ affects not only the level of endogenous bankruptcy, which is decreased, but modifies the magnitude of a change on the endogenous failure level as a consequence of an increase in risk free rate, corporate tax rate, riskiness of the firm and coupon payments. Concerning the optimal capital structure, the introduction of this payout allows to obtain results more in line with historical norms: lower optimal leverage ratios and higher yield spreads, compared to Leland's [7] results.

1 Introduction

Many firm value models have been proposed since Merton's work [11] which provides an analytical framework in which the capital structure of a firm is analyzed in terms of derivatives contracts. We focus on the corporate model proposed by Leland [7] assuming that the firm's assets value evolves as a geometric Brownian motion. The firm realizes its capital from both debt and equity. Debt is perpetual, it pays a constant coupon C per instant of time and this determines tax benefits proportional to coupon payments. Bankruptcy is determined endogenously by the inability of the firm to raise sufficient equity capital to cover its debt obligations. On the failure time T , agents which hold debt claims will get the residual value of the firm (because of bankruptcy costs), and those who hold equity will get nothing (the strict priority rule holds). This paper examines the case where the firm has net cash outflows resulting from payments to bondholders or stockholders, for instance if dividends are paid to equity holders. The interest in this problem is posed in [7] section VI-B, nevertheless the resulting optimal capital structure is not analyzed in detail. The aim of this paper is twofold: from one hand we complete the study of corporate

debt and optimal leverage in the presence of payouts in all analytical aspects, from the other hand we study numerically the effects of this variation on optimal capital structure. We will follow Leland [7] by considering only cash outflows which are proportional to firm's assets value but our analysis differs from Leland's one since we solve the optimal control problem as an optimal stopping problem (see also [3] for a similar approach). As in Leland model we assume that capital structure decisions, once made, are not changed through time. This means, for example, that the face value of debt is supposed to be constant and our analysis approaches the problem of determining the optimal amount of debt and the optimal endogenous failure level as a two-stage optimization problem (see also [1]). Equity holders have to chose both: i) the optimal amount of coupon payments C^* , ii) the optimal endogenous triggering failure level V_B^* . These decisions are strongly interrelated and not easily considered as separable ones. This interconnection is strictly related to the conflict between equity and debt holders, since they have different interests on the firm, leading also to potential agency costs arising from the model (see [8]). When equity holders have to chose the optimal amount of debt, i.e. coupon maximizing the total value of the firm, this will obviously depend on the endogenous failure level. At the same time, when choosing the optimal stopping time of the option to default, meaning finding the endogenous failure level, this will obviously depend on the amount of debt issued. An approximation to solve these complicated issues arising from equity and debt holders' conflict, will be to proceed in a two-step analysis as follows: i) the first-stage optimization problem is to determine the endogenous failure level; ii) in the second stage we determine the amount of debt which maximizes the total value of the firm, given the result about the default triggering level of the first stage.

Our findings show that the increase of the payout parameter δ affects not only the level of endogenous bankruptcy, which is decreased, but modifies the magnitude of a change on the endogenous failure level as a consequence of an increase in risk free rate, corporate tax rate, riskiness of the firm and coupon payments. Further the introduction of payouts allows to obtain lower optimal leverage ratios and higher yield spreads, compared to Leland's [7] results, which are more in line with historical norms.

The paper is organized as follows: Section 2 introduces the model and determines the optimal failure time as an optimal stopping time, getting the endogenous failure level. Then, in Section 3 the influence of coupon, payouts and corporate tax rate on all financial variables is studied. Section 4 describes optimal capital structure as a consequence of optimal coupon choice. Once determined the endogenous failure level, equity holders aim is to find the coupon payment which allows to maximize the total value of the firm.

2 A Firm's Capital Structure with Payouts

In this section we introduce the model, which is very close to Leland's [7], but we modify the drift with a parameter δ , which might represent a constant proportional cash flow

generated by the assets and distributed to security holders. We consider a firm realizing its capital from both debt and equity. Debt is perpetual and pays a constant coupon C per instant of time. On the failure time T , agents which hold debt claims will get the residual value of the firm, and those who hold equity will get nothing. We assume that the firm activities value is described by the process $V_t = Ve^{X_t}$, where X_t evolves, under the risk neutral probability measure, as

$$dX_t = \left(r - \delta - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t, \quad X_0 = 0, \quad (1)$$

where W is a standard Brownian motion, r the constant risk-free rate, r , δ and $\sigma > 0$. Following [9] we assume δ being a constant fraction of value V paid out to security holders. In line with [8], parameter δ represents the total payout rate to all security holders, thus V represents the value of the net cash flows generated by the firm's activities (excluding cash flows related to debt issuance). Moreover we assume that δ is not affected by changes in leverage (see [9] footnote 4).

When bankruptcy occurs at random time T , a fraction α ($0 \leq \alpha < 1$) of firm value is lost (for instance payed because of bankruptcy procedures), debt holders receive the rest and stockholders nothing, meaning that the strict priority rule holds. We suppose that the failure time T is a stopping time. Thus, applying contingent claim analysis in a Black-Scholes setting, for a given stopping (failure) time T , *debt value* is

$$D(V, C, T) = \mathbb{E}_V \left[\int_0^T e^{-rs} C ds + (1 - \alpha) e^{-rT} V_T \right], \quad (2)$$

where the expectation is taken with respect to the risk neutral probability and we denote

$$\mathbb{E}_V[\cdot] := \mathbb{E}[\cdot | V_0 = V].$$

We assume that from paying coupons the firm obtains tax deductions, namely τ , $0 \leq \tau < 1$, proportionally to coupon payments at a rate τC until default¹, so we get *equity value* as

$$E(V, C, T) = V - \mathbb{E}_V \left[(1 - \tau) \left(\int_0^T e^{-rs} C ds \right) + e^{-rT} V_T \right]. \quad (3)$$

The *total value* of the (levered) firm can always be expressed as sum of equity and debt value: this leads to write the total value of the levered firm as the firm's asset value (unlevered) plus tax deductions on debt payments C less the value of bankruptcy costs:

$$v(V, C, T) = V + \mathbb{E}_V \left[\tau \int_0^T e^{-rs} C ds - \alpha e^{-rT} V_T \right]. \quad (4)$$

¹Since we consider only tax benefits in this model, δV can be interpreted as the after-tax net cash flow before interest, see also [13] footnote 3.

2.1 Endogenous Failure Level

In this subsection we analyze in detail the first-stage optimization problem faced by equity holders, i.e. we find the endogenous failure level which maximizes equity value, for a fixed coupon level. Recall that equity claim in an option-like contract since there is an option to default embodied in it. The best for equity holders will be to exercise this option when V will reach a failure level V_B endogenously derived (i.e. satisfying a smooth-fit principle). For this analysis, we suppose the coupon level C being fixed.

On the set of stopping times we maximize the equity value $T \mapsto E(V, C, T)$, for an arbitrary level of the coupon rate C . From optimal stopping theory (see [4]) and following [2, 3], the failure time, “optimal stopping time”, is known to be a constant level hitting time. Hence default happens at passage time T when the value V falls to some constant level V_B . The value of V_B is endogenously derived and will be determined with an optimal rule later. Further we note that, given (1), it holds that

$$T = \inf\{t \geq 0 : V_t \leq V_B\} = \inf\{t \geq 0 : X_t \leq \log \frac{V_B}{V}\}. \quad (5)$$

Moreover it holds $V_T = V_B$, as the process V_t is continuous.

Thus, the optimal stopping problem of equity holders is turned to maximize the equity function defined in (3) as a function of V_B :

$$E : V_B \mapsto V - \frac{(1-\tau)C}{r} (1 - \mathbb{E}_V[e^{-rT}]) - V_B \mathbb{E}_V[e^{-rT}]. \quad (6)$$

In order to compute the equity value (6) it remains to determine $\mathbb{E}_V[e^{-rT}]$. To this hand we use the following formula for the Laplace transform of a constant level hitting time by a Brownian motion with drift ([5] p. 196-197):

Proposition 2.1 *Let $X_t = \mu t + \sigma W_t$ and $T_b = \inf\{s : X_s = b\}$, then, for all $\gamma > 0$, it holds*

$$\mathbb{E}[e^{-\gamma T_b}] = \exp \left[\frac{\mu b}{\sigma^2} - \frac{|b|}{\sigma} \sqrt{\frac{\mu^2}{\sigma^2} + 2\gamma} \right].$$

Since $V_t = V \exp[(r - \delta - \frac{1}{2}\sigma^2)t + \sigma W_t]$ by (1), we get $\mathbb{E}_V[e^{-rT}] = \left(\frac{V_B}{V}\right)^{\lambda(\delta)}$ where

$$\lambda(\delta) = \frac{r - \delta - \frac{1}{2}\sigma^2 + \sqrt{(r - \delta - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \quad (7)$$

Remark 2.2 *As a function of δ , the coefficient $\lambda(\delta)$ in (7) is decreasing and convex. In order to simplify the notation, we will denote $\lambda(\delta)$ as λ in the sequel.*

Finally the equity function to be optimized w.r.t. V_B is

$$E : V_B \mapsto V - \frac{(1-\tau)C}{r} + \left(\frac{(1-\tau)C}{r} - V_B \right) \left(\frac{V_B}{V} \right)^\lambda, \quad (8)$$

and the following properties must be satisfied:

$$E(V, C, T) \geq E(V, C, \infty) \quad \text{and} \quad E(V, C, T) \geq 0 \quad \text{for all } V \geq V_B. \quad (9)$$

Considering equity function in (8), the first property in (9) is equivalent to

$$E(V, C, T) - E(V, C, \infty) = \left(\frac{(1-\tau)C}{r} - V_B \right) \left(\frac{V_B}{V} \right)^\lambda \geq 0.$$

In fact this term is the *option to default* embodied in equity. Since this is an option to be exercised by the firm, this must have positive value, so it must be $\frac{(1-\tau)C}{r} - V_B \geq 0$. Finally we are led to the constraint:

$$V_B \leq \frac{(1-\tau)C}{r}. \quad (10)$$

As for the second property in (9), we observe that if V_B was chosen by the firm, then the total value of the firm v would be maximized by setting V_B as low as possible. Nevertheless, because equity has limited liability, then V_B cannot be arbitrary small, but $E(V, C, T)$ must be nonnegative. Note that $E(V, C, \infty) = V - \frac{(1-\tau)C}{r} \geq 0$ under the following constraint

$$V \geq \frac{(1-\tau)C}{r}. \quad (11)$$

A natural constraint on V_B is $V_B < V$, indeed, if not, the optimal stopping time would necessarily be $T = 0$ and then

$$E(V, C, T) = V - \frac{(1-\tau)C}{r} + \left(\frac{(1-\tau)C}{r} - V_B \right) = V - V_B < 0.$$

Finally $E(V, C, T) \geq 0$ for all $V \geq V_B$ and the problem faced by equity holders is:

$$\max_{V_B \in \left[0, \frac{(1-\tau)C}{r}\right]} E(V, V_B, C),$$

which is equivalent to

$$\max_{V_B \in \left[0, \frac{(1-\tau)C}{r}\right]} \left(\frac{(1-\tau)C}{r} - V_B \right) \left(\frac{V_B}{V} \right)^\lambda, \quad (12)$$

since the risk less component of equity value $V - \frac{(1-\tau)C}{r}$ does not depend on V_B . This last formulation (12) represents exactly the optimal stopping problem we want to solve: the economic meaning is that equity holders have to chose which is the optimal exercise time of their option to default, or, equivalently, the endogenous failure level solution of (12).

Proposition 2.3 *The endogenous failure level solution of (12) is*

$$V_B(C; \delta, \tau) = \frac{C(1-\tau)}{r} \frac{\lambda}{\lambda+1}, \quad (13)$$

where λ is given by (7).

Proof In order to obtain the endogenous failure level V_B we maximize the function (8), which turns in maximizing the *option to default* $g(V_B)$ embodied in equity given by

$$g(V_B) : V_B \mapsto \left(\frac{(1-\tau)C}{r} - V_B \right) \left(\frac{V_B}{V} \right)^\lambda.$$

We have

$$g'(V_B) = -\frac{\left(\frac{V_B}{V}\right)^\lambda f(V_B)}{rV_B}, \quad g''(V_B) = -\frac{\lambda \left(\frac{V_B}{V}\right)^\lambda (f(V_B) + C(1-\tau))}{rV_B^2},$$

with

$$f(V_B) = V_B r(1+\lambda) - \lambda C(1-\tau).$$

Function $g(V_B)$ is increasing for $0 < V_B < \widehat{V}_B$, then decreasing for $\widehat{V}_B < V_B < \frac{(1-\tau)C}{r}$, with \widehat{V}_B solution of $f(V_B) = 0$:

$$\widehat{V}_B = \frac{(1-\tau)C}{r} \frac{\lambda}{1+\lambda}.$$

Moreover $g(V_B)$ is convex for $V_B < \bar{V}_B$, with \bar{V}_B solution of $g''(V_B) = 0$

$$\bar{V}_B = \frac{(1-\tau)C}{r} \frac{\lambda-1}{1+\lambda},$$

and concave for $\bar{V}_B < V_B < \frac{(1-\tau)C}{r}$.

Observe what follows:

- i) $\frac{(1-\tau)C}{r} > \widehat{V}_B > \bar{V}_B, \quad \forall \lambda > 0$;
- i) in case $\lambda < 1$, $\bar{V}_B < 0$, meaning $g(V_B)$ concave inside $[0, \frac{C(1-\tau)}{r}]$. The maximum exists and is unique and it is achieved at point \widehat{V}_B ;
- ii) in case $\lambda > 1$, $0 < \bar{V}_B < \widehat{V}_B$, meaning $g(V_B)$ convex for $0 < V_B < \bar{V}_B$ and concave for $\bar{V}_B < V_B < \frac{C(1-\tau)}{r}$.

The sign of $g'(V_B)$ is the same of $f(V_B) = -V_B r(1 + \lambda) + \lambda C(1 - \tau)$. Moreover, we have

$$g'(\bar{V}_B) > 0, \quad g'\left(\frac{(1 - \tau)C}{r}\right) < 0,$$

since

$$f'(\bar{V}_B) = (1 - \tau)C > 0, \quad f'\left(\frac{(1 - \tau)C}{r}\right) = -(1 - \tau)(1 + \lambda)C < 0,$$

As a consequence the maximum exists and is unique inside $[0, \frac{(1 - \tau)C}{r}]$, for any value of $\lambda > 0$ and it is achieved for \widehat{V}_B such that: $g'(V_B) = 0 \Rightarrow (1 - \tau)C\lambda = (\lambda + 1)rV_B$. •

Remark 2.4 Note that (13) satisfies the smooth pasting condition (see [1], [10] footnote 60):

$$\frac{\partial E}{\partial V}\bigg|_{V=V_B} = 0.$$

This condition holds since the endogenous failure level (13) is the lowest admissible failure level equity holders can consistent with both i) limited liability of equity, ii) equity being a non-negative and increasing function of current firm's assets value V .

Observe that equity is an increasing function of V when the following constraint is satisfied:

$$\frac{\partial E}{\partial V} = 1 - \frac{\lambda}{V} \left(\frac{V_B}{V}\right)^\lambda \left(\frac{(1 - \tau)C}{r} - V_B\right) \geq 0. \quad (14)$$

Moreover, the lowest value V can assume is V_B and at point $V = V_B$ we have $E(V_B, V_B) = 0$, thus in order to have equity increasing for $V \geq V_B$ it is sufficient

$$\frac{\partial E}{\partial V}\bigg|_{V=V_B} \geq 0. \quad (15)$$

Solving for V_B gives

$$V_B \geq \frac{C(1 - \tau)}{r} \frac{\lambda}{\lambda + 1},$$

with the right hand side being exactly (13).

According to [1] we observe that the smooth pasting conditions gives an endogenous failure level which is also solution of the optimal stopping problem (12), since

$$\frac{\partial E(V, V_B, C)}{\partial V_B} \leq 0, \quad \forall \widehat{V}_B < V_B < \frac{(1 - \tau)C}{r},$$

\widehat{V}_B being (13).

We observe that the equity function is convex w.r.t. firm's current assets value V , as constraint (10) is satisfied:

$$V_B(C; \delta, \tau) < \frac{(1 - \tau)C}{r}.$$

Further (13) has to satisfy $V_B \leq V$, therefore the following inequality holds:

$$\frac{C(1 - \tau)}{r} \frac{\lambda}{\lambda + 1} \leq V. \quad (16)$$

Remark 2.5 *As a particular case when $\delta = 0$ we obtain Leland [7], where $\lambda = \frac{2r}{\sigma^2}$*

$$E(V, C, V_B) = V - \frac{(1 - \tau)C}{r} + \left(\frac{(1 - \tau)C}{r} - V_B \right) \left(\frac{V_B}{V} \right)^{2r/\sigma^2},$$

and the failure level is

$$V_B(C; 0, \tau) = \frac{C(1 - \tau)}{r + \frac{1}{2}\sigma^2}. \quad (17)$$

Since the application $\delta \mapsto \frac{\lambda}{\lambda + 1}$ is decreasing, (17) is greater than (13) for any value of τ :

$$V_B(C; \delta, \tau) = \frac{C(1 - \tau)}{r} \frac{\lambda}{\lambda + 1} < V_B(C; 0, \tau) = \frac{C(1 - \tau)}{r + \frac{1}{2}\sigma^2}. \quad (18)$$

The failure level $V_B(C; \delta, \tau)$ is decreasing with respect to τ, r, σ^2 and proportional to the coupon C , for any value of δ . We note that the dependence of $V_B(C; \delta, \tau)$ on all parameters τ, r, σ^2, C is affected by the choice of the parameter δ . In fact the application $\delta \mapsto \frac{\partial V_B(C; \delta, \tau)}{\partial \tau}$ is negative and increasing, while $\delta \mapsto \frac{\partial V_B(C; \delta, \tau)}{\partial C}$ is positive and decreasing: thus introducing a payout $\delta > 0$ implies a lower reduction (increase) of the endogenous failure level as a consequence of a higher tax rate (coupon), if compared to the case $\delta = 0$. Similarly a change in the risk free rate r or in the riskiness σ^2 of the firm has a different impact on $V_B(C; \delta, \tau)$ depending on the choice of δ . Figure 1 shows that depending on both the payout level and the riskiness of the firm, a change in σ can produce an increase (decrease) on the endogenous failure level with a very different magnitude depending on the payout. As shown in the plot, considering $V_B(C; \delta, \tau)$ as function of the volatility for different values of payout means facing different functions, i.e. the shape and the slope is not constant for each level of σ when δ changes. The distance between two curves (referring to a different δ) is not constant, meaning that payouts do not simply generate a traslation.

In line with the results in [7] the endogenous failure level $V_B(C; \delta, \tau)$ in (13) is independent of both firm's assets value V and α , the fraction of firm value which is lost in the event of bankruptcy (since the strict priority rule holds). The choice of the endogenous failure level $V_B(C; \delta, \tau)$ is a consequence of equity holders maximizing behavior: this is

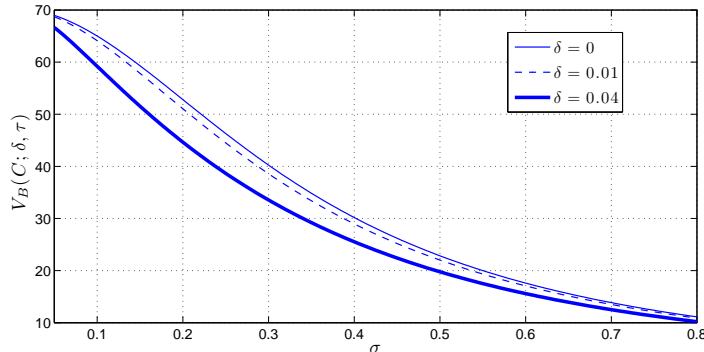


Figure 1: **Endogenous failure level as function of σ .** This plot shows the behavior of the endogenous failure level $V_B(C; \delta, \tau)$ given by (13) as function of σ , for a fixed level of coupon $C = 6.5$ satisfying (11). We assume $V = 100$, $r = 6\%$, $\tau = 0.35$, $\alpha = 0.5$.

why it is independent of α . The economic reason behind is related to the option to default embodied in equity: recall that in order to find the optimal failure level equity holders face the problem of maximizing $V_B \mapsto g(V, C, V_B)$ given by (12) and equity value is not affected by bankruptcy costs since the strict priority rule holds. The endogenous failure level will always be lower than face value of debt, hence equity holders will get nothing at bankruptcy, and in this sense their decision does not depend on parameter α . Only debt holders will bear bankruptcy costs.

2.2 Expected Time to Default

We have proved that introducing payouts has an actual influence on the endogenous failure level $V_B(C; \delta, \tau)$: in particular we showed that δ lowers the failure boundary chosen by equity holders when we consider the coupon being fixed. Also when the coupon is chosen to maximize the total value of the firm, the optimal failure level $V_B^*(V; \delta, \tau)$ reduces as a consequence of a higher payout, as we will show in subsection 4. Consistent with our base case parameters' values, Table 3 gives an idea of the magnitude of this reduction in terms of new optimal default triggering level.

An interesting point should be to analyze not only the influence of payouts on the failure level, but also on expected time to default. How long does it take for V to reach the failure level V_B ? Do payouts have a quantitatively significant effect on it or not? Since we are considering a framework with infinite horizon, it should be interesting to analyze whether introducing payouts will have a significant influence on the expected time to default. We know that firm's activities value evolves as a log-normal variable, thus the

expected time for process V_t to reach the constant failure level V_B can be studied as shown in the following Proposition.

Proposition 2.6 *Let T_b defined in Proposition 2.1. Consider $\mu := r - \delta - \frac{1}{2}\sigma^2$ and $b := \log \frac{V_B}{V}$, with $V \geq V_B$. The following holds:*

- if $\mu > 0$, $\mathbb{E}_V[T_b] = -\left(\frac{V_B}{V}\right)^{\frac{2\mu}{\sigma^2}} \frac{\log \frac{V_B}{V}}{\mu}$,
- if $\mu < 0$, $\mathbb{E}_V[T_b] = \frac{\log \frac{V_B}{V}}{\mu}$.

Proof The result follows by Proposition 2.1 and

$$\mathbb{E}_V[T_b] = -\frac{\partial \mathbb{E}[e^{-\gamma T_b}]}{\partial \gamma} \Big|_{\gamma=0}.$$

•

Proposition 2.6 is useful to show that payouts have an actual influence on the expected time to default and this impact must be analyzed from two different points of view: i) payouts influence on the drift μ of process X_t ; ii) payouts influence on the endogenous failure level. From our previous analysis we know that payouts decrease both μ and the endogenous (and also optimal) failure level chosen by equity holders. But what really matters is the interaction between these effects. We think that from an empirical point of view studying the impact of these two combined effects on the expected time to default could be an alternative way to measure the risk of default associated to the firm's capital structure also from both a qualitative and quantitative point of view.

3 Comparative Statics of Financial Variables

In this section we aim at analyzing the dependence of all financial variables on C , δ , τ at the endogenous failure level $V_B(C; \delta, \tau)$ obtained from the first-stage optimization problem (12). By substituting its expression (13) into equity, debt and total value of the firm, we obtain the following functions:

$$E : (C; \delta, \tau) \mapsto V - \frac{(1-\tau)C}{r} + \frac{(1-\tau)C}{r} \frac{1}{\lambda+1} \left(\frac{C(1-\tau)}{rV} \frac{\lambda}{\lambda+1} \right)^\lambda \quad (19)$$

$$D : (C; \delta, \tau) \mapsto \frac{C}{r} - \frac{C}{r} \left(1 - (1-\alpha)(1-\tau) \frac{\lambda}{\lambda+1} \right) \left(\frac{C(1-\tau)}{rV} \frac{\lambda}{\lambda+1} \right)^\lambda \quad (20)$$

$$v : (C; \delta, \tau) \mapsto V + \frac{\tau C}{r} - \frac{C}{r} \left(\tau + \alpha \frac{\lambda(1-\tau)}{\lambda+1} \right) \left(\frac{C(1-\tau)}{rV} \frac{\lambda}{\lambda+1} \right)^\lambda. \quad (21)$$

Figure 1 shows the behavior of equity, debt and total value of the firm given by (19)-(21) as function of the payout rate δ .

Variable	Limit as		Behaviour w.r.t.		
	$V \rightarrow \infty$	$V \rightarrow V_B$	C	δ	τ
E	$\sim V - \frac{(1-\tau)C}{r}$	0	Convex, \searrow	\nearrow	Convex, \nearrow
D	$\frac{C}{r}$	$\frac{\lambda C(1-\tau)(1-\alpha)}{r(1+\lambda)}$	Concave, \cap -Shaped	\searrow^a	\nearrow
v	$\sim V + \frac{\tau C}{r}$	$\frac{\lambda C(1-\tau)(1-\alpha)}{r(1+\lambda)}$	Concave	\searrow^b	\nearrow
R	r	$\frac{r(1+\lambda)}{\lambda(1-\alpha)(1-\tau)}$	Concave	\nearrow	\searrow
$R - r$	0	$\frac{r(1+\lambda(\alpha+\tau-\alpha\tau))}{\lambda(1-\alpha)(1-\tau)}$	Concave	\nearrow	\searrow

^a See Proposition 3.5.

^b See Proposition 3.7.

Table 1: **Comparative statics of financial variables.** The table shows the behaviour of all financial variables at $V_B(C; \delta, \tau)$ under constraint (11).

3.1 Equity

We analyze **equity's** behaviour with respect to δ .

Proposition 3.1 *The equity function (19) is decreasing and convex as function of λ .*

Proof Equity's behaviour w.r.t. λ is summarized by

$$f(\lambda) = \frac{1}{\lambda + 1} \left(\frac{C(1-\tau)}{rV} \frac{\lambda}{1+\lambda} \right)^\lambda. \quad (22)$$

The logarithmic derivative of (22) is $\log\left(\frac{C(1-\tau)}{rV} \frac{\lambda}{\lambda+1}\right)$ which is negative by (16). Moreover

$$f''(\lambda) = \frac{1}{\lambda(1+\lambda)} f(\lambda) + \left(\log \frac{C(1-\tau)}{rV} + \log \frac{\lambda}{1+\lambda} \right) f'(\lambda) > 0, \quad (23)$$

thus equity is decreasing and convex w.r.t. λ . •

As a consequence of Proposition 3.1 and Remark 2.2, equity is increasing w.r.t. δ . Concerning equity's convexity w.r.t. δ , the following result holds.

Proposition 3.2 *The equity function (19) is convex w.r.t. δ if*

$$V > V_B(C; \delta, \tau) e^{\frac{2r}{\lambda^2 \sqrt{\mu^2 + 2r\sigma^2}}} \quad (24)$$

where $V_B(C; \delta, \tau)$ is given by (13).

Proof In order to study equity's convexity w.r.t. δ , we evaluate $\partial_\delta^2 E$ using (22) and (23), obtaining:

$$\begin{aligned} \partial_\delta^2 E &= \left[\frac{1}{\lambda(1+\lambda)} f(\lambda) + \left(\log^2 \frac{C(1-\tau)}{rV} \frac{\lambda}{1+\lambda} \right) f(\lambda) \right] \frac{\lambda^2}{\mu^2 + 2r\sigma^2} \\ &+ f(\lambda) \left(\log \frac{C(1-\tau)}{rV} \frac{\lambda}{1+\lambda} \right) \frac{2r}{(\mu^2 + 2r\sigma^2)\sqrt{\mu^2 + 2r\sigma^2}} \end{aligned} \quad (25)$$

Substituting $V_B(C; \delta, \tau) = C(1-\tau)\lambda/(r(1+\lambda))$ and re-arranging terms gives:

$$\partial_\delta^2 E = \frac{f(\lambda)}{\mu^2 + 2r\sigma^2} \left[\frac{\lambda}{1+\lambda} + \log \frac{V_B}{V} \left(\lambda^2 \log \frac{V_B}{V} + \frac{2r}{\sqrt{\mu^2 + 2r\sigma^2}} \right) \right] \quad (26)$$

As $V \geq V_B$ then $\log \frac{V_B}{V} < 0$ for each parameters' choice, then a sufficient condition for equity's convexity w.r.t. δ is

$$\lambda^2 \log \frac{V_B}{V} + \frac{2r}{\sqrt{\mu^2 + 2r\sigma^2}} < 0,$$

which is equivalent to (24). •

Remark 3.3 *Equity's dependence on the payout rate δ is strictly related to equity's dependence on μ . Observe that $\partial_\delta^2 E = \partial_\mu^2 E$, since $\partial_\mu \lambda = -\partial_\delta \lambda$ and $\partial_\mu^2 \lambda = \partial_\delta^2 \lambda$. Introducing payouts has a positive effect on equity value.*

Studying equity's behavior w.r.t. C and τ , we can observe that E in (19) is a function of the product $C(1-\tau)$; thus $E(C; \delta, \tau)$ is: i) decreasing and convex w.r.t. coupon C , ii) increasing and convex w.r.t. the corporate tax rate τ .

We observe that also in the presence of a payout rate $\delta > 0$, equity holders have incentives to increase the riskiness of the firm, since λ decreases with higher volatility. These incentives are higher as the payout rate increases: Figure 3 shows the behavior of $\frac{\partial E}{\partial \sigma}$ as function of V , for three different levels of δ . For each value of V , $\frac{\partial E}{\partial \sigma}$ increases with δ , meaning that a higher payout produces greater incentives for shareholders to increase the riskiness of the firm, once debt is issued, thus rising potential agency costs due to incentive compatibility problem between equity and debt holders. The payout influence on a potential rise in agency costs is strictly related to the actual distance to default of the firm, as shown in Figure 3. Observe that we report $\frac{\partial E}{\partial \sigma}$ as function of current assets value V : if it is true that $\delta \mapsto \frac{\partial E}{\partial \sigma}$ is an increasing function $\forall V \geq V_B$, this increase in equity sensitivity to firm riskiness is strongly related to the *distance to default*. Recall that the endogenous failure level is independent of V , thus the plot in Figure 3 is built on an endogenous failure level which changes only with δ (the other parameters being fixed). The potential increase in agency costs due to a higher $\frac{\partial E}{\partial \sigma}$ is more pronounced when V is small or very high, meaning for *extreme values* of the distance to default, since V_B is endogenously given.

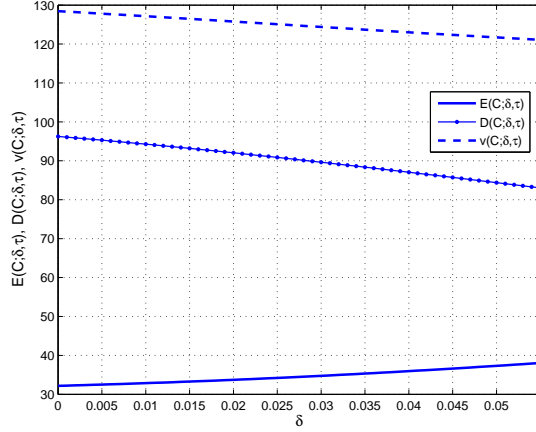


Figure 2: **Equity, debt and total value of the firm as function of δ .** This plot shows the behavior of equity (19), debt (20) and total value of the firm (21) as function of δ , for a fixed level of coupon $C = 6.5$ satisfying (11). We assume $V = 100$, $r = 6\%$, $\sigma = 0.2$, $\tau = 0.35$, $\alpha = 0.5$.

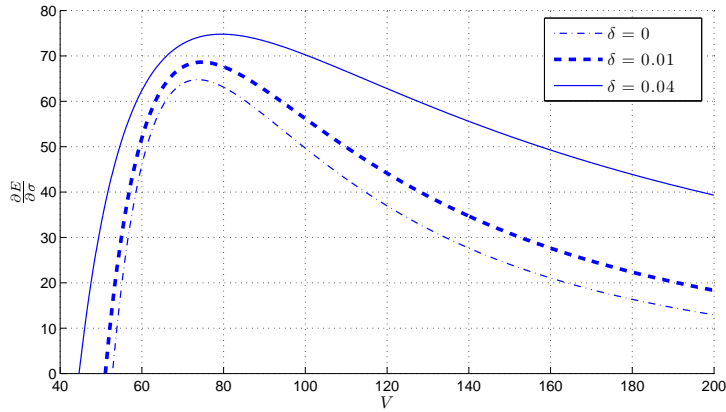


Figure 3: **Effect of a change in σ on equity value.** This plot shows the behavior of $\frac{\partial E}{\partial \sigma}$ as function of firm's current assets value V , for a fixed level of coupon $C = 6.5$ and different values of $\delta = 0, 0.04$. We consider $r = 0.06$, $\sigma = 0.2$, $\alpha = 0.5$, $\tau = 0.35$. For each level of the payout rate δ , equity function has a different support, i.e. $V \geq V_B(C; \delta, \tau)$, with $V_B(C; \delta, \tau)$ given by (13). As a consequence, each curve representing equity sensitivity w.r.t. σ is plotted in its own support.

3.2 Debt and Yield Spread

We consider now the **debt function** $D(C; \delta, \tau)$ in (20). The application $D(C; \delta, \tau)$ is concave w.r.t. coupon C , allowing to analyze the maximum capacity of debt of the firm

as it is shown in the following Proposition.

Proposition 3.4 *The application $C \mapsto D(C; \delta, \tau)$ is concave and achieves a maximum at*

$$C_{max}(V, \delta, \tau) = \frac{rV(1 + \lambda)}{\lambda(1 - \tau)} \left(\frac{1}{\lambda(\tau + \alpha(1 - \tau)) + 1} \right)^{\frac{1}{\lambda}}. \quad (27)$$

$C_{max}(V, \delta, \tau)$ represents the maximum capacity of the firm's debt. Substituting this value for the coupon into debt function $D(C; \delta, \tau)$ and simplifying yields:

$$D_{max}(V, \delta, \tau) = \frac{V}{1 - \tau} \left(\frac{1}{\lambda(\tau + \alpha(1 - \tau)) + 1} \right)^{\frac{1}{\lambda}}. \quad (28)$$

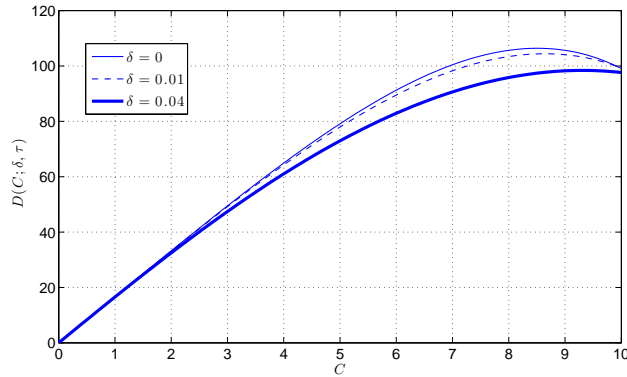


Figure 4: Debt value as function of the coupon. This plot shows the behaviour of debt value given in (20) as function of coupon payments C , for different levels of δ . We assume $V = 100$, $r = 0.06$, $\sigma = 0.2$, $\tau = 0.35$, $\alpha = 0.5$. We consider three different levels of $\delta = 0, 0.01, 0.04$. The value $\bar{C} = \frac{rV}{(1-\tau)}$ is the maximum value that coupon C can assume due to constraint $C(1 - \tau) - rV < 0$. With our base case it is approximately $\bar{C} = 9.23$.

Equation (28) represents the debt capacity of the firm: the maximum value that debt can achieve by choosing the coupon C . Not surprisingly the debt capacity of the firm is proportional to firm's current assets value V , decreases with higher bankruptcy costs α and increases if the corporate tax rate rises. In the presence of payouts, if τ changes, its effect on debt capacity is lower than in case $\delta = 0$, since $\delta \mapsto \frac{\partial D_{max}(V; \delta, \tau)}{\partial \tau}$ is decreasing.

Under constraint (11), as δ increases, debt decreases as shown in the following Proposition.

Proposition 3.5 Debt value $D(C; \delta, \tau)$ defined in (20) is a decreasing function of δ for

$$V > V_B(C, \delta, \tau) e^{\frac{\tau + \alpha(1-\tau)}{1 + \lambda(\tau + \alpha(1-\tau))}}, \quad (29)$$

with $V_B(C; \delta, \tau)$ given in (13).

