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Titre : *Liberté infinitésimale et modèles matriciels déformés*

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Résumé : Le travail effectué dans cette thèse concerne les domaines de la théorie des matrices aléatoires et des probabilités libres, dont on connaît les riches connexions depuis le début des années 90. Les résultats s'organisent principalement en deux parties : la première porte sur la liberté infinitésimale, la seconde sur les matrices aléatoires déformées.

Plus précisément, on jette les bases d'une théorie combinatoire de la liberté infinitésimale, au premier ordre d'abord, telle que récemment introduite par Belinschi et Shlyakhtenko, puis aux ordres supérieurs. On en donne un cadre simple et général, et on introduit des fonctionnelles de cumulants non-croisés, caractérisant la liberté infinitésimale. L'accent est mis sur la combinatoire et les idées d'essence différentielle qui sous-tendent cette notion.

La seconde partie poursuit l'étude des déformations de modèles matriciels, qui a été ces dernières années un champ de recherche très actif. Les résultats présentés sont originaux en ce qu'ils concernent des perturbations déterministes Hermitiennes de rang non nécessairement fini de matrices de Wigner et de Wishart. En outre, un apport de ce travail est la mise en lumière du lien entre la convergence des valeurs propres de ces modèles et les probabilités libres, plus particulièrement le phénomène de subordination pour la convolution libre. Ce lien donne une illustration de la puissance des idées des probabilités libres dans les problèmes de matrices aléatoires.

Mots clés : Probabilités ; Matrices aléatoires ; Probabilités libres ; Probabilités libres de type B ; Liberté infinitésimale ; Cumulants non-croisés infinitésimaux ; Système dual de dérivation ; Subordination ; Modèle matriciel déformé ; Plus grande valeur propre ; Valeur propre extrêmale ; Matrice de Wigner ; Matrice de covariance empirique.

Discipline : Mathématiques appliquées.

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par

Maxime Février

Infinitesimal freeness and deformed matrix models

Soutenue le 3 décembre 2010 devant le jury composé de

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Liberté infinitésimale et modèles matriciels déformés

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Infinitesimal freeness and deformed matrix models

This thesis is about Random Matrix Theory and Free Probability whose strong relation is known since the early nineties. The results mainly organize in two parts : one on infinitesimal freeness, the other on deformed matrix models.

More precisely, a combinatorial theory of first order infinitesimal freeness, as introduced by Belinschi and Shlyakhtenko, is developed and generalized to higher order. We give a simple and general framework and we introduce infinitesimal non-crossing cumulant functionals, providing a characterization of infinitesimal freeness. The emphasis is put on combinatorics and on the essentially differential ideas underlying this notion.

The second part carries further the study of deformations of matrix models, which has been a very active field of research these past years. The results we present are original in the sense they deal with non-necessarily finite rank deterministic Hermitian perturbations of Wigner and Wishart matrices. Moreover, these results shed light on the link between convergence of eigenvalues of deformed matrix models and free probability, particularly the subordination phenomenon related to free convolution. This link gives an illustration of the power of free probability ideas in random matrix problems.

Keywords: Probability; Random matrices; Free probability; Free probability of type B; Infinitesimal freeness; Infinitesimal non-crossing cumulants; Dual derivation system; Subordination; Deformed matrix model; Largest eigenvalue; Extreme eigenvalue; Wigner matrix; Sample covariance matrix.

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Contents

Notations

Throughout this document, we will use the following notations :

- $\mathbb{R}, \mathbb{C}$ respectively denote the fields of real and complex numbers.
- The real and imaginary parts of a complex number $z \in \mathbb{C}$ are denoted by $\Re z$ and $\Im z$.
- When $\mathcal{A}$ is a commutative unital complex algebra, $\mathcal{M}_N(\mathcal{A})$ is the set of square matrices of size N with entries in $\mathcal{A}$, and $\mathcal{M}_{N,p}(\mathcal{A})$ is the set of rectangular matrices with N rows and p columns, and with entries in $\mathcal{A}$.
- We will use Tr for the trace of a matrix :

  $$\forall X \in \mathcal{M}_N(\mathcal{A}), \mathrm{Tr}(X) = \sum_{i=1}^{N} X_{ii},$$

  and tr for the normalized trace :

  $$\mathrm{tr} = \frac{1}{N}\mathrm{Tr}.$$

- The (real) eigenvalues of a Hermitian matrix are enumerated in the decreasing order

  $$\{\lambda_1 \geq \ldots \geq \lambda_N\}.$$

- A probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ is given by a non-empty set Ω, a tribe $\mathcal{F}$ on Ω and a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. The expectation is denoted by $\mathbb{E}$.
- The Lebesgue measure on $\mathbb{R}^m$ is denoted by dx.
- The support of a probability measure μ is denoted by $\mathrm{supp}(\mu)$.
- $[m] := \{1, \ldots, m\}$.
- For a map $f : X \longrightarrow Y$, when $\mathcal{F}$ is a family of subsets of X, we will use the notation $f(\mathcal{F})$ for the family $\{f(A), A \in \mathcal{F}\}$ of subsets of Y.

Chapter 1

Introduction to Random Matrix Theory

1.1 General context

Random Matrix Theory was born in 1928, when J. Wishart suggested, for statistical data analysis, to study a random matrix of fixed size [Wis28]. This study has been carried further by several statisticians [Fis39],[Hsu39],[Gir39]. The interest in random matrices was renewed in the fifties by a nuclear physicist, E. Wigner. According to the principles of quantum mechanics, the energy levels of a system may be described by the eigenvalues of an operator, the *Hamiltonian*, aging on a Hilbert space of infinite dimension. In the case of a system of highly excited nuclei involved in a slow reaction, the Hamiltonian is unknown, and the computations would be anyway too complicated. Wigner proposed to replace this Hamiltonian by a symmetric random matrix of large size. This idea was confirmed by a huge amount of empirical measurements and is the origin of the study of large random matrices ([Meh04]). Since then, new applications frequently appeared in theoretical physics, to enumerate planar maps [Zvo97], in chords theory, in the study of growth models [Kön05], in information theory, particularly in wireless communication, and in every subject needing the analysis of a great amount of statistical data, etc.
The object of Random Matrix Theory is the *matrix model*, that is a measurable map from a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ into a measurable matrix space. When this latter space is the set of Hermitian matrices $\mathcal{H}_N$ or symmetric matrices $\mathcal{S}_N$, as in the following pages, one talks about Hermitian or symmetric matrix models. Nevertheless, Random Matrix Theory does not reduce to these particular models : some important matrix models are neither Hermitian nor symmetric, such as the Haar measure on the unitary and orthogonal groups, or the *Ginibre ensemble* [Gin65] and its generalizations.
Once the matrix model chosen, an important question is the asymptotic

global and local behaviors of its spectrum.
The global behavior is the study of the whole spectrum. More precisely, given a matrix model X_N, how should the spectrum be normalized for the *empirical spectral measure*

$$\mu_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$$

to converge?
The local behavior is concerned with a small amount of eigenvalues. If the spectrum is real, we will be mainly interested in the largest eigenvalues. One may also consider the smallest eigenvalues, completing the study of the *edge* of the spectrum, or the spacings between neighbouring eigenvalues picked, roughly speaking, in the *bulk* of the spectrum. The problem of spacings, which is of physical interest [Meh04], was the object of a conjecture of Wigner (*Wigner's surmise*) which was proved to be wrong [Meh60], and the results obtained ([Meh04], [Joh01a]) seem to establish an analogy with number theory, and in particular with Riemann Hypothesis on the zeros of the ζ function [KS99].

1.2 Wigner matrices

Let us begin by the study of the matrix models introduced by Wigner, for which we will adopt the following definition :

Definition 1.2.1. By *real Wigner matrix* of size N and associated to the probability measure μ, we mean a symmetric matrix model W_N of size N such that $((W_N)_{ii})_{1\leq i\leq N}$, $((\sqrt{2}W_N)_{ij})_{1\leq i<j\leq N}$ are independent random variables which are identically distributed according to μ.

We also define its complex analogue :

Definition 1.2.2. By *complex Wigner matrix* of size N and associated to the probability measure μ, we mean a Hermitian matrix model W_N of size N such that
$((W_N)_{ii})_{1\leq i\leq N}$, $\sqrt{2}\Re((W_N)_{ij})_{1\leq i<j\leq N}$, $\sqrt{2}\Im((W_N)_{ij})_{1\leq i<j\leq N}$ are independent random variables which are identically distributed according to μ.

The global behavior of Wigner matrices was the object of the first asymptotic result in Random Matrix Theory : the *semicircle law* or *Wigner's Theorem.*

Theorem 1.2.3. *[BS10] Let W_N be a (real or complex) Wigner matrix of size N and associated to a probability measure μ of finite variance σ^2. Then*

$$\mu_{\frac{1}{\sqrt{N}}W_N} \underset{N\to+\infty}{\overset{a.s.}{\Rightarrow}} \mu_\sigma,$$

where

$$\mu_\sigma(dx) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\mathbf{1}_{[-2\sigma,2\sigma]}(x)dx \tag{1.1}$$

is the semicircle distribution of parameter σ*.*

The first version of this theorem was proved by Wigner himself in 1955 [Wig55]. His result was the convergence of the mean empirical spectral measure of a real Wigner matrix associated to a Bernoulli measure. He noticed a few years later [Wig58] that his method could adapt to a more general symmetric probability distribution μ all of whose moments are finite. This method, based on the convergence of the moments of the mean empirical spectral measure towards the moments of the limiting distribution, is called the *moments method.* An important remark is the identity, for each $k \in \mathbb{N}$:

$$\int_{\mathbb{R}} x^k d\mu_{\frac{1}{\sqrt{N}}W_N}(x) = \mathrm{tr}\Big(\Big(\frac{W_N}{\sqrt{N}}\Big)^k\Big),$$

allowing to express the k-th moment of the mean empirical spectral measure as a polynomial in the moments of the entries of the matrix. One thus obtains :

$$\mathbb{E}\Big[\int_{\mathbb{R}} x^k d\mu_{\frac{1}{\sqrt{N}}W_N}(x)\Big] = \frac{1}{N^{\frac{k+2}{2}}}\sum_{i_1,\dots,i_k=1}^{N} \mathbb{E}[(W_N)_{i_1i_2}\cdots(W_N)_{i_ki_1}].$$

If k is odd, each term in this sum vanishes, by symmetry of μ. Otherwise, if $k = 2k'$ is even, then, among the terms which do not vanish by symmetry of μ, one identifies those contributing when N goes to infinity, and one proves that their contributions correspond to the k-th moment of the probability measure μ_σ :

$$\int_{\mathbb{R}} x^{2k'} d\mu_\sigma(x) = \sigma^{2k'}\frac{2k'!}{k'!(k'+1)!}.$$

The quantity $C_{k'} = \frac{2k'!}{k'!(k'+1)!}$ is known as the k'-th Catalan number for its numerous combinatorial interpretations [Sta99].

One may prove this way the convergence in probability of the empirical spectral measure [Gre63]. Arnold will considerably weaken the hypothesis made on μ and will also prove an almost sure result [Arn67],[Arn71]. The case of complex Wigner matrices is similar [Weg76].

There are other proofs of Wigner's Theorem ; one of them characterizes the empirical spectral measure by its *Stieltjes transform* :

Definition 1.2.4. The Stieltjes transform of a finite positive measure μ on $\mathbb{R}$ is the function G_μ defined by :

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z - x},$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$.

It is easy to see that G_μ is actually defined and analytic on $\mathbb{C} \setminus \mathrm{supp}(\mu)$. Its behavior in the neighbooring of the support of μ allows to retrieve all the information on μ :

Lemma 1.2.5. *Let μ be a probability measure on $\mathbb{R}$.*

(1) The nontangential limit at $x \in \mathbb{R}$ of $(z-x)G_\mu(z)$ exists and is equal to $\mu(\{x\})$.

(2) The nontangential limit at $x \in \mathbb{R}$ of $\Im G_\mu(z)$ exists almost everywhere and is equal to $-\pi f(x)$, where f is the density of the absolutely continuous part of μ (with respect to Lebesgue measure).

Another property of Stieltjes transform is its characterization of the convergence of probability measures :

Lemma 1.2.6. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$.*

(1) If $(\mu_n)_{n\in\mathbb{N}}$ weakly converges towards a probability measure μ, then one has the simple convergence of G_{μ_n} towards G_μ on $\mathbb{C} \setminus \mathbb{R}$.

(2) If G_{μ_n} simply converges on $\mathbb{C} \setminus \mathbb{R}$ towards G, then G is the Stieltjes transform of a subprobability μ, and $(\mu_n)_{n\in\mathbb{N}}$ vaguely converges towards μ.

The Stieltjes transform of the empirical spectral measure of a matrix model is the normalized trace of its resolvent : if μ_N denotes the empirical spectral measure of a Hermitian matrix model X_N, then

$$\forall z \in \mathbb{C} \setminus \mathbb{R},\ G_{\mu_N}(z) = \mathrm{tr}((z\mathrm{I}_N - X_N)^{-1}).$$

This justifies the name of *resolvent method* given to the use of Stieltjes transform in Random Matrix Theory. The proof of Wigner's Theorem by the resolvent method amounts to showing the convergence of the Stieltjes transform of the empirical spectral measure of a Wigner matrix towards the Stieltjes transform of the semicircle distribution, satisfying the remarkable following functional equation :

$$\forall z \in \mathbb{C} \setminus \mathbb{R},\ \sigma^2 G_{\mu_\sigma}(z)^2 - zG_{\mu_\sigma}(z) + 1 = 0.$$

It is interesting that the limit of the empirical spectral measure is both deterministic and universal, in the sense that this limit is the same (under the same normalization) for all the Wigner matrices associated to probability measures μ sharing the same finite variance.
The second moment plays a crucial role, since the value of the variance appears in the expression of the limiting distribution (1.1). Ben Arous and Guionnet studied the global behavior of real Wigner matrices associated to a probability measure μ whose variance is infinite [BAG08]. Such a probability measure may be obtained by choosing a sufficiently heavy tail, as follows :

Definition 1.2.7. [Sen76] A function $L : \mathbb{R}^+ \longrightarrow \mathbb{R}^{+*}$ is said to be *slowly varying* when :

$$\forall t > 0,\ \frac{L(tx)}{L(x)} \underset{x\to+\infty}{\longrightarrow} 1.$$

Definition 1.2.8. [Res07] A probability measure μ on $\mathbb{R}$ is said to be *heavy-tailed of index* $\alpha > 0$ if

$$\forall x \geq 0,\ \mu(\{t \in \mathbb{R} \mid |t| > x\}) = \frac{L(x)}{x^\alpha},$$

where $L : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a slowly varying function.

For a heavy-tailed probability measure μ of index $\alpha > 0$, one has

$$\forall k > \alpha, \int_{\mathbb{R}} |x|^k d\mu(x) = +\infty.$$

In particular, when $\alpha \in]0; 2[$, the second moment of μ is infinite.
It is proved in [BAG08] that the empirical spectral measure of Wigner matrices associated to heavy-tailed probability measures of index $\alpha \in]0; 2[$ converge under another normalization towards a new limiting distribution (both depend on the value of α).
In Wigner's Theorem, one may estimate the speed of the convergence of the empirical spectral measure towards the semicircle distribution by considering the uniform distance of their repartition functions. This speed is conjectured to be of order $\frac{1}{N}$ (see [BS10]).
A consequence of the semicircle law is that the largest eigenvalue of a Wigner matrix satisfying the assumptions of Theorem 1.2.3 cannot be with positive probability smaller that the right endpoint 2σ of the support of the limiting distribution. To be able to prove its almost sure convergence towards 2σ, one has to bound it almost surely by 2σ. Following [FK81], one uses the bound

$$\lambda_1^{2k_N} \leq \mathrm{Tr}((W_N)^{2k_N}), \tag{1.2}$$

where $(k_N)_{N\in\mathbb{N}^*}$ is a sequence of integers going slowly to infinity, and one then studies the large trace $\mathrm{Tr}((W_N)^{2k_N})$ by a refinement of the combinatorics involved in the proof of the semicircle law. This argument is sometimes called the *large traces method*. The assumption that progressively appeared to be minimal for almost sure convergence of the largest eigenvalue towards 2σ is the existence of a finite fourth moment for μ :

Theorem 1.2.9. *[BY88] Let W_N be a real or complex Wigner matrix of size N associated to a centered probability measure μ of variance σ^2 and whose fourth moment is finite. Then*

$$\frac{\lambda_1}{\sqrt{N}} \underset{N\to+\infty}{\overset{a.s.}{\longrightarrow}} 2\sigma.$$

The necessity of the assumption of finite fourth moment, proved by Bai and Yin [BY88], was illustrated by the work of Soshnikov first [Sos06] and Auffinger, Ben Arous and Péché [ABAP09] afterwards. They considered real Wigner matrices associated to a heavy-tailed probability distribution of index $\alpha \in]0;2[$ for the first named, and more generally $\alpha \in]0;4[$ for the others, and they show that the largest eigenvalues suitably renormalized converge to a Poisson process. In particular, one has :

Theorem 1.2.10. *Let W_N be a real Wigner matrix of size N associated to a heavy-tailed probability distribution μ of index $\alpha \in]0;2[$, or to a centered heavy-tailed probability distribution μ of index $\alpha \in [2;4[$, and set*

$$b_N := \inf\{x > 0 \mid \mu(]-\infty, -x[\cup]x; +\infty[) \leq \frac{2}{N(N+1)}\}.$$

Then

$$\mathbb{P}(\frac{\lambda_1}{b_N} \leq x) \underset{N \to +\infty}{\longrightarrow} \exp(-x^{-\alpha}).$$

Under the assumptions of Theorem 1.2.9, the largest eigenvalue converging almost surely towards 2σ, the question of its fluctuations around its limit is natural. They were identified by Tracy and Widom [TW94], [TW96] in the case of Wigner matrices associated to the centered Gaussian distribution of variance σ^2 (obviously satisfying the assumptions of Theorem 1.2.9).
A complex Wigner matrix of size N associated to the centered Gaussian distribution of variance σ^2 belongs to the *Gaussian Unitary Ensemble* of parameter σ^2 (abbreviated G.U.E(N, σ^2)). Its distribution on the space $\mathcal{H}_N$, up to the identifiation $\mathcal{H}_N \simeq \mathbb{R}^{N^2}$, is the following :

$$P_{N,\sigma^2,2}(dX) = \frac{1}{Z_{N,\sigma^2,2}} \exp(-\frac{1}{2\sigma^2}\mathrm{Tr}(X^2))\mathrm{dX},$$

where $Z_{N,\sigma^2,2}$ is a normalization constant to ensure that

$$\int_{\mathcal{H}_N} P_{N,\sigma^2,2}(dX) = 1.$$

Parallely, one calls *Gaussian Orthogonal Ensemble* of parameter σ^2 (abbreviated G.O.E(N, σ^2)) the real Wigner matrix of size N associated to the centered Gaussian distribution of variance σ^2, and whose distribution on the space $\mathcal{S}_N$, up to the identifiation $\mathcal{S}_N \simeq \mathbb{R}^{N(N+1)/2}$, is :

$$P_{N,\sigma^2,1}(dX) = \frac{1}{Z_{N,\sigma^2,1}} \exp(-\frac{1}{4\sigma^2}\mathrm{Tr}(X^2))\mathrm{dX},$$

where $Z_{N,\sigma^2,1}$ is a normalization constant. One analogously defines a *Gaussian Symplectic Ensemble.*
The name Gaussian Unitary (resp. Orthogonal, Symplectic) Ensemble comes

from the invariance of its distribution under the conjugation by a matrix from the unitary group $\mathcal{U}_N$ (resp. orthogonal group $\mathcal{O}_N$, symplectic group Sp_N). Using this property characterizing in a sense the Gaussian Ensembles among the Wigner matrices (see [Meh04]), Weyl's formula provides an explicit expression of the joint distribution of the eigenvalues on $\mathbb{R}^N$, that one can find in [Dei99] :

$$Q_{N,\sigma^2,\beta}(d\lambda) = \frac{1}{\tilde{Z}_{N,\sigma^2,\beta}} \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta \exp(-\frac{\beta}{4\sigma^2}\sum_{i=1}^N \lambda_i^2)\mathrm{d}\lambda,$$

where $\beta = 2$ (resp. $\beta = 1$, $\beta = 4$) corresponds to G.U.E. (resp. G.O.E., G.S.E.). The normalization constant $\tilde{Z}_{N,\sigma^2,\beta}$ may be computed using *Selberg's integral* [Sel44].
In the case of the G.U.E. (corresponding to $\beta = 2$), the point process of the eigenvalues has a determinantal structure ([Meh04], see also [Sos00]) whose kernel is expressed in terms of the orthogonal polynomials associated to the weight $\exp(-\frac{x^2}{2\sigma^2})$. These are precisely the Hermite polynomials, justifying the name of *Hermite Ensemble* sometimes attributed to the spectrum of the G.U.E.. The local behavior of the G.U.E. is thus linked to the asymptotics of Hermite polynomials, called *Plancherel-Rotach asymptotics* [PR29].
The asymptotic corresponding to the edge of the spectrum is linked to the Fredholm determinant of the operator $\mathcal{K}_{Ai}$ whose kernel is the Airy kernel :

$$\mathrm{K}_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y},$$

where Ai is the *Airy function*, unique solution on $\mathbb{R}$ of the differential equation $f''(x) = xf(x)$ satisfying $f(x) \underset{x\to+\infty}{\sim} (4\pi\sqrt{x})^{\frac{1}{2}} e^{-\frac{2}{3}x^{\frac{3}{2}}}$.

Theorem 1.2.11. *Let X_N be a G.U.E(N,σ^2) matrix, and denote by λ_1 its largest eigenvalue.*
Let $q : \mathbb{R} \longrightarrow \mathbb{R}$ be the unique solution of the differential equation

$$q''(x) = xq(x) + 2q(x)^3$$

such that $q(x) \underset{x\to+\infty}{\sim} \mathrm{Ai}(x)$. Then

$$\lim_{N\to+\infty} \mathbb{P}(N^{\frac{2}{3}}(\frac{\lambda_1}{\sqrt{N}} - 2\sigma) \leq s) = F_2(s),$$

where $F_2(s) = \exp(-\int_s^{+\infty}(x-s)q^2(x)\mathrm{d}x)$.

There is a more general result on the fluctuations of the m largest eigenvalues of the G.U.E. [TW94]. Moreover, from the complex case, Tracy et Widom [TW96] deduced that the fluctuations of the largest eigenvalue of the

G.O.E. and the G.S.E. are at the same scale than in the complex case, but with modified repartition functions

$$F_1(s) = \sqrt{F_2(s)}\exp(-\frac{1}{2}\int_s^{+\infty} q(x)\mathrm{dx})$$

and

$$F_4(s) = \sqrt{F_2(s)}\frac{1}{2}(\exp(\frac{1}{2}\int_s^{+\infty} q(x)\mathrm{dx}) + \exp(-\frac{1}{2}\int_s^{+\infty} q(x)\mathrm{dx})).$$

The distributions whose repartition functions are F_1, F_2, F_4 are now known as the Tracy-Widom distributions. They appear in the asymptotic study of percolation models ([BS05],[BM05],[PS00]), of growth models [Joh00], of the symmetric group ([BDJ99],[BDJ00],[Joh01a], [BR01]). One may consult [Kön05] for an introduction.
For a wide class of real (resp. complex) Wigner matrices, the universality of fluctuations of the largest eigenvalues around their limit are conjectured to be universal, in the sense that these fluctuations do not depend on the particular choice of μ. Soshnikov's work [Sos99] is a great advance since it gives a proof of universality of the fluctuations of the largest eigenvalues of real (resp. complex) Wigner matrices associated to symmetric probability measures μ with the same variance and satisfying the subgaussian moments condition :

$$\exists C > 0,\ \forall k \in \mathbb{N}^*,\ \int_{\mathbb{R}} x^{2k} d\mu(x) \leq (Ck)^k.$$

In this work, universality of the fluctuations is reduced to universality of the asymptotics of large traces similar as in (1.2), but with k_N going sufficiently fast to infinity ($k_N \sim N^{\frac{2}{3}}$). It remains then to refine the large traces method of [FK81] and [BY88] following [SS98b], [SS98a]. The results of [Sos99] were improved by the same strategy by Ruzmaikina [Ruz06], Péché and Soshnikov [PS07], [PS08] and Khorunzhy [KV08], [Kho09], [Kho10].
Notice in [FS08] a new promising approach of the large traces method. Recently, Tao and Vu proved a variant of Soshnikov's universality result, by a Lindeberg method, without assuming symmetry condition [TV10].
Finally, the spacings between neighbouring eigenvalues of a Wigner matrix were the object of several recent developments : Johansson [Joh01b], then Tao and Vu [TV09b], showed that the results known to hold for the G.U.E. for fifty years ([Meh04]) were quite universal ; Ben Arous and Bourgade studied the maximal spacings.

1.3 Sample covariance matrices

The interest of statisticians in Random Matrix Theory is mainly focused on *sample covariance matrices*, that we define as follows :

Definition 1.3.1. A real (resp. complex) sample covariance matrix is a symmetric (resp. Hermitian) matrix model of size N and of the form $X_N = \Sigma^{\frac{1}{2}} AA^* \Sigma^{\frac{1}{2}}$, where A is a rectangular matrix with N rows and p columns whose entries are independent identically distributed real (resp. complex) random variables and Σ is a deterministic symmetric (Hermitian) definite positive matrix of size N. The matrix Σ is called *true covariance matrix*, p the number of variables and N the sample size.

The matrix model originally considered by Wishart in [Wis28], named *Wishart matrix*, is a particular case of real sample covariance matrix, when one chooses for the entries of A independent real standard Gaussian variables.
We first deal with *white* sample covariance matrices, that is whose true covariance matrix is $\Sigma = \sigma^2 \mathrm{I_N}$.
We restrict ourselves to $p \geq N$, by simply noticing that the spectra of AA^* and A^*A only differ by $p - N$ zeros.
The distribution of white Wishart matrices, in the case $p \geq N$, is absolutely continuous with respect to the Lebesgue measure on $\mathcal{S}_N$, with support in the set of positive symmetric matrices, and density given by :

$$\frac{1}{Z_{p,N,\sigma,1}} (\det X)^{\frac{p-N-1}{2}} \exp(-\frac{1}{2\sigma^2} \mathrm{Tr}(X)),$$

where $Z_{p,N,\sigma,1}$ in a normalization constant.
This distribution is invariant under conjugation by a matrix of the orthogonal group $\mathcal{O}_p$ and one obtains an explicit expression of the joint distribution of the eigenvalues, whose support is in $(\mathbb{R}^+)^p$:

$$\frac{1}{\tilde{Z}_{p,N,\sigma^2,1}} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j| \prod_{i=1}^{N} \lambda_i^{\frac{p-N-1}{2}} \exp(-\frac{1}{2\sigma^2} \sum_{i=1}^{N} \lambda_i) \mathrm{d}\lambda.$$

Once again, the normalization constant $\tilde{Z}_{p,N,\sigma^2,1}$ may be computed using Selberg integral. The orthogonal polynomials associated to the weight

$$x^{\frac{p-N-1}{2}} \exp(-\frac{x}{2\sigma^2})$$

are explicitely known : these are the Laguerre polynomials, providing an explanation of the name *Laguerre Orthogonal Ensemble* (L.O.E.) of size N with p degrees of freedom also attributed to this model. One defines analogously the *Laguerre Unitary Ensemble* (L.U.E.) and the *Laguerre Symplectic Ensemble* (L.S.E.).
About the global behavior of white sample covariance matrices, one has the following result :

Theorem 1.3.2. *Let* $(p(N))_{N\in\mathbb{N}^*}$ *be a sequence of positive integers such that*

$$\frac{p(N)}{N} \underset{N\to+\infty}{\longrightarrow} \gamma \geq 1,$$

and let $\sigma > 0$*, consider the real or complex sample covariance matrix* $X_N = AA^*$ *where* A *is a rectangular matrix with* N *rows and* $p(N)$ *columns whose entries are independent identically distributed random variables of variance* σ^2*. Then*

$$\mu_{\frac{1}{N}X_N} \overset{a.s.}{\underset{N\to+\infty}{\Rightarrow}} \pi_{\gamma,\sigma},$$

where

$$\pi_{\gamma,\sigma}(dx) = \frac{1}{2\pi\sigma^2 x}\sqrt{(b-x)(x-a)}\mathbf{1}_{[a,b]}(x)dx, \tag{1.3}$$

setting $a = \sigma^2(\sqrt{\gamma}-1)^2$ *and* $b = \sigma^2(\sqrt{\gamma}+1)^2$*.*

The distributions $\pi_{\gamma,\sigma}$ were identified for the first time by Marchenko and Pastur in [MP67], and are for this reason called *Marchenko-Pastur distributions.* The original proof of [MP67] initiated the use of Stieltjes transform in Random Matrix theory.
In Theorem 1.3.2, the common value of the variance of the entries of A appears in the expression of the limiting distribution (1.3). Belinschi, Dembo and Guionnet studied in [BDG09] the global behavior of white real sample covariance matrices AA^* when the entries of A follow a heavy-tailed distribution of index $\alpha \in]0;2[$. The empirical spectral measure converges in this case, under a new normalization, towards a new limiting distribution (both depend on the value of α).
As in Wigner's Theorem, one may study the speed of the convergence in Theorem 1.3.2 : it seems that it is also of order $\frac{1}{N}$, even if this has not been proved yet for a general sample covariance matrix [BS10].
A consequence of Theorem 1.3.2 is that the largest eigenvalue of a sample covariance matrix satisfying the assumptions of Theorem 1.3.2 is almost surely greater than the right endpoint $b = \sigma^2(1+\sqrt{\frac{1}{\gamma}})^2$ of the support of the limiting distribution. Assuming that the distribution of the entries of A have a finite fourth moment, one gets an almost sure convergence result :

Theorem 1.3.3. *[YBK88] Let* $(p(N))_{N\in\mathbb{N}^*}$ *be a sequence of positive integers such that*

$$\frac{p(N)}{N} \underset{N\to+\infty}{\longrightarrow} \gamma \geq 1,$$

and let $\sigma > 0$*, consider a sample covariance matrix* $X_N = AA^*$*, where* A *is a rectangular matrix with* N *rows and* $p(N)$ *columns whose entries are independent random variables identically distributed according to a centered distribution of variance* σ^2 *and having a finite fourth moment. Then*

$$\frac{\lambda_1}{N} \overset{a.s.}{\underset{N\to+\infty}{\longrightarrow}} \sigma^2(\sqrt{\gamma}+1)^2.$$

The first result of convergence of the largest eigenvalue of a sample covariance matrix is due to Geman [Gem80], who proved, by the large traces method, the almost sure convergence for centered subgaussian entries. The assumptions were progressively reduced to those of Theorem 1.3.3. The necessity of the assumption of finite fourth moment to have almost sure convergence was proved in [BSY88]. Note that, to obtain the convergence in probability, one knows a weaker necessary and sufficient condition [Sil89].
In [ABAP09], real sample covariance matrices AA^*, with entries of A following a centered heavy-tailed distribution of index $\alpha \in]0;4[$, are studied. It is shown there that the largest eigenvalues, suitably normalized, converge to a Poisson process.
The fluctuations of the largest eigenvalue around its limit were identified in the cases of the L.U.E [Joh00] and of the L.O.E. [Joh01c].

Theorem 1.3.4. *Denoting by λ_1 the largest eigenvalue of a L.U.E. matrix of size N, with $p(N)$ degrees of freedom, satisfying*

$$\frac{p(N)}{N} \underset{N\to+\infty}{\longrightarrow} \gamma \geq 1,$$

one has

$$\lim_{N\to+\infty} \mathbb{P}\Big(\frac{N^{\frac{2}{3}}}{\sigma^2(1+\sqrt{\frac{p(N)}{N}})(1+\sqrt{\frac{N}{p(N)}})^{\frac{1}{3}}}\Big(\frac{\lambda_1}{N} - \sigma^2(1+\sqrt{\frac{p(N)}{N}})^2\Big) \leq s\Big) = F_2(s).$$

Theorem 1.3.5. *Denoting by λ_1 the largest eigenvalue of a L.O.E. matrix of size N, with $p(N)$ degrees of freedom, satisfying*

$$\frac{p(N)}{N} \underset{N\to+\infty}{\longrightarrow} \gamma \geq 1,$$

one has

$$\lim_{N\to+\infty} \mathbb{P}\Big(\frac{N^{\frac{2}{3}}}{\sigma^2(1+\sqrt{\frac{p(N)}{N}})(1+\sqrt{\frac{N}{p(N)}})^{\frac{1}{3}}}\Big(\frac{\lambda_1}{N} - \sigma^2(1+\sqrt{\frac{p(N)}{N}})^2\Big) \leq s\Big) = F_1(s).$$

As for Wigner matrices, it is conjectured that the fluctuations of the largest eigenvalues around their limit are universal for a wide class of real (resp. complex) white sample covariance matrices. This universality was proved for sample covariance matrices whose entries are symmetric and subgaussian in [Sos02], [Péc09], and more general entries in [TV09a].

We now focus on non-white sample covariance matrices $\Sigma^{\frac{1}{2}} AA^* \Sigma^{\frac{1}{2}}$ where Σ is more general. Although there is a direct statistical interest of this model, we rather view it as a *deformation* of the white case and we analyze the influence of Σ on the asymptotic behavior of the spectrum. We restrict ourselves to a deterministic symmetric (resp. Hermitian) definite positive Σ, even if some results adapt to random true covariance matrices independent of A. Such a random diagonal Σ, with independent identically distributed diagonal entries, is indeed considered in [MP67]. We will also make the following assumption on Σ :

$$\mu_\Sigma \underset{N\to+\infty}{\Rightarrow} \rho, \tag{1.4}$$

where ρ is a compactly supported probability measure on $\mathbb{R}^+$.
The influence of the deformation on the global behavior is the following :

Theorem 1.3.6. *[Sil95] Let $(p(N))_{N\in\mathbb{N}^*}$ be a sequence of positive integers such that*

$$\frac{p(N)}{N} \underset{N\to+\infty}{\longrightarrow} \gamma \geq 1,$$

and let Σ be a deterministic Hermitian positive definite matrix satisfying condition (1.4), consider the sample covariance matrix $X_N = \Sigma^{\frac{1}{2}} AA^* \Sigma^{\frac{1}{2}}$, where A is a rectangular matrix with N rows and N columns, with real or complex independent entries identically distributed according to a distribution of variance 1. Then

$$\mu_{\frac{1}{N}X_N} \overset{a.s.}{\underset{N\to+\infty}{\Rightarrow}} \pi_{\gamma,\rho},$$

where $\pi_{\gamma,\rho}$ is characterized by :

$$G_{\pi_{\gamma,\rho}}(z) = \int_{\mathbb{R}} \frac{1}{z - t(1+\gamma z G_{\pi_{\gamma,\rho}}(z))} \rho(dt). \tag{1.5}$$

We are now in position to examine the convergence and the fluctuations of the largest eigenvalue of a sample covariance matrix. For instance, Bai and Silverstein proved, under certain additional assumptions compared to Theorem 1.3.6, the almost sure convergence of the largest eigenvalue of $X_N = \Sigma^{\frac{1}{2}} AA^* \Sigma^{\frac{1}{2}}$ towards the right endpoint of the support of the limiting distribution $\pi_{\gamma,\rho}$ [BS98]. They put into evidence a more precise phenomenon, the *exact separation* phenomenon : to any compact interval I in the complementary of the spectrum of Σ, one associates an explicit compact interval J in the complementary of the spectrum of X_N for N sufficiently large, and moreover there are as many eigenvalues of X_N on the right of J as eigenvalues of Σ on the right of I [BS99]. In the same direction, El Karoui proved, for non-white Wishart matrices associated to a large choice of true covariance matrices, not only convergence of the largest eigenvalue towards the right edpoint of the limiting support, but also that its fluctuations were governed by Tracy-Widom distribution [EK07]. However, some choices of Σ perturb the behavior of the largest eigenvalue. This is sometimes the case with the *spiked population model*, introduced by Johnstone [Joh01c].

Definition 1.3.7. By spiked population model we mean a sample covariance matrix whose true covariance matrix Σ is a deterministic diagonal matrix of size N and of the form :

$$\Sigma = \mathrm{Diag}(\theta_1, \dots, \theta_r, 1, \dots, 1),$$

where $\theta_1 \geq \cdots \geq \theta_r$.

Baik, Ben Arous and Péché established in [BBAP05] a *phase transition* for the largest eigenvalue of the spiked population model.

Theorem 1.3.8. *Let X_N be a Gaussian spiked population model defined above, with $r, \theta_1, \dots, \theta_r$ fixed independently of N and*

$$\frac{p(N)}{N} \underset{N \to +\infty}{\longrightarrow} \gamma \geq 1,$$

then :

- *if $\theta_1 = \ldots = \theta_k > 1 + \sqrt{\gamma}$ and $\theta_k > \theta_{k+1}$,*

$$\mathbb{P}\Big(\frac{\sqrt{N}}{\sqrt{\gamma\theta_1^2(1 - \frac{1}{\gamma(\theta_1-1)^2})}}\Big(\frac{\lambda_1}{N} - \sigma^2\gamma\theta_1(1 + \frac{1}{\gamma(\theta_1 - 1)}\Big) \leq s\Big) \underset{N \to +\infty}{\to} G_k(s),$$

  where G_k is the distribution of the largest eigenvalue of a G.U.E$(k, 1)$.

- *if $\theta_1 = \ldots = \theta_k = 1 + \sqrt{\gamma}$ et $\theta_{k+1} < 1 + \sqrt{\gamma}$,*

$$\mathbb{P}\Big(\frac{N^{\frac{2}{3}}\gamma^{\frac{1}{6}}}{(1+\sqrt{\gamma})^{\frac{4}{3}}}\Big(\frac{\lambda_1}{N} - \sigma^2(1 + \sqrt{\gamma})^2\Big) \leq s\Big) \underset{N \to +\infty}{\to} F_{k+2}(s),$$

  where F_{k+2} is a modification of Tracy-Widom distribution.

- *if $\theta_1 < 1 + \sqrt{\frac{1}{\gamma}}$,*

$$\mathbb{P}\Big(\frac{N^{\frac{2}{3}}\gamma^{\frac{1}{6}}}{(1+\sqrt{\gamma})^{\frac{4}{3}}}\Big(\frac{\lambda_1}{N} - \sigma^2(1 + \sqrt{\gamma})^2\Big) \leq s\Big) \underset{N \to +\infty}{\to} F_2(s),$$

  where F_2 is the Tracy-Widom distribution.

This result was generalized in different directions : to real spiked population models by Paul (at least for the first point)[Pau07], singular spiked population models by Onatski [Ona08], non-Gaussian by Baik and Silverstein for the convergence [BS06] and Bai and Yao on the one hand [BY08b], Féral and Péché on the other hand [FP07], for the fluctuations. The arguments of Bai and Yao allow to deal with non-diagonal Σ (but still with a fixed number of eigenvalues different of 1). An improved result was recently proved by Bai and Yao, about the following generalized spiked population model :

Definition 1.3.9. By generalized spiked population model we mean a sample covariance matrix whose true covariance matrix Σ is a deterministic Hermitian positive definite matrix of size N with r fixed eigenvalues $\theta_1 > \cdots > \theta_r$ and $N-r$ other eigenvalues $(\beta_{i,N})_{1\leq i\leq N-r}$.

The result is the following :

Theorem 1.3.10. *Let X_N be a generalized spiked population model of size N defined above ; assume that $r, \theta_1, \ldots, \theta_r$ are fixed independently of N, that*

$$\frac{p(N)}{N} \underset{N\to+\infty}{\longrightarrow} \gamma \geq 1,$$

$$\mu_\Sigma \underset{N\to+\infty}{\Rightarrow} \rho,$$

$$\max_{1\leq i\leq N-r} \operatorname{dist}(\beta_{i,N}, \operatorname{supp}(\rho)) \underset{N\to+\infty}{\longrightarrow} 0$$

and $(\|\Sigma\|)_{N\in\mathbb{N}^*}$ is bounded. Then, denoting

$$\psi(\theta) = \gamma\theta + \theta\int_{\mathbb{R}} \frac{t}{\theta - t}\rho(dt),$$

- *if θ_i is the m_i-th eigenvalue of Σ (in the decreasing order) and satisfies $\psi'(\theta_i) > 0$, one has almost sure convergence of $\frac{\lambda_{m_i}}{N}$ towards $\psi(\theta_i)$ with Gaussian fluctuations.*
- *if θ_i is the m_i-th eigenvalue of Σ (in the decreasing order) and satisfies $\psi'(\theta_i) \leq 0$, there are two cases :*
  - *if the maximal interval of the complementary of the support of ρ contains a subinterval I on which ψ' is positive, then one has almost sure convergence of $\frac{\lambda_{m_i}}{N}$ towards $\psi(x)$ where x is an endpoint of I ;*
  - *otherwise, $\frac{\lambda_{m_i}}{N}$ almost surely converges to the $\rho([0;\lambda_{m_i}])$-quantile of $\pi_{\gamma,\rho}$.*

1.4 Additively deformed model

The content of the last part of the preceding Section was the study of the influence of a multiplicative perturbation on the local behavior of sample covariance matrices. We now investigate the influence of an additive perturbation on the local behavior of Hermitian matrix models.
We restrict to fixed (independently of N) rank r perturbations. In this case, the empirical spectral measure of the deformed model still almost surely converges towards the semicircle distribution (1.1).
For a deformed G.U.E.

$$\frac{W_N}{\sqrt{N}} + A_N,$$

where W_N is a G.U.E(N,σ^2) matrix and A_N is a deterministic Hermitian matrix of rank r fixed independently of N and whose r nonzero eigenvalues are fixed, Johansson, following the work of Brézin and Hikami [BH96], [BH97], proved in [Joh01b] that the spectrum still has a determinantal structure, with a modified kernel. His result, which applies to a more general deformed model, relies on Harish-Chandra-Itzykson-Zuber integral and on an interpretation in terms of nonintersecting Brownian motions. In [Péc06], Péché obtains, by a saddle point analysis, the local statistics at the edge of the spectrum of the deformed G.U.E(N,σ) via the asymptotic of the kernel. Her approach adapts to a perturbation whose rank is negligible compared to N. One observes a phase transition phenomenon analogous to Theorem 1.3.8: denoting by θ_1 the largest eigenvalue of A_N, the limiting value and the fluctuations of the largest eigenvalue λ_1 of $\frac{W_N}{\sqrt{N}}+A_N$ depend on the position of θ_1 with respect to σ :

- if $\theta_1<\sigma$, one has convergence of λ_1 towards 2σ with Tracy-Widom fluctuations.
- if $\theta_1=\sigma$, one still has convergence of λ_1 towards 2σ but with modified fluctuations.
- if $\theta_1>\sigma$, one has convergence of λ_1 towards $\rho_{\theta_1}:=\theta_1+\frac{\sigma^2}{\theta_1}>2\sigma$, with Gaussian fluctuations.

As in the multiplicative case, the phase transition phenomenon exhibits some universality. Péché and Féral proved indeed in [FP07] that the conclusions above still hold for a *deformed Wigner model*

$$\frac{W_N}{\sqrt{N}}+A_N,$$

where W_N is a real or complex Wigner matrix of size N associated to a symmetric subgaussian distribution with variance σ^2, and A_N is a deterministic matrix whose entries are all equal to $\frac{\theta}{N}$ where θ is any fixed real number. Without an explicit formula for the density of the eigenvalues, beyond the Gaussian case, Féral et Péché's strategy is the large traces method, as developed by [SS98b], [SS98a], [Sos99]. Recall that, in this method, one links the convergence and the fluctuations of the largest eigenvalue of a matrix model to the asymptotics of large traces, for which it is possible, by an ad-hoc combinatorics, to prove that they only depend on the values of θ and σ. This implies that the asymptotic behavior of the largest eigenvalue is the same in the general case as in the Gaussian case. The result of [FP07] is completed by the older result of [FK81] on the largest eigenvalue of Wigner matrices associated to a non-necessarily centered distribution. The results of [FP07] are limited by the very particular form of the perturbation A_N, of rank 1 and all of whose entries are equal.

More general finite rank perturbations are considered in the work of Capitaine, Donati-Martin and Féral in [CDMF09]. Note that the model, as well as the method used and the conclusions obtained, are a bit different than the preceding. More precisely, the model, authorizing more general finite rank perturbations, also replaces the subgaussian moments condition on the distribution μ of the entries of the Wigner matrix, by an assumption of *Poincaré inequality*, that is :

Definition 1.4.1. A probability measure μ on $\mathbb{R}$ satisfies a Poincaré inequality if there exists a positive constant C such that, for all function $f : \mathbb{R} \to \mathbb{C}$ of class $\mathcal{C}^1$ and such that f and f' are in $L^2(\mu)$,

$$\mathbf{V}(f) \leq C \int |f'|^2 d\mu,$$

where $\mathbf{V}(f) = \mathbb{E}(|f - \mathbb{E}(f)|^2)$ denotes the variance of f.

The convergence result of the largest eigenvalues in [CDMF09] is the same phase transition as in previous works.

Theorem 1.4.2. *[CDMF09] Let W_N be a real or complex Wigner matrix of size N associated to a symmetric probability measure of variance σ^2 and satisfying a Poincaré inequality, let A_N be a fixed rank r deterministic symmetric or Hermitian matrix having a fixed number J of real nonzero fixed distinct eigenvalues*

$$\theta_1 > \cdots > \theta_J,$$

of muliplicities k_i (of course $\sum_{i=1}^{J} k_i = r$), among which $J_{+\sigma}$ (resp. $J_{-\sigma}$) are greater than σ (resp. lower than $-\sigma$). Then, for $1 \leq i \leq J_{+\sigma}$ or $J - J_{-\sigma} + 1 \leq i \leq N$, the k_i eigenvalues

$$\lambda_{\sum_{j=1}^{i}-1 k_j+1}, \ldots, \lambda_{\sum_{j=1}^{i} k_j}$$

of $\frac{1}{\sqrt{N}}W_N + A_N$ (in the decreasing order) converge almost surely towards ρ_{θ_i}. Moreover,

$$\lambda_{\sum_{j=1}^{J_{+\sigma}} k_j+1} \xrightarrow[N\to+\infty]{a.s.} 2\sigma,$$

$$\lambda_{N-\sum_{j=J-J_{-\sigma}+1}^{J} k_j+1} \xrightarrow[N\to+\infty]{a.s.} -2\sigma.$$

However, another progress of [CDMF09] is to put into evidence the non-universality of fluctuations : there are perturbations, a matrix whose only nonzero entry is at first row and first column and is greater than σ for instance, which bring fluctuations of the largest eigenvalue depending on the distribution of the entries of the Wigner matrix. A more careful study of these non-universal fluctuation is carried in [CDF09].
The plan of the proof of Theorem 1.4.2 follows the proof of the result of

Baik and Silverstein on sample covariance matrices [BS06], as suggested by Féral in her PhD dissertation : they first show an almost sure inclusion, for N large enough, of the spectrum of $\frac{1}{\sqrt{N}}W_N + A_N$ in a neighborhood of the set composed of the support of the semicircle distribution and of the $\{\rho_{\theta_i}\}_{1\leq i\leq J}$. Then, they establish, an exact separation phenomenon between the spectrum of A_N and the spectrum of $\frac{1}{\sqrt{N}}W_N + A_N$. Nevertheless, the strategy differs from [BS06] in the proof of the inclusion of the spectrum, prefering a method initiated by Haagerup and Thorbjornsen in [HT05], and developed further in [Sch05], [CDM07]. This method relies on a precise estimation of the Stieltjes transform of the mean empirical spectral measure. A systematic study of the largest eigenvalues and of their associated eigenvectors is carried by Benaych-Georges and Rao in [BR09] for general unitarily invariant models deformed by finite rank perturbations. Their elegant proof mixes arguments from linear algebra and from concentration of measure theory.

Chapter 2

Introduction to free probability

2.1 From operator algebra to free probability

Free probability is rooted in the theory of operator algebras, more precisely in the problem of classification of structures named *von Neumann algebras*, because the mathematician John von Neumann introduced them in the thirties.

Definition 2.1.1. Let $\mathcal{H}$ be a Hilbert space, a von Neumann algebra in $\mathcal{H}$ is a $*$-subalgebra $\mathcal{A}$ of the algebra of bounded linear operators of $\mathcal{H}$ equal to its *bicommutant* :

$$\mathcal{A} = \mathcal{A}''.$$

Among von Neumann algebras, let us focus on those built from a discrete group G. On the Hilbert space $l^2(G)$, equipped with an orthonormal basis $(e_g)_{g\in G}$, consider the left regular representation λ on G defined by $\lambda(h)(e_g) = e_{hg}$. Then $L(G) := \lambda(G)''$ is a von Neumann algebra in $l^2(G)$, called the group von Neumann algebra of G.
The group von Neumann algebra of a discrete group G has a *state*, that is a linear functional φ which is positive ($\forall x \in L(G), \varphi(x^*x) \geq 0$) normalized ($\varphi(1) = 1$), and with the special properties of tracialty ($\forall x, y \in L(G), \varphi(xy) = \varphi(yx)$), and ultraweak continuity.
Any commutative von Neumann algebra being isomorphic to an algebra of essentially bounded random variables $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, which, equipped with the expectation $\mathbb{E}$, is the framework of classical probability, we adopt the following definition :

Definition 2.1.2. We call *W^*-probability space* the pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a von Neumann algebra in a Hilbert space $\mathcal{H}$, and φ is a ultraweakly continuous state.

We often work in the following simpler and more general structure, which is the framework of noncommutative probability :

Definition 2.1.3. We call *noncommutative probability space* the pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital complex algebra, and φ is a linear functional on $\mathcal{A}$ satisfying $\varphi(1_{\mathcal{A}}) = 1$.

In a noncommutative probability space $(\mathcal{A}, \varphi)$, $\mathcal{A}$ plays the role of the set of random variables $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ in classical probability, and φ the role of the expectation $\mathbb{E}$. The elements $a \in \mathcal{A}$ are therefore called *noncommutative random variables.*
If $n \neq m$ are two distinct positive integers, we know that the free groups $\mathbb{F}_n$ and $\mathbb{F}_m$ are not isomorphic, but the problem to decide whether their von Neumann algebras are or not isomorphic remains open. The idea of Voiculescu, motivated by this isomorphism problem, was to translate the algebraic freeness condition from the level of groups to the level of group algebras and to develop a noncommutative probability theory, on a noncommutative probability space, in which this new notion of freeness plays the role of independence in classical probability [Voi85].

Definition 2.1.4. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, unital subalgebras $(\mathcal{A}_i)_{i\in I}$ are said to be free if they satisfy the following condition:

$$\begin{gathered}\text{If } i_1, \dots, i_n \in I \text{ are such that } i_1 \neq i_2 \neq \dots \neq i_n,\\ \text{and if } a_1 \in \mathcal{A}_{i_1}, \dots, a_n \in \mathcal{A}_{i_n} \text{ are such that } \varphi(a_1) = \cdots = \varphi(a_n) = 0,\\ \text{then } \varphi(a_1 \cdots a_n) = 0.\end{gathered}$$

Subsets of $(\mathcal{A}, \varphi)$ are said to be free if the unital subalgebras they generate are free.

The program of free probability is the development of a free version of the classical theory of probability. One may distinguish two alternative approaches : the analytical approach and the combinatorial approach.

2.2 Analytical approach

In a classical probability space $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$, the distribution of a complex random variable X is a probability measure μ_X on $\mathbb{C}$, defined, for each Borel set B of $\mathbb{C}$, by

$$\mu_X(B) = \mathbb{P}(X^{-1}(B)).$$

In a W^*-probability space $(\mathcal{A}, \varphi)$, one defines the distribution of a noncommutative random variable $a \in \mathcal{A}$, under the extra assumption that it is normal, that is satisfying $a^*a = aa^*$, in the following way : using the spectral theorem, one associates to the linear functional

$$f \longmapsto \varphi(f(a)),$$

defined on the set of measurable maps on $\mathbb{C}$, a compactly supported probability measure μ_a on $\mathbb{C}$ satisfying, for each measurable map f, the relation:

$$\varphi(f(a)) = \int_{\mathbb{R}} f(x) d\mu_a(x).$$

The measure μ_a is the *distribution* of the noncommutative random variable a. For any compactly supported probability measure μ on $\mathbb{C}$, it is possible to construct a W^*-probability space $(\mathcal{A}, \varphi)$ and a normal noncommutative random variable $a \in \mathcal{A}$ with distribution μ.
In the same way as, in classical probability, one may realize in a probability space two independent random variables of given distributions, one may also realize in a W^*-probability space two free noncommutative random variables of given distributions, by the construction of a *free product* [VDN92].

2.2.1 Free additive convolution

In the correspondence above between normal noncommutative random variables and compactly supported probability measures on $\mathbb{C}$, the selfadjoint variables, satisfying $a^* = a$, correspond to compactly supported probability measures on $\mathbb{R}$. Using the free product, one defines an operation on compactly supported probability measures on $\mathbb{R}$, named free additive convolution, and denoted by $\boxplus$: if μ and ν are two compactly supported probability measures on $\mathbb{R}$, $\mu \boxplus \nu$ is the distribution of $a + b$, when a and b are free selfadjoint noncommutative random variables of respective distributions μ and ν [Voi86]. The operation $\boxplus$ was successively extended to probability measures with finite variance [Maa92], then to general probability measures on $\mathbb{R}$ [BV93].
The linearization of the classical convolution of measures is obtained by the log-Fourier transform ; its free analogue is called the R-transform [Voi86], [BV93]. For a probability measure μ on $\mathbb{R}$, the Stieltjes transform G_μ has a right inverse K_μ defined on a domain of the form

$$\bigcup_{\alpha>0} \{z = x + iy : 0 < y < \delta_\alpha, x < \alpha y\}.$$

Definition 2.2.1. Let μ be a probability measure on $\mathbb{R}$, we call R-transform of μ the function R_μ defined by

$$R_\mu(z) := K_\mu(z) - \frac{1}{z}.$$

Theorem 2.2.2. *Let μ and ν be two probability measures on $\mathbb{R}$, then there exists a domain of the complex plane where $R_{\mu \boxplus \nu}$, R_μ and R_ν are defined and related by :*

$$R_{\mu \boxplus \nu} = R_\mu + R_\nu.$$

Free additive convolution of probability measures has an important property, called subordination :

Theorem 2.2.3. *Let μ and ν be two probability measures on $\mathbb{R}$, there exists a unique analytic map $\omega_{\mu,\nu} : \mathbb{C}^+ \to \mathbb{C}^+$ such that*

$$\forall z \in \mathbb{C}^+, G_{\mu \boxplus \nu}(z) = G_\mu(\omega_{\mu,\nu}(z)),$$

$$\forall z \in \mathbb{C}^+, \Im \omega_{\mu,\nu}(z) \geq \Im z$$

and

$$\lim_{y \uparrow +\infty} \frac{\omega_{\mu,\nu}(iy)}{iy} = 1.$$

This phenomenon, first observed by Voiculescu under a generic assumption [Voi93], was proved in full generality in [Bia98]. A new proof was given later, using a fixed point theorem for analytic self-maps of the upper half-plane [BB07]. Subordination allowed to prove some remarkable regularity properties of the free additive convolution of two probability measures ([Voi93],[BV95],[Bia97a],[BV98],[BB04],[Bel06],[Bel08],[BBGG09]) or to give a new definition of free additive convolution [CG08a].
Following the analogy with classical probability, one proves free versions of the most famous limit theorems : weak law of large numbers [BP96], central limit theorem ([Voi86],[BV95],[Kar07]) or Poisson theorem. We state here a version of the free central limit theorem :

Theorem 2.2.4. *Let $(a_n)_{n \in \mathbb{N}^*}$ be a sequence of free noncommutative random variables in a noncommutative probability space $(\mathcal{A}, \varphi)$, satisfying :*

$$\forall n \in N^*, \varphi(a_n) = 0,$$

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(a_i^2) = \sigma^2 > 0,$$

$$\sup_{i \in \mathbb{N}^*} \varphi(a_i^k) < \infty.$$

Then

$$\mu_{\frac{1}{\sqrt{n}} \sum_{i=1}^n a_i} \underset{n \to +\infty}{\Rightarrow} \mu_\sigma.$$

It is remarkable that the limiting distribution in the free central limit theorem is the semicircle distribution, exactly as in Wigner's Theorem. This distribution is therefore the free analogue of the Gaussian distribution ; one sometimes calls it the *free Gaussian distribution*. The *free Poisson distributions* appear as the limits in the free version of the free Poisson theorem:

Theorem 2.2.5. *Let $\gamma \geq 0$, $\sigma > 0$, then*

$$((1 - \frac{\gamma}{n})\delta_0 + \frac{\gamma}{n}\delta_{\sigma^2})^{\boxplus n} \underset{n \to +\infty}{\Rightarrow} \pi_{\gamma,\sigma}.$$

The free Poisson distributions coincide with the Marchenko-Pastur distributions. Free Poisson and free Gaussian distributions are particular cases of $\boxplus$-infinitely divisible distributions, whose definition is the following rewriting of the classical definition :

Definition 2.2.6. A probability measure μ on $\mathbb{R}$ is said $\boxplus$-infinitely divisible if, for each $n \in \mathbb{N}^*$, there is a probability measure μ_n on $\mathbb{R}$ such that :

$$\mu = \mu_n^{\boxplus n}.$$

Infinitely divisible distributions are important for two reasons : on the one hand, they are the distributions μ generating a free additive convolution semigroup $(\mu_t)_{t\geq 0}$ (satisfying $\mu_0 = \delta_0$, $\mu_1 = \mu$, $\mu_s \boxplus \mu_t = \mu_{s+t}, \forall s, t \geq 0$ and $t \longrightarrow \mu_t$ is continuous for the weak-$*$ topology) allowing to define a process with free and stationnary increments. We denote then $\mu_t = \mu^{\boxplus t}$. Notice that, contrary to the classical case, for each probability measure μ, one may define a partial free additive convolution semigroup $(\mu_t)_{t\geq 1}$ ([BV95],[NS96]). On the other hand, infinitely divisible distributions are the possible limits in free convolution limit theorems [BP00].
The R-transform of a free additive infinitely divisible distribution has a representation, analogous to the Lévy-Hincin representation of the log-Fourier transform of classical infinitely divisible distributions [BV93] : there is a nonnegative real number $\beta \geq 0$ and a finite positive measure σ on $\mathbb{R}$ such that

$$\forall z \in \mathbb{C}^-, \; R_\mu(z) = \beta + \int_{\mathbb{R}} \frac{z+t}{1-tz} d\sigma(t).$$

The analogy is deeper, since Bercovici et Pata built a bijection between classical and free infinitely divisible distributions, with the nice property to conserve the partial domains of attraction [BP99].
The study of limit theorems, initiated by Bercovici and Pata, was generalized to infinitesimal triangular arrays by Chistyakov and Götze [CG08a] and Bercovici and Wang [BW08a].

2.2.2 Free multiplicative convolution

Contrary to the classical case, in which the exp function allows to deduce results on multiplicative convolution of probability measures from results on additive convolution, free multiplication requires a special study.
Free multiplicative convolution, denoted by $\boxtimes$, is an operation defined on two sets of probability measures : probability measures on the unit circle $\mathbb{T}$ and compactly supported probability measures on $\mathbb{R}^+$ [Voi87].
Probability measures on $\mathbb{T}$ correspond to unitary noncommutative random variables, that is satisfying $a^*a = aa^* = 1_{\mathcal{A}}$. The free multiplicative convolution associates to a pair of probability measures μ and ν on $\mathbb{T}$ the distribution $\mu \boxtimes \nu$ of ab, where a and b are free unitary noncommutative random variables

of a W^*-probability space $(\mathcal{A}, \varphi)$ with respective distributions μ and ν.
Compactly supported probability measures in $\mathbb{R}^+$ correspond to noncommutative random variables that one can write as x^*x, for an $x \in \mathcal{A}$: they are said positive. Free multiplicative convolution associates to a pair of compactly probability measures μ and ν on $\mathbb{R}^+$ the distribution $\mu \boxtimes \nu$ of $a^{\frac{1}{2}}ba^{\frac{1}{2}}$, where a and b are free positive noncommutative random variables of a W^*-probability space with distributions μ and ν. As in the additive case, free multiplicative convolution was extended to an operation on general probability measures on $\mathbb{R}^+$ [BV93]. We choose to detail the study of this extended operation.
The role of Mellin transform in classical probability is played in free probability by the S-transform [Voi87],[BV93] :

Definition 2.2.7. Let $\mu \neq \delta_0$ be a probability measure on $\mathbb{R}^+$, the function

$$\psi_\mu(z) = \int_{\mathbb{C}} \frac{zt}{1 - zt} d\mu(z)$$

is invertible on the half-plane $i\mathbb{C}^+$, with inverse χ_μ. We call S-transform of μ the function S_μ defined by :

$$S_\mu(z) = \frac{z+1}{z} \chi_\mu(z).$$

Theorem 2.2.8. *Let $\mu \neq \delta_0$ and $\nu \neq \delta_0$ be two probability measures on $\mathbb{R}^+$, then there exists a domain of the complex plane where*

$$S_{\mu \boxtimes \nu} = S_\mu S_\nu.$$

The difficulties set by centered measures are discussed in [RS07] and [AEPA09]. Free multiplicative convolution also presents a subordination phenomenon ([Bia98],[BB07]), allowing to deduce regularity results on the free multiplicative convolution of measures ([Bel03],[BB05]) :

Theorem 2.2.9. *Let $\mu \neq \delta_0$ and $\nu \neq \delta_0$ be two probability measures on $\mathbb{R}^+$, there exists an analytic map $\omega : \mathbb{C}^+ \to \mathbb{C}^+$ such that*

$$\forall z \in \mathbb{C}^+, \ \psi_{\mu \boxtimes \nu}(z) = \psi_\mu(\omega(z))$$

and

$$\forall z \in \mathbb{C}^+, \ \arg(\omega(z)) \geq \arg(z).$$

One defines $\boxtimes$-infinitely divisible distributions :

Definition 2.2.10. A probability measure μ on $\mathbb{R}^+$ is said $\boxtimes$-infinitely divisible if, for each $n \in \mathbb{N}^*$, there exists a probability measure μ_n on $\mathbb{R}^+$ such that :

$$\mu = \mu_n^{\boxtimes n}.$$

According to [BV93], a probability measure μ on $\mathbb{R}^+$ is $\boxtimes$-infinitely divisible if and only if $\mu = \delta_0$ or there is a real number $a \in \mathbb{R}$, a nonnegative number $b \geq 0$ and a positive measure σ on $\mathbb{R}$ such that

$$\forall z \in \mathbb{C}^+,\ \Sigma_\mu(z) := S_\mu(\frac{z}{1-z}) = \exp(a - bz + \int_0^{+\infty} \frac{1+tz}{z-t} d\sigma(t)).$$

The study of multiplicative limit theorems, initiated by Bercovici and Pata [BP00], was generalized to infinitesimal triangular arrays by Bercovici and Wang [BW08b] and Chistyakov and Götze [CG08b].

2.3 Combinatorial approach

For the combinatorial approach of free probability, the framework is a noncommutative probability space $(\mathcal{A}, \varphi)$ in which the distribution of a noncommutative random variable is defined as the linear functional μ_a on $\mathbb{C}[X]$ satisfying :

$$\forall P \in \mathbb{C}[X],\ \mu_a(P) := \varphi(P(a)).$$

A benefit of this approach is to take into account n-tuples of noncomutative random variables. The distribution of a n-tuple of noncommutative random variables $(a_1, \ldots, a_n) \in \mathcal{A}^n$ is simply the linear functional on the space of noncommutative polynomials in n indeterminates $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ defined by:

$$\forall P \in \mathbb{C}\langle X_1, \ldots, X_n\rangle, \mu_{(a_1,\ldots,a_n)}(P(X_1, \ldots, X_n)) := \varphi(P(a_1, \ldots, a_n)).$$

We have an algebraic free product construction, allowing to define operations of free additive and multiplicative convolution on the set of linear functionals on the space of noncommutative polynomials in n indeterminates $\mathbb{C}\langle X_1, \ldots, X_n\rangle$.