Proof It is enough to study the monotonicity of debt function with respect to λ . Debt's dependence on λ is the opposite of

$$g(\lambda) = \left(1 - (1 - \alpha)(1 - \tau) \frac{\lambda}{\lambda + 1} \right) \left(\frac{C(1 - \tau)}{rV} \frac{\lambda}{\lambda + 1} \right)^\lambda,$$

its log-derivative being

$$h(\lambda) = \frac{g'}{g}(\lambda) = \left(\log \frac{\lambda}{\lambda + 1} + \frac{\alpha + \tau - \alpha\tau}{1 + \lambda(\alpha + \tau - \alpha\tau)} + \log \frac{C(1 - \tau)}{rV} \right),$$

thus

$$h(\lambda) = \frac{g'}{g}(\lambda) = \frac{\alpha + \tau - \alpha\tau}{1 + \lambda(\alpha + \tau - \alpha\tau)} + \log \frac{V_B(C, \delta, \tau)}{V}.$$

We have $h(\lambda) > 0$ for

$$V < V_B(C, \delta, \tau) e^{\frac{\tau + \alpha(1-\tau)}{1 + \lambda(\tau + \alpha(1-\tau))}},$$

with $V_B(C; \delta, \tau)$ given in (13). •

As the payout rate δ increases, the maximum capacity of debt reduces (the application $\lambda \mapsto D_{max}(V; \delta, \tau)$ is increasing) and C_{max} increases. The economic reason is that with a higher payout rate less assets remain in the firm, thus a lower debt issuance can be supported, giving an insight for a potential influence on agency costs. And our analysis shows that increasing payouts produces important effects on agency costs. As [6] suggest, after debt is issued, equity holders can potentially extract value from debt holders by increasing the riskiness of the firm, since $\frac{\partial E}{\partial \sigma} > 0$, while the opposite happens for debt holders, $\frac{\partial D}{\partial \sigma} < 0$: a higher volatility decreases debt value under (29). The *asset substitution* problem still exists with payouts. Moreover, payouts can strongly modify the magnitude of potential agency costs arising from the model. There is an incentive compatibility problem between debt holders and equity holders: once debt is issued, shareholders will benefit from an increase in the riskiness of the firm (i.e. deciding to invest in riskier activities/projects), through transferring value from debt to equity (also if equity is not exactly an ordinary call option, see [9] footnote 29), and these incentives are higher for equity holders as δ rises, for each firm's current assets value $V \geq V_B$. Analyzing in detail from both analytic and economic point of view the asset substitution problem is beyond the scope of this paper, nevertheless we found some interesting insights.

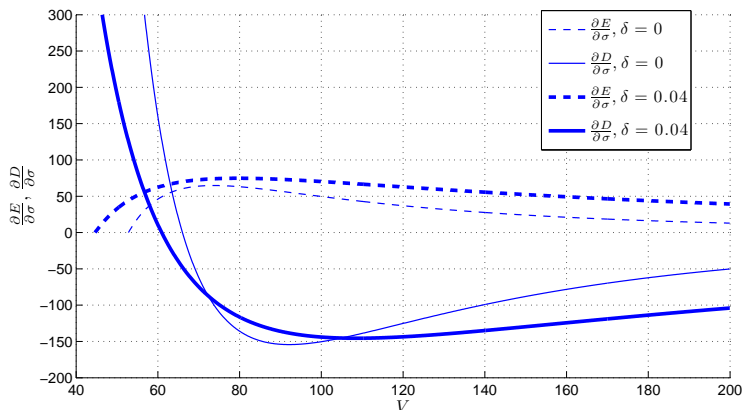


Figure 5: **Effect of a change in σ on equity and debt values.** This plot shows the behavior of $\frac{\partial E}{\partial \sigma}$ (dashed line) and $\frac{\partial D}{\partial \sigma}$ (solid line) as function of firm's current assets value V , for a fixed level of coupon $C = 6.5$. We consider $r = 0.06, \sigma = 0.2, \alpha = 0.5, \tau = 0.35$ and two different levels of $\delta = 0, 0.04$. Equity value is given by (19), debt value by (20). For each level of the payout rate δ , equity and debt functions have a different support, i.e. $V \geq V_B(C; \delta, \tau)$, with $V_B(C; \delta, \tau)$ given by (13). As a consequence, each curve representing equity and debt sensitivity w.r.t. σ is plotted in its own support.

In Figure 5 we analyze which is the effect of a change in the volatility level σ on both equity and debt sensitivity to σ as payouts increase. Following [9] we study the magnitude of this effect as function of V . Considering two different levels of the payout rate $\delta = 0, 0.04$ we compute $\frac{\partial E}{\partial \sigma}, \frac{\partial D}{\partial \sigma}$ and analyze them for different values of $V \geq V_B$. The firm will bear potential agency costs for the range of values V such that $\frac{\partial E}{\partial \sigma} > 0, \frac{\partial D}{\partial \sigma} < 0$, meaning under (29). Alternative measures of potential agency costs are: i) how wide is the range of V such that the problem exists; ii) the magnitude of the gap between $\frac{\partial E}{\partial \sigma}, \frac{\partial D}{\partial \sigma}$ (see also [9] footnote 30). We refer to agency costs only as potential, since we are assuming that capital structure decisions, once made, are not subsequently changed. Observe that $\delta \mapsto \frac{\partial D}{\partial \sigma}$ is not a monotonic function (differently from equity sensitivity), meaning that once again, the influence of payouts is strongly connected with the actual distance to default faced by the firm. Payouts have an influence also on the shape of $V \mapsto \frac{\partial D}{\partial \sigma}$. In line with [9], the incentives for increasing risk are positive for both equity and debt holders when bankruptcy is imminent, i.e. when the distance to default approaches zero. Figure 5 shows that introducing payouts increases the range of V for which potential agency costs exist, thus rising the problem of adverse incentives between debt holders and equity holders and so potential agency costs for the firm. This range is $V > 66$ in case $\delta = 0$ and $V > 60$ for a 4% payout, meaning an increase of this range around 6% of current asset's value in our base case. The range always starts when current assets value is such that debt sensitivity w.r.t. σ is null. And this will happen at point \bar{V} satisfying constraint (29)

as equality, i.e. for

$$\bar{V} := V_B(C, \delta, \tau) e^{\frac{\tau + \alpha(1-\tau)}{1 + \lambda(\tau + \alpha(1-\tau))}},$$

with $V_B(C; \delta, \tau)$ given in (13).

Notice that \bar{V} decreases with payouts in our base case, but we stress that the magnitude of this reduction can be quantitatively different depending on all parameters involved in the capital structure (i.e. bankruptcy costs, corporate tax rate, risk free rate, volatility), since they are all interrelated variables. To give an idea of this, considering a lower corporate tax rate $\tau = 0.15$ and a payout $\delta = 0.05$ the increase (w.r.t. the case $\delta = 0$) in the range of V for which potential agency costs exist is about 10% of initial assets value $V = 100$.

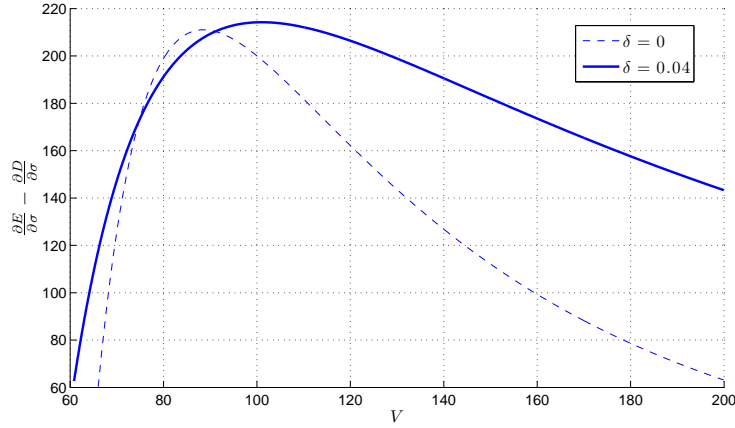


Figure 6: Effect of a change in σ on equity and debt values. This plot shows the magnitude of the gap between $\frac{\partial E}{\partial \sigma}$ and $\frac{\partial D}{\partial \sigma}$ as function of firm's current assets value V , for a fixed level of coupon $C = 6.5$. We consider $r = 0.06, \sigma = 0.2, \alpha = 0.5, \tau = 0.35$ and two different levels of $\delta = 0, 0.04$. Equity value is given by (19), debt value by (20). For each level of the payout rate δ , equity and debt functions have a different support, i.e. $V \geq V_B(C; \delta, \tau)$, with $V_B(C; \delta, \tau)$ given by (13). As a consequence, each curve representing equity and debt sensitivity w.r.t. σ is plotted in its own support.

Figure 6 analyses the magnitude of the gap between $\frac{\partial E}{\partial \sigma} > 0$ and $\frac{\partial D}{\partial \sigma} < 0$ inside the range where the conflict exists. What is interesting is that agency costs are relatively flat (or slightly decreasing) only for V being between approximately 78 – 90. As the distance to default increases (or decreases), the magnitude of the gap between the two sensibilities hardly increases, i.e. the incentive compatibility problem becomes more difficult to solve with a greater payout δ . And this is still true also considering a *pure* Modigliani-Miller framework with zero tax benefits and bankruptcy costs, where $\frac{\partial E}{\partial \sigma} = -\frac{\partial D}{\partial \sigma}$. In such a case the conflict approaches a zero sum game as found in [9] and the introduction of payouts increases its magnitude (see Figure 7). As observed previously, the payout influence on

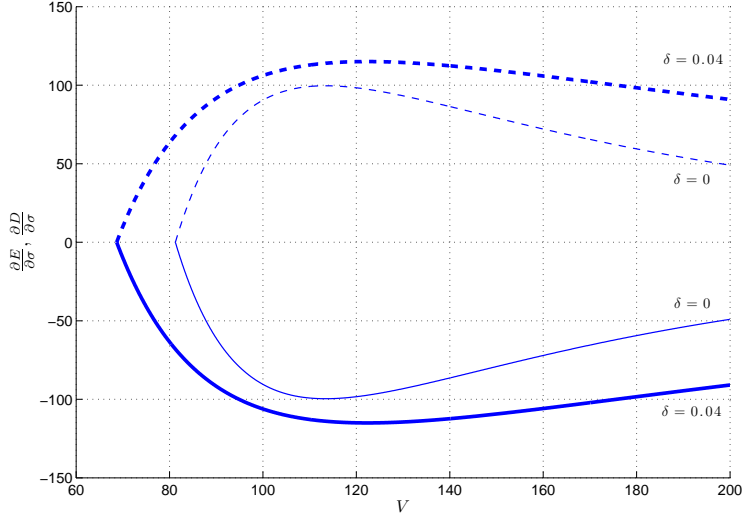


Figure 7: **Effect of a change in σ on equity and debt values when $\alpha = \tau = 0$.** This plot shows the behaviour of $\frac{\partial E}{\partial \sigma}$ (dashed line) and $\frac{\partial D}{\partial \sigma}$ (solid line) as function of firm's current assets value V , for a fixed level of coupon $C = 6.5$. We consider $r = 0.06, \sigma = 0.2, \alpha = 0.5, \tau = 0.35$ and two different levels of payout rate $\delta = 0, 0.04$. Equity value is given by (19), debt value by (20). For each level of the payout rate δ , equity function has a different support, i.e. $V \geq V_B(C; \delta, \tau)$, with $V_B(C; \delta, \tau)$ given by (13). As a consequence, each curve representing equity sensitivity w.r.t. σ is plotted in its own support.

agency costs strongly depends on firm's activities value V , thus on the distance to default. Our base case allows to show that as the distance to default increases or decreases, the payout can strongly contribute to rise *average firm risk*, by rising the magnitude of the gap between equity and debt sensitivity w.r.t. σ , increasing the incentive incompatibility problem between equity and debt holders. This will produce a direct effect on credit spreads, increasing them when average firm risk is higher in order to compensate debt holders.

Finally, a higher coupon C has a positive effect on the interest rate paid by risky debt, **yield**, defined as

$$R(C; \delta, \tau) := \frac{C}{D(C; \delta, \tau)}, \quad (30)$$

with $D(C; \delta, \tau)$ given in (20).

Actually yield $R(C; \delta, \tau)$ is increasing as function of C and decreasing as function of τ . A higher corporate tax rate τ will reduce both yield $R(C; \delta, \tau)$ and **yield spread** $R(C; \delta, \tau) - r$ by rising debt (lowering the endogenous failure level $V_B(C; \delta, \tau)$, see also [7 footnote 22): this follows by the relation $\partial_\tau R(C; \delta, \tau) = -C \frac{\partial_\tau D(C; \delta, \tau)}{(D(C; \delta, \tau))^2}$.

Proposition 3.6 *The function yield $R(C; \delta, \tau)$ defined in (30) is increasing w.r.t. δ .*

Proof As D is an increasing function of λ and $\partial_\lambda R = -C \frac{\partial_\lambda D}{D^2}$, we obtain that R is a decreasing function of λ . Thus by Remark 2.2 R is increasing w.r.t. δ . •

While the introduction of δ reduces debt, the opposite happens for yield spreads, which are higher. This is due to two main reasons: first, as δ increases, less assets remains in the firm, thus increasing the likelihood of default. Secondly, introducing the payout rate produces a direct effect of rising the average firm risk, as shown before. As a consequence, debt holders must be compensated with higher yield spreads.

Observe that R can be expressed as:

$$R : (C; \delta, \tau) \mapsto r \bar{R} \left(\frac{C}{V} \right), \quad (31)$$

with

$$\bar{R} \left(\frac{C}{V} \right) = \left[1 - \left(\frac{C}{V} \right)^\lambda \left(\frac{(1-\tau)\lambda}{r} \right)^\lambda \left(1 - (1-\alpha)(1-\tau) \frac{\lambda}{\lambda+1} \right) \right]^{-1}. \quad (32)$$

As increasing function of the ratio $\frac{C}{V}$, the term $\bar{R} \left(\frac{C}{V} \right)$ represents the *risk-adjustment factor* paid to debt holders. Introducing payouts rises debt's volatility as Figure 8 shows: as a consequence, the compensation paid by the firm to debt holders for the risk assumed must be higher, and this is why $\bar{R} \left(\frac{C}{V} \right)$ increases.

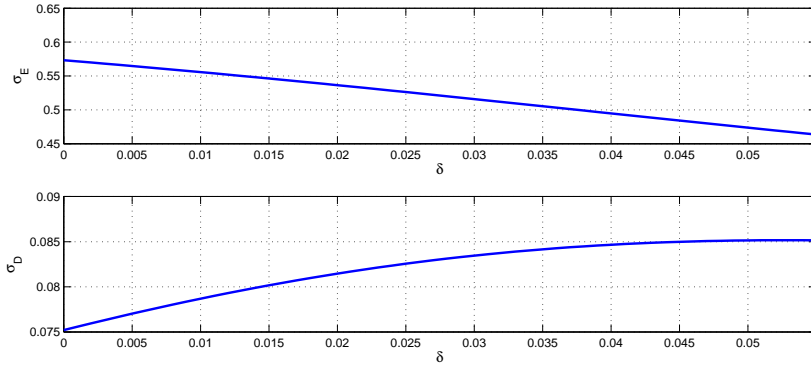


Figure 8: Volatility of Equity and Debt. This plot shows the behavior of equity and debt volatility σ_E, σ_D as function of payout rate, for a fixed level of coupon $C = 6.5$. We consider $V = 100, r = 0.06, \sigma = 0.2, \alpha = 0.5, \tau = 0.35$. By Ito calculus formula we derive the behavior of equity and debt volatility σ_E, σ_D .

When $V \rightarrow \infty$, yield spread $R - r$ approaches to r , $\frac{\partial D}{\partial \sigma} \rightarrow 0$ and debt becomes risk free: this is exactly as in [7], since in such a case, the hypothesis of debt being redeemed in full becomes quite certain and this is not affected by the choice of the payout rate δ . Introducing payouts will instead rise R in case $V \rightarrow V_B$, considering a "pure" Modigliani-Miller [12] framework: if $\alpha = \tau = 0$, as V approaches the failure level V_B , $R \rightarrow r(1 + \frac{1}{\lambda})$, while in case $\delta = 0$ we have $R \rightarrow r + \frac{1}{2}\sigma^2$. If there are no bankruptcy costs or tax benefits of debt, introducing payouts allows to have yield exceeding the risk free rate r by more than $\frac{1}{2}\sigma^2$, since $\frac{r}{\lambda} > \frac{1}{2}\sigma^2$, thus providing to bondholders a higher compensation for risk, if compared to the case $\delta = 0$.

3.3 Total Value

The **total value of the firm** $v(C; \delta, \tau)$ in (21) is a concave function of coupon C and an increasing function of corporate tax rate τ . The following proposition shows the behavior of the total value of the firm with respect to the payout rate δ .

Proposition 3.7 *The total value $v(C; \delta, \tau)$ defined in (21) is decreasing w.r.t. δ if*

$$V > V_B(C; \delta, \tau) e^{\frac{\tau + \alpha(1 - \tau)}{\tau + \lambda(\tau + \alpha(1 - \tau))}} \quad (33)$$

with $V_B(C; \delta, \tau)$ given in (13).

Proof The behavior of $v(C; \delta, \tau)$ is the one of the following:

$$G : \lambda \mapsto (\tau + \lambda(\tau + \alpha(1 - \tau))) \frac{1}{\lambda + 1} \left(\frac{C(1 - \tau)}{rV} \frac{\lambda}{\lambda + 1} \right)^\lambda,$$

which satisfies:

$$\frac{G'}{G}(\lambda) = h(\lambda) = \frac{\tau + \alpha(1 - \tau)}{\tau + \lambda(\tau + \alpha(1 - \tau))} + \log \left(\frac{C(1 - \tau)}{rV} \frac{\lambda}{\lambda + 1} \right) = \quad (34)$$

$$= \frac{\tau + \alpha(1 - \tau)}{\tau + \lambda(\tau + \alpha(1 - \tau))} + \log \frac{V_B(C; \delta, \tau)}{V}. \quad (35)$$

Actually, the behaviour of G is given by the sign of $h(\lambda)$, with $\frac{\tau + \alpha(1 - \tau)}{\tau + \lambda(\tau + \alpha(1 - \tau))} > 0$ and $\log \frac{V_B(C; \delta, \tau)}{V} < 0$.

Thus, in case

$$V > V_B(C; \delta, \tau) e^{\frac{\tau + \alpha(1 - \tau)}{\tau + \lambda(\tau + \alpha(1 - \tau))}}$$

we have $h(\lambda) < 0$ and the total value of the firm is decreasing w.r.t. δ . •

The economic intuition behind constraint (33) being satisfied, is that if the initial value of the firm V is sufficiently greater² than the failure level, $\delta \mapsto v(C; \delta, \tau)$ is decreasing due to the fact that introducing payouts makes bankruptcy more likely, since less assets remain in the firm.

3.4 Probability of Default

An important issue to take into account in this framework is the probability of the firm of going bankruptcy. In order to analyze the payout influence on default rates, we now conduct this study under the historical measure as follows.

The cumulative probability $F(s)$ of going bankruptcy in the interval $[0, s)$ is given by:

$$F(s) = N\left(\frac{b - \gamma s}{\sigma\sqrt{s}}\right) + e^{2b\gamma/\sigma^2} N\left(\frac{b + \gamma s}{\sigma\sqrt{s}}\right), \quad (36)$$

where $N(\cdot)$ is the normal cumulative probability function, $b := \log \frac{V_B}{V}$ and $\gamma := \mu_A - \delta - \frac{1}{2}\sigma^2$, where $\mu_A := r + \pi_A$ and π_A being the historical asset risk premium (see also Equation (15) in [9]). As observed in [9], footnote 27, the probability of default is quite dependent on the drift assumed for the process V_t . As a consequence, the payout rate δ has an actual influence on the probability of going bankruptcy $F(s)$. We show qualitative behavior of $F(s)$ after the introduction of payouts in Figure 9, while Table 2 provides numerical result. We study cumulative default probability for $\pi_A = 7.5\%$ (as in [9]) and $\pi_A = 5\%$, for a range of payout values.

Looking at cumulative probability over a period between 10 and 25 years shows that a higher payout makes debt riskier, through rising the likelihood of default. This effect is more pronounced as the horizon we consider is longer: going from 10 to 25 years the increase in the probability of default has a greater magnitude. And this effect is higher if the asset risk premium reduces. As extreme cases we consider that when $\pi_A = 7.5\%$, looking at a 10-years horizon the probability of default is 2.249% in case $\delta = 0$, while it becomes 2.405% when $\delta = 4\%$, thus the increase is 7% of 2.249%. When we consider $F(25)$, the probability of default is 2.537% with no cash payouts. The case $\delta = 4\%$ shows a rising in the default probability up to 3.857%, meaning an increase of around 52% of its starting value 2.537%.

A reduction in the risk premium further increases this gap: when $\pi_A = 5\%$, for instance, the 25-years probability of default goes from 3.857% to 9.686%, meaning a rising of around 150% of the 10-years value.

²Observe that at point $V = V_B(C; \delta, \tau)e^{\frac{\tau + \alpha(1-\tau)}{\tau + \lambda(\tau + \alpha(1-\tau))}}$ satisfying (33) as equality, we have $\frac{\partial E}{\partial \lambda} = -\frac{\partial D}{\partial \lambda}$. Moreover under (33), constraint (29) is always satisfied, meaning we are inside the range in which agency costs exist.

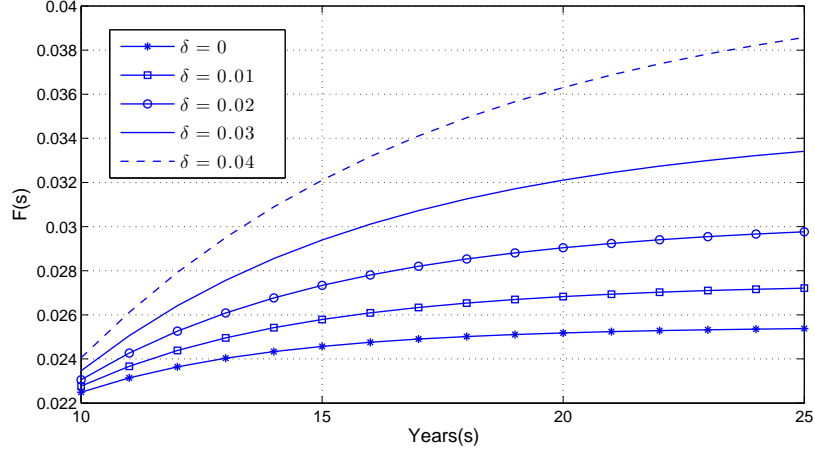


Figure 9: **Cumulative Probability of Default.** This plot shows the cumulative probability of default $F(s)$ over the period $(0, s]$, considering Equation (36). The plot shows $F(s)$ for different values of the payout rate $\delta = 0, 0.01, 0.02, 0.03, 0.04$. Base case parameters' values are $V = 100$, $r = 0.06$, $\sigma = 0.2$, $\alpha = 0.5$, $\tau = 0.35$; the coupon is chosen optimally. We consider an historical asset risk premium $\pi_A = 7.5\%$.

$\pi_A = 7.5\%$	$F(10)$	$F(15)$	$F(20)$	$F(25)$
$\delta = 0$	0.02249	0.02457	0.02518	0.02537
$\delta = 0.01$	0.02277	0.02579	0.02683	0.02721
$\delta = 0.02$	0.02306	0.02733	0.02904	0.02976
$\delta = 0.03$	0.02346	0.02939	0.03211	0.03341
$\delta = 0.04$	0.02405	0.03211	0.03630	0.03857
$\pi_A = 5\%$	$F(10)$	$F(15)$	$F(20)$	$F(25)$
$\delta = 0$	0.04487	0.05157	0.05429	0.05548
$\delta = 0.01$	0.04644	0.05577	0.06006	0.06217
$\delta = 0.02$	0.04797	0.06066	0.06721	0.07081
$\delta = 0.03$	0.04965	0.06654	0.07626	0.08212
$\delta = 0.04$	0.05158	0.07367	0.08767	0.09686

Table 2: **Cumulative Default Probability.** The table shows cumulative default probabilities for two different values of historical assets risk premium π_A analyzing the influence of the payout rate δ on the probability of going bankruptcy $F(s)$ given by Equation (36).

4 Optimal Capital Structure

We now consider the second-stage optimization problem, meaning equity holders have to find the optimal amount of debt (coupon payments) which maximizes the total value of

the firm. This will be done, taking into account the result of the first-stage optimization problem as a relation between coupon payments C and endogenous failure level V_B .

The second stage-optimization problem is:

$$\max_C v(V, V_B(C; \delta, \tau), C), \quad (37)$$

where the failure level V_B is replaced by its endogenous value $V_B(C; \delta, \tau)$ given by Equation (13).

Finally, solving (37) is equivalent to optimizing the total value of the firm $v(C; \delta, \tau)$ given by Equation (21)

$$v : (C; \delta, \tau) \mapsto V + \frac{\tau C}{r} - \frac{C}{r} \left(\tau + \alpha \frac{\lambda(1-\tau)}{\lambda+1} \right) \left(\frac{C(1-\tau)}{rV} \frac{\lambda}{\lambda+1} \right)^\lambda,$$

with respect to the coupon C .

The application $C \mapsto (C; \delta, \tau)$ is concave since $A := \frac{\tau}{r} + \alpha \frac{\lambda(1-\tau)}{r(\lambda+1)} > 0$ and $\lambda > 0$, therefore the following result holds.

Proposition 4.1 *For any fixed δ, τ , the optimal coupon is:*

$$C^*(V; \delta, \tau) = \frac{rV(\lambda+1)}{\lambda(1-\tau)} \left(\frac{\tau}{\lambda(\tau + \alpha(1-\tau)) + \tau} \right)^{\frac{1}{\lambda}}. \quad (38)$$

We observe that $C^*(V; \delta, \tau) < C_{max}(V; \delta, \tau)$, where C_{max} is defined in (27). Moreover, this max-coupon satisfies $V > \frac{(1-\tau)C_{max}}{\tau} \frac{\lambda}{\lambda+1}$.

The optimal coupon $C^*(V; \delta, \tau)$ is an increasing function of τ . In fact

$$\frac{\partial C^*(V, \delta, \tau)}{\partial \tau} = \left(\frac{1}{1-\tau} + \frac{\alpha}{\tau(\tau(1+\lambda) + \alpha\lambda(1-\tau))} \right) C^*(V, \delta, \tau) > 0.$$

Replacing (38) in (13) yields the optimal failure level

$$V_B^*(V; \delta, \tau) = V \left(\frac{\tau}{\lambda(\tau + \alpha(1-\tau)) + \tau} \right)^{\frac{1}{\lambda}}. \quad (39)$$

Remark 4.2 *In case $\delta = 0$, we have $\lambda = \frac{2r}{\sigma^2}$ and we get the same results as in [7]:*

$$V_B^*(V; 0, \tau) = V \left(\frac{\tau\sigma^2}{2r(\tau + \alpha(1-\tau)) + \tau\sigma^2} \right)^{\frac{\sigma^2}{2r}}. \quad (40)$$

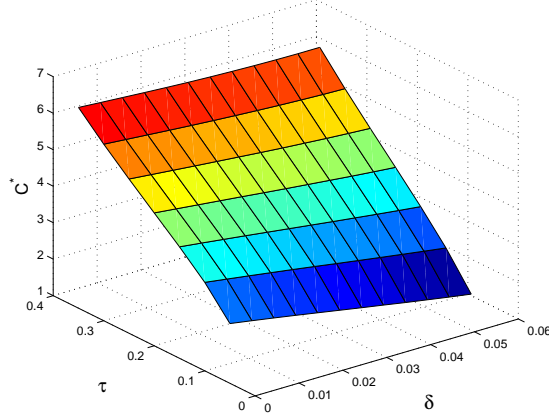


Figure 10: **Optimal Coupon.** This plot shows the behavior of optimal coupon $C^*(V; \delta, \tau)$ as function of the payout rate δ and corporate tax rate τ . We consider $V = 100$, $r = 0.06$, $\sigma = 0.2$, $\alpha = 0.5$.

Proposition 4.3 *Consider the optimal failure level (39). The following results hold:*

- i) $\delta \mapsto V_B^*(V; \delta, \tau)$ is a decreasing function;
- ii) $\tau \mapsto V_B^*(V; \delta, \tau)$ is an increasing function.

Proof i) Using Remark 2.2, it is enough to study the following function

$$F : \lambda \mapsto -\frac{1}{\lambda} \log \left(\frac{\tau + \lambda(\tau + \alpha(1 - \tau))}{\tau} \right).$$

Taking the derivative w.r.t. λ , we obtain:

$$F'(\lambda) = \frac{1}{\lambda^2} \left(\log(1 + z) - \frac{z}{1 + z} \right),$$

with $z := \lambda \left(1 + \alpha \left(\frac{1 - \tau}{\tau} \right) \right)$. It is sufficient to study the sign of

$$G : z \mapsto \log(z + 1) - \frac{z}{1 + z},$$

with $z \in [0, \frac{2r}{\sigma^2} (1 + \alpha (\frac{1 - \tau}{\tau}))]$. Since $G(0) = 0$ and $G'(z) \geq 0$, for any z the function F is increasing. Finally $\delta \mapsto V_B^*(V; \delta, \tau)$ is decreasing.

ii) The result follows by:

$$\frac{\partial V_B^*(V; \delta, \tau)}{\partial \tau} := \frac{\partial V_B^*(C^*(V; \delta, \tau))}{\partial \tau} = \frac{\partial V_B^*(C^*(V; \delta, \tau))}{\partial C^*} \frac{\partial C^*(V; \delta, \tau)}{\partial \tau} > 0.$$

We now completely describe the optimal capital structure of the firm.

Let $y := \lambda(\tau + \alpha(1 - \tau))$, the following holds:

$$C^* = \frac{rV(1 + \lambda)}{\lambda(1 - \tau)} \left(\frac{\tau}{y + \tau} \right)^{\frac{1}{\lambda}} \quad (41)$$

$$V_B^* = V \left(\frac{\tau}{y + \tau} \right)^{\frac{1}{\lambda}} \quad (42)$$

$$D^* = \frac{V}{\lambda(1 - \tau)} \left(\frac{\tau}{y + \tau} \right)^{\frac{1}{\lambda}} \left(\lambda + \frac{y(1 - \tau)}{y + \tau} \right) \quad (43)$$

$$E^* = V \left(1 - \left(\frac{\tau}{y + \tau} \right)^{\frac{1}{\lambda}} \frac{1}{\lambda} \left(1 + \lambda + \frac{\tau}{y + \tau} \right) \right) \quad (44)$$

$$R^* = \frac{r(1 + \lambda)}{\lambda + \frac{y(1 - \tau)}{y + \tau}}. \quad (45)$$

Introducing a payouts into (39) has an actual influence: $\delta \mapsto V_B^*(V; \delta, \tau)$ is a decreasing function for any value of τ , while $\tau \mapsto V_B^*(V; \delta, \tau)$ is increasing for any value of δ . Similarly optimal coupon $C^*(V; \delta, \tau)$ given by (38) will benefit from a higher corporate tax rate and decrease w.r.t. δ , as Figure 10 shows.

δ	C^*	D^*	R^*	$R^* - r$	E^*	V_B^*	v^*	L^*
0	6.501	96.275	6.753	75.257	32.167	52.821	128.442	74.956 %
0.005	6.459	94.924	6.804	80.437	32.879	51.634	127.804	74.274 %
0.010	6.419	93.547	6.862	86.177	33.602	50.422	127.149	73.573 %
0.015	6.380	92.135	6.925	92.463	34.347	49.177	126.482	72.845 %
0.020	6.344	90.706	6.994	99.401	35.097	47.918	125.803	72.102 %
0.025	6.312	89.272	7.071	107.053	35.847	46.653	125.119	71.350 %
0.030	6.283	87.826	7.154	115.389	36.606	45.377	124.432	70.582 %
0.035	6.258	86.382	7.245	124.457	37.366	44.103	123.748	69.805 %
0.040	6.239	84.960	7.343	134.342	38.111	42.850	123.072	69.033 %
0.045	6.225	83.558	7.450	144.989	38.849	41.615	122.407	68.263 %
0.050	6.217	82.189	7.564	156.424	39.570	40.411	121.759	67.502 %
0.055	6.215	80.859	7.686	168.619	40.272	39.242	121.132	66.753 %

Table 3: **Effect of payout rate δ on all financial variables at the optimal leverage ratio.** Base case parameters' values: $V = 100$, $\sigma = 0.2$, $\tau = 0.35$, $r = 0.06$, $\alpha = 0.5$. The first row of the table shows Leland's framework with his base case parameters' values, in particular with $\delta = 0$. L^* is in percentage (%), R^* , $R^* - r$ are in basis points (bps).

The capital structure of the firm is strongly affected by payouts: Tables 3 and 4 show the behavior of all financial variables at optimal leverage ratio, when the parameter δ moves away from zero. Consistent with our base case, these tables report both numerical results and a qualitative analysis.

Columns 6 and 7 of Table 3 show equity and debt values when the coupon C is chosen to maximize the total value of the firm. Optimal equity value increases with a higher payout, while optimal debt decreases (recall that δ does not include cash flows related to debt financing). These two effects have a different magnitude: δ influence on debt is in fact greater than δ influence on equity, as a consequence the optimal total value of the firm, $v^* := D^* + E^*$, reduces. Now consider optimal leverage ratio, defined as $L^* := \frac{D^*}{v^*}$. The last column of Table 3 shows that increasing payouts decreases optimal leverage ratio L^* , and this effect is more pronounced as δ is higher. Considering our base case, optimal leverage can drop from approximately 75% to 66.75% passing from $\delta = 0$ to $\delta = 0.055$. A higher payout δ will bring to a lower total value of the firm v^* , since a lower leverage ratio can be supported when less assets remain in the firm, as Leland suggests [8]. Optimal leverage ratios can be strongly affected by payouts, but their influence is related to the riskiness of the firm. Figure 9 shows optimal leverage as function of δ for three different values of σ . For each level of δ , optimal leverage ratio decreases as σ rises. Observe also that L^* is decreasing w.r.t. δ for each level of σ , but this reduction in optimal leverage is lower as the riskiness of the firm rises.

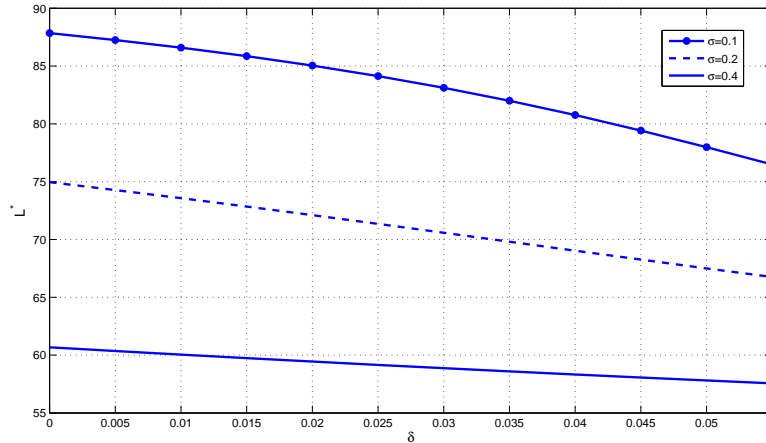


Figure 11: **Optimal leverage ratio as function of δ .** This plot show optimal leverage ratio L^* as function of δ for three different levels of volatility σ . We consider $V = 100$, $r = 0.06$, $\alpha = 0.5$, $\tau = 0.35$.

Recall from Section 2 that higher payouts rise the probability of going bankruptcy given by (36): as a consequence, when δ rises, optimal yield R^* and optimal yield spread

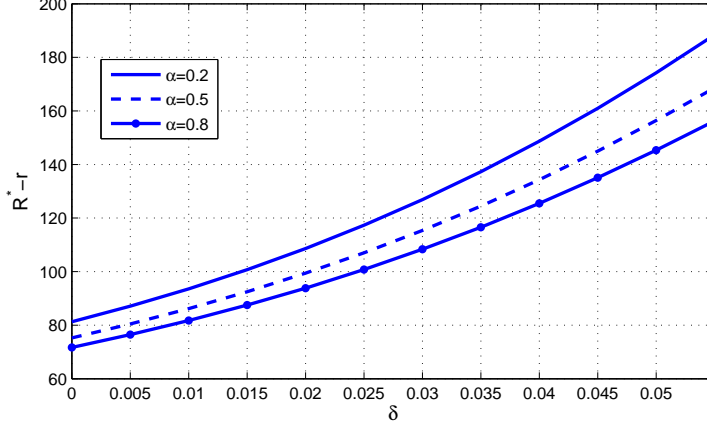


Figure 12: **Optimal spreads as function of δ .** This plot show optimal spreads $R^* - r$ as function of δ for three different levels of α . We consider $V = 100$, $r = 0.06$, $\sigma = 0.2$, $\tau = 0.35$.

$R^* - r$ increase. Two are the main reasons: first, less assets remain in the firm, thus bankruptcy is more likely; secondly, the average firm risk is higher, thus debt holders must be compensated for the higher risk assumed. Leland [7] observes that as bankruptcy costs rise, surprisingly optimal yield spread reduces when the coupon is chosen optimally. This is due to the fact that a higher α will decrease the optimal coupon. Our analysis shows that when payouts are introduced, optimal yield spreads are still decreasing w.r.t. α for each level of δ (see Figure 10). Moreover, payouts influence on optimal yield spreads is higher as α reduces. Considering as extreme cases $\delta = 0$ and $\delta = 0.055$, optimal yield spread rises from 71.6778 bps (basis points) to 156.2624 bps with $\alpha = 0.8$, from 75.2554 bps to 168.6365 bps with $\alpha = 0.5$ and from 81.2559 bps to 188.424 bps with $\alpha = 0.2$.

Financial Variables	C^*	D^*	R^*	$R^* - r$	E^*	V_B^*	v^*	L^*
Sign of change as $\delta \nearrow$	< 0	< 0	> 0	> 0	> 0	< 0	< 0	< 0

Table 4: **Effect of payout rate δ on all financial variables at the optimal leverage ratio.** The table shows for each financial variable the effect of increasing δ . Considering our base case, we report the sign of change in each variable as the payout moves away from 0.

We now turn to the study of tax deduction τ influence on all financial variables. Tables 5 and 6 show the behavior of all financial variables at optimal coupon level C^* for different values of the corporate tax rate τ when a payout $\delta = 0.01$ is introduced. As the tax deduction increases, all financial variables, except equity, will benefit from this. This result extends Table II in [7] since it allows for a payout $\delta > 0$. Concerning the optimal failure level $V_B^*(V; \delta, \tau)$ we observe that by (39) the corporate tax rate τ has no influence

on the optimal failure level at optimal leverage ratio in case $\alpha = 0$. The same holds for equity value E^* at optimal leverage ratio: a change in the corporate tax rate has no effect on equity value in the absence of bankruptcy costs.

τ	C^*	D^*	R^*	$R^* - r$	E^*	V_B^*	v^*	L^*
0.35	6.419	93.547	6.862	86.177	33.602	50.422	127.149	73.573 %
0.30	5.743	85.077	6.750	75.038	35.742	48.582	120.819	70.417 %
0.25	5.115	77.078	6.636	63.617	38.377	46.360	115.454	66.760 %
0.20	4.513	69.225	6.519	51.928	41.683	43.631	110.908	62.417 %
0.15	3.907	61.062	6.398	39.840	46.020	40.133	107.082	57.023 %

Table 5: **Effect of a change in the corporate tax rate τ on all financial variables at the optimal leverage ratio.** This table considers a case in which a payout δ is introduced ($\delta = 0.01$) and studies the effect of a change in the corporate tax τ . R^*, L^* are in percentage (%), $R^* - r$ is in basis points (bps).

Financial Variables	C^*	D^*	R^*	$R^* - r$	E^*	V_B^*	v^*	L^*
Sign of change as $\tau \nearrow$	> 0	> 0	> 0	> 0	$< 0^a$	$> 0^a$	> 0	> 0

^a No effect if $\alpha = 0$.

Table 6: **Effect of corporate tax rate τ on all financial variables at the optimal leverage ratio.** The table shows for each financial variable the effect of increasing τ for fixed $\delta = 0.01$. Considering our base case, we report the sign of change in each variable as the corporate tax rate increases.

5 Conclusions

Introducing the payout rate has an actual influence on all financial variables determining the capital structure of the firm. We are considering δ as the total payout rate to all security holders excluding cash flows related to debt issuance, see [7]. For an arbitrary coupon level C , an increasing payout will rise equity and reduce both debt and total value of the firm, making bankruptcy more likely since less assets will remain in the firm. Payouts will have a direct and indirect influence on the endogenous failure level $V_B(C; \delta, \tau)$ chosen by equity holders: this default boundary will be lower as payouts increase (direct effect). Adding to this, our results suggest that payouts strongly modify the influence of all parameters r, τ, C, σ^2 on the endogenous failure level, through affecting the magnitude of their influence (indirect effect). Potential agency costs arising from the model due to the asset substitution problem are not independent of the choice of δ . They are affected in two main directions. Equity holders will always have greater incentives to increase the riskiness of the firm when the payout rises, for each value V bigger than the endogenous

failure level, meaning that equity sensitivity to σ is an increasing and positive function of δ . At first, the range of values V for which the conflict of interests between equity and debt holders exists increases with payouts. Secondly, the gap between equity and debt sensitivity to σ increases dramatically as function of the distance to default when δ is introduced (only for a small range of values agency costs are relatively flat). Concerning optimal capital structure Leland's [7] results show too high leverage ratios (and/or too low yield spreads): assuming payouts $\delta > 0$ allows to overcome this, providing lower optimal leverage ratios and higher yield spreads. Leverage ratio reduces because when payouts increase, less assets is available inside the firm, thus only a lower leverage can be supported by the firm. Yield spreads increases with payouts due to two main reasons: i) bankruptcy is more likely; ii) average firm risk rises, since both debt and equity sensitivity to σ rise (absolute value). Debt becomes riskier and more sensible to a change in the volatility of the firm, consequently debt holders must be compensated for this. In this paper the payout δ is considered exogenous and constant through time: relaxing this assumption and making it dependent on the coupon level will allow to analyze the payout decision as an endogenous one, trying to study it as the solution of an optimal payout policy. Moreover, making the payout rate endogenous will provide an interesting framework in which asymmetric information between equity holders and debt holders can be introduced as long as a detailed analysis of the asset substitution problem, as idea for future research.

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II

Tax Benefits Asymmetry

Corporate Debt Value with Switching Tax Benefits and Payouts

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Abstract

This paper analyzes a structural model of corporate debt in the spirit of Leland model [17] within a more realistic general context where payouts and asymmetric tax-code provisions are introduced. We analytically derive the value of the tax benefit claim in this context and study the joint effect of tax asymmetry and payouts on optimal corporate financing decisions. Results show a quantitatively significant impact on both optimal debt issuance and leverage ratios, thus providing a way to explain differences in observed leverage across firms.

Keywords: structural model; corporate debt; default; optimal stopping; tax benefits of debt.

1 Introduction

The capital structure of a firm has been analyzed in terms of derivatives contracts since Merton's work [23]. The capital structure decision is a complex issue due to many factors entering in the determinacy of corporate financing policy. Riskiness of the firm, bankruptcy costs, payouts, interest rates and taxes are important factors to take into account when defining the capital structure of a firm. According to Modigliani - Miller theorem [24] the activities of the firm are assumed to be independent from the financial structure when no taxes and no bankruptcy costs are assumed. When we consider a firm subject to default risk in a framework with bankruptcy costs and taxes, the owners of the firm can choose optimally the capital structure, where optimally means the capital structure which maximizes the total value of the firm. The economic intuition is that deciding the optimal capital structure means choosing which will be the best allocation of values (effective and potential) belonging to the firm. All this can produce very different firm's values due for example to the trade-off existing between some interacting variables (i.e. debt and tax

benefits). In particular corporate tax rate are an important determinant of optimal capital structure, as early recognized by [24] and observed in more recent empirical studies (see [14, 27]). In this paper the model proposed by [17] is extended by means of a switching (even debt dependent) in tax savings and the introduction of a company's assets payout ratio (as in [3, 4]). We perform a quantitative study of the effects of both components on the optimal capital structure of the firm, obtaining analytical results in most cases. Our findings show that the combined effect of tax asymmetry and payouts produces predicted optimal leverages ratios which are more in line with historical norms (significantly reduced w.r.t. the ones in [17]) and empirical evidence. As a matter of fact, tax code provisions can vary across nations, across industries, across activity sector in which the firm is operating in, and also across time. Suppose for example the tax code being modified to encourage investments (see [27] footnote 1, [26]). The main empirical insight when analyzing tax influence on capital structure decisions is that leverage ratios are higher for firms facing higher corporate tax rates (as it is shown in [30]). The economic insight we want to analyze is how asymmetry in tax-code provisions are incorporated in firm's financing decisions and moreover the quantitative impact of this on optimal debt issuance and leverage decisions.

The total value of the firm is realized from both equity and debt. Equivalently, it can be achieved by considering firm's activities value in the unlevered case (meaning when no debt is issued) plus tax benefits of debt, less bankruptcy costs. We assume bankruptcy being determined endogenously by the inability of the firm to raise sufficient equity capital to cover its debt obligations. Following [17] we consider an infinite horizon assuming that the firm issues debt and debt is perpetual. Debt pays a constant coupon per instant of time and this determines tax benefits proportional to coupon payments. A payout rate δ is also introduced as in [3, 4]. On the failure time, agents which hold debt claims will get the residual value of the firm (because of bankruptcy costs), and those who hold equity will get nothing (the strict priority rule holds). Structural models assume constant corporate tax rates, thus tax benefits of debt are constant through time. The original assumption in [17] is that the firm has deductibility of interest payments for all firm's assets values above the failure level, producing a constant tax-sheltering value. Leland argues that default and leverage decisions might be affected by non constant corporate tax rates, because a loss of tax advantages is possible for low firm values. Thus in [17] section VI.A the author suggests that, when assets value decreases, it is more likely that profits will be lower than coupon payments and the firm will not be able to fully benefit tax savings. The empirical analysis of [14] confirms that the corporate tax schedule is asymmetric, in most cases it is convex. The quantitative impact of this asymmetry on the optimal default boundary and the leverage ratio is considered in [27] under the hypothesis of a piecewise linear tax function when the state variable is the operating income; the simulation study therein shows that the effect of tax asymmetry on the optimal leverage ratio is quantitatively significant, while it is lower on the optimal default boundary. Further [26] examines the relation between tax convexity and investment in the presence of a strictly convex (quadratic) tax function.

In this paper we extend Leland model by incorporating the possibility of two different corporate tax rates, namely τ_1, τ_2 and net cash outflows as in [3, 4]. We consider as state variable firm's current assets value. The switching from a corporate tax rate to the other is determined by the firm value crossing a critical level. We consider two alternative frameworks: at first, the switching barrier is assumed to be a constant exogenous level; then we analyze a more realistic scenario in which this level depends upon the amount of debt the firm has issued. In fact, as pointed out in [17], under U.S. tax codes, a necessary condition required to fully benefit tax savings, is that the firm's EBIT (earnings before interest and taxes) must cover payments required for coupons. We obtain an explicit form for the tax benefit claim, which allows us to study monotonicity and convexity of equity function, to find the endogenous failure level analytically in the general case with payouts and to prove its existence and uniqueness in the general case. Further, exploiting the linearity of the smooth pasting condition with respect to the coupon, we are able to study the optimal capital structure of the firm. Our approach differs from [17] since we solve the optimal control problem as an optimal problem in the set of passage times; the key method is the Laplace transform of the stopping failure time [1], [13], [16].

We introduce parameter $\theta := \frac{\tau_2}{\tau_1}$ as a measure of the degree of asymmetry. Our study shows that tax asymmetry increases the optimal failure level and reduces the optimal leverage ratio, with a more pronounced effect on optimal leverage ratios, thus confirming results in [27]. Nevertheless, we find that, as far as the magnitude is concerned, introducing a payout produces an even more significant reduction in optimal leverage ratios. Thus the joint effect of tax asymmetries and payouts drops down optimal leverage to empirically representative values and moreover seems to be a flexible way to capture differences among firms facing different tax-code provisions. For example observing firms belonging to different activity sectors, this could be a way to explain differences in observed capital structure decisions, mainly in leverage ratios. The analysis developed in [3, 4] showed that introducing payouts in a structural model with a unique corporate tax rate has the effect of reducing both optimal leverage ratio and optimal failure level. In the present paper we find that this reduction in both optimal leverage and optimal failure level increases as the degree of asymmetry of the tax schedule rises, meaning as the difference between the two corporate tax rates is higher. We study both the impact of asymmetry in tax benefits on optimal capital structure compared to what happens under a flat tax schedule (i.e. a unique constant tax rate) as benchmark model, but also how these decisions change as the asymmetry varies (i.e. for example if the switching barrier moves), showing two alternative approaches to measure the impact of asymmetry on corporate decisions. Finally, the maximum total value of the firm value, as far as optimal debt issuance decrease both with asymmetry and payouts, since less assets remain in the firm (due to payouts) and debt becomes less attractive (due to a potential loss of value). Further we study optimal capital structure when the switching barrier considered an increasing and linear function of the coupon level, in order to represent a more realistic framework in which EBIT is considered as a barrier determining a potential loss in coupon payments deductibility. In

such a case a higher profit is needed in order to fully benefit from tax savings. Given a payout rate, as the optimal coupon decreases for higher degrees of asymmetry, then also the optimal switching barrier decreases: this trade-off concerning the potential tax benefits loss leads to empirically representative value in predicted leverage ratios. One may wonder why payouts and asymmetry are considered in the same model and why their joint effect is quantitatively significant. Notice that δ and θ have a deeply different nature from an economic point of view: if it is true that in our model δ is exogenously given, we can also recognize that it can be partly modified or chosen by the firm, even when it is not a result of an endogenous decision (i.e. even if it does not depend on coupon payments, as in our case). What we mean is that, opposite to this, the corporate tax schedule is instead imposed to the firm by an external authority. Moreover, the corporate tax schedule could be very different depending for example on the sector in which the firm is operating in. Thus, we think that analyzing the joint effect of these two factors could be an interesting and flexible way to analyze and improve empirical findings inside a structural model of credit risk allowing to explain why a high dispersion in observed leverage ratios exists. We think that the joint effect is quantitatively significant since this model captures insights which are internal (payouts) and external to the firm (tax asymmetry). Factors apparently very far can produce a quantitative joint influence, as in our case.

The paper is organized as follows. In Section 2 we introduce the model where a payout rate δ is introduced and tax benefits are not constant through time, allowing for a possible switching in tax savings. We compute the tax benefit claim. In Section 3 the endogenous failure level is derived and the influence of tax asymmetry on it is analyzed. The optimal capital structure when the switching barrier is fixed and with debt dependent switching barrier is achieved in Section 4. Section 5 concludes. Proofs are in Appendix.

2 The Model

In this section we introduce a structural model of corporate debt in the spirit of [17]; nevertheless, our model exhibits two differences: the model for the firm's activities includes a parameter δ which represents a constant fraction of value paid to security holders (e.g. dividends, see also [19, 4]), further, we consider a corporate tax schedule which is not flat, meaning we suppose the corporate tax rate being not unique and constant through time. We derive the value of the tax benefit claim in this framework, following Leland [17] in modeling tax benefits of debt: asymmetry in corporate tax code provisions becomes also asymmetry in tax benefits of debt. The switching in tax benefits is due to asymmetry in the corporate tax schedule which we suppose being based on the existence of two different corporate tax rates. We assume the switching being determined by firm's assets value crossing a specified barrier, thus depending on current activities value a firm can face a different corporate tax rate.

We assume an infinite time horizon, as in [17]. This is a reasonable first approxima-

tion for long term corporate debt and enables us to have an analytic framework where all corporate securities depending on the underlying variable are time independent, thus obtaining closed forms. We consider a firm realizing its capital from both debt and equity. The firm has only one perpetual debt outstanding, which pays a constant coupon stream C per instant of time¹. This assumption can be justified, as Leland suggests, thinking about two alternative scenarios: a debt with very long maturity (in this case the return of principal has no value) or a debt which is continuously rolled over at a fixed interest rate (as in [19]). Bankruptcy is triggered endogenously by the inability of the firm to raise sufficient capital to meet its current obligations. On the failure time T , agents which hold debt claims will get the residual value of the firm, and those who hold equity will get nothing. Following Leland [17] we do not consider personal taxes, thus we model the tax benefits claim as a derivative depending directly on corporate tax rate provisions.

Suppose that firm's activities value is described by process $V_t = Ve^{X_t}$, where X_t evolves, under the risk neutral probability measure, as

$$dX_t = \left(r - \delta - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t, \quad X_0 = 0, \quad (1)$$

where W is a standard Brownian motion, r the constant risk-free rate, r , δ and $\sigma > 0$. The term δV_t represents the firm's cash flow: we can think of it as an after-tax net cash flow before interest, since we only consider tax benefits of debt. When bankruptcy occurs at stopping time T , a fraction α ($0 \leq \alpha < 1$) of firm value is lost (for instance paid to who takes care of the bankruptcy procedures), the debt holders receive the rest and the stockholders nothing, meaning that the strict priority rule holds. The failure passage time is determined when the firm value falls to some constant level V_B . The value of V_B is endogenously derived and will be determined with an optimal rule later. Define

$$T_{V_B} := \inf\{t \geq 0 : V_t = V_B\} = \inf\{t \geq 0 : X_t = \log \frac{V_B}{V}\}.$$

In the spirit of Leland we assume that from paying coupons the firm obtains tax deductions. Most structural models of credit risk assume a setting in which tax benefits are constant through time: from paying coupons a firm obtains tax deductions proportionally to the coupon payment. In [17] the corporate tax rate τ is assumed to be constant; nevertheless, in Appendix A the author derives the endogenous failure level in the case when there are no tax benefits for the assets value going under an exogenously specified level. The empirical analysis of [14] confirms that the corporate tax schedule is asymmetric. Moreover [27] assumes the hypothesis of a piecewise linear tax function and reports a quite significant impact of this asymmetry on the optimal leverage ratio. These studies motivate our extension of Leland's setting in the direction of a structural model with endogenous default boundary presenting a more general (even debt dependent) corporate tax schedule.

¹Instantaneous coupon payments can be written as $C := cF$, where F is face value of debt, supposed to be constant through time, as in Leland [17].

Following [17, 23] we can always express tax benefits as a claim on the underlying asset represented by the unlevered value of the firm V . We now describe the general scheme to determine the value of this claim for a given corporate tax schedule. On \mathbb{R}_*^+ let τ be the corporate tax function and F the tax benefits function. Tax benefits of debt can be seen as the value of a claim on V_t paying a continuous instantaneous dividend $\tau(V_t)C$ if there is no default and 0 in the event of bankruptcy. Let

$$G_t := e^{-rt}F(V_t) + \int_0^t e^{-rs}\tau(V_s)Cds,$$

represent the value of tax benefits at time t , discounted at the risk free rate r , plus cumulated tax-sheltering value of interest payment $\tau(V_t)C$ up to time t , discounted at r . To avoid arbitrage opportunities we impose such a process to be a local martingale under the risk neutral probability (see [23]). Assuming that F belongs to C_b^2 , then G is a true martingale since the martingale part of the process $e^{-rt}F(V_t)$ is $\int_0^t e^{-rs}F'(V_s)\sigma V_s dW_s$, F' is bounded and $e^{-rt}V_t$ is square integrable on $\Omega \times \mathbb{R}_*^+$. Then it follows using Doob theorem:

Lemma 2.1 *For any stopping time T the value of the tax benefits of debt is equal to:*

$$F(V) = \mathbb{E}_V \left[e^{-rT}F(V_T) + \int_0^T e^{-rs}\tau(V_s)Cds \right], \quad (2)$$

where the expectation is taken with respect to the risk neutral probability and we denote

$$\mathbb{E}_V[\cdot] := \mathbb{E}[\cdot | V_0 = V].$$

In this section the asymmetric tax benefits schedule is specified through the introduction of an exogenously given level of firm's assets value at which the tax deduction changes. We modify Leland [17] assumption about a unique constant level τ , considering a piecewise linear model in which two different corporate tax rates τ_1, τ_2 are in force. We assume that the deductibility of coupon payment generates tax benefits for all value $V \geq V_B$, but these tax savings are reduced when V falls to a specified constant value. As the firm approaches bankruptcy, it will lose tax benefits.

The corporate tax rate switches from τ_1 to τ_2 when the value of the firm reaches a certain (exogenous) barrier V_S , therefore the corporate tax function is equal to

$$\tau = \tau_1 \mathbf{1}_{(V_S, \infty)} + \tau_2 \mathbf{1}_{(V_B, V_S)} \quad (3)$$

depending on the firm's activities value V_t staying upon the prescribed level V_S . Obviously we assume $V_S > V_B$. The tax-sheltering value of interest payments will not be constant through time: it will be $\tau_1 C$ for $V \geq V_S$, and $\tau_2 C$ in case $V_B \leq V < V_S$. We assume $\tau_2 \leq \tau_1$, meaning loss of tax benefits below V_S .

The first passage time at V_S is defined

$$T_{V_S} = \inf\{t \geq 0 : V_t = V_S\} = \inf\{t \geq 0 : X_t = \log \frac{V_S}{V}\}. \quad (4)$$

Note that $V_{T_{V_B}} = V_B$ and $V_{T_{V_S}} = V_S$, as the process V_t is continuous.

Using integral representation of tax benefits we can write (2) as:

$$F(V) = \mathbb{E}_V \left[e^{-rT_{V_S} \wedge T_{V_B}} F(V_{T_{V_S} \wedge T_{V_B}}) + \int_0^{T_{V_S} \wedge T_{V_B}} e^{-rs} C \tau(V_s) ds \right]. \quad (5)$$

where $\tau(V_s)$ is specified by (3).

It is easily seen that in order to compute (5), it is enough to have explicit formulas for the Laplace transform of a double boundaries passage time. The formulas for the Laplace transforms are obtained in Appendix A, Propositions 6.2, 6.3. As V_B is a failure level, we impose that tax benefits are completely lost at failure, then the required boundary condition is $F(V_B) = 0$. Finally we can state the following result.

Proposition 2.2 *Suppose that the deduction tax function $\tau(\cdot)$ is defined by (3), then the tax benefits claim $F(V)$ in (2) is equal to:*

$$F(V) = \left(A_0 + A_1 V^{-\lambda_1} + A_2 V^{-\lambda_2} \right) \mathbf{1}_{(V_B, V_S)}(V) + \left(B_0 + B_2 V^{-\lambda_2} \right) \mathbf{1}_{(V_S, \infty)}(V), \quad (6)$$

where

$$A_0 = \frac{\tau_2 C}{r} \quad (7)$$

$$A_1 = \frac{C \lambda_2 V_S^{\lambda_1} (\tau_2 - \tau_1)}{r(\lambda_1 - \lambda_2)}, \quad (8)$$

$$A_2 = - \left(A_1 V_B^{-\lambda_1} + \frac{\tau_2 C}{r} \right) V_B^{\lambda_2}, \quad (9)$$

$$B_0 = \frac{\tau_1 C}{r}, \quad (10)$$

$$B_2 = A_2 + A_1 \frac{\lambda_1 V_S^{\lambda_2 - \lambda_1}}{\lambda_2}. \quad (11)$$

and λ_1, λ_2 are defined as

$$\lambda_1 = \frac{\mu - \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}, \quad \lambda_2 = \frac{\mu + \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2} \quad (12)$$

with

$$\mu := r - \delta - \frac{1}{2}\sigma^2. \quad (13)$$

We comment the above result in order to underline the effects of the tax asymmetry assumption on the value of the tax benefit claim $F(V)$. Under the hypothesis $\tau_2 \leq \tau_1$, it holds $A_1 > 0$, $A_2 < 0$, $B_2 < 0$. Therefore in both segments $V \geq V_S$ and $V_B \leq V < V_S$, the function $F(V)$ is strictly increasing w.r.t. firm's current assets value V . Further, we note that the tax benefits value in the segment $V \geq V_S$ is a strictly concave function of V , since B_2 is negative. If the tax rate were always τ_2 , both above and below V_S , we would have

$$F_L(V, \tau_2) = \frac{\tau_2 C}{r} - \frac{\tau_2 C}{r} V_B^{\lambda_2} V^{-\lambda_2},$$

which coincides with the result obtained in [17]. We now compare this value with $F(V)$ in our framework in case $V_B \leq V < V_S$:

$$F(V) = A_0 + A_1 V^{-\lambda_1} + A_2 V^{-\lambda_2},$$

with A_0, A_1, A_2 given by Equations (7)-(9). Notice that now, for all assets value below the switching barrier (but obviously above the failure level V_B) the value of the claim $F(V)$ exhibits three terms instead of two: while the constant term $\frac{\tau_2 C}{r}$ appears in both models, in Leland framework the term depending on $V^{-\lambda_1}$ does not appear. The presence of $A_1 V^{-\lambda_1}$ captures the effect of: i) payouts, since in case $\delta = 0$ we have $\lambda_1 = -1$; ii) most important, it captures the possible switching from τ_2 to a higher level τ_1 , thus representing the value of a possible gain in tax savings, through coefficient A_1 . This is why it is positive and increasing w.r.t. V . Coefficient A_1 reflects exactly the asymmetry in the corporate tax schedule; it is increasing w.r.t. both $\tau_2 - \tau_1$ and the switching barrier V_S . Coefficient A_2 is instead negative and depends on both the asymmetry of the corporate tax schedule and the default event.

Remark 2.3 *In Appendix A [17] the author proposes a structural model in which the instantaneous tax benefit is zero, if the firm's value V falls under a prescribed level. His approach to the problem is that of ordinary differential equations with boundary conditions. We observe that considering the particular case of $\delta = 0$ and $\tau_2 = 0$, we recover the same result as in [17].*

We are now ready to complete the description of the corporate capital structure model. Applying contingent claim analysis in a Black-Scholes setting, given the stopping (failure) time T_{V_B} , the expression for debt value is given by:

$$D(V, V_B, C) = \mathbb{E}_V \left[\int_0^{T_{V_B}} e^{-rs} C ds + (1 - \alpha) e^{-rT_{V_B}} V_B \right]. \quad (14)$$

It is important to stress that the corporate tax schedule does not affect directly debt value, thus equation (14) holds for whatever corporate tax schedule. What is important is knowing that corporate tax provisions have an influence on capital structure decisions since issuing debt allows to have some tax savings, thus a potential increase in firm's value.

But they do not affect directly debt value, which depends only on the coupon level and obviously, on the default event though bankruptcy costs α and the failure level V_B . The effect of asymmetry in the tax scheme will produce an impact on debt value only through the choice of the endogenous failure level and thus on the optimal coupon equity holders will choose.

The *total value* of the firm $v(V)$ consists of three terms: firm's assets value (unlevered), plus the value of the tax benefits claim $F(V)$ given in (6), less the value of the claim on bankruptcy costs:

$$v(V, V_B, C) = V + F(V) - \mathbb{E}_V[e^{-rT_{V_B}} \alpha V_B]. \quad (15)$$

Since an alternative but equivalent formulation for the total value of the firm is the sum of equity and debt values, finally it is possible to write *equity value* as:

$$E(V, V_B, C) = v(V, V_B, C) - D(V, V_B, C). \quad (16)$$

Equity holders have to define the capital structure of the firm. In order to do this, they have to choose both the endogenous failure level and the optimal amount of debt to issue. As stressed in [4] these are interrelated decisions which can hardly be separated. Our approach to the problem is to conduct the analysis in two stages: i) at first we determine the endogenous failure level, ii) then we find the optimal coupon, given the result about the default boundary.

3 Endogenous Failure Level

The aim of this section is to investigate the effects of introducing a different asymmetric corporate tax rate schedule on the endogenous failure level chosen by equity holders. We will conduct a detailed analytical study considering the influence of the corporate tax function in (3) on the firm's capital structure. The analysis is conducted for a given and fixed level of coupon payments, namely C .

3.1 Failure Level with exogenous switching barrier

Given the value of the tax benefits claim $F(V)$ in (6) we can write debt, equity and total value of the firm. First consider the debt function, which is not directly affected by (3), since for the moment we are considering V_B as a constant level and C is fixed. Debt value can be seen as the sum of a risk-free debt C/r plus a positive term depending on the risk of default. Using Proposition 6.1, (14) becomes:

$$D(V, V_B, C) = \frac{C}{r} + \left((1 - \alpha)V_B - \frac{C}{r} \right) \left(\frac{V_B}{V} \right)^{\lambda_2}. \quad (17)$$

Further using Proposition 2.2, the total value of the firm defined in (15) is equal to

$$v(V, V_B, C) = V + \left(\frac{\tau_2 C}{r} + A_1 V^{-\lambda_1} + A_2 V^{-\lambda_2} \right) \mathbf{1}_{(V_B, V_S)}(V) \quad (18)$$

$$+ \left(\frac{\tau_1 C}{r} + B_2 V^{-\lambda_2} \right) \mathbf{1}_{(V_S, \infty)}(V) - \alpha V_B \left(\frac{V_B}{V} \right)^{\lambda_2}.$$

Finally from (16) we obtain

$$E(V, V_B, C) = V + \left(\frac{\tau_2 C}{r} + A_1 V^{-\lambda_1} + A_2 V^{-\lambda_2} \right) \mathbf{1}_{(V_B, V_S)}(V) \quad (19)$$

$$+ \left(\frac{\tau_1 C}{r} + B_2 V^{-\lambda_2} \right) \mathbf{1}_{(V_S, \infty)}(V) - \frac{C}{r} - \left(V_B - \frac{C}{r} \right) \left(\frac{V_B}{V} \right)^{\lambda_2}.$$

Equity function must reflect its nature of an option-like contract. For any C , we have $E(V_B, V_B, C) = 0$ meaning that when V falls to V_B there is no equity to cover the firm's debt obligations, thus equity holders will chose to default. We first analyze the equity value for $V_B \leq V < V_S$:

$$E(V, V_B, C) = V - (1 - \tau_2) \frac{C}{r} + A_1 V^{-\lambda_1} + \left(-A_1 V_B^{-\lambda_1} + \frac{C}{r} (1 - \tau_2) - V_B \right) \left(\frac{V}{V_B} \right)^{-\lambda_2}, \quad (20)$$

where A_1 defined in (8) is positive if $\tau_2 < \tau_1$. Observe that the first term $V - (1 - \tau_2) \frac{C}{r}$ is nothing but equity value considering a constant tax-sheltering value of interest payments $\tau_2 C$, unless limit of time (when there is no risk of default). The term $A_1 V^{-\lambda_1}$ is strictly related to our tax benefits assumption: since in this model the corporate tax rate can switch from τ_1 to τ_2 , coefficient A_1 captures this effect. It depends on both the switching barrier V_S and the difference between the two tax levels, it disappears when considering a unique corporate tax rate (i.e. $\tau_2 = \tau_1$, as in [17]), and achieves its maximum in case $\tau_2 = 0$, meaning full loss of tax benefits below V_S . Coefficient A_1 represents the opportunity-cost of V being in $[V_B, V_S)$ instead of $[V_S, \infty)$. In fact, considering Leland [17] framework, equity value is increasing with respect to τ : our assumption about the tax deductibility scheme modifies the unique constant τ introducing an asymmetry. Suppose V being in $[V_B, V_S]$: this asymmetry becomes an opportunity, since $\tau_2 < \tau_1$. As a consequence, coefficient A_1 is positive and increasing w.r.t. $\tau_2 - \tau_1$, decreasing w.r.t. the switching barrier V_S . As the difference $\tau_2 - \tau_1$ increases, A_1 increases too: the possible gain in tax benefits in the event of $V = V_S$ is higher. As V_S becomes higher, coeteris paribus, the probability of V reaching V_S before reaching V_B is reduced, thus obtaining a gain in tax benefits becomes less likely and A_1 will be lower. The last term is exactly the option to default which is embodied in equity. Observe that in this case, the option to default will compensate equity holders also for the tax deductibility asymmetry, through the term $-A_1 V_B^{-\lambda_1}$. Therefore it must hold

$$-A_1 V_B^{-\lambda_1} + \frac{C}{r} (1 - \tau_2) - V_B > 0. \quad (21)$$

Analogously we consider the equity value for $V \geq V_S$:

$$E(V, V_B, C) = V - \frac{(1 - \tau_1)C}{r} + A_1 \frac{\lambda_1}{\lambda_2} V_S^{\lambda_2 - \lambda_1} V^{-\lambda_2} + \left(-A_1 V_B^{-\lambda_1} + \frac{C}{r}(1 - \tau_2) - V_B \right) \left(\frac{V}{V_B} \right)^{-\lambda_2}. \quad (22)$$

The first term $V - (1 - \tau_1)\frac{C}{r}$ represents equity value when there is no risk of default, with a constant corporate tax rate τ_1 . The second term captures two different effects: one due to the switching in tax benefits, the other related to default, both arising in the event of V falling to V_S . Observe that the second term $A_1 \frac{\lambda_1}{\lambda_2} V_S^{\lambda_2 - \lambda_1}$ is negative, reflecting the possible partial loss of tax benefits below V_S . Its negative effect on equity increases with an increase in the switching barrier V_S and also as a consequence of a higher difference $\tau_2 - \tau_1$. What remains is the option to default, which will be activated only if V reaches V_S . As previously, the option to default must have positive value, meaning constraint (21) being satisfied.

In the following proposition we analyze the verification of constraint (21) to get equity convex in V .

Proposition 3.1 *Suppose that $\tau_2 < \tau_1$, then condition (21) is satisfied for $V_B < \bar{V}_B$ such that*

$$\frac{1}{1 + A_1} \frac{C(1 - \tau_2)}{r} < \bar{V}_B < \frac{C(1 - \tau_2)}{r}.$$

Remark 3.2 *If $\delta = 0$, then condition (21) becomes $-A_1 V_B + \frac{C}{r}(1 - \tau_2) - V_B > 0$, therefore*

$$V_B < \frac{1}{1 + A_1} \frac{C(1 - \tau_2)}{r}. \quad (23)$$

The convexity condition required in [17] is $V_B < \frac{C(1 - \tau_2)}{r}$; note that, as $A_1 > 0$,

$$\frac{1}{1 + A_1} \frac{C(1 - \tau_2)}{r} < \frac{C(1 - \tau_2)}{r}.$$

On the other side, under the hypothesis $\delta > 0$ and a unique constant corporate tax rate $\tau_1 = \tau_2$, then $A_1 = 0$ and the condition (21) becomes $V_B < \frac{C(1 - \tau_2)}{r}$, as found in [3, 4]. This emphasizes the fact that the difference between these two convexity constraints is due to asymmetry in tax benefits: introducing asymmetry in tax benefits makes the convexity constraint on V_B more tight if compared to the case of a unique constant tax-sheltering value of $\tau_2 C$. This result does not depend on payouts, it is true for each level of δ .

Proposition 3.1 gives an upper bound for V_B ; nevertheless, due to limited liability of equity, the failure level V_B cannot be chosen arbitrarily small, but $E(V, V_B, C)$ must be non negative for all values $V \geq V_B$. To this end we write equity function (20) as:

$$E(V, V_B, C) = f(V, C) + g(V, V_B, C),$$

where

$$f(V, C) = V - (1 - \tau_2) \frac{C}{r} + A_1 V^{-\lambda_1}, \quad (24)$$

and

$$g(V, V_B, C) = \left(-A_1 V_B^{-\lambda_1} + \frac{C}{r} (1 - \tau_2) - V_B \right) \left(\frac{V}{V_B} \right)^{-\lambda_2}. \quad (25)$$

Constraint (21) is needed in order to make the option embodied in equity having positive value and its value $g(V, V_B, C)$ in (25) being a convex and decreasing function of V . The value $g(V, V_B, C)$ increases as V approaches V_B , therefore compensating equity holders for the reduction in equity due to a lower V . The function $f(V, C)$ in (24) represents equity value with no risk of default unless limit of time, when asymmetry in tax benefits is assumed. Under constraint (21) f is increasing and convex when $V \in [V_B, V_S)$. At point $V = V_B$, $g(V, V_B, C)$ and $f(V, C)$ have exactly the same value, with opposite sign. Thus the following proposition provides a condition such that an increment in V must produce an impact on equity value without default risk $f(V, C)$ higher than its effect on the option to default $g(V, V_B, C)$. This allows to keep $E(V, V_B, C) \geq 0$ when the option to default approaches its exercise instant, i.e. $V \rightarrow V_B$.

Proposition 3.3 *The function $V \mapsto E(V, V_B, C)$ is increasing and strictly convex in $V_B \leq V < V_S$ if V_B satisfies constraints (21) and*

$$V_B(1 + \lambda_2) + A_1 V_B^{-\lambda_1} (\lambda_2 - \lambda_1) \geq \frac{C(1 - \tau_2)}{r} \lambda_2, \quad (26)$$

Moreover $E(V, V_B, C) \geq 0$ for $V \geq V_B$.