2.3.1 Non-crossing partitions

There is a combinatorial approach to classical probability, due to Rota, with framework the lattices of partitions of finite sets.
A *partition* of a finite set E is a family $p = \{V_1, \ldots, V_r\}$ of non-empty and disjoint subsets of E such that

$$\bigcup_{i=1}^{r} V_i = E.$$

The subsets $V_1, \ldots, V_r$ are called the *blocks* of p ; the set of blocks of p is denoted by $\mathrm{bl}(p)$, its cardinal by $|p|$.
Two elements i, j of E belonging to the same block of a partition p are said *linked* in p and we denote this by $i \sim_p j$.
On the set Π_E of all partitions of E, one defines a partial order $\leq$, the *reverse refinement order*, by :

$$p \leq q \text{ if and only if any block of } p \text{ is included in a block of } q.$$

In the partially ordered set $(\Pi_E, \leq)$, any pair (p,q) of partitions has a least upper bound $p \vee_{\Pi_E} q$ and a greatest lower bound $p \wedge_{\Pi_E} q$: $(\Pi_E, \leq)$ is a *lattice*.
The combinatorics of free probability is governed by non-crossing partitions. Non-crossing partitions, whose study was initiated by Kreweras [Kre72], are partitions of a totally ordered set $(E, \leq)$ without crossing. More precisely, for a non-crossing partition p of $(E, \leq)$, whenever $i < j < k < l$ with $i \sim_p k$ and $j \sim_p l$, one necessarily has $i \sim_p j$.
A block W of a non-crossing partition p of $(E, \leq)$ is called an *interval-block* if it is of the form $W = \{x \in E : a \leq x \leq b\}$, for $a \leq b$ in E. Any non-crossing partition has an interval-block, and it is easy to check the stronger following statement : let p be a non-crossing partition of $(E, \leq)$, let V be a block of p, and let a and b be consecutive in V (meaning $(i,j) \cap V \neq \emptyset$). If the interval (i,j) is itself non-empty, then there exists a block-interval W in p such that $W \subseteq (i,j)$.
The set $(NC^{(A)}(E), \leq)$ of non-crossing partitions of $(E, \leq)$ equipped with the reverse refinement order is a lattice. Its maximal element 1_E has E for only block ; its minimal element 0_E has each singleton of E as a block. When $E = [m] := \{1 < \ldots < m\}$, we write $NC^{(A)}(m)$ instead of $NC^{(A)}([m])$.
It is nice to represent a non-crossing partition $p \in NC^{(A)}(m)$ in the following geometric way : represent $1, \ldots, m$ by equidistants points clockwisely ordered on a circle, and draw, for each block of p, the convex polygone whose vertices are the elements of the block. It is necessary and sufficient for a partition to be non-crossing that the polygones built this way do not intersect.
The lattice of non-crossing partitions has two remarkable properties : it is selfdual and its intervals have a canonical factorization.
The selfduality is a consequence of the existence of an anti-automorphism discovered by Kreweras [Kre72], the *Kreweras complementation map*, denoted by Kr, and defined as follows : given a non-crossing partition p of $[m]$, $\mathrm{Kr}(p)$ is the biggest partition (for the reverse refinement order) of

$$\overline{[m]} := \{\overline{1} < \ldots < \overline{m}\}$$

such that $p \cup \mathrm{Kr}(p)$ is a non-crossing partition of

$$[m] \cup \overline{[m]} = \{1 < \overline{1} < \ldots < m < \overline{m}\}.$$

There is an elegant geometric construction of the Kreweras complement of a non-crossing partition. The number of blocks of the Kreweras complement of a non-crossing partition is related to the number of blocks of this partition by the formula :

$$|p| + |\mathrm{Kr}(p)| = m + 1, \ \forall p \in NC^{(A)}(m). \tag{2.1}$$

Notice also that, for $p \in NC^{(A)}(m)$, the description of $\mathrm{Kr}^2(p)$ is easy in the geometric representation above : $\mathrm{Kr}^2(p)$ is obtained from p by a counterclockwise rotation of angle $\frac{2\pi}{m}$.
In addition to selfduality, the lattice $(NC^{(A)}(m), \leq)$ has another important property : its intervals, that is the sets of the form

$$[p,q] := \{\pi \in NC^{(A)}(m) \mid p \leq \pi \leq q\},$$

have a *canonical factorization* :

Theorem 2.3.1. *[Spe94] For $p, q \in NC^{(A)}(m)$ satisfying $p \leq q$, there is a sequence of integers $(k_1, \ldots, k_m) \in \mathbb{N}^m$ such that*

$$[p,q] \cong NC^{(A)}(1)^{k_1} \times \cdots \times NC^{(A)}(m)^{k_m}.$$

As a finite partially ordered set, $(NC^{(A)}(m), \leq)$ is the frame of a convolution operation and of a Moebius inversion formula :

Proposition 2.3.2. *There is a function*

$$M\ddot{o}b^{(A)} : \{(p,q) \in NC^{(A)}(m) \mid p \leq q\} \longrightarrow \mathbb{C},$$

such that, for $f, g : NC^{(A)}(m) \longrightarrow \mathbb{C}$, the following formulas are equivalent:

$$\forall p \in NC^{(A)}(m),\ f(p) = \sum_{q \leq p} g(q). \tag{2.2}$$

$$\forall p \in NC^{(A)}(m),\ g(p) = \sum_{q \leq p} f(q) M\ddot{o}b^{(A)}(q,p). \tag{2.3}$$

$M\ddot{o}b^{(A)}$ is called the Moebius function *of $NC^{(A)}(m)$.*

2.3.2 Non-crossing cumulant functionals

We adopt the following notation :

Notation 2.3.3. Let $\mathcal{A}$ be a unital complex algebra, let $(a_1, \ldots, a_n) \in \mathcal{A}^n$, and let $V = \{v_1 < \ldots < v_m\} \subseteq [n]$, then we denote

$$(a_1, \ldots, a_n) \mid V := (a_{v_1}, \ldots, a_{v_m}) \in \mathcal{A}^m.$$

For a family of multilinear maps $(r_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$, one defines, for each $n \in \mathbb{N}$ and each $\pi \in NC^{(A)}(n)$, the n-linear functional $r_\pi : \mathcal{A}^n \to \mathcal{C}_k$ by

$$r_\pi(a_1, \ldots, a_n) := \prod_{V \in \pi} r_{|V|}((a_1, \ldots, a_n) \mid V).$$

A fundamental object in the combinatorial approach of free probability is the family of non-crossing cumulant functionals, defined in [Spe94] by the *free moment-cumulant formula* :

Definition 2.3.4. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, the *non-crossing cumulant functionals* are the family of multilinear functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$, uniquely determined by : for each $n \geq 1$ and each $a_1, \ldots, a_n \in \mathcal{A}$,

$$\sum_{p \in NC^{(A)}(n)} \kappa_p(a_1, \ldots, a_n) = \varphi(a_1 \cdots a_n). \tag{2.4}$$

The formula (2.4) has an inverse by Moebius inversion :

$$\kappa_n(a_1, \ldots, a_n) = \sum_{p \in NC^{(A)}(n)} \text{Möb}^{(A)}(p, 1_n) \varphi_p(a_1, \ldots, a_n), \tag{2.5}$$

generalizing simply by

$$\kappa_\rho = \sum_{\pi \in NC(n),\ \pi \leq \rho} \text{Möb}^{(A)}(\pi, \rho) \cdot \varphi_\pi^{(A)}, \quad \forall \rho \in NC(n). \tag{2.6}$$

The multilinear functionals $(\varphi_n : \mathcal{A}^n \to \mathcal{C}_k)_{n=1}^{\infty}$ implicitely used in formula (2.5) are obviously defined by

$$\varphi_n(a_1, \ldots, a_n) = \varphi(a_1 \cdots a_n).$$

The non-crossing cumulant functionals have the following properties :

Proposition 2.3.5. *One has $\kappa_n(a_1, \ldots, a_n) = 0$ whenever $n \geq 2$, $a_1, \ldots, a_n \in \mathcal{A}$, and there is $1 \leq i \leq n$ such that $a_i \in \mathbb{C}1_{\mathcal{A}}$.*

Proposition 2.3.6. *[KS00] Let $x_1, \ldots, x_s$ be in $\mathcal{A}$, and consider products of the form*

$$a_1 = x_1 \cdots x_{s_1},\ a_2 = x_{s_1+1} \cdots x_{s_2},\ \ldots,\ a_n = x_{s_{n-1}+1} \cdots x_{s_n},$$

where $1 \leq s_1 < s_2 < \cdots < s_n = s$. Then

$$\kappa_n(a_1, \ldots, a_n) = \sum_{\substack{\pi \in NC(s) \text{ such} \\ \text{that } \pi \vee \theta = 1_s}} \kappa_\pi(x_1, \ldots, x_s),$$

where $\theta \in NC(s)$ is the partition :

$$\theta = \{\{1, \ldots, s_1\}, \{s_1 + 1, \ldots, s_2\}, \ldots, \{s_{n-1} + 1, \ldots, s_n\}\}.$$

The importance of non-crossing cumulant functionals in free probability comes from the following result, due to Speicher :

Theorem 2.3.7. *[Spe94] Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be subsets of $\mathcal{A}$. The following statements are equivalent:*
(1) $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are free.
(2) For each $m \geq 2$, each $i_1, \ldots, i_m \in \{1, \ldots, n\}$ not all equal, and each $a_1 \in \mathcal{M}_{i_1}, \ldots, a_m \in \mathcal{M}_{i_m}$, one has $\kappa_m(a_1, \ldots, a_n) = 0$. One says that $\mathcal{M}_1, \ldots, \mathcal{M}_n$ have vanishing mixed cumulants.

An immediate corollary of this result is the simple formula describing the addition of two free n-tuples of noncommutative random variables :

Proposition 2.3.8. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and $\mathcal{M}_1, \mathcal{M}_2$ be free subsets of $\mathcal{A}$. Then, for each $n \geq 1$, and each n-tuples $(a_1, \ldots, a_n) \in \mathcal{M}_1^n, (b_1, \ldots, b_n) \in \mathcal{M}_2^n$, one has :*

$$\kappa_n(a_1 + b_1, \ldots, a_n + b_n) = \kappa_n(a_1, \ldots, a_n) + \kappa_n(b_1, \ldots, b_n). \tag{2.7}$$

More surprisingly, non-crossing cumulants describe the multiplication of free n-tuples :

Proposition 2.3.9. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $\mathcal{M}_1, \mathcal{M}_2$ be free subsets of $\mathcal{A}$. Then, for each $n \geq 1$ and each n-tuples $(a_1, \ldots, a_n) \in \mathcal{M}_1^n, (b_1, \ldots, b_n) \in \mathcal{M}_2^n$, one has :*

$$\kappa_n(a_1 b_1, \ldots, a_n b_n) = \sum_{p \in NC^{(A)}(n)} \kappa_p(a_1, \ldots, a_n) \kappa_{Kr(p)}(b_1, \ldots, b_n). \tag{2.8}$$

These two results may be written in a more compact form by the use of formal series. For simplicity, we restrict ourselves to the case of a single variable, and we will use the following notation :

Definition 2.3.10. Let $\mathcal{C}$ be a unital complex algebra, we denote by $\Theta_{\mathcal{C}}^{(A)}$ the set of formal power series of the form

$$f(z) = \sum_{n=1}^{\infty} \alpha_n z^n,$$

where the α_n's are elements of $\mathcal{C}$.

An important example of formal power series of $\Theta_{\mathcal{C}}^{(A)}$, is the series :

$$\zeta_{\mathcal{C}}^{(A)}(z) := \sum_{n=1}^{\infty} 1_{\mathcal{C}} z^n.$$

An operation on formal power series with complex coefficients is introduced in [NS96], as a convolution on the lattices of non-crossing partitions. Here is the definition, in the case of a single variable, but with coefficients in any commutative complex unital algebra.

Definition 2.3.11. On $\Theta_{\mathcal{C}}^{(A)}$, one defines a binary associative and commutative operation $\boxed{\star}_{\mathcal{C}}^{(A)}$, as follows : if

$$f(z) = \sum_{n=1}^{\infty} \alpha_n z^n \in \Theta_{\mathcal{C}}^{(A)}$$

and

$$g(z) = \sum_{n=1}^{\infty} \beta_n z^n \in \Theta_{\mathcal{C}}^{(A)},$$

then $f\boxed{\star}_{\mathcal{C}}^{(A)}g$ is the formal power series $\sum_{n=1}^{\infty} \gamma_n z^n$, where

$$\gamma_m = \sum_{\substack{p \in NC^{(A)}(m) \\ p:=\{E_1,\ldots,E_h\} \\ \mathrm{Kr}(p):=\{F_1,\ldots,F_l\}}} (\prod_{i=1}^{h} \alpha_{\mathrm{card}(E_i)}) \cdot (\prod_{j=1}^{l} \beta_{\mathrm{card}(F_j)}).$$

A formal power series $f \in \Theta_{\mathcal{C}}^{(A)}$ is invertible with respect to $\boxed{\star}_{\mathcal{C}}^{(A)}$ if and only if its coefficient of degree 1 is itself invertible in the algebra $\mathcal{C}$. In particular, $\zeta_{\mathcal{C}}^{(A)}$ is invertible with respect to $\boxed{\star}_{\mathcal{C}}^{(A)}$, and we call its inverse the Moebius series, denoted by $\text{Möb}_{\mathcal{C}}^{(A)}$. The proofs of these statements are direct adaptations of the proofs one can find in [NS96].

Definition 2.3.12. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $a \in \mathcal{A}$ be a noncommutative random variable. We call moments generating series of a the formal power series :

$$M_a(X) = \sum_{n=1}^{\infty} \varphi(a^n) X^n \in \Theta_{\mathbb{C}}^{(A)}, \tag{2.9}$$

and R-transform of a the formal power series :

$$R_a(X) = \sum_{n=1}^{\infty} \kappa_n(a, \ldots, a) X^n \in \Theta_{\mathbb{C}}^{(A)}. \tag{2.10}$$

The free moment-cumulant formula (2.4) and its inverse (2.5) read at the level of formal power series :

Proposition 2.3.13. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $a \in \mathcal{A}$ be a noncommutative random variable. Then, one has :*

$$M_a = R_a \boxed{\star}_{\mathbb{C}}^{(A)} \zeta_{\mathbb{C}}^{(A)}$$

$$R_a = M_a \boxed{\star}_{\mathbb{C}}^{(A)} M\ddot{o}b_{\mathbb{C}}^{(A)}$$

The importance of the operation $\boxed{\star}_{\mathbb{C}}^{(A)}$ in free probability comes from the fact, proved in [NS96], that $\boxed{\star}_{\mathbb{C}}^{(A)}$ furnishes the combinatorial description of the multiplication of two free noncommutative random variables, in terms of their R-transforms.

Proposition 2.3.14. *[NS96] Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let a, b be free noncommutative random variables in $\mathcal{A}$. Then, one has*

$$R_{a+b} = R_a + R_b$$

and

$$R_{ab} = R_a \boxed{\star}_{\mathbb{C}}^{(A)} R_b.$$

Although the proposition above generalizes without difficulty to free n-tuples of noncommutative random variables, the single variable case has some special features. One may for example define a Fourier transform for multiplicative functions on non-crossing partitions [NS97] : this is a map $\mathcal{F}$ associating to a formal power series $f(X)$ in one indeterminate and with nonzero coefficient of degree 1 (to ensure invertibility with respect to composition of maps) the formal power series $\mathcal{F}(f)(X) := \frac{1}{X} f^{\langle -1 \rangle}(X)$. The map $\mathcal{F}$ has the important property to transform the operation $\boxed{\star}$ into the multiplication of formal power series : $\mathcal{F}(f \boxed{\star}^{(A)} g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$. If $a \in (\mathcal{A}, \varphi)$ is a non-centered noncommutative random variable in a noncommutative probability space, the series $\mathcal{F}(R_a)$ is the combinatorial approach to the S-transform.

Definition 2.3.15. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, the S-transform of a non-centered noncommutative random variable $a \in \mathcal{A}$ is the formal power series S_a defined by :

$$S_a(X) := \frac{1}{X} R_a^{\langle -1 \rangle}(X).$$

Proposition 2.3.16. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $a, b \in \mathcal{A}$ be two non-centered noncommutative random variables, then one has :*

$$S_{ab}(X) = S_a(X) S_b(X).$$

The R- and S-transforms of a compactly supported probability measure μ are analytic in a neighborhood of 0 and one may thus identify their power series developments to the formal power series given by the R- and S-transforms of a noncommutative random variable in a W^*-probability space having distribution μ. This is equivalent to identifying a compactly supported probability measure on $\mathbb{R}$ to the sequence of its moments : this is a simple particular case of the famous *moments problem.*

2.4 Relation to Random Matrix Theory

In the first chapter, we studied random matrices from the point of view of classical probability, as random variables with values in a matrix space, in general noncommutative. We may also take the point of view of noncommutative probability.
We have to restrict ourselves to the algebra $\mathcal{M}_N(L^{\infty-})$ of matrix models whose entries are random variables with finite moments. On this algebra, the expectation of the normalized trace

$$\varphi(M) = \mathbb{E}[\mathrm{tr}(M)]$$

is a linear normalized (and tracial) functional : we get a noncommutative probability space.
The fundamental fact, discovered by Voiculescu in 1991 [Voi91], is that freeness appears as the asymptotic relative position in this space of certain independent matrix models, when their size goes to infinity : this is *asymptotic freeness.*

Definition 2.4.1. Given a sequence of noncommutative probability spaces $(\mathcal{A}_n, \varphi_n)_{n\in\mathbb{N}}$, a sequence of families of noncommutative random variables $((a_i^{(n)})_{i\in I})_{n\in\mathbb{N}}$ converges in distribution if there exists a linear functional μ on $\mathbb{C}\langle X_i, i \in I\rangle$ such that

$$\forall P \in \mathbb{C}\langle X_i, i \in I\rangle,\ \varphi_n(P((a_i^{(n)})_{i\in I})) \underset{n\to+\infty}{\longrightarrow} \mu(P).$$

The distribution μ is called the limit distribution.

Definition 2.4.2. In the preceding notations, if

$$I = \underset{j\in J}{\cup} I_j$$

is a partition of I, the sequence of families $(((a_i^{(n)})_{i\in I_j})_{j\in J})_{n\in\mathbb{N}}$ is asymptotically free as n goes to infinity if it converges in distribution towards μ and if in addition the families $((X_i)_{i\in I_j}), j \in J$ are free in the noncommutative probability space $(\mathbb{C}\langle X_i, i \in I\rangle, \mu)$.

In the results of asymptotic freeness for random matrices we are going to state, the sequence of noncommutative probability spaces is

$$(\mathcal{M}_N(L^{\infty-}), \mathbb{E}[\mathrm{tr}(\cdot)]).$$

In this particular case, one can sometimes prove stronger asymptotic freeness results, for instance show that the convergence of the random variables $\mathrm{tr}(P((X_i^{(n)})_{i\in I}))$ holds not only in mean, but also in probability or almost surely.

Asymptotic freeness for Wigner matrices was examined by Voiculescu (see [Voi91], [Voi98]), Dykema ([Dyk93]), Capitaine and Casalis ([CC04]), Thorbjornsen ([Tho00]) and Hiai and Petz ([HP00]). We record the following theorem, stating that independent Wigner matrices and a family of deterministic Hermitian matrices are asymptotically free.

Theorem 2.4.3. *Let $(W_N(i))_{i\in I}$ be a family of independent real or complex Wigner matrices of size N and associated to a centered distribution of $L^{\infty-}$ and of variance 1, and let $(D_N(j))_{j\in J}$ be a family of deterministic Hermitian matrices converging in distribution and satisfying*

$$\sup_{k\in\mathbb{N}^*}\max_{j\in J}\sup_{N\in\mathbb{N}^*}\frac{1}{N}\mathrm{tr}(|D_N(j)|^k)^{\frac{1}{k}} < +\infty.$$

Then, the sequence of families $\{(D_N(j))_{j\in J}\}, \{\frac{W_N(i)}{\sqrt{N}}\}_{i\in I}$ is asymptotically free.

This result may be considered as a multi-matricial version of Wigner's Theorem 1.2.3, and sheds light on the ubiquity of the semicircle distribution as limiting distribution in the global behavior of Wigner matrices and in the free central limit theorem. As a corollary, one obtains a description of the limiting distribution in the global behavior of the deformed Wigner model. Such asymptotic freeness results are proved for other matrix models : independent Wishart matrices ([CC04]), independent Haar unitary or orthogonal matrices ([Voi91], [Voi98], [Col03]), independent permutation matrices, etc... These results may sometimes be precised by results of almost sure convergence of operator norms :

$$\forall P \in \mathbb{C}\langle X_i, i \in I\rangle,\ \left\|P((X_i^{(n)})_{i\in I})\right\| \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} \|P((x_i)_{i\in I})\|,$$

where $(x_i)_{i\in I}$ is a family of free noncommutative random variables with distribution μ in a W^*-probability space. From such results ([HT05], [Sch05], [CDM07], [Mal10]), one may deduce, from the point of view of random matrices, the almost sure convergence of the largest or smallest eigenvalue of deformed models towards an endpoint of the support of the limiting distribution.

The discovery of the relation between Random Matrix Theory and free probability gave birth to a new approach of random matrices, as middle ground between classical probability, whose object is independent scalar random variables, and free probability, whose object is free noncommutative random variables, realized as operators acting on a Hilbert space of infinite dimension. Let us mention that matricial realizations of the Bercovici-Pata bijections were built in [BG05] and a theory of matricial cumulants was initiated in [CC06],[CC07].

2.5 Free probability of type B

The lattices of partitions of a finite set may be reinterpreted as the intersection lattice for the hyperplane arrangement corresponding to the type A root system [Rei97]. The non-crossing partitions may be seen in particular as combinatorial objects of type A, justifying the notation $NC^{(A)}(E)$.
This inscription of the lattice of non-crossing partitions in combinatorial objects of type A was clarified by Biane, who gave in [Bia97b] a bijection t between $NC^{(A)}(m)$ and the set of points lying on a geodesic in the Cayley graph of the symmetric group S_m, when the set of generators is chosen to contain all the transpositions. This bijection associates to a non-crossing partition $p \in NC^{(A)}(m)$ the permutation $t(p) \in S_m$ whose restriction to each block V of p is the trace of the cycle $(1, \ldots, m) \in S_m$ on V. When $a \in [m]$, $t(p)(a)$ will be called the neighbour of a in p. Geometrically, this is the first point linked with a that one meets when one clockwisely describes the circle, starting from a.
There are other types of root systems, such as the type B, for which Reiner built a lattice of non-crossing partitions [Rei97]. To describe the lattice $NC^{(B)}(n)$ of non-crossing partitions of type B, we consider the totally ordered set

$$[\pm n] = \{1 < 2 < \ldots < n < -1 < -2 < \ldots < -n\}.$$

One defines $NC^{(B)}(n)$ as the subset of $NC^{(A)}([\pm n])$ consisting of non-crossing partitions that are invariant under the action of the inversion map $x \mapsto -x$.
In such a partition $\pi \in NC^{(B)}(n)$, there is at most one inversion-invariant block, that we call, whenever it exists, the *zero-block* of π. The other blocks of π come two by two : if F is a block which is not inversion-invariant, then $-F$ is another block (obviously not inversion-invariant). Let us give a special notation to the subset of non-crossing partition having a zero-block :

$$NCZ^{(B)}(n) := \{\pi \in NC^{(B)}(n) \mid \pi \text{ with a zero-block}\}, \tag{2.11}$$

Immediately, $NC^{(B)}(n)$ is a sublattice of $(NC^{(A)}([\pm n]), \leq)$, with the same minimal and maximal elements.
Moreover, $NC^{(B)}(n)$ is stable by Kreweras complementation map (considered on $NC^{(A)}([\pm n])$), which, restricted from $NC^{(A)}([\pm n])$ to $NC^{(B)}(n)$, will give an anti-automorphism of $NC^{(B)}(n)$, still named Kreweras complementation map (on $NC^{(B)}(n)$) and denoted by Kr. In this case, the important relation (2.1) becomes

$$|\pi| + |\mathrm{Kr}(\pi)| = 2n + 1, \forall \pi \in NC^{(B)}(n).$$

As a consequence of this formula, for $\pi \in NC^{(B)}(n)$, exactly one of the two partitions π and $\mathrm{Kr}(\pi)$ has a zero-block. Moreover, the Kreweras complementation map is a bijection between $NCZ^{(B)}(n)$ and its complementary

$NC^{(B)}(n) \setminus NCZ^{(B)}(n)$.
In the description of a non-crossing partition of type B, one uses the map Abs : $[\pm n] \longrightarrow [n]$ sending $\pm i$ on i. One has then the following result :

Theorem 2.5.1. *[BGN03] The map $\pi \mapsto Abs(\pi)$ is a $(n+1)$-to-1 cover from $NC^{(B)}(n)$ onto $NC^{(A)}(n)$.*

More precisely, a non-crossing partition of type B π is characterized by its image p by the map Abs, which is a non-crossing partition of $NC^{(A)}(n)$, and the choice of the block of $p \cup \mathrm{Kr}(p)$ (there are indeed $n+1$ choices for this, due to relation (2.1)) corresponding to the unique zero-block of $\pi \cup \mathrm{Kr}(\pi)$.
The lattices $NC^{(B)}(m)$ are the frame of a Moebius inversion ; the Moebius functions of type B are denoted by $\mathrm{Möb}^{(B)}$, and are related to $\mathrm{Möb}^{(A)}$ in the following way :

$$\Big(\sigma \leq \tau \text{ in } NCZ^{(B)}(n)\Big) \;\Rightarrow\; \mathrm{Möb}^{(B)}(\sigma,\tau) = \mathrm{Möb}^{(A)}\big(\mathrm{Abs}(\sigma), \mathrm{Abs}(\tau)\big). \tag{2.12}$$

To prove this, observe that Abs is an isomorphism of partially ordered sets between the interval $[\sigma,\tau] \subseteq NC^{(B)}(n)$ and $[\mathrm{Abs}(\sigma), \mathrm{Abs}(\tau)] \subseteq NC(n)$, and use the fact that the values of $\mathrm{Möb}^{(B)}(\sigma,\tau)$ and $\mathrm{Möb}^{(A)}\big(\mathrm{Abs}(\sigma), \mathrm{Abs}(\tau)\big)$ only depend on the isomorphism classes of these intervals.
The lattice $NC^{(B)}(m)$ of non-crossing partitions of type B may also be obtained as the image by Biane's bijection of the set of points lying on a geodesic in the Cayley graph of the hyperoctaedral group W_m for a relevant choice of generating set [BGN03].
One may construct a combinatorial theory of free probability of type B by replacing the occurences of the symmetrics groups and of the lattices of non-crossing partitions of type A by their type B analogues, the hyperoctaedral groups and the non-crossing partitions of type B [BGN03]. A central role is played by the type B analogue $\boxed{\star}^{(B)}$ of the operation $\boxed{\star}_{\mathbb{C}}^{(A)}$.

Definition 2.5.2. [BGN03]

1. Denote by $\Theta^{(B)}$ the set of formal power series

$$f(z) = \sum_{n=1}^{\infty} (\alpha'_n, \alpha''_n) z^n,$$

where the α'_n's and α''_n's are complex numbers.

2. Let $f(z) := \sum_{n=1}^{\infty} (\alpha'_n, \alpha''_n) z^n$ and $g(z) = \sum_{n=1}^{\infty} (\beta'_n, \beta''_n) z^n$ be in $\Theta^{(B)}$. For each $m \geq 1$, consider the scalars γ'_m and γ''_m defined by

$$\gamma'_m = \sum_{\substack{p \in NC^{(A)}(m) \\ p := \{E_1,\dots,E_h\} \\ \mathrm{Kr}(p) := \{F_1,\dots,F_l\}}} \Big(\prod_{i=1}^{h} \alpha'_{\mathrm{card}(E_i)}\Big)\Big(\prod_{j=1}^{l} \beta'_{\mathrm{card}(F_j)}\Big),$$

$$\gamma''_m = \sum_{\substack{p\in NC^{(B)}(m)\,with\,zero-block\\ p:=\{Z,X_1,-X_1,\ldots,X_h,-X_h\}\\ \mathrm{Kr}(p):=\{Y_1,-Y_1,\ldots,Y_l,-Y_l\}}} (\prod_{i=1}^{h}\alpha'_{\mathrm{card}(X_i)})\alpha''_{\mathrm{card}(Z)/2}(\prod_{j=1}^{l}\beta'_{\mathrm{card}(Y_j)})$$

$$+ \sum_{\substack{p\in NC^{(B)}(m)\,without\,zero-block\\ p:=\{X_1,-X_1,\ldots,X_h,-X_h\}\\ \mathrm{Kr}(p):=\{Z,Y_1,-Y_1,\ldots,Y_l,-Y_l\}}} (\prod_{i=1}^{h}\alpha'_{\mathrm{card}(X_i)})\beta''_{\mathrm{card}(Z)/2}(\prod_{j=1}^{l}\beta'_{\mathrm{card}(Y_j)}).$$

Then the series $\sum_{n=1}^{\infty}(\gamma'_n,\gamma''_n)z^n$ is called the type B convolution of f and g, and denoted by $f \boxed{\star}^{(B)} g$.

As noticed in [BGN03], the type B convolution operation coincides with a type A convolution operation on a certain algebra $\mathbb{G}$.

Theorem 2.5.3. $\boxed{\star}^{(B)} = \boxed{\star}^{(A)}_{\mathbb{G}}$*, where* $\mathbb{G}$ *is the two-dimensional Grassman algebra generated by an element* ε *satisfying* $\varepsilon^2 = 0$*.*
Thus $\mathbb{G}$ *is the extension of* $\mathbb{C}$ *defined by*

$$\mathbb{G} = \{\alpha + \varepsilon\beta \mid \alpha, \beta \in \mathbb{C}\}, \tag{2.13}$$

with the following multiplication rule :

$$(\alpha_1 + \varepsilon\beta_1)\cdot(\alpha_2 + \varepsilon\beta_2) = \alpha_1\alpha_2 + \varepsilon(\alpha_1\beta_2 + \beta_1\alpha_2).$$

The specificity of the operation $\boxed{\star}^{(B)}$ leads naturally to define a *type B noncommutative probability space* as a system $(\mathcal{A}, \varphi, \mathcal{V}, f, \Phi)$, where

- $(\mathcal{A}, \varphi)$ is a usual noncommutative probability space,
- $\mathcal{V}$ is a complex vector space,
- $f : \mathcal{V} \longrightarrow \mathbb{C}$ is a linear functional,
- $\Phi : \mathcal{A} \times \mathcal{V} \times \mathcal{A} \longrightarrow \mathcal{V}$ is an action of $\mathcal{A}$ on $\mathcal{V}$.

Practically, one works in the *link-algebra* $\mathcal{A} \times \mathcal{V}$, equipped with its natural vector space structure and with the multiplication given by :

$$(a_1, \xi_1)\cdot(a_2, \xi_2) := (a_1a_2, \Phi(a_1, \xi_2, 1_{\mathcal{A}}) + \Phi(1_{\mathcal{A}}, \xi_1, a_2)).$$

A type B noncommutative random variable is an element of the link-algebra $(a, \xi) \in \mathcal{A} \times \mathcal{V}$, its distribution is the linear map from $\mathbb{C}[X]$ into $\mathbb{G}$:

$$P \longrightarrow E(P((a, \xi))),$$

where $E((a,\xi)) := \varphi(a) + \varepsilon f(\xi)$.
One also defines non-crossing cumulant functionals of type B [BGN03] with values in $\mathbb{G}$ by :

$$\sum_{p \in NC^{(A)}(n)} \kappa_p^{(B)}((a_1,\xi_1),\dots,(a_n,\xi_n)) = E((a_1,\xi_1)\cdots(a_n,\xi_n)). \tag{2.14}$$

It is important to notice that the first component of a type B non-crossing cumulant in $(\mathcal{A},\varphi,\mathcal{V},f,\Phi)$ is simply a non-crossing cumulant in $(\mathcal{A},\varphi)$.
The notion of freeness of type B for $(\mathcal{A}_1,\mathcal{V}_1),\dots,(\mathcal{A}_m,\mathcal{V}_m)$ in the type B noncommutative probability space $(\mathcal{A},\varphi,\mathcal{V},f,\Phi)$ is defined in terms of moments to ensure the vanishing of mixed type B non-crossing cumulants :

Definition 2.5.4. [BGN03] Given a type B noncommutative probability space $(\mathcal{A},\varphi,\mathcal{V},f,\Phi)$, let $\mathcal{A}_1,\dots,\mathcal{A}_m$ be unital subalgebras of $\mathcal{A}$ and let $\mathcal{V}_1,\dots,\mathcal{V}_m$ subspaces of $\mathcal{V}$ such that each $\mathcal{V}_j$ is invariant by the action of $\mathcal{A}_j$, for each $1 \le j \le m$. Then, $(\mathcal{A}_1,\mathcal{V}_1),\dots,(\mathcal{A}_m,\mathcal{V}_m)$ are said to be free of type B if $\mathcal{A}_1,\dots,\mathcal{A}_m$ are free in $(\mathcal{A},\varphi)$ and

$$f(a_m\cdots a_1\xi b_1\cdots b_n) = \begin{cases} \varphi(a_1b_1)\cdots\varphi(a_nb_n)f(\xi), \\ \quad \text{if } m=n \text{ and } i_1=j_1,\dots,i_n=j_n \\ 0, \text{ else,} \end{cases} \tag{2.15}$$

whenever

- $m,n \ge 0$;
- $a_1 \in \mathcal{A}_{i_1},\dots,a_m \in \mathcal{A}_{i_m}, b_1 \in \mathcal{A}_{j_1},\dots,b_n \in \mathcal{A}_{j_n}, \xi \in \mathcal{V}_h$, where two consecutive indices among $i_m,\dots,i_1,h,j_1,\dots,j_n$ are different ;
- $\varphi(a_m) = \cdots = \varphi(a_1) = 0 = \varphi(b_1) = \cdots = \varphi(b_n)$.

Free additive convolution of type B, denoted by $\boxplus^{(B)}$, describing the distribution of the sum of two free (of type B) type B noncommutative random variables, is an operation on the set of pairs $(\mu^{(0)},\mu^{(1)})$ of linear functionals on $\mathbb{C}[X]$ satisfying $\mu^{(0)}(1)=1$ and $\mu^{(1)}(1)=0$.
Later, Popa established type B versions of the limit theorems and of the S-transform [Pop07].
Recently, the analytic aspects of free probability of type B were investigated by [BS09] ; the authors underline an interesting interpretation of free probability of type B, in terms of infinitesimal freeness :

Definition 2.5.5. Let $(\mathcal{A},\varphi)$ be a noncommutative probability space, let T be a subset of $\mathbb{R}$ having 0 as accumulation point, and let s families $\{a_u^v(t) \mid 1 \le v \le m_u\}_{t\in T}$, indexed by T, of noncommutative random variables in

$(\mathcal{A}, \varphi)$. These families are said to be infinitesimally free if, for any choice of noncommutative polynomials $P_1, \ldots, P_n$, one has

$$\varphi((P_1(a^v_{i_1}(t), 1 \leq v \leq m_{i_1}) - \varphi(P_1(a^v_{i_1}(t), 1 \leq v \leq m_{i_1}))) \cdots (P_n(a^v_{i_n}(t), 1 \leq v \leq m_{i_n}) - \varphi(P_n(a^v_{i_n}(t), 1 \leq v \leq m_{i_n})))) \underset{t \to 0}{=} o(t),$$

whenever $i_1 \neq i_2 \neq \ldots \neq i_n$.

When the family (indexed by T) of distributions $(\mu_t)_{t \in T}$ of

$$\{a^v_u(t) \mid 1 \leq v \leq m_u, 1 \leq u \leq s\}_{t \in T}$$

has zeroth and first derivatives at 0, that is a pair of linear functionals $(\mu^{(0)}, \mu^{(1)})$ satisfying

$$\mu^{(0)} = \lim_{t \to 0} \mu_t \tag{2.16}$$

and

$$\mu^{(1)} = \frac{d}{dt}_{|t=0} \mu_t = \lim_{t \to 0} \frac{1}{t}(\mu_t - \mu^{(0)}), \tag{2.17}$$

the condition of infinitesimal freeness is equivalent to the two following conditions :

$$\mu^{(0)}((P_1 - \mu^{(0)}(P_1)) \cdots (P_n - \mu^{(0)}(P_n))) = 0 \tag{2.18}$$

and

$$\mu^{(1)}((P_1 - \mu^{(0)}(P_1)) \cdots (P_n - \mu^{(0)}(P_n))) = \tag{2.19}$$

$$\sum_{j=1}^{n} \mu^{(0)}((P_1 - \mu^{(0)}(P_1)) \cdots \mu^{(1)}(P_j) \cdots (P_n - \mu^{(0)}(P_n))). \tag{2.20}$$

The link, put into light by [BS09], between infinitesimal freeness and freeness of type B has the following consequence : given two families $(\mu_t)_{t \in T}$ et $(\nu_t)_{t \in T}$ of distributions with zeroth and first derivatives at 0, the zeroth and first derivatives at 0 of $\mu_t \boxplus \nu_t$ are then given by :

$$(\lim_{t \to 0} (\mu_t \boxplus \nu_t), \frac{d}{dt}_{|t=0} (\mu_t \boxplus \nu_t)) = (\lim_{t \to 0} \mu_t, \frac{d}{dt}_{|t=0} \mu_t) \boxplus^{(B)} (\lim_{t \to 0} \nu_t, \frac{d}{dt}_{|t=0} \nu_t). \tag{2.21}$$

Chapter 3

Presentation of results

We present the main results of the following chapters, which may be organized in two distinct parts : the first one (Chapters 4 and 5) develops a combinatorial approach of infinitesimal freeness ; the second one (Chapters 6 and 7) further studies certain additively deformed models.

3.1 First order infinitesimal freeness

Infinitesimal freeness (see Definition 2.5.5) appeared in [BS09] as analytic interpretation of free probability of type B. A possible frame for the combinatorial approach of infinitesimal freeness is thus the type B noncommutative probability space introduced in [BGN03]. We propose another frame, both simpler and more general, the *infinitesimal noncommutative probability space*:

Definition 3.1.1. By infinitesimal noncommutative probability space, we mean the structure

$$\left\{ \begin{array}{l} (\mathcal{A}, \varphi, \varphi'), \quad \text{where } \mathcal{A} \text{ is a complex unital algebra} \\ \qquad \text{and } \varphi, \varphi' : \mathcal{A} \to \mathbb{C} \text{ are linear with } \varphi(1_{\mathcal{A}}) = 1,\ \varphi'(1_{\mathcal{A}}) = 0. \end{array} \right. \tag{3.1}$$

A particular case of infinitesimal noncommutative probability space is already considered in [BS09] : this is the algebra of noncommutative polynomials in m indeterminates $\mathbb{C}\langle X_1, \ldots, X_m \rangle$, equipped with linear functionals

$$\varphi = \mu^{(0)}, \varphi' = \mu^{(1)}$$

obtained as zeroth and first derivatives at 0 of a family of distributions $(\mu_t)_{t \in T}$ indexed by a subset T of real numbers having 0 as accumulation point. Reformulated in our new frame, infinitesimal freeness from [BS09] reads :

Definition 3.1.2. Let $(\mathcal{A}, \varphi, \varphi')$ be an infinitesimal noncommutative probability space and let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be unital subalgebras of $\mathcal{A}$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_m$

are said to be *infinitesimally free* with respect to (φ, φ') when they satisfy the following condition :

If $i_1, \dots, i_n \in \{1, \dots, m\}$ are such that $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{n-1} \neq i_n$, and if $a_1 \in \mathcal{A}_{i_1}, \dots, a_n \in \mathcal{A}_{i_n}$ are such that $\varphi(a_1) = \cdots = \varphi(a_n) = 0$, then $\varphi(a_1 \cdots a_n) = 0$ and

$$\varphi'(a_1 \cdots a_n) = \begin{cases} \varphi(a_1\, a_n)\varphi(a_2\, a_{n-1}) \cdots \varphi(a_{(n-1)/2}\, a_{(n+3)/2}) \cdot \varphi'(a_{(n+1)/2}), \\ \text{ if } n \text{ is odd and } i_1 = i_n, i_2 = i_{n-1}, \dots, i_{\frac{n-1}{2}} = i_{\frac{n+3}{2}}, \\ 0, \quad \text{else.} \end{cases} \tag{3.2}$$

Taking into account the original link between infinitesimal freeness and freeness of type B, it is expected that the structure of infinitesimal noncommutative probability space is related to the structure of noncommutative probability space of type B. Given a noncommutative probability space of type B $(\mathcal{A}, \varphi, \mathcal{V}, f, \Phi)$, the link-algebra $(\mathcal{A} \times \mathcal{V}, \phi, \phi')$, equipped with the linear functionals $\phi((a, \xi)) := \varphi(a)$ and $\phi'((a, \xi)) := f(\xi)$, is an infinitesimal noncommutative probability space. The infinitesimal noncommutative probability space is therefore a simpler and more general frame for free probability of type B.

Some practical difficulties may arise when making computations with the two linear functionals defining an infinitesimal noncommutative probability space $(\mathcal{A}, \varphi, \varphi')$. That is why we will often work in the equivalent structure of scarce $\mathbb{G}$-probability space, introduced in [Oan07] :

$$\begin{cases} (\mathcal{A}, \tilde{\varphi}), \quad \text{where } \mathcal{A} \text{ is a unital complex algebra} \\ \quad \text{and } \tilde{\varphi} : \mathcal{A} \to \mathbb{G} \text{ is } \mathbb{C}\text{-linear with } \tilde{\varphi}(1_{\mathcal{A}}) = 1, \end{cases} \tag{3.3}$$

where $\mathbb{G}$ is the two-dimensional Grassman algebra (see Definition 2.5.3). The correspondence between infinitesimal noncommutative probability space and scarce $\mathbb{G}$-probability space relies on the simple formula :

$$\tilde{\varphi} = \varphi + \varepsilon \varphi'.$$

Notice that the structure of scarce $\mathbb{G}$-probability space $(\mathcal{A}, \tilde{\varphi})$ differs from the frame of $\mathbb{G}$-valued free probability, in that the map $\tilde{\varphi}$ is in general not $\mathbb{G}$-linear.

One of the first fundamental facts of the free probability theory is the existence of a free product, that is the existence of a noncommutative probability space containing free copies of given noncommutative probability spaces. We present an *infinitesimal free product* construction :

Proposition 3.1.3. *Let $(\mathcal{A}_1, \varphi_1), \dots, (\mathcal{A}_m, \varphi_m)$ be noncommutative probability spaces, and consider their free product*

$$(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_m, \varphi_m).$$

Assume that, for each $1 \leq i \leq m$, one has a linear functional $\varphi'_i : \mathcal{A}_i \to \mathbb{C}$ such that $\varphi'_i(1_{\mathcal{A}}) = 0$. Then there exists a unique linear functional $\varphi' : \mathcal{A} \to \mathbb{C}$ such that $\varphi' \mid \mathcal{A}_i = \varphi'_i$, $1 \leq i \leq m$, and such that $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.

An important step in the combinatorial approach of infinitesimal free probability is the introduction, in addition to Speicher's non-crossing cumulant functionals in the noncommutative probability space $(\mathcal{A}, \varphi)$, of infinitesimal non-crossing cumulant functionals in the infinitesimal noncommutative probability space $(\mathcal{A}, \varphi, \varphi')$:

Definition 3.1.4. For each $n \geq 1$, consider the multilinear functional $\kappa'_n : \mathcal{A}^n \to \mathbb{C}$ defined by the formula

$$\kappa'_n(a_1, \ldots, a_n) = \tag{3.4}$$

$$\sum_{\substack{\pi \in NC(n) \\ V \in \pi}} \Big[\text{Möb}(\pi, 1_n) \ \varphi'_{|V|}(\,(a_1, \ldots, a_n) \mid V\,) \prod_{\substack{W \in \pi \\ W \neq V}} \varphi_{|W|}(\,(a_1, \ldots, a_n) \mid W\,) \Big],$$

for $a_1, \ldots, a_n \in \mathcal{A}$. The functionals κ'_n are called *infinitesimal non-crossing cumulant functionals* associated to $(\mathcal{A}, \varphi, \varphi')$.

We prove that these infinitesimal non-crossing cumulants play the same role as the non-crossing cumulants in usual free probability : for instance, the following infinitesimal analogue of the famous result of Speicher.

Proposition 3.1.5. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be unital subalgebras of $\mathcal{A}$. The following statements are equivalet :*
(1) $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.
(2) For each $n \geq 2$, each $i_1, \ldots, i_n \in \{1, \ldots, m\}$ not all equal, and each $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$, one has $\kappa_n(a_1, \ldots, a_n) = \kappa'_n(a_1, \ldots, a_n) = 0$.

These infinitesimal non-crossing cumulants may be interpreted from different points of view :

- When φ and φ' are obtained as zeroth and first derivatives at 0 of a family of distributions $(\varphi_t)_{t \in T}$ indexed by a subset T of real numbers having 0 as accumulation point, the n-th infinitesimal non-crossing cumulant associated to $(\mathcal{A}, \varphi, \varphi')$ is then the derivative at 0 of the family $(\kappa_n^{(t)})_{t \in T}$, where $\kappa_n^{(t)}$ is the n-th non-crossing cumulant associated to $(\mathcal{A}, \varphi_t)$.
  This gives a recipe for obtaining formula (3.4) : it may be obtained by taking a formal derivative in the usual free moment-cumulant formula (2.4).

- Given an infinitesimal noncommutative probability space $(\mathcal{A}, \varphi, \varphi')$, one may define, in the scarce $\mathbb{G}$-probability space associated $(\mathcal{A}, \tilde{\varphi})$, $\mathbb{G}$-valued non-crossing cumulant functionals by rewriting simply the usual free moment-cumulant formula :
$$\tilde{\kappa}_n = \sum_{\pi \in NC(n)} \text{Möb}^{(A)}(\pi, 1_n) \cdot \tilde{\varphi}_\pi, \quad n \geq 1. \tag{3.5}$$
The infinitesimal non-crossing cumulants associated to $(\mathcal{A}, \varphi, \varphi')$ appear then as ε-component of the cumulants $\tilde{\kappa}_n$. This correspondence between infinitesimal non-crossing cumulants on the one hand and the associated cumulants in the scarce $\mathbb{G}$-probability space indicate a method to easily prove infinitesimal analogues of classical results of free probability : it is to check that their $\mathbb{G}$-valued versions hold, which is often straightforward because the combinatorics involved is essentially the same as in usual free probability, and to take the ε-component of formulas obtained this way. We illustrate this method by the example of the formula for the multiplication of infinitesimally free n-tuples, and its consequences on compressions by infinitesimally free projections and on the construction of families of variables following an infinitesimally free Poisson distribution.

- The type B essence of infinitesimal freeness remains in certain formulas. For instance, by reindexing the sum in (3.4) by the set of type B non-crossing partitions with a zero-block, infinitesimal non-crossing cumulants may be viewed as type B non-crossing cumulants. More generally, the formulas describing the alternating products of infinitesimally free variables may be rewritten using the type B language.

After noticing that an infinitesimal noncommutative probability space $(\mathcal{A}, \varphi, \varphi')$ is obtained by adding to the usual noncommutative probability space $(\mathcal{A}, \varphi)$ another linear functional φ' satisfying $\varphi'(1_\mathcal{A}) = 0$, and that infinitesimal freeness of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$ with respect to (φ, φ') implies their freeness with respect to φ, the folowing question is natural: given a noncommutative probability space $(\mathcal{A}, \varphi)$ and unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$ that are free with respect to φ, how to construct a linear functional φ' satisfying $\varphi'(1_\mathcal{A}) = 0$ and such that $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are infinitesimally free with respect to (φ, φ') ?
In an infinitesimal noncommutative probability space $(\mathcal{A}, \varphi, \varphi')$, the relation between φ' and the infinitesimal non-crossing cumulants $(\kappa'_n)_{n \in \mathbb{N}^*}$ on the one hand and φ and the usual non-crossing cumulants $(\kappa_n)_{n \in \mathbb{N}^*}$ on the other hand may be formalized thanks to the notion of *dual derivation system*. In particular, a simple dual derivation system may be built from any derivation of the algebra $\mathcal{A}$, providing a possible answer to the question above :

Proposition 3.1.6. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be unital subalgebras of $\mathcal{A}$ that are free in $(\mathcal{A}, \varphi)$. Assume* *that $D : \mathcal{A} \to \mathcal{A}$ is a derivation such that $D(\mathcal{A}_i) \subseteq \mathcal{A}_i$ for each $1 \leq i \leq m$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$, where $\varphi' = \varphi \circ D$.*

3.2 Higher order infinitesimal freeness

For a family $(\mu_t)_{t\in T}$ of distributions indexed by a subset T of $\mathbb{R}$ having 0 as accumulation point, one has already defined its zeroth and first derivatives at 0 by formulas (2.16) and (2.17). Assuming that the limits considered exist, it is possible to define recursively higher order derivatives by :

$$\forall i \geq 2, \ \frac{\mu^{(i)}}{i!} := \lim_{t\to 0} \frac{1}{t^i}(\mu_t - \sum_{j=0}^{i-1} \frac{t^j}{j!}\mu^{(j)}).$$

When $(\mu_t)_{t\in T}$ and $(\nu_t)_{t\in T}$ are two such families with as many derivatives at 0 as necessary, we investigate the problem of the derivatives at 0 of the family $(\mu_t \boxplus \nu_t)_{t\in T}$ of their free additive convolutions. We know by [BS09] that

$$(\lim_{t\to 0}(\mu_t \boxplus \nu_t), \frac{d}{dt}_{|t=0}(\mu_t \boxplus \nu_t)) = (\lim_{t\to 0} \mu_t, \frac{d}{dt}_{|t=0}\mu_t) \boxplus^{(B)} (\lim_{t\to 0} \nu_t, \frac{d}{dt}_{|t=0}\nu_t).$$

The appearance of the type B convolution in this formula comes from the link found by Belinschi and Shlyakhtenko between free probability of type B and infinitesimal freeness defined for time-indexed families by Definition 2.5.5. To get the higher order derivatives, one generalizes the notion of infinitesimal freeness to any order $k \in \mathbb{N}$:

Definition 3.2.1. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, let T be a subset of $\mathbb{R}$ having 0 as accumulation point, and let s families $\{a_u^v(t) \mid 1 \leq v \leq m_u\}_{t\in T}$, indexed by T, of noncommutative random variables in $(\mathcal{A}, \varphi)$. These families are said infinitesimally free of order $k \in \mathbb{N}$ if, for any choice of noncommutative polynomials $P_1, \ldots, P_n$, one has

$$\varphi((P_1(a_{i_1}^v(t), 1 \leq v \leq m_{i_1}) - \varphi(P_1(a_{i_1}^v(t), 1 \leq v \leq m_{i_1}))) \cdots (P_n(a_{i_n}^v(t), 1 \leq v \leq m_{i_n}) - \varphi(P_n(a_{i_n}^v(t), 1 \leq v \leq m_{i_n})))) \underset{t\to 0}{=} o(t^k),$$

whenever $i_1 \neq i_2 \neq \ldots \neq i_n$.

The combinatorial approach of infinitesimal freness of order k has for natural frame the *infinitesimal noncommutative probability space of order k*, built by adding not only one but k linear functionals to a usual noncommutative probability space $(\mathcal{A}, \varphi)$:

Definition 3.2.2. By infinitesimal noncommutative probability space of order k we mean a structure $(\mathcal{A}, (\varphi^{(i)})_{0\leq i\leq k})$, where $\mathcal{A}$ is a unital complex algebra, $\varphi^{(0)} : \mathcal{A} \longrightarrow \mathbb{C}$ is a linear functional satisfying $\varphi^{(0)}(1_{\mathcal{A}}) = 1$, and $\varphi^{(i)} : \mathcal{A} \longrightarrow \mathbb{C}$, $1 \leq i \leq k$, are linear functionals such that $\varphi^{(i)}(1_{\mathcal{A}}) = 0$.

The distribution of a n-tuple of infinitesimal noncommutative random variables of order k is the family of $k+1$ linear functionals $(\mu^{(i)})_{0\leq i\leq k}$ defined on $\mathbb{C}\langle X_1,\ldots,X_n\rangle$ by :

$$\forall P\in\mathbb{C}\langle X_1,\ldots,X_n\rangle,\ \mu^{(i)}(P(X_1,\ldots,X_n)):=\varphi^{(i)}(P(a_1,\ldots,a_n)).$$

The practical difficulties arising when dealing with $k+1$ linear functionals lead us to work in the structure, equivalent to $(\mathcal{A},(\varphi^{(i)})_{0\leq i\leq k})$, in which the $k+1$ linear functionals $\varphi^{(i)}, 0\leq i\leq k$, are consolidated in only one $\mathbb{C}$-linear map $\tilde{\varphi}=(\varphi^{(i)})_{0\leq i\leq k}$ with values in the algebra $\mathcal{C}_k$ of dimension $k+1$, which generalizes the two-dimensional Grassman algebra $\mathbb{G}$:

Definition 3.2.3. Let $\mathcal{C}_k$ be the commutative complex $(k+1)$-dimensional algebra $\mathbb{C}^{k+1}$, equipped with its usual vector space structure and the following multiplication : if $\alpha=(\alpha^{(0)},\ldots,\alpha^{(k)})\in\mathcal{C}_k$ and $\beta=(\beta^{(0)},\ldots,\beta^{(k)})\in\mathcal{C}_k$, then

$$\alpha\cdot\beta=(\gamma^{(0)},\ldots,\gamma^{(k)})$$

is defined by

$$\gamma^{(i)}:=\sum_{j=0}^{i}C_i^j\alpha^{(j)}\beta^{(i-j)}. \tag{3.6}$$

The structure defined this way is called a *scarce $\mathcal{C}_k$-noncommutative probability space.*

Definition 3.2.4. By scarce $\mathcal{C}_k$-noncommutative probability space, we mean a pair $(\mathcal{A},\tilde{\varphi})$, where $\mathcal{A}$ is a unital complex algebra and $\tilde{\varphi}:\mathcal{A}\to\mathcal{C}_k$ is a $\mathbb{C}$-linear map satisfying $\tilde{\varphi}(1_{\mathcal{A}})=1_{\mathcal{C}_k}$.

The transposition to this frame of the definition of infinitesimal freeness of order k is not easy to formalize, and is not clear enough to be taken as a definition. We adopt the idea to first introduce the *infinitesimal non-crossing cumulant functionals of order k*, and then to define infinitesimal freeness of order k by the condition of vanishing mixed infinitesimal non-crossing cumulants of order k.

Definition 3.2.5. Let $(\mathcal{A},(\varphi^{(i)})_{0\leq i\leq k})$ be an infinitesimal noncommutative probability space of order k. The *infinitesimal non-crossing cumulant functionals of order k* are a family of multilinear functionals $(\kappa_n^{(i)}:\mathcal{A}^n\to\mathbb{C}, 0\leq i\leq k)_{n=1}^{\infty}$, uniquely determined by : for each $n\geq 1$, each $0\leq i\leq k$ and each $a_1,\ldots,a_n\in\mathcal{A}$, one has

$$\sum_{\substack{p\in NC^{(A)}(n)\\ p:=\{V_1,\ldots,V_h\}}}\ \sum_{\lambda\in\Lambda_{h,i}}C_i^{\lambda_1,\ldots,\lambda_h}\kappa_p^{(\lambda)}(a_1,\ldots,a_n)=\varphi^{(i)}(a_1\cdots a_n), \tag{3.7}$$

where

$$\Lambda_{h,i}:=\{\lambda=(\lambda_1,\ldots,\lambda_h)\in\mathbb{N}^h\mid\sum_{j=1}^{h}\lambda_j=i\}. \tag{3.8}$$

As in first order infinitesimal free probability, one may interpret the infinitesimal non-crossing cumulants of order k :

- When $(\varphi^{(i)})_{0\leq i\leq k}$ are obtained as successive derivatives at 0 of a family of distributions $(\varphi_t)_{t\in T}$ indexed by a subset T of $\mathbb{R}$ having 0 as accumulation point, the n-th infinitesimal non-crossing cumulants of order k associated to $(\mathcal{A},(\varphi^{(i)})_{0\leq i\leq k})$ are then the derivatives at 0 of the family $(\kappa_{(n,t)})_{t\in T}$, where $\kappa_{(n,t)}$ is the n-th non-crossing cumulant associated to $(\mathcal{A},\varphi_t)$.
  Thus, the formula (3.7) may be obtained by taking i times the formal derivative in usual free moment-cumulant formula (2.4).

- On the other hand, given $(\mathcal{A},(\varphi^{(i)})_{0\leq i\leq k})$ an infinitesimal noncommutative probability space of order k, one may define, in the associated scarce $\mathcal{C}_k$-probability space $(\mathcal{A},\tilde{\varphi})$ the $\mathcal{C}_k$-valued non-crossing cumulants by simply copying the usual free moment-cumulant formula :

$$\tilde{\kappa}_n=\sum_{\pi\in NC(n)}\text{Möb}^{(A)}(\pi,1_n)\cdot\tilde{\varphi}_\pi,\quad n\geq 1. \tag{3.9}$$

  The n-th infinitesimal non-crossing cumulants of order k associated to $(\mathcal{A},(\varphi^{(i)})_{0\leq i\leq k})$ appear then as the components of the cumulant functionals $\tilde{\kappa}_n$. As in the first order, one may use this correspondence between infinitesimal non-crossing cumulants of order k and cumulants of the associated scarce $\mathcal{C}_k$-probability space to write direct proofs of infinitesimal analogues of classical results of free probability.

- The last possible point of view on infinitesimal non-crossing cumulants of order k is related to their combinatorial nature : to develop it, we are led to introduce and study a new set of non-crossing partitions likely to index the sums defining these infinitesimal cumulants. This is the set $NC^{(k)}(n)$ of non-crossing partitions of type k geeralizing the set of type B non-crossing partitions (which correspond to the particular case $k=1$). It still has the important property to be a cover of the set of non-crossing partitions of type A.

As announced, we use the infinitesimal non-crossing cumulants of order k to define the infinitesimal freeness of order k via a condition of vanishing mixed cumulants :

Definition 3.2.6. Subsets $\mathcal{M}_1,\ldots,\mathcal{M}_n$ of an infinitesimal noncommutative probability space of order k are *infinitesimally free of order k* if they satisfy the *vanishing of mixed infinitesimal cumulants* condition, that is, for each $0\leq i\leq k$,

$$\kappa_m^{(i)}(a_1,\ldots,a_m)=0$$

whenever $a_1\in\mathcal{M}_{i_1},\ldots,a_m\in\mathcal{M}_{i_m}$ and $\exists 1\leq s<t\leq m$, such that $i_s\neq i_t$.

Defined this way, infinitesimal freeness of order k coincides with Definition 3.2.1, providing a characterization in terms of moments.

Theorem 3.2.7. *Let $(\mathcal{A}, (\varphi^{(i)})_{0\leq i\leq k})$ be an infinitesimal noncommutative probability space of order k, and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ unital subalgebras of $\mathcal{A}$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are infinitesimally free of order k if and only if, for each $l \in \mathbb{N}^*$, and each $a_1 \in \mathcal{A}_{i_1}, \ldots, a_l \in \mathcal{A}_{i_l}$, one has*

$$\varphi_t((a_1 - \varphi_t(a_1)) \cdots (a_l - \varphi_t(a_l))) = o(t^k), \tag{3.10}$$

whenever $i_1 \neq \ldots \neq i_l$, where $\varphi_t := \sum_{i=0}^{k} \frac{\varphi^{(i)}}{i!} t^i$. The condition (3.10) is equivalent to the $k+1$ following equations :

$$\forall i \in \{0, \ldots, k\}, \sum_{j=0}^{i} \sum_{\lambda \in \Lambda_{l,i-j}} (-1)^{\#\{m\geq 1, \lambda_m > 0\}} \mu^{(j)}(\hat{\mu}^{(\lambda_1)}(P_1) \cdots \hat{\mu}^{(\lambda_l)}(P_l)) = 0, \tag{3.11}$$

where $\hat{\mu}^{(\lambda)}(P) := P - \mu^{(0)}(P)$ if $\lambda = 0$, and $\hat{\mu}^{(\lambda)}(P) := \mu^{(\lambda)}(P)$ else.

Infinitesimally free convolutions $\boxplus^{(k)}$ and $\boxtimes^{(k)}$, describing the distributions of the sum and of the multiplication of two variables that are infinitesimally free of order k, provide the k first derivatives of the free convolutions of time-indexed families of distributions :

Proposition 3.2.8. *Let $\{\mu_t\}_{t\in T}$ (resp. $\{\nu_t\}_{t\in T}$) be a family of linear functionals on $\mathbb{C}\langle X_u, 1 \leq u \leq m\rangle$ (resp. $\mathbb{C}\langle Y_u, 1 \leq u \leq m\rangle$) such that $\mu^{(i)} = \frac{d^i}{dt^i}_{|t=0}\mu_t$ (resp. $\nu^{(i)} = \frac{d^i}{dt^i}_{|t=0}\nu_t$) exist for $0 \leq i \leq k$. Let us set*

$$(\eta^{(i)})_{0\leq i\leq k} := (\mu^{(i)})_{0\leq i\leq k} \boxplus^{(k)} (\nu^{(i)})_{0\leq i\leq k},$$

$$(\theta^{(i)})_{0\leq i\leq k} := (\mu^{(i)})_{0\leq i\leq k} \boxtimes^{(k)} (\nu^{(i)})_{0\leq i\leq k}.$$

Then $\eta^{(i)} = \frac{d^i}{dt^i} |_{t=0}\, \mu_t \boxplus \nu_t$ and $\theta^{(i)} = \frac{d^i}{dt^i} |_{t=0}\, \mu_t \boxtimes \nu_t$.

3.3 Eigenvalues of spiked deformations of Wigner matrices

The results on Wigner models deformed by a deterministic Hermitian matrix of finite rank, recalled in Section 1.4, may be interpreted in terms of free probability.

First, the asymptotic freeness result stated in Theorem 2.4.3 gives a new proof to the global behavior of the deformed Wigner model

$$M_N := \frac{W_N}{\sqrt{N}} + A_N,$$

where W_N is a complex Wigner matrix of size N associated to a centered distribution in $L^{\infty-}$ and of variance σ^2 and A_N is a deterministic fixed rank Hermitian matrix : since the empirical spectral distributions of $\frac{W_N}{\sqrt{N}}$ and A_N converge respectively towards the semicircle distribution μ_σ and the distribution δ_0, one obtains the convergence of the empirical spectral distribution of M_N towards $\mu_\sigma \boxplus \delta_0 = \mu_\sigma$.
For the convergence of the largest eigenvalue of M_N, one distinguishes in Theorem 1.4.2, and actually in the results of [Péc06],[FP07],[CDMF09], two situations : denoting by θ_1 the largest eigenvalue of A_N and λ_1 the largest eigenvalue of M_N, one has :

- if $\theta_1 \leq \sigma$, λ_1 converges towards 2σ.
- si $\theta_1 > \sigma$, λ_1 converges towards $\rho_{\theta_1} := \theta_1 + \frac{\sigma^2}{\theta_1} > 2\sigma$.

The semicircular distribution μ_σ being infinitely divisible for free additive convolution, the subordination function of $\mu_\sigma \boxplus \nu$ with respect to any compactly supported probability measure ν satisfies the conclusions of the following proposition :

Proposition 3.3.1. *[Bia97a] Let μ, ν two probability measures, μ being $\boxplus$-infinitely divisible, one defines $H_{\nu,\mu}$ on $\mathbb{C}^+$ by :*

$$\forall z \in \mathbb{C}^+, H_{\nu,\mu}(z) = z + R_\mu(G_\nu(z)).$$

Then the subordination function $\omega_{\nu,\mu}$ is a conformal bijection from $\mathbb{C}^+$ onto a domain $\Omega_{\nu,\mu}$ of $\mathbb{C}^+$, whose inverse is the restriction of $H_{\nu,\mu}$ to $\Omega_{\nu,\mu}$.