Remark 3.4 *If $\delta = 0$, constraint (26) becomes*

$$V_B \geq \frac{2(1 - \tau_2)V_S C}{V_S(\sigma^2 + 2r) + 2C(\tau_1 - \tau_2)}. \quad (27)$$

Recalling also constraint (23), in order to have equity increasing and convex w.r.t. V , the endogenous failure level has to satisfy:

$$\frac{2(1 - \tau_2)V_S C}{V_S(\sigma^2 + 2r) + 2C(\tau_1 - \tau_2)} \leq V_B \leq \left(2 + \frac{\sigma^2}{r} \right) \frac{(1 - \tau_2)V_S C}{V_S(\sigma^2 + 2r) + 2C(\tau_1 - \tau_2)}.$$

We consider the coupon rate C being fixed and maximize equity value in order to find the endogenous failure level. To this end we impose the smooth-pasting condition (see [17] footnote 20 and [22] footnote 60):

$$\frac{\partial E}{\partial V} \Big|_{V=V_B} = 0. \quad (28)$$

Theorem 3.5 *Suppose constraint (21) holds, then the endogenous failure level $V_B(C; \tau_1, \tau_2; \delta)$ which satisfies (28) exists and is unique, under the condition*

$$V_S \geq \frac{(1 - \tau_1)C}{r} \frac{\lambda_2}{1 + \lambda_2}. \quad (29)$$

We note that condition (28) is equivalent to

$$\frac{\partial f}{\partial V} \Big|_{V=V_B} = - \frac{\partial g}{\partial V} \Big|_{V=V_B},$$

where f and g are defined in (24) and (25). Thus a solution of (28) is an implicit solution of ²

$$(1 + \lambda_2) = \frac{\lambda_2 C}{r} V_B^{-1} \left\{ \left(\frac{V_S}{V_B} \right)^{\lambda_1} (\tau_2 - \tau_1) + (1 - \tau_2) \right\}. \quad (31)$$

Remark 3.6 *Note that this choice of bankruptcy level also optimizes $V_B \mapsto E(V, V_B, C)$.*

Under constraints (21) and (29), equity is increasing (and convex) w.r.t. assets value V if $V_B \geq V_B(C; \tau_1, \tau_2; \delta)$, where $V_B(C; \tau_1, \tau_2; \delta)$ satisfies the smooth pasting condition, being the minimum failure level that equity holders can choose due to limited liability of equity. Consider the function g in (25) which is the option to default embodied in equity. The endogenous failure level $V_B(C; \tau_1, \tau_2; \delta)$ satisfies $\frac{\partial E(V, V_B, C)}{\partial V_B} = 0$ if and only if $\frac{\partial g(V, V_B, C)}{\partial V_B} = 0$. Further $\frac{\partial g(V, V_B, C)}{\partial V_B} = 0$ is equivalent to equation (31), which is exactly the optimal stopping problem equation. It gives the optimal exercise time of the option to default embodied in equity. Differentiating equity value w.r.t. V_B , we have $\partial_{V_B} E = \partial_{V_B} g$ which has the same sign as the decreasing function

$$V_B \mapsto A_1 V_B^{-\lambda_1} (\lambda_1 - \lambda_2) + \lambda_2 \frac{C}{r} (1 - \tau_2) - V_B (1 + \lambda_2).$$

This function is positive then negative and cancel at point V_B exactly solution to Equation (31).

If $\delta = 0$, explicit solution can be obtained by solving equation (31) with respect to V_B

$$V_B(C; \tau_1, \tau_2; 0) = \frac{2CV_S(1 - \tau_2)}{V_S(\sigma^2 + 2r) + 2C(\tau_1 - \tau_2)}, \quad (32)$$

²In particular if we assume a switch to zero tax level, i.e. $\tau_2 = 0$, we obtain the optimal failure $V_B(C; \tau_1, 0; \delta)$ as implicit solution of

$$(1 + \lambda_2) = \frac{\lambda_2 C}{r} V_B^{-1} \left(1 - \tau_1 \left(\frac{V_S}{V_B} \right)^{\lambda_1} \right). \quad (30)$$

This result completes the analysis in [19] Appendix B, in the case where we let the maturity $T \rightarrow \infty$ since the authors analyze the switch to zero tax level only in the no-dividend case.

thus extending [17] Appendix A (case $\tau_2 = 0$).

Finally in case $\tau_2 = \tau_1 =: \tau$ we obtain:

$$V_B(C; \tau, \tau; \delta) = \frac{\lambda_2 C(1 - \tau)}{r(1 + \lambda_2)}, \quad (33)$$

which is the endogenous failure level in case of a unique constant tax-sheltering value of interest payment τC with a payout rate δ . This result extends that found in [17] equation (14) to the case when the firm's assets value model accounts for a non zero payout rate. See [4] for a detailed analysis of this case.

Remark 3.7 *Observe that under constraint (29), the endogenous failure level $V_B(C; \tau_1, \tau_2; 0)$ is lower than the switching barrier exogenously given V_S . The following inequality*

$$\frac{2(1 - \tau_2)V_S C}{V_S(\sigma^2 + 2r) + 2C(\tau_1 - \tau_2)} \leq V_S \quad (34)$$

holds under

$$V_S \geq \frac{2C(1 - \tau_1)}{(\sigma^2 + 2r)} \quad (35)$$

which is exactly constraint (29) in case $\delta = 0$.

3.1.1 Effect of corporate tax rate asymmetry on the endogenous failure level

In this paragraph we analyze the impact of the corporate tax function $\tau(\cdot)$ defined in (3) on the endogenous failure level. We fix τ_1 and V_S , and we study the influence of τ_2 on the endogenous failure level, given the coupon C . Introducing asymmetry makes debt more or less attractive hence it should increase or decrease the optimal leverage ratio. Asymmetry also makes more or less attractive to keep a loss-making firm alive, hence it should raise or decrease the endogenous failure level and bring default closer or farther.

In order to work with explicit formulas we consider the no-payout case $\delta = 0$: therefore $\lambda_1 = -1$ and $\lambda_2 = \frac{2r}{\sigma^2}$. In the general case we will resort to numerical comparisons.

Let us consider: the endogenous failure level obtained with the constant instantaneous tax benefits in [17]

$$V_{B_L}(C; \tau_1; 0) = \frac{2C(1 - \tau_1)}{\sigma^2 + 2r}, \quad (36)$$

the level (32) obtained in the case of switching between two tax levels and, as a particular case of (32) with $\tau_2 = 0$, the switch to zero tax benefits (as in [17] Appendix A)

$$V_B(C; \tau_1, 0; 0) = \frac{2CV_S}{V_S(\sigma^2 + 2r) + 2\tau_1 C}. \quad (37)$$

Proposition 3.8 *The function*

$$\tau_2 \mapsto V_B(C; \tau_1, \tau_2; 0)$$

defined in (32) is decreasing. In particular $V_B(C; \tau_1, 0; 0)$ in (37) is greater than $V_B(C; \tau_1, \tau_2; 0)$ in (32). Further for any $\tau_1 > \tau_2$

$$V_B(C; \tau_1, \tau_2; 0) > V_{B_L}(C; \tau_1; 0), \quad V_{B_L}(C; \tau_2; 0) > V_B(C; \tau_1, \tau_2; 0).$$

We can observe that under our asymmetric corporate tax schedule, a higher τ_2 will increase equity value (for each coupon level C), and reduce the endogenous failure level $V_B(C; \tau_1, \tau_2; 0)$. Thus, increasing tax deductions could be a way to support firms in crisis times.

Finally we conclude that in the model where $\delta = 0$ a higher asymmetry in the tax deductibility will increase the failure level endogenously chosen. We can introduce $\theta := \frac{\tau_2}{\tau_1}$ as a measure of the degree of asymmetry of the corporate tax schedule: $\theta = 1$ represents Leland framework (no asymmetry), $\theta = 0$ is the maximum asymmetry case, meaning full loss of tax shelter below V_S . Any other case $0 \leq \theta \leq 1$ represent nothing but an intermediate asymmetric scenario. As asymmetry increases, meaning $\theta := \frac{\tau_2}{\tau_1}$ closer to 0, the endogenous failure level $V_B(C; \tau_1, \tau_2; 0)$ increases for any value of the exogenous switching barrier. In such a case, in fact, below V_S the firm will have less tax benefits, due to the lower τ_2 , bringing the endogenous failure level higher.

Remark 3.9 *Suppose for a moment τ_2 being fixed. Note that the application $\tau_1 \mapsto V_B(C; \tau_1, \tau_2; 0)$ is decreasing. This is in line with a reduction in the degree of asymmetry of the corporate tax schedule (i.e. θ closer to 1). A higher τ_1 allows the firm to have greater tax savings above V_S , bringing equity value higher both above and below (coefficient A_1 will be higher) the switching barrier V_S , thus bringing down the endogenous failure level.*

The impact of the deductibility asymmetry affects the endogenous failure level also through the exogenous switching barrier V_S . In the no-payout case it is easily seen the following.

Proposition 3.10 *The following result holds: the failure level $V_B(C; \tau_1, \tau_2; 0)$ in (32):*
i) is decreasing with respect to the exogenous barrier V_S if $\tau_2 > \tau_1$,
ii) is increasing with respect to the exogenous barrier V_S if $\tau_2 < \tau_1$.

Assume that $\tau_2 < \tau_1$. Given a certain degree of asymmetry, meaning $\theta := \frac{\tau_2}{\tau_1}$ being fixed, suppose to change the exogenous switching barrier V_S . This means that a higher V_S will increase the endogenous default boundary. Starting from $V \geq V_S$, the switching from τ_1 to τ_2 will be more likely, thus it will be more likely losing some tax benefits. As

extreme case we can consider what happens as V_S increases: the model will approach Leland framework with a unique constant tax-sheltering value on interest payments $\tau_2 C$. Equity holders will lose the opportunity to switch from τ_2 to a higher tax savings region, thus equity will be lower and the endogenous failure level higher, as shown in Proposition 3.8. All this holds since for the moment we are conducting an analysis supposing the coupon C being fixed. Notice that these results can also be seen as those ones obtained in a limit-model with constant τ_2 : when the coupon is fixed, the endogenous failure level is decreasing w.r.t. the corporate tax rate (see [17]), confirming our results in this subsection.

A change in θ , as far as a change in V_S , should be an alternative way to analyze how the impact of a variation in the asymmetry of the corporate tax schedule can affect optimal capital structure decisions, as we will do in Section 4. Observe that a change in θ or a change in V_S produce a different effect on the asymmetry of the tax schedule: we propose to interpret θ as a *vertical* measure of asymmetry, V_S as *horizontal* measure. What we mean is that θ modifies the degree of asymmetry, by acting on the distance between the two corporate tax rates, thus measuring the potential instantaneous loss of tax benefits at point $V = V_S$. A change in V_S represents an *horizontal* measure of asymmetry since it modifies the range of firm's values for which the firm faces a higher (lower) deductibility. When $\theta \rightarrow 1$, or $V_S \rightarrow V_B$, the limit-model is a framework with a flat corporate tax schedule with a constant corporate tax rate τ_1 , but the economic intuition behind is completely different. Consider a coeteris paribus analysis in which all variables except θ are constant: as θ moves, what is changing is only the measure of the potential loss in tax benefits, meaning the distance between the two levels. As opposite case, when V_S is the only variable to move, the potential loss in tax benefits is still the same, what changes is the probability of reaching the barrier, thus the likelihood of the potential loss.

3.2 Failure level with debt dependent switching barrier

In section 3.1 we assumed that the switching barrier V_S being exogenously given. Nevertheless this hypothesis is not completely realistic, and we expect that V_S will depend on the amount of debt issued by the firm (see [17] section VI.A). If assets value falls, it is more likely that profits will be lower than coupon payments, thus the firm will not fully benefit tax savings. Under U.S. tax codes, a necessary condition required to fully benefit tax savings, is that the firm's EBIT (earnings before interest and taxes) must cover payments required for coupons (see [17]). We now introduce the rate of EBIT and suppose it is related³ to assets value in the following way:

$$EBIT := aV - k, \tag{38}$$

with $0 < a < 1, k > 0$, where k represents costs and a is a fraction of firm's current assets value. In this case the gross profit falls to 0 when V equals $\frac{k}{a}$. We assume that the

³In [17] EBIT is modeled as a linear function of V , and also in [18] is supposed to be equal to a constant fraction of firm's assets value.

corporate tax rate is τ_1 in case $EBIT - C \geq 0$, and τ_2 otherwise, with $\tau_2 \leq \tau_1$. Under this specification the switching barrier V_S depends upon the amount of debt issued by the firm in the following way:

$$V_S = k + \frac{1}{a}C. \quad (39)$$

In this scenario the switching barrier is not exogenously given: it depends on a constant term k due to costs needed to determine the rate of EBIT, then it is a linear function of the coupon level. The switching barrier V_S increases with both k, C : a higher profit is required to cover higher costs k and/or higher interest payments, in order to benefit tax savings from issuing debt.

We now analyze how this different definition of the switching barrier V_S affects the endogenous failure level. The endogenous failure level is optimally chosen by equity holders by applying the smooth pasting condition; when applying the smooth pasting condition, we differentiate equity w.r.t. V and then evaluate this derivative at point V_B . We stress that definition (39) makes the switching barrier dependent and linear on C , but V_S does not depend on firm's current assets value V . Thus we can use results from Section 3.1 about equity value in order to find the default boundary in this case. In the case $\delta = 0$ we consider equation (32) and modify it according to the debt dependent switching barrier (39), the endogenous failure level becomes:

$$V_B^c(C; \tau_1, \tau_2; 0; k, a) = \frac{2C(ak + C)(1 - \tau_2)}{(ak + C)(\sigma^2 + 2r) + 2aC(\tau_1 - \tau_2)}. \quad (40)$$

Look at increasing costs k or reducing a , the fraction of firm's value necessary to determine the rate of EBIT: this will bring default closer, rising the endogenous failure level in (40) optimally chosen by equity holders.

Proposition 3.11 *The endogenous failure level $V_B^c(C; \tau_1, \tau_2; 0; k, a)$ defined in (40) is:*

- i) increasing and concave w.r.t. k ;*
- ii) decreasing and convex w.r.t. a ;*
- iii) increasing and concave w.r.t. C .*

Consider a comparative static analysis: if k increases and/or a reduces, EBIT is lower for each firm's assets value V , thus the default boundary is higher since debt has a greater likelihood of losing its tax benefits, meaning for the firm is more likely to lose potential value.

We analyze the relation between total coupon payments supported by the firm and the endogenous failure level chosen by equity holders under this debt dependent asymmetry framework. A comparison between (32), (36) and (40) shows that under the assumption of tax benefits asymmetry, the endogenous failure level is an increasing and concave function of the coupon level, instead of being a linear increasing function of C (in case of a unique constant corporate tax rate). It is still true that the endogenous failure level is independent

of firm's current assets value V and the fraction (of firm's value) α which is lost because of bankruptcy procedures⁴. When the corporate tax rate is unique, a change in the coupon level affects the optimal equity holders' choice in the same way for all coupon levels. A debt-dependent asymmetry, meaning the switching barrier depending on coupon, introduces a different effect through modifying the shape of the endogenous failure level as function of C . As a consequence, a change in C modifies the endogenous failure level with different magnitudes, depending on the value of outstanding debt. If the firm is supporting very high interest payments, a reduction (increase) in the coupon level will produce a small effect on the failure level, while in case of low coupon payments, a variation in C will strongly affect the endogenous default boundary, producing a bigger impact on it.

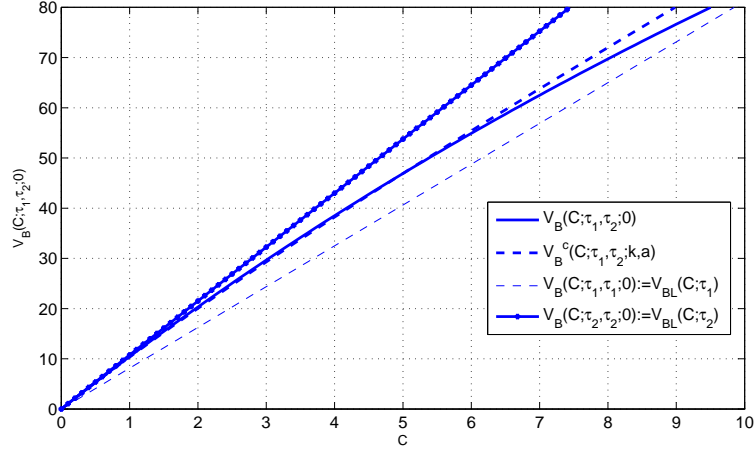


Figure 1: **Endogenous failure level.** This plot shows the behavior of the endogenous failure level w.r.t. coupon level C . We consider four alternative scenarios: two with a constant corporate tax rate representing Leland framework respectively with τ_1 , τ_2 alternatively. Then we consider asymmetric tax benefits cases: $V_B(C; \tau_1, \tau_2; 0)$ when the switching barrier is exogenous, $V_B(C; \tau_1, \tau_2; 0; k, a)$ when it is debt dependent. Parameters values are: $\sigma = 0.2$, $r = 0.05$, $\delta = 0$, $\tau_1 = 0.35$, $\theta = 0.4$, $V_S = 90$. We then consider $k = 60$, $a = \frac{1}{6}$, giving $V_S^c = 60 + 6C$. Recall that $\theta := \frac{\tau_2}{\tau_1}$ represents the degree of asymmetry in tax benefits.

Figure 1 shows the behavior of the endogenous failure level when different frameworks are considered: two constant tax benefits cases (alternatively a unique constant τ_1 or τ_2), two switching scenarios, one with V_S exogenous ($V_S = 90$), the other with the switching barrier debt dependent ($V_S^c = 60 + 6C$). Leland frameworks with constant τ_1 , τ_2 represent

⁴The independence w.r.t. α means that bankruptcy costs does not *directly* affect the endogenous failure level, since the strict priority rule holds. Bankruptcy costs will instead affect the optimal failure level through the choice of the optimal coupon C^* which maximizes total firm value.

two extreme boundaries between which both the endogenous failure levels obtained under asymmetric tax benefits lie. We now compare (32) and (40): they are both increasing and concave w.r.t. coupon level C . When coupon payments are low, (32) is greater than (40), though their difference is very small. As coupon increases, the behavior completely changes: the debt dependent switching barrier causes the failure level to be higher than in case V_S constant, and the difference between the two levels increases too. The reason is that the debt dependent switching barrier (39) increases with coupon level, so a firm paying a high coupon C is facing a higher switching barrier, thus a greater probability of losing tax benefits, since now EBIT must cover a greater value of interest payments.

Proposition 3.12 *Consider $V_B(C; \tau_1, \tau_2; 0)$ in (32) and $V_B^c(C; \tau_1, \tau_2; 0; k, a)$ in (40). The following holds:*

$$\begin{aligned} V_B(C; \tau_1, \tau_2; 0) &> V_B^c(C; \tau_1, \tau_2; 0; k, a), \quad \text{if } V_S > \frac{C}{a} + k \\ V_B(C; \tau_1, \tau_2; 0) &= V_B^c(C; \tau_1, \tau_2; 0; k, a), \quad \text{if } V_S = \frac{C}{a} + k \\ V_B(C; \tau_1, \tau_2; 0) &< V_B^c(C; \tau_1, \tau_2; 0; k, a), \quad \text{if } V_S < \frac{C}{a} + k. \end{aligned}$$

4 Optimal Capital Structure

In this section we determine the optimal capital structure within the model assuming the corporate tax function (3) in both cases of exogenous and debt dependent switching barrier. In both cases we give numerical results in the general framework with both asymmetry and payouts. As particular case, we show also what happen for $\delta = 0$ in order to isolate the asymmetry effect on corporate financing decisions.

Before considering optimal capital structure, the coupon was supposed to be fixed. We now turn to the optimization of the total value of the firm depending on the endogenous failure level solution of the optimal stopping problem faced by equity holders. Once determined this default boundary as an endogenous one, equity holders will incorporate this decision into the total value of the firm. Then maximize it w.r.t. C in order to find the optimal amount of debt to issue, that one which guarantees the maximum total value of the firm due to the limited liability constraint. Thus the optimal coupon, namely C^* , maximize total firm value. Once found, we replace C^* in all expressions of previous subsections in order to fully describe the optimal capital structure.

We then consider and show numerical results analyzing each financial variable at its optimal level and study effects of both i) corporate tax asymmetry and ii) payout rate on optimal coupon C^* , optimal debt value D^* , optimal equity value E^* , optimal default boundary V_B^* and optimal total value of the firm v^* . We also analyze the optimal yield spread $R^* - r$ where $R^* := \frac{C^*}{D^*}$, and the optimal leverage ratio, defined as the ratio between optimal debt and optimal total value $L^* := \frac{D^*}{v^*}$ (when coupon is at its optimal level C^*).

4.1 Exogenous switching barrier

Here we show how optimal capital structure is affected by both asymmetry and payouts when the switching barrier is exogenously given.

The optimal coupon C^* must be chosen in order to maximize

$$C \mapsto v(V, V_B(C; \tau_1, \tau_2; \delta), C),$$

where $v(V, V_B(C; \tau_1, \tau_2; \delta), C)$ is defined in (18). The optimal failure level is not given in closed form, nevertheless the following result allows us to study the optimal capital structure.

Proposition 4.1 *The function $V_B \mapsto C(V_B; \tau_1, \tau_2; \delta)$ is increasing, where V_B is implicitly given by equation (31).*

It follows that the study of the influence of the coupon C on the total value of the firm v , is equivalent to study v as function of V_B . Thus, optimizing $C \mapsto v(V, V_B(C; \tau_1, \tau_2; \delta), C)$ is equivalent to optimize $V_B \mapsto v(V, V_B, C(V_B; \tau_1, \tau_2; \delta))$.

Remark 4.2 *Even if we are not able to get an analytical expression of the optimal coupon C^* , we obtain that it has to satisfy constraints (21) and (26). We numerically determine the optimal coupon and verify that these constraints are satisfied for our case studies.*

In the case $\delta = 0$ the endogenous failure level is given in closed form by equation (32). Then we study the optimal capital structure by maximizing the application

$$C \mapsto v(V, V_B(C; \tau_1, \tau_2; 0), C).$$

Proposition 4.3 *The function $C \mapsto v(V, V_B(C; \tau_1, \tau_2; 0), C)$ is a concave function achieving a maximum at point C^* as solution of $\frac{\partial v(V, V_B(C; \tau_1, \tau_2; 0), C)}{\partial C} = 0$, under the condition⁵ $\tau_1 < \frac{2}{3} + \frac{\tau_2}{3}$. Thus an optimal capital structure exists and is unique.*

Finally the optimal failure level $V_B^*(V; \tau_1, \tau_2; \delta)$ is obtained by plugging the optimal coupon C^* into the endogenous failure level given by Theorem 3.5, that is $V_B(C^*(V); \tau_1, \tau_2; \delta)$.

In order to study the asymmetry effect on these variables we consider $\theta := \frac{\tau_2}{\tau_1}$, $\theta \geq 0$, measuring the *vertical degree of asymmetry* of the corporate tax schedule.

Notice that $\theta = 1$ represents a non asymmetric case: moreover, considering also $\delta = 0$ will lead exactly to results obtained by Leland [17]. The case $\delta \neq 0$, $\theta = 1$ is the one noted in Remark (33), which is comparable with our analysis in [4], where a detailed analysis

⁵This conditions is always satisfied with our parameter values, since we always consider $\tau_1 < \frac{2}{3}$.

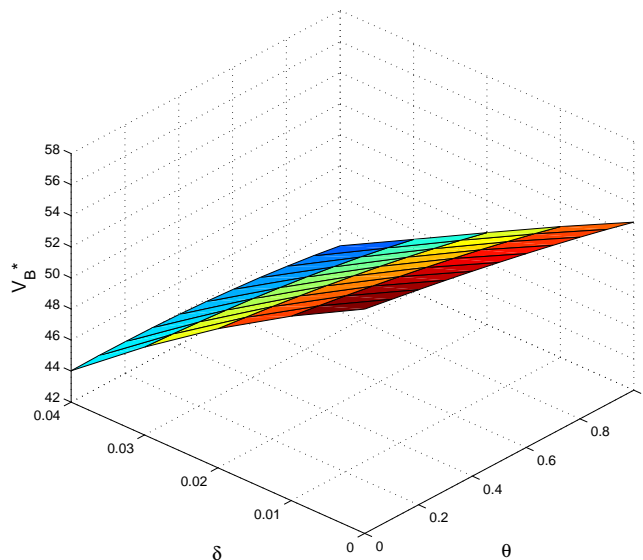


Figure 2: **Optimal failure level as function of δ, θ .** This plot shows the behavior of the optimal failure level V_B^* as function of dividend δ and degree of asymmetry θ . The switching barrier V_S is exogenous and parameters values are: $\sigma = 0.2, r = 0.06, \tau_1 = 0.35, \alpha = 0.5, V_S = 90, V = 100$. Recall that $\theta := \frac{\tau_2}{\tau_1}$ represents the degree of asymmetry in tax benefits.

about the influence of payouts on optimal capital structure decisions is conducted from both a qualitative and quantitative point of view. Asymmetry increases as θ goes to 0, achieving its maximum for $\theta = 0$ (maximum asymmetry case). This scenario represents a switching from a tax level τ_1 to zero-tax benefits: this happens when tax benefits are completely lost for $V < V_S$.

First notice that tax asymmetry raises the optimal failure level $V_B^*(V; \tau_1, \tau_2; \delta)$: for any value of δ (and for values of $V_S < V$), as the tax asymmetry increases then the optimal failure level $V_B^*(V; \tau_1, \tau_2; \delta)$ increases. The opposite happens when considering the payout influence, given a degree of asymmetry θ . For any fixed value of θ , the optimal failure level decreases as δ increases from 0 to 0.04. Results are in Table 1, while Figure 2 shows the behavior of the optimal failure level as function of both δ, θ . From [4] we know that only introducing payouts will bring to a lower optimal failure level when the corporate tax rate is unique and constant through time. This more general scenario, where both payouts and asymmetry in tax benefits are in force, shows through our numerical analysis that the final joint effect can be quantitatively significant. Consider as extreme cases $\theta = 1, \delta = 0$

and $\theta = 0, \delta = 0.04$: passing from no asymmetry and no payouts, to a payout rate equal to a 4% of current assets value, brings to a reduction in optimal failure level of around 8.8%, passing from 52.82 to 44.02.

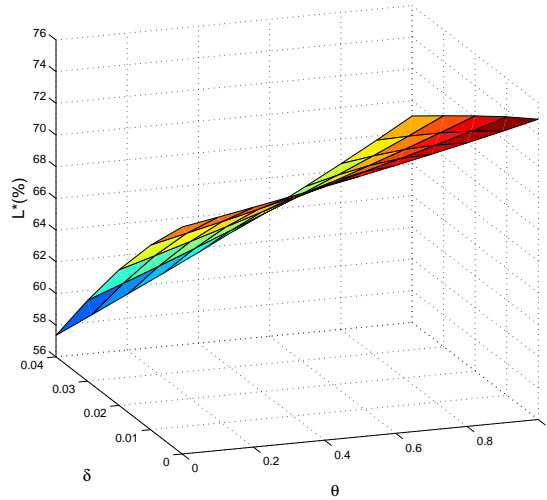


Figure 3: **Optimal Leverage as function of δ, θ .** This plot shows the behavior of optimal leverage ratio L^* as function of dividend δ and degree of asymmetry θ . The switching barrier V_S is exogenous and parameters values are: $\sigma = 0.2, r = 0.06, \tau_1 = 0.35, \alpha = 0.5, V_S = 90, V = 100$. Recall that $\theta := \frac{\tau_2}{\tau_1}$ represents the degree of asymmetry in tax benefits.

Further optimal leverage ratio is strongly affected by asymmetry in the corporate tax schedule as shown in Figure 3. In order to isolate the asymmetry effect, consider Table 1 in case $\delta = 0$: results are due only to the switching in tax benefits and bring to a reduction in optimal debt, optimal total value of the firm and also optimal leverage ratios.

Extending the analysis by considering the general case in which both $\delta > 0, 0 \leq \theta < 1$ shows that for each level of payouts, increasing the degree of asymmetry reduces optimal leverage and this effect is stronger when the the payout rate is higher. Consider the last column of Table 1: comparing the two extreme cases $\theta = 1$ and $\theta = 0$, the difference in optimal leverage ratio is 4.5% when $\delta = 0$, 6% when $\delta = 0.01$ and 12% when $\delta = 0.04$. Tax asymmetry has a negative effect on optimal leverage ratios L^* : for any value of δ considered, L^* decrease as the degree of asymmetry increases, that is as $\theta \rightarrow 0$. The decrease of the optimal leverage is quantitatively more significant as the payout rate rises. Analogously, as observed in [3, 4] the capital structure of a firm is strongly affected by payouts. From [3, 4] we know that introducing payouts in a structural model with a unique corporate tax rate τ has the effect of reducing optimal leverage ratios. Table 1 allows us to

confirm this result also when the tax schedule is asymmetric. We consider as extreme cases to compare $\delta = 0$ and $\delta = 0.04$. Considering a unique corporate tax rate means $\theta = 1$: in such a case we know from [3, 4] that the difference in optimal leverage ratio is quite 6%. We now introduce convexity and look at $\theta = 0.8, \theta = 0.4, \theta = 0$: this difference in optimal leverage becomes respectively 7%, 11% and 13%. A higher payout will lower the optimal total value of the firm v^* , since a lower debt issuance can be supported because less assets remain in the firm. As a consequence, this will bring down leverage ratios. But also the asymmetry effect is to reduce leverage ratios, since the potential loss in tax benefits due to the existence of the switching barrier makes debt less attractive: considering a scenario where both effects exist, will bring to a strong reduction in predicted optimal total value of the firm, optimal debt and optimal leverage ratios. We now consider Leland [17] case, i.e. $\theta = 1, \delta = 0$, and compare it with a scenario in which both payouts and asymmetry exists, meaning $\theta = 0, \delta = 0.04$ in order to capture the joint effect of these two realistic generalizations. Observe that passing from no asymmetry and no payouts, to a payout rate equal to a 4% of current assets value V , brings to a dramatic reduction in optimal leverage: in such a case, this joint influence of δ, θ brings to an optimal leverage ratio of 57.36%, with a significant reduction of 17% from Leland result of a 75%-leveraged firm, leading to a value which is more in line with historical norms⁶. This strong impact on optimal leverage ratios suggests that asymmetry and payouts seem to be important factors involved in the determinacy of corporate capital structure decisions.

Figure 4 shows the behavior of optimal coupon C^* as function of δ and θ . Observe that for each degree of convexity $0 \leq \theta < 1$ the optimal coupon is decreasing w.r.t. δ , extending results in [3, 4] to the case of asymmetric corporate tax schedule. What we stress now in this general framework where both payouts and an asymmetric tax scheme interact, is that this negative effect of payouts on C^* is greater as the asymmetry in tax benefits increases, i.e. as $\theta \rightarrow 0$. From Figure 4 we can also observe that the optimal coupon C^* decreases as $\theta \rightarrow 0$ for each level of the payout rate δ . The economic reason is that introducing asymmetry in tax benefits makes debt less attractive for the firm, thus leading to a not negligible reduction in the optimal coupon level choice. The decrease in C^* due to the asymmetric tax benefits scheme will be higher as payouts increase, as we can note considering the slope in Figure 4 w.r.t. θ for each level of δ . Payouts and asymmetry in tax benefits influence each other by increasing the magnitude of their own effects on the optimal coupon, bringing to a joint influence on optimal coupon which is quantitative significant. To analyze the interaction between δ and θ on C^* consider for example three alternative scenarios $\theta = 1, 0.4, 0$: when δ goes from 0 to 0.04, the optimal coupon reduces from 6.5% to 6.23% in case $\theta = 1$, from 6.03% to 5.2% in case $\theta = 0.4$, from 5.78% to 4.32% in case $\theta = 0$. The reduction in C^* due to an increased payout is considerably higher as asymmetry in tax benefits increases: when tax benefits are completely lost under V_S the reduction of optimal coupon is more than 5 times the reduction in case of a constant τ . Our analysis in this paper confirms our results in [3, 4] and moreover extend their

⁶Leland [17] in his Section D observes that a leverage of 52% is quite in line with historical norms.

$\delta = 0$							
θ	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*
0	5.784	84.149	87.389	35.519	56.435	119.668	70.318 %
0.1	5.844	85.136	86.467	35.253	56.142	120.389	70.717 %
0.2	5.906	86.163	85.489	34.975	55.837	121.138	71.128 %
0.3	5.971	87.234	84.454	34.683	55.518	121.918	71.552 %
0.4	6.038	88.352	83.357	34.378	55.186	122.730	71.989 %
0.5	6.107	89.519	82.195	34.057	54.838	123.576	72.441 %
0.6	6.179	90.741	80.965	33.719	54.474	124.461	72.908 %
0.7	6.254	92.022	79.660	33.363	54.092	125.386	73.391 %
0.8	6.333	93.367	78.278	32.987	53.691	126.355	73.893 %
0.9	6.415	94.783	76.812	32.590	53.267	127.372	74.414 %
1.0	6.501	96.274	75.256	32.167	52.820	128.442	74.956 %
$\delta = 0.01$							
θ	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*
0	5.539	79.524	96.489	38.047	53.868	117.571	67.639 %
0.1	5.613	80.658	95.868	37.683	53.604	118.341	68.157 %
0.2	5.689	81.838	95.167	37.306	53.325	119.144	68.688 %
0.3	5.768	83.069	94.384	36.914	53.030	119.983	69.234 %
0.4	5.850	84.355	93.515	36.506	52.720	120.861	69.795 %
0.5	5.935	85.701	92.554	36.080	52.391	121.781	70.373 %
0.6	6.024	87.112	91.497	35.634	52.043	122.746	70.969 %
0.7	6.116	88.594	90.336	35.166	51.673	123.760	71.585 %
0.8	6.212	90.155	89.067	34.674	51.281	124.829	72.223 %
0.9	6.313	91.802	87.680	34.154	50.865	125.957	72.884 %
1.0	6.419	93.545	86.169	33.604	50.420	127.149	73.571 %
$\delta = 0.04$							
θ	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*
0	4.637	63.933	125.230	47.526	44.023	111.459	57.360 %
0.1	4.772	65.630	127.059	46.690	44.048	112.321	58.431 %
0.2	4.911	67.390	128.708	45.844	44.043	113.234	59.514 %
0.3	5.054	69.220	130.171	44.981	44.007	114.201	60.612 %
0.4	5.203	71.129	131.439	44.100	43.942	115.229	61.729 %
0.5	5.357	73.129	132.503	43.194	43.845	116.323	62.867 %
0.6	5.517	75.230	133.352	42.261	43.716	117.491	64.031 %
0.7	5.684	77.447	133.974	41.292	43.553	118.739	65.224 %
0.8	5.860	79.794	134.354	40.284	43.356	120.078	66.452 %
0.9	6.044	82.291	134.477	39.227	43.121	121.518	67.719 %
1	6.239	84.957	134.326	38.114	42.847	123.072	69.031 %

Table 1: **Effect of payouts and asymmetry in the tax schedule on all financial variables at their optimal level when the switching barrier V_S is exogenous.** Base case parameter values: $V_0 = 100$, $\sigma = 0.2$, $\tau_1 = 0.35$, $r = 6\%$, $\alpha = 0.5$. This table reports results considering an exogenous barrier $V_S = 90$ and three different cases: $\delta = 0$, $\delta = 0.01$, $\delta = 0.04$. Remember that asymmetry increases (decreases) as θ decreases (increases). The last row of the table shows the case $\theta = 1$ for each level of δ : the tax rate is unique (no asymmetry). This row represents the model presented in [3, 4]. In case $\delta = 0$ it represents Leland [17] results. Leverage is in percentage (%), spreads in basis points (bps).

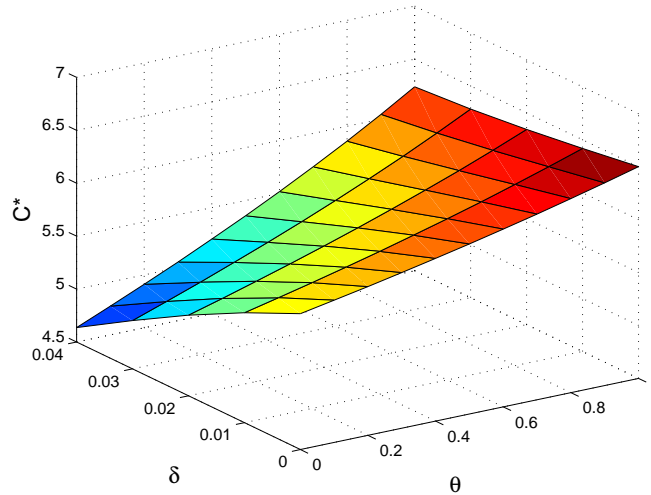


Figure 4: **Optimal Coupon as function of δ, θ .** This plot shows the behavior of the optimal coupon C^* as function of payout rate δ and degree of asymmetry θ . The switching barrier V_S is exogenous and parameters values are: $\sigma = 0.2, r = 0.05, \tau_1 = 0.35, \alpha = 0.5, V_S = 90, V = 100$. Recall that $\theta := \frac{\tau_2}{\tau_1}$ represents the degree of asymmetry in tax benefits.

validity under asymmetry in tax benefits. Adding to this, the contribution of the present work is also to show how optimal capital structure is much more affected by payouts when considering a more realistic framework allowing also for asymmetry in the corporate tax schedule.

As it concerns optimal equity value and optimal spreads we note that the joint effect of asymmetry and payouts raises both optimal equity and optimal spreads. We can explain this as a consequence of two main insights arising from the model.

i) First, when payouts are introduced, less assets remain in the firm, thus making possible only a lower optimal debt issuance. Adding to this, asymmetry makes debt less attractive, due to a possible switching to lower tax benefits, thus a potential loss of value has to be taken into account. As a consequence, the joint effect is to reduce both the optimal coupon C^* and the optimal amount⁷ of debt D^* . Equity value increases at its optimal level due to the joint effect of δ, θ on both C^*, V_B^* : recall that the optimal (endogenous) failure level increases as the degree of asymmetry is higher.

⁷As noted in [17] for $\delta = 0$, and also supported by results in [4] for $\delta > 0$, the firm will always chose a coupon level which is lower than that one corresponding to the maximum capacity of debt. As a consequence, a lower coupon means a lower debt value. Moreover, as in [17] we are assuming the face value of debt being constant.

ii) Secondly, we can also think about the joint effect of payouts and asymmetry in tax benefits as something which contributes to increase the *average riskiness* of the firm and moreover makes bankruptcy more likely. This is why, despite lower optimal leverage ratios, optimal spreads increase, in line with [17] suggestions. When $\theta \rightarrow 0$, the potential loss in tax benefits due to passing from τ_1 to τ_2 increases.

Optimal debt will be lower, equity higher but the first effect will always dominate the second one, bringing to lower optimal total values of the firm. This is because payouts and corporate tax asymmetry increase the likelihood of default. Debt holders must be compensated, this is why optimal spreads predicted in this model are considerably higher w.r.t. the case $\theta = 1, \delta = 0$, capturing all these economic insights.

Table 2 reports the qualitative behavior of financial variables at their optimal level when the exogenous switching barrier V_S and the payout rate δ are fixed, while the *vertical* degree of asymmetry increases, i.e. $\theta \rightarrow 0$, aiming at isolating and capturing only this asymmetry influence on optimal capital structure decisions made by the firm in a comparative static analysis.

Fin. Var.	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*
$\theta \rightarrow 0$	< 0	< 0	> 0	> 0	> 0	< 0	< 0

Table 2: **Effect of a change in the vertical degree of asymmetry of the corporate tax schedule on financial variables at optimal leverage ratio as $\theta \rightarrow 0$ when V_S is exogenous.** The table shows for each financial variable the effect of increasing asymmetry in the corporate tax schedule, i.e. for $\theta \rightarrow 0$, given the payout rate δ and the exogenous switching barrier V_S . We report the sign of change in each variable as the degree of asymmetry increases.

Concerning the asymmetry of the corporate tax schedule a similar analysis could be done analyzing how optimal capital structure decisions are affected when the exogenous switching barrier V_S moves, meaning when the *horizontal degree of asymmetry* changes, fixing both θ and the payout rate δ . Numerical results show that different values of the barrier can significantly modify optimal choices, meaning the corporate tax schedule is an important determinant in leverage decisions. Table 3 shows numerical result for this case, Table 4 reports only the qualitative behavior of all financial variables at their optimal level as V_S increases. The switching barrier being exogenously given, results show in which direction a higher V_S will move optimal capital structure decisions.

A possible explanation of what we observe in Table 4 could be that, coeteris paribus, as V_S rises (decreases), the *horizontal* degree of asymmetry changes. As extreme case, our framework tends to a limit-model in which the tax sheltering value of interest payments is constant and equal to the lower $\tau_2 C$ (higher $\tau_1 C$). And this represents the limit-model for each degree of asymmetry θ and payout level δ . A reduction (increase) in the corporate tax rate produces exactly the effects shown by our results: each variable at its own optimal level decreases (increases), except equity value which instead rises (reduces). And this

$\delta = 0, \theta = 0.8$							
V_S	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*
85	6.371	93.831	78.983	32.818	53.810	126.649	74.087 %
86	6.363	93.736	78.845	32.855	53.787	126.591	74.046 %
87	6.356	93.642	78.705	32.891	53.763	126.533	74.006 %
88	6.348	93.549	78.564	32.925	53.739	126.474	73.967 %
89	6.340	93.458	78.422	32.957	53.715	126.415	73.930 %
90	6.333	93.367	78.278	32.987	53.691	126.355	73.893 %
91	6.325	93.278	78.132	33.017	53.665	126.295	73.857 %
92	6.318	93.189	77.985	33.044	53.640	126.234	73.823 %
93	6.311	93.102	77.837	33.070	53.614	126.172	73.789 %
94	6.303	93.015	77.686	33.095	53.587	126.110	73.757 %
95	6.296	92.929	77.534	33.119	53.560	126.047	73.725 %
$\delta = 0.01, \theta = 0.4$							
V_S	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*
85	5.971	85.731	96.453	35.908	53.168	121.640	70.480 %
86	5.946	85.448	95.868	36.037	53.080	121.485	70.336 %
87	5.922	85.168	95.281	36.161	52.992	121.330	70.196 %
88	5.898	84.894	94.694	36.280	52.902	121.174	70.059 %
89	5.874	84.622	94.106	36.395	52.812	121.018	69.926 %
90	5.850	84.355	93.515	36.506	52.720	120.861	69.795 %
91	5.827	84.091	92.923	36.613	52.626	120.704	69.667 %
92	5.804	83.830	92.328	36.716	52.531	120.546	69.542 %
93	5.781	83.572	91.732	36.816	52.435	120.388	69.419 %
94	5.758	83.316	91.132	36.912	52.338	120.228	69.299 %
95	5.736	83.063	90.530	37.005	52.238	120.068	69.179 %
$\delta = 0.04, \theta = 0$							
V_S	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*
85	4.848	66.084	133.629	46.254	45.218	112.339	58.826 %
86	4.804	65.635	131.884	46.525	44.976	112.160	58.519 %
87	4.760	65.196	130.174	46.787	44.737	111.982	58.220 %
88	4.718	64.766	128.497	47.040	44.498	111.806	57.927 %
89	4.677	64.346	126.850	47.286	44.260	111.632	57.641 %
90	4.637	63.933	125.230	47.526	44.023	111.459	57.360 %
91	4.597	63.528	123.636	47.759	43.786	111.287	57.085 %
92	4.558	63.129	122.065	47.987	43.549	111.116	56.814 %
93	4.520	62.736	120.515	48.210	43.312	110.946	56.546 %
94	4.483	62.348	118.985	48.428	43.075	110.776	56.283 %
95	4.446	61.965	117.473	48.643	42.838	110.608	56.022 %

Table 3: **Effect of asymmetry in the tax schedule on all financial variables at optimal level.** Base case parameter values: $V_0 = 100$, $\sigma = 0.2$, $\tau_1 = 0.35$, $r = 6\%$, $\alpha = 0.5$. This table reports results considering an exogenous barrier V_S varying from 80 to 95 and different scenarios for payouts and degree of asymmetry θ : $\delta = 0, \theta = 0.8$, $\delta = 0.01, \theta = 0.4$, $\delta = 0.04, \theta = 0$. Leverage is in percentage (%), spreads in basis points (bps).

result is robust w.r.t. each payout level. The behavior we find in this limit-model is in line with [17] Table II, where $\delta = 0$: all variables except equity are increasing w.r.t. the

constant corporate tax rate τ . Moreover, when payouts are introduced in a flat corporate tax schedule model, results are still in line with [4] Table 5. This behavior of financial variables holds for each level of payout δ and each *vertical* degree of asymmetry θ : what is different among different scenarios is only the magnitude of the effect, obviously depending on the joint influence. And the joint influence is higher as payouts increase and $\theta \rightarrow 0$. Consider as an example a switching barrier of 95: in case $\delta = 0.04$ and $\theta = 0$ the model predicts a 56% optimal leverage, while in case $\delta = 0$ and $\theta = 0.8$ this optimal ratio is around 73%. This second case is very close to Leland's results [17] with constant τ_2 , and the difference in leverage is quite negligible, i.e. 2%, while the first case brings to a huge 19%-reduction in leverage ratios.

Fin. Var.	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*
$V_S \nearrow$	< 0	< 0	< 0	> 0	< 0	< 0	< 0

Table 4: **Effect of a change in the asymmetry of the corporate tax schedule on financial variables at optimal leverage ratio as V_S increases.** The table shows for each financial variable the effect of changing the asymmetry in the corporate tax schedule when the exogenous switching barrier V_S increases. These results are given for a fixed θ and payout rate δ . We report the sign of change in each variable as V_S increases.

4.2 Debt dependent switching barrier

In this section we study optimal capital structure when the switching barrier considered is increasing with coupon C , being defined as $V_S := k + \frac{1}{a}C$. We will show not only the case $\delta = 0$ for which we have closed form solution for the endogenous failure level, but also numerical results for the general case $\delta > 0$ where a payout rate is considered.

In case $\delta = 0$, we define the optimal capital structure by substituting the endogenous failure level $V_B^c(C; \tau_1, \tau_2; 0; k, a)$ obtained in (40) into the total value of the firm v and then maximizing it w.r.t. C . This will give the optimal coupon C^* , allowing to analyze optimal leverage and optimal capital structure decisions as reported in Table 5. In the general case $\delta > 0$ we do not have a closed form for the endogenous failure level, and the smooth pasting condition is not linear w.r.t. C , since also the switching barrier depends on C . Thus we numerically analyze the existence of an optimal capital structure and show in this subsection our results.

The peculiarity of this model is that as the optimal coupon decreases for higher *vertical degree of asymmetry* $\theta \rightarrow 0$, then also the optimal switching barrier V_S^* decreases, meaning that also the *horizontal degree of asymmetry* is changed. From the opposite point of view, we observe that as $\theta \rightarrow 1$ the debt dependent switching barrier approaches current firm's activities value V : in the limit, for $\theta = 1$, asymmetry disappears, meaning the corporate tax schedule tends to a flat one.

$\delta = 0$								
θ	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*	V_S^*
0	5.079	76.816	61.192	42.294	50.969	119.110	64.491%	90.474
0.1	5.184	78.286	62.185	41.474	51.086	119.760	65.369%	91.104
0.2	5.296	79.847	63.270	40.608	51.221	120.455	66.288%	91.776
0.3	5.414	81.487	64.399	39.712	51.361	121.199	67.234%	92.484
0.4	5.539	83.219	65.592	38.779	51.511	121.998	68.213%	93.234
0.5	5.673	85.065	66.903	37.795	51.683	122.860	69.238%	94.038
0.6	5.815	87.015	68.275	36.776	51.861	123.791	70.292%	94.890
0.7	5.968	89.102	69.795	35.700	52.066	124.802	71.394%	95.808
0.8	6.133	91.336	71.475	34.568	52.298	125.905	72.544%	96.798
0.9	6.310	93.720	73.283	33.392	52.547	127.112	73.730%	97.860
1.0	6.501	96.274	75.256	32.167	52.820	128.442	74.956%	99.006
$\delta = 0.01$								
θ	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*	V_S^*
0	4.855	72.545	69.236	44.808	48.414	117.353	61.818%	89.130
0.1	4.968	74.100	70.447	43.922	48.542	118.022	62.785%	89.808
0.2	5.088	75.745	71.729	42.993	48.682	118.738	63.792%	90.528
0.3	5.216	77.492	73.102	42.016	48.835	119.508	64.843%	91.296
0.4	5.352	79.342	74.545	40.995	48.998	120.338	65.933%	92.112
0.5	5.498	81.318	76.108	39.917	49.180	121.236	67.075%	92.988
0.6	5.655	83.431	77.804	38.780	49.383	122.211	68.268%	93.930
0.7	5.823	85.683	79.599	37.592	49.598	123.275	69.506%	94.938
0.8	6.006	88.115	81.611	36.326	49.848	124.441	70.808%	96.036
0.9	6.203	90.720	83.755	35.006	50.114	125.726	72.157%	97.218
1.0	6.419	93.545	86.169	33.604	50.420	127.149	73.571%	98.514
$\delta = 0.04$								
θ	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*	V_S^*
0	4.121	58.897	99.700	53.172	39.483	112.068	52.554%	84.726
0.1	4.263	60.709	102.201	52.060	39.714	112.769	53.835%	85.578
0.2	4.416	62.651	104.860	50.877	39.964	113.528	55.185%	86.496
0.3	4.581	64.732	107.682	49.620	40.233	114.352	56.608%	87.486
0.4	4.758	66.955	110.622	48.296	40.511	115.252	58.095%	88.548
0.5	4.951	69.361	113.804	46.877	40.818	116.238	59.672%	89.706
0.6	5.161	71.959	117.210	45.364	41.148	117.323	61.334%	90.966
0.7	5.391	74.782	120.897	43.744	41.510	118.526	63.093%	92.346
0.8	5.644	77.858	124.910	42.008	41.905	119.866	64.954%	93.864
0.9	5.925	81.235	129.362	40.135	42.349	121.370	66.932%	95.550
1.0	6.239	84.957	134.326	38.114	42.847	123.072	69.031%	97.434

Table 5: **Effect of payouts and asymmetry in the tax schedule on all financial variables at their optimal level when the switching barrier V_S is debt dependent.** Base case parameter values: $V_0 = 100$, $\sigma = 0.2$, $\tau_1 = 0.35$, $r = 6\%$, $\alpha = 0.5$, $k = 60$, $a = 1/6$. This table reports results considering an exogenous barrier $V_S = 90$ and three different cases: $\delta = 0$, $\delta = 0.01$, $\delta = 0.04$. Remember that asymmetry increases (decreases) as θ decreases (increases). The last row of the table shows the case $\theta = 1$ for each level of δ : the tax rate is unique (no asymmetry). This row represents the model presented in [3, 4]. In case $\delta = 0$ it represents Leland [17] results.

If compared to the case considered in previous subsection, where V_S is exogenously given, this more realistic framework allows to analyze the joint effect of a change in both the *vertical* and *horizontal* degrees of asymmetry of the corporate tax schedule. As θ moves, the optimal coupon changes, and this in turns modifies the optimal switching barrier V_S^* . A change in the *vertical* degree of asymmetry will affect optimal capital structure decisions, both directly and indirectly, in this last case by changing the range of firm's values for which tax benefits depend alternatively on τ_1, τ_2 .

Fin. Var.	C^*	D^*	$R^* - r$	E^*	V_B^*	v^*	L^*
$\theta \rightarrow 0$	< 0	< 0	< 0	> 0	< 0	< 0	< 0

Table 6: **Effect of a change in the degree of asymmetry of the corporate tax schedule on financial variables at optimal leverage ratio as $\theta \rightarrow 0$ when V_S is debt dependent.** The table shows for each financial variable the effect of changing the degree of asymmetry in the corporate tax schedule as $\theta \rightarrow 0$ and the switching barrier is debt dependent. Results hold for each level of the payout rate $\delta \geq 0$. We report the sign of change in each variable as $\theta \rightarrow 0$.

When the switching barrier depends on the amount of debt issued, a higher profit is needed in order to have higher coupon payments fully deductible. Recall that we are assuming EBIT has to cover coupon payments in order to benefit from tax savings. In this even simplified but more realistic framework, greater debt has a greater likelihood of losing its tax benefits, and optimal leverage drops significantly. This reduction is quantitative higher than what we found in case V_S being exogenously given, for each level of payout. The decrease in optimal leverage is a 19%-reduction in case $\delta = 0.04$, much more than 12% in case of V_S fixed. Leverage can reach a 52% in line with historical norms (see [17]). Table 6 shows that optimal credit spreads decreases in this scenario, reflecting the lesser leverage, in line with suggestions in [17]. In this simplified framework we model EBIT as a linear function of V and this allows to show that operational costs could be another variable to analyze in order to explain observed leverage ratios. As k and/or a rise, this will affect the optimal amount of debt issued, since a higher profit is necessary to fully benefit from coupon deductibility. An increase in k and/or a will drop predicted leverage.

The economic insight we want to give is that this simple model is flexible to analyze the impact of many factors on optimal capital structure decisions, providing a framework to develop in the direction of a more empirical research, allowing to explain differences in observed leverages among firms facing different tax-code provisions. What we find is in line with what [14] observe: when the corporate tax rate is higher, observed leverage ratios are higher. Nevertheless our model also analyzes how a change in tax-code provisions can affect a single firm's corporate decisions. The general result is that asymmetry always lowers optimal debt, optimal leverage ratios and the maximum total value of the firm since less tax savings (actual and/or potential) are available, meaning there is always a loss of potential value for the firm⁸.

⁸As in Leland [17] this model does not consider tax loss carryforwards which could be an interesting

5 Conclusions

Corporate capital structure is a complex decision since it is affected by a large number of factors. We analyze a structural model with endogenous bankruptcy starting from Leland framework [17] and extending it in two main directions. We consider a more general model in which we introduce a payout rate δ and an asymmetric corporate tax schedule. Rather than considering a flat tax scheme, i.e. a unique corporate tax rate, we analyze asymmetric tax code provisions allowing for a switching in corporate tax rates. The switching from a corporate tax rate to the other is determined by the firm value crossing i) an exogenous barrier, ii) a debt dependent switching barrier (allowing to model EBIT). We investigate the joint effects of this corporate tax scheme and payouts on optimal default level and optimal capital structure. We propose alternative ways to measure the degree of asymmetry of the corporate tax schedule: i) *the degree of vertical asymmetry*, related to the gap between tax rates; ii) *the degree of horizontal asymmetry*, related to the range of firm's values above and/or below the switching barrier. We derive the value of the tax benefit claim in this framework, following Leland [17] about how to model tax benefits of debt. Asymmetry in corporate tax code provisions becomes also asymmetry in tax benefits of debt. Optimal capital structure is analyzed through the derivation of the endogenous failure level chosen by equity holders, and optimal coupon (maximizing total firm's value). Our results support [14], [27] suggestion that tax-code provisions should be considered when studying corporate financing decisions. We observe that all financial variables at their optimal level are affected by this asymmetric tax schedule, if compared to Leland [17] results with a flat tax-code (a unique constant corporate tax rate, meaning no asymmetry). Moreover, we analyze the joint effect on optimal capital structure of payouts and asymmetry in tax benefits. Results show that payouts and asymmetry always lowers optimal debt, optimal leverage ratios and the maximum total value of the firm. This is because less assets remain in the firm (due to payouts), in line with [3, 4], and also less tax savings (actual and/or potential) are available for the firm, meaning a loss of potential value for the firm. Leverage ratios are significantly affected from a quantitative point of view by both factors: they drop down from 75% in Leland case to a 52% in case of a 4% payout rate and maximum asymmetry case (when we consider the debt dependent switching barrier). These are two extreme cases, but we find a huge impact on leverage ratios for most cases between these two scenarios. Our analysis in this paper confirms our results in [4] and shows how optimal capital structure is much more affected by the introduction of a payout rate $\delta > 0$ inside a more realistic framework allowing for asymmetry in tax-code provisions. These two factors influence each other with a resulting quantitative huge joint effect on optimal debt and leverage. The degree of *vertical* asymmetry in the corporate tax schedule ($\theta := \frac{\tau_2}{\tau_1}$) is a parameter imposed to the firm by external authorities. Moreover, this parameter can vary strongly depending on the sector in which the firm is operating in, and can also vary in time, for example to encourage

point to develop, since they will introduce path dependence, making the model even more realistic.

investments. Thus, our analysis provides a way to measure the economic influence (from both a qualitative and quantitative point of view) of such an important and external factor on internal and endogenous optimal choices made by the firm. Moreover, we also analyze effects on corporate decisions produced by its joint influence with payout rates, which are supposed to be constant (but we know that they should be even partly modified by the firm). We think that this framework should be a flexible way to follow in order to better explain differences in observed leverage ratios across firms belonging to different activity sectors, facing different tax-code provisions, as an idea to apply for a future more empirical research.

6 Appendix A

The following result contains the formula for the Laplace transform of a constant level hitting time by a Brownian motion with drift ([15] p. 196-197):

Proposition 6.1 *Let $X_t = \mu t + \sigma W_t$ and $T_b = \inf\{s : X_s = b\}$, then for all $\gamma > 0$,*

$$\mathbb{E}[e^{-\gamma T_b}] = \exp\left[\frac{\mu b}{\sigma^2} - \frac{|b|}{\sigma} \sqrt{\frac{\mu^2}{\sigma^2} + 2\gamma}\right].$$

The computation of the Laplace transform of double passage times, $T_{V_S} \wedge T_{V_B}$ is contained in the following result ([15] p. 99-100).

Proposition 6.2 *Let $T_b = \inf\{t, W_t = b\}$. Then if $b < 0 < c$,*

$$\mathbb{E}_0[e^{-\gamma T_b} \mathbf{1}_{\{T_b < T_c\}}] = \frac{\sinh(c\sqrt{2\gamma})}{\sinh((c-b)\sqrt{2\gamma})} ; \mathbb{E}_0[e^{-\gamma T_c} \mathbf{1}_{\{T_c < T_b\}}] = \frac{\sinh(-b\sqrt{2\gamma})}{\sinh((c-b)\sqrt{2\gamma})}.$$

We now turn to compute such expectation for a geometric Brownian motion with drift.

Proposition 6.3 *Let $T = \inf\{t, \log V + \mu t + \sigma W_t = \log V_B\}$ and $T_S = \inf\{t, \log V + \mu t + \sigma W_t = \log V_S\}$. Then if $V_B < V < V_S$,*

$$\mathbb{E}_V[e^{-rT} \mathbf{1}_{\{T < T_S\}}] = \left(\frac{V_B}{V}\right)^{\mu/\sigma^2} \frac{\sinh(\log[(V_S/V) \frac{\sqrt{2r+\mu^2/\sigma^2}}{\sigma}])}{\sinh(\log[(V_S/V_B) \frac{\sqrt{2r+\mu^2/\sigma^2}}{\sigma}])} \text{ noted as } g(V, V_S, V_B),$$

$$\mathbb{E}_V[e^{-rT_S} \mathbf{1}_{\{T_S < T\}}] = \left(\frac{V_S}{V}\right)^{\mu/\sigma^2} \frac{\sinh(\log[(V/V_B) \frac{\sqrt{2r+\mu^2/\sigma^2}}{\sigma}])}{\sinh(\log[(V_S/V_B) \frac{\sqrt{2r+\mu^2/\sigma^2}}{\sigma}])} \text{ noted as } f(V, V_S, V_B).$$

Proof It follows by Proposition 6.2 using a change of probability measure. The process $t \mapsto \tilde{W}_t = \mu t + \sigma W_t$ is a Q Brownian motion with $Q = L\mathbb{P}$, with the \mathbb{P} -martingale defined as

$$dL_t = -L_t \frac{\mu}{\sigma} dW_t.$$

Equivalently, $\mathbb{P} = Z \cdot Q$, with $dZ_t = Z_t \frac{\mu}{\sigma} d\tilde{W}_t$. Remark that $Z_t = \exp(\frac{\mu}{\sigma} \tilde{W}_t - \frac{1}{2}(\frac{\mu}{\sigma})^2 t)$. Thus $\mathbb{E}_V[e^{-rT} \mathbf{1}_{\{T < T_S\}}] = \mathbb{E}_0^Q[Z_T e^{-rT} \mathbf{1}_{\{T < T_S\}}]$ with now

$$T = \inf\{t, \tilde{W}_t = \log(\frac{V_B}{V})^{1/\sigma}\}.$$

So we get

$$\mathbb{E}_V[e^{-rT} \mathbf{1}_{\{T < T_S\}}] = \mathbb{E}_0[\exp(\frac{\mu}{\sigma} \tilde{W}_T - \frac{1}{2}(\frac{\mu}{\sigma})^2 T - rT) \mathbf{1}_{\{T < T_S\}}].$$

We can use the fact that, by continuity, $\tilde{W}_T = \log(\frac{V_B}{V})^{1/\sigma}$, so

$$\mathbb{E}_V[e^{-rT} \mathbf{1}_{\{T < T_S\}}] = (\frac{V_B}{V})^{\mu/\sigma^2} \mathbb{E}_0[\exp(-(\frac{1}{2}(\frac{\mu}{\sigma})^2 + r)T) \mathbf{1}_{\{T < T_S\}}].$$

Finally, we use Proposition 6.2 with $\gamma = \frac{1}{2}(\frac{\mu}{\sigma})^2 + r$, $b = \log(\frac{V_B}{V})^{1/\sigma}$, $c = \log(\frac{V_S}{V})^{1/\sigma}$ to conclude:

$$\mathbb{E}_V[e^{-rT} \mathbf{1}_{\{T < T_S\}}] = (\frac{V_B}{V})^{\mu/\sigma^2} \frac{\sinh((\log \frac{V_S}{V})^{1/\sigma} \sqrt{(\frac{\mu}{\sigma})^2 + 2r})}{\sinh((\log \frac{V_S}{V_B})^{1/\sigma} \sqrt{(\frac{\mu}{\sigma})^2 + 2r})}.$$

The second proof is quite similar. •

Corollary 6.4 Using $\sinh \circ \log(x^\beta) = \frac{1}{2}(x^\beta - x^{-\beta})$, we get:

$$\mathbb{E}_V[e^{-rT} \mathbf{1}_{\{T < T_S\}}] = \frac{V_B^{\lambda_2} (V_S)^{\lambda_2 - \lambda_1} V^{-\lambda_2} - V_B^{\lambda_2} V^{-\lambda_1}}{(V_S)^{\lambda_2 - \lambda_1} - V_B^{\lambda_2 - \lambda_1}} \quad (41)$$

$$\mathbb{E}_V[e^{-rT_S} \mathbf{1}_{\{T_S < T\}}] = \frac{(V_S)^{\lambda_2} (V_B)^{\lambda_2 - \lambda_1} V^{-\lambda_2} - (V_S)^{\lambda_2} V^{-\lambda_1}}{V_B^{\lambda_2 - \lambda_1} - (V_S)^{\lambda_2 - \lambda_1}} \quad (42)$$

with

$$\lambda_1 = \frac{\mu - \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}, \quad \lambda_2 = \frac{\mu + \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}.$$

and

$$\mu = r - \delta - \frac{1}{2}\sigma^2.$$

7 Appendix B

Proof (of Proposition 2.2) Using integral representation of tax benefits we get:

- if $V \geq V_S$

$$F(V) = \mathbb{E}_V \left[e^{-rT_{V_S}} F(V_{T_{V_S}}) + \int_0^{T_{V_S}} e^{-rs} \tau_1 C \mathbf{1}_{(V_S, \infty)}(V_s) ds \right]$$

- if $V_B \leq V < V_S$:

$$\begin{aligned} F(V) &= \mathbb{E}_V \left[e^{-rT_{V_S \wedge T_{V_B}}} F(V_{T_{V_S \wedge T_{V_B}}}) + \int_0^{T_{V_S \wedge T_{V_B}}} e^{-rs} \tau_2 C \mathbf{1}_{(V_B, V_S)}(V_s) ds \right] \\ &= \frac{\tau_2 C}{r} + (F(V_S) - \frac{\tau_2 C}{r}) \mathbb{E}_V \left[e^{-rT_{V_S}} \mathbf{1}_{T_{V_S} \leq T_{V_B}} \right] + \\ &+ (F(V_B) - \frac{\tau_2 C}{r}) \mathbb{E}_V \left[e^{-rT_{V_B}} \mathbf{1}_{T_{V_B} < T_{V_S}} \right] \end{aligned}$$

From boundary condition $F(V_B) = 0$ we get

$$F(V) = \frac{\tau_2 C}{r} \left(1 - \mathbb{E}_V \left[e^{-rT_{V_B}} \mathbf{1}_{T_{V_B} < T_{V_S}} \right] \right) + (F(V_S) - \frac{\tau_2 C}{r}) \mathbb{E}_V \left[e^{-rT_{V_S}} \mathbf{1}_{T_{V_S} \leq T_{V_B}} \right]$$

Therefore we obtain

$$\begin{aligned} F(V) &= \left[\frac{\tau_1 C}{r} + \left(F(V_S) - \frac{\tau_1 C}{r} \right) \left(\frac{V_S}{V} \right)^{\lambda_2} \right] \mathbf{1}_{(V_S, \infty)} \\ &+ \left[\frac{\tau_2 C}{r} + \frac{(F(V_S) - \frac{\tau_2 C}{r}) V_S^{\lambda_2} + \frac{\tau_2 C}{r} V_B^{\lambda_2}}{V_S^{\lambda_2 - \lambda_1} - V_B^{\lambda_2 - \lambda_1}} V^{-\lambda_1} - \frac{V_B^{-\lambda_1} (F(V_S) - \frac{\tau_2 C}{r}) + V_S^{-\lambda_1} \frac{\tau_2 C}{r}}{V_S^{\lambda_2 - \lambda_1} - V_B^{\lambda_2 - \lambda_1}} (V_B V_S)^{\lambda_2} V^{-\lambda_2} \right] \mathbf{1}_{(V_B, V_S)} \end{aligned}$$

Further F is a C^1 function, thus imposing continuity of the derivative at point V_S yields:

$$\begin{aligned} -\lambda_2 V_S^{-1} (F(V_S) - \frac{\tau_1 C}{r}) &= -\lambda_1 V_S^{-1} \frac{(F(V_S) - \frac{\tau_2 C}{r}) V_S^{\lambda_2 - \lambda_1}}{V_S^{\lambda_2 - \lambda_1} - V_B^{\lambda_2 - \lambda_1}} + \lambda_1 V_S^{-1} \frac{-\frac{\tau_2 C}{r} V_B^{\lambda_2} V_S^{-\lambda_1}}{V_S^{\lambda_2 - \lambda_1} - V_B^{\lambda_2 - \lambda_1}} \\ &+ \lambda_2 V_S^{-1} \frac{(F(V_S) - \frac{\tau_2 C}{r}) V_B^{\lambda_2 - \lambda_1}}{V_S^{\lambda_2 - \lambda_1} - V_B^{\lambda_2 - \lambda_1}} - \lambda_2 V_S^{-1} \frac{-\frac{\tau_2 C}{r} V_S^{-\lambda_1} V_B^{\lambda_2}}{V_S^{\lambda_2 - \lambda_1} - V_B^{\lambda_2 - \lambda_1}}. \end{aligned}$$

Finally

$$F(V_S) = \frac{C(\lambda_2 \tau_1 - \lambda_1 \tau_2)}{r(\lambda_2 - \lambda_1)} + \frac{\lambda_2 C(\tau_2 - \tau_1)}{r(\lambda_2 - \lambda_1)} V_S^{\lambda_1 - \lambda_2} V_B^{\lambda_2 - \lambda_1} - \frac{\tau_2 C}{r} V_S^{-\lambda_2} V_B^{\lambda_2}.$$

•

Proof (of Proposition 3.1) Condition (21) can be written as

$$f(V_B) > g(V_B)$$

with $f(V_B) = -A_1 V_B^{-\lambda_1}$ and $g(V_B) = V_B - \frac{C}{r}(1 - \tau_2)$ and we study the problem for $V_B \geq 0$.

Observe that:

- $f(0) = 0$, $f(V_B) \leq 0$, $\forall V_B \geq 0$ since $A_1 > 0$ for $\tau_2 < \tau_1$,
- $f'(V_B) = \lambda_1 A_1 V_B^{-\lambda_1 - 1} < 0$,
- $g(0) = -\frac{C}{r}(1 - \tau_2) < 0$,
- $g' = 1$.

Since $f(V_B)$ is negative, f is decreasing function of V_B , g is an increasing function of V_B and $g(0) < f(0) = 0$, we can state that a solution \bar{V}_B of $f(\bar{V}_B) = g(\bar{V}_B)$ exists and it is unique and $\bar{V}_B < \frac{C}{r}(1 - \tau_2)$, since $g(\frac{C}{r}(1 - \tau_2)) = 0$.

Further we obtain a lower bound for \bar{V}_B , by studying f as function of δ . As $\frac{\partial f(V_B, \delta)}{\partial \delta} = \frac{\partial f(V_B, \delta)}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial \delta}$ and

$$\frac{\partial f(V_B, \delta)}{\partial \lambda_1} = - \left(\frac{1}{\lambda_2 - \lambda_1} + \log \frac{V_S}{V_B} \right) f(V_B, \delta) < 0; \quad \frac{\partial \lambda_1}{\partial \delta} = \frac{\lambda_1}{\sqrt{\mu^2 + 2r\sigma^2}} < 0,$$

thus $\delta \mapsto f(V_B, \delta)$ is increasing. This implies that the intercept \bar{V}_B of f and g is increasing, so that $\bar{V}_B > \bar{V}_B(0)$, where we denoted by $\bar{V}_B(0)$ the bound resulting from condition (21) in the special case $\delta = 0$. In fact in this case (21) solves explicitly giving $\bar{V}_B(0) = \frac{1}{1+A_1} \frac{C(1-\tau_2)}{r}$ (see also Remark 3.2). •

Proof (of Proposition 3.3) Under constraint (21) the option to default has positive value. In order to prove that equity be is increasing, we split the equity function (20) as the sum of two terms

$$f(V, C) = V - (1 - \tau_2) \frac{C}{r} + A_1 V^{-\lambda_1}, \quad (43)$$

and

$$g(V, V_B, C) = \left(-A_1 V_B^{-\lambda_1} + \frac{C}{r}(1 - \tau_2) - V_B \right) \left(\frac{V}{V_B} \right)^{-\lambda_2}. \quad (44)$$

Observe that:

- i) the application $V \mapsto f(V, C)$ is increasing and convex for $V \geq V_B$;
- ii) the application $V \mapsto g(V, V_B, C)$ is decreasing and convex for $V \geq V_B$.

As a consequence, in order to ensure $V \mapsto E(V, V_B, C)$ to be increasing for $V \geq V_B$, we need that

$$\frac{\partial f(V, C)}{\partial V} \Big|_{V=V_B} \geq - \frac{\partial g(V, V_B, C)}{\partial V} \Big|_{V=V_B} \quad (45)$$

which leads to constraint (26). In such a case, by i) and ii), $V \mapsto E(V, V_B, C)$ is also convex. •

Proof (of Theorem 3.5) Since the optimal failure level, if it exists, has to be lower than the switching barrier V_S , we study the existence and uniqueness of the solution $V_B(C; \tau_1, \tau_2; \delta)$ of Equation (31) in $[0, V_S]$. Imposing the smooth pasting condition gives that $V_B(C; \tau_1, \tau_2; \delta)$ has to satisfy

$$f(V_B) = g(V_B) \quad (46)$$

where $f(V_B) = (1 + \lambda_2)V_B$ and $g(V_B) = A + BV_B^{-\lambda_1}$, with the constants A and B given by

$$A = \frac{\lambda_2 C}{r}(1 - \tau_2) \quad (47)$$

$$B = V_S^{\lambda_1} \frac{\lambda_2 C}{r}(\tau_2 - \tau_1). \quad (48)$$

Observe that $f(V_B)$ is a linear and increasing function of V_B since $\lambda_2 > 0$, going from 0 to $(1 + \lambda_2)V_S$ in $[0, V_S]$. On the other side $g(V_B)$ is decreasing as $g'(V_B) = -\lambda_1 B V_B^{-\lambda_1 - 1} < 0$ since $\lambda_1 < 0$ and $B < 0$ for $\tau_2 < \tau_1$. Observe that $g(V_S) = \frac{\lambda_2 C(1 - \tau_1)}{r}$.

Therefore there exists a solution to (46) if and only if $(1 + \lambda_2)V_S > \frac{\lambda_2 C(1 - \tau_1)}{r}$, or equivalently $V_S > \frac{\lambda_2 C(1 - \tau_1)}{r(1 + \lambda_2)}$.