In particular,

$$\forall z \in \mathbb{C}^+, \; H_{\delta_0,\mu_\sigma}(z) = z + \frac{\sigma^2}{z}.$$

The condition $\theta > \sigma$ rewrites then :

$$\theta \notin \operatorname{supp}(\delta_0) = 0 \quad , \quad H'_{\delta_0,\mu_\sigma}(\theta) > 0.$$

Moreover, when this condition is satisfied, the limit of the largest eigenvalue is

$$\rho_\theta = H_{\delta_0,\mu_\sigma}(\theta).$$

For the Gaussian spiked population model (see Definition 1.3.7), the asymptotic freeness result for a Wishart matrix and a deterministic Hermitian positive matrix, combined with Marchenko-Pastur Theorem 1.3.2, offers a new point of view on the global behavior : the empirical spectral measure converges towards the distribution $\pi_{\gamma,1} \boxtimes \delta_1 = \pi_{\gamma,1}$. The subordination equation of $\pi_{\gamma,1} \boxtimes \delta_1$ with respect to δ_1, written in terms of Stieltjes transform, gives :

$$\forall z \in \mathbb{C}^+, \; G_{\pi_{\gamma,1}}(z) = G_{\delta_1}\Big(\frac{1}{1 - \frac{1}{zG_{\pi_{\gamma,1}}(z)}}\Big).$$

The inverse of the function $z \longmapsto \frac{1}{1-\frac{1}{zG_{\pi_{\gamma,1}}(z)}}$ is $\psi : z \longmapsto \gamma z + \frac{z}{z-1}$ and the condition

$$\theta \notin \mathrm{supp}(\delta_1) = 1 \quad , \quad \psi'(\theta) > 0$$

is equivalent to the condition

$$\theta > 1 + \sqrt{\gamma}$$

appearing in the work of [BBAP05], [Ona08],[Pau07],[BS06], as necessary and sufficient condition for the largest eigenvalue to converge outside of the support of $\pi_{\gamma,1}$. Moreover, the limit of the largest eigenvalue when this condition is satisfied is exactly $\psi(\theta)$.

The relevance of this interpretation in terms of subordination function is confirmed by the results of Theorem 1.3.10 on the generalized spiked population model : the function ψ in the statement is indeed the inverse of the function involved when one writes the subordination equation of $\pi_{\gamma,1} \boxtimes \rho$ with respect to ρ in terms of Stieltjes transforms.

In Chapter 6, we use this interpretation of the behavior of eigenvalues of deformed models in terms of free probability to establish a more precise result on the eignevalues of the deformed Wigner model $M_N = \frac{1}{\sqrt{N}} W_N + A_N$ where

- W_N is a complex Wigner matrix of size N associated to a symmetric distribution μ with variance σ^2 and satisfying a Poincaré inequality.
- A_N is a deterministic Hermitian matrix converging in distribution towards a compactly supported probability measure ν. We also assume that there exists a fixed integer $r \geq 0$ (independently of N) such that A_N has $N - r$ eigenvalues $\beta_{j,N}$ satisfying

  $$\max_{1 \leq j \leq N-r} \mathrm{dist}(\beta_j(N), \mathrm{supp}(\nu)) \underset{\mathrm{N}\to\infty}{\longrightarrow} 0.$$

  The other eigenvalues are J fixed real numbers $\theta_1 > \ldots > \theta_J$ outside the support of ν such that θ_i is an eigenvalue of A_N with multiplicity k_i ($\sum_i k_i = r$). The θ_i are called the spikes of A_N.

In this particular case, the condition

$$\theta \notin \mathrm{supp}(\nu) \quad , \quad H'_{\nu,\mu_\sigma}(\theta) > 0,$$

is equivalent to

$$\theta \notin \overline{U_{\nu,\mu_\sigma}},$$

where U_{ν,μ_σ} is the open set introduced by Biane in [Bia97a] :

$$U_{\nu,\mu_\sigma} := \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\}.$$

The main result we prove gives a precise description of the convergence of eigenvalues of M_N according to the position of the θ_i's with respect to $\overline{U_{\nu,\mu_\sigma}}$ and to the connected components of the support of ν :

Theorem 3.3.2. *For each spike θ_i, let $n_{i-1}+1, \ldots, n_{i-1}+k_i$, be the descending ranks of θ_i among the eigenvalues of A_N.*

1) *If $\theta_i \notin \overline{U_{\nu,\mu_\sigma}}$, the k_i eigenvalues $(\lambda_{n_{i-1}+j}(M_N), 1 \leq j \leq k_i)$ converge almost surely outside the support of $\mu_\sigma \boxplus \nu$ towards $\rho_{\theta_i} = H_{\nu,\mu_\sigma}(\theta_i)$.*

2) *If $\theta_i \in \overline{U_{\nu,\mu_\sigma}}$, then we denote $[s_{l_i}, t_{l_i}]$ $(1 \leq l_i \leq m)$ the connected component of $\overline{U_{\nu,\mu_\sigma}}$ containing θ_i.*

   a) *If θ_i is on the right (resp. on the left) of every connected component of* supp(ν) *included in $[s_{l_i}, t_{l_i}]$, then the k_i eigenvalues $\lambda_{n_{i-1}+j}(M_N)$, $1 \leq j \leq k_i$, almost surely converge towards the endpoint $H_{\nu,\mu_\sigma}(t_{l_i})$ (resp. $H_{\nu,\mu_\sigma}(s_{l_i})$) of the support of $\mu_\sigma \boxplus \nu$.*

   b) *If θ_i is between two connected components of* supp(ν) *included in $[s_{l_i}, t_{l_i}]$, then the k_i eigenvalues $(\lambda_{n_{i-1}+j}(M_N), 1 \leq j \leq k_i)$ almost surely converge towards the α_i-quantile of $\mu_\sigma \boxplus \nu$ (i.e. q_{α_i} defined by $\alpha_i = (\mu_\sigma \boxplus \nu)(]-\infty, q_{\alpha_i}])$) where $\alpha_i = 1 - \lim_N (n_{i-1}/N) = \nu(]-\infty, \theta_i])$.*

As a byproduct of our analysis, we obtain the following proposition in the non-spiked case :

Proposition 3.3.3. *If the model M_N is non-spiked, i.e. $r = 0$, the largest (resp. smallest) eigenvalues $\lambda_{1+k}(M_N)$ (resp. $\lambda_{N-k}(M_N)$) almost surely converge towards the right (resp. left) endpoint of the support of $\mu_\sigma \boxplus \nu$.*

The heuristic underlying our results is the following : the eigenvalues of M_N are the poles of $G_{\mu_{M_N}}$, which is well approximated by $G_{\mu_\sigma \boxplus \mu_{A_N}}$. This last Stieltjes transform has poles outside the support of the limiting distribution given, using subordination equation

$$G_{\mu_\sigma \boxplus \mu_{A_N}} = G_{\mu_{A_N}} \circ \omega_{\mu_{A_N}, \mu_\sigma},$$

by some $H_{\mu_{A_N}, \mu_\sigma}(\theta_i)$.
The proof mainly follows the overall plan of [BS06], as in [CDMF09] :

1. we show, for N large enough, an almost sure inclusion of the spectrum of M_N in a neighborhood of the set formed by the support of $\mu_\sigma \boxplus \nu$ and the ρ_{θ_i};

2. then, we establish an exact separation phenomenon between the spectra of M_N and A_N.

More precisely, to obtain the first point, we give a precise estimation of $g_N(z) - \tilde{g}_N(z)$, where $g_N = \mathbb{E}(G_{\mu_{M_N}})$ and $\tilde{g}_N = G_{\mu_\sigma \boxplus \mu_{A_N}}$:

Proposition 3.3.4. *For $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\left| g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N} \right| \leq \frac{P(|\Im z|^{-1})}{N^2} \tag{3.12}$$

where P is a polynomial with nonnegative coefficients and $E_N(z)$ is given by

$$E_N(z) = \{\sigma^2 \tilde{g}_N'(z) - 1\} \frac{\kappa_4}{2N^2}$$

$$\sum_{i,l} [(G_{A_N}(z - \sigma^2 g_N(z)))^2]_{ii} [G_{A_N}(z - \sigma^2 g_N(z))]_{ii} ([G_{A_N}(z - \sigma^2 g_N(z))]_{ll})^2.$$

The proof relies on a "approximate matricial subordination equation", thanks to an integration by parts lemma, exact in the Gaussian case, approximate in general.
We deduce the required inclusion in two steps : on the one hand, an adaptation of the method used by Haagerup and Thorbjornsen in [HT05], and developed further in [Sch05], [CDM07], [CDMF09], [Mal10], leads to the inclusion of the spectrum of M_N in a neighborhood of the support of $\mu_\sigma \boxplus \mu_{A_N}$; on the other hand, the study of the support of a free additive convolution by a semicircular distribution carried by Biane in [Bia97a] allows to show that:

Theorem 3.3.5. *For each $\epsilon > 0$,*

$$\mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N}) \subset \mathrm{supp}(\mu_\sigma \boxplus \nu) \bigcup \{\rho_{\theta_i},\, \theta_i \in \mathbb{R} \setminus \overline{U_{\nu,\mu_\sigma}}\} + (-\epsilon, \epsilon),$$

for N large enough.

Finally, the second point is the following result :

Theorem 3.3.6. *If i_N satisfies*

$$\lambda_{i_N+1}(A_N) < \omega_{\sigma,\nu}(a) \quad \textit{and} \quad \lambda_{i_N}(A_N) > \omega_{\sigma,\nu}(b), \tag{3.13}$$

one has

$$\mathbb{P}[\lambda_{i_N+1}(M_N) < a \ \textit{ and } \ \lambda_{i_N}(M_N) > b, \textit{for all large } N] = 1. \tag{3.14}$$

Let us insist on the fact that the inclusion of the spectrum of the deformed model in a neighborhood of the support of a certain free convolution of measures and the exact separation phenomenon are the additive analogues of those described in [BS98], [BS99] for sample covariance matrices. Note however that Bai and Silverstein do not use the language of free probability and obtain the inclusion of the spectrum by considering Stieltjes transforms of not averaged empirical spectral measures.

3.4 Largest eigenvalues of additive deformations of Wishart matrices

In Chapter 7, we adapt the same method to the study of the largest eigenvalues of the *additively deformed Wishart model* :

$$M_N(p(N)) = \frac{1}{N}X_N + A_N$$

- X_N is a white complex Wishart matrix of size N and $p(N)$ degrees of freedom,
- $\lim_{N\to+\infty} \frac{p(N)}{N} = \alpha > 0$,
- A_N is a deterministic diagonal matrix converging in distribution to a compactly supported probability measure ν. We also assume that there is a fixed integer $r \geq 0$ (independent of N) such that A_N has $N-r$ eigenvalues $\beta_{j,N}$ satisfying

  $$\max_{1\leq j\leq N-r} \text{dist}(\beta_j(N), \text{supp}(\nu)) \underset{N\to\infty}{\longrightarrow} 0.$$

  The other eigenvalues are J fixed real numbers $\theta_1 > \ldots > \theta_J$ independent of N outside the support of ν and such that θ_i is an eigenvalue of A_N with multiplicity k_i ($\sum_i k_i = r$). The θ_i are called the spikes of A_N.

Our study follows the same scheme as in the deformed Wigner model: one writes an "approximated matricial subordination equation" by integrating by parts, which provides a precise estimate of $g_N(z) - \tilde{g}_N(z)$, where $g_N = \mathbb{E}(G_{\mu_{M_N(p(N))}})$ and $\tilde{g}_N = G_{\pi_{\alpha,1}\boxplus\mu_{A_N}}$:

Proposition 3.4.1. *$\forall z \in \mathbb{C}^+$,*

$$|g_N(z) - \tilde{g}_N(z)| \leq (|z| + K)^a \frac{P(|\Im z|^{-1})}{N^2} \tag{3.15}$$

where K and a are positive numbers and P is a polynomial with nonnegative coefficients.

One deduces the almost sure inclusion of the spectrum of $M_N(p(N))$ in a neighborhood of the support of $\pi_{\alpha,1} \boxplus \mu_{A_N}$, by a variant of the method of Haagerup and Thorbjornsen, then one studies this support, to obtain :

Theorem 3.4.2. *$\forall \epsilon > 0$,*

$$\mathbb{P}(\textit{For } N \textit{ sufficiently large}, \text{Spect}(M_N(p(N))) \subset$$

$$\{x, \text{dist}(x, \text{supp}(\pi_{\alpha,1} \boxplus \nu) \cup \{H_{\nu,\pi_{\alpha,1}}(\theta_i),\, 1 \leq i \leq J\}) \leq \epsilon\}) = 1.$$

Notice that we cannot rely, in the study of the support of a free additive convolution by a Marchenko-Pastur distribution, on the work of Biane [Bia97a], like we did in the semicircular case, because his work only focuses on the semicircular distribution. Among the original results of the Chapter 7, one gives a descritption of $\mathrm{supp}(\pi_{\alpha,1} \boxplus \nu)$, analogous to this of $\mathrm{supp}(\mu_\sigma \boxplus \nu)$ as the image by a homeomorphism of the closure of the open set U_{ν,μ_σ} :

Proposition 3.4.3. *$H_{\tau,\pi_{\gamma,1}}$ is an increasing diffeomorphism from $\mathbb{R} \setminus F_{\tau,\pi_{\gamma,1}}$ onto $\mathbb{R} \setminus \mathrm{supp}(\pi_{\gamma,1} \boxplus \tau)$, where*

$$F_{\tau,\pi_{\gamma,1}} = \overline{\mathbb{R} \setminus \mathbb{R} \cap \partial\Omega_{\tau,\pi_{\gamma,1}}} \cup \{x \in \mathbb{R}; \tau(\{x\}) > \gamma\}.$$

Finally, one proves a weak exact separation phenomenon, only allowing to conclude the convergence of the largest eigenvalue of $M_N(p(N))$:

Theorem 3.4.4. *(1) If $\theta_1 > \max F_{\nu,\pi_{\alpha,1}}$, the k_1 eigenvalues*

$$(\lambda_j(M_N(p(N)), 1 \leq j \leq k_1)$$

almost surely converge outside the support of $\pi_{\alpha,1} \boxplus \nu$ towards $\rho_{\theta_1} = H_{\nu,\pi_{\alpha,1}}(\theta_1)$.

(2) If $\theta_1 \leq \max F_{\nu,\pi_{\alpha,1}}$, then, for each $k \in \mathbb{N}$, the first k eigenvalues $(\lambda_j(M_N(p(N)), 1 \leq j \leq k)$ almost surely converge to the right end-point of the support of $\pi_{\alpha,1} \boxplus \nu$.

Chapter 4

First order infinitesimal freeness

This chapter is the text of the article "Infinitesimal non-crossing cumulants and free probability of type B" [FN10], written in collaboration with A. Nica and published in Journal of Functional Analysis.

4.1 Introduction

4.1.1 The framework of the chapter

This chapter is concerned with a form of free independence for noncommutative random variables, which can be called "freeness of type B" or "infinitesimal freeness", and occurs in relation to objects of the form

$$\begin{cases} (\mathcal{A}, \varphi, \varphi'), \quad \text{where } \mathcal{A} \text{ is a unital algebra over } \mathbb{C} \\ \quad \text{and } \varphi, \varphi' : \mathcal{A} \to \mathbb{C} \text{ are linear with } \varphi(1_{\mathcal{A}}) = 1,\ \varphi'(1_{\mathcal{A}}) = 0. \end{cases} \tag{4.1}$$

The motivation for considering objects as in (4.1) is three-fold.

(a) This framework generalizes the link-algebra associated to a noncommutative probability space of type B, in the sense introduced by Biane, Goodman and Nica [BGN03]. One can thus take the point of view that (4.1) provides us with an enlarged framework for doing "free probability of type B". This point of view is justified by the fact that lattices of non-crossing partitions of type B do indeed appear in the underlying combinatorics – see e.g. Theorem 4.5.4 below, concerning alternating products of infinitesimally free random variables.

(b) It turns out to be beneficial to consolidate the functionals φ, φ' from (4.1) into only one functional

$$\tilde{\varphi} : \mathcal{A} \to \mathbb{G}, \quad \tilde{\varphi} := \varphi + \varepsilon\varphi', \tag{4.2}$$

where $\mathbb{G}$ denotes the two-dimensional Grassman algebra generated by an element ε which satisfies $\varepsilon^2 = 0$. Thus $\mathbb{G}$ is the extension of $\mathbb{C}$ defined as

$$\mathbb{G} = \{\alpha + \varepsilon\beta \mid \alpha, \beta \in \mathbb{C}\}, \tag{4.3}$$

with multiplication given by $(\alpha_1 + \varepsilon\beta_1) \cdot (\alpha_2 + \varepsilon\beta_2) = \alpha_1\alpha_2 + \varepsilon(\alpha_1\beta_2 + \beta_1\alpha_2)$, and the structure from (4.1) could equivalently be treated as

$$\begin{cases} (\mathcal{A}, \tilde{\varphi}), & \text{where } \mathcal{A} \text{ is a unital algebra over } \mathbb{C} \\ & \text{and } \tilde{\varphi} : \mathcal{A} \to \mathbb{G} \text{ is } \mathbb{C}\text{-linear with } \tilde{\varphi}(1_{\mathcal{A}}) = 1. \end{cases} \tag{4.4}$$

The framework (4.4) was discussed in the PhD Thesis of Oancea [Oan07], under the name of "scarce [1] $\mathbb{G}$-probability space". Specifically, Chapter 7 of [Oan07] studies a concept of $\mathbb{G}$-freeness for a family of unital subalgebras in a $\mathbb{G}$-probability space, which is defined via a vanishing condition for mixed $\mathbb{G}$-valued cumulants, and generalizes the concept of freeness of type B from [BGN03].

(c) The recent paper [BS09] by Belinschi and Shlyakhtenko discusses a concept of "infinitesimal distribution" $(\mathbb{C}\langle X_1, \ldots, X_k\rangle, \mu, \mu')$ which is exactly as in (4.1), with $\mathbb{C}\langle X_1, \ldots, X_k\rangle$ denoting the algebra of polynomials in non-commuting indeterminates $X_1, \ldots, X_k$. This remarkable paper brings forth the idea that interesting infinitesimal distributions arise when μ is the limit at 0 and μ' is the derivative at 0 for a family of k-variables distributions $(\mu_t : \mathbb{C}\langle X_1, \ldots, X_k\rangle \to \mathbb{C})_{t \in T}$, where T is a set of real numbers having 0 as accumulation point. As we will show below, this ties in really nicely with the $\mathbb{G}$-valued cumulant considerations mentioned in (b); indeed, one could say that [BS09] puts the ε from (4.3) in its right place – it is a sibling of the ε's from calculus, only that instead of just having "ε^2 much smaller than ε" one goes for the radical requirement that $\varepsilon^2 = 0$.

Upon consideration, it seems that what goes best with the framework from (4.1) is the "infinitesimal" terminology from (c), which is in particular adopted in the next definition. Throughout the paper some terminology inspired from (a) and (b) will also be used, in the places where it is suggestive to do so (e.g. when talking about "soul companions for φ" in subsection 4.1.3 below).

Definition 4.1.1. 1^o A structure $(\mathcal{A}, \varphi, \varphi')$ as in (4.1) will be called an *infinitesimal noncommutative probability space* (abbreviated as *incps*).

2^o Let $(\mathcal{A}, \varphi, \varphi')$ be an incps and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$. We will say that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are *infinitesimally free* with respect to (φ, φ') when they satisfy the following condition:

[1] The adjective "scarce" is used in order to distinguish from the concept of "$\mathbb{G}$-probability space" from operator-valued free probability, where one would require the functional $\tilde{\varphi}$ to be $\mathbb{G}$-linear.

If $i_1, \ldots, i_n \in \{1, \ldots, k\}$ are such that $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$, and if $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ are such that $\varphi(a_1) = \cdots = \varphi(a_n) = 0$, then $\varphi(a_1 \cdots a_n) = 0$ and

$$\varphi'(a_1 \cdots a_n) = \begin{cases} \varphi(a_1\, a_n)\varphi(a_2\, a_{n-1}) \cdots \varphi(a_{\frac{n-1}{2}}\, a_{\frac{n-1}{2}})\varphi'(a_{(n+1)/2}), \\ \quad \text{if } n \text{ is odd and } i_1 = i_n, i_2 = i_{n-1}, \ldots, i_{\frac{n-1}{2}} = i_{\frac{n+3}{2}}, \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.5}$$

Recall that in the free probability literature it is customary to use the name *noncommutative probability space* for a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is linear with $\varphi(1_{\mathcal{A}}) = 1$. Thus the concept of *infinitesimal* noncommutative probability space is obtained by adding to $(\mathcal{A}, \varphi)$ another functional φ' as in (4.1). It is also immediate that Definition 4.1.1.2^o of infinitesimal freeness is obtained by adding the condition (4.5) to the "usual" definition for the freeness of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ in $(\mathcal{A}, \varphi)$ (as appearing e.g. in [VDN92], Definition 2.5.1).

Definition 4.1.1.2^o is a reformulation of the concept with the same name from Definition 13 of [BS09]. The relations with [BS09], [BGN03] are discussed more precisely in Section 4.2 (cf. Remarks 4.2.8, 4.2.9). Section 4.2 also collects a few miscellaneous properties of infinitesimal freeness that follow directly from the definition. Most notable among them is that one can easily extend to infinitesimal framework the well-known free product construction of noncommutative probability spaces $(\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k)$, as presented e.g. in Lecture 6 of [NS06]. More precisely: if $(\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k) =: (\mathcal{A}, \varphi)$ and if we are given linear functionals $\varphi_i' : \mathcal{A}_i \to \mathbb{C}$ such that $\varphi_i'(1_{\mathcal{A}}) = 0$, $1 \leq i \leq k$, then there exists a unique linear functional $\varphi' : \mathcal{A} \to \mathbb{C}$ such that $\varphi' \mid \mathcal{A}_i = \varphi_i'$, $1 \leq i \leq k$, and such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$. (See Proposition 4.2.4 below.) The resulting incps $(\mathcal{A}, \varphi, \varphi')$ can thus be taken, by definition, as the *free product* of $(\mathcal{A}, \varphi_i, \varphi_i')$ for $1 \leq i \leq k$.

4.1.2 Non-crossing cumulants for $(\mathcal{A}, \varphi, \varphi')$

An important tool in the combinatorics of free probability is the family of non-crossing cumulant functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n \geq 1}$ associated to a noncommutative probability space $(\mathcal{A}, \varphi)$. These functionals were introduced in [Spe94]; for a detailed presentation of their basic properties, see Lecture 11 of [NS06]. For every $n \geq 1$, the multilinear functional $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ is defined via the summation formula (2.5) over the lattice $NC(n)$ of non-crossing partitions of $\{1, \ldots, n\}$. Here we only pick a special value of n that we use for illustration, e.g. $n = 3$. In this special case one has, for all $a_1, a_2, a_3 \in \mathcal{A}$,

$$\begin{aligned} \kappa_3(a_1, a_2, a_3) = \; & \varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2a_3) - \varphi(a_2)\varphi(a_1a_3) \\ & -\varphi(a_3)\varphi(a_1a_2) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3). \end{aligned} \tag{4.6}$$

The expression on the right-hand side of (4.6) has 5 terms (premultiplied by integer coefficients [2] such as $1, -1$, or 2), corresponding to the fact that $|NC(3)| = 5$.

Let now $(\mathcal{A}, \varphi, \varphi')$ be an incps as in Definition 4.1.1. Then in addition to the non-crossing cumulant functionals $\kappa_n : \mathcal{A}^n \to \mathbb{C}$ associated to φ we will define another family of multilinear functionals $(\kappa'_n : \mathcal{A}^n \to \mathbb{C})_{n \geq 1}$, which involve both φ and φ'. For every $n \geq 1$, the functional κ'_n is obtained by taking a *formal derivative* in the formula for κ_n, where we postulate that the derivative of φ is φ' and we invoke linearity and the Leibnitz rule for derivatives. For instance for $n = 3$ the term $\varphi(a_1 a_2 a_3)$ on the right-hand side of (4.6) is derivated into $\varphi'(a_1 a_2 a_3)$, the term $\varphi(a_1)\varphi(a_2 a_3)$ is derivated into $\varphi'(a_1)\varphi(a_2 a_3) + \varphi(a_1)\varphi'(a_2 a_3)$, etc, yielding the formula for κ'_3 to be

$$\begin{aligned}\kappa'_3(a_1, a_2, a_3) = \; & \varphi'(a_1 a_2 a_3) - \varphi'(a_1)\varphi(a_2 a_3) - \varphi(a_1)\varphi'(a_2 a_3) \\ & -\varphi'(a_2)\varphi(a_1 a_3) - \varphi(a_2)\varphi'(a_1 a_3) - \varphi'(a_3)\varphi(a_1 a_2) \\ & -\varphi(a_3)\varphi'(a_1 a_2) + 2\varphi'(a_1)\varphi(a_2)\varphi(a_3) \\ & +2\varphi(a_1)\varphi'(a_2)\varphi(a_3) + 2\varphi(a_1)\varphi(a_2)\varphi'(a_3).\end{aligned} \tag{4.7}$$

We will refer to the functionals κ'_n as *infinitesimal non-crossing cumulants* associated to $(\mathcal{A}, \varphi, \varphi')$. The precise formula defining them appears in Definition 4.3.7 below. The passage from the formula for κ_n to the one for κ'_n is related to a concept of *dual derivation system* on a space of multilinear functionals on $\mathcal{A}$, which is discussed in Section 4.7 of the chapter.

The role of infinitesimal non-crossing cumulants in the study of infinitesimal freeness is described in the next theorem.

Theorem 4.1.2. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$. The following statements are equivalent:*
(1) $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free.
(2) For every $n \geq 2$, for every $i_1, \ldots, i_n \in \{1, \ldots, k\}$ which are not all equal to each other, and for every $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$, one has that $\kappa_n(a_1, \ldots, a_n) = \kappa'_n(a_1, \ldots, a_n) = 0$.

Theorem 4.1.2 provides an infinitesimal version for the basic result of Speicher (Theorem 2.3.7) which describes the usual freeness of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ in $(\mathcal{A}, \varphi)$ in terms of the cumulants κ_n.

In the remaining part of this subsection we point out some other interpretations of the formula defining κ'_n (all corresponding to one or another of the points of view (a), (b), (c) listed at the beginning of subsection 4.1.1). The easy verifications required by these alternative descriptions of κ'_n are shown at the beginning of Section 4.4.

First of all one can consider, as in [BS09], the situation when φ, φ' in (4.1) are obtained as the *infinitesimal limit* of a family of functionals $\{\varphi_t \mid t \in T\}$.

[2] The meaning of these coefficients is that they are special values of the Möbius function of $NC(3)$, see Proposition 2.3.2.

Here T is a subset of $\mathbb{R}$ which has 0 as an accumulation point, every φ_t is linear with $\varphi_t(1_{\mathcal{A}}) = 1$, and we have

$$\varphi(a) = \lim_{t \to 0} \varphi_t(a) \text{ and } \varphi'(a) = \lim_{t \to 0} \frac{\varphi_t(a) - \varphi(a)}{t}, \quad \forall\, a \in \mathcal{A}. \tag{4.8}$$

(Note that such families $\{\varphi_t \mid t \in T\}$ can in fact always be found, e.g. by simply taking $\varphi_t = \varphi + t\varphi'$, $t \in (0, \infty)$.) In such a situation, the formal derivative which leads from κ_n to κ'_n turns out to have the same effect as a "$\frac{d}{dt}$" derivative. Consequently, we get the alternative formula

$$\kappa'_n(a_1, \ldots, a_n) = \Big[\frac{d}{dt} \kappa_n^{(t)}(a_1, \ldots, a_n) \Big] \Big|_{t=0}, \tag{4.9}$$

where $\kappa_n^{(t)}$ denotes the nth non-crossing cumulant functional of φ_t.

Second of all, it is possible to take a direct combinatorial approach to the functionals κ'_n, and identify precisely a set of non-crossing partitions which indexes the terms in the summation defining $\kappa'_n(a_1, \ldots, a_n)$. This set turns out to be the set $NCZ^{(B)}(n)$ defined by (2.11). Hence in a terminology focused on types of non-crossing partitions, one could call the functionals κ_n and κ'_n "non-crossing cumulants of type A and of type B", respectively. The idea put forth here is that, in some sense, summations over $NCZ^{(B)}(n)$ appear as "derivatives for summation over $NC(n)$". A more refined formula supporting this idea is shown in Proposition 4.6.6 below, in connection to the concept of dual derivation sytem.

In the case $n = 3$ that we are using for illustration, the 10 terms appearing on the right-hand side of (4.7) are indexed by the 10 partitions with zero-block in $NC^{(B)}(3)$. The relation between a partition τ and the corresponding term is easy to follow: the zero-block Z of τ produces the $\varphi'(\cdots)$ factor, and every pair $V, -V$ of non-zero-blocks of τ produces a $\varphi(\cdots)$ factor.

Finally (third of all) one can also give a description of κ'_n which corresponds to the "$\mathbb{G}$-valued" point of view appearing as (b) on the list from subsection 4.1.1. This goes as follows. Let $\tilde{\varphi} = \varphi + \varepsilon\varphi' : \mathcal{A} \to \mathbb{G}$ be as in (4.2), and consider the family of $\mathbb{C}$-multilinear functionals $(\tilde{\kappa}_n : \mathcal{A}^n \to \mathbb{G})_{n \geq 1}$ defined by the same summation formula as for the usual non-crossing cumulant functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n \geq 1}$, only that now we use $\tilde{\varphi}$ instead of φ in the summations. So, for example, for $n = 3$ we have, for each $a_1, a_2, a_3 \in \mathcal{A}$,

$$\begin{aligned} \tilde{\kappa}_3(a_1, a_2, a_3) = \ & \tilde{\varphi}(a_1 a_2 a_3) - \tilde{\varphi}(a_1)\tilde{\varphi}(a_2 a_3) - \tilde{\varphi}(a_2)\tilde{\varphi}(a_1 a_3) \\ & - \tilde{\varphi}(a_3)\tilde{\varphi}(a_1 a_2) + 2\tilde{\varphi}(a_1)\tilde{\varphi}(a_2)\tilde{\varphi}(a_3) \in \mathbb{G}. \end{aligned} \tag{4.10}$$

It then turns out that the functional κ'_n can be obtained by reading the ε-component of $\tilde{\kappa}_n$.

We take the opportunity to introduce here a piece of terminology from the literature on Grassman algebras (see e.g. [DeW92], pp. 1-2): the complex numbers α, β which give the two components of a Grassman number $\gamma =$

$\alpha+\varepsilon\beta \in \mathbb{G}$ will be called the *body* and respectively the *soul* of γ; it will come in handy throughout the paper to denote them [3] as

$$\alpha = \mathrm{Bo}(\gamma), \quad \beta = \mathrm{So}(\gamma). \tag{4.11}$$

This notation will also be used in connection to a $\mathbb{G}$-valued function f defined on some set $\mathcal{S}$ – we define functions $\mathrm{Bo}\, f$ and $\mathrm{So}\, f$ from $\mathcal{S}$ to $\mathbb{C}$ by

$$(\mathrm{Bo}\, f)(x) = \mathrm{Bo}\,(f(x)), \quad (\mathrm{So}\, f)(x) = \mathrm{So}\,(f(x)), \quad \forall\, x \in \mathcal{S}. \tag{4.12}$$

Returning then to the functionals $\tilde{\kappa}_n : \mathcal{A}^n \to \mathbb{G}$ from the preceding paragraph, their connection to the κ_n' (and also to the κ_n) can be recorded as

$$\mathrm{Bo}\ \tilde{\kappa}_n = \kappa_n, \quad \mathrm{So}\ \tilde{\kappa}_n = \kappa_n', \quad \forall\, n \geq 1. \tag{4.13}$$

Due to (4.13), $\tilde{\kappa}_n$ can be used as a simplifying tool in calculations with κ_n' (in the sense that it may be easier to run the corresponding calculation with $\tilde{\kappa}_n$, in $\mathbb{G}$, and only pick soul parts at the end of the calculation). In particular, this will be useful when proving Theorem 4.1.2, since the condition $\kappa_n(a_1, \ldots, a_n) = \kappa_n'(a_1, \ldots, a_n) = 0$ from Theorem 4.1.2(2) amounts precisely to $\tilde{\kappa}_n(a_1, \ldots, a_n) = 0$.

4.1.3 Using derivations to find "soul companions"

When studying infinitesimal freeness it may be of interest to consider the situation where we have fixed a noncommutative probability space $(\mathcal{A}, \varphi)$ and a family $\mathcal{A}_1, \ldots, \mathcal{A}_k$ of unital subalgebras of $\mathcal{A}$ which are free in $(\mathcal{A}, \varphi)$. In this situation we can ask: how do we find interesting examples of functionals $\varphi' : \mathcal{A} \to \mathbb{C}$ with $\varphi'(1_{\mathcal{A}}) = 0$ and such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ become infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$? A nice name for such functionals φ' is suggested by the $\mathbb{G}$-valued point of view described in subsection 4.1.1 : since φ and φ' are the body part and respectively the soul part of the consolidated functional $\tilde{\varphi} : \mathcal{A} \to \mathbb{G}$, one may say that we are looking for a suitable *soul companion* φ' for the given "body functional" φ (and in reference to the given subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$).

Let us note that the remark made at the end of subsection 4.1.1 can be interpreted as a statement about soul companions. Indeed, this remark says that if $(\mathcal{A}, \varphi)$ is the free product of $(\mathcal{A}_1, \varphi_1), \ldots, (\mathcal{A}_k, \varphi_k)$, then a φ' from the desired set of soul companions is parametrized precisely by a family of linear functionals $\varphi_i' : \mathcal{A}_i \to \mathbb{C}$ such that $\varphi_i'(1_{\mathcal{A}}) = 0$, $1 \leq i \leq k$.

The point we follow here, with inspiration from [BS09], is that some interesting recipes to construct "soul companions" for a given $\varphi : \mathcal{A} \to \mathbb{C}$

[3] Besides being amusing, "Bo" and "So" give a faithful analogue for the common notations "Re" and "Im" used when one introduces $\mathbb{C}$ as a 2-dimensional algebra over $\mathbb{R}$.

arise from ideas pertaining to differentiability. This is intimately related to the fact that κ_n' is a formal derivative for κ_n, hence to equations of the form

$$d_n(\kappa_n) = \kappa_n', \quad \forall\, n \geq 1,$$

where $(d_n)_{n\geq 1}$ is a dual derivation system on $\mathcal{A}$. Indeed, suppose we are given a derivation $D : \mathcal{A} \to \mathcal{A}$; then one has a natural dual derivation system associated to it, which acts by

$$(d_n f)(a_1, \ldots, a_n) = \sum_{m=1}^{n} f\big(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\big), \tag{4.14}$$

for $f : \mathcal{A}^n \to \mathbb{C}$ multilinear and $a_1, \ldots, a_n \in \mathcal{A}$. By using the d_n from (4.14), we obtain the following theorem.

Theorem 4.1.3. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps, and let κ_n and κ_n' be the non-crossing cumulant functionals associated to it. Suppose $D : \mathcal{A} \to \mathcal{A}$ is a derivation with the property that $\varphi' = \varphi \circ D$. Then for every $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$ one has*

$$\kappa_n'(a_1, \ldots, a_n) = \sum_{m=1}^{n} \kappa_n(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n). \tag{4.15}$$

Moreover, when combined with Theorem 4.1.2, the formula for infinitesimal cumulants obtained in (4.15) has the following immediate consequence.

Corollary 4.1.4. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ which are free in $(\mathcal{A}, \varphi)$. Suppose we found a derivation $D : \mathcal{A} \to \mathcal{A}$ such that $D(\mathcal{A}_i) \subseteq \mathcal{A}_i$ for every $1 \leq i \leq k$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$, where $\varphi' = \varphi \circ D$.*

For comparison, let us also look at the parallel statement arising in connection to infinitesimal limits. This is essentially the same as Remark 15 from [BS09], and goes as follows.

Proposition 4.1.5. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ which are free in $(\mathcal{A}, \varphi)$. Suppose we found a family of linear functionals $(\varphi_t : \mathcal{A} \to \mathbb{C})_{t\in T}$ with $\varphi_t(1_{\mathcal{A}}) = 1$ for every $t \in T$ and such that:*

(i) $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are free in $(\mathcal{A}, \varphi_t)$ for every $t \in T$.

(ii) $\lim_{t\to 0} \varphi_t(a) = \varphi(a)$, for every $a \in \mathcal{A}$.

(iii) The limit $\varphi'(a) := \lim_{t\to 0}(\varphi_t(a) - \varphi(a))/t$ exists, for every $a \in \mathcal{A}$.

Then $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$, where $\varphi' : \mathcal{A} \to \mathbb{C}$ is defined by condition (iii).

A natural example accompanying Proposition 4.1.5 comes in connection to $\boxplus$-convolution powers of joint distributions of k-tuples (cf. Example 4.7.9 below). In Section 4.8 we also discuss a couple of natural situations when Corollary 4.1.4 applies (cf. Example 4.7.7).

4.1.4 Outline of the rest of the chapter

Besides the introduction, the chapter has six other sections. In Section 4.2 we collect some basic properties of infinitesimal freeness, and we discuss the relations between Definition 4.1.1 and the frameworks of [BS09], [BGN03]. In Section 4.3 we introduce the non-crossing infinitesimal cumulants, we verify the equivalence between their various alternative descriptions, and we prove Theorem 4.1.2.

Sections 4.4 and 4.5 address the topic of alternating products of infinitesimally free random variables. Section 4.4 uses this topic to illustrate a "generic" method to obtain infinitesimal analogues for known results in usual free probability: one replaces $\mathbb{C}$ by $\mathbb{G}$ in the proof of the original result, then one takes the soul part in the $\mathbb{G}$-valued statement that comes out. By using this method we obtain the infinitesimal versions of two important facts related to alternating products that were originally found in [NS96] – one of them is about compressions by free projections, the other concerns a method of constructing free families of free Poisson elements. In Section 4.5 we remember that the concept of incps has its origins in the considerations "of type B" from [BGN03], and we look at how the essence of these considerations persists in the framework of the present paper. The main point of the section is that, when taking the soul part of the $\mathbb{G}$-valued formulas for alternating products of infinitesimally free random variables, one does indeed obtain nice analogues of type B (with summations over $NC^{(B)}(n)$) for the type A formulas. In particular, this offers another explanation for why the infinitesimal cumulant functional κ_n' can be described by using a summation formula over $NCZ^{(B)}(n)$.

In Section 4.6 we return to the point of view of treating κ_n' as a derivative of the usual non-crossing cumulant functional κ_n, and we discuss the related concept of dual derivation system on a unital algebra $\mathcal{A}$. Finally, Section 4.7 elaborates on the discussion about soul companions from the above subsection 4.1.3. In particular, we show how the dual derivation system provided by a derivation $D : \mathcal{A} \to \mathcal{A}$ leads to the setting for infinitesimal freeness from Corollary 4.1.4. Section 4.7 (and the chapter) concludes with a couple of examples related to the settings of Corollary 4.1.4 and of Proposition 4.1.5.

4.2 Basic properties of infinitesimal freeness

In this section we collect some basic properties of infinitesimal freeness, and we discuss the relations between Definition 4.1.1 and the frameworks from [BS09], [BGN03].

Definition 4.2.1. Here are some standard variations of Definition 4.1.1.

1^o The concept of infinitesimal freeness carries over to $*$-algebras. More precisely, we will use the name *$*$-incps* for an incps $(\mathcal{A}, \varphi, \varphi')$ where $\mathcal{A}$ is a unital $*$-algebra and where

(i) φ is positive definite, that is, $\varphi(a^*a) \geq 0, \ \forall\, a \in \mathcal{A}$;

(ii) φ' is selfadjoint, that is, $\varphi'(a^*) = \overline{\varphi'(a)}, \ \forall\, a \in \mathcal{A}$.

2^o Another standard variation of the definitions is that infinitesimal freeness can be considered for arbitrary subsets of $\mathcal{A}$ (which don't have to be subalgebras). So if $(\mathcal{A}, \varphi, \varphi')$ is an incps (respectively a $*$-incps) and if $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are subsets of $\mathcal{A}$, then we will say that $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are *infinitesimally free* (respectively *infinitesimally $*$-free*) when the unital subalgebras (respectively $*$-subalgebras) generated by $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are so.

Remark 4.2.2. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ which are free in $(\mathcal{A}, \varphi)$. It is very easy to see (cf. Remark 2.5.2 in [VDN92] or Examples 5.15 in [NS06]) that the way how φ acts on $\mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$ can be reconstructed from the restrictions $\varphi \mid \mathcal{A}_i$, $1 \leq i \leq k$. The simplest illustration for how this works is provided by the formula

$$\varphi(ab) = \varphi(a)\varphi(b), \quad \forall\, a \in \mathcal{A}_{i_1},\, b \in \mathcal{A}_{i_2}, \text{ with } i_1 \neq i_2, \tag{4.16}$$

which is obtained by expanding the product and then collecting terms in the equation $\varphi\Big((a - \varphi(a)1_{\mathcal{A}}) \cdot (b - \varphi(b)1_{\mathcal{A}}) \Big) = 0$.

A similar phenomenon turns out to take place when dealing with infinitesimal freeness: the way how φ' acts on $\mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$ can be reconstructed from the restrictions of φ and of φ' to $\mathcal{A}_i$, $1 \leq i \leq k$. For example, the counterpart of Equation (4.16) says that

$$\varphi'(ab) = \varphi'(a)\varphi(b) + \varphi(a)\varphi'(b), \quad \forall\, a \in \mathcal{A}_{i_1},\, b \in \mathcal{A}_{i_2}, \text{ where } i_1 \neq i_2. \tag{4.17}$$

This is obtained by expanding the product and then collecting terms in the equation $\varphi'\Big((a - \varphi(a)1_{\mathcal{A}}) \cdot (b - \varphi(b)1_{\mathcal{A}}) \Big) = 0$ (which is a particular case of Equation (4.5)), and by taking into account that $\varphi'(1_{\mathcal{A}}) = 0$.

We leave it as an easy exercise to the reader to verify that the similar calculation for an alternating product of 3 factors (which makes a more involved use of Equation (4.5)) leads to the formula

$$\varphi'(a_1 b a_2) = \varphi'(a_1 a_2)\varphi(b) + \varphi(a_1 a_2)\varphi'(b), \tag{4.18}$$

for $a_1, a_2 \in \mathcal{A}_{i_1}$, $b \in \mathcal{A}_{i_2}$, with $i_1 \neq i_2$.

Remark 4.2.3. *(Traciality.)* Another well-known fact in usual free probability is that if the unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}$ are free in $(\mathcal{A}, \varphi)$ and if $\varphi \mid \mathcal{A}_i$ is a trace for every $1 \leq i \leq k$, then φ is a trace on $\mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$. This too extends to the infinitesimal framework: if $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$ and if $\varphi \mid \mathcal{A}_i, \varphi' \mid \mathcal{A}_i$ are traces for every $1 \leq i \leq k$, then φ and φ' are traces on $\mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$. Rather than writing an ad-hoc proof of this fact based directly on Definition 4.1.1, we find it more instructive to do this by using cumulants – see Proposition 4.3.16 below.

We next move to describing the free product of infinitesimal noncommutative probability spaces announced at the end of Section 4.1.1.

Proposition 4.2.4. *Let $(\mathcal{A}_1, \varphi_1), \ldots, (\mathcal{A}_k, \varphi_k)$ be noncommutative probability spaces, and consider the free product $(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k)$ (as described e.g. in Lecture 6 of [NS06]). Suppose that for every $1 \le i \le k$ we are given a linear functional $\varphi_i' : \mathcal{A}_i \to \mathbb{C}$ such that $\varphi_i'(1_{\mathcal{A}}) = 0$. Then there exists a unique linear functional $\varphi' : \mathcal{A} \to \mathbb{C}$ such that $\varphi' \mid \mathcal{A}_i = \varphi_i'$, $1 \le i \le k$, and such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.*

Proof. We start by reviewing a few basic facts and notations related to $(\mathcal{A}, \varphi)$. Each of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ is identified as a unital subalgebra of $\mathcal{A}$, such that $\varphi \mid \mathcal{A}_i = \varphi_i$. For $1 \le i \le k$ we denote $\mathcal{A}_i^o = \{a \in \mathcal{A}_i \mid \varphi(a) = 0\}$, and for every $n \ge 1$ and $1 \le i_1, \ldots, i_n \le k$ such that $i_1 \ne i_2, \ldots, i_{n-1} \ne i_n$ we put

$$\mathcal{W}_{i_1,\ldots,i_n} := \mathrm{span}\big\{a_1 \cdots a_n \mid a_1 \in \mathcal{A}_{i_1}^o, \ldots, a_n \in \mathcal{A}_{i_n}^o\big\}. \tag{4.19}$$

It is known that $\mathcal{W}_{i_1,\ldots,i_n}$ is canonically isomorphic to the tensor product $\mathcal{A}_{i_1}^o \otimes \cdots \otimes \mathcal{A}_{i_n}^o$, via the identification $a_1 \cdots a_n \simeq a_1 \otimes \cdots \otimes a_n$, for $a_1 \in \mathcal{A}_{i_1}^o, \ldots, a_n \in \mathcal{A}_{i_n}^o$. Moreover it is known that the spaces $\mathcal{W}_{i_1,\ldots,i_n}$ defined in (4.19) realize a direct sum decomposition of the kernel of φ. (See [NS06], pp. 81-84.)

Due to the direct sum decomposition mentioned above, we may define the required functional φ' by separately prescribing its behaviour at $1_{\mathcal{A}}$ and on each of the subspaces $\mathcal{W}_{i_1,\ldots,i_n}$. We put $\varphi'(1_{\mathcal{A}}) := 0$. We also prescribe φ' to be 0 on $\mathcal{W}_{i_1,\ldots,i_n}$ whenever n is even, and whenever n is odd but it is not true that $i_m = i_{n+1-m}$ for all $1 \le m \le (n-1)/2$. Suppose next that $n = 2m-1$, odd, and that the indices $i_1, \ldots, i_n$ are such that $i_1 = i_{2m-1}$, $i_2 = i_{2m-2}, \ldots, i_{m-1} = i_{m+1}$. By using the identification $\mathcal{W}_{i_1,\ldots,i_n} \simeq \mathcal{A}_{i_1}^o \otimes \cdots \otimes \mathcal{A}_{i_n}^o$ it is immediate that we can define a linear map on $\mathcal{W}_{i_1,\ldots,i_n}$ by the requirement that

$$a_1 \cdots a_{2m-1} \mapsto \varphi_{i_1}(a_1 a_{2m-1}) \varphi_{i_2}(a_2 a_{2m-2}) \cdots \varphi_{i_{m-1}}(a_{m-1} a_{m+1}) \cdot \varphi_{i_m}'(a_m),$$

for every $a_1 \in \mathcal{A}_{i_1}^o, \ldots, a_n \in \mathcal{A}_{i_n}^o$; we take this as the prescription for how φ' is to act on $\mathcal{W}_{i_1,\ldots,i_n}$.

Directly from Definition 4.1.1 it is immediate that, with $\varphi' : \mathcal{A} \to \mathbb{C}$ defined as in the preceding paragraph, $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$. The uniqueness of φ' with this property is also immediate. □

Definition 4.2.5. Let $(\mathcal{A}_1, \varphi_1, \varphi_1'), \ldots, (\mathcal{A}_k, \varphi_k, \varphi_k')$ be infinitesimal noncommutative probability spaces. We define their *free product* to be $(\mathcal{A}, \varphi, \varphi')$ where $(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k)$ and where $\varphi' : \mathcal{A} \to \mathbb{C}$ is the functional provided by Proposition 4.2.4.

Remark 4.2.6. In the context of Proposition 4.2.4, suppose that the probability spaces $(\mathcal{A}_1, \varphi_1), \ldots, (\mathcal{A}_k, \varphi_k)$ are $*$-noncommutative probability spaces. Then so is the free product $(\mathcal{A}, \varphi)$ (see [NS06], Theorem 6.13). If moreover each of the functionals $\varphi_i' : \mathcal{A}_i \to \mathbb{C}$ given in Proposition 4.2.4 is selfadjoint, then it is easily checked that the resulting functional $\varphi' : \mathcal{A} \to \mathbb{C}$ is selfadjoint too. Hence, if in Definition 4.2.5 each of $(\mathcal{A}_i, \varphi_i, \varphi_i')$ is a $*$-incps, then the free product $(\mathcal{A}, \varphi, \varphi')$ is a $*$-incps as well.

Example 4.2.7. For an illustration of the above, we look at a simple example where the spaces $\mathcal{W}_{i_1,\ldots,i_n}$ are all 1-dimensional. Consider the k-fold free product group $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$ and let φ be the canonical trace on the group algebra $\mathcal{A} := \mathbb{C}[\mathbb{Z}_2 * \cdots * \mathbb{Z}_2]$. So $\mathcal{A}$ is a unital $*$-algebra freely generated by k unitaries $u_1, \ldots, u_k$ of order 2, and has a linear basis $\mathcal{B}$ given by

$$\mathcal{B} = \{1_{\mathcal{A}}\} \cup \left\{ u_{i_1} \cdots u_{i_n} \;\middle|\; \begin{array}{c} n \geq 1,\ 1 \leq i_1, \ldots, i_n \leq k, \\ \text{with } i_1 \neq i_2, \ldots, i_{n-1} \neq i_n \end{array} \right\}. \tag{4.20}$$

The linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ acts on the basis $\mathcal{B}$ by

$$\varphi(1_{\mathcal{A}}) = 1, \quad \text{and} \quad \varphi(b) = 0, \ \ \forall\, b \in \mathcal{B} \setminus \{1_{\mathcal{A}}\}.$$

It is easy to verify (see e.g. Lecture 6 in [NS06]) that we have $(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k)$, where for $1 \leq i \leq k$ we denote $\mathcal{A}_i = \text{span}\{1_{\mathcal{A}}, u_i\}$ (2-dimensional $*$-subalgebra of $\mathcal{A}$), and where $\varphi_i := \varphi \mid \mathcal{A}_i$. The direct sum decomposition of $\mathcal{A}$ with respect to this free product structure simply has

$$\mathcal{W}_{i_1,\ldots,i_n} = \text{1-dimensional space spanned by } u_{i_1} \cdots u_{i_n},$$

for every $n \geq 1$ and every alternating sequence $i_1, \ldots, i_n$ as described in (4.20).

Now let $\varphi_i' : \mathcal{A}_i \to \mathbb{C}$ be linear functionals such that $\varphi_i'(1_{\mathcal{A}}) = 0$, $1 \leq i \leq k$. Clearly, these functionals are determined by the values

$$\varphi_1'(u_1) =: \alpha_1', \ldots, \varphi_k'(u_k) =: \alpha_k'.$$

The free product extension $\varphi' : \mathcal{A} \to \mathbb{C}$ then acts by

$$\varphi'(u_{i_1} \cdots u_{i_n}) = \begin{cases} \alpha_{i_m}', & \text{if } n \text{ is odd, } n = 2m-1, \\ & \text{and } i_1 = i_{2m-1}, \ldots, i_{m-1} = i_{m+1} \\ 0, & \text{otherwise.} \end{cases} \tag{4.21}$$

Note that formula (4.21) looks particularly nice in the case when $k = 2$ – indeed, in this case the requirement that $i_1 = i_{2m-1}, \ldots, i_{m-1} = i_{m+1}$ is automatically satisfied whenever $n = 2m-1$ and $i_1, \ldots, i_n$ are as in (4.20).

Remark 4.2.8. *(Relation to [BS09]).* Definition 13 of [BS09] introduces a concept of infinitesimal freeness for unital subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ in an incps $(\mathcal{A}, \mu, \mu')$. As explained there (immediately following to Definition 13), and reviewed in (2.18), (2.19), this amounts to two requirements: that $\mathcal{A}_1, \mathcal{A}_2$ are free in $(\mathcal{A}, \mu)$, and that they satisfy the following additional condition:

$$\mu'\Big((p_1 - \mu(p_1)1_{\mathcal{A}}) \cdots (p_n - \mu(p_n)1_{\mathcal{A}}) \Big) = \tag{4.22}$$

$$\sum_{m=1}^{n} \mu\Big((p_1 - \mu(p_1)1_{\mathcal{A}}) \cdots \mu'(p_m) \cdots (p_n - \mu(p_n)1_{\mathcal{A}}) \Big)$$

for $p_1 \in \mathcal{A}_{i_1}, \ldots, p_n \in \mathcal{A}_{i_n}$, where $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$. By denoting $p_m - \mu(p_m)1_{\mathcal{A}} =: q_m$ and by taking into account that $\mu'(q_m) = \mu'(p_m)$, $1 \leq m \leq n$, one sees that condition (4.22) is equivalent to its particular case requesting that

$$\mu'(q_1 \cdots q_n) = \sum_{m=1}^{n} \mu(q_1 \cdots q_{m-1} q_{m+1} \cdots q_n) \cdot \mu'(q_m) \tag{4.23}$$

for $q_1 \in \mathcal{A}_{i_1}, \ldots, q_n \in \mathcal{A}_{i_n}$, where $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$ and where $\mu(q_1) = \cdots = \mu(q_n) = 0$.

But now, let $\mathcal{A}_1, \mathcal{A}_2$ be unital subalgebras of $\mathcal{A}$ which are free in $(\mathcal{A}, \mu)$. A standard calculation from usual free probability (see e.g. Lemma 5.18 on page 73 of [NS06]) says that, with $q_1, \ldots, q_n$ as in (4.23), one has

$$\mu(q_1 \cdots q_{m-1} q_{m+1} \cdots q_n) = 0$$

unless it is true that $m - 1 = n - m$ and that $i_{m-1} = i_{m+1}, i_{m-2} = i_{m+2}, \ldots, i_1 = i_n$; moreover, if the latter conditions are satisfied, then

$$\mu(q_1 \cdots q_{m-1} q_{m+1} \cdots q_n) = \mu(q_{m-1} q_{m+1})\, \mu(q_{m-2} q_{m+2}) \cdots \mu(q_1 q_n).$$

This clearly implies that the sum on the right-hand side of (4.23) has at most one term which is different from 0; and moreover, when such a term exists, it is exactly as described in Equation (4.5) of Definition 4.1.1.

Hence, modulo an immediate reformulation, the concept of infinitesimal freeness from [BS09] is the same as the one used in this chapter (which justifies the fact that we are calling it by the same name).

Remark 4.2.9. *(Relation to [BGN03]).* A *noncommutative probability space of type B* is defined in [BGN03] (see also Section 2.5) as a system

$$(\mathcal{A}, \varphi, \mathcal{V}, f, \Phi),$$

where $(\mathcal{A}, \varphi)$ is a noncommutative probability space, $\mathcal{V}$ is a complex vector space, $f : \mathcal{V} \to \mathbb{C}$ is a linear functional, and $\Phi : \mathcal{A} \times \mathcal{V} \times \mathcal{A} \to \mathcal{V}$ is a two-sided action. We will write for short $a\xi b$ and respectively $a\xi$, ξb instead of $\Phi(a, \xi, b)$ and respectively $\Phi(a, \xi, 1_{\mathcal{A}})$, $\Phi(1_{\mathcal{A}}, \xi, b)$, for $a, b \in \mathcal{A}$ and $\xi \in \mathcal{V}$. Let

$\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ and let $\mathcal{V}_1, \ldots, \mathcal{V}_k$ be linear subspaces of $\mathcal{V}$, such that $\mathcal{V}_i$ is closed under the two-sided action of $\mathcal{A}_i$, $1 \leq i \leq k$. Definition 7.2 of [BGN03] (Definition 2.5.4) introduces a concept of what it means for $(\mathcal{A}_1, \mathcal{V}_1), \ldots, (\mathcal{A}_k, \mathcal{V}_k)$ to be free in $(\mathcal{A}, \varphi, \mathcal{V}, f, \Phi)$. This amounts to two requirements: that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are free in $(\mathcal{A}, \varphi)$, and that the following additional condition is satisfied:

$$f(a_m \ldots a_1 \xi b_1 \ldots b_n) = \begin{cases} \varphi(a_1 b_1) \cdots \varphi(a_n b_n) f(\xi), \\ \quad \text{if } m = n \text{ and } i_1 = j_1, \ldots, i_n = j_n \\ 0, \text{ otherwise,} \end{cases} \tag{4.24}$$

holding for $m, n \geq 0$ and $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}, b_1 \in \mathcal{A}_{j_1}, \ldots, b_n \in \mathcal{A}_{j_n}, \xi \in \mathcal{V}_h$, where any two consecutive indices among $i_m, \ldots, i_1, h, j_1, \ldots, j_n$ are different from each other, and where $\varphi(a_m) = \cdots = \varphi(a_1) = 0 = \varphi(b_1) = \cdots = \varphi(b_n)$.

Now, to $(\mathcal{A}, \varphi, \mathcal{V}, f, \Phi)$ as above one associates a *link-algebra*, which is simply the direct product $\mathcal{M} = \mathcal{A} \times \mathcal{V}$ endowed with the natural structure of complex vector space and with multiplication

$$(a, \xi) \cdot (b, \eta) = (ab, a\eta + \xi b), \quad \forall\, a, b \in \mathcal{A},\ \xi, \eta \in \mathcal{V}. \tag{4.25}$$

If we define $\psi, \psi' : \mathcal{M} \to \mathbb{C}$ by

$$\psi(\,(a, \xi)\,) := \varphi(a), \quad \psi'(\,(a, \xi)\,) := f(\xi), \quad \forall\,(a, \xi) \in \mathcal{M}, \tag{4.26}$$

then $(\mathcal{M}, \psi, \psi')$ becomes an incps. Let again $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ and $\mathcal{V}_1, \ldots, \mathcal{V}_k$ be linear subspaces of $\mathcal{V}$ such that $\mathcal{V}_i$ is closed under the two-sided action of $\mathcal{A}_i$, $1 \leq i \leq k$. Then $\mathcal{M}_1 := \mathcal{A}_1 \times \mathcal{V}_1, \ldots, \mathcal{M}_k := \mathcal{A}_k \times \mathcal{V}_k$ are unital subalgebras of the link-algebra $\mathcal{M}$, and we claim that

$$\left(\begin{array}{c} (\mathcal{A}_1, \mathcal{V}_1), \ldots, (\mathcal{A}_k, \mathcal{V}_k) \\ \text{are free in } (\mathcal{A}, \varphi, \mathcal{V}, f, \Phi), \\ \text{in the sense of [BGN03]} \end{array} \right) \Leftrightarrow \left(\begin{array}{c} \mathcal{M}_1, \ldots, \mathcal{M}_k \text{ are free} \\ \text{in } (\mathcal{M}, \psi, \psi'), \text{ in the} \\ \text{sense of Definition 4.1.1} \end{array} \right). \tag{4.27}$$

In order to prove the implication "$\Leftarrow$" in (4.27), we only have to write

$$f(a_m \ldots a_1 \xi b_1 \ldots b_n) = \psi'\big(\,(a_m, 0_{\mathcal{V}}) \cdots (a_1, 0_{\mathcal{V}}) \cdot (0_{\mathcal{A}}, \xi) \cdot (b_1, 0_{\mathcal{V}}) \cdots (b_n, 0_{\mathcal{V}})\,\big)$$

and then invoke Equation (4.5). For the implication "$\Rightarrow$", consider some elements $(a_1, \xi_1) \in \mathcal{M}_{i_1}, \ldots, (a_n, \xi_n) \in \mathcal{M}_{i_n}$ where $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$ and where $\psi(\,(a_1, \xi_1)\,) = \cdots = \psi(\,(a_n, \xi_n)\,) = 0$ (which just means that $\varphi(a_1) = \cdots = \varphi(a_n) = 0$). By using how the multiplication on $\mathcal{M}$ and how ψ' are defined, we see that

$$\psi'\big(\,(a_1, \xi_1) \cdots (a_n, \xi_n)\,\big) = \sum_{m=1}^{n} f(a_1 \cdots a_{m-1} \xi_m a_{m+1} \cdots a_n). \tag{4.28}$$

But because of (4.24), at most one term in the sum on the right-hand side of (4.28) can be different from 0; moreover such a term can only occur for $m = (n+1)/2$, if (n is odd and) $i_1 = i_{2m-1}, \ldots, i_{m-1} = i_{m+1}$. Finally, if the latter equalities of indices are satisfied, then the unique term left in the sum from (4.28) is $\varphi(a_1 a_{2m-1}) \cdots \varphi(a_{m-1} a_{m+1}) f(\xi_m)$, and the conditions defining the infinitesimal freeness of $\mathcal{M}_1, \ldots, \mathcal{M}_k$ in $(\mathcal{M}, \psi, \psi')$ follow.

Hence, by focusing on the link-algebra, one can incorporate the freeness of type B from [BGN03] into the framework of this chapter.

4.3 Infinitesimal cumulants

Remark 4.3.1. We will work with the Grassman algebra $\mathbb{G}$ from subsection 4.1.1, and with the maps $\mathrm{Bo}, \mathrm{So} : \mathbb{G} \to \mathbb{C}$ defined in subsection 4.1.2. It is immediate that the multiplication of $\mathbb{G}$ is commutative, and that the "body" map $\mathrm{Bo} : \mathbb{G} \to \mathbb{C}$ is a homomorphism of unital algebras. Concerning how the "soul" map So behaves with respect to multiplication, we record the immediate formula

$$\mathrm{So}(\gamma_1 \cdots \gamma_n) = \sum_{i=1}^{n} \Big(\mathrm{So}(\gamma_i) \cdot \prod_{\substack{1 \leq j \leq n, \\ j \neq i}} \mathrm{Bo}(\gamma_j) \Big), \quad \forall\, n \geq 1,\ \forall\, \gamma_1, \ldots, \gamma_n \in \mathbb{G}. \tag{4.29}$$

Notation 4.3.2. *In the following, we fix a pair* $(\mathcal{A}, \tilde{\varphi})$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\tilde{\varphi} : \mathcal{A} \to \mathbb{G}$ is $\mathbb{C}$-linear with $\tilde{\varphi}(1_{\mathcal{A}}) = 1$. In connection to this $\tilde{\varphi}$ we will repeat all the constructions of functionals described in subsection 2.3.2, with the only difference that the range space of these functionals is now $\mathbb{G}$. So for every $n \geq 1$ we put

$$\tilde{\varphi}_n(a_1, \ldots, a_n) = \tilde{\varphi}(a_1 \cdots a_n).$$

Then for every $\pi \in NC(n)$ we define $\tilde{\varphi}_\pi : \mathcal{A}^n \to \mathbb{G}$ by

$$\tilde{\varphi}_\pi(a_1, \ldots, a_n) := \prod_{V \in \pi} \tilde{\varphi}_{|V|}\big((a_1, \ldots, a_n) \mid V \big), \quad a_1, \ldots, a_n \in \mathcal{A}. \tag{4.30}$$

This is followed by defining a family of cumulant functionals $(\, \tilde{\kappa}_n : \mathcal{A}^n \to \mathbb{G}\,)_{n \geq 1}$, where

$$\tilde{\kappa}_n = \sum_{\pi \in NC(n)} \mathrm{Möb}^{(A)}(\pi, 1_n) \cdot \tilde{\varphi}_\pi, \quad n \geq 1. \tag{4.31}$$

Finally, for every $\pi \in NC(n)$ we define $\tilde{\kappa}_\pi : \mathcal{A}^n \to \mathbb{G}$ by

$$\tilde{\kappa}_\pi(a_1, \ldots, a_n) := \prod_{V \in \pi} \tilde{\kappa}_{|V|}\big((a_1, \ldots, a_n) \mid V \big), \quad a_1, \ldots, a_n \in \mathcal{A}. \tag{4.32}$$

It is easily seen that, exactly as in the $\mathbb{C}$-valued case, the families of functionals $\{\tilde{\kappa}_\pi \mid \pi \in NC(n)\}$ and $\{\tilde{\varphi}_\pi \mid \pi \in NC(n)\}$ are related by moment-cumulant formulas (i.e. by summation formulas as shown in Equations (2.4), (2.5). We only record here the special case of moment-cumulant formula which expresses $\tilde{\varphi}_{1_n}$ as a sum of cumulant functionals, and thus says that

$$\tilde{\varphi}(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \tilde{\kappa}_\pi(a_1, \ldots, a_n) \in \mathbb{G}, \quad \forall\, a_1, \ldots, a_n \in \mathcal{A}. \tag{4.33}$$

Remark 4.3.3. The next thing to do concerning $(\mathcal{A}, \tilde{\varphi})$ is to see if the analogue of Theorem 2.3.7 holds. Both conditions (1) and (2) from the statement of that theorem can be faithfully transcribed to the $\mathbb{G}$-valued framework, but it turns out they are no longer equivalent to each other – the implication (2) ⇒ (1) still holds, but its converse does not. This is discussed in more detail in Remark 4.3.14 below.