Similarly a solution $V_B(C; \tau_1, \tau_2; \delta)$ exists and is unique in case $\tau_2 > \tau_1$. •

Proof (of Proposition 3.8) Compute

$$\frac{\partial V_B(C; \tau_1, \tau_2; 0)}{\partial \tau_2} = -\frac{\lambda_2 C V_S (r V_S (1 + \lambda_2) - C \lambda_2 (1 - \tau_1))}{(-r V_S (1 + \lambda_2) - C \lambda_2 (\tau_1 - \tau_2))^2} \quad (49)$$

$$= -\frac{2 C V_S (V_S (\sigma^2 + 2r) + 2 C (\tau_1 - 1))}{(V_S (\sigma^2 + 2r) + 2 C (\tau_1 - \tau_2))^2}, \quad (50)$$

then, by condition (35), it follows that $\frac{\partial V_B(C; \tau_1, \tau_2; 0)}{\partial \tau_2} < 0$. This proves both the monotonicity of $\tau_2 \mapsto V_B(C; \tau_1, \tau_2; 0)$ and the inequality $V_B(C; \tau_1, 0; 0) > V_B(C; \tau_1, \tau_2; 0)$. •

Proof (of Proposition 3.10) Consider Equation (32); the derivative of this endogenous failure level $V_B(C; \tau_1, \tau_2; 0)$ with respect to V_S is:

$$\frac{\partial V_B(C; \tau_1, \tau_2; 0)}{\partial V_S} = \frac{4 C^2 (1 - \tau_2) (\tau_1 - \tau_2)}{(V_S (\sigma^2 + 2r) + 2 C (\tau_1 - \tau_2))^2}.$$

Proof (of Proposition 3.11) i) It is sufficient to consider:

$$\frac{\partial V_B^c(C; \tau_1, \tau_2; 0; k, a)}{\partial k} = \frac{4 C^2 a^2 (\tau_2 - 1) (\tau_2 - \tau_1)}{[(ak + C)(\sigma^2 + 2r) + 2aC(\tau_1 - \tau_2)]^2} > 0$$

$$\frac{\partial^2 V_B^c(C; \tau_1, \tau_2; 0; k, a)}{\partial k^2} = \frac{8C^2 a^3 (1 - \tau_2)(\tau_2 - \tau_1)(\sigma^2 + 2r)}{[(ak + C)(\sigma^2 + 2r) + 2aC(\tau_1 - \tau_2)]^3} < 0.$$

ii) Evaluating the first and second derivative of the failure level w.r.t. a :

$$\frac{\partial V_B^c(C; \tau_1, \tau_2; 0; k, a)}{\partial a} = \frac{4C^3(\tau_2 - 1)(\tau_1 - \tau_2)}{[(ak + C)(\sigma^2 + 2r) + 2aC(\tau_1 - \tau_2)]^2} < 0,$$

$$\frac{\partial^2 V_B^c(C; \tau_1, \tau_2; 0; k, a)}{\partial a^2} = \frac{8C^3(1 - \tau_2)(\tau_1 - \tau_2)(2C(\tau_1 - \tau_2) + k(\sigma^2 + 2r))}{[(ak + C)(\sigma^2 + 2r) + 2aC(\tau_1 - \tau_2)]^3} > 0.$$

iii) The following holds:

$$\frac{\partial V_B^c(C; \tau_1, \tau_2; 0; k, a)}{\partial C} = \frac{2(1 - \tau_2)[(\sigma^2 + 2r)(C + ak)^2 + 2aC^2(\tau_1 - \tau_2)]}{[(ak + C)(\sigma^2 + 2r) + 2aC(\tau_1 - \tau_2)]^2} > 0,$$

$$\frac{\partial^2 V_B^c(C; \tau_1, \tau_2; 0; k, a)}{\partial C^2} = \frac{8a^3 k^2 (1 - \tau_2)(\sigma^2 + 2r)(\tau_2 - \tau_1)}{[(ak + C)(\sigma^2 + 2r) + 2aC(\tau_1 - \tau_2)]^3} < 0.$$

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Proof (of Proposition 3.12) From Proposition 3.10 when an asymmetric tax benefits scheme is introduced, the endogenous failure level increases with the switching barrier. Thus, it is sufficient to compare the two switching barriers, since they are both independent on firm's current assets value V . •

Proof (of Proposition 4.1) Solving Equation (31) with respect to C we have:

$$C(V_B; \tau_1, \tau_2; \delta) = \frac{V_B r (1 + \lambda_2)}{\lambda_2 (V_S^{\lambda_1} V_B^{-\lambda_1} (\tau_2 - \tau_1) + (1 - \tau_2))} \quad (51)$$

In this case, taking the derivative of C with respect to the failure level V_B we get:

$$\frac{\partial C(V_B; \tau_1, \tau_2; \delta)}{\partial V_B} = \frac{r(1 + \lambda_2)}{\lambda_2} \cdot \frac{(V_S^{\lambda_1} V_B^{-\lambda_1} (\tau_2 - \tau_1)(1 + \lambda_1) + (1 - \tau_2))}{(V_S^{\lambda_1} V_B^{-\lambda_1} (\tau_2 - \tau_1) + (1 - \tau_2))^2} \quad (52)$$

Notice that $(V_S^{\lambda_1} V_B^{-\lambda_1} (\tau_2 - \tau_1)(1 + \lambda_1) + 1 - \tau_2)$ is greater than

$$\inf (1 - \tau_2, (\tau_2 - \tau_1)(1 + \lambda_1) + (1 - \tau_2))$$

since $V_S^{\lambda_1} V_B^{-\lambda_1} \in [0, 1]$. Obviously $1 - \tau_2 > 0$, and

$$(\tau_2 - \tau_1)(1 + \lambda_1) + 1 - \tau_2 > 0$$

since it is a linear decreasing function of λ_1 , with $\lambda_1 < 0$. This function decreases from $+\infty$ to $1 - \tau_1 > 0$, as $\lambda_1 \rightarrow -\infty$ and $\lambda_1 \rightarrow 0$ respectively. •

Proof (of Proposition 4.3) Here we consider the case $\delta = 0$. From the smooth pasting condition we have:

$$V_B(C; \tau_1, \tau_2; 0) = \frac{2CV_S(1 - \tau_2)}{V_S(\sigma^2 + 2r) + (\tau_1 - \tau_2)C}.$$

The application $C \mapsto V_B(C; \tau_1; \tau_2; 0)$ is increasing and concave. Substituting this expression for $V_B(C; \tau_1, \tau_2; 0)$ and setting $\lambda_2 = \frac{2r}{\sigma^2}$ we have that v is the sum of a linear function of C and the following non linear function of C :

$$f(C) := - \left(\frac{2C(\tau_1 - \tau_2)}{V_S(\sigma^2 + 2r)} V_B(C; \tau_1; \tau_2; 0) + \frac{\tau_2 C}{r} + \alpha V_B(C; \tau_1; \tau_2; 0) \right) \left(\frac{V_B(C; \tau_1; \tau_2; 0)}{V} \right)^{\frac{2r}{\sigma^2}}. \quad (53)$$

We want to prove is that the application $C \mapsto v(V_B(C; \tau_1; \tau_2; 0))$ is a concave function. It is sufficient to prove that the above function f is concave. It is useful to rewrite Equation (53) in the following way, studying separately the three terms:

$$f(C) = -(f_1(C) + f_2(C) + f_3(C)) V^{-\frac{2r}{\sigma^2}}$$

with

$$\begin{aligned} f_1(C) &= \frac{2(\tau_1 - \tau_2)}{V_S(\sigma^2 + 2r)} C V_B(C; \tau_1; \tau_2; 0)^{\frac{2r}{\sigma^2} + 1} \\ f_2(C) &= \frac{\tau_2 C}{r} V_B(C; \tau_1; \tau_2; 0)^{\frac{2r}{\sigma^2}} \\ f_3(C) &= \alpha V_B(C; \tau_1; \tau_2; 0)^{\frac{2r}{\sigma^2} + 1} \end{aligned} \quad (54)$$

Applications $C \mapsto f_1(C)$ and $C \mapsto f_2(C)$ are convex, since:

$$\frac{\partial^2 f_1(C)}{\partial C^2} = \frac{2(\tau_1 - \tau_2)}{V_S(\sigma^2 + 2r)} \frac{2V_B(C; \tau_1; \tau_2; 0)^{\frac{2r}{\sigma^2} + 1} (\sigma^2 + 2r)^2 V_S^2(\sigma^4 + 3r\sigma^2 + 2r^2)}{\sigma^4 (V_S(\sigma^2 + 2r) + (\tau_1 - \tau_2)C)^2 C} > 0 \quad (55)$$

$$\frac{\partial^2 f_2(C)}{\partial C^2} = \frac{\tau_2}{r} \frac{2V_B(C; \tau_1; \tau_2; 0)^{\frac{2r}{\sigma^2}} r V_S^2(\sigma^2 + 2r) (\sigma^4 + 4r\sigma^2 + 4r^2)}{\sigma^4 (V_S(\sigma^2 + 2r) + (\tau_1 - \tau_2)C)^2 C} > 0. \quad (56)$$

The application $C \mapsto f_3(C)$ is convex:

$$\frac{\partial^2 f_3(C)}{\partial C^2} = \frac{2\alpha V_B(C; \tau_1; \tau_2; 0)^{\frac{2r}{\sigma^2} + 1} (\sigma^2 + 2r)^2 (V_S(\sigma^2 + 2r) + (\tau_2 - \tau_1)C)}{\sigma^4 (V_S(\sigma^2 + 2r) + (\tau_1 - \tau_2)C)^2 C^2} > 0$$

if

$$V_S > \frac{(\tau_1 - \tau_2)C}{\sigma^2 + 2r} \quad (57)$$

On the other side by (35) it holds that

$$V_S > \frac{2(1 - \tau_1)C}{(\sigma^2 + 2r)},$$

therefore constraint (57) holds if $2 - 2\tau_1 > \tau_1 - \tau_2$, which is equivalent to $\tau_1 < \frac{2}{3} + \frac{\tau_2}{3}$. Under this last condition, constraint (57) is always satisfied for $0 \leq \tau_2 \leq \tau_1$, meaning that $f_3(C)$ is convex $\forall \tau_2 : 0 \leq \tau_2 \leq \tau_1$. Finally the application $f(C) : C \mapsto -(f_1(C) + f_2(C) + f_3(C))$ is concave since it is the sum of three concave functions.

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III

Volatility Risk

Optimal Capital Structure with Endogenous Default and Volatility Risk

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Abstract

This paper analyzes the capital structure of a firm in an infinite time horizon framework following Leland [12] under the more general hypothesis that the firm's assets value process belongs to a fairly large class of stochastic volatility models. In such a scheme, we describe and analyze the effects of stochastic volatility on all variables which constitute the capital structure. The endogenous failure level is derived in order to exploit the optimal amount of debt chosen by the firm. To this aim we derive and propose a *corrected* version of the smooth-fit principle under volatility risk in order to determine the optimal stopping problem solution. Exploiting optimal capital structure we found that the stochastic volatility framework seems to be a robust way to improve results in the direction of both higher spreads and lower leverage ratios in a quantitatively significant way.

Keywords: structural model; volatility risk; volatility time scales; endogenous default; optimal stopping.

1 Introduction

In this paper we extend the study of the optimal capital structure with endogenous default proposed by Leland [12] assuming that the firm value process belongs to a fairly large class of stochastic volatility models. The main empirical results in credit risk literature have emphasized a poor job of structural models in predicting credit spreads [5]; this weakness of the modeling could be related to the diffusion assumptions made in the papers by [12, 13]. Thus Leland suggests in [14] that a possible improvement to these results could ensue from introducing jumps and/or removing the assumption of constant volatility in the underlying firm value stochastic evolution. The former extension has been addressed

in [10] who extend the analysis to allow the value of the firm's assets to make downward jumps, in particular they suppose that the dynamics of the firm's assets is driven by the exponential of a Levy process; the authors find explicit expression for the bankruptcy level, while for the value of the firm and the value of its debt they no longer have closed form expressions. In [4] models the firm's asset as a double exponential jump-diffusion process and [11] study Black-Cox credit framework under the assumption that the log-leverage ratio is a time changed Brownian motion. Further [6] consider a pure jump process of the Variance-Gamma type. To the best of our knowledge, the latter extension of [12] to a general class of stochastic volatility models has not be addressed. Thus the aim of this paper is to study the optimal capital structure of a firm inside a structural model with endogenous bankruptcy in the spirit of [12], but assuming a stochastic volatility for the firm's assets.

We introduce a process describing the dynamic of the diffusion coefficient driven by a one factor mean-reverting process of Orstein-Uhlenbeck type, negatively correlated with firm's assets value evolution. Differently from the classical Leland framework and even from the more general context with payouts and asymmetric corporate tax rates studied in [1, 2, 3], the key point relies in the fact that inside this framework we cannot obtain explicit expressions for all the variables involved in the capital structure by means of the Laplace transform of the stopping failure time, because this transform, which was the key tool, is not available in closed form in our stochastic volatility framework. Nevertheless debt, equity, bankruptcy costs and tax benefits are claims on the firm's assets, thus we apply ideas and techniques developed in [7] for the pricing of derivatives securities whose underlying asset price's volatility is characterize by means of its time scales fluctuations. This approach has been applied in [8] to the pricing of a defaultable zero coupon bond. Here we consider a one-factor stochastic volatility model and apply single perturbation theory as in [7] in order to find approximate closed form solutions for variables involved in our problem. We analyze all financial variables and study the effects of the stochastic volatility assumption on the endogenous failure level determined by equity holders maximizing behavior. All claims have a more complicated expression with respect to the constant volatility case, depending not only on the process describing firm's activities value, but also on the process driving the diffusion coefficient. Equity holders still face the problem of optimizing equity value w.r.t. the failure level. Nevertheless under our approach, the failure level derived from standard smooth pasting principle is not the solution of the optimal stopping problem, but only represents a lower bound which has to be satisfied due to limited liability of equity. Choosing that failure level would mean an early exercise of the option to default. A *corrected smooth pasting condition* must be applied in order to find the endogenous failure level solution of the optimal stopping problem. Moreover, we show the convergence of our results to Leland case [12] as the particular case of zero-perturbation.

Pricing and hedging problems related to equity markets suggest that a pricing model with stochastic volatility is seen a fundamental feature in modeling the underlying assets

dynamic. Empirical findings about structural models of credit risk show that this kind of models usually underestimates spreads and default probabilities, while predicted leverage ratios are too high. In [1, 2, 3], we showed how to modify a pure Leland model in order to obtain a more realistic framework and also empirical predictions about default rates, leverage and spreads more in line with historical norms.

In this paper by taking into account the stochastic volatility risk component of the firm's asset dynamic, our aim is to better capture extreme returns behavior which could be a robust way to improve empirical predictions about spreads and leverage. Introducing randomness in volatility allows to deal with a structural model in which the distribution of stock prices returns is not symmetric: in our mind this seems to be the right way for capturing what structural models are not able to explain with a constant diffusion coefficient. The numerical results show that the assumption of stochastic volatility model produces relevant effects on the optimal capital structure in terms of higher credit spreads and lower leverage ratios, if compared with the original Leland case. Moreover the *corrected smooth-fit principle* seems to be an issue to develop in order to analyze the optimal exercise time of American-style options under a stochastic volatility pricing model (see also [15]).

The paper is organized as follows. Section 2 describes the stochastic volatility pricing model. Section 3 provides a detailed analysis of defaultable claims valuation in all mathematical aspects. In Section 4 we fully exploit the capital structure of the firm under volatility risk providing approximate values for all derivatives depending on firm's current assets value. Section 5 shows numerical results about optimal capital structure, then Section 6 gives some concluding remarks.

2 The model

We introduce a process describing the dynamic of the diffusion coefficient driven by a one factor mean-reverting process of Orstein-Uhlenbeck type, negatively correlated with firm's assets value evolution. From an economic point of view how to chose the diffusion coefficient, i.e. how to model volatility is a fundamental issue. Assuming the diffusion coefficient being constant means assuming that the riskiness of the firm does not change through time. While volatility is not an observable variable, market data suggest that the riskiness of the firm can deeply vary in time. Therefore the economic intuition is that analyzing the capital structure of a firm in a stochastic volatility pricing framework should be a robust and flexible way to improve structural models predictions bringing them closer to empirical evidence. We stress robust and flexible since we will suppose firm's activities value belonging to a fairly large class of stochastic volatility models.

We consider a firm whose (unlevered) activities value dynamic is described by process V_t . Process Y_t is introduced to describe the evolution of the diffusion coefficient. Under

the *physical* measure \mathbf{P} the dynamics of the model are described by the following SDEs in \mathbb{R}^2 :

$$dV_t = \mu V_t dt + f(Y_t) V_t dW_t, \quad (1)$$

$$dY_t = \eta(m - Y_t) dt + \beta d\widetilde{W}_t, \quad (2)$$

where $d\langle W, \widetilde{W} \rangle_t = \rho dt$ and $\beta^2 = 2\nu^2\eta$, $\eta = 1/\epsilon$, and $f(\cdot)$ is supposed to be bounded and Lipschitz. Parameter μ represents the expected rate of return of firm's assets value. We also suppose $f(Y) \in L^2(\Omega \times [0, \infty])$ and that there exist two constants a and A such that $0 < a \leq f \leq A$ to avoid explosion. The above SDE admits a unique strong solution. Y_t is a Gaussian process, which is explicitly known given the initial condition $Y_0 = y$:

$$Y_t = m + (y - m)e^{-\eta t} + \beta \int_0^t e^{-\eta(t-s)} d\widetilde{W}_s, \quad (3)$$

The invariant distribution of Y_t , obtained as $t \rightarrow \infty$ is $N(m, \frac{\beta^2}{2\eta})$ and the important feature to stress is that it does not depend on the initial condition y .

Substituting $\beta^2 = 2\nu^2\eta$ and $\eta = \frac{1}{\epsilon}$, we have:

$$Y_t^\epsilon = m + (y - m)e^{-\frac{t}{\epsilon}} + \nu \sqrt{\frac{2}{\epsilon}} \int_0^t e^{-\frac{(t-s)}{\epsilon}} d\widetilde{W}_s.$$

If we consider $\epsilon \rightarrow 0$, then almost surely $Y_t^\epsilon \rightarrow m$, so does diffusion parameter $f(Y_t) \rightarrow f(m)$. Moreover, following [7] (pg. 40-41) we assume $\rho < 0$. Analyzing financial data suggests the existence of a negative *leverage effect* between stock prices and volatility, i.e. $\rho < 0$, since real data shows that prices tend to decrease when volatility rises. In our model we consider this correlation ρ being constant, also if we know that it could be varying through time.

From structural models theory we know that each component of the capital structure of the firm can be seen as a claim on the underlying assets represented by firm's activities value V . Thus, in order to find the values of these claims, the pricing problem is addressed under a *risk neutral* probability measure \mathbf{Q} , where the asset's evolution follows the SDEs (cf. [7] p. 31):

$$dV_t = rV_t dt + f(Y_t) V_t dW_t, \quad (4)$$

$$dY_t = (\eta(m - Y_t) - \beta\Lambda(Y_t)) dt + \beta d\widetilde{W}_t, \quad (5)$$

where r is the constant risk free rate and $\Lambda(Y_t)$ is defined as:

$$\Lambda(Y_t) = \rho \frac{\mu - r}{f(Y_t)} + \gamma(Y_t) \sqrt{1 - \rho^2}, \quad (6)$$

where $\frac{\mu - r}{f(Y_t)}$ is the *excess return-to-risk ratio*, and $\gamma(Y_t)$ is the *risk premium factor* or *market price of volatility risk* which allows to take into account the second source of randomness \widetilde{W}_t driving the volatility process. Following [8] we assume $\gamma(\cdot)$ being bounded and a function of y only¹.

¹As particular case we will also consider γ being a constant when facing the hedging problem.

From now on the aim is to develop a pricing model for the capital structure of a firm whose underlying assets value is V_t , without specifying a particular function $f(\cdot)$. This will allow to present results which are not strictly dependent on a specific volatility process but instead related to a fairly general class of one-factor processes and in this sense they are robust, i.e. model-independent.

Following structural models approach the aim is to analyze the capital structure of a firm in terms of derivative contracts. In the spirit of [12] we consider an infinite time horizon and a firm which is issuing both equity and debt. The firm issues debt and debt is perpetual. Debt holders receive a constant coupon C per instant of time. We assume that from issuing debt the firm obtains tax deductions proportional to coupon payments. The corporate tax rate τ is assumed to be constant, thus the firm will benefit of a tax-sheltering value of interest payments τC . The firm is subject to the risk of default, thus: when coupon payments are low, the total value of the firm rises with an increase in C due to tax benefits of debt, but as C reaches a certain level, the total value of the firm decreases, due to bankruptcy costs. This is the trade-off between taxes and bankruptcy costs. Default is endogenously triggered. The economic intuition is that default arrives when the firm is not able to cover its debt obligations, meaning when equity value is null (due to limited liability). The mathematical tool to treat default is that the failure passage time is determined when firm's activities value falls to some constant level V_B . The value of V_B is endogenously derived by equity holders in order to maximize equity value.

We define the stopping time

$$T_{V_B} = \inf\{t \geq 0 : V_t = V_B\},$$

moreover, since process V is right continuous, it holds $V_{T_B} = V_B$.

Following [12, 17] contingent claim valuation can be used, so it is possible to express each component of the capital structure as a claim on the underlying assets representing firm activities value (see also [1, 2, 3]).

3 Pricing Defaultable Claims under Volatility Risk

In this section we apply ideas and techniques developed in [7] for the pricing of derivatives securities whose underlying asset price's volatility is characterized by means of its time scales fluctuations. We consider a one-factor stochastic volatility model and apply single perturbation theory as in [7] in order to find approximate values for defaultable derivatives involved in our problem. This approach has been applied in [8] to the pricing of a defaultable zero coupon bond. We will conduct the analysis considering the more general case of defaultable paying-dividend derivatives.

We are assuming that firm's activities value belongs to a fairly large class of stochastic volatility models. In this framework we obtain very general expressions for all claims

describing the components of the capital structure, each one of them depending on both processes V and Y . Actually both processes V and Y depend on the parameter ϵ but we will omit it to have a simpler notation. The aim is to determine how the capital structure is affected by volatility risk, meaning both a qualitative and quantitative analysis of such an influence.

Let the price of a general claim be $F^\epsilon(V_t, Y_t)$, function F^ϵ being supposed to be C_b^2 , depending on the parameter ϵ . We consider this general claim as a dividend-paying contract, thus by no arbitrage hypothesis, the process

$$t \mapsto e^{-rt} F^\epsilon(V_t, Y_t) + d \int_0^t e^{-rs} ds$$

which represents the value of the claim at time t , discounted at the risk free rate r , plus cumulated dividends up to time t , discounted at r and d is the constant dividend paid by the claim, has to be a local martingale. Moreover, being a true martingale $\forall T$ stopping time, yields:

$$E \left[e^{-rT} F^\epsilon(V_T, Y_T) + d \int_0^T e^{-rs} ds | V_0 = x, Y_0 = y \right] = F^\epsilon(x, y), \quad (7)$$

where the expectation is taken in the risk neutral measure (see [17]).

Thus, the function $F^\epsilon(x, y)$ has to satisfy the following partial derivatives equation:

$$\begin{aligned} d - rF^\epsilon(x, y) + rx\partial_x F^\epsilon(x, y) + \left(\frac{1}{\epsilon}(m - y) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(y) \right) \partial_y F^\epsilon(x, y) + \quad (8) \\ + \frac{1}{2}x^2 f^2(y) \partial_{x^2}^2 F^\epsilon(x, y) + \rho\nu\sqrt{\frac{2}{\epsilon}} f(y)x \partial_{xy}^2 F^\epsilon(x, y) + \nu^2 \frac{1}{\epsilon} \partial_{y^2}^2 F^\epsilon(x, y) = 0. \end{aligned}$$

Re-arranging terms we have:

$$\begin{aligned} d - rF^\epsilon(x, y) + rx\partial_x F^\epsilon(x, y) + \frac{1}{2}x^2 f^2(y) \partial_{x^2}^2 F^\epsilon(x, y) \quad (9) \\ + \frac{1}{\epsilon} \left((m - y) \partial_y F^\epsilon(x, y) + \nu^2 \partial_{y^2}^2 F^\epsilon(x, y) \right) \\ + \rho\nu\sqrt{\frac{2}{\epsilon}} f(y)x \partial_{xy}^2 F^\epsilon(x, y) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \Lambda(y) \partial_y F^\epsilon(x, y) = 0. \end{aligned}$$

Since differential equation (9) involves terms of order $\frac{1}{\epsilon}$, $\frac{1}{\sqrt{\epsilon}}$, 1, we introduce the following notation for any C_b^2 function $g(x, y)$:

$$\begin{aligned} \mathcal{L}_0 g(x, y) &= (m - y) \partial_y g(x, y) + \nu^2 \partial_{y^2}^2 g(x, y), \\ \mathcal{L}_1 g(x, y) &= \rho\sqrt{2}\nu f(y)x \partial_{xy}^2 g(x, y) - \sqrt{2}\nu\Lambda(y) \partial_y g(x, y), \quad (10) \\ (\mathcal{L}_{BS}(f))g(x, y) &= d - rg(x, y) + rx\partial_x g(x, y) + \frac{1}{2}x^2 f^2(y) \partial_{x^2}^2 g(x, y). \end{aligned}$$

where

- \mathcal{L}_0 is the infinitesimal generator of an ergodic Markov process, involving only y variable;
- \mathcal{L}_1 is the operator depending on the mixed derivative ∂_{xy} , since we are supposing a correlation ρ between the Brownian motions of processes V and Y ;
- $\partial_t f + \mathcal{L}_{BS}(f)$ is the Black-Scholes operator when the volatility level is $f(y)$ and the claim pays a constant dividend d .

It is now possible to write differential equation (9) in the following way:

$$\left(\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_{BS}(f) \right) F^\epsilon(x, y) = 0, \quad x \in \mathbb{R}_*^+, \quad y \in \mathbb{R}. \quad (11)$$

We now discuss the boundary conditions of this problem. Since we are considering an infinite horizon, a terminal condition requires $\frac{F^\epsilon(x, y)}{x}$ being bounded $\forall \epsilon$ when $x \rightarrow \infty$ in order to avoid bubbles (see [4]). Further a boundary condition at default is given by $F^\epsilon(x_B, y) = f_*(x_B)$ where $f_*(x_B)$ depends on the specific claim we are considering and x_B is the failure level. This last boundary condition is strictly related to the economic meaning of each specific claim (i.e. for equity we have $f_E(x_B) = 0$, while for debt $f_D(x_B) = (1 - \alpha)x_B$ since the strict priority rule holds). From an economic point of view this last boundary condition simply means that tax benefits are completely lost in the event of bankruptcy. The solution $F^\epsilon(x, y)$ of Equation (11) exists and is unique $\forall \epsilon > 0$.

Following [7], we expand the solution F^ϵ in powers of $\sqrt{\epsilon}$:

$$F^\epsilon = P_0 + \sqrt{\epsilon} P_1 + \epsilon P_2 + \epsilon \sqrt{\epsilon} P_3 + \dots, \quad (12)$$

where P_0, P_1, P_2, \dots are functions of (x, y) to be determined such that $\frac{P_0(x, y)}{x}$ will be bounded as $x \rightarrow \infty$ and $P_0(x_B, y) = 0$.

Substituting Equation (12) into Equation (11) we have

$$\left(\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_{BS}(f) \right) (P_0 + \sqrt{\epsilon} P_1 + \epsilon P_2 + \epsilon \sqrt{\epsilon} P_3 + \dots) = 0.$$

So we obtain:

$$\begin{aligned} & \frac{1}{\epsilon} \mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 P_0 + \mathcal{L}_{BS}(f) P_0 + \sqrt{\epsilon} \mathcal{L}_{BS}(f) P_1 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_0 P_1 + \mathcal{L}_1 P_1 \\ & \epsilon \mathcal{L}_0 P_2 + \sqrt{\epsilon} \mathcal{L}_1 P_2 + \epsilon \mathcal{L}_{BS}(f) P_2 + \sqrt{\epsilon} \mathcal{L}_0 P_3 + \epsilon \mathcal{L}_1 P_3 + \epsilon \sqrt{\epsilon} \mathcal{L}_{BS}(f) P_3 + \dots = 0 \end{aligned}$$

Re-arranging terms (ref. Eq. (5.16) pag. 90):

$$\frac{1}{\epsilon} \mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}} (\mathcal{L}_1 P_0 + \mathcal{L}_0 P_1) + \quad (13)$$

$$\begin{aligned} & + (\mathcal{L}_{BS}(f) P_0 + \mathcal{L}_1 P_1 + \mathcal{L}_0 P_2) \\ & + \sqrt{\epsilon} (\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_{BS}(f) P_1) \\ & + \dots \\ & = 0. \end{aligned} \quad (14)$$

Equating terms of order $\frac{1}{\epsilon}$, we must have

$$\mathcal{L}_0 P_0 = 0.$$

Since the operator \mathcal{L}_0 in Equation (10) is the infinitesimal generator of an ergodic Markov process acting only on y variable, (and P_0 has to be bounded) P_0 must depend only on x variable.

Similarly, to eliminate the term of order $\frac{1}{\sqrt{\epsilon}}$, we must have:

$$\mathcal{L}_1 P_0 + \mathcal{L}_0 P_1 = 0.$$

Observe that the operator \mathcal{L}_1 in Equation (10) involves only the derivative w.r.t. y variable. Since P_0 only depends on x variable, as a consequence we have $\mathcal{L}_1 P_0 = 0$ and what remains is $\mathcal{L}_0 P_1 = 0$. Using the same argument as above for the operator \mathcal{L}_0 , we have to find P_1 as function of x variable only.

Remark 3.1 *Observe that if we consider only P_0 and P_1 terms, this implies that we are looking for an approximate value of the claim*

$$F^\epsilon(x) \approx P_0(x) + \sqrt{\epsilon} P_1(x),$$

and this solution does not depend on y variable, meaning that it does not depend on the present volatility $f^2(y)$.

Now recall Equation (13). What we have to do is continuing to eliminate terms of order $1, \sqrt{\epsilon}, \dots$ and so on. The idea is to study asymptotic approximations for F^ϵ which become more accurate as $\epsilon \rightarrow 0$. At this point the problem is to solve the following equation:

$$\mathcal{L}_{BS}(f)P_0 + \mathcal{L}_1 P_1 + \mathcal{L}_0 P_2 = 0. \tag{15}$$

Since \mathcal{L}_1 involves the mixed derivative ∂_{xy} and P_1 must depend only on x variable, we are sure that

$$\mathcal{L}_1 P_1 = 0.$$

What remains is:

$$\mathcal{L}_{BS}(f)P_0 + \mathcal{L}_0 P_2 = 0 \tag{16}$$

The variable x being fixed, focusing on the dependence of $\mathcal{L}_{BS}(f)$ on y , Equation (16) is a Poisson equation (cf.[7], p. 91) which admits a unique solution P_2 only if

$$\langle \mathcal{L}_{BS}(f)P_0 \rangle = 0,$$

where

$$\langle g \rangle := \int_{\mathbb{R}} g(y) \Phi(y) dy, \tag{17}$$

and $\Phi(y)$ denotes the density function of the Gaussian distribution $\mathcal{N}(m, \nu^2)$.

Following [7] we define the effective volatility as

$$\bar{\sigma}^2 = \langle f^2 \rangle. \quad (18)$$

As a consequence, $\langle \mathcal{L}_{BS}(f) \rangle = \mathcal{L}_{BS}(\bar{\sigma})$, and the zero-order term P_0 will be the solution of

$$\mathcal{L}_{BS}(\bar{\sigma})P_0 = 0 \quad (19)$$

as shown in the following Proposition.

Proposition 3.2 *Equation (19) admits as unique solution the zero-order term P_0 with the following form:*

$$P_0(x) = k_* + l_* \left(\frac{x_B}{x}\right)^\lambda \quad (20)$$

with

$$\lambda = \frac{2r}{\bar{\sigma}^2}, \quad (21)$$

and k_*, l_* depend on the specific dividend and boundary conditions of each claim.

Proof We are looking for a solution P_0 satisfying the following ODE

$$\mathcal{L}_{BS}(\bar{\sigma})P_0 = 0,$$

with boundary conditions: $\frac{P_0(x)}{x}$ to be bounded as $x \rightarrow \infty$, and $P_0(x_B) = f_*(x_B)$, where $f_*(x_B)$ depends on the specific claim we are considering.

Recalling Equation (10) for Black-Scholes operator $\mathcal{L}_{BS}(\bar{\sigma})$, we have to find $P_0(x)$ depending only on x variable, as solution of

$$d - rP_0(x) + rxP_0'(x) + \frac{1}{2}\bar{\sigma}^2x^2P_0''(x) = 0, \quad (22)$$

where d is the dividend paid by the claim considered. Equation (22) is exactly what we have to solve when considering Leland framework assuming a constant volatility $\bar{\sigma}$. Boundary conditions for each claim are needed to determine k_*, l_* as we will show later.

•

The following Proposition provides the value of each specific claim describing the capital structure of the firm, i.e. equity value, debt, tax benefits, bankruptcy costs and total value of the firm, under a pricing model with constant volatility equal to $\bar{\sigma}$.

Proposition 3.3 *Under our assumption of stochastic volatility, the capital structure of the firm has the following P_0 terms:*

$$P_0^{TB}(x, x_B) = \frac{\tau C}{r} - \frac{\tau C}{r} \left(\frac{x_B}{x}\right)^\lambda, \quad (23)$$

$$P_0^D(x, x_B) = \frac{C}{r} + \left((1 - \alpha)x_B - \frac{C}{r}\right) \left(\frac{x_B}{x}\right)^\lambda, \quad (24)$$

$$P_0^{BC}(x, x_B) = \alpha x_B \left(\frac{x_B}{x}\right)^\lambda, \quad (25)$$

$$P_0^E(x, x_B) = x - \frac{(1 - \tau)C}{r} + \left(\frac{(1 - \tau)C}{r} - x_B\right) \left(\frac{x_B}{x}\right)^\lambda, \quad (26)$$

$$P_0^V(x, x_B) = x - \frac{(1 - \tau)C}{r} + \left(\frac{(1 - \tau)C}{r} - x_B\right) \left(\frac{x_B}{x}\right)^\lambda. \quad (27)$$

Proof From Proposition 3.2 we know that each claim will be of the form

$$P_0(x) = k_* + l_* \left(\frac{x_B}{x}\right)^\lambda$$

with

$$\lambda = \frac{2r}{\sigma^2}.$$

We will define a claim on x for each component of the capital structure: equity (E), tax benefits (TB), debt (D), bankruptcy costs (BC) and the total value of the firm (V). Recall that d denote the dividend paid by each claim. Boundary conditions specific for each claim will give k_* and l_* as shown in the table below. •

Claim *	TB	D	BC	E	V
$P_{0*}(x_B, x_B)$	0	$(1 - \alpha)x_B$	αx_B	0	$(1 - \alpha)x_B$
$k_* = \lim_{x \rightarrow \infty} \frac{P_{0*}(x, x_B)}{x}$	$\frac{\tau C}{r}$	$\frac{C}{r}$	0	$x - \frac{(1 - \tau)C}{r}$	$x + \frac{\tau C}{r}$
l_*	$-\frac{\tau C}{r}$	$(1 - \alpha)x_B - \frac{C}{r}$	αx_B	$\frac{(1 - \tau)C}{r} - x_B$	$-(\alpha x_B + \frac{\tau C}{r})$
d_*	τC	C	0	$-(1 - \tau)C$	τC

Table 1: This table shows boundary conditions for each specific claim: row 1 describes boundary conditions at default, i.e. for $x \rightarrow x_B$; row 2 describes boundary conditions as $x \rightarrow \infty$.

We now search for the the second order correction term P_2 , in order to be able to define an equation allowing us to find P_1 . The following proposition holds.