In the remaining part of this subsection we will point out two other facts from the theory of usual non-crossing cumulants where (unlike for Theorem 2.3.7) both the statement and the proof can be transcribed without any problems from the $\mathbb{C}$-valued to the $\mathbb{G}$-valued framework.

Proposition 4.3.4. *One has that $\tilde{\kappa}_n(a_1, \ldots, a_n) = 0$ whenever $n \geq 2$, $a_1, \ldots, a_n \in \mathcal{A}$, and there exists $1 \leq m \leq n$ such that $a_m \in \mathbb{C}1_{\mathcal{A}}$.*

Proof. This is the analogue of Proposition 11.15 in [NS06]. It is straightforward (left to the reader) to see that the proof shown on p. 182 of [NS06] goes without any changes to the $\mathbb{G}$-valued framework. □

Proposition 4.3.5. *Let $x_1, \ldots, x_s$ be in $\mathcal{A}$ and consider some products of the form*

$$a_1 = x_1 \cdots x_{s_1},\ a_2 = x_{s_1+1} \cdots x_{s_2},\ \ldots,\ a_n = x_{s_{n-1}+1} \cdots x_{s_n},$$

where $1 \leq s_1 < s_2 < \cdots < s_n = s$. Then

$$\tilde{\kappa}_n(a_1, \ldots, a_n) = \sum_{\substack{\pi \in NC(s) \text{ such} \\ \text{that } \pi \vee \theta = 1_s}} \tilde{\kappa}_\pi(x_1, \ldots, x_s), \tag{4.34}$$

where $\theta \in NC(s)$ is the partition with interval blocks $\{1, \ldots, s_1\}$, $\{s_1 + 1, \ldots, s_2\}, \ldots, \{s_{n-1} + 1, \ldots, s_n\}$.

Proof. This is the analogue of Theorem 11.20 in [NS06], and the proof of this theorem (as shown on pp. 178-180 of [NS06]) goes without any changes to the $\mathbb{G}$-valued framework. □

Notation 4.3.6. *Throughout this whole section we fix an incps* $(\mathcal{A}, \varphi, \varphi')$. We will use the notation "κ_n" for the non-crossing cumulant functionals associated to φ, as described in Section 2.3.2. Moreover, we will denote, same as in the introduction:

$$\tilde{\varphi} = \varphi + \varepsilon\varphi' : \mathcal{A} \to \mathbb{G}$$

and we will consider the family of non-crossing cumulant functionals $(\tilde{\kappa}_n : \mathcal{A}^n \to \mathbb{G})_{n\geq 1}$ which are associated to $\tilde{\varphi}$ as in formula (4.31).

Definition 4.3.7. For every $n \geq 1$, consider the multilinear functional $\kappa'_n : \mathcal{A}^n \to \mathbb{C}$ defined by the formula

$$\kappa'_n(a_1, \ldots, a_n) = \tag{4.35}$$

$$\sum_{\substack{\pi\in NC(n)\\ V\in\pi}} \Big[\text{Möb}(\pi, 1_n)\ \varphi'_{|V|}(\,(a_1, \ldots, a_n) \mid V\,) \prod_{\substack{W\in\pi\\ W\neq V}} \varphi_{|W|}(\,(a_1, \ldots, a_n) \mid W\,)\ \Big],$$

for $a_1, \ldots, a_n \in \mathcal{A}$. The functionals κ'_n will be called *infinitesimal non-crossing cumulant functionals* associated to $(\mathcal{A}, \varphi, \varphi')$.

A moment's thought shows that Equation (4.35) is indeed obtained from the fomula (2.5) defining κ_n, where one uses the formal derivation procedure announced in subsection 4.1.2 of the introduction.

We next state precisely (in Propositions 4.3.8, 4.3.10 and Remark 4.3.9) the equivalence between Definition 4.3.7 and the other facets of κ'_n that were mentioned in subsection 4.1.2.

Proposition 4.3.8. *Suppose that* φ, φ' *are the infinitesimal limit of a family* $\{\varphi_t \mid t \in T\}$, *in the sense described in Equation (4.8). Let us use the notation* $\kappa_n^{(t)}$ *for the non-crossing cumulant functional of* φ_t, *for* $t \in T$ *and* $n \geq 1$. *Then for every* $n \geq 1$ *and every* $a_1, \ldots, a_n \in \mathcal{A}$ *one has that*

$$\kappa_n(a_1, \ldots, a_n) = \lim_{t\to 0} \kappa_n^{(t)}(a_1, \ldots, a_n),$$

and

$$\kappa'_n(a_1, \ldots, a_n) = \Big[\ \frac{d}{dt}\kappa_n^{(t)}(a_1, \ldots, a_n)\Big]\ \Big|_{t=0}.$$

Proof. Fix $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$. For every $t \in T$ we have that

$$\kappa_n^{(t)}(a_1, \ldots, a_n) = \sum_{\pi\in NC(n)} \text{Möb}^{(A)}(\pi, 1_n) \cdot \prod_{V\in\pi} \varphi_t\big(\,(a_1, \ldots, a_n) \mid V\,\big). \tag{4.36}$$

From (4.36) it is clear that $\lim_{t\to 0} \kappa_n^{(t)}(a_1, \ldots, a_n) = \kappa_n(a_1, \ldots, a_n)$. Moreover, it is immediate that the function of t appearing on the right-hand side of (4.36) has a derivative at 0; and upon using linearity and the Leibnitz formula to compute this derivative, one obtains precisely the formula (4.35) that defined $\kappa'_n(a_1, \ldots, a_n)$. □

Remark 4.3.9. As observed in Section 2.5, the set

$$\{(\pi, V) \mid \pi \in NC(n), V \in \pi\}$$

which indexes the sum on the right-hand side of Equation (4.35) is the image of $NCZ^{(B)}(n)$ via the bijection

$$\big(\tau \in NCZ^{(B)}(n) \text{ with zero-block } Z\big) \mapsto (\,\mathrm{Abs}(\tau), \mathrm{Abs}(Z)\,).$$

When τ and (π, V) correspond to each other via this bijection, we have that $\mathrm{Möb}^{(B)}(\tau, 1_{\pm n}) = \mathrm{Möb}^{(A)}(\pi, 1_n)$ (cf. implication (2.12)) ; moreover, the rest of the product indexed by (π, V) on the right-hand side of Equation (4.35) is precisely equal to $\varphi_\tau^{(B)}(a_1, \ldots, a_n)$, where we anticipate here the notation $\varphi_\tau^{(B)}$ from Equation (4.77). In conclusion, the change of variable from (V, π) to τ converts (4.35) into a summation formula "of type B",

$$\kappa'_n = \sum_{\tau \in NCZ^{(B)}(n)} \mathrm{Möb}^{(B)}(\tau, 1_{\pm n}) \cdot \varphi_\tau^{(B)}. \tag{4.37}$$

It is easy to see that (4.37) is equivalent to a plain summation formula which writes $\varphi'(a_1 \cdots a_n)$ in terms of cumulants (cf. Remark 4.5.5 below, where one also sees that the absence of terms indexed by partitions from $NC^{(B)}(n) \setminus NCZ^{(B)}(n)$ is caused by the fact that $\varphi'(1_{\mathcal{A}}) = 0$).

Proposition 4.3.10. *For every $n \geq 1$ one has that Bo $\tilde{\kappa}_n = \kappa_n$ and So $\tilde{\kappa}_n = \kappa'_n$.*

Proof. For the first statement we only have to take the body part on both sides of Equation (4.31) and use the fact that $\mathrm{Bo} : \mathbb{G} \to \mathbb{C}$ is a homomorphism of unital algebras. For the second statement we take soul parts in (4.31) and then use the multiplication formula (4.29). □

We now go to Theorem 4.1.2. Note that, in view of Proposition 4.3.10, the equalities "$\kappa_n(a_1, \ldots, a_n) = \kappa'_n(a_1, \ldots, a_n) = 0$" from condition (2) of Theorem 4.1.2 may be replaced with "$\tilde{\kappa}_n(a_1, \ldots, a_n) = 0$". We will prove Theorem 4.1.2 in this alternative form, which is stated below as Proposition 4.3.12.

Lemma 4.3.11. *Suppose that n is a positive integer and π is a partition in $NC(n)$, such that the following two properties hold:*
(i) For every $1 \leq i \leq n-1$, the numbers i and $i+1$ do not belong to the same block of π.
(ii) π has at most one block of cardinality 1.
Then n is odd, and π is the partition

$$\Big\{ \ \{1, n\},\ \{2, n-1\}, \ldots, \{(n-1)/2, (n+3)/2\},\ \{(n+1)/2\} \ \Big\}.$$

Proof. We will use the observation about interval-blocks of non-crossing partitions that was recorded in Subsection 2.3.1. Clearly, condition (i) implies that π cannot have interval-blocks V with $|V| \geq 2$; by also taking (ii) into account we thus see that π has a unique interval-block V_o, of the form $V_o = \{p\}$ for some $1 \leq p \leq n$.

Let V be a block of π, distinct from V_o. We claim that

$$| V \cap [1,p) | \leq 1, \quad | V \cap (p,n] | \leq 1. \tag{4.38}$$

Indeed, assume for instance that we had $| V \cap [1,p) | \geq 2$. Then we could find $i, j \in V$ such that $i < j < p$ and $(i,j) \cap V = \emptyset$. Note that $j \neq i+1$, due to condition (i); but then, as observed in Subsection 2.3.1, the partition π must have an interval-block $W \cap (i,j)$, in contradiction to the fact that the unique interval-block of π is V_o.

For every block $V \neq V_o$ of π it then follows that

$$| V \cap [1,p) | = | V \cap (p,n] | = 1.$$

Indeed, if in (4.38) one of the sets $V \cap [1,p)$, $V \cap (p,n]$ would be empty, then it would follow that $|V| = 1$ and hypothesis (ii) would be contradicted.

The list of blocks of π which are distinct from V_o can thus be written in the form

$$\begin{cases} V_1 = \{i_1, j_1\}, \ldots, V_m = \{i_m, j_m\}, \quad \text{where} \\ i_1 < p < j_1, \ldots, i_m < p < j_m, \quad \text{and} \quad i_1 < i_2 < \cdots < i_m. \end{cases} \tag{4.39}$$

Observe that in (4.39) we must have $j_1 > j_2 > \cdots > j_m$. Indeed, if it was true that $j_s < j_t$ for some $1 \leq s < t \leq m$, then it would follow that $i_s < i_t < p < j_s < j_t$, and the blocks V_s, V_t would cross. Hence we have obtained $i_1 < \cdots < i_m < p < j_m < \cdots < j_1$; together with (4.39), this implies that $n = 2m+1$ and that π is precisely the partition indicated in the lemma. □

Proposition 4.3.12. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$. The following statements are equivalent:*
(1) $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.
(2) For every $n \geq 2$, for every $i_1, \ldots, i_n \in \{1, \ldots, k\}$ which are not all equal to each other, and for every $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$, one has that $\tilde{\kappa}_n(a_1, \ldots, a_n) = 0$.

Proof. *"(1) ⇒ (2)".* We prove the required statement about cumulants by induction on n. For the base case $n = 2$, consider elements $a_1 \in \mathcal{A}_{i_1}$ and $a_2 \in \mathcal{A}_{i_2}$, where $i_1 \neq i_2$. By using the formulas which define κ_2 and κ_2' and by invoking Equations (4.16) and (4.17) from Remark 4.2.2 we find that

$$\begin{cases} \kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2) = 0 \quad \text{and} \\ \kappa_2'(a_1, a_2) = \varphi'(a_1 a_2) - \varphi'(a_1)\varphi(a_2) - \varphi(a_1)\varphi'(a_2) = 0, \end{cases}$$

hence $\tilde{\kappa}_2(a_1, a_2) = \kappa_2(a_1, a_2) + \varepsilon\kappa_2'(a_1, a_2) = 0$.

We now prove the induction step: assume that the vanishing of mixed cumulants is already proved for $1, 2, ..., n-1$, where $n \geq 3$. We consider elements $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ where not all indices $i_1, \ldots, i_n$ are equal to each other, and we want to prove that $\tilde{\kappa}_n(a_1, \ldots, a_n) = 0$. By invoking Proposition 4.3.4 we may replace every a_m with $a_m - \varphi(a_m)1_{\mathcal{A}}$, $1 \leq m \leq n$, and therefore assume without loss of generality that $\varphi(a_1) = \cdots = \varphi(a_n) = 0$. Observe that this implies $\tilde{\varphi}(a_p)\tilde{\varphi}(a_q) = (\varepsilon\varphi'(a_p)) \cdot (\varepsilon\varphi'(a_q)) = 0$, hence that

$$\tilde{\kappa}_2(a_p, a_q) = \tilde{\varphi}(a_p a_q) - \tilde{\varphi}(a_p)\tilde{\varphi}(a_q) = \tilde{\varphi}(a_p a_q), \quad \forall\, 1 \leq p < q \leq n. \tag{4.40}$$

Another assumption that can be made without loss of generality is that $i_m \neq i_{m+1}, \forall\, 1 \leq m < n$. Indeed, if there exists $1 \leq m < n$ such that $i_m = i_{m+1}$, then we invoke the special case of Proposition 4.3.5 which states that

$$\begin{aligned}\tilde{\kappa}_{n-1}(a_1, \ldots, a_m a_{m+1}, \ldots, a_n) = \qquad & \tilde{\kappa}_n(a_1, \ldots, a_n) \\ + \sum\nolimits_{\substack{\pi \in NC(n) \text{ with } |\pi|=2 \\ \pi \text{ separates } m \text{ and } m+1}} & \tilde{\kappa}_\pi(a_1, \ldots, a_n).\end{aligned} \tag{4.41}$$

The induction hypothesis gives us that the left-hand side and every term in the sum on the right-hand side of Equation (4.41) are equal to 0, and it follows that $\tilde{\kappa}_n(a_1, \ldots, a_n)$ must be 0 as well.

Hence for the rest of the proof of this induction step we will assume that $\varphi(a_1) = \cdots = \varphi(a_n) = 0$ and that $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$. This makes $a_1, \ldots, a_n$ be exactly as in Definition 4.1.1, so we get that $\varphi(a_1 \cdots a_n) = 0$ and that $\varphi'(a_1 \cdots a_n)$ is as described in Equation (4.5). In terms of the functional $\tilde{\varphi}$, we have

$$\tilde{\varphi}(a_1 \cdots a_n) = \varepsilon\varphi'(a_1 \cdots a_n) = \tag{4.42}$$

$$= \begin{cases} \varepsilon\varphi(a_1\, a_n)\varphi(a_2\, a_{n-1}) \cdots \varphi(a_{(n-1)/2}\, a_{(n+3)/2}) \cdot \varphi'(a_{(n+1)/2}), \\ \qquad \text{if } n \text{ is odd and } i_1 = i_n, i_2 = i_{n-1}, \ldots, i_{(n-1)/2} = i_{(n+3)/2}, \\ 0, \quad \text{otherwise.} \end{cases}$$

Now let us consider the relation (4.33), written in the equivalent form

$$\tilde{\kappa}_n(a_1, \ldots, a_n) = \tilde{\varphi}(a_1 \cdots a_n) - \sum_{\substack{\pi \in NC(n), \\ \pi \neq 1_n}} \tilde{\kappa}_\pi(a_1, \ldots, a_n). \tag{4.43}$$

Observe that if a partition $\pi \in NC(n)$ has two distinct blocks $\{p\}, \{q\}$ of cardinality one, then the term indexed by π on the right-hand side of (4.43) vanishes, because it contains the subproduct $\tilde{\kappa}_1(a_p)\tilde{\kappa}_1(a_q) = \tilde{\varphi}(a_p)\tilde{\varphi}(a_q) = 0$. On the other hand if $\pi \in NC(n)$ has a block V which contains two consecutive numbers i and $i+1$, then the term indexed by π on the right-hand side of (4.43) vanishes as well, due to the induction hypothesis. Hence the sum subtracted on the right-hand side of (4.43) can only get non-zero contributions from partitions $\pi \in NC(n)$ which satisfy the hypotheses of Lemma 4.3.11; from the lemma it then follows that the sum in question is 0 for n even, and is equal to

$$\tilde{\kappa}_2(a_1, a_n)\tilde{\kappa}_2(a_2, a_{n-1}) \cdots \tilde{\kappa}_2(a_{(n-1)/2}, a_{(n+3)/2}) \cdot \tilde{\kappa}_1(a_{(n+1)/2}) \tag{4.44}$$

for n odd.

Let us focus for a moment on the quantity that appeared in (4.44). The vanishing of mixed cumulants of order 2 (which is part of our induction hypothesis) implies that this quantity vanishes unless $i_1 = i_n$, $i_2 = i_{n-1}, \ldots, i_{(n-1)/2} = i_{(n+3)/2}$. In the case that the latter equalities of indices hold, we can continue (4.44) with

$$= \tilde{\varphi}(a_1 a_n)\tilde{\varphi}(a_2 a_{n-1}) \cdots \tilde{\varphi}(a_{(n-1)/2} a_{(n+3)/2}) \cdot \tilde{\varphi}(a_{(n+1)/2}) \quad \text{(due to (4.40))}$$

$$= \varepsilon\varphi(a_1 a_n)\varphi(a_2 a_{n-1}) \cdots \varphi(a_{(n-1)/2} a_{(n+3)/2}) \cdot \varphi'(a_{(n+1)/2}). \tag{4.45}$$

(The equality (4.45) holds because $\tilde{\varphi}(a_{(n+1)/2}) = \varepsilon\varphi'(a_{(n+1)/2})$, and due to how the multiplication on $\mathbb{G}$ works.)

So all in all, what we have obtained is that

$$\tilde{\kappa}_n(a_1, \ldots, a_n) = \tag{4.46}$$

$$\begin{cases} \tilde{\varphi}(a_1 \cdots a_n) - \varepsilon\varphi(a_1 a_n)\varphi(a_2 a_{n-1}) \cdots \varphi(a_{\frac{n-1}{2}} a_{\frac{n+3}{2}}) \cdot \varphi'(a_{\frac{n+1}{2}}), \\ \quad \text{if } n \text{ is odd and } i_1 = i_n, i_2 = i_{n-1}, \ldots, i_{\frac{n-1}{2}} = i_{\frac{n+3}{2}}, \\ \tilde{\varphi}(a_1 \cdots a_n), \quad \text{otherwise.} \end{cases}$$

By comparing Equations (4.46) and (4.42) we see that, in all cases, we have $\tilde{\kappa}_n(a_1, \ldots, a_n) = 0$. This concludes the induction argument, and the proof of the implication (1) $\Rightarrow$ (2) of the proposition.

"(2) $\Rightarrow$ (1)". Consider indices $i_1, \ldots, i_n \in \{1, \ldots, k\}$ such that

$$i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$$

and elements $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ such that

$$\varphi(a_1) = \cdots = \varphi(a_n) = 0.$$

We have to prove that $\varphi(a_1 \cdots a_n) = 0$ and that $\varphi'(a_1 \cdots a_n)$ is as described in formula (4.5) from Definition 4.1.1. To this end we consider the $\mathbb{G}$-valued moment $\tilde{\varphi}(a_1 \cdots a_n) = \varphi(a_1 \cdots a_n) + \varepsilon\varphi'(a_1 \cdots a_n)$, and write it in terms of $\mathbb{G}$-valued cumulants as in the beginning of the section :

$$\tilde{\varphi}(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \prod_{V \in \pi} \tilde{\kappa}_{|V|}((a_1, \ldots, a_n) \mid V). \tag{4.47}$$

An argument very similar to the one used in the proof of the implication (1) $\Rightarrow$ (2) above shows that the sum on the right-hand side of (4.47) can only get non-zero contributions from partitions $\pi \in NC(n)$ which satisfy the hypotheses of Lemma 4.3.11. If n is even then there is no such partition, and we obtain $\tilde{\varphi}(a_1 \cdots a_n) = 0$. If n is odd, then the sum in (4.47) reduces to only one term and we obtain that

$$\tilde{\varphi}(a_1 \cdots a_n) = \tilde{\kappa}_2(a_1, a_n)\tilde{\kappa}_2(a_2, a_{n-1}) \cdots \tilde{\kappa}_2(a_{(n-1)/2}, a_{(n+3)/2}) \cdot \tilde{\kappa}_1(a_{(n+1)/2}). \tag{4.48}$$

Moreover, in the case when n is odd, the hypothesis that mixed cumulants vanish gives us that the right-hand side of (4.48) is equal to 0 unless we have $i_1 = i_n, \ldots, i_{(n-1)/2} = i_{(n+3)/2}$. And finally, if the latter equalities of indices hold, then the right-hand side of (4.48) gets converted into $\varepsilon\varphi(a_1 a_n)\varphi(a_2 a_{n-1}) \cdots \varphi(a_{(n-1)/2} a_{(n+3)/2}) \cdot \varphi'(a_{(n+1)/2})$, by the same argument that led to (4.45) in the proof of the implication (1) $\Rightarrow$ (2). The conclusion is that $\varphi(a_1 \cdots a_n) = 0$ (in all cases), and that $\varphi'(a_1 \cdots a_n)$ is as in Equation (4.5), as required. □

Corollary 4.3.13. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_k$ be subsets of $\mathcal{A}$. The following statements are equivalent:*
(1) $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.
(2) For every $n \geq 2$, for every $i_1, \ldots, i_n \in \{1, \ldots, k\}$ which are not all equal to each other, and for every $x_1 \in \mathcal{X}_{i_1}, \ldots, x_n \in \mathcal{X}_{i_n}$, one has that $\tilde{\kappa}_n(x_1, \ldots, x_n) = 0$.

Proof. This is a faithful copy of the proof giving the analogous result over $\mathbb{C}$ (cf. Theorem 11.20 in [NS06]). For the reader's convenience, we repeat here the highlights of the argument. Let $\mathcal{A}_i$ denote the unital subalgebra of $\mathcal{A}$ generated by $\mathcal{X}_i$, $1 \leq i \leq k$. The infinitesimal freeness of $\mathcal{X}_1, \ldots, \mathcal{X}_k$ is by definition equivalent to the one of $\mathcal{A}_1, \ldots, \mathcal{A}_k$, hence to the fact that condition (2) from Proposition 4.3.12 holds. We must thus prove that "(2) in Proposition 4.3.12" is equivalent to "(2) in Corollary 4.3.13". The implication "$\Rightarrow$" is trivial. For "$\Leftarrow$" it suffices (by multilinearity of $\tilde{\kappa}_n$ and Proposition 4.3.4) to prove that $\tilde{\kappa}_n(a_1, \ldots, a_n) = 0$ when

$$a_1 = x_1 \cdots x_{s_1},\ a_2 = x_{s_1+1} \cdots x_{s_2},\ \ldots,\ a_n = x_{s_{n-1}+1} \cdots x_{s_n} \tag{4.49}$$

for $n \geq 2$ and $s_1 < s_2 < \cdots < s_n$, with $x_1, \ldots, x_{s_1} \in \mathcal{X}_{i_1},\ x_{s_1+1}, \ldots, x_{s_2} \in \mathcal{X}_{i_2}, \ldots, x_{s_{n-1}+1}, \ldots, x_{s_n} \in \mathcal{X}_{i_n}$, and where the indices $i_1, \ldots, i_n$ are not all equal to each other. But for $a_1, \ldots, a_n$ as in (4.49), Proposition 4.3.5 gives us the cumulant $\tilde{\kappa}_n(a_1, \ldots, a_n)$ as a sum of cumulants $\tilde{\kappa}_\pi(x_1, \ldots, x_{s_n})$; and a direct combinatorial analysis (exactly as on p. 186 of [NS06]) shows that all the latter cumulants vanish because of condition (2) form Corollary 4.3.13. □

Remark 4.3.14. Since the functional $\tilde{\varphi} : \mathcal{A} \to \mathbb{G}$ and its associated cumulants $\tilde{\kappa}_n$ play such a central role in the proof of Theorem 4.1.2, it is natural to ask: can't one actually characterize infinitesimal freeness by the same kind of moment condition as in the definition of usual freeness, with the only modification that one now uses $\tilde{\varphi}$ instead of φ? To be precise, consider the following condition which a family of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}$ may or may not satisfy:

$$\begin{cases} \text{For every } n \geq 1 \text{ and } 1 \leq i_1, \ldots, i_n \leq k \text{ such that } i_1 \neq i_2, \ldots, i_{n-1} \neq i_n, \\ \text{and every } a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n} \text{ such that } \tilde{\varphi}(a_1) = \cdots = \tilde{\varphi}(a_n) = 0, \\ \text{one has that } \tilde{\varphi}(a_1 \cdots a_n) = 0. \end{cases} \tag{4.50}$$

Isn't then condition (4.50) equivalent to infinitesimal freeness?

On the positive side it is immediate, directly from Definition 4.1.1, that (4.50) is indeed implied by infinitesimal freeness. However, the converse statement is not true : it may happen that (4.50) is satisfied and yet $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are not infinitesimally free. What causes this to happen is that one cannot generally "center" an element $a \in \mathcal{A}$ with respect to $\tilde{\varphi}$ (the scalars available are from $\mathbb{C}$, and there may be no $\lambda \in \mathbb{C}$ such that $\tilde{\varphi}(a - \lambda 1_{\mathcal{A}}) = 0$). This limits the scope of condition (4.50), and makes it insufficient for recomputing $\tilde{\varphi}$ on $\mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k)$ from the restrictions $\tilde{\varphi} \mid \mathcal{A}_i$, $1 \leq i \leq k$.

For a simple concrete example showing how (4.50) may fail to imply infinitesimal freeness, suppose we are in the situation from Example 4.2.7, with $\mathcal{A} = \mathbb{C}[\mathbb{Z}_2 * \cdots * \mathbb{Z}_2]$ and where $\mathcal{A}_1 = \mathrm{span}\{1_{\mathcal{A}}, u_1\}, \ldots, \mathcal{A}_k = \mathrm{span}\{1_{\mathcal{A}}, u_k\}$ are the k copies of $\mathbb{C}[\mathbb{Z}_2]$ canonically embedded into $\mathcal{A}$. Suppose moreover that the linear functionals $\varphi, \varphi' : \mathcal{A} \to \mathbb{C}$ are such that $\tilde{\varphi} = \varphi + \varepsilon \varphi'$ satisfies

$$\tilde{\varphi}(1_{\mathcal{A}}) = 1, \quad \tilde{\varphi}(u_1) = \cdots = \tilde{\varphi}(u_k) = \varepsilon. \tag{4.51}$$

Then, no matter how $\tilde{\varphi}$ acts on words of length ≥ 2 made with $u_1, \ldots, u_k$, it will be true that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ satisfy condition (4.50) with respect to $\tilde{\varphi}$; this is due to the simple reason that the restrictions $\tilde{\varphi} \mid \mathcal{A}_i$ $(1 \leq i \leq k)$ are one-to-one. But on the other hand, Remark 4.2.2 tells us that if $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are to be infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$, then $\tilde{\varphi}$ is uniquely determined by (4.51); for example, the formulas given for illustration in Equations (4.16), (4.17) imply that we must have $\tilde{\varphi}(u_1 u_2) = \tilde{\varphi}(u_1)\tilde{\varphi}(u_2) = \varepsilon^2 = 0$. Hence any choice of $\tilde{\varphi}$ as in (4.51) and with $\tilde{\varphi}(u_1 u_2) \neq 0$ provides an example for how condition (4.50) does not imply infinitesimal freeness.

We conclude this section by establishing the fact about traciality that was announced in Remark 4.2.3.

Lemma 4.3.15. *Let $\mathcal{B}$ be a unital subalgebra of $\mathcal{A}$, and suppose that $\tilde{\varphi} \mid \mathcal{B}$ is a trace. Then*

$$\tilde{\kappa}_n(b_1, b_2, \ldots, b_n) = \tilde{\kappa}_n(b_2, b_n, \ldots, b_1), \quad \forall\, n \geq 2,\ b_1, \ldots, b_n \in \mathcal{B}. \tag{4.52}$$

Proof. Let Γ be the cyclic permutation of $\{1, \ldots, n\}$ defined by $\Gamma(1) = 2, \ldots, \Gamma(n-1) = n, \Gamma(n) = 1$. It is easy to see (cf. Exercise 9.41 on p. 153 of [NS06]) that Γ induces an automorphism of the lattice $NC(n)$ which maps $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ to $\Gamma \cdot \pi := \{\Gamma(V_1), \ldots, \Gamma(V_p)\}$.

Now let some $b_1, \ldots, b_n \in \mathcal{B}$ be given. The right-hand side of (4.52) is $\tilde{\kappa}_n(b_{\Gamma(1)}, \ldots, b_{\Gamma(n)})$, which is by definition equal to

$$\sum_{\pi \in NC(n)} \text{Möb}^{(A)}(\pi, 1_n) \cdot \tilde{\varphi}_\pi(b_{\Gamma(1)}, \ldots, b_{\Gamma(n)}). \tag{4.53}$$

By taking into account the traciality of $\tilde{\varphi}$ on $\mathcal{B}$ it is easily verified that $\tilde{\varphi}_\pi(b_{\Gamma(1)}, \ldots, b_{\Gamma(n)}) = \tilde{\varphi}_{\Gamma \cdot \pi}(b_1, \ldots, b_n)$, $\forall \pi \in NC(n)$. Since $\text{Möb}^{(A)}(\Gamma \cdot \pi, 1_n)$ $= \text{Möb}^{(A)}(\Gamma \cdot \pi, \Gamma \cdot 1_n) = \text{Möb}^{(A)}(\pi, 1_n)$, $\forall \pi \in NC(n)$, it becomes clear that the change of variable $\Gamma \cdot \pi =: \rho$ will convert the sum from (4.53) into the one which defines $\tilde{\kappa}_n(b_1, \ldots, b_n)$. □

Proposition 4.3.16. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ that are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$. If $\varphi \mid \mathcal{A}_i$ and $\varphi' \mid \mathcal{A}_i$ are traces for every $1 \leq i \leq k$, then φ and φ' are traces on Alg($\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$).*

Proof. The given hypothesis and the required conclusion can be rephrased by saying that $\tilde{\varphi}$ is a trace on every $\mathcal{A}_i$, and respectively that $\tilde{\varphi}$ is a trace on Alg($\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$). Clearly, the rephrased conclusion will follow if we prove that

$$\tilde{\varphi}(x_1 \cdots x_{n-1} x_n) = \tilde{\varphi}(x_n x_1 \cdots x_{n-1}) \tag{4.54}$$

where $x_1 \in \mathcal{A}_{i_1}, \ldots, x_n \in \mathcal{A}_{i_n}$ with $n \geq 2$ and $1 \leq i_1, \ldots, i_n \leq k$. Let us fix such n, $i_1, \ldots, i_n$ and $x_1, \ldots, x_n$. It is moreover convenient to denote $y_1 := x_n, y_2 := x_1, \ldots, y_n := x_{n-1}$, so that (4.54) takes the form $\tilde{\varphi}(x_1 \cdots x_n)$ $= \tilde{\varphi}(y_1 \cdots y_n)$.

Let π_o be the partition of $\{1, \ldots, n\}$ defined by the requirement that for $1 \leq p < q \leq n$ we have $\big(\, p, q$ in the same block of $\pi_o \,\big) \Leftrightarrow i_p = i_q$. The hypothesis that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free and Proposition 4.3.12 imply that

$$\tilde{\varphi}(x_1 \cdots x_n) = \sum_{\substack{\pi \in NC(n) \text{ such} \\ \text{that } \pi \leq \pi_o}} \tilde{\kappa}_\pi(x_1, \ldots, x_n). \tag{4.55}$$

(Note that π_o may not belong to $NC(n)$, but the inequality $\pi \leq \pi_o$ still makes sense, in reverse refinement order.) Now, by using Lemma 4.3.15 it is easily checked that for every $\pi \in NC(n)$ such that $\pi \leq \pi_o$ one has

$$\tilde{\kappa}_\pi(x_1, \ldots, x_n) = \tilde{\kappa}_{\Gamma \cdot \pi}(y_1, \ldots, y_n), \tag{4.56}$$

where "$\Gamma \cdot \pi$" has the same significance as in the proof of Lemma 4.3.15. If we combine (4.55) with (4.56) and then make the change of variable $\Gamma \cdot \pi =: \rho$, we arrive to

$$\tilde{\varphi}(x_1 \cdots x_n) = \sum_{\substack{\rho \in NC(n) \text{ such} \\ \text{that } \rho \leq \Gamma \cdot \pi_o}} \tilde{\kappa}_\rho(y_1, \ldots, y_n). \tag{4.57}$$

Finally, we invoke once more the infinitesimal freeness of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ and Proposition 4.3.12, to conclude that the right-hand side of (4.57) is precisely the moment-cumulant expansion for $\tilde{\varphi}(y_1 \cdots y_n)$. □

4.4 Alternating products of infinitesimally free random variables

In Proposition 4.3.12 we saw that infinitesimal freeness can be described as a vanishing condition for mixed $\mathbb{G}$-valued cumulants. Because of this fact and because $\mathbb{G}$ is commutative, (which makes practically all calculations with non-crossing cumulants go without any change from $\mathbb{C}$-valued to $\mathbb{G}$-valued framework) we get a "generic method" for proving infinitesimal versions of various results presented in the monograph [NS06] – replace $\mathbb{C}$ by $\mathbb{G}$ in the proof of the original result, then take the soul part of what comes out. Note that the infinitesimal results so obtained do not have $\mathbb{G}$ in their statement, hence could also be attacked by using other approaches to infinitesimal freeness (in which case, however, proving them may be more than a straightforward routine).

In this section we show how the generic method suggested above works when applied to the topic of alternating products of infinitesimally free random variables. In particular, we will obtain the infinitesimal versions for two important facts related to this topic, that were originally found in [NS96] – one of them is about compressions by free projections, the other concerns a method of constructing free families of free Poisson elements. Since the proofs of the $\mathbb{G}$-valued formulas that we need are identical to those of their $\mathbb{C}$-valued counterparts, we will not give them here, but we will merely indicate where in [NS06] can the $\mathbb{C}$-valued proofs be exactly found. The starting point is provided by the following formulas, obtained by doing the $\mathbb{C}$-to-$\mathbb{G}$ change in Theorem 14.4 of [NS06].

Proposition 4.4.1. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps and let $\mathcal{A}_1, \mathcal{A}_2$ be unital subalgebras of $\mathcal{A}$ which are infinitesimally free. Consider the functional $\tilde{\varphi} = \varphi + \varepsilon\varphi' : \mathcal{A} \to \mathbb{G}$ and the associated cumulant functionals $(\tilde{\kappa}_n : \mathcal{A}^n \to \mathbb{G})_{n \geq 1}$. Recall that for every $n \geq 1$ and $\pi \in NC(n)$ we also have functionals $\tilde{\varphi}_\pi, \tilde{\kappa}_\pi : \mathcal{A}^n \to \mathbb{G}$, as defined in Notation 4.3.2.*

1º For every $a_1, \ldots, a_n \in \mathcal{A}_1$ and $b_1, \ldots, b_n \in \mathcal{A}_2$ one has that

$$\tilde{\varphi}(a_1 b_1 \cdots a_n b_n) = \sum_{\pi \in NC(n)} \tilde{\kappa}_\pi(a_1, \ldots, a_n) \cdot \tilde{\varphi}_{Kr(\pi)}(b_1, \ldots, b_n). \tag{4.58}$$

2º For every $a_1, \ldots, a_n \in \mathcal{A}_1$ and $b_1, \ldots, b_n \in \mathcal{A}_2$ one has that

$$\tilde{\kappa}_n(a_1 b_1, \ldots, a_n b_n) = \sum_{\pi \in NC(n)} \tilde{\kappa}_\pi(a_1, \ldots, a_n) \cdot \tilde{\kappa}_{Kr(\pi)}(b_1, \ldots, b_n). \tag{4.59}$$

□

We now start on the application to free compressions.

Definition 4.4.2. Let $(\mathcal{A}, \varphi, \varphi')$ be an incps, and let $p \in \mathcal{A}$ be an idempotent element such that $\varphi(p) \neq 0$. We denote $\varphi(p) =: \alpha$ and $\varphi'(p) = \alpha'$. The *compression* of $(\mathcal{A}, \varphi, \varphi')$ by p is then defined to be the incps $(\mathcal{B}, \psi, \psi')$ where

$$\mathcal{B} := p\mathcal{A}p = \{b \in \mathcal{A} \mid pb = b = bp\} \tag{4.60}$$

and where $\psi, \psi' : \mathcal{B} \to \mathbb{C}$ are defined by

$$\psi(b) = \frac{1}{\alpha}\varphi(b), \quad \psi'(b) = \frac{1}{\alpha}\varphi'(b) - \frac{\alpha'}{\alpha^2}\varphi(b), \quad b \in \mathcal{B}. \tag{4.61}$$

Remark 4.4.3. 1º In the preceding definition, note that the consolidated functional $\tilde{\psi} = \psi + \varepsilon\psi' : \mathcal{B} \to \mathbb{G}$ is given by the simple formula

$$\tilde{\psi}(b) = \frac{1}{\widetilde{\alpha}}\tilde{\varphi}(b), \quad b \in \mathcal{B}, \tag{4.62}$$

where $\widetilde{\alpha} := \alpha + \varepsilon\alpha' \in \mathbb{G}$.

2º If in the preceding definition $(\mathcal{A}, \varphi, \varphi')$ is a $*$-incps and p is a projection, then by using the relations $p = p^* = p^2$ we immediately infer that $0 < \alpha \leq 1$ and $\alpha' \in \mathbb{R}$. As a consequence, $(\mathcal{B}, \psi, \psi')$ defined there is a $*$-incps as well.

Theorem 4.4.4. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps. Let $p \in \mathcal{A}$ be an idempotent element such that $\varphi(p) \neq 0$. Denote $\varphi(p) =: \alpha$, $\varphi'(p) =: \alpha'$, and consider the compressed incps $(\mathcal{B}, \psi, \psi')$ from Definition 4.4.2. For every $n \geq 1$ let $\kappa_n, \kappa'_n : \mathcal{A}^n \to \mathbb{C}$ and $\underline{\kappa}_n, \underline{\kappa}'_n : \mathcal{B}^n \to \mathbb{C}$ be the nth non-crossing cumulant and infinitesimal cumulant functional associated to $(\mathcal{A}, \varphi, \varphi')$ and to $(\mathcal{B}, \psi, \psi')$, respectively. Let $\mathcal{X}$ be a subset of $\mathcal{A}$ which is infinitesimally free from $\{p\}$. Then we have*

$$\underline{\kappa}_n(px_1p, \ldots, px_np) = \frac{1}{\alpha}\kappa_n(\alpha x_1, \ldots, \alpha x_n), \quad \forall\, n \geq 1,\ x_1, \ldots, x_n \in \mathcal{X} \tag{4.63}$$

and

$$\begin{cases} \underline{\kappa}'_1(px_1p) = \kappa'_1(x_1), \forall\, x_1 \in \mathcal{X}, \\ \underline{\kappa}'_n(px_1p, \ldots, px_np) = \frac{(n-1)\alpha'}{\alpha^2}\kappa'_n(\alpha x_1, \ldots, \alpha x_n), n \geq 2, x_1, \ldots, x_n \in \mathcal{X}. \end{cases} \tag{4.64}$$

Proof. It is easily verified that Equations (4.63) and (4.64) are the body part and respectively the soul part for the formula

$$\underline{\widetilde{\kappa}}_n(px_1p, \ldots, px_np) = \tilde{\alpha}^{n-1} \cdot \tilde{\kappa}_n(x_1, \ldots, x_n) \in \mathbb{G}, \quad \forall\, n \geq 1, \ x_1, \ldots, x_n \in \mathcal{X}, \tag{4.65}$$

where the "tilde" notations have their usual meaning ($\underline{\widetilde{\kappa}}_n = \underline{\kappa}_n + \varepsilon \cdot \underline{\kappa}'_n$, $\tilde{\alpha} = \alpha + \varepsilon \cdot \alpha'$). But the latter formula is just the $\mathbb{G}$-valued counterpart for Theorem 14.10 in [NS06]; its proof is obtained by faithfully doing the $\mathbb{C}$-to-$\mathbb{G}$ transcription of the proof of that theorem in [NS06], with the minor change that the powers of $\tilde{\alpha}$ must be kept outside the cumulant functionals (one cannot write "$\tilde{\kappa}_n(\tilde{\alpha}x_1, \ldots, \tilde{\alpha}x_n)$", since $\mathcal{A}$ is only a $\mathbb{C}$-algebra). Note that the argument obtained in this way is indeed an application of Proposition 4.4.1, in the same way as Theorem 14.10 is an application of Theorem 14.4 in [NS06]. □

Corollary 4.4.5. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps. Let $p \in \mathcal{A}$ be an idempotent element with $\varphi(p) \neq 0$, and consider the compressed incps $(\mathcal{B}, \psi, \psi')$ defined as above. Let $\mathcal{X}_1, \ldots, \mathcal{X}_k$ be subsets of $\mathcal{A}$ such that $\{p\}, \mathcal{X}_1, \ldots, \mathcal{X}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$. Put $\mathcal{Y}_i = p\mathcal{X}_ip \subseteq \mathcal{B}$, $1 \leq i \leq k$. Then $\mathcal{Y}_1, \ldots, \mathcal{Y}_k$ are infinitesimally free in $(\mathcal{B}, \psi, \psi')$.*

Proof. This is an immediate consequence of Corollary 4.3.13, where the needed vanishing of mixed cumulants follows from the explicit formulas found in Theorem 4.4.4. □

We now go to the construction of families of infinitesimally free Poisson elements. We will use the infinitesimal (a.k.a "type B") versions of semicircular and of free Poisson elements that appeared in [Pop07] in connection to limit theorems of type B, and are discussed in detail in Sections 4 and 5 of [BS09]. For the present chapter it is most convenient to introduce these elements in terms of their infinitesimal cumulants, as stated in Definitions 4.4.6 and 4.4.8 below.

Definition 4.4.6. Let $(\mathcal{A}, \varphi, \varphi')$ be a $*$-incps. A selfadjoint element $x \in \mathcal{A}$ will be called *infinitesimally semicircular* when it satisfies

$$\kappa_n(x, \ldots, x) = \kappa'_n(x, \ldots, x) = 0, \quad \forall\, n \geq 3. \tag{4.66}$$

If in addition to that we also have

$$\kappa_1(x) = 0, \quad \kappa_2(x, x) = 1, \tag{4.67}$$

then we will say that x is a *standard* infinitesimally semicircular element.

Remark 4.4.7. 1^o By using the multilinearity of κ_n, κ'_n and Proposition 4.3.4, it is immediately seen that if x is infinitesimally semicircular then so is $\alpha(x - \beta 1_{\mathcal{A}})$ for any $\alpha > 0$ and $\beta \in \mathbb{R}$. Moreover, leaving aside the trivial case when $\kappa_2(x,x) = 0$, one can always pick α and β so that $\alpha(x - \beta 1_{\mathcal{A}})$ is standard.

2^o Let x be standard infinitesimally semicircular in $(\mathcal{A}, \varphi, \varphi')$. Then all moments $\varphi(x^n)$ and $\varphi'(x^n)$ for $n \geq 1$ are completely determined by the real parameters [4] α_1', α_2' defined by

$$\alpha_1' := \kappa_1'(x) = \varphi'(x), \quad \text{and} \quad \alpha_2' := \kappa_2'(x,x) = \varphi'(x^2). \tag{4.68}$$

It is in fact very easy to calculate what these moments are. Indeed, one can calculate the $\mathbb{G}$-valued moments $\tilde{\varphi}(x^n) = \varphi(x^n) + \varepsilon\varphi'(x^n)$ by using the moment-cumulant formula (4.33), where one takes into account that

$$\tilde{\kappa}_1(x) = \varepsilon\alpha_1', \ \tilde{\kappa}_2(x,x) = 1 + \varepsilon\alpha_2', \text{ and } \tilde{\kappa}_n(x,\ldots,x) = 0 \text{ for all } n \geq 3.$$

The expansion of $\tilde{\varphi}(x^n)$ in terms of $\{\tilde{\kappa}_\pi(x,\ldots,x) \mid \pi \in NC(n)\}$ can get non-zero contributions only from such partitions π where every block V of π has $|V| \leq 2$ and where there is at most one block of π of cardinality 1 (the latter condition coming from the fact that $(\tilde{\kappa}_1(x))^2 = 0$). We distinguish two cases, depending on the parity of n.

Case 1. n is even, $n = 2m$. We get a sum extending over non-crossing pairings in $NC(n)$, which gives us

$$\tilde{\varphi}(x^{2m}) = C_m \cdot (1 + \varepsilon\alpha_2')^m = C_m \cdot (1 + \varepsilon m\alpha_2'),$$

or in other words

$$\varphi(x^{2m}) = C_m, \quad \varphi'(x^{2m}) = \alpha_2' \cdot (mC_m), \tag{4.69}$$

where C_m stands for the mth Catalan number.

Case 2. n is odd, $n = 2m+1$. Here we get a sum extending over the partitions $\pi \in NC(n)$ which have one block of 1 element and m blocks of 2 elements. There are $(2m+1)C_m$ such partitions; so we obtain

$$\tilde{\varphi}(x^{2m+1}) = (2m+1)C_m \cdot \Big((\varepsilon\alpha_1')\,(1 + \varepsilon\alpha_2')^m \Big),$$

leading to

$$\varphi(x^{2m+1}) = 0, \quad \varphi'(x^{2m+1}) = \alpha_1' \cdot \big((2m+1)C_m\big). \tag{4.70}$$

[4] Any two numbers $\alpha_1', \alpha_2' \in \mathbb{R}$ can appear here. Indeed, Example 4.7.7 shows situations where one has $\alpha_1' = 1, \alpha_2' = 0$ and respectively $\alpha_1' = 0, \alpha_2' = 2$. One can rescale the functionals φ' of these two special cases to get standard infinitesimal semicirculars x_1, x_2 having any pairs of parameters $\alpha_1', 0$ and respectively $0, \alpha_2'$; then due to Proposition 4.2.4 one may assume that x_1, x_2 are infinitesimally free, and form the average $(x_1 + x_2)/\sqrt{2}$, which is standard infinitesimally semicircular with generic parameters in (4.68).

Definition 4.4.8. Let $(\mathcal{A}, \varphi, \varphi')$ be a $*$-incps, and let λ, β', γ' be real parameters, where $\lambda > 0$. A selfadjoint element $y \in \mathcal{A}$ will be called *infinitesimally free Poisson* of parameter λ and [5] infinitesimal parameters β', γ' when it has non-crossing cumulants given by

$$\begin{cases} \kappa_n(y, \ldots, y) &= \lambda, \\ \kappa_n'(y, \ldots, y) &= \beta' + n\gamma', \quad \forall\, n \geq 1. \end{cases} \tag{4.71}$$

Theorem 4.4.9. *Let $(\mathcal{A}, \varphi, \varphi')$ be a $*$-incps. Let $x \in \mathcal{A}$ be a standard infinitesimally semicircular element, and let $\mathcal{S}$ be a subset of $\mathcal{A}$ which is infinitesimally free from $\{x\}$. Then for every $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{S}$ we have*

$$\kappa_n(xa_1x, \ldots, xa_nx) = \varphi(a_1 \cdots a_n) \tag{4.72}$$

and

$$\kappa_n'(xa_1x, \ldots, xa_nx) = \varphi'(a_1 \cdots a_n) + n\, \varphi'(x^2) \cdot \varphi(a_1 \cdots a_n). \tag{4.73}$$

Proof. Equations (4.72) and (4.73) are the body part and respectively the soul part for the formula

$$\tilde{\kappa}_n(xa_1x, \ldots, xa_nx) = \big(\tilde{\kappa}_2(x, x)\big)^n \cdot \tilde{\varphi}(a_1 \cdots a_n) \in \mathbb{G}. \tag{4.74}$$

The proof of the latter formula is obtained by doing the $\mathbb{C}$-to-$\mathbb{G}$ transcription either for the arguments used in Proposition 12.18 and Example 12.19 on pp. 207-208 of [NS06], or for the arguments in Propositions 17.20 and 17.21 on pp. 283-284 of [NS06]. $\square$

The ensuing construction of families of infinitesimally free Poisson elements is stated in the next corollary. Part 2^o of the corollary has also appeared as Corollary 36 of [BS09].

Corollary 4.4.10. *Let $(\mathcal{A}, \varphi, \varphi')$ be a $*$-incps, and let $x \in \mathcal{A}$ be a standard infinitesimally semicircular element. Let $e_1, \ldots, e_k \in \mathcal{A}$ be projections such that $e_i \perp e_j$ for $1 \leq i < j \leq k$ and such that $\{e_1, \ldots, e_k\}$ is infinitesimally free from $\{x\}$. Then*

1^o *The elements $xe_1x, \ldots, xe_kx$ form an infinitesimally free family in $(\mathcal{A}, \varphi, \varphi')$.*

2^o *For every $1 \leq i \leq k$, xe_ix is infinitesimally free Poisson with parameter λ_i and infinitesimal parameters β_i', γ_i' given by $\lambda_i = \varphi(e_i)$, $\beta_i' = \varphi'(e_i)$, $\gamma_i' = \varphi'(x^2) \cdot \varphi(e_i)$.*

[5] A more complete definition of these elements would also use a 4th parameter $r > 0$, and have each of λ, β', γ' multiplied by r^n in Equations (4.71). For the sake of simplicity, here we have set this additional parameter to $r = 1$.

Proof. 1^o This is an immediate consequence of Corollary 4.3.13, where the needed vanishing of mixed cumulants follows from the explicit formulas found in Theorem 4.4.9.

2^o By putting $a_1 = \cdots = a_n := e_i$ in (4.72) and (4.73) we see that the cumulants of xe_ix have the form required in Definition 4.4.8, with parameters $\lambda_i, \beta'_i, \gamma'_i$ as stated. □

4.5 Relations with the lattices $NC^{(B)}(n)$

In this section we remember that the concept of incps has its origins in the considerations "of type B" from [BGN03], and we look at how the essence of these considerations persists in the framework of the present chapter.

The strategy of [BGN03] was to study the type B analogue for the operation of *boxed convolution* $\boxed{\star}^{(A)}$ from Definition 2.3.11. The focus on $\boxed{\star}^{(A)}$ was motivated by the fact that it provides in some sense a "middle ground" between alternating products of free random variables and the structure of intervals in the lattices $NC(n)$ (see discussion on pp. 2282-2283 of [BGN03]). The key point discovered in [BGN03] (stated in the form of the equation $\boxed{\star}^{(B)} = \boxed{\star}_{\mathbb{G}}^{(A)}$ in Theorem 2.5.3) was that boxed convolution of type B can still be defined by the formulas from type A, provided that one uses scalars from $\mathbb{G}$.

For a detailed discussion on $\boxed{\star}^{(A)}$ we refer the reader to Lecture 17 of [NS06]. What is important for us here is that the formula used to define $\boxed{\star}^{(A)}$ (cf. Equation (17.1) on p. 273 of [NS06]) has already made an appearance, in $\mathbb{G}$-valued context, in Equations (4.58), (4.59) of the preceding section. So then, the present incarnation of the "$\boxed{\star}^{(B)} = \boxed{\star}_{\mathbb{G}}^{(A)}$" principle from Theorem 2.5.3 should just amount to the following fact: if one takes the soul parts of Equations (4.58) and (4.59), then summations over $NC^{(B)}(n)$ must arise. This is stated precisely in Theorem 4.5.4 below, which is actually an easy application of the fact that the absolute value map Abs : $NC^{(B)}(n) \to NC(n)$ is an $(n+1)$-to-1 cover (see Theorem 2.5.1).

We start by introducing some notations that will be used in Theorem 4.5.4, namely the type B analogues for the functionals $\varphi_\pi^{(A)}$ and $\kappa_\pi^{(A)}$ from subsection 3.2.

Notation 4.5.1. Let $(\mathcal{A}, \varphi, \varphi')$ be an incps and consider the families of non-crossing cumulant functionals $(\kappa_n, \kappa'_n)_{n\geq 1}$. For every $n \geq 1$ and every $\tau \in NC^{(B)}(n)$, define a multilinear functional $\kappa_\tau^{(B)} : \mathcal{A}^n \to \mathbb{C}$, as follows.

Case 1. If $\tau \in NCZ^{(B)}(n)$, $\tau = \{Z, V_1, -V_1, \ldots, V_p, -V_p\}$, then we put

$$\kappa_\tau^{(B)}(a_1, \ldots, a_n) := \kappa'_{\frac{|Z|}{2}}\big((a_1, \ldots, a_n) \mid \mathrm{Abs}(Z)\big) \cdot \prod_{j=1}^{p} \kappa_{|V_j|}\big((a_1, \ldots, a_n) \mid \mathrm{Abs}(V_j)\big), \qquad (4.75)$$

for every $a_1, \ldots, a_n \in \mathcal{A}$.

Case 2. If $\tau \in NC^{(B)}(n) \setminus NCZ^{(B)}(n)$, $\tau = \{V_1, -V_1, \ldots, V_p, -V_p\}$, then we put

$$\kappa_\tau^{(B)}(a_1, \ldots, a_n) := \prod_{j=1}^{p} \kappa_{|V_j|}\big((a_1, \ldots, a_n) \mid \mathrm{Abs}(V_j) \big), \qquad (4.76)$$

for $a_1, \ldots, a_n \in \mathcal{A}$.

Notation 4.5.2. Let $(\mathcal{A}, \varphi, \varphi')$ be an incps. Consider the families of multilinear functionals $(\varphi_n, \varphi'_n : \mathcal{A}^n \to \mathbb{C})_{n \geq 1}$ defined by

$$\varphi_n(a_1, \ldots, a_n) = \varphi(a_1 \cdots a_n),$$

$$\varphi'_n(a_1, \ldots, a_n) = \varphi'(a_1 \cdots a_n).$$

Then for every $n \geq 1$ and every $\tau \in NC^{(B)}(n)$ we define a multilinear functional $\varphi_\tau^{(B)} : \mathcal{A}^n \to \mathbb{C}$ by the same recipe as in Notation 4.5.1 (with discussion separated in 2 cases), where every occurrence of κ_m (respectively κ'_m) is replaced by φ_m (respectively φ'_m). For example, the analogue of Case 1 is like this: for $n \geq 1$ and for $\tau = \{Z, V_1, -V_1, \ldots, V_p, -V_p\}$ in $NCZ^{(B)}(n)$ we define $\varphi_\tau^{(B)} \mathcal{A}^n \to \mathbb{C}$ by putting

$$\varphi_\tau^{(B)}(a_1, \ldots, a_n) := \varphi'_{\frac{|Z|}{2}}\big(a_1, \ldots, a_n) \mid \mathrm{Abs}(Z) \big) \cdot \prod_{j=1}^{p} \varphi_{|V_j|}\big((a_1, \ldots, a_n) \mid \mathrm{Abs}(V_j) \big), \qquad (4.77)$$

for $a_1, \ldots, a_n \in \mathcal{A}$.

Remark 4.5.3. 1^o It is immediate that for $\tau \in NC^{(B)}(n) \setminus NCZ^{(B)}(n)$ one has

$$\kappa_\tau^{(B)} = \kappa_{Abs(\tau)}^{(A)}, \quad \varphi_\tau^{(B)} = \varphi_{Abs(\tau)}^{(A)}. \qquad (4.78)$$

2^o The functionals introduced in Notation 4.5.1 extend both families κ_n and κ'_n. Indeed, we have that $\kappa'_n = \kappa_{1_{\pm n}}^{(B)}$ and that $\kappa_n = \kappa_{1_n}^{(A)} = \kappa_\tau^{(B)}$ for every $n \geq 1$ and any $\tau \in NC^{(B)}(n)$ such that $\mathrm{Abs}(\tau) = 1_n$ (e.g. $\tau = \{\{1, \ldots, n\}, \{-1, \ldots, -n\}\}$). A similar remark holds in connection to the functionals $\varphi_\tau^{(B)}$ – they extend both families φ_n and φ'_n.

Theorem 4.5.4. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps, and consider multilinear functionals on $\mathcal{A}$ as in Notations 4.5.1, 4.5.2. Let $\mathcal{A}_1, \mathcal{A}_2$ be unital subalgebras of $\mathcal{A}$ which are infinitesimally free. Then for every $a_1, \ldots, a_n \in \mathcal{A}_1$ and $b_1, \ldots, b_n \in \mathcal{A}_2$ one has*

$$\varphi'(a_1 b_1 \cdots a_n b_n) = \sum_{\sigma \in NC^{(B)}(n)} \kappa_\sigma^{(B)}(a_1, \ldots, a_n) \cdot \varphi_{Kr(\sigma)}^{(B)}(b_1, \ldots, b_n) \quad (4.79)$$

and

$$\kappa'_n(a_1 b_1, \ldots, a_n b_n) = \sum_{\sigma \in NC^{(B)}(n)} \kappa_\sigma^{(B)}(a_1, \ldots, a_n) \cdot \kappa_{Kr(\sigma)}^{(B)}(b_1, \ldots, b_n). \quad (4.80)$$

Proof. Let π be a partition in $NC(n)$, and consider the expression

$$\mathrm{So}\Big(\tilde{\kappa}_\pi(a_1, \ldots, a_n)\, \tilde{\kappa}_{Kr(\pi)}(b_1, \ldots, b_n) \Big)$$

$$= \mathrm{So}\Big(\prod_{V \in \pi} \tilde{\kappa}_{|V|}\big((a_1, \ldots, a_n) \mid V\big) \cdot \prod_{W \in Kr(\pi)} \tilde{\kappa}_{|W|}\big((b_1, \ldots, b_n) \mid W\big) \Big).$$

In view of the formula (4.29) describing the soul part of a product, the latter expression is equal to a sum of $n+1$ terms, some of them indexed by the blocks $V \in \pi$, and the others indexed by the blocks W of $\mathrm{Kr}(\pi)$. We leave it as a straightforward exercise to the reader to write these $n+1$ terms explicitly, and verify that the natural correspondence to the $n+1$ partitions in $\{\tau \in NC^{(B)}(n) \mid \mathrm{Abs}(\tau) = \pi\}$ leads to the formula

$$\mathrm{So}\Big(\tilde{\kappa}_\pi(a_1, \ldots, a_n)\, \tilde{\kappa}_{Kr(\pi)}(b_1, \ldots, b_n) \Big) \tag{4.81}$$

$$= \sum_{\substack{\tau \in NC^{(B)}(n) \text{ such} \\ \text{that } \mathrm{Abs}(\tau) = \pi}} \kappa_\tau^{(B)}(a_1, \ldots, a_n) \cdot \varphi^{(B)}_{Kr(\tau)}(b_1, \ldots, b_n).$$

(Note: the Kreweras complement $\mathrm{Kr}(\tau)$ from (4.81) is taken in the lattice $NC^{(B)}(n)$; we use here the fact that $\mathrm{Abs}(\tau) = \pi \Rightarrow \mathrm{Abs}(\mathrm{Kr}(\tau)) = \mathrm{Kr}(\pi)$ – cf. Lemma 1.4 in [BGN03].)

By summing over $\pi \in NC(n)$ on both sides of (4.81), we obtain that

$$\mathrm{So}\Big(\text{right-hand side of } (4.58) \Big) = (\text{right-hand side of } (4.81)).$$

Since the soul part of the left-hand side of Equation (4.58) is $\varphi'(a_1 b_1 \cdots a_n b_n)$, this proves that (4.79) holds. The verification of (4.80) is done in exactly the same way, by starting from Equation (4.59) of Proposition 4.4.1. □

Remark 4.5.5. If in the preceding theorem we make $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_2 = \mathbb{C}1_{\mathcal{A}}$, and if we take $b_1 = \cdots = b_n = 1_{\mathcal{A}}$, then we obtain the formula

$$\varphi'(a_1 \cdots a_n) = \sum_{\sigma \in NCZ^{(B)}(n)} \kappa_\sigma^{(B)}(a_1, \ldots, a_n), \quad \forall\, a_1, \ldots, a_n \in \mathcal{A}. \tag{4.82}$$

The terms indexed by $\sigma \in NC^{(B)}(n) \setminus NCZ^{(B)}(n)$ have disappeared in (4.82), due to the fact that $\varphi'(1_{\mathcal{A}}) = 0$. This formula could also be obtained, by a suitable Möbius inversion argument, directly from the formula (4.35) defining κ_n'.

4.6 Dual derivation systems

Notation 4.6.1. Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$, and for every $n \geq 1$ let $\mathfrak{M}_n$ denote the vector space of multilinear functionals from $\mathcal{A}^n$ to $\mathbb{C}$. If $\pi = \{V_1, \ldots, V_p\}$ is a partition in $NC(n)$ where the blocks $V_1, \ldots, V_p$ are listed in increasing order of their minimal elements, then we define a multilinear map

$$J_\pi : \mathfrak{M}_{|V_1|} \times \cdots \times \mathfrak{M}_{|V_p|} \ni (f_1, \ldots, f_p) \to f \in \mathfrak{M}_n, \tag{4.83}$$

where

$$f(a_1, \ldots, a_n) := \prod_{j=1}^{p} f_j\big((a_1, \ldots, a_n) \mid V_j \big), \quad \forall\, a_1, \ldots, a_n \in \mathcal{A}. \tag{4.84}$$

Remark 4.6.2. 1^o The formula (4.84) from the preceding notation is the same as those used to define the families of functionals $\{\varphi_\pi^{(A)} \mid \pi \in NC(n)\}$ and $\{\kappa_\pi^{(A)} \mid \pi \in NC(n)\}$ in Subsection 2.3.2. Hence if $(\mathcal{A}, \varphi)$ is a non-commutative probability space and if $(\kappa_n)_{n \geq 1}$ are the non-crossing cumulant functionals associated to φ, then for $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ as in Notation 4.6.1 we get that

$$J_\pi(\, \kappa_{|V_1|}, \ldots, \kappa_{|V_p|}\,) = \kappa_\pi^{(A)}. \tag{4.85}$$

Likewise, for the same $(\mathcal{A}, \varphi)$ and π we get

$$J_\pi(\, \varphi_{|V_1|}, \ldots, \varphi_{|V_p|}\,) = \varphi_\pi^{(A)}. \tag{4.86}$$

2^o Let $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ be as in Notation 4.6.1, and let $1 \leq j \leq p$ be such that V_j is an interval-block of π. Denote $|V_j| =: m$ and let $\overset{\vee}{\pi} \in NC(n-m)$ be the partition obtained by removing the block V_j out of π and by redenoting the elements of $\{1, \ldots, n\} \setminus V_j$ as $1, \ldots, n-m$, in increasing order. On the other hand, let us denote by $\gamma \in NC(n)$ the partition of $\{1, \ldots, n\}$ into the two blocks V_j and $\{1, \ldots, n\} \setminus V_j$. It is then immediate that for every $f_1 \in \mathfrak{M}_{|V_1|}, \ldots, f_p \in \mathfrak{M}_{|V_p|}$ we can write

$$J_\pi(f_1, \ldots, f_p) = J_\gamma(g, f_j) \quad \text{where } g := J_{\overset{\vee}{\pi}}(f_1, \ldots f_{j-1}, f_{j+1}, \ldots, f_p). \tag{4.87}$$

Due to this observation and to the fact that every non-crossing partition has interval-blocks, considerations about the multilinear functions J_π from Notation 4.6.1 can sometimes be reduced (via an induction argument on $|\pi|$) to discussing the case when $|\pi| = 2$.