Proposition 3.4 *The solution of Equation (16) is the second-order correction term P_2 depending on both x and y variables as follows:*

$$P_2(x, y) = -\frac{1}{2}\phi(y)x^2\frac{\partial^2 P_0(x)}{\partial x^2}, \quad (28)$$

with $P_0(x)$ given by Equation (20) and $\phi(y)$ being solution of

$$\nu^2\phi''(y) + (m - y)\phi'(y) = f(y)^2 - \bar{\sigma}^2, \quad (29)$$

or equivalently

$$\nu^2\Phi(y)\phi'(y) = \int_{-\infty}^y (f^2(z) - \bar{\sigma}^2)\Phi(z)dz. \quad (30)$$

Proof We are searching the second order correction term P_2 as solution of Equation (16). Since $\mathcal{L}_{BS}(\bar{\sigma})P_0 = 0$, we can write

$$\mathcal{L}_{BS}(f)P_0(x) = \mathcal{L}_{BS}(f)P_0(x) - \mathcal{L}_{BS}(\bar{\sigma})P_0(x) = \frac{1}{2}(f(y)^2 - \bar{\sigma}^2)x^2\frac{\partial^2 P_0(x)}{\partial x^2}.$$

So, from Equation (16), it remains to find P_2 such that

$$\mathcal{L}_0 P_2 = -\frac{1}{2}(f(y)^2 - \bar{\sigma}^2)x^2\frac{\partial^2 P_0(x)}{\partial x^2},$$

where \mathcal{L}_0 is given by Equation (10) being an operator involving only the derivative w.r.t. y variable. Thus

$$P_2(x, y) = -\frac{1}{2}\phi(y)x^2\frac{\partial^2 P_0(x)}{\partial x^2},$$

with $\phi(y)$ being solution of

$$\mathcal{L}_0\phi(y) = (f(y)^2 - \bar{\sigma}^2).$$

Following the same methodology as before, we have to impose the coefficient of $\sqrt{\epsilon}$ in Equation (13) being null:

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_{BS}(f)P_1 = 0. \quad (31)$$

We now have a Poisson equation which admits a unique solution P_3 only if

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_{BS}(f)P_1 \rangle = 0. \quad (32)$$

Equation (32) and (28) will lead us to find the first order correction term $P_1(x)$ for each claim defining a specific component of firm's capital structure (i.e. equity, debt, tax benefits of debt, bankruptcy costs, total value) as we show in the following Proposition.

Proposition 3.5 *Under our assumption of stochastic volatility, assuming boundary conditions $P_1(x_B, x_B, C) = 0$ and $\lim_{x \rightarrow \infty} P_1(x, x_B, C) = 0$, the capital structure of the firm has the following component for first correction terms $P_1(x, x_B)$:*

$$P_1^{TB}(x, x_B) = -l^{TB} H \cdot \left(\frac{x_B}{x}\right)^\lambda \log \frac{x_B}{x} \quad (33)$$

$$P_1^D(x, x_B) = -l^D H \cdot \left(\frac{x_B}{x}\right)^\lambda \log \frac{x_B}{x} \quad (34)$$

$$P_1^{BC}(x, x_B) = -l^{BC} H \cdot \left(\frac{x_B}{x}\right)^\lambda \log \frac{x_B}{x} \quad (35)$$

$$P_1^E(x, x_B) = -l^E H \cdot \left(\frac{x_B}{x}\right)^\lambda \log \frac{x_B}{x} \quad (36)$$

$$P_1^V(x, x_B) = -l^V H \cdot \left(\frac{x_B}{x}\right)^\lambda \log \frac{x_B}{x} \quad (37)$$

where

$$H = \frac{4r}{\sigma^4} \left(\frac{\sqrt{2}}{2} \nu \langle \Lambda \phi' \rangle + \frac{2r}{\sigma^2} v_3 \right), \quad v_3 = \rho \frac{\sqrt{2}}{2} \nu \langle f \phi' \rangle, \quad (38)$$

and

$$\begin{aligned} l^{TB} &= -\frac{\tau C}{r}, \quad l^D = (1 - \alpha)x_B - \frac{C}{r}, \quad l^{BC} = \alpha x_B, \\ l^E &= \frac{(1 - \tau)C}{r} - x_B, \quad l^V = -\left(\alpha x_B + \frac{\tau C}{r}\right). \end{aligned} \quad (39)$$

Proof We can rewrite Equation (32) in this way

$$(\mathcal{L}_{BS}(\bar{\sigma})P_1)(x) = - \int_{\mathbb{R}} \mathcal{L}_1 P_2(x, y) \Phi(y) dy. \quad (40)$$

Recall equation (28) for the second order correction term P_2 :

$$P_2(x, y) = -\frac{1}{2} \phi(y) x^2 \frac{\partial^2 P_0(x)}{\partial x^2}$$

and Definition (10) for the operator \mathcal{L}_1 . We have

$$\mathcal{L}_1 P_2(x, y) = \rho \sqrt{2} \nu f(y) x \partial_{xy}^2 \left(-\frac{1}{2} x^2 \frac{\partial^2 P_0}{\partial x^2} \phi(y) \right) - \sqrt{2} \nu \Lambda(y) \partial_y \left(-\frac{1}{2} x^2 \frac{\partial^2 P_0}{\partial x^2} \phi(y) \right) \quad (41)$$

$$= \rho \sqrt{2} \nu f(y) x \partial_x \left(-\frac{1}{2} x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \phi'(y) - \sqrt{2} \nu \Lambda(y) \phi'(y) \left(-\frac{1}{2} x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \quad (42)$$

In order to find P_1 we have to solve the following equation

$$\mathcal{L}_{BS}(\bar{\sigma})P_1(x, x_B, C) = v_3 x^3 \frac{\partial^3 P_0}{\partial x^3} + v_2 x^2 \frac{\partial^2 P_0}{\partial x^2}, \quad (43)$$

$$\text{with } v_2 = \sqrt{2}\rho\nu \langle f\phi' \rangle - \frac{\sqrt{2}}{2}\nu \langle \Lambda\phi' \rangle = 2v_3 - \frac{\sqrt{2}}{2}\nu \langle \Lambda\phi' \rangle, \quad (44)$$

$$v_3 = \rho \frac{\sqrt{2}}{2} \nu \langle f\phi' \rangle, \quad (45)$$

meaning equation

$$\mathcal{L}_{BS}(\bar{\sigma})P_1(x, x_B, C) = \rho \frac{\sqrt{2}}{2} \nu \langle f\phi' \rangle x^3 \frac{\partial^3 P_0}{\partial x^3} + \left(\sqrt{2}\rho\nu \langle f\phi' \rangle - \frac{\sqrt{2}}{2}\nu \langle \Lambda\phi' \rangle \right) x^2 \frac{\partial^2 P_0}{\partial x^2}. \quad (46)$$

Recall the general structure given by Equations (20) for the zero order terms $P_0(x)$:

$$P_0(x) = k_* + l_* \left(\frac{x_B}{x} \right)^\lambda,$$

with

$$\lambda = \frac{2r}{\bar{\sigma}^2},$$

where k_*, l_* are given in Table 3. Observe that k_* is constant w.r.t. x for all claims except equity and for equity value it is linear w.r.t. x .

The following holds for each claim, namely *:

$$\frac{\partial P_{0*}(x)}{\partial x} = \frac{\partial k_*}{\partial x} - l_* \lambda x_B^\lambda x^{-\lambda-1} \quad (47)$$

$$\frac{\partial^2 P_{0*}(x)}{\partial x^2} = l_* \lambda (\lambda + 1) x_B^\lambda x^{-\lambda-2} \quad (48)$$

$$\frac{\partial^3 P_{0*}(x)}{\partial x^3} = -l_* \lambda (\lambda + 1) (\lambda + 2) x_B^\lambda x^{-\lambda-3} \quad (49)$$

Finally

$$\begin{aligned} \mathcal{L}_{BS}(\bar{\sigma})P_1(x, x_B, C) &= -l^* \lambda^2 (\lambda + 1) \frac{\sqrt{2}}{2} \rho \nu \langle f\phi' \rangle \left(\frac{x_B}{x} \right)^\lambda - \frac{\sqrt{2}}{2} \nu \langle \Lambda\phi' \rangle l^* \lambda (\lambda + 1) \left(\frac{x_B}{x} \right)^\lambda \\ &= l^* \lambda (\lambda + 1) (v_2 - (\lambda + 2)v_3) \left(\frac{x_B}{x} \right)^\lambda \\ &= l^* \lambda (\lambda + 1) \left(2v_3 - \frac{\sqrt{2}}{2} \nu \langle \Lambda\phi' \rangle - (\lambda + 2)v_3 \right) \left(\frac{x_B}{x} \right)^\lambda \\ \mathcal{L}_{BS}(\bar{\sigma})P_1(x, x_B, C) &= -l^* \lambda (\lambda + 1) \left(\frac{\sqrt{2}}{2} \nu \langle \Lambda\phi' \rangle + \lambda v_3 \right) \left(\frac{x_B}{x} \right)^\lambda. \end{aligned} \quad (50)$$

Recall Definition (10) of the Black-Scholes operator:

$$\mathcal{L}_{BS}(\bar{\sigma})P_1(x, x_B, C) = d - rP_1(x, x_B, C) + rxP_1'(x) + \frac{1}{2}\bar{\sigma}^2x^2P_1''(x)$$

Thus P_1 is the solution of:

$$d - rP_1(x) + rxP_1'(x) + \frac{1}{2}\bar{\sigma}^2x^2P_1''(x) = Bx^{-\lambda} \quad (51)$$

with

$$B = l_* \lambda(\lambda + 1) (v_2 - (\lambda + 2)v_3) x_B^\lambda, \quad (52)$$

$$= -l_* \lambda(\lambda + 1) \left(\frac{\sqrt{2}}{2} \nu \langle \Lambda \phi' \rangle + \lambda v_3 \right) x_B^\lambda. \quad (53)$$

This second order ODE admits P_1 given by Equation (36) as unique solution, taking into account boundary conditions

$$\lim_{x \rightarrow \infty} P_1(x, x_B) = 0, \quad (54)$$

$$P_1(x_B, x_B) = 0. \quad (55)$$

The solution is

$$P_1(x, x_B) = A_0 + A_1x + A_2x^{-\lambda} + A_3x^{-\lambda} \log x,$$

From boundary condition (54)-(55) we have respectively :

$$A_0 = 0, A_1 = 0; A_2 = -A_3 \log x_B,$$

thus the P_1 component of the first correction term is of the form

$$P_1(x, x_B) = -A_3x^{-\lambda} \log \frac{x_B}{x},$$

with $A_3 = -\frac{2B}{2r + \bar{\sigma}^2}$, meaning :

$$\begin{aligned} A_3 &= -\frac{2l^* \lambda(\lambda + 1) (v_2 - (\lambda + 2)v_3) x_B^\lambda}{2r + \bar{\sigma}^2} \\ &= \frac{2l^* \lambda(\lambda + 1) \left(\frac{\sqrt{2}}{2} \nu \langle \Lambda \phi' \rangle + \lambda v_3 \right) x_B^\lambda}{2r + \bar{\sigma}^2} \\ &= l^* \frac{4r}{\bar{\sigma}^4} \left(\frac{\sqrt{2}}{2} \nu \langle \Lambda \phi' \rangle + \frac{2r}{\bar{\sigma}^2} v_3 \right) x_B^\lambda. \end{aligned} \quad (56)$$

$$P_1 = -\frac{l^* \lambda^2 (\lambda + 1) \sqrt{2} \rho \nu \langle f \phi' \rangle x_B^\lambda}{2r + \bar{\sigma}^2} x^{-\lambda} \log \frac{x_B}{x}.$$

Finally we can write P_1 in terms of v_3 as:

$$P_1 = -l^* \frac{4r}{\bar{\sigma}^4} \left(\frac{\sqrt{2}}{2} \nu \langle \Lambda \phi' \rangle + \frac{2r}{\bar{\sigma}^2} v_3 \right) \left(\frac{x_B}{x} \right)^\lambda \log \frac{x_B}{x}.$$

The final result with this method is:

$$P_1(x, x_B) = -l^* \rho \sqrt{2} \nu \langle f \phi' \rangle \frac{4r^2}{\bar{\sigma}^6} \left(\frac{x_B}{x} \right)^\lambda \log \left(\frac{x_B}{x} \right). \quad (57)$$

Therefore we can write:

$$P_1^*(x) = -l^* H \cdot \left(\frac{x_B}{x} \right)^\lambda \log \frac{x_B}{x}, \quad (58)$$

where

$$H = -\frac{2\lambda(\lambda+1)(v_2 - (\lambda+2)v_3)}{2r + \bar{\sigma}^2}, \quad (59)$$

which simplifies to

$$H = \frac{4r}{\bar{\sigma}^4} \left(\frac{\sqrt{2}}{2} \nu \langle \Lambda \phi' \rangle + \frac{2r}{\bar{\sigma}^2} v_3 \right), \quad (60)$$

and for each specific claim, the term l^* is given in Table 3, row 3:

$$l^{TB} = -\frac{\tau C}{r}, \quad l^D = (1 - \alpha)x_B - \frac{C}{r}, \quad l^{BC} = \alpha x_B,$$

$$l^E = \frac{(1 - \tau)C}{r} - x_B, \quad l^V = -\left(\alpha x_B + \frac{\tau C}{r} \right).$$

•

4 Capital Structure of the Firm under Volatility Risk

We now analyze in detail all financial variables defining the capital structure of the firm by using previous results in order to determine how their values are affected by volatility risk. We can interpret each financial variable as a derivative contract, thus the aim is to understand how the price of each claim must be *corrected* due to volatility risk. We are assuming the driving Ornstein-Uhlenbeck process for the diffusion coefficient being a fast mean reverting process, with $\eta := \frac{1}{\epsilon}$ as speed of mean-reversion. By applying singular perturbation theory, each component of the capital structure will be defined as a claim whose value can be found through an asymptotic expansion of the price in power of $\sqrt{\epsilon}$.

One of the main problems concerning perturbation theory is that, once we have explicit expressions for the *approximate value* of our claims, we have to calibrate parameters. The main issue is how to estimate $\bar{\sigma}$ from market data since a volatility process is not observable in financial markets. Usually the approach is to estimate $\bar{\sigma}$ from historical data and then

to calibrate the small parameters V_2, V_3 . In this paper we do not address the issue of how to calibrate parameters, but we take results from [7], [8], [9] where a specific and detailed analysis of this topic is conducted.

The value of each claim $F^\epsilon(x, x_B)$ will be approximate with the following corrected pricing formula:

$$\tilde{F}(x, x_B) = P_0(x, x_B) + \sqrt{\epsilon}P_1(x, x_B), \quad (61)$$

where $P_0(x, x_B)$ and $P_1(x, x_B)$ are obtained, respectively, as solutions of Equations (19), (46).

Remark 4.1 *i) Let $\tilde{P}_1(x, x_B) := \sqrt{\epsilon}P_1(x, x_B)$. Observe that an equivalent formulation for $\tilde{F}(x, x_B)$ is to write directly each approximate claim as solution of*

$$\mathcal{L}_{BS}(\bar{\sigma})(P_0(x, x_B) + \tilde{P}_1(x, x_B)) = V_2x^2\frac{\partial^2 P_0}{\partial x^2} + V_3x^3\frac{\partial^3 P_0}{\partial x^3}, \quad (62)$$

with

$$V_2 = \sqrt{\epsilon}v_2, \quad V_3 = \sqrt{\epsilon}v_3,$$

and v_2, v_3 are given by Equations (44)-(45). Coefficients V_2, V_3 correspond to the notation used in [7] (Equations (5.39-5.40), page 95 where parameter α in the book corresponds to our $\eta = \frac{1}{\epsilon}$).

This result follows by applying $\mathcal{L}_{BS}(\bar{\sigma})$ to function $P_0 + \sqrt{\epsilon}P_1$ (for P_1 look at Equation (46)). Following [7] we can interpret the right hand side of Equation (62) as a path dependent payment stream which corrects the price accounting for volatility randomness during the path before the default time. Notice that it may be positive or negative depending on the specific P_0 claim we are dealing with, since it involves its second and third derivatives w.r.t. x , meaning that it will be strictly related to the value of each claim in a pure Leland [12] model. More specifically, this expression involves Greeks of the corresponding price in a Black and Scholes setting with constant volatility $\bar{\sigma}$. While the second derivative is well known to be the Γ of the P_0 derivative, [7] propose to name the third derivative Epsilon defined as $\frac{\partial \Gamma}{\partial x}$.

ii) As a consequence of i), we can observe what follows about the sign of coefficient H given in Equation (59):

$$H = -\frac{2\lambda(\lambda+1)(v_2 - (\lambda+2)v_3)}{2r + \bar{\sigma}^2} = \frac{2r}{\bar{\sigma}^4}\sqrt{2\nu} \left(\langle \Lambda\phi' \rangle + \frac{2r}{\bar{\sigma}^2}\rho\langle f\phi' \rangle \right), \quad (63)$$

thus the sign of H is the same as $\langle \Lambda\phi' \rangle + \frac{2r}{\bar{\sigma}^2}\rho\langle f\phi' \rangle$.

iii) Following [8] we will assume $\rho < 0$, $V_2 < 0$ meaning $\rho\langle f\phi' \rangle < \langle \Lambda\phi' \rangle$ and $V_3 > 0$ meaning $\rho\langle f\phi' \rangle > 0$ (see [7] and [9] for a detailed analysis about parameters calibration), thus obtaining $H > 0$.

From previous results we can now write each approximate claim, namely $*$, on x with the following structure:

$$\tilde{F}_*(x, x_B) = P_{0*}(x, x_B) - \sqrt{\epsilon} l_* H \left(\frac{x_B}{x} \right)^\lambda \log \frac{x_B}{x}, \quad (64)$$

with

$$H = \frac{2r}{\bar{\sigma}^4} \sqrt{2\nu} \left(\langle \Lambda \phi' \rangle + \frac{2r}{\bar{\sigma}^2} \rho \langle f \phi' \rangle \right) > 0,$$

and l_* given in Table 3 depending for each specific claim on its own boundary conditions.

Notice that:

i) assuming $H > 0$, the sign of the correction term $\tilde{P}_1(x, x_B)$ is the same as l_* , thus can be positive or negative depending only the specific boundary conditions of each derivative contract. More specifically, from an economic point of view, it will depend on the final payoff of each derivative. An important feature to stress is that this term allows to correct the price accounting for randomness in volatility during the path of process x_t , i.e. for $x > x_B$ in a model-independent way. Results hold for a large class of processes, since we are not specifying a particular function $f(\cdot)$ for the diffusion coefficient.

ii) H -dependence on V_2, V_3 is fundamental in order to analyze the stochastic volatility effects on all defaultable claims defining the capital structure of the firm, since both coefficients V_2 and V_3 involved in our pricing problem have a precise economic interpretation. From [8] we know that calibrating V_2, V_3 from market data suggests to assume these small parameters being respectively: $V_2 < 0, V_3 > 0$. Typically coefficient $V_2 < 0$ since it represents a correction for the price in terms of volatility level, while V_3 is the skew effect related to the third moment of stock prices returns. Assuming a negative correlation $\rho < 0$ in this pricing model with stochastic volatility will produce its effect on the distribution of stock prices returns, making it asymmetric. In particular it will strongly modify the left tail of returns distributions, making it fatter.

An important issue to consider is the accuracy of these approximations. Let $F^\epsilon(x, x_B)$ be the true unknown value of the claim under stochastic volatility with terminal condition $F^\epsilon(x, x_B) = z(x)$. As analyzed in detail in [7] (see Chapter 5), it can be shown that when $z(x)$ is smooth and bounded, we have

$$|F^\epsilon(x, x_B) - (P_0(x, x_B) + \tilde{P}_1(x, x_B))| \leq k \cdot \epsilon,$$

where k is a constant which does not depend on the speed of mean reversion of the volatility process but may depend on its current level y .

Remark 4.2 *It is important to stress that as $\epsilon \rightarrow 0$ the model becomes a pure Leland model with constant volatility $\bar{\sigma}$, while V_3 in Equation (62) vanishes as ρ becomes 0, meaning the uncorrelated case becomes a pure Leland model with constant volatility equal to the*

corrected effective volatility $\sigma^* = \sqrt{\bar{\sigma}^2 - 2V_2} \neq \bar{\sigma}$. The difference between the two cases will be only a volatility level correction due to the market price of risk which exists also if we assume $\rho = 0, \Lambda \neq 0$: V_2 will still correct claims values for the market price of volatility risk. Typically, the effective corrected volatility σ^* is higher than the historical average volatility $\bar{\sigma}$ and this is why market data suggest to assume $V_2 < 0$.

These two cases coincide only if we assume $\rho = 0, \Lambda = 0$.

All these features will produce important economic implications on the analysis of the capital structure of the firm, as shown in what follows.

4.1 Equity

We now consider equity value under stochastic volatility.

Using singular perturbation theory we have approximate equity value $\tilde{E}(x, x_B)$ with the following form:

$$\tilde{E}(x, x_B) = P_0^E(x, x_B) + \sqrt{\epsilon} P_1^E(x, x_B),$$

with P_0^E and P_1^E given by Equations (26), (36). Its explicit expression is:

$$\tilde{E}(x, x_B) = x - \frac{(1-\tau)C}{r} + \left(\frac{(1-\tau)C}{r} - x_B \right) \left(\frac{x_B}{x} \right)^\lambda \left(1 - \sqrt{\epsilon} H \log \frac{x_B}{x} \right), \quad (65)$$

with

$$H = \frac{2r}{\bar{\sigma}^4} \sqrt{2\nu} \left(\langle \Lambda \phi' \rangle + \frac{2r}{\bar{\sigma}^2} \rho \langle f \phi' \rangle \right) > 0.$$

Consider now Equation (65): re-arranging terms we can always express approximate equity value as the sum of two different components by isolating equity value without risk of default and the option to default embodied in equity under this pricing model with stochastic volatility:

$$\tilde{E}(x, x_B) = f(x, C) + \tilde{g}(x, x_B),$$

with

$$\begin{aligned} f(x, C) &= x - \frac{(1-\tau)C}{r}, \\ \tilde{g}(x, x_B) &= \left(\frac{(1-\tau)C}{r} - x_B \right) \left(\frac{x_B}{x} \right)^\lambda h_\epsilon(x, x_B), \end{aligned} \quad (66)$$

and

$$h_\epsilon(x, x_B) := \left(1 + \sqrt{\epsilon} H \log \frac{x}{x_B} \right). \quad (67)$$

Since $H > 0$, we have

$$h_\epsilon(x, x_B) > 1 \quad \forall x \geq x_B, \quad x_B \in [0, x]. \quad (68)$$

We can interpret approximate equity $\tilde{E}(x, x_B)$ as a 'new' derivative contract whose value derives from two different sources. From an economic point of view $f(x, C)$ is equity value without risk of default and unless limit of time, and it is exactly the same value we have in a pure Leland [12] framework because this function doesn't depend directly on the failure level x_B . As a consequence, the stochastic volatility assumption does not produce any effect on it, since this term is independent of the probability of x reaching the barrier x_B . Its value is always positive under

$$x > \frac{(1 - \tau)C}{r}. \quad (69)$$

Function $\tilde{g}(x, x_B)$ depends directly on firm's current assets value x , on coupon payments C , on the failure level x_B and also on all parameters describing the volatility mean reverting process. And this because $\tilde{g}(x, x_B)$ represents nothing but a defaultable contract: since it is an option, it must have positive value, thus we must impose

$$x_B < \frac{(1 - \tau)C}{r}. \quad (70)$$

We can interpret $\tilde{g}(x, x_B)$ as a *corrected option to default embodied in equity* having positive value $\forall x \geq x_B$, with $x_B \in [0, \frac{(1-\tau)C}{r}]$. Notice that constraint (69) and (70) are the same we have in a pure Leland model. We have no more constraints considering separately the two components $f(x, C), \tilde{g}(x, x_B)$ since the randomness in volatility does not produce any effect when i) there is no risk of default, i.e. on $f(x, C)$ value; ii) at default, i.e. when $x = x_B$, thus on the payoff $l^E = \frac{(1-\tau)C}{r} - x_B$ taken by equity holders at the exercise time.

Observe that we can write the option to default as $\tilde{g}(x, x_B) = g(x, x_B)h_\epsilon(x, x_B)$, where $g(x, x_B) := \left(\frac{(1-\tau)C}{r} - x_B\right) \left(\frac{x_B}{x}\right)^\lambda$ is the value of the option to default in a pure Leland framework with constant volatility $\bar{\sigma}$. As a consequence we can interpret $h_\epsilon(x, x_B)$ as a *path-dependent correction for the price* due to our stochastic volatility assumption. The idea is to interpret it as a *correction for the price* due to randomness in volatility when $x > x_B$, meaning in each instant before the default event: only when the option is 'alive' but still not exercised.

In this case it will be a correction for the option value *only before default*, while at failure $x = x_B$ the option value is the same as in Leland, since the correction term $h_\epsilon(x, x_B)$ becomes 1 and the first order correction term P_1 disappears at boundary. This interpretation seems to be analogous to the European case treated in [7]: the stochastic volatility effect is null at maturity. Results presented in the book, also for barrier options (pg. 128), seem to be in the direction of constructing and studying a new claim with the

same structure of the P_0 term, only corrected for stochastic volatility *before the exercise time*, in our case meaning before the barrier is touched, thus before default is triggered. Let's think about a single bond under a stochastic volatility pricing model. What happens is that as the maturity date approaches, bond price volatility tends to zero, becoming exactly zero at expiration date. And this must hold: otherwise, to keep bond price equal to its (fixed) face value at maturity, the underlying assets process should have an indefinitely increasing drift (i.e. tending to $+\infty$) in order to compensate a potential volatility effect at expiration date. And this why the correction term $h_\epsilon(x, x_B)$ becomes 1 for $x = x_B$.

In this framework with volatility risk what is strongly affected by this assumption is the probability of x reaching x_B . Let's think for example about the present value of one unit of money obtained at failure. Under volatility risk, this value is different from its present value in a constant volatility case. In a pure Leland setting with constant volatility $\bar{\sigma}$, such a value is given by

$$f_1(x, x_B) = \left(\frac{x_B}{x}\right)^\lambda, \quad (71)$$

while in this model the *corrected present value* of 1 unit of money at default is

$$f_2(x, x_B) = f_1(x, x_B)h_\epsilon(x, x_B) > f_1(x, x_B), \quad (72)$$

since the holder of the contract has to be compensated for randomness of volatility, i.e. the uncertainty of the riskiness of his contract. Figure 1 shows how stochastic volatility affects these values *during the path*.

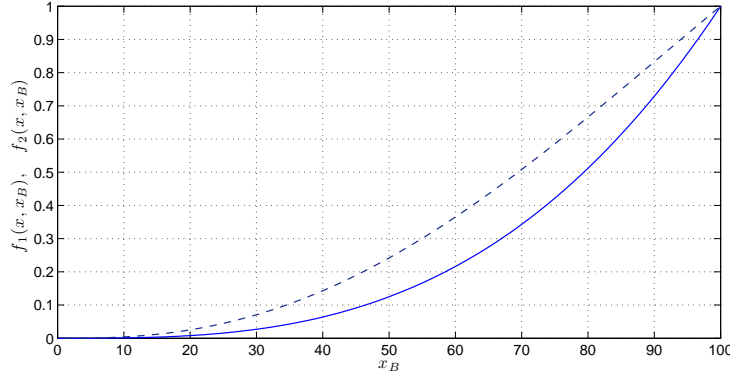


Figure 1: **Behavior of Functions** $f_1(x, x_B), f_2(x, x_B)$. The plot shows $f_1(x, x_B), f_2(x, x_B)$ given by (71)-(72) as functions of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0, r = 0.06, \bar{\sigma} = 0.2, \alpha = 0.5, \tau = 0.35, C = 6.5, V_3 = 0.003, V_2 = 2V_3, \rho < 0$.

Remark 4.3 *Analyzing this option to default embodied in equity we can also observe what*

follows:

$$\tilde{g}(x, x_B, C, \epsilon) = \left(\frac{(1-\tau)C}{r} - x_B \right) \left(\frac{x_B}{x} \right)^\lambda h_\epsilon(x, x_B, C),$$

which is equal to

$$\tilde{g}(x, x_B, C, \epsilon) = \left(\frac{(1-\tau)C}{r} - x_B \right) \left(\frac{x_B}{x} \right)^\lambda \left(1 + \sqrt{\epsilon} H \log \frac{x}{x_B} \right).$$

Indicating with $P_0^g(x, x_B) := g(x, x_B)$ the option to default in Leland framework with volatility $\bar{\sigma}$, we can write the approximate option to default as:

$$\tilde{g}(x, x_B, C, \epsilon) = P_0^g(x, x_B) + \tilde{P}_1^g(x, x_B),$$

where

$$\tilde{P}_1^g(x, x_B) = -\sqrt{\epsilon} H \cdot \left(\frac{(1-\tau)C}{r} - x_B \right) \log \frac{x_B}{x} \left(\frac{x_B}{x} \right)^\lambda.$$

As expected, the first correction term $\tilde{P}_1^g(x, x_B)$ is the same as $\tilde{P}_1^E(x, x_B)$, since this is exactly the option to default embodied in equity, meaning the defaultable contract.

From Remark 4.1 we know that $H > 0$. As a consequence, observe that the first correction term $P_1^E(x, x_B)$ for the price of equity claim due to stochastic volatility has positive value $\forall x > x_B$, meaning that it always has the effect of increasing equity claim value. The economic reason is that holding equity claim is now riskier since randomness in volatility is introduced, thus modifying the riskiness of the firm. Equity holders have to be compensated for this.

From now on, we will use the following formulation for approximate equity:

$$\tilde{E}(x, x_B) = x - \frac{(1-\tau)C}{r} + \left(\frac{(1-\tau)C}{r} - x_B \right) \left(\frac{x_B}{x} \right)^\lambda h_\epsilon(x, x_B), \quad (73)$$

with $h_\epsilon(x, x_B) = (1 - \sqrt{\epsilon} H \log \frac{x_B}{x})$.

4.1.1 First Order Correction

We now want to analyze which is the effect of the first-order correction term $\tilde{P}_1^E(x, x_B)$ on the behavior of approximate equity w.r.t. current firm's assets value x , as shown in the following Proposition.

Proposition 4.4 *The first correction term*

$$\tilde{P}_1^E(x, x_B) = -\sqrt{\epsilon} H \cdot \left(\frac{(1-\tau)C}{r} - x_B \right) \log \frac{x_B}{x} \left(\frac{x_B}{x} \right)^\lambda,$$

increases equity value for $x > x_B$. The maximum correction effect is achieved when the distance to default is $\log \frac{x_B}{x} = -\frac{1}{\lambda}$.

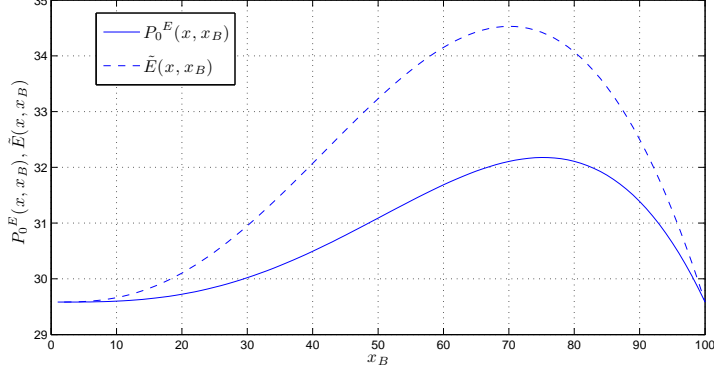


Figure 2: **Approximate Equity.** The plot shows approximate equity value $\tilde{E}(x, x_B)$ and $P_0^E(x, x_B)$ term as function of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $\rho < 0$, $x = 100$.

Proof We now consider the behavior of $\tilde{P}_1^E(x, x_B) > 0$ studying its partial derivative w.r.t. x , as follows:

$$\frac{\partial \tilde{P}_1^E(x, x_B)}{\partial x} = \frac{\sqrt{\epsilon}H \cdot \left(\frac{(1-\tau)C}{r} - x_B \right)}{x} \left(\frac{x_B}{x} \right)^\lambda \left(\lambda \log \frac{x_B}{x} + 1 \right),$$

The sign of $\frac{\partial \tilde{P}_1^E(x, x_B)}{\partial x}$ is the same as $f(x, x_B) := \lambda \log \frac{x_B}{x} + 1$. The application $x \mapsto f(x, x_B)$ is a decreasing function going from $f(x_B, x_B) = 1$ to $-\infty$. So the sign of $\frac{\partial \tilde{P}_1^E(x, x_B)}{\partial x}$ is :

$$\begin{aligned} \frac{\partial \tilde{P}_1^E(x, x_B)}{\partial x} &\geq 0 \quad \text{for } x \leq x_B e^{\frac{1}{\lambda}}, \\ \frac{\partial \tilde{P}_1^E(x, x_B)}{\partial x} &< 0 \quad \text{for } x > x_B e^{\frac{1}{\lambda}}. \end{aligned}$$

Thus, the application $x \mapsto \tilde{P}_1^E(x, x_B)$ admits a unique maximum at point $x = x_B e^{\frac{1}{\lambda}}$.

4.1.2 Endogenous Failure Level

We now turn to analyze the endogenous failure level chosen by equity holders. The economic problem is that equity holders want to maximize equity value, but due to limited

liability of equity they cannot chose an arbitrary small failure level. A natural constraint on x_B can be found by imposing:

$$\frac{\partial \tilde{E}(x, x_B)}{\partial x} \Big|_{x=x_B} \geq 0 \quad \forall x \geq x_B,$$

which guarantees equity being a non-negative and increasing function of firm's current assets value x for $x \geq x_B$. The following Proposition determines the failure level which satisfies this condition.

Proposition 4.5 *Under $\lambda > \sqrt{\epsilon}H$, the endogenous failure level chosen by equity holders in order to maximize $x_B \mapsto \tilde{E}(x, x_B)$ has to belong to the following interval:*

$$\left[\bar{x}_B, \frac{(1-\tau)C}{r} \right], \quad (74)$$

where

$$\bar{x}_B := \frac{(1-\tau)C}{r} \frac{\lambda - \sqrt{\epsilon}H}{1 + \lambda - \sqrt{\epsilon}H} > 0 \quad (75)$$

is solution of the traditional smooth-pasting condition:

$$\frac{\partial \tilde{E}(x, x_B)}{\partial x} \Big|_{x=x_B} = 0.$$

Proof Observe that

$$\frac{\partial \tilde{E}(x, x_B)}{\partial x} \Big|_{x=x_B} = \frac{\partial (P_0^E(x, x_B) + \tilde{P}_1^E(x, x_B))}{\partial x} \Big|_{x=x_B},$$

thus in order to have equity a non-negative and increasing function of current assets value x it is sufficient to impose:

$$\frac{\partial \tilde{E}(x, x_B)}{\partial x} \Big|_{x=x_B} \geq 0 \quad \forall x \geq x_B.$$

This in turns lead to

$$1 - \frac{l^E}{x_B} (\lambda - \sqrt{\epsilon}H) \geq 0, \quad (76)$$

with $l^E = \frac{(1-\tau)C}{r} - x_B > 0$ due to (70).

Finally, under $\lambda - \sqrt{\epsilon}H > 0$, (76) is satisfied for $x_B \geq \bar{x}_B$ with \bar{x}_B solution of:

$$1 - \frac{l^E}{x_B} (\lambda - \sqrt{\epsilon}H) = 0. \quad (77)$$

This solution \bar{x}_B satisfies also $\bar{x}_B < \frac{(1-\tau)C}{r}$, thus the interval $\left[\bar{x}_B, \frac{(1-\tau)C}{r} \right]$ is not empty.

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Remark 4.6 Notice that this lower bound $\bar{x}_B > 0$ for the endogenous failure level depends on all parameters involved in the volatility diffusion process, since it depends on ϵ , and H , thus on V_2, V_3 . To be more precise, this lower bound is strictly related to the market price of risk, to the skew effect and also to the speed of convergence chosen for the volatility mean reverting process.

Observe that as particular case, for $\epsilon \rightarrow 0$, this lower bound is exactly the lower bound arising from a pure Leland framework x_{BL} with constant volatility $\bar{\sigma}$:

$$x_{BL} := \frac{(1 - \tau)C}{r} \frac{\lambda}{1 + \lambda}.$$

In our more general case, we stress that \bar{x}_B reduces as the speed of mean reversion increases, since the application $\sqrt{\epsilon} \mapsto \bar{x}_B$ is decreasing and the speed of mean reversion is $\frac{1}{\epsilon}$. This means that under our stochastic volatility framework equity holders can chose a failure level which is lower than in a pure Leland model.

The lower bound \bar{x}_B is also decreasing w.r.t. $H > 0$, thus its dependence on the market price of volatility and on the leverage effect ρ has the same sign of H -dependence on the same parameters.

Remark 4.6 is useful to formulate the optimal stopping problem faced by equity holders in this framework of stochastic volatility:

$$\max_{x_B \in [\bar{x}_B, \frac{(1-\tau)C}{r}]} \tilde{E}(x, x_B, C, \epsilon), \quad (78)$$

with \bar{x}_B given in (75).

From traditional optimal stopping theory we know that, under appropriate hypothesis, applying the smooth pasting condition to the function which has to be optimized, will give a failure level which is exactly the solution of the optimal stopping problem. What we showed in all analytical aspects in [2, 3], is an example of a framework in which applying the smooth pasting condition (which is a low contact condition) gives the same failure level we obtain by maximizing equity value w.r.t. the constant default barrier. An analogous relation exists when we analyze American-style options under Black and Scholes model (see also [7]). Working inside a Black and Scholes pricing framework means that the smooth pasting condition applied to equity value gives not only a lower bound for the failure level chosen by equity holders, but directly the endogenous failure level solution of the optimal stopping problem.

In this stochastic volatility pricing model the *traditional smooth pasting condition* applied to approximate equity $\frac{\partial \tilde{E}(x, x_B)}{\partial x} \Big|_{x=x_B} = 0$ gives only a lower bound $\bar{x}_B > 0$ for the endogenous failure level which has to be satisfied (due to limited liability of equity) in order to have equity as an increasing function of firm's assets value.

In the constant volatility case, this lower bound is also solution of the optimal stopping

problem, i.e. $\max_{x_B} E(x, x_B)$, meaning that equity holders will always choose the lowest admissible failure level. But this is *not* still true when volatility risk is introduced in the model. The failure level \bar{x}_B is not the solution of the optimal stopping problem (78). We show this in the following Proposition.

Proposition 4.7 *The endogenous failure level solution of the optimal stopping problem*

$$\max_{x_B \in [\bar{x}_B, \frac{(1-\tau)C}{r}]} \tilde{E}(x, x_B, C, \epsilon), \text{ where } \bar{x}_B = \frac{(1-\tau)C}{r} \frac{\lambda - H\sqrt{\epsilon}}{1 + \lambda - H\sqrt{\epsilon}},$$

is \tilde{x}_B , solution of

$$\left(1 + \sqrt{\epsilon}H \log \frac{x}{x_B}\right) \left(\frac{\lambda(1-\tau)C}{r} - (\lambda+1)x_B\right) - H\sqrt{\epsilon} \left(\frac{(1-\tau)C}{r} - x_B\right) = 0. \quad (79)$$

Proof Equity holders face the following optimal stopping problem:

$$\max_{x_B \in [\bar{x}_B, \frac{(1-\tau)C}{r}]} \tilde{E}(x, x_B, C, \epsilon). \quad (80)$$

Recall that $\tilde{E}(x, x_B, C, \epsilon) = f(x, C) + \tilde{g}(x, x_B, C, \epsilon)$, where $f(x, C)$ does not depend on the failure level x_B .

As a consequence, our problem (80) is equivalent to

$$\max_{x_B \in [\bar{x}_B, \frac{(1-\tau)C}{r}]} \tilde{g}(x, x_B, C, \epsilon),$$

where

$$\tilde{g}(x, x_B, C, \epsilon) := \left(\frac{(1-\tau)C}{r} - x_B\right) \left(\frac{x_B}{x}\right)^\lambda \left(1 + \sqrt{\epsilon}H \log \frac{x}{x_B}\right)$$

is the *corrected option to default* embodied in equity. To study \tilde{g} behavior, we turn to its derivative computation writing it in this compact formulation:

$$\partial_{x_B} \tilde{g}(x, x_B, C) = \left(\frac{x_B}{x}\right)^\lambda \left(h_\epsilon(x, x_B) \left(-1 + \frac{\lambda l^E}{x_B}\right) - \sqrt{\epsilon} \frac{l^E H}{x_B}\right),$$

where $l^E = \frac{(1-\tau)C}{r} - x_B$. Actually, the sign of partial derivative of \tilde{g} w.r. t. x_B is the one of function

$$g_1(x_B) = -x_B h_\epsilon(x, x_B) + l^E (\lambda h_\epsilon(x, x_B) - \sqrt{\epsilon}H), \quad (81)$$

explicitly given by

$$g_1(x_B) = -x_B \left(1 + \sqrt{\epsilon}H \log \frac{x}{x_B}\right) \left(\frac{(1-\tau)C}{r} - x_B\right) \left(\lambda \left(1 + \sqrt{\epsilon}H \log \frac{x}{x_B}\right) - \sqrt{\epsilon}H\right).$$

Remark that the smallest value of x_B , i.e. the lower bound $\bar{x}_B = \frac{(1-\tau)C}{r} \frac{\lambda - H\sqrt{\epsilon}}{1 + \lambda - H\sqrt{\epsilon}}$ is solution of:

$$1 - \frac{l^E}{x_B}(\lambda - \sqrt{\epsilon}H) = 0,$$

thus, letting $\bar{l}^E := \frac{(1-\tau)C}{r} - \bar{x}_B \Rightarrow 0$, at point $x_B = \bar{x}_B$ we have

$$\sqrt{\epsilon} \frac{H\bar{l}^E}{\bar{x}_B} = -1 + \frac{\lambda\bar{l}^E}{\bar{x}_B}.$$

As a consequence,

$$g_1(\bar{x}_B) = \sqrt{\epsilon}H\bar{l}^E (h_\epsilon(x, \bar{x}_B) - 1) > 0,$$

in explicit form

$$g_1(\bar{x}_B) = H\sqrt{\epsilon} \log \frac{x}{\bar{x}_B} \frac{(1-\tau)C}{r} \left(\frac{H\sqrt{\epsilon}}{1 + \lambda - H\sqrt{\epsilon}} \right) > 0,$$

Concerning the biggest value $\widehat{x}_B := \frac{(1-\tau)C}{r}$, we have

$$g_1(\widehat{x}_B) = -\widehat{x}_B h_\epsilon(x, \widehat{x}_B) < 0.$$

Now, we turn to:

$$g_1'(x_B) = -h_\epsilon(x, x_B)(1 + \lambda) + h_\epsilon'(x, x_B)(\lambda l^E - x_B) + \sqrt{\epsilon}H,$$

$$g_1''(x_B) = -h_\epsilon'(x, x_B) \left(1 + \frac{\lambda l^E}{x_B} + 2\lambda \right) > 0,$$

since $h_\epsilon'(x, x_B) < 0$. Respectively, their explicit form is:

$$g_1'(x_B) = - \left(1 + \sqrt{\epsilon}H \log \frac{x}{x_B} \right) (1 + \lambda) - \sqrt{\epsilon}H \left(\frac{\lambda(1-\tau)C}{rx_B} - (\lambda + 1) \right) + \sqrt{\epsilon}H,$$

$$g_1''(x_B) = \frac{\sqrt{\epsilon}H}{x_B} \left(1 + \lambda \left(1 + \frac{(1-\tau)C}{rx_B} \right) \right) > 0.$$

This function $g_1'(x_B)$ is increasing, so $g_1(x_B)$ is convex, starting from positive value to negative value, thus there exists a unique solution \tilde{x}_B which realizes the maximum of equity and it is solution of $g_1(x_B) = 0$, whose equation is explicitly given by

$$\left(1 + \sqrt{\epsilon}H \log \frac{x}{x_B} \right) \left(\frac{\lambda(1-\tau)C}{r} - (\lambda + 1)x_B \right) - H\sqrt{\epsilon} \left(\frac{(1-\tau)C}{r} - x_B \right) = 0. \quad (82)$$

•

Remark 4.8 *i) Let $x_{BL} := \frac{(1-\tau)C}{r} \frac{\lambda}{1+\lambda}$ being Leland endogenous failure level. Observe that our corrected option to default embodied in equity has a negative first derivative w.r.t. x_B at point x_{BL} since the sign of g_1 given in (81) at that point is:*

$$g_1(x_{BL}) = -\sqrt{\epsilon}H \frac{(1-\tau)C}{r(1+\lambda)} < 0,$$

meaning that the endogenous failure level \tilde{x}_B satisfies (using g_1 convexity)

$$\bar{x}_B < \tilde{x}_B < x_{BL}.$$

Notice that these three points coincides as $\epsilon \rightarrow 0$.

ii) We can rewrite Equation (79) as:

$$h_\epsilon(x, x_B) - \frac{l^E}{x_B} (\lambda h_\epsilon(x, x_B) - \sqrt{\epsilon}H) = 0,$$

and observe that the only term which depends on coupon is $l^E = \frac{(1-\tau)C}{r} - x_B$ which is linear w.r.t. C . Solve it w.r.t. the endogenous coupon \tilde{C} as:

$$\tilde{C} = \frac{rx_B}{1-\tau} \frac{(\lambda+1)h_\epsilon(x, x_B) - \sqrt{\epsilon}H}{\lambda h_\epsilon(x, x_B) - \sqrt{\epsilon}H}, \quad (83)$$

explicitly given by

$$\tilde{C} = \frac{rx_B}{1-\tau} \frac{(\lambda+1)(1 + \sqrt{\epsilon}H \log \frac{x}{x_B}) - \sqrt{\epsilon}H}{\lambda(1 + \sqrt{\epsilon}H \log \frac{x}{x_B}) - \sqrt{\epsilon}H}.$$

The application $x_B \mapsto \tilde{C}$ is increasing.

We now analyze Equation (79) which gives the endogenous failure level \tilde{x}_B in order to define a *corrected smooth-pasting condition* inside a pricing model of stochastic volatility.

Proposition 4.9 *Corrected Smooth-Pasting.*

At point \tilde{x}_B implicit solution of (79), the following 'corrected smooth-pasting' condition holds:

$$\frac{\partial P_0^E}{\partial x} \Big|_{x=x_B} h_\epsilon(x, x_B) + \frac{\partial \tilde{P}_1^E}{\partial x} \Big|_{x=x_B} = 0, \quad (84)$$

where $\tilde{P}_1^E = \sqrt{\epsilon}P_1^E$.

Proof What we want to show is that finding the solution of $\frac{\partial \tilde{g}(x, x_B, C, \epsilon)}{\partial x_B} = 0$ is equivalent to find the solution of what we define a *corrected smooth-pasting condition*. In this case also the smooth pasting condition must take into account the perturbation given by the

stochastic volatility effect, and in this sense we call it 'corrected'. Consider Equation (79) and observe that we can rewrite it under the following equivalent formulation:

$$h_\epsilon(x, x_B) \left(-1 + \frac{\lambda l^E}{x_B} \right) - \sqrt{\epsilon} \frac{l^E H}{x_B} = 0.$$

Moreover, we have:

$$\frac{\partial P_0^E}{\partial x} \Big|_{x=x_B} = 1 - \frac{\lambda l^E}{x_B}, \quad \frac{\partial \tilde{P}_1^E}{\partial x} \Big|_{x=x_B} = \sqrt{\epsilon} H \frac{l^E}{x_B}.$$

•

As in a pure Leland model, the endogenous failure level chosen by equity holders \tilde{x}_B does not depend on bankruptcy costs, since the strict priority rule still holds, but instead depends on coupon, risk free rate and tax rate. Coeteris paribus, it is increasing w.r.t the coupon level and decreasing w.r.t. the corporate tax rate τ .

When volatility risk is introduced, equity holders will choose the default barrier which maximizes equity value depending on both the market price of volatility and the leverage effect, since coefficient H in Equation (79) captures both these effects.

Moreover, what is completely new for an endogenous failure level derived inside a structural model framework is its dependence on the initial firm's assets value x . This dependence could be related to the fact that also the 'standard' smooth-pasting condition $\frac{\partial \tilde{E}(x, x_B)}{\partial x} \Big|_{x=x_B} = 0$ does not give the endogenous failure level, but only a lower bound for it. And this lower bound is not the solution of the optimal stopping problem. Equation (84) suggests that the endogenous failure level is the solution of a *corrected smooth-pasting condition* in the sense that we must take into account the randomness introduced in volatility. As we observed in Proposition 4.4 the correction for the price is not constant through the path, achieving a maximum correction effect when the distance to default is $\log \frac{x_B}{x} = -\frac{1}{\lambda}$. Figure 2 shows the behavior of both approximate equity $\tilde{E}(x, x_B)$ and $P_0^E(x, x_B)$ w.r.t. the failure level x_B . As we can see, $\tilde{E}(x, x_B)$ achieves its maximum before $P_0^E(x, x_B)$, meaning for a lower failure level.

Remark 4.10 Recall Equation (84). Observe that we can interpret its solution \tilde{x}_B as the failure level at which the following conditions hold:

$$h_\epsilon(x, \tilde{x}_B) = -\frac{\frac{\partial \tilde{P}_1^E}{\partial x} \Big|_{x=\tilde{x}_B}}{\frac{\partial P_0^E}{\partial x} \Big|_{x=\tilde{x}_B}} = -\frac{\frac{\partial \tilde{P}_1^E}{\partial x} \Big|_{x=\tilde{x}_B}}{\Delta_{P_0^E}(\tilde{x}_B)}, \quad (85)$$

where Δ denotes the Greek of the P_0^E derivative evaluated at point $x = \tilde{x}_B$, or

$$\sqrt{\epsilon} H \log \frac{\tilde{x}_B}{x} = \frac{\frac{\partial \tilde{E}}{\partial x} \Big|_{x=\tilde{x}_B}}{\Delta_{P_0^E}(\tilde{x}_B)}. \quad (86)$$

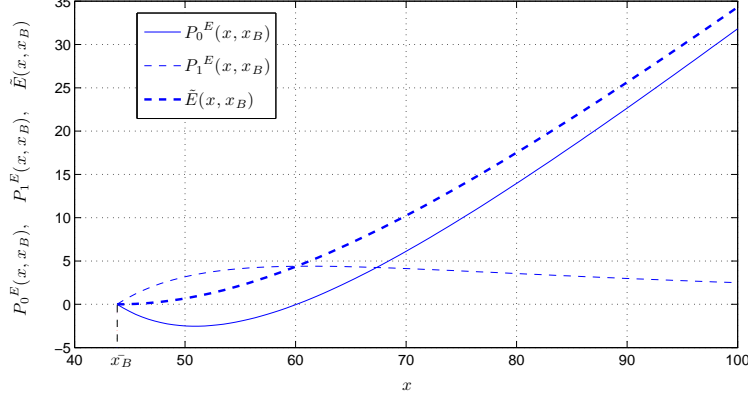


Figure 3: . The plot shows approximate equity value $\tilde{E}(x, x_B)$, and $P_0^E(x, x_B)$, $\tilde{P}_1^E(x, x_B)$ terms as function of current assets' value x . The support of each function is $[\tilde{x}_B, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $\rho < 0$. The endogenous failure level \tilde{x}_B is determined for $x = 100$.

i) First recall that the lower bound \tilde{x}_B solution of $\frac{\partial \tilde{E}(x, x_B)}{\partial x} \Big|_{x=\tilde{x}_B} = 0$ is independent of current assets value x . Equivalently, it guarantees:

$$1 = - \frac{\frac{\partial \tilde{P}_1^E}{\partial x} \Big|_{x=\tilde{x}_B}}{\Delta_{P_0^E}(\tilde{x}_B)},$$

meaning the slope of P_0^E and \tilde{P}_1^E have the same absolute value, but opposite sign. This only guarantees equity being an increasing and non-negative function of x as shown in Figure 3.

ii) Secondly, observe that the two equivalent formulations (85)-(86) for equation (84) are useful to better understand the economic optimality of the endogenous failure level \tilde{x}_B and its dependence on firm's activities value x . The left hand side of both equation is the only one depending on x . Choosing \tilde{x}_B means choosing the endogenous level corresponding to the optimal exercise time: due to current assets value x , before and after \tilde{x}_B the correction for the price $h_\epsilon(x, x_B)$ does not exactly compensate the ratio between the instantaneous variations (due to an instantaneous variation in x) in the first order correction term and the Δ of the corresponding Black and Scholes contract $P_0^E(x, x_B)$. Equation (86) suggests the same dependence relating the distance to default and the ratio between the instantaneous variations in approximate claim $\tilde{E}(x, x_B)$ and, again, the Δ -sensitivity of $P_0^E(x, x_B)$. The endogenous failure level can be seen as an equilibrium level which increases as current assets value x rises.

In this sense, looking at Equation 85, the correction term $h_\epsilon(x, x_B)$ can be interpret as an elasticity measure at equilibrium.

By simply applying $\frac{\partial \tilde{E}(x, x_B)}{\partial x} |_{x=x_B} = 0$ will give a failure level \bar{x}_B which is independent of firm's current activities value x . Choosing this failure barrier corresponds to a *non-optimal exercise* of the option to default embodied in equity. The *corrected smooth-pasting condition* leads to an *endogenous* failure level \tilde{x}_B which is greater than its lower bound, and moreover depends on firm's assets value x . This is a new insight arising from a structural modeling approach.

A possible explanation of this dependence is that under volatility risk, assuming $\rho < 0$ makes the distribution of stock price returns not symmetric. In particular, extreme returns behavior is better captured, since with our assumptions the *left tail* of their distribution is fatter: this is why it is not optimal to exercise at the standard smooth pasting level but choosing a failure level greater than this. The starting point of process x_t now matters in the choice of the the optimal stopping time, since volatility is not constant. The *correction for the price* represented by $h_\epsilon(x, x_B)$ is a path-dependent correction depending on current assets value x : therefore, the optimal exercise time must consider this.

Inside this framework the riskiness of the firm is taken into account from two different points of view: i) its market price through Λ , ii) the leverage effect through ρ . Observe that even in the uncorrelated case $\rho = 0$, the endogenous failure level still depend on x and it is still different from the lower bound \bar{x}_B . In such a case the correction for the volatility level due to coefficient V_2 is still in force. As $\epsilon \rightarrow 0$ the dependence of the endogenous failure level on x disappears (and also in case $\rho = 0, \Lambda = 0$).

4.2 Debt, Tax Benefits, Bankruptcy Costs and Total Value of the Firm

Using contingent claim valuation, debt value can be expressed as a claim on the underlying asset describing firm's activities value x . Under stochastic volatility assumption we have the following expressions for approximate values of debt, tax benefits and bankruptcy costs:

$$\tilde{D}(x, x_B) = \frac{C}{r} + \left((1 - \alpha)x_B - \frac{C}{r} \right) \left(\frac{x_B}{x} \right)^\lambda h_\epsilon(x, x_B), \quad (87)$$

$$\tilde{T}B(x, x_B) = \frac{\tau C}{r} - \frac{\tau C}{r} \left(\frac{x_B}{x} \right)^\lambda h_\epsilon(x, x_B), \quad (88)$$

$$\tilde{B}C(x, x_B) = \alpha x_B \left(\frac{x_B}{x} \right)^\lambda h_\epsilon(x, x_B), \quad (89)$$

with the correction term $h_\epsilon(x, x_B) = \left(1 + \sqrt{\epsilon} H \log \frac{x}{x_B} \right)$, and $H = \frac{2r}{\sigma^4} \sqrt{2\nu} (\langle \Lambda \phi' \rangle + \frac{2r}{\sigma^2} \rho \langle f \phi' \rangle)$.

Due (only) to the assumption of infinite horizon, in this setting debt holders receive $\frac{C}{r}$ without limit of time in the event of no default and $(1 - \alpha)x_B - \frac{C}{r}$ in case of x reaching

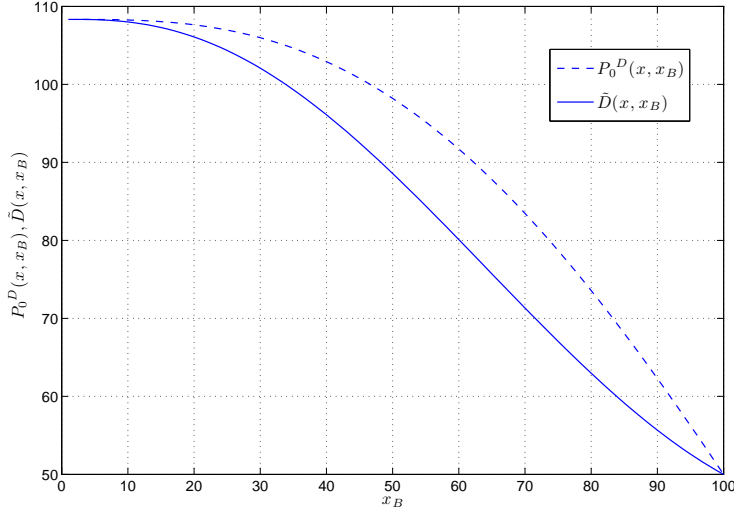


Figure 4: **Approximate Debt.** The plot shows approximate debt value $\tilde{D}(x, x_B)$ and $P_0^D(x, x_B)$ term as functions of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $x = 100$, $\rho < 0$.

x_B , meaning $\frac{C}{r}$ the component of debt value without default risk. Observe that this is the same as in Leland framework, meaning that at boundaries stochastic volatility does not produce effect. What is different from Leland framework is that the correction term P_1 modifies the claim value for $x > x_B$. The value of *approximate* tax benefits of debt has a *downward correction*: the first-order correction term $P_1^{TB}(x, x_B)$ is negative for all values $x > x_B$, and this is due to the volatility risk influence on the likelihood of default.

In order to completely describe the capital structure we can write the total value of the firm as the sum of equity and debt value or equivalently as current assets value plus tax benefits of debt, less bankruptcy costs.

Alternative and equivalent definitions for the total value of the firm are:

$$\tilde{v}(x, x_B, C, \epsilon) := \tilde{E}(x, x_B, C, \epsilon) + \tilde{D}(x, x_B, C, \epsilon) = x + \tilde{T}B(x, x_B) - \tilde{B}C(x, x_B). \quad (90)$$

Approximate total value of the firm is given by:

$$\tilde{v}(x, x_B, C) = x + \frac{\tau C}{r} - \left(\alpha x_B + \frac{\tau C}{r} \right) h_\epsilon(x, x_B) \left(\frac{x_B}{x} \right)^\lambda. \quad (91)$$

The correction term $h_\epsilon(x, x_B) = 1 + \sqrt{\epsilon} H \log \frac{x}{x_B}$ is a fundamental quantity in this framework of stochastic volatility. Its contribution is that one of a path dependent correction

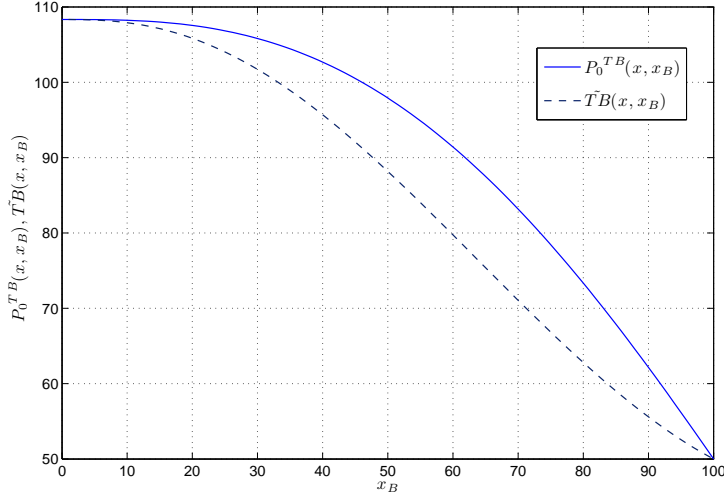


Figure 5: **Approximate Tax Benefits of Debt.** The plot shows approximate tax benefits of debt $TB(x, x_B)$ and $P_0^{TB}(x, x_B)$ term as functions of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $x = 100$, $\rho < 0$.

for prices due to randomness in volatility.

Our focus is to understand if the stochastic volatility effect can increase credit spreads and reduce leverage ratios predicted by the model. Recall that credit spreads are defined as

$$R(x, x_B) := \frac{C}{D(x, x_B)} - r,$$

where $D(x, x_B) := P_0^D(x, x_B)$. We will denote with $R(x, x_B)$ the credit spread under a pure Leland framework and with

$$\tilde{R}(x, x_B) := \frac{C}{\tilde{D}(x, x_B)} - r$$

approximate credit spread under stochastic volatility. Figure 6 shows that randomness in volatility moves credit spreads exactly in the expected direction, rising them before the default time in order to compensate investors for the new source of risk.

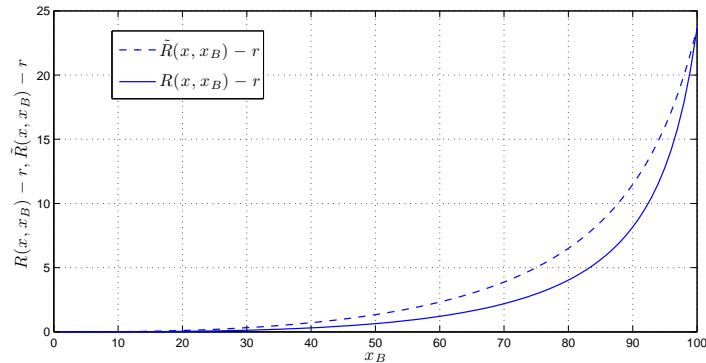


Figure 6: **Approximate Credit Spreads.** The plot shows approximate credit spreads $\tilde{R}(x, x_B) - r$ and $R(x, x_B) - r$ as functions of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $x = 100$, $\rho < 0$.

5 Optimal Capital Structure

We now analyze the capital structure of the firm considering that equity holders will chose the coupon C in order to maximize the total value of the firm. We recall our approximate total value of the firm, given by Equation (91).

$$\tilde{v}(x, x_B, C) = x + \frac{\tau C}{r} - \left(\alpha x_B + \frac{\tau C}{r} \right) \left(1 + \sqrt{\epsilon} H \log \frac{x}{x_B} \right) \left(\frac{x_B}{x} \right)^\lambda. \quad (92)$$

The problem is now to optimize:

$$g : C \mapsto \tilde{v}(x, \tilde{x}_B(C), C)$$

where $\tilde{x}_B(C)$ is the endogenous failure level solution of (79). Since we do not have \tilde{x}_B is explicit form, by exploiting the linearity of (79) w.r.t. coupon C as noted in Remark 4.8, we will proceed following the same idea proposed in [3]. Consider $x_B \mapsto \tilde{C}(x_B)$, where \tilde{C} is solution of (79). Thus, an equivalent problem will be to optimize w.r.t. x_B the following function:

$$x_B \mapsto \tilde{v}(x, x_B, \tilde{C}(x_B)).$$

We now numerically compute the optimal capital structure of the firm, i.e. all financial variables at their optimal level in order to analyze the volatility risk influence.

We consider Leland [12] results as a benchmark in order to understand the effect produced by volatility risk on all financial variables. The aim is to analyze both sources

of risk induced by the model: at first only the *skew effect*, captured by ρ , then its joint influence with the *correction for the volatility level*, related to the difference between the effective volatility σ^* and the average volatility $\bar{\sigma}$. Table 2 shows how optimal corporate

ρ	\tilde{C}^*	\tilde{D}^*	\tilde{R}^*	$\tilde{R}^* - r$	\tilde{E}^*	\tilde{x}_B^*	\tilde{v}^*	\tilde{L}^*
0	6.501	96.274	6.753	75.255	32.168	52.820	128.442	74.956 %
-0.05	5.910	84.158	7.022	102.197	39.581	44.916	123.739	68.012 %
-0.06	5.741	81.452	7.049	104.862	41.417	43.413	122.870	66.292 %
-0.07	5.569	78.796	7.067	106.744	43.255	41.961	122.051	64.560 %
-0.08	5.397	76.233	7.080	107.991	45.054	40.575	121.287	62.853 %
-0.09	5.230	73.792	7.087	108.748	46.787	39.264	120.578	61.198 %
-0.1	5.203	73.398	7.088	108.835	47.068	39.053	120.465	60.929 %

Table 2: **Skew effect on optimal capital structure.** The table shows financial variables at their optimal level when only the *skew effect* is considered, i.e. $\rho < 0, \Lambda = 0$. The first row of the table reports Leland [12] results as benchmark, as particular case of $\rho = 0, \Lambda = 0$. We consider $r = 0.06, \bar{\sigma} = 0.2, \alpha = 0.5, \tau = 0.35$. Recall $V_3 := \sqrt{\epsilon\rho\frac{\sqrt{2}}{2}}\nu\langle f\phi' \rangle$. We consider $V_3 = -0.06\rho, V_2 = 2V_3$, see also [8]. L^*, R^* are in percentage (%), $R^* - r$ in basis points (bps).

decisions are influenced by the introduction of a negative correlation ρ between assets value dynamic and the volatility process. We leave ρ varying from $\rho = -0.05$ to $\rho = -0.1$, in order to capture its effects on corporate decisions. The first step is to conduct the analysis by assuming $\Lambda = 0$, meaning the correction for the volatility level being null. Numerical results show that only the *skew effect* induced by $\rho < 0$ produces a significant impact on corporate financing decisions. Skewness in the underlying dynamics makes debt less attractive. What emerges is that optimal coupon, debt, total value of the firm and leverage ratios drop down. And in some cases, this reduction is significant. Only a slightly negative correlation $\rho = -0.05$ bring down leverage of around 8%, while a 15%-reduction is achieved with $\rho = -0.1$. The maximum total value of the firm is also reduced, since the increase in optimal equity value is always lower than the reduction in optimal debt. The coupon level chosen to maximize total firm value is decreasing with the *skew effect*, generating a downward jump in optimal debt from 96.3 in case $\rho = 0$ to 73.4 in case $\rho = -0.1$, which is absolutely a not negligible one. Despite lower leverage ratios, yield spreads are increasing with $|\rho|$. This behavior can be explained thinking about the likelihood of default, which should be increased by the skew effect. The market perception about the credit risk of the firm changes and the firm becomes a riskier activity. There is uncertainty about its volatility, and its riskiness moves in time. Investing in this firm requires a higher compensation.

Letting $\rho = 0$, and introducing the correction for the volatility level will bring the model to a pure Black and Scholes setting with constant volatility σ^* . This is not the case we are interested in. As a second step we consider both sources of risk associated to firm

value. Table 3 shows how financial variables are modified when also the *correction for the volatility level* is considered in a framework with negative leverage effect, i.e. $\rho < 0$. The relation existing between the corrected effective volatility σ^* and the average volatility $\bar{\sigma}$ is:

$$\sigma^* = \sqrt{\bar{\sigma}^2 - 2V_2},$$

with $V_2 = -\frac{\sqrt{2}}{2}\nu \langle \Lambda \phi' \rangle < 0$ under $\rho = 0$. As noted in [7], markets data suggest that usually the corrected effective volatility is higher than the average volatility, this is why we consider $\sigma^* > \sqrt{\bar{\sigma}}$. As example we consider a negative correlation $\rho = -0.05$ and a gap between σ^* and $\bar{\sigma}$ of 1%, 2%, respectively. The skew effect and the volatility level correction seem to represent an interesting feature to develop applied to credit risk models. Optimal financing decisions move w.r.t. a pure Leland model, and where both sources of risk are considered together, their joint influence is strong. The optimal amount of debt is reduced and leverage ratios can drop down from 75% to 62% only with a slightly negative correlation $\rho = -0.05$ and a volatility level correction of 2%.

σ^*	\bar{C}^*	\bar{D}^*	\bar{R}^*	$\bar{R}^* - r$	\bar{E}^*	\tilde{x}_B^*	\tilde{v}^*	\bar{L}^*
$\bar{\sigma}$	6.501	96.274	6.753	75.255	32.168	52.820	128.442	74.956 %
$\bar{\sigma} + 0.01$	5.701	80.266	7.103	110.252	42.011	42.955	123.260	65.120 %
$\bar{\sigma} + 0.02$	5.597	77.350	7.236	123.597	43.495	41.887	124.041	62.358 %

Table 3: Skew effect and volatility level correction: influence on optimal capital structure. The table shows financial variables at their optimal level when $\rho = -0.05$ and also a volatility correction is considered. Recall that $\sigma^* = \sqrt{\bar{\sigma}^2 - 2V_2}$. We consider $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $V_3 = 0.003$. L^* , R^* are in percentage (%), $R^* - r$ in basis points (bps).

Numerical results emphasize interesting insights arising from a model where a negative correlation among assets returns and shocks in volatility and a volatility level correction coexist. Thus suggesting a possible direction to follow aiming at improving empirical predictions inside a structural model with endogenous bankruptcy. Introducing a process describing the evolution of assets volatility makes possible to capture how prices are modified due to the market's perception of firm's credit risk: there is uncertainty about the volatility level and its evolution in time. Due to possible shocks in volatility, the firm becomes a riskier activity for the market. Investors will require higher compensations for this, thus yield spreads must be higher despite lower leverage ratios.

6 Conclusions

The focus in this paper is to introduce volatility risk inside a structural model of credit risk with endogenous default. The capital structure of a firm is analyzed in a framework of infinite time horizon following Leland's idea [12] but assuming a stochastic volatility

model for the firm’s assets. In particular we consider a stochastic volatility pricing model where a one-factor process of Orstein-Uhlenbeck type describing the dynamic of the diffusion coefficient is introduced. Moreover, following [7], assets value process belongs to a fairly large class of stochastic volatility models and from this point of view results are model-independent. The idea of the paper is to better capture extreme returns behavior, making stock prices returns distribution asymmetric and with fatter left tails. Inside this framework we describe and analyze the effects of volatility risk on all financial variables describing the capital structure to understand if introducing volatility risk could be a way to improve empirical predictions of structural models (i.e. higher spreads, lower leverage ratios). We analyze each component of the capital structure as a derivative contract whose value can be derived by applying singular perturbation techniques as in [7]. If compared to the classical Leland model [12], the value of each claim must be *corrected* to compensate the holder of each derivative contract for volatility risk. This correction acts only during the path of processes, i.e. not when the contract is riskless, not when default arrives. This *correction* is due to two main sources of risk associated with the introduced randomness in volatility: i) the *skew effect*, arising by assuming a correlation ρ between assets and volatility dynamics; ii) a *volatility level correction*. Equity holders still face the problem of optimizing equity value w.r.t. the failure level. Under this approach, the failure level derived from standard smooth pasting principle is not the solution of the optimal stopping problem, but only represents a lower bound which has to be satisfied due to limited liability of equity in order to have equity an increasing function of x . Choosing that failure level is not optimal since it would mean an early exercise of the option to default embodied in equity. A *corrected smooth pasting condition* must be applied in order to find the endogenous failure level solution of the optimal stopping problem. What is new in our approach, is that we are dealing in a structural model framework and the failure level solution of the optimal stopping problem depends on current firm’s activities value. An idea is to better exploit this insight trying to understand how it is affected by the mean-reversion speed, meaning studying the distance between the lower bound \bar{x}_B and the optimal solution \tilde{x}_B under our framework. Under volatility risk, there is uncertainty about the current riskiness of the firm, thus the actual value of firm’s activities matters. Numerical results obtained by exploiting optimal capital structure suggest that this stochastic volatility pricing model seems to be a robust way to improve results in the direction of both higher spreads and lower leverage ratios in a quantitatively significant way. The market perception of the credit risk associated to the firm is captured by *approximate corrected prices*: yield spreads are higher, despite lower leverage, since the required compensation for risk increases.

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CONCLUSIONS

The thesis analyzes credit risk modeling following a structural model approach with endogenous default. We extend the classical Leland [5] framework in three main directions with the aim at obtaining results more in line with empirical evidence. We introduce *payouts* in [2, 3], and then also consider corporate tax rate *asymmetry* [4]: numerical results show that these lead to predicted leverage ratios closer to historical norms, through their joint influence on optimal capital structure. Finally, in [1], we introduce *volatility risk*. Following Leland suggestions in [6], we consider a framework in which the assumption of constant volatility in the underlying firm's assets value stochastic evolution is removed. Analyzing defaultable claims involved in the capital structure of the firm we derive their *corrected* prices under a fairly large class of stochastic volatility models by applying singular perturbation theory as in [9]. Exploiting optimal capital structure, the stochastic volatility framework seems to be a robust way to improve results in the direction of both higher spreads and lower leverage ratios in a quantitatively significant way.

Some ideas for future research deal with both theoretical and empirical issues. At first, we would like to extend Leland [5] model in the direction of a dynamic capital structure framework with endogenous default. Allowing for variations in the coupon payments level should be a way to understand how equity holders can *adjust* the capital structure as a more realistic feature to analyze.

The main empirical results in credit risk literature emphasize a poor job of structural models in predicting credit spreads for short maturities. In our minds the stochastic volatility framework is able to better capture the credit exposure of the firm and this is why we would like to work more to develop our application in [1]. For example, studying how volatility time scales (fast and slow factors) can interact and affect spreads and default probability inside a structural model framework with finite horizon.

In the same direction, but from a different point of view, we would like to improve results obtained by calibrating Merton-like structural models under a stochastic volatility framework (see in [12], [8]). In [7] the classical Merton model [10] is studied inside an empirical (regression) analysis showing that, supporting partly findings in [11], even this simple setting is able to predict bond returns sensitivity with respect to changes in stock returns (*hedge ratios*). Nevertheless, volatility risk seems to be a fundamental issue to deal with. Among existing literature about credit risk, [12] propose to consider credit default swap (CDS) premiums as a direct measure of credit spreads, suggesting that the volatility and jump risks of a firm are able to predict most of the variation in their levels. Our aim, following this idea, is to use high-frequency equity prices in order to better capture the *volatility risk* component through a volatility estimator robust with respect to the micro-structure noise.

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