Definition 4.6.3. Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and let the spaces $(\mathfrak{M}_n)_{n \geq 1}$ and the multilinear functions $\{J_\pi \mid \pi \in \cup_{n=1}^{\infty} NC(n)\}$ be as in Notation 4.6.1. We will call *dual derivation system* a family of linear maps

$(d_n : \mathfrak{D}_n \to \mathfrak{M}_n)_{n \geq 1}$ where, for every $n \geq 1$, $\mathfrak{D}_n$ is a linear subspace of $\mathfrak{M}_n$, and where the following two conditions are satisfied.

(i) Let $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ be as in Notation 4.6.1. Then for every $f_1 \in \mathfrak{D}_{|V_1|}, \ldots, f_p \in \mathfrak{D}_{|V_p|}$ one has that $J_\pi(f_1, \ldots, f_p) \in \mathfrak{D}_n$ and that

$$d_n\big(J_\pi(f_1, \ldots, f_p) \big) = \sum_{j=1}^{p} J_\pi(f_1, \ldots, f_{j-1}, d_{|V_j|}(f_j), f_{j+1}, \ldots, f_p). \quad (4.88)$$

(ii) For every $f \in \mathfrak{D}_1$ and every $n \geq 1$ one has that $f \circ \mathrm{Mult}_n \in \mathfrak{D}_n$ and that

$$d_n\big(f \circ \mathrm{Mult}_n\big) = (d_1\, f) \circ \mathrm{Mult}_n, \quad (4.89)$$

where $\mathrm{Mult}_n : \mathcal{A}^n \to \mathcal{A}$ is the multiplication map.

Remark 4.6.4. 1^o When verifying condition (i) in Definition 4.6.3, it suffices to check the particular case when $|\pi| = 2$. Indeed, the general case of Equation (4.88) can then be obtained by induction on $|\pi|$, where one invokes the argument from (4.87).

2^o In the setting of Definition 4.6.3, let us use the notation $f \times g$ for the functional obtained by "concatenating" $f \in \mathfrak{M}_m$ and $g \in \mathfrak{M}_n$. So $f \times g \in \mathfrak{M}_{m+n}$ acts simply by

$$(f \times g)(a_1, \ldots, a_m, b_1, \ldots, b_n) = f(a_1, \ldots, a_m) g(b_1, \ldots, b_n),$$

for all $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathcal{A}$.
Clearly one can write $f \times g = J_\gamma(f, g)$ where $\gamma \in NC(m+n)$ is the partition with two blocks $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$. By using Equation (4.88) we thus obtain that

$$d_{m+n}(f \times g) = \big(d_m(f) \times g\big) + \big(f \times d_n(g)\big), \quad \forall\, m, n \geq 1,\ f \in \mathfrak{M}_m,\ g \in \mathfrak{M}_n. \quad (4.90)$$

So a dual derivation system gives in particular a derivation on the algebra structure defined by using concatenation on $\oplus_{n=1}^{\infty} \mathfrak{M}_n$. Note however that Equation (4.90) alone is not sufficient to ensure condition (i) from Definition 4.6.3 (since it cannot control J_π for partitions such as $\pi = \{\, \{1,3\},\, \{2\}\, \} \in NC(3)$).

Proposition 4.6.5. *Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and let $(d_n : \mathfrak{D}_n \to \mathfrak{M}_n)_{n \geq 1}$ be a dual derivation system on $\mathcal{A}$. Let φ be a linear functional in $\mathfrak{D}_1$, and denote $d_1(\varphi) =: \varphi'$. Consider the incps $(\mathcal{A}, \varphi, \varphi')$, and let $(\kappa_n, \kappa_n')_{n \geq 1}$ be the non-crossing cumulant and infinitesimal cumulant functionals associated to this incps. Then for every $n \geq 1$ we have that*

$$\kappa_n \in \mathfrak{D}_n \text{ and } d_n(\kappa_n) = \kappa_n'. \quad (4.91)$$

Proof. Denote as usual $\varphi_n := \varphi \circ \mathrm{Mult}_n$, $\varphi'_n := \varphi' \circ \mathrm{Mult}_n$, $n \geq 1$. Since $\varphi \in \mathfrak{D}_1$, condition (ii) from Definition 4.6.3 implies that $\varphi_n \in \mathfrak{D}_n$ and $d_n(\varphi_n) = \varphi'_n$ for every $n \geq 1$.

Now let $\pi = \{V_1, \ldots, V_p\}$ be a partition in $NC(n)$, with $V_1, \ldots, V_p$ written in increasing order of their minimal elements. By using Equation (4.86) from Remark 4.6.2 and condition (i) in Definition 4.6.3 we find that

$$d_n(\varphi_\pi^{(A)}) = \sum_{j=1}^{p} J_\pi\big(\varphi_{|V_1|}, \ldots, \varphi_{|V_{j-1}|}, \varphi'_{|V_j|}, \varphi_{|V_{j+1}|}, \ldots, \varphi_{|V_p|} \big) \tag{4.92}$$

(where the latter formula incorporates the fact that $d_{|V_j|}(\varphi_{|V_j|}) = \varphi'_{|V_j|}$).

We next consider the formula (2.5) which expresses a cumulant functional κ_n in terms of the functionals $\{\varphi_\pi^{(A)} \mid \pi \in NC(n)\}$. From this formula it follows that $\kappa_n \in \mathfrak{D}_n$ and that

$$d_n(\kappa_n) = \sum_{\substack{\pi \in NC(n) \\ \pi = \{V_1, \ldots, V_p\}}} \mathrm{M\ddot{o}b}(\pi, 1_n) \Big(\sum_{j=1}^{p} J_\pi\big(\varphi_{|V_1|}, \ldots, \varphi'_{|V_j|}, \ldots, \varphi_{|V_p|} \big) \Big). \tag{4.93}$$

It is immediate that on the right-hand side of (4.93) we have obtained precisely the sum over $\{(\pi, V) \mid \pi \in NC(n),\ V \text{ block of } \pi\}$ which was used to introduce κ'_n in Definition 4.3.7. □

Proposition 4.6.6. *Let $(\mathcal{A}, \varphi, \varphi')$ be an incps, and consider the multilinear functionals $\varphi_\pi^{(A)}$ ($\pi \in NC(n)$, $n \geq 1$) which were introduced in Subsection 2.3.2. Suppose that for every $n \geq 1$ the set $\{\varphi_\pi^{(A)} \mid \pi \in NC(n)\}$ is linearly independent in $\mathfrak{M}_n$; let $\mathfrak{D}_n$ denote its span, and let $d_n : \mathfrak{D}_n \to \mathfrak{M}_n$ be the linear map defined by the requirement that*

$$d_n(\, \varphi_\pi^{(A)} \,) = \sum_{\substack{\tau \in NCZ^{(B)}(n) \text{ such} \\ \text{that } Abs(\tau) = \pi}} \varphi_\tau^{(B)}, \quad \forall\, \pi \in NC(n), \tag{4.94}$$

with $\varphi_\tau^{(B)}$ as in Notation 4.5.2. Then $(d_n)_{n \geq 1}$ is a dual derivation system, and $d_1(\varphi) = \varphi'$.

Proof. It is obvious that the unique partition $\tau \in NCZ^{(B)}(n)$ such that $\mathrm{Abs}(\tau) = 1_n$ is $\tau = 1_{\pm n}$. Thus if we put $\pi = 1_n$ in Equation (4.94) we obtain that $d_n(\varphi_{1_n}^{(A)}) = \varphi_{1_{\pm n}}^{(B)}$; in other words, this means that

$$d_n(\varphi \circ \mathrm{Mult}_n) = \varphi' \circ \mathrm{Mult}_n, \quad \forall\, n \geq 1. \tag{4.95}$$

The particular case $n = 1$ of (4.95) gives us that $d_1(\varphi) = \varphi'$. Moreover, it becomes clear that

$$d_n(f \circ \mathrm{Mult}_n) = (d_1 f) \circ \mathrm{Mult}_n, \quad \forall\, n \geq 1 \text{ and } f \in \mathbb{C}\varphi;$$

since in this proposition we have $\mathfrak{D}_1 = \mathbb{C}\varphi$, we thus see that condition (ii) from Definition 4.6.3 is verified.

The rest of the proof is devoted to verifying (i) from Definition 4.6.3. We fix a partition $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ for which we will prove that Equation (4.88) holds. Both sides of (4.88) behave multilinearly in the arguments $f_1 \in \mathfrak{D}_{|V_1|}, \ldots, f_p \in \mathfrak{D}_{|V_p|}$; hence, due to how $\mathfrak{D}_{|V_1|}, \ldots, \mathfrak{D}_{|V_p|}$ are defined, it suffices to prove the following statement: for every $\pi_1 \in NC(|V_1|), \ldots, \pi_p \in NC(|V_p|)$ we have that $J_\pi(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_p}) \in \mathfrak{D}_n$ and that

$$d_n\big(\, J_\pi(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_p})\,\big) = \tag{4.96}$$
$$\sum_{j=1}^{p} J_\pi(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_{j-1}}, d_{|V_j|}(\varphi^{(A)}_{\pi_j}), \varphi^{(A)}_{\pi_{j+1}}, \ldots, \varphi^{(A)}_{\pi_p}).$$

In what follows we fix some partitions $\pi_1 \in NC(|V_1|), \ldots, \pi_p \in NC(|V_p|)$, for which we will prove that this statement holds.

Observe that, in view of how the maps $d_{|V_j|}$ are defined, on the right-hand side of (4.96) we have

$$\sum_{j=1}^{p} \sum_{\substack{\tau \in NCZ^{(B)}(n) \text{ such} \\ \text{that } Abs(\tau) = \pi_j}} J_\pi(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_{j-1}}, \varphi^{(B)}_{\tau}, \varphi^{(A)}_{\pi_{j+1}}, \ldots, \varphi^{(A)}_{\pi_p}).$$

But let us recall from Section 2.5 that the partitions in $\{\tau \in NCZ^{(B)}(n) \mid \mathrm{Abs}(\tau) = \pi_j\}$ are indexed by the set of blocks of π_j. More precisely, for every $1 \le j \le p$ and $V \in \pi_j$ let us denote by $\tau(j,V)$ the unique partition in $NCZ^{(B)}(n)$ such that $\mathrm{Abs}(\tau) = \pi_j$ and such that the zero-block Z of τ has $\mathrm{Abs}(Z) = V$; then the double sum written above for the right-hand side of Equation (4.96) becomes

$$\sum_{j=1}^{p} \sum_{V \in \pi_j} J_\pi(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_{j-1}}, \varphi^{(B)}_{\tau(j,V)}, \varphi^{(A)}_{\pi_{j+1}}, \ldots, \varphi^{(A)}_{\pi_p}). \tag{4.97}$$

Now to the left-hand side of (4.96). For every $1 \le j \le p$ let $\widehat{\pi}_j$ be the partition of V_j obtained by transporting the blocks of π_j via the unique order preserving bijection from $\{1, \ldots, |V_j|\}$ onto V_j. Then $\widehat{\pi}_1, \ldots, \widehat{\pi}_p$ form together a partition $\rho \in NC(n)$ which refines π, and it is immediate that $J_\pi(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_p}) = \varphi^{(A)}_\rho$. In particular this shows of course that

$$J_\pi(\varphi^{(A)}_{\pi_1}, \ldots, \varphi^{(A)}_{\pi_p}) \in \mathfrak{D}_n.$$

Moreover, by using how $d_n(\varphi^{(A)}_\rho)$ is defined, we obtain that the left-hand side of (4.96) is equal to $\sum_{W \in \rho} \varphi^{(B)}_{\sigma(W)}$, where for every $W \in \rho$ we denote by $\sigma(W)$ the unique partition in $NCZ^{(B)}(n)$ such that $\mathrm{Abs}(\sigma(W)) = \rho$ and such that the zero-block Z of $\sigma(W)$ has $\mathrm{Abs}(Z) = W$.

Finally, we observe that the set of blocks of ρ is the disjoint union of the sets of blocks of the partitions $\widehat{\pi}_1, \ldots, \widehat{\pi}_p$, and is hence in natural bijection with $\{(j, V) \mid 1 \le j \le p \text{ and } V \in \pi_j\}$. We leave it as a straightforward (though somewhat notationally tedious) exercise to the reader to verify that when $W \in \rho$ corresponds to (j, V) via this bijection, then the term indexed by (j, V) in (4.97) is precisely equal to $\varphi^{(B)}_{\sigma(W)}$. Hence the double sum from (4.97) is identified term by term to $\sum_{W \in \rho} \varphi^{(B)}_{\sigma(W)}$ via the bijection $W \leftrightarrow (j, V)$, and the required formula (4.96) follows. ☐

Remark 4.6.7. The linear independence hypothesis in Proposition 4.6.6 is necessary, otherwise we need some relations to be satisfied by φ and φ'. Indeed, suppose for example that the set $\{\varphi^{(A)}_\pi \mid \pi \in NC(2)\}$ is linearly dependent in $\mathfrak{M}_2$. It is immediately verified that this is equivalent to the fact that φ is a character of $\mathcal{A}$ ($\varphi(ab) = \varphi(a)\varphi(b), \forall\, a, b \in \mathcal{A}$). Hence $\kappa_2 = 0$, so if Proposition 4.6.6 is to work then we should have $\kappa'_2 = d_2(\kappa_2) = 0$ as well, implying that φ' satisfies the condition $\varphi'(ab) = \varphi(a)\varphi'(b) + \varphi'(a)\varphi(b), \quad \forall\, a, b \in \mathcal{A}$.

4.7 Soul companions for a given φ

In this section we elaborate on the facts announced in the Subsection 4.1.3 of the introduction. We start by recording some basic properties of the set of functionals φ' which can appear as soul-companions for φ, when $(\mathcal{A}, \varphi)$ and $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are given.

Proposition 4.7.1. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ which are freely independent in $(\mathcal{A}, \varphi)$.*

1^o *The set of linear functionals*

$$\mathcal{F}' := \left\{ \varphi' : \mathcal{A} \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi' \text{ linear, } \varphi'(1_{\mathcal{A}}) = 0, \text{ and } \mathcal{A}_1, \ldots, \mathcal{A}_k \\ \text{are infinitesimally free in } (\mathcal{A}, \varphi, \varphi') \end{array} \right\} \tag{4.98}$$

is a linear subspace of the dual of $\mathcal{A}$.

2^o *Suppose that $Alg(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k) = \mathcal{A}$, and consider the linear map*

$$\mathcal{F}' \ni \varphi' \mapsto (\varphi' \mid \mathcal{A}_1, \ldots, \varphi' \mid \mathcal{A}_k) \in \mathcal{F}'_1 \times \cdots \times \mathcal{F}'_k, \tag{4.99}$$

where $\mathcal{F}'$ is as in (4.98) and where for $1 \le i \le k$ we denote $\mathcal{F}'_i = \{\varphi' : \mathcal{A}_i \to \mathbb{C} \mid \varphi' \text{ linear, } \varphi'(1_{\mathcal{A}}) = 0\}$. The map from (4.99) is one-to-one.

Proof. 1^o This is immediate from Definition 4.1.1, and specifically from the fact that φ' makes a linear appearance on the right-hand side of Equation (4.5).

2^o Let $\varphi' \in \mathcal{F}'$ be such that $\varphi' \mid \mathcal{A}_i = 0, \forall\, 1 \leq i \leq k$. Then from Equation (4.5) it is immediate that $\varphi'(a_1 \cdots a_n) = 0$ for all choices of $a_1, \ldots, a_n \in \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$. The linear span of the products $a_1 \cdots a_n$ formed in this way is the algebra generated by $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$, hence is all of $\mathcal{A}$, and the conclusion that $\varphi' = 0$ follows. □

Remark 4.7.2. In the framework of Proposition 4.7.1, the linear map (4.99) may not be surjective. For an example, consider the full Fock space over $\mathbb{C}^2$,

$$\mathcal{T} = \mathbb{C}\Omega \oplus \mathbb{C}^2 \oplus (\mathbb{C}^2 \otimes \mathbb{C}^2) \oplus \cdots \oplus (\mathbb{C}^2)^{\otimes n} \oplus \cdots ,$$

and let $L_1, L_2 \in B(\mathcal{T})$ be the left-creation operators associated to the two vectors in the canonical orthonormal basis of $\mathbb{C}^2$. Then L_1, L_2 are isometries with mutually orthogonal ranges; this is recorded in algebraic form by the relations

$$L_1^* L_1 = L_2^* L_2 = 1 \text{ (identity operator on } \mathcal{T}), \quad L_1^* L_2 = 0.$$

For $i = 1, 2$ let $\mathcal{A}_i$ denote the unital $*$-subalgebra of $B(\mathcal{T})$ generated by L_i, and let $\mathcal{A} = \mathrm{Alg}(\mathcal{A}_1 \cup \mathcal{A}_2)$, the unital $*$-algebra generated by L_1 and L_2 together. It is well-known (see e.g. Lecture 7 of [NS06]) that $\mathcal{A}_1$ and $\mathcal{A}_2$ are free in $(\mathcal{A}, \varphi)$ where φ is the vacuum-state on $\mathcal{A}$. Let $\varphi_2' : \mathcal{A}_2 \to \mathbb{C}$ be any linear functional such that $\varphi_2'(1_{\mathcal{A}}) = 0$ and $\varphi_2'(L_2) = 1$. Then there exists no linear functional $\varphi' : \mathcal{A} \to \mathbb{C}$ such that $\varphi' \mid \mathcal{A}_2 = \varphi_2'$ and such that $\mathcal{A}_1, \mathcal{A}_2$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$. Indeed, if such φ' would exist then from Equation (4.18) of Remark 4.2.2 it would follow that

$$\varphi'(L_1^* L_2 L_1) = \varphi(L_1^* L_1)\varphi'(L_2) + \varphi'(L_1^* L_1)\varphi(L_2) = 1 \cdot 1 + 0 \cdot 0 = 1,$$

which is not possible, since $L_1^* L_2 L_1 = 0$.

Remark 4.7.3. The example from the above remark shows that we can't always extend a given system of functionals φ_i' in order to get a soul companion φ' for φ. But Proposition 4.2.4 gives us an important case when we are sure this is possible, namely the one when $(\mathcal{A}, \varphi)$ is the free product $(\mathcal{A}_1, \varphi_1) * \cdots * (\mathcal{A}_k, \varphi_k)$.

In the remaining part of this section we will look at the two recipes for obtaining a soul companion that were stated in Corollary 4.1.4 and Proposition 4.1.5. For the first of them, we start by verifying that a derivation on $\mathcal{A}$ does indeed define a dual derivation system as indicated in Equation (4.14).

Proposition 4.7.4. *Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and let $D : \mathcal{A} \to \mathcal{A}$ be a derivation. For every $n \geq 1$ let $\mathfrak{M}_n$ denote the space of multilinear functionals from $\mathcal{A}^n$ to $\mathbb{C}$, and define $d_n : \mathfrak{M}_n \to \mathfrak{M}_n$ by putting*

$$(d_n f)(a_1, \ldots, a_n) := \sum_{m=1}^{n} f\Big(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\Big), \qquad (4.100)$$

for $f \in \mathfrak{M}_n$ *and* $a_1, \ldots, a_n \in \mathcal{A}$. *Then* $(d_n)_{n\geq 1}$ *is a dual derivation system on* $\mathcal{A}$.

Proof. We first do the immediate verification of condition (ii) from Definition 4.6.3. Let f be a functional in $\mathfrak{M}_1$, let n be a positive integer, and denote $g = f \circ \mathrm{Mult}_n \in \mathfrak{M}_n$. Then for every $a_1, \ldots, a_n \in \mathcal{A}$ we have

$$\begin{aligned}(d_n g)(a_1, \ldots, a_n) &= \sum_{m=1}^{n} f\Big(a_1 \cdots a_{m-1} \cdot D(a_m) \cdot a_{m+1} \cdots a_n\Big) \\ &= f\Big(D(a_1 \cdots a_n) \Big)\end{aligned}$$

(where at the first equality sign we used the definitions of d_n and of g, and at the second equality sign we used the derivation property of D). Since $d_1 f$ is just $f \circ D$, it is clear that we have obtained $d_n g = (d_1 f) \circ M_n$, as required.

For the remaining part of the proof we fix $\pi = \{V_1, \ldots, V_p\} \in NC(n)$ and $f_1 \in \mathfrak{M}_{|V_1|}, \ldots, f_p \in \mathfrak{M}_{|V_p|}$ as in (i) of Definition 4.6.3, and we verify that the formula (4.88) holds. Denote $f := J_\pi(f_1, \ldots, f_p) \in \mathfrak{M}_n$. In the summation which defines $d_n f$ in Equation (4.100) we group the terms by writing

$$\sum_{j=1}^{p} \Big(\sum_{m \in V_j} f\Big(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\Big) \Big). \tag{4.101}$$

It will clearly suffice to prove that, for every $1 \leq j \leq p$, the term indexed by j in the sum (4.101) is equal to the term indexed by j on the right-hand side of (4.88).

So then let us also fix a j, $1 \leq j \leq p$. We write explicitly the block V_j of π as $\{v_1, \ldots, v_s\}$ with $v_1 < \cdots < v_s$. From the definition of f as $J_\pi(f_1, \ldots, f_p)$ it is then immediate that for $m = v_r \in V_j$ we have

$$f\Big(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\Big) = \tag{4.102}$$

$$= \Big(\prod_{\substack{1 \leq i \leq p, \\ i \neq j}} f_i\big((a_1, \ldots, a_n) \mid V_i \big) \Big) \cdot \; f_j\big(a_{v_1}, \ldots, D(a_{v_r}), \ldots, a_{v_s} \big).$$

When summing over $1 \leq r \leq s$ in (4.102), the sum only affects the last factor of the product on the right-hand side, which sums to $(d_s f_j)\big(a_{v_1}, \ldots, a_{v_s} \big)$. The result of this summation is hence that

$$\sum_{m \in V_j} f\Big(a_1, \ldots, D(a_m), \ldots, a_n\Big) = J_\pi\big(f_1, \ldots, d_{|V_j|}(f_j), \ldots, f_p\big),$$

as required. □

Corollary 4.7.5. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $D : \mathcal{A} \to \mathcal{A}$ be a derivation. Define $\varphi' := \varphi \circ D$. Let the non-crossing and the infinitesimal non-crossing cumulant functionals associated to $(\mathcal{A}, \varphi, \varphi')$ be denoted by κ_n and respectively by κ_n', in the usual way. Then for every $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$ one has*

$$\kappa_n'(a_1, \ldots, a_n) = \sum_{m=1}^{n} \kappa_n\Big(a_1, \ldots, a_{m-1}, D(a_m), a_{m+1}, \ldots, a_n\Big). \tag{4.103}$$

Proof. This follows from Proposition 4.6.5, where we use the specific dual derivation system put into evidence in Proposition 4.7.4. □

Corollary 4.7.6. *In the notations of Corollary 4.7.5, let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ which are freely independent with respect to φ, and such that $D(\mathcal{A}_i) \subseteq \mathcal{A}_i$ for $1 \leq i \leq k$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.*

Proof. We verify that condition (2) from Theorem 4.1.2 is satisfied. The vanishing of mixed cumulants κ_n follows from the hypothesis that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are free in $(\mathcal{A}, \varphi)$. But then the specific formula obtained for the infinitesimal cumulants κ_n' in Corollary 4.7.5, together with the hypothesis that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are invariant under D, imply that the mixed infinitesimal cumulants κ_n' vanish as well. □

Example 4.7.7. Consider the situation where $\mathcal{A}$ is the algebra of noncommutative polynomials in k indeterminates, denoted by $\mathbb{C}\langle X_1, \ldots, X_k\rangle$. We will view $\mathcal{A}$ as a $*$-algebra, with $*$-operation uniquely determined by the requirement that each of $X_1, \ldots, X_k$ is selfadjoint. Consider the unital $*$-subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}$ where $\mathcal{A}_i = \text{span}\{X_i^n \mid n \geq 0\}$, $1 \leq i \leq k$. We will look at two natural derivations on $\mathcal{A}$ that leave $\mathcal{A}_1, \ldots, \mathcal{A}_k$ invariant, and we will examine some examples of infinitesimal freeness given by these derivations.

(a) Let $D : \mathcal{A} \to \mathcal{A}$ be the linear operator defined by putting $D(1) = 0$, $D(X_i) = 1 \; \forall \, 1 \leq i \leq k$, and

$$D(X_{i_1} \cdots X_{i_n}) = \sum_{m=1}^{n} X_{i_1} \cdots X_{i_{m-1}} X_{i_{m+1}} \cdots X_{i_n}, \tag{4.104}$$

for all $n \geq 2$, and all $1 \leq i_1, \ldots, i_n \leq k$.
It is immediate that D is a derivation on $\mathcal{A}$, which is selfadjoint (in the sense that $D(P^*) = D(P)^*$, $\forall \, P \in \mathcal{A}$). For every $1 \leq i \leq k$ we have that $D(\mathcal{A}_i) \subseteq \mathcal{A}_i$ and that D acts on $\mathcal{A}_i$ as the usual derivative (in the sense that $D(\, P(X_i)\,) = P'(X_i)$, $\forall \, P \in \mathbb{C}[X]$).

Now let $\mu : \mathcal{A} \to \mathbb{C}$ be a positive definite functional with $\mu(1) = 1$ and such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are free in $(\mathcal{A}, \mu)$. Then Corollary 4.7.6 implies that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in the $*$-incps $(\mathcal{A}, \mu, \mu')$, where $\mu' := \mu \circ D$.

Note that in this special example we actually have

$$\kappa'_n(X_{i_1}, \ldots, X_{i_n}) = 0, \quad \forall\, n \geq 2,\ \forall\, 1 \leq i_1, \ldots, i_n \leq k; \tag{4.105}$$

this is an immediate consequence of the the formula (4.103), combined with the fact that a non-crossing cumulant vanishes when one of its arguments is a scalar.

Equation (4.105) gives in particular that

$$\kappa'_n(X_i, \ldots, X_i) = 0, \quad \forall\, n \geq 2 \text{ and } 1 \leq i \leq k.$$

So if μ is defined such that every X_i has a standard semicircular distribution in $(\mathcal{A}, \mu)$, then every X_i will become a standard infinitesimal semicircular element in $(\mathcal{A}, \mu, \mu')$, in the sense of Remark 4.4.7, and where in Equation (4.68) we take $\alpha'_1 = 1$, $\alpha'_2 = 0$.

(b) Let $D_\# : \mathcal{A} \to \mathcal{A}$ be the linear operator defined by putting $D_\#(1) = 0$ and

$$D_\#(X_{i_1} \cdots X_{i_n}) = n\, X_{i_1} \cdots X_{i_n}, \quad \forall\, n \geq 1,\ \forall\, 1 \leq i_1, \ldots, i_n \leq k. \tag{4.106}$$

Then $D_\#$ is a selfadjoint derivation, sometimes called "the number operator" on $\mathcal{A}$. It is clear that $D_\#$ leaves every $\mathcal{A}_i$ invariant, $1 \leq i \leq k$. Hence if $\mu : \mathcal{A} \to \mathbb{C}$ is as in part (a) above (such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are free in $(\mathcal{A}, \mu)$), then Corollary 4.7.6 implies that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in the $*$-incps $(\mathcal{A}, \mu, \mu'_\#)$, where $\mu'_\# := \mu \circ D_\#$.

Since $D_\#(X_i) = X_i$ for $1 \leq i \leq k$, the formula (4.103) for infinitesimal non-crossing cumulants now gives

$$\kappa'_n(X_{i_1}, \ldots, X_{i_n}) = n \cdot \kappa_n(X_{i_1}, \ldots, X_{i_n}), \quad \forall\, n \geq 1,\ \forall\, 1 \leq i_1, \ldots, i_n \leq k. \tag{4.107}$$

In the particular case when μ is such that every X_i is standard semicircular in $(\mathcal{A}, \mu)$, it thus follows that every X_i becomes a standard infinitesimal semicircular element in $(\mathcal{A}, \mu, \mu'_\#)$, where we set the parameters from Equation (4.68) to be $\alpha'_1 = 0$ and $\alpha'_2 = 2$. On the other hand, if μ is defined such that every X_i has a standard free Poisson distribution in $(\mathcal{A}, \mu)$ (with $\kappa_n(X_i, \ldots, X_i) = 1$ for all $n \geq 1$), then the X_i will become infinitesimal free Poisson elements in $(\mathcal{A}, \mu, \mu'_\#)$, in the sense of Definition 4.4.8 and where we take $\beta' = 0$, $\gamma' = 1$ in Equation (4.71).

We now move to the situation described in Proposition 4.1.5. Clearly, this is just an immediate consequence of Proposition 4.3.8.

Corollary 4.7.8. *In the notations of Proposition 4.3.8, suppose that the unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$ of $\mathcal{A}$ are freely independent with respect to φ_t for every $t \in T$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \varphi, \varphi')$.*

Proof. Consider elements $a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n}$ where the indices $i_1, \ldots, i_n$ are not all equal to each other. The freeness of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ in $(\mathcal{A}, \varphi_t)$ implies that $\kappa_n^{(t)}(a_1, \ldots, a_n) = 0$ for every $t \in T$. The limit and derivative at 0 of the function $t \mapsto \kappa_n^{(t)}(a_1, \ldots, a_n)$ must therefore vanish, which means (by Proposition 4.3.8) that $\kappa_n(a_1, \ldots, a_n) = \kappa_n'(a_1, \ldots, a_n) = 0$. Hence condition (2) from Theorem 4.1.2 is satisfied, and the conclusion follows. □

Example 4.7.9. Consider again the situation where $\mathcal{A}$ is the $*$-algebra $\mathbb{C}\langle X_1, \ldots, X_k\rangle$, as in Example 4.7.7, and where $\mu : \mathcal{A} \to \mathbb{C}$ is a positive definite functional with $\mu(1) = 1$. Let $(\kappa_n)_{n\geq 1}$ be the non-crossing cumulant functionals of μ, and let $\{\kappa_\pi^{(A)} \mid \pi \in \cup_{n=1}^{\infty} NC(n)\}$ be the extended family of multilinear functionals.

For every $t > 0$, let $\mu_t : \mathcal{A} \to \mathbb{C}$ be the linear functional defined by putting $\mu_t(1) = 1$ and

$$\mu_t(X_{i_1}, \ldots, X_{i_n}) = \sum_{\pi \in NC(n)} (t+1)^{|\pi|} \cdot \kappa_\pi(X_{i_1}, \ldots, X_{i_n}), \tag{4.108}$$

for all $n \geq 1$ and $1 \leq i_1, \ldots, i_n \leq k$. As is easily seen, μ_t is uniquely determined by the fact that its non-crossing cumulant functionals $(\kappa_n^{(t)})_{n\geq 1}$ satisfy

$$\kappa_n^{(t)}(X_{i_1}, \ldots, X_{i_n}) = (t+1) \cdot \kappa_n(X_{i_1}, \ldots, X_{i_n}), \quad \forall\, n \geq 1,\ 1 \leq i_1, \ldots, i_n \leq k. \tag{4.109}$$

Due to this fact, μ_t is called the "$(t+1)$-th convolution power of μ" with respect to the operation $\boxplus$ of free additive convolution – see pp. 231-233 of [NS06] for details.

From (4.108) it is clear that the family $\{\mu_t \mid t > 0\}$ has infinitesimal limit (μ, μ') at $t = 0$, where μ is the functional we started with, while μ' is defined by putting $\mu'(1) = 0$ and

$$\mu'(X_{i_1} \cdots X_{i_n}) = \sum_{\pi \in NC(n)} |\pi| \cdot \kappa_\pi(X_{i_1}, \ldots, X_{i_n}), \quad \forall\, n \geq 1,\ 1 \leq i_1, \ldots, i_n \leq k. \tag{4.110}$$

Note also that by using Equation (4.109) and by invoking Proposition 4.3.8 we get

$$\kappa_n'(X_{i_1}, \ldots, X_{i_n}) = \kappa_n(X_{i_1}, \ldots, X_{i_n}), \quad \forall\, n \geq 1,\ 1 \leq i_1, \ldots, i_n \leq k. \tag{4.111}$$

Now let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be the unital $*$-subalgebras of $\mathcal{A}$ that were also considered in Example 4.7.7, $\mathcal{A}_i = \text{span}\{X_i^n \mid n \geq 0\}$ for $1 \leq i \leq k$. Suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are free in $(\mathcal{A}, \mu)$. Then they are free in $(\mathcal{A}, \mu_t)$ for every $t > 0$; this follows from Equation (4.109) and the description of freeness in terms of non-crossing cumulants (cf. Theorem 2.3.7), where we take into account that $\mathcal{A}_i$ is the unital algebra generated by X_i. Hence this is a situation where Corollary 4.7.8 applies, and we conclude that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are infinitesimally free in $(\mathcal{A}, \mu, \mu')$.

Note also that if X_i has a standard semicircular distribution in $(\mathcal{A}, \mu)$, then Equation (4.111) implies that X_i becomes an infinitesimal semicircular element in $(\mathcal{A}, \mu, \mu')$, where the parameters α'_1, α'_2 from Remark 4.4.7 are taken to be $\alpha'_1 = 0$, $\alpha'_2 = 1$. Likewise, if X_i is a standard free Poisson in $(\mathcal{A}, \mu)$, then Equation (4.111) implies that X_i becomes an infinitesimal free Poisson element in $(\mathcal{A}, \mu, \mu')$, where the parameters β', γ' from Definition 4.4.8 are taken to be $\beta' = 1$, $\gamma' = 0$.

Chapter 5

Higher order infinitesimal freeness

This chapter is the text of the paper "Higher order infinitesimal freeness" [Fev10], submitted for publication.

5.1 Framework of higher order infinitesimal freeness

Throughout this chapter, the integer $k \in \mathbb{N}$ is fixed. In this section, we introduce the two equivalent structures of infinitesimal noncommutative probability space and of scarce $\mathcal{C}_k$-noncommutative probability space and we discuss their relations to previously defined structures.

5.1.1 Infinitesimal probability space of order k

The object of this subsection is to introduce the structure which is the framework for our notion of infinitesimal freeness of order k, namely the *infinitesimal noncommutative probability space of order k*.

Definition 5.1.1. We call infinitesimal noncommutative probability space of order k a structure $(\mathcal{A}, (\varphi^{(i)})_{0\leq i\leq k})$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$, $\varphi^{(0)} : \mathcal{A} \longrightarrow \mathbb{C}$ is a linear map with $\varphi^{(0)}(1_{\mathcal{A}}) = 1$, and $\varphi^{(i)} : \mathcal{A} \longrightarrow \mathbb{C}$, $1 \leq i \leq k$, are linear maps with $\varphi^{(i)}(1_{\mathcal{A}}) = 0$.

Remark 5.1.2. The notion of infinitesimal noncommutative probability space of order 1 coincides with the notion of infinitesimal noncommutative probability space introduced in Definition 4.1.1. The structure defined above is therefore a generalization of this latter object, and the use of the adjective infinitesimal is justified.

An element $a \in (\mathcal{A}, (\varphi^{(i)})_{0\leq i\leq k})$ of an infinitesimal noncommutative probability space of order k is called an *infinitesimal noncommutative random* *variable of order k*. The *infinitesimal distribution of order k* of a n-tuple $(a_1, \dots, a_n) \in \mathcal{A}^n$ of infinitesimal noncommutative random variables of order k is the $(k+1)$-tuple $(\mu^{(i)})_{0 \le i \le k}$ of linear functionals on $\mathbb{C}\langle X_1, \dots, X_n\rangle$ defined by :

$$\mu^{(i)}(P(X_1, \dots, X_n)) := \varphi^{(i)}(P(a_1, \dots, a_n)).$$

The range of infinitesimal distributions is the set of *infinitesimal laws of order k*, introduced below.

Definition 5.1.3. An infinitesimal law (of order k) on n variables is a $(k+1)$-tuple of linear functionals $(\mu^{(i)})_{0 \le i \le k}$, where $\mu^{(i)} : \mathbb{C}\langle X_1, \dots, X_n\rangle \to \mathbb{C}$ is defined on the algebra of noncommutative polynomials and satisfies $\mu^{(i)}(1) = \delta_i^0$.

For some purposes, it is handy to consider, instead of $k+1$ linear functionals as in Definition 5.1.1, an equivalent unique linear map with values in a $(k+1)$-dimensional algebra. The relevant algebra, denoted by $\mathcal{C}_k$, is described below.

5.1.2 The algebra $\mathcal{C}_k$

In Subsection 4.1.1, the two linear maps φ and φ' of an infinitesimal noncommutative probability space $(\mathcal{A}, \varphi, \varphi')$ are consolidated in a single linear map $\tilde{\varphi}$ on $\mathcal{A}$ with values in the two-dimensional Grassman algebra $\mathbb{G}$ generated by an element ε which satisfies $\varepsilon^2 = 0$:

$$\mathbb{G} = \{\alpha + \varepsilon\beta \mid \alpha, \beta \in \mathbb{C}\}.$$

This algebra has a quite natural $(k+1)$-dimensional generalization introduced below.

Definition 5.1.4. Let $\mathcal{C}_k$ denote the $(k+1)$-dimensional complex algebra $\mathbb{C}^{k+1}$ with usual vector space structure and multiplication given by the following rule: if $\alpha = (\alpha^{(0)}, \dots, \alpha^{(k)}) \in \mathcal{C}_k$ and $\beta = (\beta^{(0)}, \dots, \beta^{(k)}) \in \mathcal{C}_k$, then

$$\alpha \cdot \beta = (\gamma^{(0)}, \dots, \gamma^{(k)})$$

is defined by

$$\gamma^{(i)} := \sum_{j=0}^{i} C_i^j \alpha^{(j)} \beta^{(i-j)}. \tag{5.1}$$

The algebra $\mathcal{C}_k$ is a unital complex commutative algebra. Its unit is $1_{\mathcal{C}_k} = (1, 0, \dots, 0)$. An element is invertible in the algebra $\mathcal{C}_k$ if and only if its first coordinate is non-zero.
The analogy between formula (5.1) defining the product in $\mathcal{C}_k$ and the well-known Leibniz rule giving the recipe for computing the derivatives of the product of two smooth functions makes it easy to establish the formula

for the product $\beta = \alpha_1 \cdots \alpha_n$ of n elements $\alpha_1, \ldots, \alpha_n \in \mathcal{C}_k$. Precisely, if $\alpha_j = (\alpha_j^{(0)}, \ldots, \alpha_j^{(k)})$ and $\beta = (\beta^{(0)}, \ldots, \beta^{(k)})$, one has :

$$\beta^{(i)} = \sum_{\lambda \in \Lambda_{n,i}} C_i^{\lambda_1, \ldots, \lambda_n} \prod_{j=1}^{n} \alpha_j^{(\lambda_j)},$$

where

$$C_i^{\lambda_1, \ldots, \lambda_n} = \frac{i!}{\lambda_1! \cdots \lambda_n!}$$

and

$$\Lambda_{n,i} := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \sum_{j=1}^{n} \lambda_j = i\}. \tag{5.2}$$

There is an alternative description of the algebra $\mathcal{C}_k$: it may be identified with the algebra of $(k+1)$-by-$(k+1)$ upper triangular Toeplitz matrices (with usual matricial operations) as follows :

$$(\alpha^{(0)}, \ldots, \alpha^{(k)}) \simeq \begin{pmatrix} \alpha^{(0)} & \alpha^{(1)} & \ldots & \frac{\alpha^{(k-1)}}{(k-1)!} & \frac{\alpha^{(k)}}{k!} \\ 0 & \alpha^{(0)} & \ldots & \ldots & \frac{\alpha^{(k-1)}}{(k-1)!} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \alpha^{(0)} & \alpha^{(1)} \\ 0 & 0 & \ldots & \ldots & \alpha^{(0)} \end{pmatrix}.$$

Consider

$$\varepsilon := \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & 1 \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}.$$

It is easy to compute the values of ε^i for $0 \leq i \leq k+1$; in particular $\varepsilon^{k+1} = 0_{\mathcal{C}_k}$ and any element $\alpha = (\alpha^{(0)}, \ldots, \alpha^{(k)}) \in \mathcal{C}_k$ may be uniquely decomposed

$$\alpha = \sum_{i=0}^{k} \alpha^{(i)} \frac{\varepsilon^i}{i!}. \tag{5.3}$$

The family $(\frac{\varepsilon^i}{i!}, 0 \leq i \leq k)$ is thus a linear basis of $\mathcal{C}_k$, to which we will refer as the canonical basis of $\mathcal{C}_k$. In particular, $\mathcal{C}_k \simeq \mathbb{C}[\varepsilon] = \mathbb{C}_k[\varepsilon] \simeq \mathbb{C}[X]/(X^{k+1})$. In the definition of a usual noncommutative probability space, if one asks for the state to be $\mathcal{C}_k$-valued, one obtains a slightly different structure, introduced in the next section.

5.1.3 Scarce $\mathcal{C}_k$-noncommutative probability space

Definition 5.1.5. By *scarce $\mathcal{C}_k$-noncommutative probability space*, we mean a couple $(\mathcal{A}, \tilde{\varphi})$, where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\tilde{\varphi} : \mathcal{A} \to \mathcal{C}_k$ is a $\mathbb{C}$-linear map satisfying $\tilde{\varphi}(1_{\mathcal{A}}) = 1_{\mathcal{C}_k}$.

Remark 5.1.6. The notion of scarce noncommutative probability space was introduced in [Oan07], but only in the particular case of scarce $\mathbb{G}$-noncommutative probability space.

Remark 5.1.7. To any infinitesimal noncommutative probability space of order k, denoted $(\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})$, we may associate a natural scarce $\mathcal{C}_k$-noncommutative probability space $(\mathcal{A}, \tilde{\varphi})$, by putting

$$\tilde{\varphi} := \sum_{i=0}^{k} \varphi^{(i)} \frac{\varepsilon^i}{i!}. \tag{5.4}$$

Reciprocally, given a scarce $\mathcal{C}_k$-noncommutative probability space $(\mathcal{A}, \tilde{\varphi})$, the linear decomposition of $\tilde{\varphi}$ in the canonical basis of $\mathcal{C}_k$ gives rise to $k+1$ linear functionals $(\varphi^{(i)})_{0 \leq i \leq k}$, and consequently to an infinitesimal noncommutative probability space of order k : $(\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})$.
This equivalence between an infinitesimal noncommutative probability space of order k and its associated scarce $\mathcal{C}_k$-noncommutative probability space $(\mathcal{A}, \tilde{\varphi})$ is fundamental in what follows. Indeed, we will continuously switch from one structure to the other, according to the principle that our interest is in the infinitesimal structure whereas the computations are easier in the scarce $\mathcal{C}_k$ structure, in the sense that they mimetize those from usual free probability.

An element a of a scarce $\mathcal{C}_k$-noncommutative probability space $(\mathcal{A}, \tilde{\varphi})$ is called a *$\mathcal{C}_k$-noncommutative random variable*. We associate to such an $a \in \mathcal{A}$ the sequence of its *$\mathcal{C}_k$-valued moments* $(\tilde{\varphi}(a^n))_{n \in \mathbb{N}^*}$. We call *$\mathcal{C}_k$-valued distribution* of a the whole sequence of its moments, or equivalently, the linear map from $\mathbb{C}[X]$ into $\mathcal{C}_k$ which maps any polynomial P to $\tilde{\varphi}(P(a))$. One may find easier to collect all the $\mathcal{C}_k$-valued moments in a formal power series, as follows :

Definition 5.1.8. Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-noncommutative probability space. The *$\mathcal{C}_k$-valued moment series* of $a \in \mathcal{A}$ is the power series $\tilde{M}_a \in \Theta_{\mathcal{C}_k}^{(A)}$ defined as follows :

$$\tilde{M}_a(z) := \sum_{n=1}^{\infty} \tilde{\varphi}(a^n) z^n.$$

The notion of $\mathcal{C}_k$-valued distribution is easily generalized to n-tuples of variables :

Definition 5.1.9. The $\mathcal{C}_k$*-valued distribution* of a n-tuple $(a_1, \ldots, a_n) \in \mathcal{A}^n$ of $\mathcal{C}_k$-noncommutative random variables in a scarce $\mathcal{C}_k$-noncommutative probability space $(\mathcal{A}, \tilde{\varphi})$ is the linear map $\tilde{\mu}_{(a_1,\ldots,a_n)} : \mathbb{C}\langle X_1, \ldots, X_n\rangle \to \mathcal{C}_k$ defined by

$$\tilde{\mu}_{(a_1,\ldots,a_n)}(P(X_1, \ldots, X_n)) := \tilde{\varphi}(P(a_1, \ldots, a_n)).$$

As mentioned in Remark 4.2.9, scarce $\mathbb{G}$-noncommutative probability spaces and infinitesimal noncommutative probability spaces provide a nice framework to do free probability of type B. The equivalent structures defined above are therefore the natural setting for generalizing free probability of type B. There is another structure linked to free probability of type B that one may find interesting to generalize here : the noncommutative probability space of type B, proposed in the original work on free probability of type B [BGN03], and reviewed in Section 2.5. Its natural generalization is the *noncommutative probability space of type* k :

Definition 5.1.10. By a noncommutative probability space of type k we understand a system $(\mathcal{V}^{(0)}, f^{(0)}, \ldots, \mathcal{V}^{(k)}, f^{(k)}, (\Phi_{i,j})_{0\le i,j\le k})$, where $(\mathcal{V}^{(0)}, f^{(0)})$ is a noncommutative probability space of type A, $\mathcal{V}^{(i)}$, $1 \le i \le k$, are complex vector spaces, $f^{(i)} : \mathcal{V}^{(i)} \longrightarrow \mathbb{C}$, $1 \le i \le k$, are linear maps, $\Phi_{i,j} : \mathcal{V}^{(i)} \times \mathcal{V}^{(j)} \longrightarrow \mathcal{V}^{(i+j)}$,$0 \le i, j \le k$, are bilinear maps satisfying

$$\Phi_{h+i,j}(\Phi_{h,i}(x, y), z) = \Phi_{h,i+j}(x, \Phi_{i,j}(y, z)),$$

$\forall h, i, j \in \mathbb{N}, \forall x \in \mathcal{V}^{(h)}, \forall y \in \mathcal{V}^{(i)}, \forall z \in \mathcal{V}^{(j)}$.

To make the preceding definition work, we put $\mathcal{V}^{(i)} = \{0\}$, when $i \ge k + 1$. The following fact noticed in Remark 4.2.9 still holds : noncommutative probability spaces of type k are particular cases of scarce $\mathcal{C}_k$-noncommutative probability spaces. Indeed, given a noncommutative probability space of type k $(\mathcal{V}^{(0)}, f^{(0)}, \ldots, \mathcal{V}^{(k)}, f^{(k)}, (\Phi_{i,j})_{0\le i,j\le k})$, the direct product $\prod_{i=0}^{k} \mathcal{V}^{(i)}$ can be endowed with a complex unital algebra structure, via the maps $(\Phi_{i,j})_{0\le i,j\le k}$. This algebra, together with the linear map $\tilde{\varphi}(x_0, \ldots, x_k) := (f^{(0)}(x_0), \ldots, f^{(k)}(x_k))$, forms a scarce $\mathcal{C}_k$-noncommutative probability space.

There are natural equivalent notions of freeness on the structures introduced above, generalizing both infinitesimal freeness from Definition 4.1.1 and freeness of type B from [BGN03] (see Definition 2.5.4). In Definition 4.1.1, infinitesimal freeness in $(\mathcal{A}, \varphi, \varphi')$ is defined by two conditions on the linear functionals φ, φ' ; its generalization to an infinitesimal noncommutative probability space of order k denoted by $(\mathcal{A}, (\varphi^{(i)})_{0\le i\le k})$ would require $k + 1$ conditions on the linear functionals $(\varphi^{(i)})_{0\le i\le k}$. Infinitesimal freeness being also equivalent to the vanishing of the infinitesimal non-crossing cumulants (cf. Theorem 4.1.2), we adopt this approach and define the infinitesimal freeness of order k by the vanishing of some multilinear functionals, called infinitesimal non-crossing cumulant functionals of order k and introduced in the next section.

5.2 Infinitesimal non-crossing cumulants of order k

We begin this section by introducing a total order on the set of blocks of a non-crossing partition, which will be useful in the sequel.

Definition 5.2.1. Let $p \in NC^{(A)}(m)$, and $V, W \in \mathrm{bl}(p)$.
1^o V is said to be nested in W if $\min W < \min V \leq \max V < \max W$.
2^o V is said to be on the left of W if $\max V < \min W$.
3^o $V \sqsubset W \Leftrightarrow V$ is nested in W or V is on the left of W.

The proof of the next proposition is trivial and left to the reader.

Proposition 5.2.2. $\sqsubset$ *is a total order on bl*(p).

If $p \in NC^{(A)}(m)$, we have seen that $p \cup \mathrm{Kr}(p)$ is a non-crossing partition of $[m] \cup \overline{[m]}$ in $m+1$ blocks. These blocks will be listed in two different ways. The first way is to list them all together in the increasing order $\sqsubset$: we will write $\mathrm{Mix}(p, i)$ for the i-th block of $p \cup \mathrm{Kr}(p)$ in the increasing order $\sqsubset$, for $1 \leq i \leq m+1$.
For some purposes, it is nice to list separately the blocks of p and of $\mathrm{Kr}(p)$, and we will write $\mathrm{Sep}(p, i)$ to denote the i-th block of p in the increasing order $\sqsubset$ if $1 \leq i \leq |p|$ and to denote the $(i - |p|)$-th block of $\mathrm{Kr}(p)$ in the increasing order $\sqsubset$ if $|p| + 1 \leq i \leq m+1$.
It is interesting to look at the first blocks in the two resulting lists : $\mathrm{Mix}(p, 1)$ is a singleton in $[m] \cup \overline{[m]}$, $\mathrm{Sep}(p, 1)$ is an interval in $[m]$. In particular, we can deduce the well-known fact that a non-crossing partition always owns an interval-block.

5.2.1 $\mathcal{C}_k$-non-crossing cumulant functionals

In this subsection, we define non-crossing cumulant functionals in the framework of a scarce $\mathcal{C}_k$-noncommutative probability space by the free moment-cumulant formula from usual free probability, with the only difference that the computations take place in the algebra $\mathcal{C}_k$ instead of the field of complex numbers $\mathbb{C}$.

Definition 5.2.3. Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-noncommutative probability space. The *$\mathcal{C}_k$-non-crossing cumulant functionals* are a family of multilinear maps $(\tilde{\kappa}_n : \mathcal{A}^n \to \mathcal{C}_k)_{n=1}^{\infty}$, uniquely determined by the following equation : for every $n \geq 1$ and every $a_1, \ldots, a_n \in \mathcal{A}$,

$$\sum_{p \in NC^{(A)}(n)} \tilde{\kappa}_p(a_1, \ldots, a_n) = \tilde{\varphi}(a_1 \cdots a_n). \tag{5.5}$$

In usual free probability, the formula above is known as the free moment-cumulant formula (see (2.4)). The only difference is that computations here take place in the unital commutative complex algebra $\mathcal{C}_k$ instead of $\mathbb{C}$. However, as already mentioned in Section 4.3, the proofs (see [NS06]) of the following classical results remain valid in this setting. That is why we record them without proof.
For every $n \geq 1$ and every $a_1, \dots, a_n \in \mathcal{A}$ we have that:

$$\tilde{\kappa}_n(a_1, \dots, a_n) = \sum_{p \in NC^{(A)}(n)} \text{Möb}^{(A)}(p, 1_n) \tilde{\varphi}_p(a_1, \dots, a_n). \tag{5.6}$$

Obviously, the multilinear maps $(\tilde{\varphi}_n : \mathcal{A}^n \to \mathcal{C}_k)_{n=1}^{\infty}$ implicitely used in formula (5.6) are defined by $\tilde{\varphi}_n(a_1, \dots, a_n) = \tilde{\varphi}(a_1 \cdots a_n)$.

Proposition 5.2.4. *One has that $\tilde{\kappa}_n(a_1, \dots, a_n) = 0$ whenever $n \geq 2$, $a_1, \dots, a_n \in \mathcal{A}$, and there exists $1 \leq i \leq n$ such that $a_i \in \mathbb{C}1_{\mathcal{A}}$.*

Proposition 5.2.5. *Let $x_1, \dots, x_s$ be in $\mathcal{A}$ and consider some products of the form*

$$a_1 = x_1 \cdots x_{s_1}, \; a_2 = x_{s_1+1} \cdots x_{s_2}, \; \dots, \; a_n = x_{s_{n-1}+1} \cdots x_{s_n},$$

where $1 \leq s_1 < s_2 < \cdots < s_n = s$. Then

$$\tilde{\kappa}_n(a_1, \dots, a_n) = \sum_{\substack{\pi \in NC(s) \text{ such} \\ \text{that } \pi \vee \theta = 1_s}} \tilde{\kappa}_\pi(x_1, \dots, x_s),$$

where $\theta \in NC(s)$ is the partition :

$$\theta = \{\{1, \dots, s_1\}, \{s_1 + 1, \dots, s_2\}, \dots, \{s_{n-1} + 1, \dots, s_n\}\}.$$

Given a $\mathcal{C}_k$-noncommutative random variable $a \in (\mathcal{A}, \tilde{\varphi})$, the quantities $\tilde{\kappa}_n(a, \dots, a)$ are called its *$\mathcal{C}_k$-valued cumulants*, and they are collected in a power series :

Definition 5.2.6. Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-noncommutative probability space. The *$\mathcal{C}_k$-valued R-transform* of $a \in \mathcal{A}$ is the power series $\tilde{R}_a \in \Theta_{\mathcal{C}_k}^{(A)}$ defined as follows :

$$\tilde{R}_a(z) := \sum_{n=1}^{\infty} \tilde{\kappa}_n(a, \dots, a) z^n.$$

Following the well-known result of [Spe94] stating roughly that, in a usual noncommutative probability space, subsets are free if and only if they satisfy the vanishing of mixed cumulants condition (see Theorem 2.3.7), we generalize this condition to our setting :

Definition 5.2.7. Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-noncommutative probability space and $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be subsets of $\mathcal{A}$. We say that $\mathcal{M}_1, \ldots, \mathcal{M}_n$ have *vanishing mixed $\mathcal{C}_k$-cumulants* if

$$\tilde{\kappa}_m(a_1, \ldots, a_m) = 0$$

whenever $a_1 \in \mathcal{M}_{i_1}, \ldots, a_m \in \mathcal{M}_{i_m}$ and $\exists 1 \leq s < t \leq m$ such that $i_s \neq i_t$.

As announced, infinitesimal freeness of order k is defined by the vanishing of mixed $\mathcal{C}_k$-cumulants condition. More precisely :

Definition 5.2.8. We will say that subsets $\mathcal{M}_1, \ldots, \mathcal{M}_n \subseteq \mathcal{A}$ of a scarce $\mathcal{C}_k$-noncommutative probability space $(\mathcal{A}, \tilde{\varphi})$ are *infinitesimally free of order k* if they have vanishing mixed $\mathcal{C}_k$-cumulants.

Remark 5.2.9. Using a classical argument in free probability, one can prove that, if $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are unital subalgebras which are infinitesimally free of order k in a scarce $\mathcal{C}_k$-noncommutative probability space $(\mathcal{A}, \tilde{\varphi})$, then one has :

$$\tilde{\varphi}(a_1 \cdots a_m) = 0$$

whenever $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$ with $i_1 \neq \ldots \neq i_m$ satisfy $\tilde{\varphi}(a_1) = \ldots = \tilde{\varphi}(a_m) = 0$.
The converse in our $\mathcal{C}_k$-valued situation is not true, because one cannot use the nice "centering trick", as noticed in Remark 4.3.14.

In the next subsection, we switch to the infinitesimal framework, and define infinitesimal non-crossing cumulant functionals, with the intuition that they should appear as the coefficients in the decomposition of the $\mathcal{C}_k$-non-crossing cumulant functionals in the canonical basis of $\mathcal{C}_k$.

5.2.2 Infinitesimal non-crossing cumulant functionals

In this short subsection, we focus on an infinitesimal noncommutative probability space of order k structure $(\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})$. The aim is to define cumulants and freeness in this setting, in a consistent way with the last subsection. For convenience, we will use the following notation :

Notation 5.2.10. For a family of multilinear maps $(r_n^{(i)} : \mathcal{A}^n \to \mathbb{C}, 0 \leq i \leq k)_{n=1}^{\infty}$, we define for any $n \in \mathbb{N}$, any $\pi = \{V_1 \sqsubset \cdots \sqsubset V_h\} \in NC^{(A)}(n)$ and any $\lambda \in \Lambda_{n,h}$ (defined by (5.2)) the n-linear functional $r_\pi^{(\lambda)} : \mathcal{A}^n \to \mathbb{C}$ by

$$r_\pi^{(\lambda)}(a_1, \ldots, a_n) := \prod_{i=1}^{h} r_{|V_i|}^{(\lambda_i)}((a_1, \ldots, a_n) \mid V_i).$$

The underlying idea is to consider the $\mathcal{C}_k$-non-crossing cumulant functionals $(\tilde{\kappa}_n : \mathcal{A}^n \to \mathcal{C}_k)_{n=1}^{\infty}$ in the associated scarce $\mathcal{C}_k$-noncommutative probability space $(\mathcal{A}, \tilde{\varphi})$ (see Remark 5.1.7), and then to define the required n-th infinitesimal non-crossing cumulant functionals as the n-linear forms appearing as coefficients in the linear decomposition of $\tilde{\kappa}_n : \mathcal{A}^n \to \mathcal{C}_k$ in the canonical basis of $\mathcal{C}_k$. This leads to the following definition:

Definition 5.2.11. Let $(\mathcal{A}, (\varphi^{(i)})_{0\le i\le k})$ be an infinitesimal noncommutative probability space of order k. The *infinitesimal non-crossing cumulant functionals of order k* are a family of multilinear maps $(\kappa_n^{(i)} : \mathcal{A}^n \to \mathbb{C}, 0 \le i \le k)_{n=1}^{\infty}$, uniquely determined by the following equation : for every $n \ge 1$, every $0 \le i \le k$ and every $a_1, \dots, a_n \in \mathcal{A}$ we have that:

$$\sum_{\substack{p\in NC^{(A)}(n)\\ p:=\{V_1,\dots,V_h\}}} \sum_{\lambda\in\Lambda_{h,i}} C_i^{\lambda_1,\dots,\lambda_h} \kappa_p^{(\lambda)}(a_1,\dots,a_n) = \varphi^{(i)}(a_1\cdots a_n). \tag{5.7}$$

Infinitesimal freeness in the framework of an infinitesimal noncommutative probability space of order k is obviously defined by the vanishing of mixed infinitesimal cumulants.

Definition 5.2.12. We will say that subsets $\mathcal{M}_1, \dots, \mathcal{M}_n$ of an infinitesimal noncommutative probability space of order k are *infinitesimally free of order k* if they have *vanishing mixed infinitesimal cumulants*, which means that, for each $0 \le i \le k$,

$$\kappa_m^{(i)}(a_1, \dots, a_m) = 0$$

whenever $a_1 \in \mathcal{M}_{i_1}, \dots, a_m \in \mathcal{M}_{i_m}$ and $\exists 1 \le s < t \le m$ such that $i_s \ne i_t$.

Remark 5.2.13. It is straightforward to check, using formula (5.7), that the infinitesimal non-crossing cumulant functionals of an infinitesimal noncommutative probability space of order k are indeed linked to the $\mathcal{C}_k$-non-crossing cumulant functionals of the associated scarce $\mathcal{C}_k$-noncommutative probability space by :

$$\tilde{\kappa}_n = \sum_{i=0}^{k} \kappa_n^{(i)} \frac{\varepsilon^i}{i!}. \tag{5.8}$$

A consequence of formulas (5.6) and (5.8) is the validity of the following inverse formula:

$$\kappa_n^{(i)}(a_1,\dots,a_n) = \sum_{\substack{p\in NC^{(A)}(n)\\ p:=\{V_1,\dots,V_h\}}} \sum_{\lambda\in\Lambda_{h,i}} \text{Möb}^{(A)}(p, 1_n) C_i^{\lambda_1,\dots,\lambda_h} \varphi_p^{(\lambda)}(a_1,\dots,a_n), \tag{5.9}$$

and of the following proposition :

Proposition 5.2.14. *One has that $\kappa_n^{(i)}(a_1, \dots, a_n) = 0$ whenever $0 \le i \le k$, $n \ge 2$, $a_1, \dots, a_n \in \mathcal{A}$, and there exists $1 \le j \le n$ such that $a_j \in \mathbb{C}1_{\mathcal{A}}$.*

Another consequence of relation (5.8) is that subsets $\mathcal{M}_1, \ldots, \mathcal{M}_n$ of an infinitesimal noncommutative probability space of order k are infinitesimally free of order k if and only if they are infinitesimally free of order k in the associated scarce $\mathcal{C}_k$-noncommutative probability space.

Remark 5.2.15. Let $(\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})$ be an infinitesimal noncommutative probability space of order k, and consider its infinitesimal non-crossing cumulant functionals $(\kappa_n^{(i)} : \mathcal{A}^n \to \mathbb{C}, 0 \leq i \leq k)_{n=1}^{\infty}$. It is interesting to notice that the multilinear maps $(\kappa_n^{(0)} : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ and $(\kappa_n^{(1)} : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ are respectively the usual non-crossing cumulant functionals in the noncommutative probability space $(\mathcal{A}, \varphi^{(0)})$ and the infinitesimal non-crossing cumulant functionals of Definition 4.3.7 in the infinitesimal noncommutative probability space $(\mathcal{A}, \varphi^{(0)}, \varphi^{(1)})$. This implies that subsets that are infinitesimally free of order k are in particular free in $(\mathcal{A}, \varphi^{(0)})$ and infinitesimally free in $(\mathcal{A}, \varphi^{(0)}, \varphi^{(1)})$ in the sense of Definition 4.1.1.
Infinitesimal freeness of unital subalgebras in Definition 4.1.1, as well as freeness of type B in [BGN03], is defined in terms of moments. Section 8 will provide such a characterization of the infinitesimal freeness of order k of unital subalgebras of an infinitesimal noncommutative probability space of order k in terms of moments.

As stated in Remark 5.2.15, infinitesimal freeness of order k of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq (\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})$ of an infinitesimal noncommutative probability space of order k implies freeness of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ in the noncommutative probability space $(\mathcal{A}, \varphi^{(0)})$. Conversely, is it possible to "upgrade" freeness of given unital subalgebras of a noncommutative probability space to infinitesimal freeness of order k ? This question is discussed in the next subsection.

5.2.3 Upgrading freeness to infinitesimal freeness of order k

Given a noncommutative probability space $(\mathcal{A}, \varphi)$ and free unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of $\mathcal{A}$, the question of how to build a linear form φ' on $\mathcal{A}$ such that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are infinitesimally free in the infinitesimal noncommutative probability space $(\mathcal{A}, \varphi, \varphi')$ is adressed in Section 4.7. Among the answers given there, there is the idea to define $\varphi' := \varphi \circ D$, where D is a derivation of $\mathcal{A}$ (a linear map $D : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying $\forall a, b \in \mathcal{A}, D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$) such that $D(\mathcal{A}_j) \subseteq \mathcal{A}_j$ for each $1 \leq j \leq n$. We examine the question of how to build linear forms $\varphi^{(1)}, \ldots, \varphi^{(k)}$ on $\mathcal{A}$ such that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are infinitesimally free of order k in the infinitesimal noncommutative probability space $(\mathcal{A}, \varphi, \varphi^{(1)}, \ldots, \varphi^{(k)})$. The natural idea consisting in defining $\varphi^{(i)} := \varphi \circ D^i$ where D is a derivation of $\mathcal{A}$ such that $D(\mathcal{A}_j) \subseteq \mathcal{A}_j$ for each $1 \leq j \leq n$ is a possible answer, as proved below :

Proposition 5.2.16. *Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space and let $D : \mathcal{A} \to \mathcal{A}$ be a derivation. Define $\varphi^{(i)} := \varphi \circ D^i$. Let the infinitesimal non-crossing cumulant functionals associated to $(\mathcal{A}, \varphi, \varphi^{(1)}, \ldots, \varphi^{(k)})$ be denoted by $(\kappa_n^{(i)} : \mathcal{A}^n \to \mathbb{C}, 0 \leq i \leq k)_{n=1}^{\infty}$. Then, for every $n \geq 1, 0 \leq i \leq k$ and $a_1, \ldots, a_n \in \mathcal{A}$ one has*

$$\kappa_n^{(i)}(a_1, \ldots, a_n) = \sum_{\lambda \in \Lambda_{n,i}} C_i^{\lambda_1, \ldots, \lambda_n} \kappa_n(D^{\lambda_1}(a_1), \ldots, D^{\lambda_n}(a_n)).$$

Proof. Define the family of multilinear functionals $(\eta_n^{(i)} : \mathcal{A}^n \to \mathbb{C}, 0 \leq i \leq k)_{n=1}^{\infty}$ by the following formulas : for every $n \geq 1, 0 \leq i \leq k$ and $b_1, \ldots, b_n \in \mathcal{A}$

$$\eta_n^{(i)}(b_1, \ldots, b_n) = \sum_{\lambda \in \Lambda_{n,i}} C_i^{\lambda_1, \ldots, \lambda_n} \kappa_n(D^{\lambda_1}(b_1), \ldots, D^{\lambda_n}(b_n)).$$

Our aim is then to prove that, for every $n \geq 1, 0 \leq i \leq k$, $\eta_n^{(i)} = \kappa_n^{(i)}$. We verify that the functionals $(\eta_n^{(i)}, 0 \leq i \leq k)_{n=1}^{\infty}$ satisfy the equations (5.7) defining the infinitesimal non-crossing cumulant functionals. The left-hand side of this formula writes :

$$\sum_{\substack{p \in NC^{(A)}(n) \\ p := \{V_1, \ldots, V_h\}}} \sum_{\lambda \in \Lambda_{h,i}} C_i^{\lambda_1, \ldots, \lambda_h} \eta_p^{(\lambda)}(a_1, \ldots, a_n). \tag{5.10}$$

Each $\eta_{|V_j|}^{(\lambda_j)}((a_1, \ldots, a_n) \mid V_j)$ in the latter is a sum indexed by $\Lambda_{|V_j|, \lambda_j}$, involving variables $a_i, i \in V_j$. Given $p := \{V_1, \ldots, V_h\} \in NC^{(A)}(n)$, there is a very natural bijection between $\{(\lambda, (\lambda^1, \ldots, \lambda^h)) \in \Lambda_{h,i} \times \Lambda_{n,i} \mid \lambda^j \in \Lambda_{|V_j|, \lambda_j}\}$ and the set $\Lambda_{n,i}$. Thus :

$$(5.10) = \sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_{n,i}} C_i^{\lambda_1, \ldots, \lambda_n} \kappa_p(D^{\lambda_1}(a_1), \ldots, D^{\lambda_n}(a_n)).$$

By exchanging the summation signs, the usual free moment-cumulant formula appears, and one obtains :

$$(5.10) = \sum_{\lambda \in \Lambda_{n,i}} C_i^{\lambda_1, \ldots, \lambda_n} \varphi(D^{\lambda_1}(a_1) \cdots D^{\lambda_n}(a_n)). \tag{5.11}$$

Using Leibniz rule in the right-hand side of (5.11), one may conclude :

$$\begin{aligned} (5.10) &= \varphi\Big(\sum_{\lambda \in \Lambda_{n,i}} C_i^{\lambda_1, \ldots, \lambda_n} D^{\lambda_1}(a_1) \cdots D^{\lambda_n}(a_n)\Big) \\ &= \varphi(D^i(a_1 \cdots a_n)) \\ &= \varphi^{(i)}(a_1 \cdots a_n). \end{aligned}$$

□

Corollary 5.2.17. *In the notations of Proposition 5.2.16, let* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *be unital subalgebras of* $\mathcal{A}$ *which are free in* $(\mathcal{A}, \varphi)$*, and such that* $D(\mathcal{A}_j) \subseteq \mathcal{A}_j$ *for* $1 \leq j \leq n$*. Then* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *are infinitesimally free of order* k *in* $(\mathcal{A}, \varphi, \varphi^{(1)}, \ldots, \varphi^{(k)})$*.*

5.3 Addition and multiplication of infinitesimally free variables

In this section, we consider n-tuples of infinitesimal noncommutative random variables $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathcal{A}^n$ (where $(\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})$ is an infinitesimal noncommutative probability space of order k), with respective infinitesimal distributions $(\mu^{(i)})_{0 \leq i \leq k}$ and $(\nu^{(i)})_{0 \leq i \leq k}$. We assume that the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are infinitesimally free of order k and we are interested in the distributions of the sum $(a_1, \ldots, a_n) + (b_1, \ldots, b_n)$ and of the product $(a_1 b_1, \ldots, a_n b_n)$.

5.3.1 Addition of infinitesimally free random variables

We do not provide a proof of the following result, which is a straightforward calculation using multilinearity of the infinitesimal cumulant functionals and definition of infinitesimal freeness.

Proposition 5.3.1. *Let* $(\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})$ *be an infinitesimal noncommutative probability space of order* k*. Consider subsets* $\mathcal{M}_1, \mathcal{M}_2$ *of* $\mathcal{A}$ *that are infinitesimally free of order* k*. Then, one has, for each* $n \geq 1$*, each* n*-tuples* $(a_1, \ldots, a_n) \in \mathcal{M}_1^n, (b_1, \ldots, b_n) \in \mathcal{M}_2^n$ *and each* $0 \leq i \leq k$ *:*

$$\kappa_n^{(i)}(a_1 + b_1, \ldots, a_n + b_n) = \kappa_n^{(i)}(a_1, \ldots, a_n) + \kappa_n^{(i)}(b_1, \ldots, b_n). \tag{5.12}$$

Using formulas (5.7) and (5.9), we see that the quantities

$$\kappa_m^{(i)}(a_{i_1}, \ldots, a_{i_m}), \kappa_m^{(i)}(b_{j_1}, \ldots, b_{j_m}),$$

where $0 \leq i \leq k, m \geq 1, \{i_1, \ldots, i_m\}, \{j_1, \ldots, j_m\} \subseteq [n]$, called respectively infinitesimal cumulants of $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ completely determine and are completely determined by the infinitesimal distributions of $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. Proposition 5.3.1 thus has the following corollary.

Corollary 5.3.2. *Let* $(\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})$ *be an infinitesimal noncommutative probability space of order* k*, and* $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ *be* n*-tuples of elements of* $\mathcal{A}$ *with respective infinitesimal distributions* $(\mu^{(i)})_{0 \leq i \leq k}, (\nu^{(i)})_{0 \leq i \leq k}$*. If the sets* $\{a_1, \ldots, a_n\}$ *and* $\{b_1, \ldots, b_n\}$ *are infinitesimally free of order* k*, then the infinitesimal distribution of* $(a_1, \ldots, a_n) + (b_1, \ldots, b_n)$ *only depends on* $(\mu^{(i)})_{0 \leq i \leq k}$ *and* $(\nu^{(i)})_{0 \leq i \leq k}$*. It is called the infinitesimal free additive convolution of order* k *of* $(\mu^{(i)})_{0 \leq i \leq k}$ *and* $(\nu^{(i)})_{0 \leq i \leq k}$ *and denoted by* $(\mu^{(i)})_{0 \leq i \leq k} \boxplus^{(k)} (\nu^{(i)})_{0 \leq i \leq k}$*.*

The corollary above means that the infinitesimal free additive convolution of order k defines an operation on infinitesimal laws. The practical way to compute the infinitesimal free additive convolution of order k of two infinitesimal laws is to use consecutively the inverse of the infinitesimal version of the free moment-cumulant formula (formula (5.9)), the additivity of infinitesimal cumulants (formula (5.12)), and finally the infinitesimal version of the free moment-cumulant formula (formula (5.7)). One may find easier to make the computations in a scarce $\mathcal{C}_k$-noncommutative probability space. Taking into account the link (5.8) between infinitesimal cumulant functionals and $\mathcal{C}_k$-non-crossing cumulant functionals, Proposition 5.3.1 admits the following corollaries :

Corollary 5.3.3. *Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-probability space. Consider subsets $\mathcal{M}_1, \mathcal{M}_2$ of $\mathcal{A}$ that are infinitesimally free of order k. Then, one has, for each $n \geq 1$ and each n-tuples $(a_1, \dots, a_n) \in \mathcal{M}_1^n, (b_1, \dots, b_n) \in \mathcal{M}_2^n$*

$$\tilde{\kappa}_n(a_1 + b_1, \dots, a_n + b_n) = \tilde{\kappa}_n(a_1, \dots, a_n) + \tilde{\kappa}_n(b_1, \dots, b_n).$$

Corollary 5.3.4. *Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-probability space. Consider $a, b \in \mathcal{A}$ that are infinitesimally free of order k, then*

$$\tilde{R}_{a+b} = \tilde{R}_a + \tilde{R}_b.$$

Remark 5.3.5. Using Corollary 5.3.3, it is possible to state and prove $\mathcal{C}_k$-valued versions of some famous limit theorems of free probability. We discuss this without going into the details ; for a more complete discussion of limit theorems in free probability of type B, we refer to [Pop07] and [BS09]. In a scarce $\mathcal{C}_k$-noncommutative probability space $(\mathcal{A}, \tilde{\varphi})$, consider a sequence $(a_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ of centered infinitesimally free identically distributed $\mathcal{C}_k$-valued noncommutative random variables. Then the moments of the rescaled sum

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n$$

converge to a $\mathcal{C}_k$-valued distribution characterized by the vanishing of all of its cumulants except the second one : this is the $\mathcal{C}_k$-valued version of the free central limit theorem. The distributions that appear as limits in the preceding result deserve to be named $\mathcal{C}_k$-valued semicircular elements. Their moments may be computed using the $\mathcal{C}_k$-valued free moment-cumulant formula. Paralelly, a $\mathcal{C}_k$-valued version of the free Poisson theorem may also be stated and proved, and thus a $\mathcal{C}_k$-valued Poisson distribution may be defined.

5.3.2 Multiplication of infinitesimally free random variables

We now investigate the distribution of the product of n-tuples of noncommutative random variables that are infinitesimally free of order k. We first focus on a $\mathcal{C}_k$-noncommutative probability space because, the combinatorics being the same in this setting as in usual free probability, the proofs and results will be straightforward adaptations of the usual ones, which can be found in [NS06] for instance.

Proposition 5.3.6. *Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-noncommutative probability space. Consider subsets $\mathcal{M}_1, \mathcal{M}_2$ of $\mathcal{A}$ that are infinitesimally free of order k. Then, one has, for each $n \geq 1$ and each n-tuples $(a_1, \ldots, a_n) \in \mathcal{M}_1^n, (b_1, \ldots, b_n) \in \mathcal{M}_2^n$*

$$\tilde{\kappa}_n(a_1b_1, \ldots, a_nb_n) = \sum_{p \in NC^{(A)}(n)} \tilde{\kappa}_p(a_1, \ldots, a_n)\tilde{\kappa}_{Kr(p)}(b_1, \ldots, b_n). \quad (5.13)$$

Proof. Using Proposition 5.2.5, the left-hand side of (5.13) is equal to

$$\sum_{\substack{\pi \in NC(2n) \text{ such} \\ \text{that } \pi \vee \theta = 1_s}} \tilde{\kappa}_\pi(a_1, b_1, a_2, \ldots, b_{n-1}, a_n, b_n),$$

where θ is the partition $\{\{1,2\}, \ldots, \{2n-1, 2n\}\}$.
By the vanishing of mixed cumulants condition, the only contributing terms are those indexed by non-crossing partitions π which are reunion of a non-crossing partition p of $\{1, 3, \ldots, 2n-1\}$ and a non-crossing partition q of $\{2, 4, \ldots, 2n\}$. The condition $\pi \vee \theta = 1_s$ for such a partition π may be reinterpreted as $q = \mathrm{Kr}(p)$ (up to the identifications $\{1, 3, \ldots, 2n-1\} \leftrightarrow [n]$ and $\{2, 4, \ldots, 2n\} \leftrightarrow \overline{[n]}$). □

Switching to the infinitesimal framework, one can state the following result.

Corollary 5.3.7. *Let $(\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})$ be an infinitesimal noncommutative probability space of order k, and $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathcal{A}^n$ be n-tuples with respective infinitesimal distributions $(\mu^{(i)})_{0 \leq i \leq k}$ and $(\nu^{(i)})_{0 \leq i \leq k}$.*
If the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are infinitesimally free of order k, then the infinitesimal distribution of $(a_1b_1, \ldots, a_nb_n)$ only depends on $(\mu^{(i)})_{0 \leq i \leq k}$ and $(\nu^{(i)})_{0 \leq i \leq k}$. It is denoted by $(\mu^{(i)})_{0 \leq i \leq k} \boxtimes^{(k)} (\nu^{(i)})_{0 \leq i \leq k}$ and called the infinitesimal free multiplicative convolution of order k of $(\mu^{(i)})_{0 \leq i \leq k}$ and $(\nu^{(i)})_{0 \leq i \leq k}$.

If $a, b \in \mathcal{A}$ are $\mathcal{C}_k$-noncommutative random variables that are infinitesimally free of order k in a scarce $\mathcal{C}_k$-noncommutative probability space, the $\mathcal{C}_k$-valued R-transform of $a \cdot b$ is $\tilde{R}_{a \cdot b} = \tilde{R}_a \boxed{\star}_{\mathcal{C}_k} \tilde{R}_b$, where $\boxed{\star}_{\mathcal{C}_k}$ is the version of the boxed convolution operation introduced in [NS96], but with scalars in $\mathcal{C}_k$ (see Definition 2.3.11) :

Proposition 5.3.8. *Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-noncommutative probability space. Consider $a, b \in \mathcal{A}$ that are infinitesimally free of order k, then*

$$\tilde{R}_{a\cdot b} = \tilde{R}_a \boxed{\star}^{(A)}_{\mathcal{C}_k} \tilde{R}_b.$$

Proposition 2.3.13 also has an infinitesimal analogue, as stated in the next proposition. It is indeed straightforward to check that, in the particular case of single variables, the formulas (5.5) and (5.6) may be read at the level of power series as follows :

Proposition 5.3.9. *Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-noncommutative probability space and consider a $\mathcal{C}_k$-noncommutative random variable $a \in \mathcal{A}$. Then the $\mathcal{C}_k$-valued moment series $\tilde{M}_a$ and the $\mathcal{C}_k$-valued R-transform $\tilde{R}_a$ of a are related by the equivalent formulas : $\tilde{M}_a = \tilde{R}_a \boxed{\star}^{(A)}_{\mathcal{C}_k} \zeta^{(A)}_{\mathcal{C}_k}$, $\tilde{R}_a = \tilde{M}_a \boxed{\star}^{(A)}_{\mathcal{C}_k} Möb^{(A)}_{\mathcal{C}_k}$.*

As noticed in [Pop07], the combinatorial proofs remaining valid for series with $\mathcal{C}_k$-valued coefficients such that the coefficient of degree one is invertible, one has a Fourier transform construction leading to the $\mathcal{C}_k$-valued S-transform.

Definition 5.3.10. Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-noncommutative probability space. The $\mathcal{C}_k$-valued S-transform of an infinitesimal noncommutative random variable $a \in \mathcal{A}$ such that $\tilde{\varphi}(a)$ is invertible in $\mathcal{C}_k$ is the power series $\tilde{S}_a \in \Theta^{(A)}_{\mathcal{C}_k}$ defined as follows:

$$\tilde{S}_a(z) := \frac{1}{z} \tilde{R}_a^{\langle -1 \rangle}(z).$$

Proposition 5.3.11. *Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $\mathcal{C}_k$-noncommutative probability space. Consider $a, b \in \mathcal{A}$ that are infinitesimally free of order k, and such that $\tilde{\varphi}(a)$ and $\tilde{\varphi}(b)$ are invertible in $\mathcal{C}_k$, then the $\mathcal{C}_k$-valued S-transform $\tilde{S}_{a\cdot b}$ of $a \cdot b$ satisfies:*

$$\tilde{S}_{a\cdot b}(z) = \tilde{S}_a(z)\tilde{S}_b(z).$$

Practically speaking, the computation of the distribution of the product of two infinitesimally free infinitesimal noncommutative random variables requires a good understanding of the $\mathcal{C}_k$-valued version of the boxed convolution. More precisely, in the notations of Definition 2.3.11, it would be of interest to have a formula for $\gamma_m^{(i)}$ as a function of the $\alpha_n^{(j)}$'s and the $\beta_n^{(j)}$'s. As mentioned before, the version of the boxed convolution with scalars in $\mathcal{C}_0 = \mathbb{C}$ is a classical operation in free probability. The version of the boxed convolution with scalars in $\mathcal{C}_1 = \mathbb{G}$ has already been considered in [BGN03], where it is shown to coincide with the boxed convolution based on non-crossing partitions of type B, in connection with free probability of type B (see Theorem 5.4.19). This leads to the natural question : does the operation $\boxed{\star}^{(A)}_{\mathcal{C}_k}$ coincide with a boxed convolution based on a certain set of special non-crossing partitions. In Section 5.4, we will give a positive answer to this problem, by introducing the non-crossing partitions of type k.

5.4 Non-crossing partitions of type k

This section is devoted to the introduction and study of a set of non-crossing partitions, namely the set of non-crossing partitions of type k, which has to be a cover of $NC^{(A)}(n)$ related to the version of the boxed convolution with scalars in $\mathcal{C}_k$.

5.4.1 Definition and first properties

Definition 5.4.1. Let n be a positive integer. We call *reduction mod n map* the map

$$\mathrm{Red}_n^{(k)} : [(k+1)n] \to [n]$$

sending each $i \in [(k+1)n]$ to its congruence class mod n.

Remark 5.4.2. For $k = 0$, the map $\mathrm{Red}_n^{(0)}$ is simply the identity map on $[n]$.
For $k = 1$, up to identifying $[2n]$ with $[\pm n]$, the map $\mathrm{Red}_n^{(1)}$ is identified with the map Abs defined in Section 2.5.

Definition 5.4.3. A non-crossing partition π of $[(k+1)n]$ is said to satisfy *the mod n reduction property* if $\mathrm{Red}_n^{(k)}(\pi)$ is a non-crossing partition of $[n]$ and if $\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$ is a non-crossing partition of $\overline{[n]}$.

Non-crossing partitions of type k are the non-crossing partitions of the set $[(k+1)n]$ satisfying the mod n reduction property.

Definition 5.4.4. We write $NC^{(k)}(n)$ for the set of non-crossing partitions of type k, that is non-crossing partitions of $[(k+1)n]$ satisfying the mod n reduction property.

Remark 5.4.5. All non-crossing partitions of $[n]$ trivially satisfy the mod n reduction property (since $\mathrm{Red}_n^{(0)}$ is simply the identity map). Hence $NC^{(0)}(n) = NC^{(A)}(n)$.

The next proposition states that the non-crossing partitions of type k are a generalization of the non-crossing partitions of type B.

Proposition 5.4.6. *If we identify $[\pm n]$ with $[2n]$ and* Abs *with* $\mathrm{Red}_n^{(1)}$, *then $NC^{(B)}(n) = NC^{(1)}(n)$.*

Proof. That $\pi \in NC^{(B)}(n)$ satisfies the mod n reduction property is a corollary of Theorem 2.5.1 and Lemma 1 in [BGN03].
For the converse, let $\pi \in NC^{(1)}(n)$ satisfy the mod n reduction property, and assume that there exist two elements $x, y \in [\pm n]$ such that $x \sim_\pi y$, $-x \not\sim_\pi -y$. By reduction mod n property, we necessarily have $-y \sim_\pi x \sim_\pi y \sim_\pi -x$, which is a contradiction. $\square$

Remark 5.4.7. The proof above and Lemma 1 in [BGN03] show that, for a non-crossing partition π of $[2n]$, the mod n reduction property is equivalent to the only requirement that $\mathrm{Red}_n^{(1)}(\pi)$ is a non-crossing partition of $[n]$.

In Definition 5.4.3, the reduction mod n property for a non-crossing partition π of $[(k+1)n]$ consists of two requirements : $\mathrm{Red}_n^{(k)}(\pi)$ has to be a non-crossing partition of $[n]$ and $\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$ has to be a non-crossing partition of $\overline{[n]}$. Actually, there is a slightly stronger characterization stated in the next proposition.

Proposition 5.4.8. *A non-crossing partition π of $[(k+1)n]$ satisfies the reduction mod n property if and only if $\mathrm{Red}_n^{(k)}(\pi \cup \mathrm{Kr}(\pi))$ is a non-crossing partition of $[n] \cup \overline{[n]}$.*

Proof. If $\mathrm{Red}_n^{(k)}(\pi \cup \mathrm{Kr}(\pi))$ is a non-crossing partition of $[n] \cup \overline{[n]}$, since $\mathrm{Red}_n^{(k)}(\pi)$ is a family of subsets of $[n]$ and $\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$ is a family of subsets of $\overline{[n]}$, they have to be non-crossing partitions of $[n]$ and $\overline{[n]}$ respectively ; in other words π has to satisfy the reduction mod n property.
We assume now that π is a non-crossing partition of $[(k+1)n]$ satisfying the reduction mod n property, and aim at proving that $\mathrm{Red}_n^{(k)}(\pi \cup \mathrm{Kr}(\pi))$ is a non-crossing partition of $[n] \cup \overline{[n]}$.
By the reduction property, $\mathrm{Red}_n^{(k)}(\pi \cup \mathrm{Kr}(\pi)) = \mathrm{Red}_n^{(k)}(\pi) \cup \mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$ is the union of a partition of $[n]$ and of a partition of $\overline{[n]}$, and hence a partition of $[n] \cup \overline{[n]}$. To prove that this partition is non-crossing, consider four elements $a < b < c < d$ of $[n] \cup \overline{[n]}$, such that $a \sim_{Red_n^{(k)}(\pi \cup Kr(\pi))} c$ and $b \sim_{Red_n^{(k)}(\pi \cup Kr(\pi))} d$. We have to show that $a \sim_{Red_n^{(k)}(\pi \cup Kr(\pi))} b$.
Let $1 \leq i_0 \leq (k+1)n+1$ be minimal with the property that $\mathrm{Mix}(\pi, i_0)$ contains an element x such that $\mathrm{Red}_n^{(k)}(x) \in \{a, b, c, d\}$. Choose also the smallest such x. We may assume that $\mathrm{Red}_n^{(k)}(x) = a$ (the other cases are similar). By assumption, $c \in \mathrm{Red}_n^{(k)}(\mathrm{Mix}(\pi, i_0))$: there is an element $z \in \mathrm{Mix}(\pi, i_0)$ such that $\mathrm{Red}_n^{(k)}(z) = c$. Our choice of x ensures that $x < z$ and there is necessarily an element $x < y < z$ such that $\mathrm{Red}_n^{(k)}(y) = b$. By minimality of i_0, $y \in \mathrm{Mix}(\pi, i_0)$, hence $b \in \mathrm{Red}_n^{(k)}(\mathrm{Mix}(\pi, i_0))$ is linked to a in $\mathrm{Red}_n^{(k)}(\pi \cup \mathrm{Kr}(\pi))$ and we are done. □

Remark 5.4.9. When $k = 0$, the reduction mod n property is satisfied by any non-crossing partition of $[n]$ and is in particular equivalent to the only empty requirement : $\pi \in NC^{(A)}(n)$ satisfies the reduction mod n property if and only if $\mathrm{Red}_n^{(0)}(\pi)$ is a non-crossing partition of $[n]$.
As explained in Remark 5.4.7, this is also the case when $k = 1$: $\pi \in NC^{(A)}(2n)$ satisfies the reduction mod n property if and only if $\mathrm{Red}_n^{(1)}(\pi)$ is a non-crossing partition of $[n]$.

Assume now that $k \geq 2$; the situation then is different.
As an example, for $k = 2$ and $n = 2$, consider the partition

$$\pi := \{\{1, 2, 3\}, \{4, 5, 6\}\} \in NC^{(A)}(6).$$

It is straightforward to check that $\mathrm{Red}_2^{(2)}(\pi) = \{1, 2\}$ is a non-crossing partition of $[2]$. However, from the easy computation

$$\mathrm{Kr}(\pi) = \{\{\overline{1}\}, \{\overline{2}\}, \{\overline{4}\}, \{\overline{5}\}, \{\overline{3}, \overline{6}\}\},$$

we deduce that $\mathrm{Red}_2^{(2)}(\mathrm{Kr}(\pi))$ is not a partition of $\overline{[6]}$ and consequently that π does not satisfy the reduction mod 2 property.

The following proposition states that the Kreweras complementation map may be considered as an order-reversing bijection of $NC^{(k)}(n)$.

Proposition 5.4.10. *The restrictions from $NC^{(A)}((k+1)n)$ to $NC^{(k)}(n)$ of* Kr *is an order-reversing bijection of $NC^{(k)}(n)$.*

The name of Kreweras complementation map and the notation Kr will be conserved as there is no ambiguity about the meaning of $\mathrm{Kr}(\pi)$ whether π is viewed as an element of $NC^{(k)}(n)$ or of $NC^{(A)}((k+1)n)$.

Proof. It is clearly sufficient to prove that the non-crossing partition $\mathrm{Kr}(\pi)$ of $\overline{[(k+1)n]}$ satisfies the reduction mod n property whenever π does. Assume that the non-crossing partition π of $[(k+1)n]$ satisfies the reduction mod n property. By assumption, $\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$ is a non-crossing partition of $\overline{[n]}$. It remains to prove that $\mathrm{Red}_n^{(k)}(\mathrm{Kr}^2(\pi))$ is a non-crossing partition of $\overline{\overline{[n]}}$.
From the geometric description of $\mathrm{Kr}^2(\pi)$ given in Subsection 2.3.1, we deduce that $\mathrm{Red}_n^{(k)}(\mathrm{Kr}^2(\pi))$ is obtained from $\mathrm{Red}_n^{(k)}(\pi)$ by a rotation. By reduction mod n property, $\mathrm{Red}_n^{(k)}(\pi)$ is a non-crossing partition of $[n]$, so $\mathrm{Red}_n^{(k)}(\mathrm{Kr}^2(\pi))$ is itself a non-crossing partition of $\overline{\overline{\overline{[n]}}}$. Thus the proof is complete. ☐

Given $\pi \in NC^{(k)}(n)$, $\mathrm{Kr}(\mathrm{Red}_n^{(k)}(\pi))$ and $\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$ are thus two non-crossing partitions of $\overline{[n]}$. The following lemma, generalizing Lemma 1 of [BGN03], states that these two partitions coincide.

Proposition 5.4.11. $\forall \pi \in NC^{(k)}(n), \mathrm{Kr}(\mathrm{Red}_n^{(k)}(\pi)) = \mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi)).$

Proof. Let π be a non-crossing partition of type k. By Proposition 5.4.8, $\mathrm{Red}_n^{(k)}(\pi) \cup \mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi)) = \mathrm{Red}_n^{(k)}(\pi \cup \mathrm{Kr}(\pi))$ is a non-crossing partition of $[n] \cup \overline{[n]}$. Since $\mathrm{Kr}(\mathrm{Red}_n^{(k)}(\pi))$ is maximal with the property that $\mathrm{Red}_n^{(k)}(\pi) \cup \mathrm{Kr}(\mathrm{Red}_n^{(k)}(\pi))$ is non-crossing, it follows that

$$\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi)) \leq \mathrm{Kr}(\mathrm{Red}_n^{(k)}(\pi)).$$

There is equality if, for any $\overline{x}$ having a neighbour $\overline{y} > \overline{x}$ in $\mathrm{Kr}(\mathrm{Red}_n^{(k)}(\pi))$, $\overline{y}$ is linked to $\overline{x}$ in $\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$. For such elements $\overline{x}, \overline{y} \in \overline{[n]}$, we call V the block of π containing $x+1$. The reduction property implies that $\mathrm{Red}_n^{(k)}(V)$ is a block of the partition $\mathrm{Red}_n^{(k)}(\pi)$. By construction of the Kreweras complement, $x+1$ is the smallest element of both V and $\mathrm{Red}_n^{(k)}(V)$, and y is the greatest element of $\mathrm{Red}_n^{(k)}(V)$. Consider now the greatest element z of V. Notice that $x+1 \leq \mathrm{Red}_n^{(k)}(z) \leq y$. By construction of the Kreweras complement again, $\overline{x}$ is linked to $\overline{z}$ in $\mathrm{Kr}(\pi)$, then $\overline{x}$ is linked to $\mathrm{Red}_n^{(k)}(\overline{z}) = \overline{\mathrm{Red}_n^{(k)}(z)}$ in $\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$ and therefore in $\mathrm{Kr}(\mathrm{Red}_n^{(k)}(\pi))$. This means that, if $\mathrm{Red}_n^{(k)}(z) < y$, $\overline{y}$ would not be the neighbour of $\overline{x}$ in $\mathrm{Kr}(\mathrm{Red}_n^{(k)}(\pi))$, which is a contradiction. So $\mathrm{Red}_n^{(k)}(z) = y$ or, in other words, $\overline{x}$ is linked to $\overline{y}$ in $\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$. □

A deeper description of non-crossing partitions of type k is given in the next subsection.

5.4.2 Structure of non-crossing partitions of type k

The goal of this subsection is to describe the structure of a non-crossing partition of type k. In the next proposition, t denotes the bijection between non-crossing partitions and permutations lying on a geodesic in the Cayley graph of the symmetric group, introduced by Biane in [Bia97b], and described in Section 2.5. We warn the reader that we choose to use the same notation t for this bijection, defined either on $NC(n)$ or $NC((k+1)n)$. We hope that this choice, made in the sake of simplicity, will not be a source of confusion in the reader's mind. The content of this proposition is, roughly speaking, that a type k non-crossing partition π is characterized by the two requirements : $\mathrm{Red}_n^{(k)}(\pi)$ is a non-crossing partition of $NC^{(A)}(n)$ and the elements of each of the blocks of π come in the same order as their congruence classes in its reduction $\mathrm{Red}_n^{(k)}(\pi)$.

Proposition 5.4.12. *For $\pi \in NC^{(A)}((k+1)n)$ such that $\mathrm{Red}_n^{(k)}(\pi) \in NC^{(A)}(n)$, $\pi \in NC^{(k)}(n)$ if and only if*

$$\forall x \in [(k+1)n], \mathrm{Red}_n^{(k)}(t(\pi)(x)) = t(\mathrm{Red}_n^{(k)}(\pi))(\mathrm{Red}_n^{(k)}(x)). \qquad (5.14)$$

Proof. Assume first that $\pi \in NC^{(k)}(n)$ and fix $x \in [(k+1)n]$. Set $y := t(\pi)(x)$.
By construction of t, y is the neighbour of x in π and $t(\mathrm{Red}_n^{(k)}(\pi))(\mathrm{Red}_n^{(k)}(x))$ is the neighbour of $\mathrm{Red}_n^{(k)}(x)$ in $\mathrm{Red}_n^{(k)}(\pi)$. By construction of the Kreweras complement, $\overline{x}$ is the neighbour of $\overline{y-1}$ in $\mathrm{Kr}(\pi)$, and $\overline{\mathrm{Red}_n^{(k)}(x)}$ is the neighbour of $\overline{t(\mathrm{Red}_n^{(k)}(\pi))(\mathrm{Red}_n^{(k)}(x)) - 1}$ in $\mathrm{Kr}(\mathrm{Red}_n^{(k)}(\pi)) = \mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$

(the latter equality holds because of Proposition 5.4.11). By reduction property, $\mathrm{Red}_n^{(k)}(y-1)$ is linked to $\mathrm{Red}_n^{(k)}(x)$. It follows that the neighbour of $\mathrm{Red}_n^{(k)}(x)$ in $\mathrm{Red}_n^{(k)}(\pi)$, $t(\mathrm{Red}_n^{(k)}(\pi))(\mathrm{Red}_n^{(k)}(x))$, is the first point coming after $\mathrm{Red}_n^{(k)}(y-1)$ linked to $\mathrm{Red}_n^{(k)}(x)$: it is $\mathrm{Red}_n^{(k)}(y)$ and we are done. For the converse, let $\pi \in NC^{(A)}((k+1)n)$ be such that $\mathrm{Red}_n^{(k)}(\pi) \in NC^{(A)}(n)$ and assume that condition (5.14) holds. We have to prove that $\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi))$ is a non-crossing partition of $[n]$. Let $\overline{x} \in [(k+1)n]$, its neighbour in $\mathrm{Kr}(\pi)$ is $t(\pi)^{-1}(x+1)$, by construction of the Kreweras complement. It follows of condition (5.14) that $\mathrm{Red}_n^{(k)}(t(\pi)^{-1}(x+1)) = t(\mathrm{Red}_n^{(k)}(\pi))^{-1}(\mathrm{Red}_n^{(k)}(x+1))$. Hence the congruence class of the neighbour of $\overline{x}$ in $\mathrm{Kr}(\pi)$ only depends on the congruence class of x, and moreover $\mathrm{Red}_n^{(k)}(\mathrm{Kr}(\pi)) = \mathrm{Kr}(\mathrm{Red}_n^{(k)}(\pi))$ and we are done. □

The preceding proposition has some important consequences.

Corollary 5.4.13. *Let $\pi \in NC^{(A)}((k+1)n)$ and V be a block of $\pi \cup \mathrm{Kr}(\pi)$. The cardinal of $\mathrm{Red}_n^{(k)}(V)$ divides the cardinal of V. We call* multiplicity *of V the quotient*

$$\mathrm{mult}_{\pi\cup Kr(\pi)}(V) := \frac{\mathrm{card}(V)}{\mathrm{card}(\mathrm{Red}_n^{(k)}(V))}.$$

This is a positive integer lower or equal than $k+1$. The blocks of multiplicity 1 will be called simple.

Proof. For $x \in V$, the cardinal of V is the smallest positive i verifying

$$(t(\pi))^i(x) = x.$$

A repeated use of Proposition 5.4.12 gives that, for such an i,

$$(t(\mathrm{Red}_n^{(k)}(\pi)))^i(\mathrm{Red}_n^{(k)}(x)) = \mathrm{Red}_n^{(k)}(x). \tag{5.15}$$

Thus i is a multiple of the cardinal of $\mathrm{Red}_n^{(k)}(V)$, which is also characterized by the fact that it is the smallest positive i veriying condition (5.15). □

It is not so difficult to see that, if there is a block of multiplicity $k+1$ in $\pi \cup \mathrm{Kr}(\pi)$, for $\pi \in NC^{(A)}((k+1)n)$, the other blocks are necessarily simple, because one cannot link two elements of the same congruence class without crossing the block of multiplicity $k+1$. This is in fact a particular case of the following result :

Corollary 5.4.14. *For $\pi \in NC^{(k)}(n)$,*

$$\sum_{V \in \mathrm{bl}(\pi\cup Kr(\pi))} (\mathrm{mult}_{\pi\cup Kr(\pi)}(V) - 1) = k.$$

Proof. This is a simple computation. First notice that

$$\sum_{V\in \mathrm{bl}(\pi\cup Kr(\pi))} (\mathrm{mult}_{\pi\cup Kr(\pi)}(V)-1) = \sum_{V\in \mathrm{bl}(\pi\cup Kr(\pi))} \mathrm{mult}_{\pi\cup Kr(\pi)}(V) - |\pi\cup \mathrm{Kr}(\pi)|. \tag{5.16}$$

The first term in (5.16) is

$$\sum_{W\in \mathrm{bl}(Red_n^{(k)}(\pi\cup Kr(\pi)))} \sum_{V\in \mathrm{bl}(\pi\cup Kr(\pi)):Red_n^{(k)}(V)=W} \mathrm{mult}_{\pi\cup Kr(\pi)}(V).$$

But for any block W of $\mathrm{Red}_n^{(k)}(\pi\cup\mathrm{Kr}(\pi))$, one has

$$\sum_{V\in \mathrm{bl}(\pi\cup Kr(\pi)):Red_n^{(k)}(V)=W} \mathrm{mult}_{\pi\cup Kr(\pi)}(V) = k+1.$$

Applying twice formula (2.1), we get

$$\begin{aligned}\sum_{V\in \mathrm{bl}(\pi\cup Kr(\pi))} (\mathrm{mult}_{\pi\cup Kr(\pi)}(V)-1) &=(k+1)(n+1) - |\pi\cup \mathrm{Kr}(\pi)| \\ &= (k+1)(n+1) - ((k+1)n+1) \\ &= k.\end{aligned}$$

□

For a partition $\pi\in NC^{(k)}(n)$, one may define a vector λ_π with integer coordinates as follows :

$$(\lambda_\pi)_i = \sum_{V\in \mathrm{bl}(\pi\cup Kr(\pi)):Red_n^{(k)}(V)=Mix(Red_n^{(k)}(\pi\cup Kr(\pi)),i)} (\mathrm{mult}_{\pi\cup Kr(\pi)}(V)-1).$$

The vector $\lambda_\pi\in\Lambda_{n+1,k}$ is called the *shape* of π.

Remark 5.4.15. A type B non-crossing partition π is determined by its absolute value $p := \mathrm{Abs}(\pi)$ and the choice of the block $Z\in \mathrm{bl}(p\cup\mathrm{Kr}(p))$, which has to be lifted to the zero-block of π. This latter choice is encoded in the shape λ_π of π. Indeed, type B corresponds to the case $k=1$ of non-crossing partitions of type k and therefore the shape λ_π belongs to the set $\Lambda_{n+1,1}$ consisting of the $n+1$ vectors $e_i=(\delta_i^j)_{1\le j\le n+1}, 1\le i\le n+1$. That $\lambda_\pi=e_i$ means exactly that we have to choose the block $Mix(p,i)$ as the absolute value of the zero-block. The conclusion is that a type B non-crossing partition, considered as a non-crossing partition of type 1, is determined by its reduction (or absolute value in the type B language) and its shape. Unfortunately, this is not the case when $k\ge 2$. It is interesting to ask how to determine a general non-crossing partition of type k. This question is investigated in the proof of the next proposition.

Proposition 5.4.16. *Let $\lambda \in \Lambda_{n+1,k}$. The number of $\pi \in NC^{(k)}(n)$ having shape λ and reduction a fixed non-crossing partition $p \in NC^{(A)}(n)$ is the same for any choice of $p \in NC^{(A)}(n)$. We will denote this quantity by $r(\lambda)$.*

Proof. As announced, we investigate how to determine a type k non-crossing partition $\pi \in NC^{(k)}(n)$, once its reduction $p \in NC^{(A)}(n)$ and its shape $\lambda \in \Lambda_{n+1,k}$ are given. We know that $\mathrm{Mix}(p,1)$ is a singleton of $[n] \cup \overline{[n]}$. For simplicity, we assume that it is a singleton $\{i\}$ of $[n]$. We need to know how to form the blocks of π reducing to $\{i\}$. The number of admissible ways to form these blocks depends on the value of λ_1 but of course not on p, because the actual value of i does not come into the game. Assume that these blocks are formed; this gives a decomposition of $[(k+1)n] \setminus \{x \mid \mathrm{Red}_n^{(k)}(x) = i\} \cup \overline{[(k+1)n]}$ into $\lambda_1 + 1$ sets, according to the following process : let us denote by $\{i + l_1 n, \dots, i + l_m n\}$ the smallest (with respect to $\sqsubset$) of the blocks we have just formed that is not simple (if there is no such block, i.e. when $\lambda_1 = 0$, our decomposition is trivial) ; each of the $\{\overline{i + l_j n}, \dots, \overline{i - 1 + l_{j+1} n}\}$ becomes a set in our decomposition after erasing the $i + ln, l_j \leq l \leq l_{j+1}$, for each $1 \leq j \leq m-1$. Then remove all elements x such that $i + l_1 n \leq x \leq i + l_m n$ and repeat the process by considering the new smallest block with respect to $\sqsubset$ among the remaining blocks that are not simple. Notice that the sets obtained this way may be identified with sets of the form $[l(n-1)] \cup \overline{[l(n-1)]}$, for some $l \leq k+1$, up to identifying the first and last elements of the sets. This can be done, because these elements are necessarily linked by construction of the Kreweras complement. On each of these sets, π induces a non-crossing partition that belongs to $NC^{(l)}(n-1)$. All such induced non-crossing partitions have the same reduction $\tilde{p}$ obtained by erasing in $p \cup \mathrm{Kr}(p)$ the element i and by identifying $\overline{i-1}$ with $\bar{i}$ (which are also necessarily linked in $\mathrm{Kr}(p)$). The shapes of the induced partitions sum to the shape λ of π. Hence a non-crossing partition of type k is determined by its reduction p, its shape λ, an admissible way to form the blocks reducing to $\mathrm{Mix}(p,1)$, an admissible decomposition of λ and the choice of the induced non-crossing partitions in sets $NC^{(l)}(n-1)$, having reduction $\tilde{p}$ and shape the summands in the decomposition of λ.

Our argument goes by induction on n. For $n = 1$ and any k, there is only one possible reduction, because $\mathrm{card}(NC^{(A)}(1)) = 1$ and consequently there is nothing to prove in that case. Assume that, for any l, the number of partitions in $NC^{(l)}(n-1)$ with given shape and reduction does not depend on the choice of the reduction. According to our analysis of the first part of the proof, the number of partitions $\pi \in NC^{(k)}(n)$ with given shape λ and reduction p does not depend on the choice of the reduction, because we noticed that the number of admissible ways to form the blocks reducing to $\mathrm{Mix}(p,1)$ does not depend on p, the shape decomposition depend only on λ and the way the latter blocks are formed, and by induction, the numbers of choices for the induced partitions only depend on their shapes. $\square$

Remark 5.4.17. For small values of k, one may easily compute the values of $r(\lambda)$ for each $\lambda \in \Lambda_{n+1,k}$.
In the simplest case $k = 0$,

$$\Lambda_{n+1,0} = \{(0, \dots, 0)\},$$

$$r((0, \dots, 0)) = 1.$$

For $k = 1$,

$$\Lambda_{n+1,1} = \{e_i\}_{i=1,\dots,n+1},$$

and one has

$$\forall 1 \leq i \leq n+1, r(e_i) = 1.$$

For $k = 2$,

$$\Lambda_{n+1,2} = \{e_i + e_j\}_{i,j=1,\dots,n+1}.$$

The value of $r(e_i + e_j)$ depends on whether $i = j$ or not:

$$\forall 1 \leq i \leq n+1, r(2e_i) = 1.$$

$$\forall 1 \leq i < j \leq n+1, r(e_i + e_j) = 3.$$

We investigate in the next subsection some properties of the set $NC^{(k)}(n)$.

5.4.3 Study of the poset $NC^{(k)}(n)$

The set $NC^{(k)}(n)$, being a subset of $(NC^{(A)}((k+1)n), \leq)$, inherits its partially ordered set (abbreviated poset) structure. Contrary to $NC^{(B)}(n)$, which is a sublattice of $(NC^{(A)}(2n), \leq)$ (up to the identification $[\pm n] = [2n]$), $(NC^{(k)}(n), \leq)$ is unfortunately not a sublattice of $(NC^{(A)}((k+1)n), \leq)$, when $k \geq 2$.

Remark 5.4.18. When $k = 2$ and $n = 2$, consider the partitions

$$\pi := \{\{2, 3, 4, 5\}, \{1, 6\}\} \in NC^{(2)}(2)$$

and

$$\rho := \{\{1, 2\}, \{3, 4, 5, 6\}\} \in NC^{(2)}(2).$$

It is an easy exercise to determine the meet of these two partitions in the lattice $(NC^{(A)}(6), \leq)$:

$$\pi \wedge_{NC^{(A)}(6)} \rho = \{\{1\}, \{2\}, \{3, 4, 5\}, \{6\}\}.$$

It is immediate that $\pi \wedge_{NC^{(A)}(6)} \rho$ is not an element of $NC^{(2)}(2)$ which is consequently not a sublattice of $(NC^{(A)}(6), \leq)$; the same kind of argument would prove that $NC^{(k)}(n)$ is never a sublattice of $(NC^{(A)}((k+1)n), \leq)$, as soon as $k \geq 2$.

It is natural to ask whether $NC^{(k)}(n)$ is or not a lattice in its own right for the reverse refinement order $\leq$. We do not know the answer to this question.

We now state and prove the main result of this section.

Theorem 5.4.19. *$\pi \mapsto \mathrm{Red}_n^{(k)}(\pi)$ is a $\frac{1}{(k+1)n+1}C_{(n+1)(k+1)}^{k+1}$-to-1 map from $NC^{(k)}(n)$ onto $NC^{(A)}(n)$.*

Proof. We fix $p \in NC^{(A)}(n)$. The shape λ_π of a $\pi \in NC^{(k)}(n)$ satisfying $\mathrm{Red}_n^{(k)}(\pi) = p$ is an element of the set $\Lambda_{n+1,k}$, and for each $\lambda \in \Lambda_{n+1,k}$, there are exactly $r(\lambda)$ non-crossing partitions of type k with reduction p and shape λ. Hence there are $\sum_{\lambda\in\Lambda_{n+1,k}} r(\lambda)$ non-crossing partitions of type k with reduction p, and we know by Proposition 5.4.16 that this number does not depend on p. It remains to prove that $\sum_{\lambda\in\Lambda_{n+1,k}} r(\lambda) = \frac{1}{(k+1)n+1}C_{(n+1)(k+1)}^{k+1}$, by counting the non-crossing partitions of type k with reduction $1_{[n]}$. The set formed by these partitions is precisely the set $NC_n(k)$ of non-crossing partitions of $[(k+1)n]$ having blocks of size divisible by n. The latter set appears in [BBCC07], where it is proved that its cardinal is $\frac{1}{(k+1)n+1}C_{(n+1)(k+1)}^{k+1}$. □

We end this section by defining a subset of $NC^{(k)}(n)$ that will be used in Section 5.5.

Definition 5.4.20. We write $NC_*^{(k)}(n)$ for the set of non-crossing partitions of type k without non-simple blocks in their Kreweras complement.

Remark 5.4.21. In the shape of a non-crossing partition $\pi \in NC_*^{(k)}(n)$, the coordinates corresponding to blocks of $\mathrm{Kr}(\pi)$ are zero ; there is therefore a straightforward bijection between the set of shapes of non-crossing partitions $\pi \in NC_*^{(k)}(n)$ satisfying $\mathrm{Red}_n^k(\pi) = p$ and the set $\Lambda_{|p|,k}$. Notice also that, given $p \in NC^{(A)}(n)$ and $\lambda \in \Lambda_{|p|,k}$, there are exactly $r(\lambda)$ non-crossing partitions $\pi \in NC_*^{(k)}(n)$ with reduction p and, with a small abuse of language, shape λ.

Non-crossing partitions of type k give a combinatorial description of the version of the boxed convolution with scalars in $\mathcal{C}_k$, as explained in the next section.

5.5 Boxed convolution of type k

As for type A and B, there is a boxed convolution operation associated to the non-crossing partitions of type k. It is defined on formal power series with coefficients in $\mathbb{C}^{k+1}$, as follows.

Definition 5.5.1. 1. We denote by $\Theta^{(k)}$ the set of power series of the form

$$f(z) = \sum_{n=1}^{\infty} (\alpha_n^{(0)}, \ldots, \alpha_n^{(k)}) z^n,$$

where, for each $n \geq 1$ and $0 \leq i \leq k$, $\alpha_n^{(i)}$ is a complex number.

2. Let $f(z) = \sum_{n=1}^{\infty} (\alpha_n^{(0)}, \ldots, \alpha_n^{(k)}) z^n$ and $g(z) = \sum_{n=1}^{\infty} (\beta_n^{(0)}, \ldots, \beta_n^{(k)}) z^n$ be in $\Theta^{(k)}$. For every $m \geq 1$ and every $0 \leq i \leq k$, consider the numbers $\gamma_m^{(i)}$ defined by

$$\gamma_m^{(i)} = \sum_{\pi \in NC^{(i)}(m)} \frac{C_i^{(\lambda_\pi)_1, \ldots, (\lambda_\pi)_{m+1}}}{r(\lambda_\pi)} \prod_{j=1}^{|Red_m^{(i)}(\pi)|} \alpha^{((\lambda_\pi)_j)}_{card(Sep(Red_m^{(i)}(\pi), j))} \cdot \prod_{j=|Red_m^{(i)}(\pi)|+1}^{m+1} \beta^{((\lambda_\pi)_j)}_{card(Sep(Red_m^{(i)}(\pi)), j)}.$$

Then the series $\sum_{n=1}^{\infty} (\gamma_n^{(0)}, \ldots, \gamma_n^{(k)}) z^n$ is called the boxed convolution of type k of f and g, and is denoted by $f \boxed{\star}^{(k)} g$.

It turns out that, up to identifying the two sets $\Theta^{(k)}$ and $\Theta_{\mathcal{C}_k}^{(A)}$, the two operations $\boxed{\star}^{(k)}$ and $\boxed{\star}_{\mathcal{C}_k}^{(A)}$ are actually the same, as stated in the next theorem.

Theorem 5.5.2. $\boxed{\star}^{(k)} = \boxed{\star}_{\mathcal{C}_k}^{(A)}$

Proof. Let

$$f(z) = \sum_{n=1}^{\infty} (\alpha_n^{(0)}, \ldots, \alpha_n^{(k)}) z^n \in \Theta^{(k)}$$

and

$$g(z) = \sum_{n=1}^{\infty} (\beta_n^{(0)}, \ldots, \beta_n^{(k)}) z^n \in \Theta^{(k)}.$$

Write

$$f \boxed{\star}^{(k)} g = \textstyle\sum_{n=1}^{\infty} (\gamma_n^{(0)}, \ldots, \gamma_n^{(k)}) z^n$$

and

$$f \boxed{\star}_{\mathcal{C}_k}^{(A)} g = \textstyle\sum_{n=1}^{\infty} (\delta_n^{(0)}, \ldots, \delta_n^{(k)}) z^n.$$

We fix a positive integer n, for which we will show that

$$(\gamma_n^{(0)}, \ldots, \gamma_n^{(k)}) = (\delta_n^{(0)}, \ldots, \delta_n^{(k)}).$$

Let us look at $\gamma_n^{(i)}$. First, we have

$$\gamma_n^{(i)} = \sum_{\pi \in NC^{(i)}(n)} \frac{C_i^{(\lambda_\pi)_1,\ldots,(\lambda_\pi)_{n+1}}}{r(\lambda_\pi)} \prod_{j=1}^{|Red_m^{(i)}(\pi)|} \alpha^{((\lambda_\pi)_j)}_{card(Sep(Red_m^{(i)}(\pi),j))} \cdot \prod_{j=|Red_m^{(i)}(\pi)|+1}^{n+1} \beta^{((\lambda_\pi)_j)}_{card(Sep(Red_m^{(i)}(\pi)),j)}.$$

For every $\pi \in NC^{(i)}(n)$, $1 \le j \le n+1$ and $0 \le \lambda \le k$, we put $p = \mathrm{Red}_n^{(i)}(\pi)$ and

$$\theta^{(\lambda)}(p,j) := \begin{cases} \alpha^{(\lambda)}_{card(Sep(p,j))} & if \quad j \le |p|, \\ \beta^{(\lambda)}_{card(Sep(p,j))} & if \quad j > |p|, \end{cases}$$

The summation over $NC^{(i)}(n)$ can be reduced to one over $NC^{(A)}(n)$, by using the cover $\mathrm{Red}_n^{(i)} : NC^{(i)}(n) \to NC^{(A)}(n)$. When doing so, and taking into account the explicit description of $(\mathrm{Red}_n^{(i)})^{-1}(p), p \in NC^{(A)}(n)$ provided by the proof of Theorem 5.4.19, one gets

$$\gamma_n^{(i)} = \sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_{n+1,i}} C_i^{\lambda_1,\ldots,\lambda_{n+1}} \prod_{j=1}^{n+1} \theta^{(\lambda_j)}(p,j).$$

On the other hand, by recalling the definition of the operation $\boxed{\star}^{(A)}_{\mathcal{C}_k}$, we see that $\delta_n^{(i)}$ equals

$$\sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_{n+1,i}} C_i^{\lambda_1,\ldots,\lambda_{n+1}} \prod_{j=1}^{n+1} \theta^{(\lambda_j)}(p,j).$$

By comparing, we obtain $(\gamma_n^{(0)}, \ldots, \gamma_n^{(k)}) = (\delta_n^{(0)}, \ldots, \delta_n^{(k)})$, as desired. $\square$

Corollary 5.5.3. *The operation $\boxed{\star}^{(k)}$ is associative, commutative and the series $\Delta^{(k)}(z) = \Delta^{(A)}_{\mathcal{C}_k}(z)$ is its unit element. A series $f \in \Theta^{(k)}$ is invertible with respect to $\boxed{\star}^{(k)}$ if and only if its coefficient of degree one has a non-zero first component.*

Remark 5.5.4. Theorem 5.5.2 tells us that the operation $\boxed{\star}^{(k)}$ is a boxed convolution of type A, for which one may define a generalization to power series in several noncommuting indeterminates. This means that there exists an operation $\boxed{\star}^{(k)}$ on power series in several noncommuting indeterminates. We do not find interesting to record here the formulas involved in this operation.

Non-crossing partitions of type k are thus the combinatorial objects describing the version of the boxed convolution of type A with scalars in the algebra $\mathcal{C}_k$.

It is now easy to rewrite the main formulas involving infinitesimal non-crossing cumulants with sums indexed by the set of non-crossing partitions of type k. This is the content of the next proposition :

Proposition 5.5.5. *Let $(\mathcal{A}, (\varphi^{(i)})_{0\le i\le k})$ be an infinitesimal noncommutative probability space of order k. The infinitesimal non-crossing cumulant functionals satisfy, for every $n \ge 1$, every $0 \le i \le k$ and every $a_1, \dots, a_n \in \mathcal{A}$,*

$$\varphi^{(i)}(a_1 \cdots a_n) = \sum_{\pi \in NC_*^{(i)}(n)} \frac{C_i^{(\lambda_\pi)_1, \dots, (\lambda_\pi)_{|\pi|}}}{r(\lambda_\pi)} \kappa^{(\lambda_\pi)}_{Red_n^{(i)}(\pi)}(a_1, \dots, a_n).$$

Proposition 5.5.6. *Let $(\mathcal{A}, (\varphi^{(i)})_{0\le i\le k})$ be an infinitesimal noncommutative probability space of order k. Consider subsets $\mathcal{M}_1, \mathcal{M}_2$ of $\mathcal{A}$ that are infinitesimally free of order k. Then, one has, for each $n \ge 1$, each n-tuples $(a_1, \dots, a_n) \in \mathcal{M}_1^n, (b_1, \dots, b_n) \in \mathcal{M}_2^n$ and each $0 \le i \le k$:*

$$\kappa_n^{(i)}(a_1 b_1, \dots, a_n b_n) =$$

$$\sum_{\pi \in NC^{(i)}(n)} \frac{C_i^{(\lambda_\pi)_1, \dots, (\lambda_\pi)_{n+1}}}{r(\lambda_\pi)} \kappa^{(\lambda_\pi)}_{Red_n^{(i)}(\pi \cup Kr(\pi))}(a_1, b_1, \dots, a_n, b_n).$$

We move to the main application of infinitesimal freeness.

5.6 Application to derivatives of the free convolution

In this final section, we give an application of infinitesimal freeness of order k. We consider the situation already examined in [BS09] : let $\{a_u^v(t) \mid 1 \le v \le m_u\}_{t\in T}$ be s families of noncommutative random variables in a (usual) noncommutative probability space $(\mathcal{A}, \varphi)$. These families are indexed by a subset T of $\mathbb{R}$ having zero as an accumulation point, and we are interested in the joint distribution μ_t of $\{a_u^v(t) \mid 1 \le v \le m_u, 1 \le u \le s\}$ when t is going to 0, in other words for infinitesimal values of t. Recall that μ_t is the linear functional on $\mathbb{C}\langle X_u^v, 1 \le v \le m_u, 1 \le u \le s\rangle$ defined by :

$$\mu_t(P((X_u^v)_{1\le v\le m_u, 1\le u\le s})) = \varphi(P((a_u^v(t))_{1\le v\le m_u, 1\le u\le s})).$$

In what follows, we will consider a family $\{\mu_t\}_{t\in T}$ of linear functionals on $\mathbb{C}\langle X_u^v, 1 \le v \le m_u, 1 \le u \le s\rangle$ without any further reference to the variables $\{a_u^v(t) \mid 1 \le v \le m_u, 1 \le u \le s\}_{t\in T}$. For each value of $t \in T$, one may obviously define the non-crossing cumulant functionals $((\kappa_t)_n : (\mathbb{C}\langle X_u^v, 1 \le v \le$

$m_u, 1 \leq u \leq s\rangle)^n \to \mathbb{C})_{n=1}^{\infty}$ associated to the noncommutative probability space $(\mathbb{C}\langle X_u^v, 1 \leq v \leq m_u, 1 \leq u \leq s\rangle, \mu_t)$. A way to capture the behavior of μ_t for infinitesimal values of t is to introduce recursively its derivatives at 0 by :

$$\mu^{(0)} := \lim_{t\to 0} \mu_t, \tag{5.17}$$

$$\frac{\mu^{(i)}}{i!} := \lim_{t\to 0} \frac{1}{t^i}(\mu_t - \sum_{j=0}^{i-1} \frac{t^j}{j!}\mu^{(j)}), 1 \leq i \leq k. \tag{5.18}$$

We will assume that the limits in formulas (5.17) and (5.18) exist and use the notation $\mu^{(i)} = \frac{d^i}{dt^i}_{|t=0}\mu_t$. Notice that, in [BS09], only $\mu^{(0)}$ and $\mu^{(1)}$ were studied. It follows from formulas (5.17) and (5.18) that

$$\mu_t = \sum_{i=0}^{k} \frac{\mu^{(i)}}{i!} t^i + o(t^k).$$

Notice that $(\mu^{(i)})_{0\leq i\leq k}$ is an infinitesimal law (of order k) on $\sum_{u=1}^{s} m_u$ variables and therefore $(\mathbb{C}\langle X_u^v, 1 \leq v \leq m_u, 1 \leq u \leq s\rangle, (\mu^{(i)})_{0\leq i\leq k})$ is an infinitesimal noncommutative probability space of order k. Associated to this infinitesimal noncommutative probability space of order k, we have infinitesimal non-crossing cumulant functionals $(\kappa_n^{(i)} : \mathcal{A}^n \to \mathbb{C}, 0 \leq i \leq k)_{n=1}^{\infty}$, as defined by formula (5.7). These infinitesimal cumulant functionals are linked to $((\kappa_t)_n)_{n=1}^{\infty}$ as follows :

Proposition 5.6.1. *For every $n \geq 1$ and $0 \leq i \leq k$,*

$$\kappa_n^{(i)} = \frac{d^i}{dt^i}_{|t=0}(\kappa_t)_n.$$

Proof. By the inverse of the free moment-cumulant formula, one has

$$\forall t \in K, (\kappa_t)_n = \sum_{p\in NC^{(A)}(n)} \text{Möb}^{(A)}(p, 1_n)(\mu_t)_p. \tag{5.19}$$

By the assumption made above, the right-hand side of formula (5.19) has k derivatives at 0, hence $\frac{d^i}{dt^i}_{|t=0}(\kappa_t)_n$ is well-defined and, using linearity of derivation and Leibniz rule, one obtains :

$$\frac{d^i}{dt^i}_{|t=0}(\kappa_t)_n = \sum_{\substack{p\in NC^{(A)}(n)\\ p:=\{V_1,\ldots,V_h\}}} \sum_{\lambda\in\Lambda_{h,i}} \text{Möb}^{(A)}(p, 1_n) C_i^{\lambda_1,\ldots,\lambda_h} \mu_p^{(\lambda)}.$$

One recognizes in the right-hand side above the right-hand side of formula (5.9), and we are done. □

This proposition will be the main tool to characterize infinitesimal freeness of order k in terms of moments in Theorem 5.6.3. We first give a recipe to deduce the infinitesimal behaviour of the free convolution of two families of distributions from their individual infinitesimal behaviours.

Proposition 5.6.2. *Let $\{\mu_t\}_{t\in K}$ (resp. $\{\nu_t\}_{t\in K}$) be a family of linear functionals on $\mathbb{C}\langle X_u, 1 \leq u \leq m\rangle$ (resp $\mathbb{C}\langle Y_u, 1 \leq u \leq m\rangle$) such that $\mu^{(i)} = \frac{d^i}{dt^i}_{|t=0}\mu_t$ (resp. $\nu^{(i)} = \frac{d^i}{dt^i}_{|t=0}\nu_t$) exist for $0 \leq i \leq k$. Set :*

$$(\eta^{(i)})_{0\leq i\leq k} := (\mu^{(i)})_{0\leq i\leq k} \boxplus^{(k)} (\nu^{(i)})_{0\leq i\leq k},$$

$$(\theta^{(i)})_{0\leq i\leq k} := (\mu^{(i)})_{0\leq i\leq k} \boxtimes^{(k)} (\nu^{(i)})_{0\leq i\leq k}.$$

Then $\eta^{(i)} = \frac{d^i}{dt^i} |_{t=0} \mu_t \boxplus \nu_t$ and $\theta^{(i)} = \frac{d^i}{dt^i} |_{t=0} \mu_t \boxtimes \nu_t$.

Proof. For each $t \in K$, we consider the free product

$$(\mathbb{C}\langle X_u, Y_u, 1 \leq u \leq m\rangle, \mu_t \star \nu_t).$$

Since $\frac{d^i}{dt^i}_{|t=0}\mu_t$ and $\frac{d^i}{dt^i}_{|t=0}\nu_t$ exist by assumption for each $0 \leq i \leq k$, we obtain the existence of $\frac{d^i}{dt^i}_{|t=0}(\mu_t \star \nu_t)$ for each $0 \leq i \leq k$ and these functionals are completely determined by the $\mu^{(i)}$'s and the $\nu^{(i)}$'s. In the infinitesimal noncommutative probability space $(\mathbb{C}\langle X_u, Y_u, 1 \leq u \leq m\rangle, (\frac{d^i}{dt^i}_{|t=0}(\mu_t \star \nu_t))_{0\leq i\leq k})$, the unital subalgebras $\mathcal{A}_1 = \mathbb{C}\langle X_u, 1 \leq u \leq m\rangle$ and $\mathcal{A}_2 = \mathbb{C}\langle Y_u, 1 \leq u \leq m\rangle$ are infinitesimally free of order k : indeed, if $n \geq 1, 0 \leq i \leq k$ and $P_1 \in \mathcal{A}_{i_1}, \ldots, P_n \in \mathcal{A}_{i_n}$ are such that $i_1, \ldots, i_n$ are not all equal, then

$$\kappa_n^{(i)}(P_1, \ldots, P_l) = \frac{d^i}{dt^i}_{|t=0}(\kappa_t)_n(P_1, \ldots, P_l),$$

where $(\kappa_t)_n$ is the n-th non-crossing cumulant functional in the noncommutative probability space $(\mathbb{C}\langle X_u, Y_u, 1 \leq u \leq m\rangle, \mu_t \star \nu_t)$, by Proposition 5.6.1. But it follows from the construction of the free product that $(\kappa_t)_n(P_1, \ldots, P_l) = 0$ for each $t \in K$. In particular $\kappa_n^{(i)}(P_1, \ldots, P_l) = 0$. The infinitesimal distribution of the m-tuple $(X_1 + Y_1, \ldots, X_m + Y_m)$ (resp. $(X_1 \cdot Y_1, \ldots, X_m \cdot Y_m)$) is, on the one hand $(\frac{d^i}{dt^i}_{|t=0}(\mu_t \boxplus \nu_t))_{0\leq i\leq k}$ (resp. $(\frac{d^i}{dt^i}_{|t=0}(\mu_t \boxtimes \nu_t))_{0\leq i\leq k}$) by construction of the free product and, on the other hand, $(\eta^{(i)})_{0\leq i\leq k}$ (resp. $(\theta^{(i)})_{0\leq i\leq k}$) by the argument above. □

We conclude by a characterization of infinitesimal freeness of order k in terms of moments. Its formulation and proof rely on the Proposition 5.6.1.

Theorem 5.6.3. *Let $(\mathcal{A}, (\varphi^{(i)})_{0\leq i\leq k})$ be an infinitesimal noncommutative probability space of order k, and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be unital subalgebras of $\mathcal{A}$. Then*

$\mathcal{A}_1, \ldots, \mathcal{A}_n$ are infinitesimally free of order k if and only if for any positive integer $l \in \mathbb{N}^*$, and any $a_1 \in \mathcal{A}_{i_1}, \ldots, a_l \in \mathcal{A}_{i_l}$, one has

$$\varphi_t((a_1 - \varphi_t(a_1)) \cdots (a_l - \varphi_t(a_l))) = o(t^k), \tag{5.20}$$

whenever $i_1 \neq \ldots \neq i_l$, where $\varphi_t := \sum_{i=0}^{k} \frac{\varphi^{(i)}}{i!} t^i$. The condition (5.20) translates into $k+1$ requirements :

$$\forall i \in \{0, \ldots, k\}, \sum_{j=0}^{i} \sum_{\lambda \in \Lambda_{l,i-j}} (-1)^{\#\{m \geq 1, \lambda_m > 0\}} \mu^{(j)}(\hat{\mu}^{(\lambda_1)}(P_1) \cdots \hat{\mu}^{(\lambda_l)}(P_l)) = 0, \tag{5.21}$$

where $\hat{\mu}^{(\lambda)}(P) := P - \mu^{(0)}(P)$ if $\lambda = 0$, and $\hat{\mu}^{(\lambda)}(P) := \mu^{(\lambda)}(P)$ else.

Proof. We assume that condition (5.20) holds and prove that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ satisfy the vanishing of mixed infinitesimal cumulants condition. Using Proposition 5.6.1, it is equivalent to prove that for $l \geq 2$, and $a_1 \in \mathcal{A}_{i_1}, \ldots, a_l \in \mathcal{A}_{i_l}$

$$(\kappa_t)_l(a_1, \ldots, a_l) = o(t^k) \tag{5.22}$$

whenever $\exists r \neq s, i_r \neq i_s$, where $(\kappa_t)_l$ is the l-th non-crossing cumulant functional in $(\mathcal{A}, \varphi_t)$.
We proceed by induction on $l \geq 2$.
It is easy to see that

$$(\kappa_t)_2(a_1, a_2) = \varphi_t((a_1 - \varphi_t(a_1))((a_2 - \varphi_t(a_2)). \tag{5.23}$$

If $a_1 \in \mathcal{A}_{i_1}, a_2 \in \mathcal{A}_{i_2}$ with $i_1 \neq i_2$, the right-hand side of (5.23) is $o(t^k)$ by assumption. We assume then that the vanishing of mixed infinitesimal cumulants is proved for $2, 3, \ldots, l-1$ variables, and consider $(\kappa_t)_l(a_1, \ldots, a_l)$ with $a_1 \in \mathcal{A}_{i_1}, \ldots, a_l \in \mathcal{A}_{i_l}$ such that $\exists r \neq s, i_r \neq i_s$. By Propositions 5.2.4, 5.2.5 and the induction hypothesis, we may assume that $\varphi_t(a_1) = \ldots = \varphi_t(a_l) = 0$ and $i_1 \neq \ldots \neq i_l$. Write then the free moment-cumulant formula:

$$\forall t \in K, \varphi_t(a_1 \cdots a_l) - \sum_{\substack{p \in NC^{(A)}(l) \\ p \neq 1_l}} (\kappa_t)_p(a_1, \ldots, a_l) = (\kappa_t)_l(a_1, \ldots, a_l).$$

By assumption, $(\varphi_t)(a_1 \cdots a_l) = o(t^k)$. Any non-crossing partition $p \neq 1_l$ owns an interval-block V_0, as noticed in Section 3. If V_0 is a singleton,

$$\begin{aligned} (\kappa_t)_p(a_1, \ldots, a_l) &= (\varphi_t)_{|V_0|}((a_1, \ldots, a_l) \mid V_0) \prod_{V \neq V_0} (\kappa_t)_{|V|}((a_1, \ldots, a_l) \mid V) \\ &= 0. \end{aligned}$$

Otherwise, V_0 contains two following, hence distinct, indices, and, by induction hypothesis,

$$(\kappa_t)_{|V_0|}((a_1, \ldots, a_l) \mid V_0) = o(t^k).$$

Since, for each $V \in \mathrm{bl}(p)$, $(\kappa_t)_{|V|}((a_1, \ldots, a_l) \mid V)$ is bounded in a neighborhood of 0, one may affirm that

$$(\kappa_t)_p(a_1, \ldots, a_l) = o(t^k).$$

We conclude that

$$(\kappa_t)_l(a_1, \ldots, a_l) = o(t^k),$$

as required.

For the converse, we assume that the vanishing of mixed infinitesimal cumulants is satisfied, or equivalently that equation (5.22) holds. We write then the free moment-cumulant formula :

$$\forall t \in K, (\varphi_t)(a_1 - \varphi_t(a_1) \cdots a_l - \varphi_t(a_l)) = \tag{5.24}$$

$$\sum_{p \in NC^{(A)}(l)} (\kappa_t)_p(a_1 - \varphi_t(a_1), \ldots, a_l - \varphi_t(a_l)). \tag{5.25}$$

If $a_1 \in \mathcal{A}_{i_1}, \ldots, a_l \in \mathcal{A}_{i_l}$ with $i_1 \neq \ldots \neq i_n$, the same argument as above gives that (5.25) is $o(t^k)$. This concludes the proof. □

Chapter 6

Eigenvalues of spiked deformations of Wigner matrices

This chapter is the text of the paper "Free convolution with a semi-circular distribution and eigenvalues of spiked deformations of Wigner matrices", written in collaboration with M. Capitaine, C. Donati-Martin and Délphine Féral [CDFF10], and submitted for publication.

In this chapter, we consider the following general deformed Wigner model $M_N = \frac{1}{\sqrt{N}} W_N + A_N$ such that :

- W_N is a complex Wigner matrix of size N associated to a symmetric distribution μ of variance σ^2 satisfying a Poincaré inequality (see Definition 1.4.1).

- A_N is a deterministic Hermitian matrix whose eigenvalues $\gamma_i^{(N)}$, denoted for simplicity by γ_i, are such that the spectral measure $\mu_{A_N} := \frac{1}{N}\sum_{i=1}^N \delta_{\gamma_i}$ converges to some probability measure ν with compact support. We assume that there exists a fixed integer $r \geq 0$ (independent from N) such that A_N has $N - r$ eigenvalues $\beta_j(N)$ satisfying

$$\max_{1\leq j\leq N-r} \operatorname{dist}(\beta_j(N), \operatorname{supp}(\nu)) \underset{N\to\infty}{\to} 0,$$

where $\operatorname{supp}(\nu)$ denotes the support of ν. We also assume that there are J fixed real numbers $\theta_1 > \ldots > \theta_J$ independent of N which are outside the support of ν and such that each θ_j is an eigenvalue of A_N with a fixed multiplicity k_j (with $\sum_{j=1}^J k_j = r$). The θ_j's will be called the spikes or the spiked eigenvalues of A_N.

Throughout this chapter, we will use the following notations.

- G_N denotes the resolvent of M_N
- g_N denotes the mean of the Stieltjes transform of the spectral measure of M_N, that is
$$g_N(z) = \mathbb{E}(\mathrm{tr} G_N(z)),\ z \in \mathbb{C} \setminus \mathbb{R}$$
- μ_σ denotes the semicircle distribution of parameter σ defined by (1.1).
- $\tilde{g}_N$ denotes the Stieltjes transform of the probability measure $\mu_\sigma \boxplus \mu_{A_N}$.
- When we state that some quantity $\Delta_N(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, is $O(\frac{1}{N^p})$, this means precisely that:
$$|\Delta_N(z)| \leq (|z| + K)^a \frac{P(|\Im z|^{-1})}{N^p}$$
for some $K \geq 0$, $a > 0$ and some polynomial P with nonnegative coefficients.
- For any set S in $\mathbb{R}$, we denote the set $\{x \in \mathbb{R}, \mathrm{dist}(x, S) \leq \varepsilon\}$ (resp. $\{x \in \mathbb{R}, \mathrm{dist}(x, S) < \varepsilon\}$) by $S + [-\varepsilon, +\varepsilon]$ (resp. $S + (-\varepsilon, +\varepsilon)$).

We recall some useful properties of the resolvent (see [KKP96], [CDM07]).

Lemma 6.0.4. *For a $N \times N$ Hermitian or symmetric matrix M, for any $z \in \mathbb{C} \setminus \mathrm{Spect}(M)$, we denote by $G(z) := (zI_N - M)^{-1}$ the resolvent of M. Let $z \in \mathbb{C} \setminus \mathbb{R}$,*

(i) $\|G(z)\| \leq |\Im z|^{-1}$ *where $\|.\|$ denotes the operator norm.*

(ii) $|G(z)_{ij}| \leq |\Im z|^{-1}$ *for all $i, j = 1, \ldots, N$.*

(iii) For $p \geq 2$,
$$\frac{1}{N} \sum_{i,j=1}^{N} |G(z)_{ij}|^p \leq (|\Im z|^{-1})^p. \tag{6.1}$$

(iv) The derivative with respect to M of the resolvent $G(z)$ satisfies:
$$G'_M(z).B = G(z)BG(z) \quad \text{for any matrix } B.$$

(v) Let $z \in \mathbb{C}$ such that $|z| > \|M\|$; we have
$$\|G(z)\| \leq \frac{1}{|z| - \|M\|}.$$

6.1 Free convolution by a semicircular distribution

In [Bia97a], P. Biane provides a deep study of the free convolution by a semicircular distribution. We first recall here some of his results that will be useful in our approach.
Let ν be a probability measure on $\mathbb{R}$. P. Biane [Bia97a] introduces the set

$$\Omega_{\sigma,\nu} := \{u + iv \in \mathbb{C}^+, v > v_{\sigma,\nu}(u)\},$$

where the function $v_{\sigma,\nu} : \mathbb{R} \to \mathbb{R}^+$ is defined by

$$v_{\sigma,\nu}(u) = \inf \left\{ v \geq 0, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} \leq \frac{1}{\sigma^2} \right\}$$

and proves the following

Proposition 6.1.1. *[Bia97a] The map*

$$H_{\sigma,\nu} : z \longmapsto z + \sigma^2 G_\nu(z)$$

is a homeomorphism from $\overline{\Omega_{\sigma,\nu}}$ *to* $\mathbb{C}^+ \cup \mathbb{R}$ *which is conformal from* $\Omega_{\sigma,\nu}$ *onto* $\mathbb{C}^+$. *Let* $F_{\sigma,\nu} : \mathbb{C}^+ \cup \mathbb{R} \to \overline{\Omega_{\sigma,\nu}}$ *be the inverse function of* $H_{\sigma,\nu}$. *One has,*

$$\forall z \in \mathbb{C}^+, \; G_{\mu_\sigma \boxplus \nu}(z) = G_\nu(F_{\sigma,\nu}(z))$$

and then

$$F_{\sigma,\nu}(z) = z - \sigma^2 G_{\mu_\sigma \boxplus \nu}(z). \tag{6.2}$$

Note that in particular the Stieltjes transform $\tilde{g}_N$ of $\mu_\sigma \boxplus \mu_{A_N}$ satisfies

$$\forall z \in \mathbb{C}^+, \; \tilde{g}_N(z) = G_{\mu_{A_N}}(z - \sigma^2 \tilde{g}_N(z)). \tag{6.3}$$

Considering $H_{\sigma,\nu}$ as an analytic map defined in the whole upper half-plane $\mathbb{C}^+$, P. Biane identifies $\Omega_{\sigma,\nu}$ as the connected component of the set $H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$ which contains iy for large y. In fact, it is proved in [BB05] that $\Omega_{\sigma,\nu} = H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$, or equivalently, that $H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$ is connected. We give here another proof of this result in the particular case of the free convolution by a semicircular distribution:

Proposition 6.1.2.

$$\Omega_{\sigma,\nu} = H_{\sigma,\nu}^{-1}(\mathbb{C}^+).$$

Proof of Proposition 6.1.2: It is clear from the discussion above that $\Omega_{\sigma,\nu}$ is included in $H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$.
Fix $u + iv \in H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$; we have to prove that $v > v_{\sigma,\nu}(u)$.
Since $u + iv \in H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$, we have:

$$\Im H_{\sigma,\nu}(u + iv) = v(1 - \sigma^2 \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2}) > 0.$$

This, together with $v > 0$, implies

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} < \frac{1}{\sigma^2}$$

and consequently $v \geq v_{\sigma,\nu}(u)$. If we assume that $v = v_{\sigma,\nu}(u)$, then $v_{\sigma,\nu}(u) > 0$ and finally

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} = \frac{1}{\sigma^2}$$

by Lemma 2 in [Bia97a]. This is a contradiction : necessarily $v > v_{\sigma,\nu}(u)$ or, in other words, $u + iv \in \Omega_{\sigma,\nu}$ and we are done. □

The previous results of P. Biane allow him to conclude that $\mu_\sigma \boxplus \nu$ is absolutely continuous with respect to the Lebesgue measure and to obtain the following description of the support.

Theorem 6.1.3. *[Bia97a] Define* $\Psi_{\sigma,\nu} : \mathbb{R} \to \mathbb{R}$ *by:*

$$\Psi_{\sigma,\nu}(u) = H_{\sigma,\nu}(u + iv_{\sigma,\nu}(u)) = u + \sigma^2 \int_{\mathbb{R}} \frac{(u-x)d\nu(x)}{(u-x)^2 + v_\sigma(u)^2}.$$

$\Psi_{\sigma,\nu}$ *is a homeomorphism and, at the point* $\Psi_{\sigma,\nu}(u)$*, the measure* $\mu_\sigma \boxplus \nu$ *has a density given by*

$$p_{\sigma,\nu}(\Psi_{\sigma,\nu}(u)) = \frac{v_{\sigma,\nu}(u)}{\pi\sigma^2}.$$

Define the set

$$U_{\sigma,\nu} := \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\} = \{u \in \mathbb{R}, v_{\sigma,\nu}(u) > 0\}.$$

The support of the measure $\mu_\sigma \boxplus \nu$ *is the image of the closure of the open set* $U_{\sigma,\nu}$ *by the homeomorphism* $\Psi_{\sigma,\nu}$*.* $\Psi_{\sigma,\nu}$ *is strictly increasing on* $U_{\sigma,\nu}$*.*

Hence,

$$\mathbb{R} \setminus \operatorname{supp}(\mu_\sigma \boxplus \nu) = \Psi_{\sigma,\nu}(\mathbb{R} \setminus \overline{U_{\sigma,\nu}}).$$

One has $\Psi_{\sigma,\nu} = H_{\sigma,\nu}$ on $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$ and $\Psi_{\sigma,\nu}^{-1} = F_{\sigma,\nu}$ on $\mathbb{R} \setminus \operatorname{supp}(\mu_\sigma \boxplus \nu)$. In particular, we have the following description of the complement of the support:

$$\mathbb{R} \setminus \operatorname{supp}(\mu_\sigma \boxplus \nu) = H_{\sigma,\nu}(\mathbb{R} \setminus \overline{U_{\sigma,\nu}}). \tag{6.4}$$

Let ν be a compactly supported probability measure. We are going to establish a characterization of the complement of the support of $\mu_\sigma \boxplus \nu$ involving the support of ν and $H_{\sigma,\nu}$. We will need the following preliminary lemma.

Lemma 6.1.4. *The support of* ν *is included in* $\overline{U_{\sigma,\nu}}$*.*

Proof of Lemma 6.1.4: Let x_0 be in $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$. Then, there is some $\varepsilon > 0$ such that $[x_0 - \varepsilon, x_0 + \varepsilon] \subset \mathbb{R} \setminus \overline{U_{\sigma,\nu}}$. For any integer $n \geq 1$, we define $\alpha_k = x_0 - \varepsilon + 2k\varepsilon/n$ for all $0 \leq k \leq n$. Then, as the sets $[\alpha_k, \alpha_{k+1}]$ are trivially contained in $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$, one has that:

$$\forall u \in [\alpha_k, \alpha_{k+1}], \quad \frac{1}{\sigma^2} \geq \int_{\alpha_k}^{\alpha_{k+1}} \frac{d\nu(x)}{(u-x)^2} \geq \frac{\nu([\alpha_k, \alpha_{k+1}])}{(\alpha_{k+1} - \alpha_k)^2}.$$

This readily implies that

$$\nu([x_0 - \varepsilon, x_0 + \varepsilon]) \leq \sum_{k=0}^{n-1} \nu([\alpha_k, \alpha_{k+1}]) \leq \frac{(2\varepsilon)^2}{\sigma^2 n}.$$

Letting $n \to \infty$, we get that $\nu([x_0 - \varepsilon, x_0 + \varepsilon]) = 0$, which implies that $x_0 \in \mathbb{R} \setminus \operatorname{supp}(\nu)$. $\square$

From the continuity and strict convexity of the function $u \mapsto \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2}$ on $\mathbb{R} \setminus \operatorname{supp}(\nu)$, it follows that

$$\overline{U_{\sigma,\nu}} = \operatorname{supp}(\nu) \cup \{u \in \mathbb{R} \setminus \operatorname{supp}(\nu), \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} \geq \frac{1}{\sigma^2}\} \tag{6.5}$$

and

$$\mathbb{R} \setminus \overline{U_{\sigma,\nu}} = \{u \in \mathbb{R} \setminus \operatorname{supp}(\nu), \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} < \frac{1}{\sigma^2}\}.$$

Now, as $H_{\sigma,\nu}$ is analytic on $\mathbb{R} \setminus \operatorname{supp}(\nu)$, the following characterization readily follows:

$$\mathbb{R} \setminus \overline{U_{\sigma,\nu}} = \{u \in \mathbb{R} \setminus \operatorname{supp}(\nu), H'_{\sigma,\nu}(u) > 0\}.$$

and thus, according to (6.4), we get

Proposition 6.1.5.

$$x \in \mathbb{R} \setminus \operatorname{supp}(\mu_\sigma \boxplus \nu) \Leftrightarrow \exists u \in \mathbb{R} \setminus \operatorname{supp}(\nu) \text{ such that } x = H_{\sigma,\nu}(u),\ H'_{\sigma,\nu}(u) > 0.$$

Remark 6.1.6. Note that $H_{\sigma,\nu}$ is strictly increasing on $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$ since, if $a < b$ are in $\mathbb{R} \setminus \operatorname{supp}(\nu)$, one has, by Cauchy-Schwarz inequality, that

$$\begin{aligned} H_{\sigma,\nu}(b) - H_{\sigma,\nu}(a) &= (b-a)\left[1 - \sigma^2 \int_{\mathbb{R}} \frac{d\nu(x)}{(a-x)(b-x)}\right] \\ &\geq (b-a)\left[1 - \sigma^2 \sqrt{(-g'_\nu(a))(-g'_\nu(b))}\right]. \end{aligned}$$

which is nonnegative if a and b belong to $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$.

Remark 6.1.7. Each connected component of $\overline{U_{\sigma,\nu}}$ contains at least one connected component of $\operatorname{supp}(\nu)$.

Indeed, let $[s_l, t_l]$ be a connected component of $\overline{U_{\sigma,\nu}}$. If s_l or t_l is in $\operatorname{supp}(\nu)$, $[s_l, t_l]$ contains at least a connected component of $\operatorname{supp}(\nu)$ since $\operatorname{supp}(\nu)$ is included in $\overline{U_{\sigma,\nu}}$. Now, if neither s_l nor t_l is in $\operatorname{supp}(\nu)$, according to (6.5), we have

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(s_l - x)^2} = \int_{\mathbb{R}} \frac{d\nu(x)}{(t_l - x)^2} = \frac{1}{\sigma^2}.$$

Assume that $[s_l, t_l] \subset \mathbb{R} \setminus \operatorname{supp}(\nu)$, then, by strict convexity of the function $u \longmapsto \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2}$ on $\mathbb{R} \setminus \operatorname{supp}(\nu)$, one obtains that, for any $u \in]s_l, t_l[$,

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(u - x)^2} < \frac{1}{\sigma^2},$$

which leads to a contradiction. □

Remark 6.1.8. One can readily see that

$$\overline{U_{\sigma,\nu}} \subset \{u, \operatorname{dist}(u, \operatorname{supp}(\nu)) \leq \sigma\}$$

and deduce, since $\operatorname{supp}(\nu)$ is compact, that $U_{\sigma,\nu}$ is a relatively compact open set. Hence, $\overline{U_{\sigma,\nu}}$ has a finite number of connected components and may be written as the following finite disjoint union

$$\overline{U_{\sigma,\nu}} = \bigcup_{l=m}^{1} [s_l, t_l] \quad \text{with } s_m < t_m < \ldots < s_1 < t_1. \tag{6.6}$$

We close this section with a proposition pointing out a relationship between the distribution functions of ν and $\mu_\sigma \boxplus \nu$.

Proposition 6.1.9. *Let $[s_l, t_l]$ be a connected component of $\overline{U_{\sigma,\nu}}$, then*

$$(\mu_\sigma \boxplus \nu)([\Psi_{\sigma,\nu}(s_l), \Psi_{\sigma,\nu}(t_l)]) = \nu([s_l, t_l]).$$

Proof of Proposition 6.1.9: Let $]a, b[$ be a connected component of $U_{\sigma,\nu}$. Since a and b are not atoms of ν and $\mu_\sigma \boxplus \nu$ is absolutely continuous, it is enough to show

$$(\mu_\sigma \boxplus \nu)([\Psi_{\sigma,\nu}(a), \Psi_{\sigma,\nu}(b)]) = \nu([a, b]).$$

From Cauchy's inversion formula, $\mu_\sigma \boxplus \nu$ has a density given by $p_\sigma(x) = -\frac{1}{\pi}\Im(G_\nu(F_{\nu,\sigma}(x))$ and

$$(\mu_\sigma \boxplus \nu)([\Psi_{\sigma,\nu}(a), \Psi_{\sigma,\nu}(b)]) = -\frac{1}{\pi}\Im\left(\int_{\Psi_{\sigma,\nu}(a)}^{\Psi_{\sigma,\nu}(b)} G_\nu(F_{\nu,\sigma}(x))dx\right).$$

We set $z = F_{\sigma,\nu}(x)$, then $x = H_{\sigma,\nu}(z)$ and $z = u + iv_{\sigma,\nu}(u)$. Note that $v_{\sigma,\nu}(u) > 0$ for $u \in]a, b[$ and $v_{\sigma,\nu}(a) = v_{\sigma,\nu}(b) = 0$ (see [Bia97a]). Then,

$$\begin{aligned}
&(\mu_\sigma \boxplus \nu)([\Psi_{\sigma,\nu}(a), \Psi_{\sigma,\nu}(b)]) \\
&= -\frac{1}{\pi}\Im\left(\int_a^b G_\nu(u + iv_{\sigma,\nu}(u))H'_{\sigma,\nu}(u + iv_{\sigma,\nu}(u))(1 + iv'_{\sigma,\nu}(u))du\right) \\
&= -\frac{1}{\pi}\Im\left(\int_a^b G_\nu(u + iv_{\sigma,\nu}(u))(1 + \sigma^2 G'_\nu(u + iv_{\sigma,\nu}(u)))(1 + iv'_{\sigma,\nu}(u))du\right) \\
&= -\frac{1}{\pi}\Im\left(\int_a^b G_\nu(u + iv_{\sigma,\nu}(u))(1 + iv'_{\sigma,\nu}(u))du + \frac{\sigma^2}{2}[G_\nu^2(u + iv_{\sigma,\nu}(u))]_a^b\right) \\
&= -\frac{1}{\pi}\Im\int_a^b G_\nu(u + iv_{\sigma,\nu}(u))(1 + iv'_{\sigma,\nu}(u))du = -\frac{1}{\pi}\Im\int_\gamma G_\nu(z)dz,
\end{aligned}$$

where

$$\gamma = \{z = u + iv_{\sigma,\nu}(u), u \in [a, b]\}.$$

Now, we recall that, since a and b are points of continuity of the distribution function of ν,

$$\nu([a, b]) = \lim_{\varepsilon\to 0} -\frac{1}{\pi}\Im\left(\int_a^b G_\nu(u + i\varepsilon)du\right) = \lim_{\varepsilon\to 0} -\frac{1}{\pi}\Im\left(\int_{\gamma_\varepsilon} G_\nu(z)dz\right),$$

where $\gamma_\varepsilon = \{z = u + i\varepsilon, u \in [a, b]\}$. Thus, it remains to prove that:

$$\lim_{\varepsilon\to 0}\left(\Im\left(\int_\gamma G_\nu(z)dz\right) - \Im\left(\int_{\gamma_\varepsilon} G_\nu(z)dz\right)\right) = 0. \tag{6.7}$$

Let $\varepsilon > 0$ such that $\varepsilon < \sup_{[a,b]} v_{\sigma,\nu}(u)$. We introduce the contour

$$\hat{\gamma}_\varepsilon = \{z = u + i(v_{\sigma,\nu}(u) \wedge \varepsilon), u \in [a, b]\}.$$

From the analyticity of G_ν on $\mathbb{C}^+$, we have

$$\int_\gamma G_\nu(z)dz = \int_{\hat{\gamma}_\varepsilon} G_\nu(z)dz.$$

Let $I_\varepsilon = \{u \in [a, b], v_{\sigma,\nu}(u) < \varepsilon\} = \cup C_i(\varepsilon)$, where $C_i(\varepsilon)$ are the connected components of I_ε. Then, $I_\varepsilon \downarrow_{\varepsilon\to 0} \{a, b\}$. For $u \in I_\varepsilon$,

$$|\Im G_\nu(u + i\varepsilon)| = \varepsilon\int\frac{d\nu(x)}{(u - x)^2 + \varepsilon^2} \leq \varepsilon\int\frac{d\nu(x)}{(u - x)^2 + v_{\sigma,\nu}^2(u)} \leq \frac{\varepsilon}{\sigma^2}$$

and

$$\int_{I_\varepsilon} |\Im G_\nu(u + i\varepsilon)|du \leq \frac{\varepsilon}{\sigma^2}(b - a).$$

On the other hand, for $u \in I_\varepsilon$,

$$|\Im G_\nu(u+iv_{\sigma,\nu}(u))| = v_{\sigma,\nu}(u)\int \frac{d\nu(x)}{(u-x)^2+v_{\sigma,\nu}(u)^2} \leq \frac{\varepsilon}{\sigma^2}.$$

Moreover,

$$\Re G_\nu(u+iv_{\sigma,\nu}(u))v'_{\sigma,\nu}(u) = \frac{\Psi_{\sigma,\nu}(u)-u}{\sigma^2}v'_{\sigma,\nu}(u)$$

and

$$\begin{aligned}\int_{I_\varepsilon}\Re G_\nu(u+iv_{\sigma,\nu}(u))v'_{\sigma,\nu}(u)du &= \int_{I_\varepsilon}\frac{\Psi_{\sigma,\nu}(u)-u}{\sigma^2}v'_{\sigma,\nu}(u)du\\ &= \frac{1}{\sigma^2}\sum_i[(\Psi_{\sigma,\nu}(u)-u)v_{\sigma,\nu}(u)]_{C_i(\varepsilon)}\\ &\quad -\frac{1}{\sigma^2}\int_{I_\varepsilon}(\Psi'_{\sigma,\nu}(u)-1)v_{\sigma,\nu}(u)du,\end{aligned}$$

by integration by parts. Now (see [Bia97a] or Theorem 6.1.3),

$$\int_{I_\varepsilon}\Psi'_{\sigma,\nu}(u)v_{\sigma,\nu}(u)du = \pi\sigma^2(\mu_\sigma\boxplus\nu)(\Psi_{\sigma,\nu}(I_\varepsilon)) \underset{\varepsilon\to 0}{\to} 0.$$

$$\int_{I_\varepsilon}v_{\sigma,\nu}(u)du \leq \varepsilon(b-a).$$

Since $\Psi_{\sigma,\nu}$ is increasing on $[a,b]$,

$$\sum_i[\Psi_{\sigma,\nu}(u)v_{\sigma,\nu}(u)]_{C_i(\varepsilon)} \leq \varepsilon(\Psi_{\sigma,\nu}(b)-\Psi_{\sigma,\nu}(a))$$

and

$$\sum_i[uv_{\sigma,\nu}(u)]_{C_i(\varepsilon)} \leq \varepsilon(b-a).$$

The above inequalities imply (6.7). □

6.2 Approximate subordination equation for g_N

We look for an approximative equation for $g_N(z)$ of the form (6.3). To estimate $g_N(z)$, we first handle the simplest case where W_N is a GUE matrix and then see how the equation is modified in the general Wigner case. We shall rely on an integration by parts formula. The first integration by parts formula concerns the Gaussian case; the distribution μ associated to W_N is a centered Gaussian distribution with variance σ^2 and the resulting distribution of $X_N = W_N/\sqrt{N}$ is denoted by $\mathrm{GUE}(N,\sigma^2/N)$. Then, the integration by parts formula can be expressed in a matricial form.

Lemma 6.2.1. *Let Φ be a complex-valued $\mathcal{C}^1$ function on $(M_N(\mathbb{C})_{sa})$ and $X_N \sim GUE(N, \frac{\sigma^2}{N})$. Then,*

$$\mathbb{E}[\Phi'(X_N).H] = \frac{N}{\sigma^2}\mathbb{E}[\Phi(X_N)\mathrm{Tr}(X_N H)], \tag{6.8}$$

for any Hermitian matrix H, or by linearity for $H = E_{jk}$, $1 \leq j,k \leq N$, where $(E_{jk})_{1\leq j,k\leq N}$ is the canonical basis of the complex space of $N \times N$ matrices.

For a general distribution μ, we shall use an "approximative" integration by parts formula, applied to the variable $\xi = \sqrt{2}\Re((X_N)_{kl})$ or $\sqrt{2}\Im((X_N)_{kl})$, $k < l$, or $(X_N)_{kk}$. Note that for $k < l$ the derivative of $\Phi(X_N)$ with respect to $\sqrt{2}\Re((X_N)_{kl})$ (resp. $\sqrt{2}\Im((X_N)_{kl})$) is $\Phi'(X_N).e_{kl}$ (resp. $\Phi'(X_N).f_{kl}$), where $e_{kl} = \frac{1}{\sqrt{2}}(E_{kl}+E_{lk})$ (resp. $f_{kl} = \frac{i}{\sqrt{2}}(E_{kl}-E_{lk})$) and for any k, the derivative of $\Phi(X_N)$ with respect to $(X_N)_{kk}$ is $\Phi'(X_N).E_{kk}$.

Lemma 6.2.2. *Let ξ be a real-valued random variable such that $\mathbb{E}(|\xi|^{p+2}) < \infty$. Let ϕ be a function from $\mathbb{R}$ to $\mathbb{C}$ such that the first $p+1$ derivatives are continuous and bounded. Then,*

$$\mathbb{E}(\xi\phi(\xi)) = \sum_{a=0}^{p} \frac{\kappa_{a+1}}{a!}\mathbb{E}(\phi^{(a)}(\xi)) + \varepsilon, \tag{6.9}$$

where κ_a are the cumulants of ξ, $|\varepsilon| \leq C \sup_t |\phi^{(p+1)}(t)|\mathbb{E}(|\xi|^{p+2})$, C only depends on p.

Let U be a unitary matrix such that

$$A_N = U^*\mathrm{diag}(\gamma_1, \ldots, \gamma_N)U$$

and let G stand for $G_N(z)$. Consider $\tilde{G} = UGU^*$. We describe the approach in the Gaussian case and present the corresponding results in the general Wigner case but detail some technical proofs in the Appendix.

a) Gaussian case: We apply (6.8) to $\Phi(X_N) = G_{jl}$, $H = E_{il}$, $1 \leq i,j,l \leq N$, and then take $\frac{1}{N}\sum_l$ to obtain, using the resolvent equation $GX_N = -I + zG - GA_N$ (see [CDMF09]),

$$Z_{ji} := \sigma^2\mathbb{E}[G_{ji}\mathrm{tr}(G)] + \delta_{ij} - z\mathbb{E}(G_{ji}) + \mathbb{E}[(GA_N)_{ji}] = 0.$$

Now, let $1 \leq k,p \leq N$ and consider the sum $\sum_{i,j} U^*_{ik}U_{pj}Z_{ji}$. We obtain from the previous equation

$$\sigma^2\mathbb{E}[\tilde{G}_{pk}\mathrm{tr}(G)] + \delta_{pk} - z\mathbb{E}(\tilde{G}_{pk}) + \gamma_k\mathbb{E}[\tilde{G}_{pk}] = 0. \tag{6.10}$$

Hence, using Lemma 6.8.2 in the Appendix stating that

$$|\mathbb{E}[\tilde{G}_{pk}\mathrm{tr}(G)] - \mathbb{E}[\tilde{G}_{pk}]\mathbb{E}[\mathrm{tr}(G)]| = O(\frac{1}{N^2}),$$

we finally get the following estimation

$$\mathbb{E}(\tilde{G}_{pk}) = \frac{\delta_{pk}}{(z-\sigma^2 g_N(z)-\gamma_k)} + O(\frac{1}{N^2}), \tag{6.11}$$

where we use that $|\frac{1}{z-\sigma^2 g_N(z)-\gamma_i}| \leq |\Im z|^{-1}$, and then

$$\begin{aligned} g_N(z) &= \frac{1}{N}\sum_{k=1}^N \mathbb{E}[\tilde{G}_{kk}] = \frac{1}{N}\sum_{i=1}^N \frac{1}{z-\sigma^2 g_N(z)-\gamma_k} + O(\frac{1}{N^2}) \\ &= \int_{\mathbb{R}} \frac{1}{z-\sigma^2 g_N(z)-x} d\mu_{A_N}(x) + O(\frac{1}{N^2}) \\ &= G_{\mu_{A_N}}(z-\sigma^2 g_N(z)) + O(\frac{1}{N^2}). \end{aligned}$$

In the Gaussian case, we have thus proved:

Proposition 6.2.3. *For $z \in \mathbb{C}^+$, $g_N(z)$ satisfies:*

$$g_N(z) = G_{\mu_{A_N}}(z-\sigma^2 g_N(z)) + \frac{P(|\Im z|^{-1})}{N^2}. \tag{6.12}$$

b) Non-Gaussian case: In this case, the integration by parts formula gives the following generalization of (6.11):

Lemma 6.2.4.

$$\mathbb{E}(\tilde{G}_{pk}) = \frac{\delta_{pk}}{(z-\sigma^2 g_N(z)-\gamma_k)} + \frac{\kappa_4}{2N^2}\frac{\mathbb{E}[\tilde{A}(p,k)]}{(z-\sigma^2 g_N(z)-\gamma_k)} + O(\frac{1}{N^2}), \tag{6.13}$$

where

$$\begin{aligned} \tilde{A}(p,k) &= \sum_{i,j} U^*_{ik} U_{pj} \Big\{ \sum_l G_{jl} G^3_{il} + \sum_l G_{ji} G_{il} G_{li} G_{ll} \\ &\quad + \sum_l G_{jl} G_{ii} G_{li} G_{ll} + \sum_l G_{ji} G_{ii} G^2_{ll} \Big\} \end{aligned} \tag{6.14}$$

and $\frac{1}{N^2}\tilde{A}(p,k) \leq C\frac{|\Im z|^{-4}}{N}$.

Proof Lemma 6.2.4 readily follows from (6.72), Lemma 6.8.2 and (6.71) established in the Appendix. □

Thus,

$$\begin{aligned} g_N(z) &= \frac{1}{N}\sum_{k=1}^N \mathbb{E}[\tilde{G}_{kk}] = \frac{1}{N}\sum_{k=1}^N \frac{1}{z-\sigma^2 g_N(z)-\gamma_k} \\ &+ \frac{\kappa_4}{2N^3}\sum_{k=1}^N \frac{\mathbb{E}[\tilde{A}(k,k)]}{z-\sigma^2 g_N(z)-\gamma_k} + O(\frac{1}{N^2}). \end{aligned}$$

Let us show that the first three terms in $\frac{1}{N}\sum_k \mathbb{E}[\tilde{A}(k,k)]/(z-\sigma^2 g_N(z)-\gamma_k)$ coming from the decomposition (6.14) are bounded and thus give a $O(\frac{1}{N^2})$ contribution in $g_N(z)$. We denote by G_D the diagonal matrix with k-th diagonal entry equal to $\frac{1}{z-\sigma^2 g_N(z)-\gamma_k}$.

$$\begin{aligned}\left|\sum_{i,j,k} U^*_{ik}U_{kj}\frac{1}{z-\sigma^2 g_N(z)-\gamma_k}\mathbb{E}[\sum_l G_{jl}G^3_{il}]\right| &= \left|\mathbb{E}[\sum_{i,l}(U^*G_DUG)_{il}G^3_{il}]\right| \\ &\leq |\Im z|^{-2}\mathbb{E}[\sum_{i,l}|G^3_{il}|] \\ &\leq |\Im z|^{-5}N,\end{aligned}$$

using Lemma 6.0.4. The second term is of the same kind. For the third term, we obtain

$$|\sum_i (U^*G_DUG^2G^{(d)})_{ii}G_{ii}| \leq |\Im z|^{-5}N$$

where $G^{(d)}$ is the diagonal matrix with l-th diagonal entry equal to G_{ll}. It follows that

$$g_N(z) = G_{\mu_{A_N}}(z-\sigma^2 g_N(z)) + \frac{1}{N}\hat{L}_N(z) + O(\frac{1}{N^2}),$$

where

$$\hat{L}_N(z) = \frac{\kappa_4}{2N^2}\sum_{i,j,k,l} U^*_{ik}U_{kj}\frac{1}{z-\sigma^2 g_N(z)-\gamma_k}\mathbb{E}[G_{ji}G_{ii}G^2_{ll}]. \tag{6.15}$$

It is easy to see that $\hat{L}_N(z)$ is bounded by $C|\Im z|^{-5}$.

Proposition 6.2.5. *$\hat{L}_N$ defined by* (6.15) *can be written as*

$$\hat{L}_N(z) = L_N(z) + O(\frac{1}{N}), \text{ where } L_N(z) =$$

$$\frac{\kappa_4}{2N^2}\sum_{i,l}[(G_{A_N}(z-\sigma^2 g_N(z)))^2]_{ii}[G_{A_N}(z-\sigma^2 g_N(z))]_{ii}([G_{A_N}(z-\sigma^2 g_N(z))]_{ll})^2. \tag{6.16}$$

Proof of Proposition 6.2.5:
Step 1: We first show that for $1 \leq a, b \leq N$,

$$\mathbb{E}[G_{ab}] = [G_{A_N}(z-\sigma^2 g_N(z))]_{ab} + O(\frac{1}{N}). \tag{6.17}$$

From Lemma 6.2.4, for any $1 \leq p, k \leq N$,

$$\mathbb{E}[\tilde{G}_{pk}] = \frac{\delta_{pk}}{(z-\sigma^2 g_N(z)-\gamma_k)} + \frac{\kappa_4}{2N^2}\frac{\mathbb{E}[\tilde{A}(p,k)]}{(z-\sigma^2 g_N(z)-\gamma_k)} + O(\frac{1}{N^2}).$$

Let $1 \leq a, b \leq N$,

$$\begin{aligned}
\mathbb{E}[G_{ab}] &= \sum_{p,k} U^*_{ap}\mathbb{E}[\tilde{G}_{pk}]U_{kb} \\
&= \sum_{k} U^*_{ak}\frac{1}{(z-\sigma^2 g_N(z)-\gamma_k)}U_{kb} \\
&+ \frac{\kappa_4}{2N^2}\sum_{p,k} U^*_{ap}\frac{\mathbb{E}[\tilde{A}(p,k)]}{(z-\sigma^2 g_N(z)-\gamma_k)}U_{kb} \\
&+ O(\frac{1}{N}),
\end{aligned}$$

since $\sum_{p,k}|U^*_{ap}U_{kb}| \leq N$. The first term in the right-hand side of the above equation is equal to $[G_{A_N}(z-\sigma^2 g_N(z))]_{ab}$. It remains to show that the term involving $\mathbb{E}[\tilde{A}(p,k)]$ is of order $\frac{1}{N}$. Let us consider the "worst term" in the decomposition (6.14) of $\tilde{A}(p,k)$, namely the last one.

$$\begin{aligned}
\frac{1}{2N^2}\sum_{p,k,i,j,l} & U^*_{ap}\frac{1}{(z-\sigma^2 g_N(z)-\gamma_k)}U_{kb}U^*_{ik}U_{pj}\mathbb{E}[G_{ji}G_{ii}G^2_{ll}] \\
&= \frac{1}{2N^2}\mathbb{E}[\sum_{k,i,l}\frac{1}{(z-\sigma^2 g_N(z)-\gamma_k)}U_{kb}U^*_{ik}G_{ai}G_{ii}G^2_{ll}] \\
&= \frac{1}{2N^2}\mathbb{E}[\sum_{i,l}(U^*G_DU)_{ib}G_{ai}G_{ii}G^2_{ll}] \\
&= \frac{1}{2N^2}\mathbb{E}[\sum_{l}(GG^{(d)}U^*G_DU)_{ab}G^2_{ll}] \leq \frac{1}{2N}|\Im z|^{-5}.
\end{aligned}$$

Step 2: $\hat{L}_N$ defined by (6.15) can be written as

$$\frac{\kappa_4}{2N^2}\sum_{i,l}\mathbb{E}[(U^*G_DUG)_{ii}G_{ii}G^2_{ll}].$$

First notice the following bound (see Appendix)

$$\mathbb{E}[(U^*G_DUG)_{ii}G_{ii}G^2_{ll}] - \mathbb{E}[(U^*G_DUG)_{ii}]\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2 = O(\frac{1}{N}). \quad (6.18)$$

Thus,

$$\hat{L}_N(z) = \frac{\kappa_4}{2N^2}\sum_{i,l}\mathbb{E}[(U^*G_DUG)_{ii}]\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2 + O(\frac{1}{N}).$$

Now, note that $\mathbb{E}[(U^*G_DUG)_{ii}] = \mathbb{E}[(U^*G_D\tilde{G}U)_{ii}]$ and, according to Lemma

6.2.4,

$$\begin{aligned}\mathbb{E}[(U^*G_D\tilde{G}U)_{ii}] &= \sum_{p,k}(U^*G_D)_{ip}\mathbb{E}[\tilde{G}_{pk}]U_{ki}\\ &= (U^*G_D^2U)_{ii} + \frac{\kappa_4}{2N^2}\sum_{p,k}(U^*G_D)_{ip}\mathbb{E}[\tilde{A}(p,k)](G_DU)_{ki}\\ &\quad + \sum_{p,k}(U^*G_D)_{ip}O_{pk}(\frac{1}{N^2})U_{ki}.\end{aligned}$$

Thus
$\frac{\kappa_4}{2N^2}\sum_{i,l}\mathbb{E}[(U^*G_DUG)_{ii}]\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2$

$$= \frac{\kappa_4}{2N^2}\sum_{i,l}[(G_{A_N}(z-\sigma^2 g_N(z)))^2]_{ii}\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2 \tag{6.19}$$

$$+\frac{\kappa_4^2}{4N^4}\sum_{i,l,p,k}(U^*G_D)_{ip}\mathbb{E}[\tilde{A}(p,k)](G_DU)_{ki}\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2 \tag{6.20}$$

$$+\frac{1}{N^2}\sum_{i,l,p,k}(U^*G_D)_{ip}O_{pk}(\frac{1}{N^2})U_{ki}\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2. \tag{6.21}$$

The last term (6.21) can be rewritten as

$$\frac{1}{N^2}\sum_{l,p,k}(U\mathbb{E}[G^{(d)}]U^*G_D)_{kp}O_{pk}(\frac{1}{N^2})\mathbb{E}[G_{ll}]^2,$$

so that one can easily see that it is a $O(\frac{1}{N})$.
The second term (6.20) can be rewritten as
$\frac{\kappa_4^2}{4N^4}\sum_{t,l,s}\mathbb{E}[G_{ll}]^2$

$$\times\Big\{[U^*G_DU\mathbb{E}[G^{(d)}]U^*G_DUG]_{ts}[G_{ts}^3+G_{tt}G_{st}G_{ss}]$$

$$+[U^*G_DU\mathbb{E}[G^{(d)}]U^*G_DUG]_{tt}[G_{ts}G_{st}G_{ss}+G_{tt}G_{ss}^2]\Big\},$$

which is obviously a $O(\frac{1}{N})$.
Hence, Proposition 6.2.5 follows by rewriting the first term (6.19) using (6.17). □

From the above computations, we can state the following :

Proposition 6.2.6. *For $z\in\mathbb{C}^+$, $g_N(z)$ satisfies:*

$$g_N(z) = G_{\mu_{A_N}}(z-\sigma^2 g_N(z)) + \frac{1}{N}L_N(z) + \frac{P(|\Im z|^{-1})}{N^2} \tag{6.22}$$

where $L_N(z)$ is given by (6.16).

6.3 Estimation of $g_N - \tilde{g}_N$

Proposition 6.3.1. *For $z \in \mathbb{C}^+$,*

$$g_N(z) - \tilde{g}_N(z) + \frac{\tilde{E}_N(z)}{N} = O(\frac{1}{N^2}), \tag{6.23}$$

where $\tilde{E}_N(z)$ is given by

$$\tilde{E}_N(z) = \{\sigma^2 \tilde{g}'_N(z) - 1\}\tilde{L}_N(z) \tag{6.24}$$

with $\tilde{L}_N(z) =$

$$\frac{\kappa_4}{2N^2}\sum_{i,l}[(G_{A_N}(z-\sigma^2\tilde{g}_N(z)))^2]_{ii}[G_{A_N}(z-\sigma^2\tilde{g}_N(z))]_{ii}([G_{A_N}(z-\sigma^2\tilde{g}_N(z))]_{ll})^2. \tag{6.25}$$

Proof of proposition 6.3.1: First, we are going to prove that for $z \in \mathbb{C}^+$,

$$g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N} = O(\frac{1}{N^2}), \tag{6.26}$$

where $E_N(z)$ is given by

$$E_N(z) = \{\sigma^2 \tilde{g}'_N(z) - 1\}L_N(z). \tag{6.27}$$

For a fixed $z \in \mathbb{C}^+$, one may write the subordination equation (6.3) :

$$\tilde{g}_N(z) = G_{\mu_{A_N}}(F_{\sigma,\mu_{A_N}}(z)) = G_{\mu_{A_N}}(z - \sigma^2\tilde{g}_N(z)),$$

and the approximative matricial subordination equation (6.22) :

$$g_N(z) = G_{\mu_{A_N}}(z - \sigma^2 g_N(z)) + \frac{1}{N}L_N(z) + \frac{P(|\Im z|^{-1})}{N^2}.$$

The main idea is to simplify the difference $g_N(z) - \tilde{g}_N(z)$ by introducing a complex number z' likely to satisfy

$$F_{\sigma,\mu_{A_N}}(z') = z - \sigma^2 g_N(z). \tag{6.28}$$

We know by Proposition 6.1.1 that $F_{\sigma,\mu_{A_N}}$ is a homeomorphism from $\mathbb{C}^+$ to $\Omega_{\sigma,\mu_{A_N}}$ whose inverse $H_{\sigma,\mu_{A_N}}$ has an analytic continuation to the whole upper half-plane $\mathbb{C}^+$. Since $z - \sigma^2 g_N(z) \in \mathbb{C}^+$, $z' \in \mathbb{C}$ is well-defined by the formula :

$$z' := H_{\sigma,\mu_{A_N}}(z - \sigma^2 g_N(z)).$$

One has

$$\begin{aligned} z' - z &= -\sigma^2(g_N(z) - G_{\mu_{A_N}}(z - \sigma^2 g_N(z))) \\ &= -\sigma^2\frac{L_N(z)}{N} + O(\frac{1}{N^2}) \\ &= O(\frac{1}{N}) \end{aligned}$$

There exists thus a polynomial P with nonnegative coefficients such that

$$|z' - z| \leq \frac{P(|\Im z|^{-1})}{N}.$$

On the one hand, if

$$\frac{P(|\Im z|^{-1})}{N} \geq \frac{|\Im z|}{2},$$

or equivalently

$$1 \leq \frac{2|\Im z|^{-1} P(|\Im z|^{-1})}{N}, \tag{6.29}$$

it is enough to prove that

$$g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N} = O(1). \tag{6.30}$$

Indeed, if we assume that (6.29) and (6.30) hold, then there exists a polynomial Q with nonnegative coefficients such that

$$\begin{aligned} |g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N}| &\leq Q(|\Im z|^{-1}) \\ &\leq Q(|\Im z|^{-1}) \frac{2|\Im z|^{-1} P(|\Im z|^{-1})}{N} \\ &\leq Q(|\Im z|^{-1}) (\frac{2|\Im z|^{-1} P(|\Im z|^{-1})}{N})^2. \end{aligned}$$

Hence,

$$g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N} = O(\frac{1}{N^2}).$$

To prove (6.30), one can notice that both $g_N(z)$ and $\tilde{g}_N(z)$ are bounded by $\frac{1}{|\Im z|}$, and that

$$|E_N(z)| \leq \left\{ \frac{\sigma^2}{|\Im z|^2} + 1 \right\} |L_N(z)|,$$

where $L_N(z) = O(1)$.
On the other hand, if

$$\frac{P(|\Im z|^{-1})}{N} \leq \frac{|\Im z|}{2},$$

one has :

$$|\Im z' - \Im z| \leq |z' - z| \leq \frac{|\Im z|}{2}$$

which implies $\Im z' \geq \frac{\Im z}{2}$ and therefore $z' \in \mathbb{C}^+$. As a consequence of Proposition 6.1.2, $z - \sigma^2 g_N(z) \in \Omega_{\sigma,\mu_{A_N}}$ and (6.28) is satisfied. Thus,

$$|g_N(z) - \tilde{g}_N(z') - \frac{L_N(z)}{N}| \leq \frac{P(|\Im z|^{-1})}{N^2},$$

or, in other words,

$$g_N(z) - \tilde{g}_N(z') - \frac{L_N(z)}{N} = O(\frac{1}{N^2}). \tag{6.31}$$

On the other hand,

$$\begin{aligned}
\tilde{g}_N(z') - \tilde{g}_N(z) &= (z-z') \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z'-x)(z-x)} \\
&= (z-z') \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z-x)^2} \\
&\quad + (z-z')^2 \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z'-x)(z-x)^2}.
\end{aligned}$$

Taking into account the estimation of $z' - z$ above, one has :

$$(z-z') \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z-x)^2} = -\sigma^2 \tilde{g}'_N(z) \frac{L_N(z)}{N} + O(\frac{1}{N^2})$$

and

$$(z-z')^2 \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z'-x)(z-x)^2} = O(\frac{1}{N^2}).$$

Hence

$$\tilde{g}_N(z') - \tilde{g}_N(z) + \sigma^2 \tilde{g}'_N(z) \frac{L_N(z)}{N} = O(\frac{1}{N^2}). \tag{6.32}$$

(6.26) follows from (6.31) and (6.32) since

$$\begin{aligned}
|g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N}| \leq \quad & |g_N(z) - \tilde{g}_N(z') - \tfrac{L_N(z)}{N}| \\
& + |\tilde{g}_N(z') - \tilde{g}_N(z) + \sigma^2 \tilde{g}'_N(z) \tfrac{L_N(z)}{N}|.
\end{aligned}$$

Now, since $E_N(z) = O(1)$, we can deduce from (6.26) that $g_N(z) - \tilde{g}_N(z) = O(\frac{1}{N})$ and then that $E_N(z) - \tilde{E}_N(z) = O(\frac{1}{N})$. (6.23) readily follows. □

Remark 6.3.2. By combining the estimation proved above for the difference between g_N and the Stieltjes transform of $\mu_\sigma \boxplus \mu_{A_N}$ with some classical arguments developed in [PL03], one can recover the almost sure convergence of the spectral distribution of M_N to the free convolution $\mu_\sigma \boxplus \nu$.

6.4 Inclusion of the spectrum in a neighborhood of $\mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N})$

The purpose of this section is to prove the following Theorem 6.4.1.

Theorem 6.4.1. $\forall \varepsilon > 0$,

$$\mathbb{P}(\textit{ For all large } N, \mathrm{Spect}(M_N) \subset \{x, \mathrm{dist}(x, \mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N})) \leq \varepsilon\}) = 1.$$

The proof still uses the ideas of [HT05] and [Sch05] but, since $\mu_\sigma \boxplus \mu_{A_N}$ depends on N, we need here to apply the inverse Stieltjes tranform to functions depending on N. Therefore we give the details of the proof to convince the reader that the approach still holds.

Lemma 6.4.2. *For any fixed large N, $\tilde{E}_N$ defined in Proposition 6.3.1 is the Stieltjes transform of a compactly supported distribution Λ_N on $\mathbb{R}$ whose support is included in the support of $\mu_\sigma \boxplus \mu_{A_N}$.*

The proof relies on the following characterization already used in [Sch05].

Theorem 6.4.3. *[Til53]*

- *Let Λ be a distribution on $\mathbb{R}$ with compact support. Define the Stieltjes transform of Λ, $l : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ by*

$$l(z) = \Lambda\left(\frac{1}{z-x}\right).$$

  Then l is analytic on $\mathbb{C} \setminus \mathbb{R}$ and has an analytic continuation to $\mathbb{C} \setminus \mathrm{supp}(\Lambda)$. Moreover

  (c_1) $l(z) \to 0$ as $|z| \to \infty$,

  (c_2) there exists a constant $C > 0$, an integer $n \in \mathbb{N}$ and a compact set $K \subset \mathbb{R}$ containing $\mathrm{supp}(\Lambda)$, such that for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$|l(z)| \leq C \max\{\mathrm{dist}(z, K)^{-n}, 1\},$$

  (c_3) for any $\phi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ with compact support

$$\Lambda(\phi) = -\frac{1}{\pi} \lim_{y \to 0^+} \Im \int_{\mathbb{R}} \phi(x) l(x+iy) dx.$$

- *Conversely, if K is a compact subset of $\mathbb{R}$ and if $l : \mathbb{C} \setminus K \to \mathbb{C}$ is an analytic function satisfying (c_1) and (c_2) above, then l is the Stieltjes transform of a compactly supported distribution Λ on $\mathbb{R}$. Moreover, $\mathrm{supp}(\Lambda)$ is exactly the set of singular points of l in K.*

We use here the notations and results of Section 6.1. If $u \in \mathbb{R}$ is not in the support of $\mu_\sigma \boxplus \mu_{A_N}$, according to (6.4), $u - \sigma^2 \tilde{g}_N(u) = F_{\sigma,\mu_{A_N}}(u)$ belongs to $\mathbb{R} \setminus \overline{U_{\sigma,\mu_{A_N}}}$ and then cannot belong to $\mathrm{Spect}(A_N)$ since $\mathrm{Spect}(A_N) \subset U_{\sigma,\mu_{A_N}}$. Hence the singular points of $\tilde{E}_N$ are included in the support of $\mu_\sigma \boxplus \mu_{A_N}$. Now, we are going to show that for any fixed large N, $\tilde{E}_N$ satisfies (c_1) and (c_2) of Theorem 6.4.3. Let $C > 0$ be such that, for all large N, $\mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N}) \subset [-C; C]$ and $\mathrm{supp}(\mu_{A_N}) \subset [-C; C]$.

Let $\alpha > C + \sigma$. For any $z \in \mathbb{C}$ such that $|z| > \alpha$,

$$|\sigma^2 \tilde{g}_N(z)| \leq \frac{\sigma^2}{|z| - C} \leq \frac{\sigma^2}{\alpha - C} < \frac{(\alpha - C)^2}{\alpha - C} = \alpha - C$$

and
$$|z-\sigma^2\tilde{g}_N(z)| \geq \Big||z|-|\sigma^2\tilde{g}_N(z)|\Big| > |z|-(\alpha-C) > C.$$
Thus we get that for any $z \in \mathbb{C}$ such that $|z| > \alpha$,
$$\begin{aligned} \|G_{A_N}(z-\sigma^2\tilde{g}_N(z))\| &\leq \frac{1}{|z-\sigma^2\tilde{g}_N(z)|-C} \\ &< \frac{1}{|z|-(\alpha-C)-C} \\ &< \frac{1}{|z|-\alpha}. \end{aligned}$$
We get readily that, for $|z| > \alpha$,
$$|\tilde{E}_N(z)| \leq \frac{\kappa_4}{2}\frac{1}{(|z|-\alpha)^5}\left(\frac{\sigma^2}{(|z|-C)^2}+1\right).$$
Then, it is clear than $|\tilde{E}_N(z)| \to 0$ when $|z| \to +\infty$ and (c_1) is satisfied. Now we are going to prove (c_2) using the approach of [Sch05](Lemma 5.5). Denote by $\mathcal{E}_N$ the convex envelope of the support of $\mu_\sigma \boxplus \mu_{A_N}$ and define
$$K_N := \{x \in \mathbb{R}; \operatorname{dist}(x, \mathcal{E}_N) \leq 1\}$$
and
$$D_N = \{z \in \mathbb{C}; 0 < \operatorname{dist}(z, K_N) \leq 1\}.$$

- Let $z \in D_N \cap (\mathbb{C}\setminus\mathbb{R})$ with $\Re(z) \in K_N$. We have $\operatorname{dist}(z, K_N) = |\Im z| \leq 1$. We have
$$|\tilde{E}_N(z)| \leq \frac{\kappa_4}{2}\left(\sigma^2\frac{1}{|\Im z|^2}+1\right)\frac{1}{|\Im z|^5}.$$
Noticing that $1 \leq \frac{1}{|\Im z|^2}$, we easily deduce that there exists some constant C_0 such that for any $z \in D_N \cap \mathbb{C}\setminus\mathbb{R}$ with $\Re(z) \in K_N$,
$$\begin{aligned} |\tilde{E}_N(z)| &\leq C_0|\Im z|^{-7} \\ &\leq C_0 \operatorname{dist}(z, K_N)^{-7} \\ &\leq C_0 \max(\operatorname{dist}(z, K_N)^{-7}; 1). \end{aligned}$$
- Let $z \in D_N \cap (\mathbb{C}\setminus\mathbb{R})$ with $\Re(z) \notin K_N$. Then $\operatorname{dist}(z, \operatorname{supp}(\mu_\sigma \boxplus \mu_{A_N})) \geq 1$. Since $\tilde{E}_N$ is bounded on compact subsets of $\mathbb{C}\setminus\operatorname{supp}(\mu_\sigma \boxplus \mu_{A_N})$, we easily deduce that there exists some constant $C_1(N)$ such that for any $z \in D_N$ with $\Re(z) \notin K_N$,
$$|\tilde{E}_N(z)| \leq C_1(N) \leq C_1(N)\max(\operatorname{dist}(z, K_N)^{-7}; 1).$$
- Since $|\tilde{E}_N(z)| \to 0$ when $|z| \to +\infty$, $\tilde{E}_N$ is bounded on $\mathbb{C}\setminus\overline{D_N}$. Thus, there exists some constant $C_2(N)$ such that for any $z \in \mathbb{C}\setminus\overline{D_N}$,
$$|\tilde{E}_N(z)| \leq C_2(N) = C_2(N)\max(\operatorname{dist}(z, K_N)^{-7}; 1).$$

Hence (c_2) is satisfied with $C(N) = \max(C_0, C_1(N), C_2(N))$ and $n = 7$ and Lemma 6.4.2 follows from Theorem 6.4.3. □

Proof of Theorem 6.4.1: Using the inverse Stieltjes tranform, we get respectively that, for any φ_N in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ with compact support,

$$\mathbb{E}[\operatorname{tr}(\varphi_N(M_N))] - \int_{\mathbb{R}} \varphi_N d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(\varphi_N)}{N}$$
$$= \frac{1}{\pi} \lim_{y \to 0^+} \Im \int_{\mathbb{R}} \varphi_N(x) r_N(x+iy) dx,$$

where $r_N(z) = \tilde{g}_N(z) - g_N(z) + \frac{1}{N}\tilde{E}_N(z)$ satisfies, according to Proposition 6.3.1, for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$|r_N(z)| \leq \frac{1}{N^2} P(|\Im z|^{-1})$$

for some integer k. We refer the reader to the Appendix of [CDM07] where it is proved using the ideas of [HT05] that

$$\limsup_{y \to 0^+} \left| \int_{\mathbb{R}} \varphi_N(x) h(x+iy) dx \right|$$
$$\leq \int_{\mathbb{R}} \int_0^{+\infty} |(1+D)^{k+1} \varphi_N(x)| (|x| + \sqrt{2}t + K)^\alpha Q(t) \exp(-t) dt dx,$$

when h is an analytic function on $\mathbb{C} \setminus \mathbb{R}$ which satisfies

$$|h(z)| \leq (|z| + K)^\alpha P(|\Im z|^{-1}).$$

Hence, if there exists $K > 0$ such that, for all large N, the support of φ_N is included in $[-K, K]$ and $\sup_N \sup_{x \in [-K,K]} |D^p \varphi_N(x)| = C_p < \infty$ for any $p \leq k+1$, dealing with $h(z) = N^2 r_N(z)$, we deduce that for all large N,

$$\limsup_{y \to 0^+} \left| \int_{\mathbb{R}} \varphi_N(x) r_N(x+iy) dx \right| \leq \frac{C}{N^2}$$

and then

$$\mathbb{E}[\operatorname{tr}(\varphi_N(M_N))] - \int_{\mathbb{R}} \varphi_N d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(\varphi_N)}{N} = O(\frac{1}{N^2}). \qquad (6.33)$$

Let $\rho \geq 0$ be in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that its support is included in $\{|x| \leq 1\}$ and $\int \rho(x) dx = 1$. Let $0 < \varepsilon < 1$. Define

$$\rho_{\frac{\varepsilon}{2}}(x) = \frac{2}{\varepsilon} \rho(\frac{2x}{\varepsilon}),$$

$$K_N(\varepsilon) = \{x, \operatorname{dist}(x, \operatorname{supp}(\mu_\sigma \boxplus \mu_{A_N})) \leq \varepsilon\}$$

and

$$f_N(\varepsilon)(x) = \int_{\mathbb{R}} \mathbf{1}_{K_N(\varepsilon)}(y)\rho_{\frac{\varepsilon}{2}}(x-y)dy.$$

the function $f_N(\varepsilon)$ is in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, $f_N(\varepsilon) \equiv 1$ on $K_N(\frac{\varepsilon}{2})$; its support is included in $K_N(2\varepsilon)$. Since there exists K such that, for all large N, the support of $\mu_\sigma \boxplus \mu_{A_N}$ is included in $[-K;K]$, for all large N the support of $f_N(\varepsilon)$ is included in $[-K-2;K+2]$ and for any $p>0$,

$$\sup_{x\in[-K-2;K+2]} |D^p f_N(\varepsilon)(x)| \leq \sup_{x\in[-K-2;K+2]} \int_{-K-1}^{K+1} |D^p \rho_{\frac{\varepsilon}{2}}(x-y)|dy \leq C_p(\varepsilon).$$

Thus, according to (6.33),

$$\mathbb{E}[\mathrm{tr}(f_N(\varepsilon)(M_N))] - \int_{\mathbb{R}} f_N(\varepsilon)d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(f_N(\varepsilon))}{N} = O_\varepsilon(\frac{1}{N^2}) \quad (6.34)$$

and

$$\mathbb{E}[\mathrm{tr}((f_N'(\varepsilon))^2(M_N))] - \int_{\mathbb{R}} (f_N'(\varepsilon))^2 d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N((f_N'(\varepsilon))^2)}{N} = O_\varepsilon(\frac{1}{N^2}). \quad (6.35)$$

Moreover, following the proof of Lemma 5.6 in [Sch05], one can show that $\Lambda_N(1)=0$. Then, the function $\psi_N(\varepsilon) \equiv 1 - f_N(\varepsilon)$ also satisfies

$$\mathbb{E}[\mathrm{tr}(\psi_N(\varepsilon)(M_N))] - \int_{\mathbb{R}} \psi_N(\varepsilon)d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(\psi_N(\varepsilon))}{N} = O_\varepsilon(\frac{1}{N^2}). \quad (6.36)$$

Moreover, since $\psi_N'(\varepsilon) = -f_N'(\varepsilon)$, it comes readily from (6.35) that

$$\mathbb{E}[\mathrm{tr}((\psi_N'(\varepsilon))^2(M_N))] - \int_{\mathbb{R}} (\psi_N'(\varepsilon))^2 d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N((\psi_N'(\varepsilon))^2)}{N} = O_\varepsilon(\frac{1}{N^2}).$$

Now, since $\psi_N(\varepsilon) \equiv 0$ on the support of $\mu_\sigma \boxplus \mu_{A_N}$, we deduce that

$$\mathbb{E}[\mathrm{tr}(\psi_N(\varepsilon)(M_N))] = O_\varepsilon(\frac{1}{N^2}) \quad (6.37)$$

and

$$\mathbb{E}[\mathrm{tr}((\psi_N'(\varepsilon))^2(M_N))] = O_\varepsilon(\frac{1}{N^2}). \quad (6.38)$$

By Lemma 6.8.1 (sticking to the proof of Proposition 4.7 in [HT05]), we have

$$\mathbf{V}[\mathrm{tr}(\psi_N(\varepsilon)(M_N))] \leq \frac{C_\varepsilon}{N^2}\mathbb{E}\left[\mathrm{tr}\{(\psi_N'(\varepsilon)(M_N))^2\}\right].$$

Hence, using (6.38), one can deduce that

$$\mathbf{V}[\mathrm{tr}(\psi_N(\varepsilon)(M_N))] = O_\varepsilon(\frac{1}{N^4}) \quad (6.39)$$

Set
$$Z_{N,\varepsilon} := \mathrm{tr}(\psi_N(\varepsilon)(M_N))$$
and
$$\Omega_{N,\varepsilon} = \{Z_{N,\varepsilon} > N^{-\frac{4}{3}}\}$$
From (6.37) and (6.39), we deduce that
$$\mathbb{E}\{|Z_{N,\varepsilon}|^2\} = O_\varepsilon(\frac{1}{N^4}).$$
Hence
$$P(\Omega_{N,\varepsilon}) \leq N^{\frac{8}{3}}\mathbb{E}\{|Z_{N,\varepsilon}|^2\} = O_\varepsilon(\frac{1}{N^{\frac{4}{3}}}).$$
By Borel-Cantelli lemma, we deduce that, almost surely for all large N, $Z_{N,\varepsilon} \leq N^{-\frac{4}{3}}$. Since $Z_{N,\varepsilon} \geq \mathbf{1}_{\mathbb{R}\setminus K_N(2\varepsilon)}$, it follows that, almost surely for all large N, the number of eigenvalues of M_N which are in $\mathbb{R} \setminus K_N(2\varepsilon)$ is lower than $N^{-\frac{1}{3}}$ and thus obviously has to be equal to zero. The proof of Theorem 6.4.1 is complete.□

6.5 Study of $\mu_\sigma \boxplus \mu_{A_N}$

The aim of this section is to show the following inclusion of the support of $\mu_\sigma \boxplus \mu_{A_N}$ (see Theorem 6.5.1 below). To this aim, we will use the notations and results of Section 6.1. We define
$$\Theta = \{\theta_j,\, 1 \leq j \leq J\} \quad \text{and} \quad \Theta_{\sigma,\nu} = \Theta \cap (\mathbb{R} \setminus \overline{U_{\sigma,\nu}}). \tag{6.40}$$
Furthermore, for all $\theta_j \in \Theta_{\sigma,\nu}$, we set
$$\rho_{\theta_j} := H_{\sigma,\nu}(\theta_j) = \theta_j + \sigma^2 G_\nu(\theta_j) \tag{6.41}$$
which is outside the support of $\mu_\sigma \boxplus \nu$ according to (6.4), and we define
$$K_{\sigma,\nu}(\theta_1, \ldots, \theta_J) := \mathrm{supp}(\mu_\sigma \boxplus \nu) \bigcup \left\{\rho_{\theta_j},\, \theta_j \in\, \Theta_{\sigma,\nu}\right\}. \tag{6.42}$$

Theorem 6.5.1. *For any $\varepsilon > 0$,*
$$\mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N}) \subset K_{\sigma,\nu}(\theta_1, \ldots, \theta_J) + (-\varepsilon, \varepsilon),$$
when N is large enough.

Let us decompose μ_{A_N} as
$$\mu_{A_N} = \hat{\mu}_{\beta,N} + \hat{\mu}_{\Theta,N},$$
$$\text{where } \hat{\mu}_{\beta,N} = \frac{1}{N}\sum_{j=1}^{N-r} \delta_{\beta_j(N)} \quad \text{and} \quad \hat{\mu}_{\Theta,N} = \frac{1}{N}\sum_{j=1}^{J} k_j \delta_{\theta_j}.$$
In the following, we will denote by $D(x, \delta)$ the open disk centered on x and with radius δ. We begin with a trivial technical lemma we will need in the following.

Lemma 6.5.2. *Let $\mathcal{K}$ be a compact set included in $\mathbb{R}\setminus\mathrm{supp}(\nu)$. Then $G'_{\hat{\mu}_{\beta,N}}$ (which is well defined on $\mathcal{K}$ for large N) converges to G'_ν uniformly on $\mathcal{K}$.*

Proof of Lemma 6.5.2: We first prove that for all $u \in \mathcal{K}$,

$$-G'_{\hat{\mu}_{\beta,N}}(u) = \frac{1}{N}\sum_{j=1}^{N-r}\frac{1}{(u-\beta_j)^2} \underset{N\to+\infty}{\longrightarrow} \int \frac{d\nu(x)}{(u-x)^2} = -G'_\nu(u). \quad (6.43)$$

Let $\varepsilon > 0$ be such that $\mathrm{dist}(\mathcal{K}, \mathrm{supp}(\nu)) \geq \varepsilon$. For all $u \in \mathcal{K}$, let h_u be a bounded continuous function defined on $\mathbb{R}$ which coincides with $f_u(x) = 1/(u-x)^2$ on $\mathrm{supp}(\nu)+[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]$. As $\max_{1\leq j\leq N-r}\mathrm{dist}(\beta_j(N),\mathrm{supp}(\nu))$ tends to zero as $N\to\infty$, one can find N_0 such that, for all $N \geq N_0$, $\beta_j(N) \in \mathrm{supp}(\nu)+[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]$ for all $1 \leq j \leq N-r$. Since the sequence of measures $\hat{\mu}_{\beta,N}$ weakly converges to ν, (6.43) follows, observing that $-G'_{\hat{\mu}_{\beta,N}}(u) = \int h_u(x)\, d\hat{\mu}_{\beta,N}(x)$ and $-G'_\nu(u) = \int h_u(x)d\nu(x)$.
The uniform convergence follows from Montel's theorem, since $G'_{\hat{\mu}_{\beta,N}}$ and G'_ν are analytic on $D = \{z \in \mathbb{C}, \mathrm{dist}(z, \mathrm{supp}(\nu)) > \frac{\varepsilon}{2}\}$ and uniformly bounded on D by $\frac{4}{\varepsilon^2}$ for $N \geq N_0$. □

We are now in position to give the proof of Theorem 6.5.1. We recall that, from (6.4),

$$\mathbb{R}\setminus\mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N}) = H_{\sigma,\mu_{A_N}}(\mathbb{R}\setminus\overline{U_{\sigma,\mu_{A_N}}}). \quad (6.44)$$

In the proofs, we will write for simplicity U_N, H_N and F_N instead of $U_{\sigma,\mu_{A_N}}$, $H_{\sigma,\mu_{A_N}}$ and $F_{\sigma,\mu_{A_N}}$ respectively.
The main step of the proof consists in observing the following inclusion of the open set $U_{\sigma,\mu_{A_N}}$.

Lemma 6.5.3. *For any $\varepsilon' > 0$,*

$$U_{\sigma,\mu_{A_N}} \subset \{u,\, \mathrm{dist}(u, \overline{U_{\sigma,\nu}}) < \varepsilon'\} \cup \{u,\, \mathrm{dist}(u, \Theta_{\sigma,\nu}) < \varepsilon'\}, \quad (6.45)$$

for all large N (since the compact sets $\overline{U_{\sigma,\nu}}$ and $\Theta_{\sigma,\nu}$ are disjoint, the previous union is disjoint once ε' is small enough).

Proof of Lemma 6.5.3: Define

$$\mathcal{F}_{\varepsilon'} = \{u,\, \mathrm{dist}(u, \overline{U_{\sigma,\nu}}) \geq \varepsilon'\} \cap \{u,\, \mathrm{dist}(u, \Theta_{\sigma,\nu}) \geq \varepsilon'\}.$$

We shall show that for all large N, $\mathcal{F}_{\varepsilon'} \subset \mathbb{R}\setminus\overline{U_N}$.
Since $\max_{1\leq j\leq N-r}\mathrm{dist}(\beta_j(N), \mathrm{supp}(\nu)) \underset{N\to\infty}{\longrightarrow} 0$, there exists N_0 such that for all $N \geq N_0$, the $\beta_j(N)$'s are in $\mathrm{supp}(\nu)+(-\varepsilon',\varepsilon')$. Since $\mathrm{supp}(\nu) \subset \overline{U_{\sigma,\nu}}$, it is clear that for all $N \geq N_0$, $\mathcal{F}_{\varepsilon'}$ is included in $\mathbb{R}\setminus\mathrm{Spect}A_N$. Moreover, one can readily observe that if u satisfies

$$\mathrm{dist}(u, \mathrm{supp}(\nu)+(-\varepsilon',\varepsilon')) \geq \sigma$$

and

$$\mathrm{dist}(u, \Theta) \geq \sigma$$

then, for all $N \geq N_0$, $-G'_{\mu_{A_N}}(u) \leq \frac{1}{\sigma^2}$. This implies that, for all $N \geq N_0$, the open set U_N is included in the compact set

$$\mathcal{F}'_{\varepsilon'} = \{u, \ \mathrm{dist}(u, \mathrm{supp}(\nu) + (-\varepsilon', \varepsilon')) \leq \sigma\} \cup \{u, \ \mathrm{dist}(u, \Theta) \leq \sigma\}.$$

Hence, it is sufficient to show that for N large enough, the compact set $\mathcal{K}_{\varepsilon'} := \mathcal{F}_{\varepsilon'} \cap \mathcal{F}'_{\varepsilon'}$ is contained in $\mathbb{R} \setminus \overline{U_N}$.
As ν is compactly supported, the function $u \mapsto -G'_\nu(u) = \int_\mathbb{R} d\nu(x)/(u-x)^2$ is continuous on $\mathbb{R} \setminus \mathrm{supp}(\nu)$. Hence it reaches its bounds on the compact set $\mathcal{K}_{\varepsilon'}$ (which is obviously included in $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$) so that there exists $\alpha > 0$ such that $-G'_\nu(u) \leq \frac{1}{\sigma^2} - 2\alpha$ for any u in $\mathcal{K}_{\varepsilon'}$.
According to Lemma 6.5.2, there exists N_0 such that for all $N \geq N_0$ and for all u in $\mathcal{K}_{\varepsilon'}$,

$$|G'_{\hat{\mu}_{\beta,N}}(u) - G'_\nu(u)| \leq \frac{3\alpha}{4}. \tag{6.46}$$

At last, one can notice that N_0 may be chosen large enough so that

$$\forall N \geq N_0, \quad -G'_{\hat{\mu}_{\Theta,N}}(u) = \frac{1}{N}\sum_{j=1}^{J} \frac{k_j}{(u-\theta_j)^2} \leq \frac{\alpha}{4}. \tag{6.47}$$

This is just because for all $u \in \mathcal{F}_{\varepsilon'}$, one has that: $-G'_{\hat{\mu}_{\Theta,N}}(u) \leq \frac{r}{N\varepsilon'^2}$ which converges uniformly on $\mathcal{K}'_{\varepsilon'}$ to 0 as N goes to infinity.
Combining all the preceding gives that, on $\mathcal{K}_{\varepsilon'}$, the function $-G'_{\mu_{A_N}}$ is bounded from above by $\frac{1}{\sigma^2} - \alpha$. This implies that $\mathcal{K}_{\varepsilon'}$ is included in $\mathbb{R} \setminus \overline{U_{\sigma,\mu_{A_N}}}$ which is what we wanted to show. □

Now we shall establish the following inclusion.

Lemma 6.5.4. *For all $\varepsilon > 0$, for all $\varepsilon' > 0$ small enough,*

$$\mathbb{R} \setminus (K_\sigma(\theta_1, \ldots, \theta_J) + [-\varepsilon, \varepsilon]) \subset H_N\left(\{u, \ \mathrm{dist}(u, \Theta_{\sigma,\nu} \cup \overline{U_{\sigma,\nu}}) > \varepsilon'\}\right), \tag{6.48}$$

when N is large enough.

Combined with Lemma 6.5.3, this result leads to Theorem 6.5.1.

Proof of Lemma 6.5.4: According to (6.4), (6.6) and Remark 6.1.6, we have that

$$\mathbb{R} \setminus \mathrm{supp}(\mu_\sigma \boxplus \nu) =$$

$$]-\infty, H_{\sigma,\nu}(s_m)[\ \bigcup \left(\bigcup_{l=m}^{2}]H_{\sigma,\nu}(t_l), H_{\sigma,\nu}(s_{l-1})\Big[\right) \bigcup]H_{\sigma,\nu}(t_1), +\infty[$$

i.e.

$$\mathrm{supp}(\mu_\sigma \boxplus \nu) = \bigcup_{l=m}^{1} \Big[H_{\sigma,\nu}(s_l), H_{\sigma,\nu}(t_l) \Big]. \tag{6.49}$$

Note that there exists some finite integer q such that, for ε small enough, $\mathbb{R} \setminus (K_\sigma(\theta_1, \dots, \theta_J) + [-\varepsilon, \varepsilon])$ is the following disjoint union of intervals

$$]-\infty, h_0[\bigcup_{i=1,\dots,q}]k_i, h_i[\cup]k_{q+1}, +\infty[,$$

where $h_i = H_{\sigma,\nu}(s_{p_i}) - \varepsilon$ and $k_{i+1} = H_{\sigma,\nu}(t_{p_i}) + \varepsilon$ for some p_i or $h_i = H_{\sigma,\nu}(\theta_{j_i}) - \varepsilon$ and $k_{i+1} = H_{\sigma,\nu}(\theta_{j_i}) + \varepsilon$ for some θ_{j_i} in $\Theta_{\sigma,\nu}$.
For such an $\varepsilon > 0$, since $H_{\sigma,\nu}$ coincides on $\mathbb{R} \setminus U_{\sigma,\nu}$ with the homeomorphism $\Psi_{\sigma,\nu}$ defined in Theorem 6.1.3, we can deduce in particular that $H_{\sigma,\nu}$ is right-continuous (resp. left-continuous) at each t_l (resp. s_l) for $1 \le l \le m$, and $H_{\sigma,\nu}$ is continuous at each θ_i in $\Theta_{\sigma,\nu}$. Thus, there exists $\varepsilon' > 0$ such that: for all $1 \le l \le m$,

$$H_{\sigma,\nu}(s_l - \varepsilon') \ge H_{\sigma,\nu}(s_l) - \frac{\varepsilon}{2} \quad \text{and} \quad H_{\sigma,\nu}(t_l + \varepsilon') \le H_{\sigma,\nu}(t_l) + \frac{\varepsilon}{2} \tag{6.50}$$

and for all θ_j in $\Theta_{\sigma,\nu}$,

$$H_{\sigma,\nu}(\theta_j - \varepsilon') \ge H_{\sigma,\nu}(\theta_j) - \frac{\varepsilon}{2} \quad \text{and} \quad H_{\sigma,\nu}(\theta_j + \varepsilon') \le H_{\sigma,\nu}(\theta_j) + \frac{\varepsilon}{2}. \tag{6.51}$$

Now H_N being increasing on $\mathbb{R} \setminus \overline{U_N}$, for N large enough, the image by H_N of

$$\{u,\, d(u, \Theta_{\sigma,\nu}) > \varepsilon'\} \cap \{u,\, d(u, \overline{U_{\sigma,\nu}}) > \varepsilon'\} \subseteq \mathbb{R} \setminus \overline{U_N}$$

is the following disjoint union of intervals

$$]-\infty, h_0(N)[\bigcup_{i=1,\dots,q}]k_i(N), h_i(N)[\cup]k_{q+1}(N), +\infty[,$$

where $h_i(N) = H_N(s_{p_i} - \varepsilon')$ and $k_{i+1}(N) = H_N(t_{p_i} + \varepsilon')$ or $h_i(N) = H_N(\theta_{j_i} - \varepsilon')$ and $k_{i+1}(N) = H_N(\theta_{j_i} + \varepsilon')$.
One can see that it only remains to state that for all large N: $\forall 1 \le l \le m$,

$$H_N(s_l - \varepsilon') \ge H_{\sigma,\nu}(s_l) - \varepsilon \quad \text{and} \quad H_N(t_l + \varepsilon') \le H_{\sigma,\nu}(t_l) + \varepsilon. \tag{6.52}$$

$$H_N(\theta_i - \varepsilon') \ge H_{\sigma,\nu}(\theta_i) - \varepsilon \quad \text{and} \quad H_N(\theta_i + \varepsilon') \le H_{\sigma,\nu}(\theta_i) + \varepsilon. \tag{6.53}$$

Moreover, as μ_{A_N} weakly converges to ν, it is not hard to see that for all $1 \le l \le m$, and all θ_i in $\Theta_{\sigma,\nu}$, $H_N(s_l - \varepsilon')$, $H_N(t_l + \varepsilon')$, $H_N(\theta_i - \varepsilon')$ and $H_N(\theta_i + \varepsilon')$ converge as $N \to \infty$ to $H_{\sigma,\nu}(s_l - \varepsilon')$, $H_{\sigma,\nu}(t_l + \varepsilon')$, $H_{\sigma,\nu}(\theta_i - \varepsilon')$ and $H_{\sigma,\nu}(\theta_i + \varepsilon')$ respectively. So, there exists N_0 such that for all $N \ge N_0$: $H_N(s_l - \varepsilon') \ge H_{\sigma,\nu}(s_l - \varepsilon') - \frac{\varepsilon}{2}$ and $H_N(t_l + \varepsilon') \le H_{\sigma,\nu}(t_l + \varepsilon') + \frac{\varepsilon}{2}$ as well as $H_N(\theta_i - \varepsilon') \ge H_{\sigma,\nu}(\theta_i - \varepsilon') - \frac{\varepsilon}{2}$ and $H_N(\theta_i + \varepsilon') \le H_{\sigma,\nu}(\theta_i + \varepsilon') + \frac{\varepsilon}{2}$. We can then deduce (6.52) and (6.53) from (6.50) and (6.51). □

6.6 Exact separation of eigenvalues

Before stating the fundamental exact separation phenomenon between the spectrum of M_N and the spectrum of A_N, we need a preliminary lemma (see Lemma 6.6.3 below).
From Section 6.1, we readily deduce the following

Proposition 6.6.1.

$$\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J) = \{x \in \mathbb{R}, F_{\sigma,\nu}(x) \in \mathbb{R} \setminus \{\overline{U_{\sigma,\nu}} \cup \Theta\}\}$$

and $F_{\sigma,\nu}$ *is a homeomorphism from* $\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J)$ *onto* $\mathbb{R} \setminus \{\overline{U_{\sigma,\nu}} \cup \Theta\}$ *with inverse* $H_{\sigma,\nu}$.

Remark 6.6.2. : For all $\hat{\sigma} < \sigma$, $\mathbb{R} \setminus \overline{U_{\sigma,\nu}} \subset \mathbb{R} \setminus \overline{U_{\hat{\sigma},\nu}}$ so that it makes sense to consider the following composition of homeomorphism

$$H_{\hat{\sigma},\nu} \circ F_{\sigma,\nu} : \mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J) \to \mathbb{R} \setminus K_{\hat{\sigma},\nu}(\theta_1, \ldots, \theta_J),$$

which is increasing on each connected component of $\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J)$.

Lemma 6.6.3. *Let* $[a, b]$ *be a compact set contained in* $\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J)$. *Then,*

(i) For all large N, $[F_{\sigma,\nu}(a), F_{\sigma,\nu}(b)] \subset \mathbb{R} \setminus \mathrm{Spect}(A_N)$.

(ii) For all $0 < \hat{\sigma} < \sigma$, *the interval* $[H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(a)), H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(b))]$ *is contained in* $\mathbb{R} \setminus K_{\hat{\sigma},\nu}(\theta_1, \ldots, \theta_J)$ *and* $H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(b)) - H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(a)) \geq b - a$.

Proof of Lemma 6.6.3: For simplicity, we define $K^{\varepsilon}_{\sigma,J} = K_{\sigma}(\theta_1, \ldots, \theta_J) + [-\varepsilon, \varepsilon]$. As $[a, b]$ is a compact set, there exist $\varepsilon > 0$ and $\alpha > 0$ such that

$$[a - \alpha, b + \alpha] \subset \mathbb{R} \setminus K^{\varepsilon}_{\sigma,J} \quad \text{and} \quad \mathrm{dist}([a - \alpha, b + \alpha]; K^{\varepsilon}_{\sigma,J}) \geq \alpha.$$

As before, we let $\tilde{\mu}_N = \mu_\sigma \boxplus \mu_{A_N}$. According to Theorem 6.5.1, there exists some N_0 such that for all $N \geq N_0$, $\mathrm{supp}(\tilde{\mu}_N)$ is contained in $K^{\varepsilon}_{\sigma,J}$. Thus, using (6.4) and since F_N is continuous strictly increasing on $[a - \alpha, b + \alpha]$, we have

$$\forall N \geq N_0, \quad [F_N(a - \alpha), F_N(b + \alpha)] \subset \mathbb{R} \setminus \overline{U}_N \subset \mathbb{R} \setminus \mathrm{Spect}(A_N). \quad (6.54)$$

As $F_{\sigma,\nu}$ is strictly increasing on the compact set $[a-\alpha, b+\alpha]$ ($\mathrm{supp}(\mu_\sigma \boxplus \nu) \subset K^{\varepsilon}_{\sigma,J}$), one can consider $\delta > 0$ such that

$$F_{\sigma,\nu}(a - \alpha) \leq F_{\sigma,\nu}(a) - \delta \quad \text{and} \quad F_{\sigma,\nu}(b + \alpha) \geq F_{\sigma,\nu}(b) + \delta. \quad (6.55)$$

Now, the weak convergence of the probability measures $\tilde{\mu}_N$ to $\mu_\sigma \boxplus \nu$ will lead to the result, recalling from the definition of the subordination functions that for all $x \in [a-\alpha, b+\alpha]$: $F_{\sigma,\nu}(x) = x - \sigma^2 G_{\mu_\sigma \boxplus \nu}(x)$ and $F_N(x) = x - \sigma^2 G_{\tilde{\mu}_N}(x)$

(at least for all $N \geq N_0$). Indeed, observing that for any x in $[a-\alpha, b+\alpha]$, the map $h : t \mapsto \frac{1}{x-t}$ is bounded on $K^{\varepsilon}_{\sigma,J}$, one readily gets the simple convergence of $G_{\tilde{\mu}_N}$ to $G_{\mu_\sigma \boxplus \nu}$ as well as the one of the corresponding subordination functions, by considering a bounded continuous function which coincides with h on $K^{\varepsilon}_{\sigma,J}$. We then deduce that there exists $N_0' \geq N_0$ such that, for all $N \geq N_0'$,

$$F_N(a-\alpha) \leq F_{\sigma,\nu}(a-\alpha) + \delta \quad \text{and} \quad F_N(b+\alpha) \geq F_{\sigma,\nu}(b+\alpha) - \delta. \tag{6.56}$$

Combining (6.54), (6.55) and (6.56) proves that the inclusion of point (i) holds true for all $N \geq N_0'$.

The first part of (ii) is obvious from Remark 6.6.2. The second part mainly follows from the fact that $F_{\sigma,\nu}$ is strictly increasing on $\mathbb{R}\backslash \mathrm{supp}(\mu_\sigma \boxplus \nu)$. More precisely, if we set $a' = H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(a))$ and $b' = H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(b))$, then

$$\begin{aligned} b' - a' &= F_{\sigma,\nu}(b) - F_{\sigma,\nu}(a) + \hat{\sigma}^2\big(G_\nu(F_{\sigma,\nu}(b)) - G_\nu(F_{\sigma,\nu}(a))\big) \\ &\geq F_{\sigma,\nu}(b) - F_{\sigma,\nu}(a) + \sigma^2\big(G_\nu(F_{\sigma,\nu}(b)) - G_\nu(F_{\sigma,\nu}(a))\big) \\ &\geq H_{\sigma,\nu}(F_{\sigma,\nu}(b)) - H_{\sigma,\nu}(F_{\sigma,\nu}(a)) = b - a \end{aligned}$$

since $F_{\sigma,\nu}(a) < F_{\sigma,\nu}(b)$ and then $G_\nu(F_{\sigma,\nu}(b)) - G_\nu(F_{\sigma,\nu}(a)) < 0$. □

The exact separation result involving the subordination function related to the free convolution of μ_σ and ν can now be stated. Let $[a, b]$ be a compact interval contained in $\mathbb{R}\backslash K_\sigma(\theta_1, \ldots, \theta_J)$. By Theorems 6.4.1 and 6.5.1, almost surely for all large N, $[a, b]$ is outside the spectrum of M_N. Moreover, from Lemma 6.6.3 (i), it corresponds an interval $I = [a', b']$ outside the spectrum of A_N for all large N i.e., with the convention that $\lambda_0(M_N) = \lambda_0(A_N) = +\infty$ and $\lambda_{N+1}(M_N) = \lambda_{N+1}(A_N) = -\infty$, there is $i_N \in \{0, \ldots, N\}$ such that

$$\lambda_{i_N+1}(A_N) < F_{\sigma,\nu}(a) := a' \quad \text{and} \quad \lambda_{i_N}(A_N) > F_{\sigma,\nu}(b) := b'. \tag{6.57}$$

The numbers a and a' (resp. b and b') are linked as follows:

$$a = \rho_{a'} := H_{\sigma,\nu}(a') = a' + \sigma^2 G_\nu(a'),$$

$$b = \rho_{b'} := H_{\sigma,\nu}(b') = b' + \sigma^2 G_\nu(b').$$

We claim that $[a, b]$ splits the spectrum of M_N exactly as I splits the spectrum of A_N. In other words,

Theorem 6.6.4. *With i_N satisfying (6.57), one has*

$$\mathbb{P}[\lambda_{i_N+1}(M_N) < a \ \textit{and} \ \lambda_{i_N}(M_N) > b, \textit{for all large } N] = 1. \tag{6.58}$$

The proof closely follows the proof of Theorem 4.5 in [CDMF09] by introducing in a fit way the subordination functions or their inverses. For the reader's convenience, we rewrite the whole proof. The key idea is to introduce a continuum of matrices $M_N^{(k)}$ interpolating from M_N to A_N:

$$M_N^{(k)} := \frac{\sigma_k}{\sigma} \frac{W_N}{\sqrt{N}} + A_N,$$

where

$$\sigma_k^2 = \sigma^2 (\frac{1}{1 + kC_{a,b}}),$$

and $C_{a,b}$ being a positive constant which has to be chosen small enough to ensure that the matrices $M_N^{(k)}$ and $M_N^{(k+1)}$ are close enough to each other. More precisely, $C_{a,b}$ is chosen such that

$$\max\left(\sigma^2 C_{a,b} |G_{\mu_\sigma \boxplus \nu}(a)|; \sigma^2 C_{a,b} |G_{\mu_\sigma \boxplus \nu}(b)|; 3\sigma C_{a,b}\right) < \frac{b-a}{4}. \tag{6.59}$$

In particular, $\sigma_0 = \sigma$ and $\sigma_k \to 0$ when k goes to infinity.
We first prove that the intervals $[H_{\sigma_k,\nu}(F_{\sigma,\nu}(a)), H_{\sigma_k,\nu}(F_{\sigma,\nu}(b))]$ split respectively the spectrum of $M_N^{(k)}$ in exactly the same way. Moreover, we also prove that for k large enough, the interval $[H_{\sigma_k,\nu}(F_{\sigma,\nu}(a)), H_{\sigma_k,\nu}(F_{\sigma,\nu}(b))]$ splits the spectrum of $M_N^{(k)}$ as $[F_{\sigma,\nu}(a), F_{\sigma,\nu}(b)]$ splits the spectrum of A_N, this means roughly that we extend the first statement to $k = \infty$ and the result follows.

As in [CDMF09], this proof is inspired by the work [BS99] and mainly relies on results on eigenvalues of the rescaled Wigner matrix X_N combined with the following classical result (due to Weyl).

Lemma 6.6.5. *(cf. Theorem 4.3.7 of [HJ90]) Let B and C be two $N \times N$ Hermitian matrices. For any pair of integers j, k such that $1 \leq j, k \leq N$ and $j + k \leq N + 1$, we have*

$$\lambda_{j+k-1}(B + C) \leq \lambda_j(B) + \lambda_k(C).$$

For any pair of integers j, k such that $1 \leq j, k \leq N$ and $j + k \geq N + 1$, we have

$$\lambda_j(B) + \lambda_k(C) \leq \lambda_{j+k-N}(B + C).$$

Proof of Theorem 6.6.4: Given $k \geq 0$, define

$$a_k = H_{\sigma_k}(F_{\sigma,\nu}(a)) \,\text{and}\, b_k = H_{\sigma_k}(F_{\sigma,\nu}(b)).$$

Remark 6.6.6. Note that in [CDMF09] where $\nu = \delta_0$, we considered $a_k = z_{\sigma_k}(g_\sigma(a))$ where g_σ denoted the Stieltjes transform of μ_σ and z_{σ_k} the inverse of g_{σ_k}. Actually, when $\nu = \delta_0$, then $H_{\sigma_k,\nu}(z) = z + \sigma_k^2/z = z_{\sigma_k}(1/z)$ and $F_{\sigma,\nu} = 1/g_\sigma$ so that $z_{\sigma_k}(g_\sigma) = H_{\sigma_k,\nu}(F_{\sigma,\nu})$. This very interpretation of the composition $z_{\sigma_k} \circ g_\sigma$ in terms of subordination function allows us to extend the result of exact separation to non-finite rank perturbations.

The last point of (ii) in Lemma 6.6.3 yields $b_k - a_k \geq b - a$. Moreover

$$\begin{aligned} a_{k+1} - a_k &= (\sigma_{k+1}^2 - \sigma_k^2) G_{\mu_\sigma \boxplus \nu}(a) \\ &= -C_{a,b} \frac{\sigma^2}{(1 + kC_{a,b})(1 + (k+1)C_{a,b})} G_{\mu_\sigma \boxplus \nu}(a), \end{aligned}$$

so that $|a_{k+1} - a_k| \leq \sigma^2 C_{a,b} |G_{\mu_\sigma \boxplus \nu}(a)|$. One gets similarly that $|b_{k+1} - b_k| \leq \sigma^2 C_{a,b} |G_{\mu_\sigma \boxplus \nu}(b)|$. Hence, we deduce from (6.59) that

$$|a_{k+1} - a_k| < \frac{b-a}{4} \quad \text{and} \quad |b_{k+1} - b_k| < \frac{b-a}{4}. \tag{6.60}$$

Now, we shall show by induction on k that, with probability 1, for large N, the $M_N^{(k)}$ have respectively the same amount of eigenvalues to the left sides of the interval $[a_k, b_k]$. For all $k \geq 0$, set

$$\mathrm{E}_k = \{\text{no eigenvalues of } M_N^{(k)} \text{ in } [a_k, b_k], \text{ for all large } N\}.$$

By Lemma 6.6.3 (ii) and Theorems 6.4.1 and 6.5.1, we know that $\mathbb{P}(\mathrm{E}_k) = 1$ for all k. In particular, one has for all $\omega \in \mathrm{E}_0$ and for all large N,

$$\exists j_N(\omega) \in \{0, \ldots, N\} \text{ such that } \lambda_{j_N(\omega)+1}(M_N) < a \text{ and } \lambda_{j_N(\omega)}(M_N) > b. \tag{6.61}$$

Extending the random variable j_N, by setting for instance $j_N := -1$ on the complementary of E_0, we want to show that for all k,

$$\mathbb{P}[\lambda_{j_N+1}(M_N^{(k)}) < a_k \text{ and } \lambda_{j_N}(M_N^{(k)}) > b_k, \text{ for all large } N] = 1. \tag{6.62}$$

We proceed by induction. By (6.61), this is true for $k = 0$. Now, let us assume that (6.62) holds true. Since

$$M_N^{(k+1)} = M_N^{(k)} + \left(\frac{1}{\sqrt{1 + (k+1)C_{a,b}}} - \frac{1}{\sqrt{1 + kC_{a,b}}}\right) X_N,$$

we can deduce from Lemma 6.6.5 that

$$\lambda_{j_N+1}(M_N^{(k+1)}) \leq \lambda_{j_N+1}(M_N^{(k)}) + (-\lambda_N(X_N)) C_{a,b}.$$

Since, for N large enough, $0 < -\lambda_N(X_N) \leq 3\sigma$ almost surely, it follows using (6.59) that

$$\lambda_{j_N+1}(M_N^{(k+1)}) < a_k + \frac{b-a}{4} := \hat{a}_k \quad \text{a.s..}$$

Similarly, one can show that

$$\lambda_{j_N}(M_N^{(k+1)}) > b_k - \frac{b-a}{4} := \hat{b}_k \quad \text{a.s..}$$

Inequalities (6.60) ensure that

$$[\hat{a}_k, \hat{b}_k] \subset [a_{k+1}, b_{k+1}].$$

As $\mathbb{P}(\mathrm{E}_{k+1}) = 1$, we deduce that, with probability 1,

$$\lambda_{j_N+1}(M_N^{(k+1)}) < a_{k+1} \text{ and } \lambda_{j_N}(M_N^{(k+1)}) > b_{k+1}, \quad \text{for all large } N.$$

This completes the proof by induction of (6.62).

Now, we are going to show that there exists K large enough so that, for all $k \geq K$, there is exact separation of the eigenvalues of the matrices A_N and $M_N^{(k)}$ i.e.

$$\mathbb{P}\big[\lambda_{i_N+1}(M_N^{(k)}) < a_k \text{ and } \lambda_{i_N}(M_N^{(k)}) > b_k, \quad \text{for all large } N\big] = 1. \quad (6.63)$$

There exists $\alpha > 0$ such that $[a-\alpha; b+\alpha] \subset \mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J)$. Thus according to Lemma 6.6.3 (i) for all large N,

$$[F_{\sigma,\nu}(a-\alpha); F_{\sigma,\nu}(b+\alpha)] \subset \mathbb{R} \setminus \mathrm{Spect}(A_N).$$

Now, there exists $\varepsilon' > 0$ such that $F_{\sigma,\nu}(a-\alpha) < F_{\sigma,\nu}(a) - \varepsilon'$ and $F_{\sigma,\nu}(b+\alpha) > F_{\sigma,\nu}(b) + \varepsilon'$. It follows that, for all large N,

$$\lambda_{i_N+1}(A_N) < F_{\sigma,\nu}(a) - \varepsilon' \quad \text{and} \quad \lambda_{i_N}(A_N) > F_{\sigma,\nu}(b) + \varepsilon'. \quad (6.64)$$

Using Lemma 6.6.5, (6.64) and the fact that, almost surely, for all large N,

$$0 < \max(-\lambda_N(X_N), \lambda_1(X_N)) < 3\sigma,$$

we get the following inequalities.
If $i_N < N$, for all large N,

$$\begin{aligned}
\lambda_{i_N+1}(M_N^{(k)}) &\leq \lambda_{i_N+1}(A_N) + \frac{\sigma_k}{\sigma}\lambda_1(X_N) \\
&< F_{\sigma,\nu}(a) - \varepsilon' + \frac{\sigma_k}{\sigma}\lambda_1(X_N) \\
&= a_k - \sigma_k^2 G_{\mu_\sigma \boxplus \nu}(a) + \frac{\sigma_k}{\sigma}\lambda_1(X_N) - \varepsilon' \\
&< a_k - \sigma_k^2 G_{\mu_\sigma \boxplus \nu}(a) + 3\sigma_k - \varepsilon'.
\end{aligned}$$

If $i_N > 0$, for all large N,

$$\begin{aligned}
\lambda_{i_N}(M_N^{(k)}) &\geq \lambda_{i_N}(A_N) + \frac{\sigma_k}{\sigma}\lambda_N(X_N) \\
&> F_{\sigma,\nu}(b) + \varepsilon' + \frac{\sigma_k}{\sigma}\lambda_N(X_N) \\
&= b_k - \sigma_k^2 G_{\mu_\sigma \boxplus \nu}(b) + \frac{\sigma_k}{\sigma}\lambda_N(X_N) + \varepsilon' \\
&> b_k - \sigma_k^2 G_{\mu_\sigma \boxplus \nu}(b) - 3\sigma_k + \varepsilon'.
\end{aligned}$$

As $\sigma_k \to 0$ when $k \to +\infty$, there is K large enough such that for all $k \geq K$,

$$\max(|-\sigma_k^2 G_{\mu_\sigma \boxplus \nu}(a) + 3\sigma_k|, |-\sigma_k^2 G_{\mu_\sigma \boxplus \nu}(b) - 3\sigma_k|) < \varepsilon'$$

and then, almost surely, for all N large enough

$$\lambda_{i_N+1}(M_N^{(k)}) < a_k \text{ if } i_N < N, \tag{6.65}$$

$$\text{and} \quad \lambda_{i_N}(M_N^{(k)}) > b_k \text{ if } i_N > 0. \tag{6.66}$$

Since $\lambda_{N+1}(M_N^{(k)}) = -\lambda_0(M_N^{(k)}) = -\infty$, (6.65) (resp. (6.66)) is obviously satisfied if $i_N = N$ (resp. $i_N = 0$). Thus, we have established that for any $i_N \in \{0, \ldots, N\}$ satisfying (6.57), (6.63) holds for all $k \geq K$ when K is large enough. Comparing this with (6.62), we deduce that $j_N = i_N$ almot surely and

$$\mathbb{P}\big[\lambda_{i_N+1}(M_N) < a \text{ and } \lambda_{i_N}(M_N) > b, \quad \text{for all large } N\big] = 1.$$

This ends the proof of Theorem 6.6.4. □

We readily deduce the following

Corollary 6.6.7. *Let $\varepsilon > 0$. Let us fix u in $\Theta_{\sigma,\nu} \cup \{t_l, l = 1, \ldots, m\}$ (resp. in $\Theta_{\sigma,\nu} \cup \{s_l, l = 1, \ldots, m\}$). Let us choose $\delta > 0$ small enough so that for large N, $[u + \delta; u + 2\delta]$ (resp. $[u - 2\delta; u - \delta]$) is included in $(\mathbb{R} \setminus \overline{U_{\sigma,\nu}}) \cap (\mathbb{R} \setminus \mathrm{Spect}(A_N))$ and for any $0 \leq \delta' \leq 2\delta$, $H_{\sigma,\nu}(u + \delta') - H_{\sigma,\nu}(u) < \varepsilon$ (resp. $H_{\sigma,\nu}(u) - H_{\sigma,\nu}(u - \delta') < \varepsilon$). Let $i_N = i_N(u)$ be such that*

$$\lambda_{i_N+1}(A_N) < u + \delta \text{ and } \lambda_{i_N}(A_N) > u + 2\delta$$

(resp. $\lambda_{i_N+1}(A_N) < u - 2\delta$ and $\lambda_{i_N}(A_N) > u - \delta$). Then

$$\mathbb{P}\big[\lambda_{i_N+1}(M_N) < H_{\sigma,\nu}(u) + \varepsilon \text{ and } \lambda_{i_N}(M_N) > H_{\sigma,\nu}(u), \text{ for all large } N\big] = 1.$$

(resp. $\mathbb{P}\big[\lambda_{i_N+1}(M_N) < H_{\sigma,\nu}(u)$ and $\lambda_{i_N}(M_N) > H_{\sigma,\nu}(u) - \varepsilon$ for large $N\big] = 1$.)

6.7 Convergence of eigenvalues

In the non-spiked case $\Theta = \emptyset$ i.e. $r = 0$, the results of Theorems 6.5.1 and 6.4.1 read as: $\forall \varepsilon > 0$,

$$\mathbb{P}[\mathrm{Spect}(M_N) \subset \mathrm{supp}(\mu_\sigma \boxplus \nu) + (-\varepsilon, \varepsilon), \text{ for all } N \text{ large}] = 1. \tag{6.67}$$

This readily leads to the following asymptotic result for the extremal eigenvalues.

Proposition 6.7.1. *Assume that the deformed model M_N is without spike i.e. $r = 0$. Let $k \geq 0$ be a fixed integer.*
The largest (resp. smallest) eigenvalues $\lambda_{1+k}(M_N)$ (resp. $\lambda_{N-k}(M_N)$) converge almost surely to the right (resp. left) endpoint of the support of $\mu_\sigma \boxplus \nu$.

Proof of Proposition 6.7.1: We here only focus on the convergence of the first largest eigenvalues since the other case is similar. Recalling that $\operatorname{supp}(\mu_\sigma \boxplus \nu) = \cup_{l=m}^{1}[H_{\sigma,\nu}(s_l), H_{\sigma,\nu}(t_l)]$, from (6.67), one has that, for all $\varepsilon > 0$,

$$\mathbb{P}[\limsup_N \lambda_1(M_N) \leq H_{\sigma,\nu}(t_1) + \varepsilon] = 1.$$

But as $H_{\sigma,\nu}(t_1)$ is a boundary point of $\operatorname{supp}(\mu_\sigma \boxplus \nu)$, the number of eigenvalues of M_N falling into $[H_{\sigma,\nu}(t_1) - \varepsilon, H_{\sigma,\nu}(t_1) + \varepsilon]$ tends almost surely to infinity as $N \to \infty$. Thus, almost surely,

$$\liminf_N \lambda_{1+k}(M_N) \geq H_{\sigma,\nu}(t_1) - \varepsilon.$$

The result then follows by letting $\varepsilon \to 0$. □

In the spiked case where $r \geq 1$ ($\Theta \neq \emptyset$), the spectral measure μ_{M_N} still converges almost surely to $\mu_\sigma \boxplus \nu$. We shall study the impact of the spiked eigenvalues θ_i's on the local behavior of some eigenvalues of M_N.
In particular, we shall prove that once the largest spike θ_1 is sufficiently big, the largest eigenvalue of M_N jumps almost surely above the right endpoint $H_{\sigma,\nu}(t_1)$. Once $m \geq 2$, that is when $\operatorname{supp}(\mu_\sigma \boxplus \nu)$ has at least two connected components, we prove that there may also exist some jumps into the gap(s) of this support. This phenomenon holds for any $\theta_j \in \Theta_{\sigma,\nu}$.
For $\theta_j \notin \Theta_{\sigma,\nu}$, that is if $\theta_j \in \overline{U_{\sigma,\nu}}$, two situations may occur. To explain this, let us consider the connected component $[s_{l_j}, t_{l_j}]$ of $\overline{U_{\sigma,\nu}}$ which contains θ_j. If $\operatorname{supp}(\nu) \cap [\theta_j, t_{l_j}] = \emptyset$ (resp. $\operatorname{supp}(\nu) \cap [s_{l_j}, \theta_j] = \emptyset$) then the k_j corresponding eigenvalues of M_N converge almost surely to the corresponding boundary point $H_{\sigma,\nu}(t_{l_j})$ (resp. $H_{\sigma,\nu}(s_{l_j})$) of the support of $\mu_\sigma \boxplus \nu$. Otherwise, namely when θ_j is between two connected components of $\operatorname{supp}(\nu)$ included in $[s_{l_j}, t_{l_j}]$, the convergence occurs towards a point inside the (interior) of $\operatorname{supp}(\mu_\sigma \boxplus \nu)$.
Here is the precise formulation of our result. This is the additive analogue of the main result of [BY08a] on the almost sure convergence of the eigenvalues generated by the spikes in a generalized spiked population model.

Theorem 6.7.2. *For each spiked eigenvalue θ_j, we denote the descending ranks of θ_j among the eigenvalues of A_N by $n_{j-1} + 1, \ldots, n_{j-1} + k_j$.*

1) *If $\theta_j \in \mathbb{R} \setminus \overline{U_{\sigma,\nu}}$ (i.e. $\in \Theta_{\sigma,\nu}$), the k_j eigenvalues $(\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j)$ converge almost surely outside the support of $\mu_\sigma \boxplus \nu$ towards $\rho_{\theta_j} = H_{\sigma,\nu}(\theta_j)$.*

2) If $\theta_j \in \overline{U_{\sigma,\nu}}$ then we let $[s_{l_j}, t_{l_j}]$ (with $1 \leq l_j \leq m$) be the connected component of $\overline{U_{\sigma,\nu}}$ which contains θ_j.

a) If θ_j is on the right (resp. on the left) of any connected component of $\operatorname{supp}(\nu)$ which is included in $[s_{l_j}, t_{l_j}]$ then the k_j eigenvalues $(\lambda_{n_{j-1}+i}(M_N),\ 1 \leq i \leq k_j)$ converge almost surely to $H_{\sigma,\nu}(t_{l_j})$ (resp. $H_{\sigma,\nu}(s_{l_j})$) which is a boundary point of the support of $\mu_\sigma \boxplus \nu$.

b) If θ_j is between two connected components of $\operatorname{supp}(\nu)$ which are included in $[s_{l_j}, t_{l_j}]$ then the k_j eigenvalues $(\lambda_{n_{j-1}+i}(M_N),\ 1 \leq i \leq k_j)$ converge almost surely to the α_j-th quantile of $\mu_\sigma \boxplus \nu$ (that is to q_{α_j} defined by $\alpha_j = (\mu_\sigma \boxplus \nu)(]-\infty, q_{\alpha_j}])$) where α_j is such that $\alpha_j = 1 - \lim_N \frac{n_{j-1}}{N} = \nu(]-\infty, \theta_j])$.

Proof of Theorem 6.7.2: 1) Choosing $u = \theta_j$ in Corollary 6.6.7 gives, for any $\varepsilon > 0$,

$$\rho_{\theta_j} - \varepsilon \leq \lambda_{n_{j-1}+k_j}(M_N) \leq \cdots \leq \lambda_{n_{j-1}+1}(M_N) \leq \rho_{\theta_j} + \varepsilon, \text{ for large } N \tag{6.68}$$

holds almost surely. Hence

$$\forall 1 \leq i \leq k_j, \quad \lambda_{n_{j-1}+i}(M_N) \xrightarrow{a.s.} \rho_{\theta_j}.$$

2) a) We only focus on the case where θ_j is on the right of any connected component of $\operatorname{supp}(\nu)$ which is included in $[s_{l_j}, t_{l_j}]$ since the other case may be considered with similar arguments. Let us consider the set $\{\theta_{j_0} > \ldots > \theta_{j_p}\}$ of all the θ_i's being in $[s_{l_j}, t_{l_j}]$ and on the right of any connected component of $\operatorname{supp}(\nu)$ which is included in $[s_{l_j}, t_{l_j}]$. Note that we have for all large N, for any $0 \leq h \leq p$,

$$n_{j_h - 1} + k_{j_h} = n_{j_h}$$

and θ_{j_0} is the largest eigenvalue of A_N which is lower than t_{l_j}. Let $\varepsilon > 0$. Applying Corollary 6.6.7 with $u = t_{l_j}$, we get that, almost surely,

$$\lambda_{n_{j_0-1}+1}(M_N) < H_{\sigma,\nu}(t_{l_j}) + \varepsilon \text{ and } \lambda_{n_{j_0-1}}(M_N) > H_{\sigma,\nu}(t_{l_j}) \text{ for all large } N.$$

Now, almost surely, the number of eigenvalues of M_N being in the interval $]H_{\sigma,\nu}(t_{l_j}) - \varepsilon, H_{\sigma,\nu}(t_{l_j})]$ should tend to infinity when N goes to infinity. Since almost surely for all large N, $\lambda_{n_{j_0-1}}(M_N) > H_{\sigma,\nu}(t_{l_j})$ and $\lambda_{n_{j_0-1}+1}(M_N) < H_{\sigma,\nu}(t_{l_j}) + \varepsilon$, we should have

$$H_{\sigma,\nu}(t_{l_j}) - \varepsilon \leq \lambda_{n_{j_p-1}+k_{j_p}}(M_N) \leq \ldots \leq \lambda_{n_{j_0-1}+1}(M_N) < H_{\sigma,\nu}(t_{l_j}) + \varepsilon.$$

Hence, we deduce that: $\forall 0 \leq l \leq p$ and $\forall 1 \leq i \leq k_{j_p}$, $\quad \lambda_{n_{j_p-1}+i}(M_N) \xrightarrow{a.s.} H_{\sigma,\nu}(t_{l_j})$. The result then follows since $j \in \{j_0, \ldots, j_p\}$.
b) Let $\alpha_j = 1 - \lim_N \frac{n_{j-1}}{N} = \nu(]-\infty, \theta_j])$. Denote by Q (resp. Q_N) the distribution function of $\mu_\sigma \boxplus \nu$ (resp. of the spectral measure of M_N). Since

$\mu_\sigma \boxplus \nu$ is absolutely continuous, Q is continuous on $\mathbb{R}$ and strictly increasing on each interval $[\Psi_{\sigma,\nu}(s_l), \Psi_{\sigma,\nu}(t_l)], 1 \leq l \leq m$.
From Proposition 6.1.9 and the hypothesis made on θ_j, one has that $\alpha_j \in]Q(\Psi_{\sigma,\nu}(s_{l_j})), Q(\Psi_{\sigma,\nu}(t_{l_j}))[$ and there exists a unique $q_j \in]\Psi_{\sigma,\nu}(s_{l_j}), \Psi_{\sigma,\nu}(t_{l_j})[$ such that $Q(q_j) = \alpha_j$. Moreover, Q is strictly increasing in a neighborhood of q_i.
Let $\varepsilon > 0$. From the almost sure convergence of μ_{M_N} to $\mu_\sigma \boxplus \nu$, we deduce

$$Q_N(q_j + \varepsilon) \underset{N\to\infty}{\to} Q(q_j + \varepsilon) > \alpha_j, \quad \text{a.s.}.$$

From the definition of α_j, it follows that for large N, $N, N-1, \ldots, n_{j-1} + k_j, \ldots, n_{j-1} + 1$ belong to the set $\{k, \lambda_k(M_n) \leq q_j + \varepsilon\}$ and thus,

$$\limsup_{N\to\infty} \lambda_{n_{j-1}+1}(M_N) \leq q_j + \varepsilon.$$

In the same way, since $Q_N(q_j - \varepsilon) \underset{N\to\infty}{\to} Q(q_j - \varepsilon) < \alpha_j$,

$$\liminf_{N\to\infty} \lambda_{n_{j-1}+k_j}(M_N) \geq q_j - \varepsilon.$$

Thus, the k_j eigenvalues $(\lambda_{n_{j-1}+i}(M_N),\ 1 \leq i \leq k_j)$ converge almost surely to q_j. $\square$

6.8 Appendix

We present in this appendix the different estimates on the variance used throughout the paper. They rely on the Poincaré hypothesis on the distribution μ of the entries of the Wigner matrix W_N. We assume that μ satisfies a Poincaré inequality, that is there exists a positive constant C such that for any $\mathcal{C}^\infty$ function $f : \mathbb{R} \to \mathbb{C}$ such that f and f' are in $L^2(\mu)$,

$$\mathbf{V}(f) \leq C \int |f'|^2 d\mu,$$

with $\mathbf{V}(f) = \mathbb{E}(|f - \mathbb{E}(f)|^2)$.
We refer the reader to [BG99] for a characterization of such measures on $\mathbb{R}$.
This inequality translates in the matricial case as follows:
For any matrix M, define $||M||_2 = (\text{Tr}(M^*M))^{\frac{1}{2}}$ the Hilbert-Schmidt norm.
Let $\Psi : (M_N(\mathbb{C})_{sa}) \to \mathbb{R}^{N^2}$ (resp. $\Psi : (M_N(\mathbb{R})_s) \to \mathbb{R}^{N(N+1)/2}$) be the canonical isomorphism which maps a Hermitian (resp. symmetric) matrix M to the real parts and the imaginary parts of its entries (resp. to the entries) $M_{ij}, i \leq j$.

Lemma 6.8.1. *Let M_N be the complex (resp. real) Wigner Deformed matrix introduced at the beginning of the chapter. For any $\mathcal{C}^\infty$ function f :*

$\mathbb{R}^{N^2}$(*resp.* $\mathbb{R}^{N(N+1)/2}$) $\to \mathbb{C}$ *such that* f *and its gradient* $\nabla(f)$ *are both polynomially bounded,*

$$\mathbf{V}[f \circ \Psi(M_N)] \leq \frac{C}{N}\mathbb{E}\{\|\nabla\left[f \circ \Psi(M_N)\right]\|_2^2\}. \tag{6.69}$$

From this Lemma and the properties of the resolvent G (see Lemma 6.0.4), we obtain:

- $\mathbf{V}((G_N(z))_{ij}) \leq \frac{C}{N}P(|\Im z|^{-1})$
- $\mathbf{V}((G_N(z))_{ii}^2) \leq \frac{C}{N}P(|\Im z|^{-1})$
- Let H be a deterministic Hermitian matrix with norm $\|H\|$, then,

$$\mathbf{V}((HG_N(z))_{ii}) \leq \frac{C}{N}\|H\|^2 P(|\Im z|^{-1})$$

- $\mathbf{V}(\mathrm{tr}(G_N(z))) \leq \frac{C}{N^2}P(|\Im z|^{-1})$

where P is a polynomial. It follows that:

$$\mathbb{E}[(U^*G_DUG)_{ii}G_{ii}G_{ll}^2] \quad = \quad \mathbb{E}[(U^*G_DUG)_{ii}]\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2 + \frac{1}{N}P(|\Im z|^{-1}),$$

proving (6.18).

We now prove

Lemma 6.8.2. *Let* $z \in \mathbb{C} \setminus \mathbb{R}$. *Then,*

$$|\mathbb{E}[\tilde{G}_{pk}\mathrm{tr}(G)] - \mathbb{E}[\tilde{G}_{pk}]\mathbb{E}[\mathrm{tr}(G)]| \leq \frac{P(|\Im z|^{-1})}{N^2}.$$

Proof: The cumulant expansion gives

$$z\mathbb{E}(G_{ji}) = \sigma^2\mathbb{E}(\mathrm{tr}(G)G_{ji}) + \delta_{ij} + \mathbb{E}[(GA_N)_{ji}] + \frac{\kappa_4}{2N^2}\mathbb{E}[T(i,j)] + O_{ji}(\frac{1}{N^2}),$$

where

$$\begin{aligned} T(i,j) \quad = \quad & \frac{1}{3}\left\{\frac{1}{\sqrt{2}}\sum_{l<i}\left(G_{jl}^{(3)}.(e_{li},e_{li},e_{li}) + \sqrt{-1}G_{jl}^{(3)}.(f_{li},f_{li},f_{li})\right)\right. \\ & + \frac{1}{\sqrt{2}}\sum_{l>i}\left(G_{jl}^{(3)}.(e_{il},e_{il},e_{il}) - \sqrt{-1}G_{jl}^{(3)}.(f_{il},f_{il},f_{il})\right) \\ & \left. + G_{jl}^{(3)}.(E_{ii},E_{ii},E_{ii})\right\}. \end{aligned}$$

Straightforward computations give that

$$\begin{aligned} T(i,j) = \quad & \textstyle\sum_l G_{jl}G_{li}^3 + \sum_l G_{ji}G_{il}G_{li}G_{ll} \\ & + \textstyle\sum_l G_{jl}G_{ii}G_{li}G_{ll} + \sum_l G_{ji}G_{ii}G_{ll}^2 - 2G_{ii}^3G_{ji}. \end{aligned}$$

We now compute the sum $\sum U^*_{ik}U_{pj}\dots$ to obtain:

$$\begin{aligned}(z-\gamma_k)\mathbb{E}[\tilde{G}_{pk}] = \quad & \sigma^2\mathbb{E}[\mathrm{tr}(G)\tilde{G}_{pk}] + \delta_{pk} + \tfrac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p,k)] \\ & -\tfrac{\kappa_4}{N^2}\textstyle\sum_{i,j} U^*_{ik}U_{pj}\mathbb{E}[G^3_{ii}G_{ji}] + \sum_{i,j} U^*_{ik}U_{pj}O_{ji}(\tfrac{1}{N^2}) \qquad (6.70)\end{aligned}$$

where

$$\tilde{A}(p,k) = \sum_{i,j} U^*_{ik}U_{pj}A(i,j)$$

and

$$\begin{aligned}A(i,j) = \quad & \textstyle\sum_l G_{jl}G^3_{li} + \sum_l G_{ji}G_{il}G_{li}G_{ll} \\ & + \textstyle\sum_l G_{jl}G_{ii}G_{li}G_{ll} + \sum_l G_{ji}G_{ii}G^2_{ll}.\end{aligned}$$

Since $\frac{\kappa_4}{N^2}\sum_{i,j} U^*_{ik}U_{pj}G^3_{ii}G_{ji} = \frac{\kappa_4}{N^2}(UG(G^{(d)})^3U^*)_{pk}$, this term is obviously a $O(\frac{1}{N^2})$.
Let us verify the following bound for $\tilde{A}$:

$$|\frac{1}{N^2}\tilde{A}(p,k)| \leq C\frac{|\Im z|^{-4}}{N}. \qquad (6.71)$$

Such a bound for the first term in the decomposition of A can be readily deduced from (6.1). We write the computation for the fourth term in the decomposition of A, the other two terms are similar:

$$\begin{aligned}&\frac{1}{N^2}\sum_{i,j,l} U^*_{ik}U_{pj}G_{ji}G_{ii}G^2_{ll} \\ &\quad = \frac{1}{N^2}\sum_l (UGG^{(d)}U^*)_{pk}G^2_{ll} = O(\frac{1}{N}).\end{aligned}$$

We prove now that the last term in (6.70) is of order $O(\frac{1}{N^2})$. This term is a linear combination of terms of the form:

$$\frac{\kappa_6}{N^3}\sum_{i,j,l} U^*_{ik}U_{pj}\mathbb{E}[G^{(5)}_{jl}.(v_1,\dots,v_5)],$$

where $v_u = E_{mn}$ with $(m,n)=(i,l)$ or $(m,n)=(l,i)$. The fifth derivative is a product of six G. If there are G^2_{il} or $G_{il}G_{li}$ in the product, we can conclude thanks to Lemma 6.0.4. The only term without any G_{il} is

$$G_{ji}G_{ll}G_{ii}G_{ll}G_{ii}G_{ll}$$

which gives the contribution

$$\frac{1}{N^3}\sum_l (UG(G^{(d)})^2U^*)_{pk}G^3_{ll} = O(\frac{1}{N^2}).$$

The term with one G_{il} (or G_{li}) will also give a contribution in $\frac{1}{N^2}$. Hence

$$(z-\gamma_k)\mathbb{E}[\tilde{G}_{pk}] = \sigma^2\mathbb{E}[\mathrm{tr}(G)\tilde{G}_{pk}] + \delta_{pk} + \frac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p,k)] + O(\frac{1}{N^2}). \quad (6.72)$$

We now apply (6.8) (or its extension (6.9)) to $\Phi(X_N) = G_{jl}G_{qq}$ and $H = E_{il}$ and take the sum in l. We obtain

$$\begin{aligned} z\mathbb{E}(G_{ji}G_{qq}) = \quad & \sigma^2\mathbb{E}(\mathrm{tr}(G)G_{ji}G_{qq}) + \tfrac{\sigma^2}{N}\mathbb{E}[G_{qi}(G^2)_{jq}] + \mathbb{E}[G_{qq}\delta_{ij}] \\ & +\mathbb{E}[(GA_N)_{ji}G_{qq}] + \tfrac{\kappa_4}{2N^2}\mathbb{E}[T(i,j)G_{qq}] \\ & +\tfrac{\kappa_4}{2N^2}\mathbb{E}[B(i,j,q)] + O_{j,i}(\tfrac{1}{N^2}), \end{aligned}$$

where $B(i,j,q)$ stands for all the terms coming from the third derivative of the product $(G_{jl}G_{qq})$ except $G_{qq}G^{(3)}_{jl}$. Now, we consider $\frac{1}{N}\sum_q$ of the above equalities to obtain:

$$\begin{aligned} z\mathbb{E}(G_{ji}\mathrm{tr}(G)) = \quad & \sigma^2\mathbb{E}(\mathrm{tr}(G)^2G_{ji}) + \tfrac{\sigma^2}{N^2}\mathbb{E}[(G^3)_{ji}] + \mathbb{E}[\mathrm{tr}(G)\delta_{ij}] \\ & +\mathbb{E}[(GA_N)_{ji}\mathrm{tr}(G)] + \tfrac{\kappa_4}{2N^2}\mathbb{E}[T(i,j)\mathrm{tr}(G)] \\ & +\tfrac{\kappa_4}{2N^2}\tfrac{1}{N}\textstyle\sum_q \mathbb{E}[B(i,j,q)] + O_{j,i}(\tfrac{1}{N^2}). \end{aligned}$$

We now compute the sum $\sum U^*_{ik}U_{pj}\ldots$ and obtain

$$\begin{aligned} (z-\gamma_k)\mathbb{E}(\tilde{G}_{pk}\mathrm{tr}(G)) = \quad & \sigma^2\mathbb{E}(\mathrm{tr}(G)^2\tilde{G}_{pk}) + \tfrac{\sigma^2}{N^2}\mathbb{E}[(UG^3U^*)_{pk}] \\ & +\mathbb{E}[\mathrm{tr}(G)\delta_{pk}] + \tfrac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p,k)\mathrm{tr}(G)] \\ & +\tfrac{\kappa_4}{2N^2}\tfrac{1}{N}\textstyle\sum_q \mathbb{E}[\tilde{B}(p,k,q)] + O(\tfrac{1}{N^2}), \end{aligned}$$

where

$$\tilde{B}(p,k,q) = \sum U^*_{ik}U_{pj}B(i,j,q)$$

and we notice that the terms $\frac{\kappa_4}{2N^2}\sum U^*_{ik}U_{pj}\mathbb{E}[(T(i,j)-A(i,j))\mathrm{tr}(G)]$ and $\sum U^*_{ik}U_{pj}O_{j,i}(\frac{1}{N^2})$ remain a $O(\frac{1}{N^2})$ by the same arguments used to handle the analogue terms in (6.70).

Now, consider the difference between the above equation and $g_N(z)\times$(6.70):

$$\begin{aligned} & (z-\gamma_k)\mathbb{E}[(\tilde{G}_{pk}(\mathrm{tr}(G)-\mathbb{E}[\mathrm{tr}(G)])] = \\ & \frac{\sigma^2}{N^2}\mathbb{E}[(UG^3U^*)_{pk}] + \sigma^2\mathbb{E}[\mathrm{tr}(G)(\mathrm{tr}(G)-\mathbb{E}[\mathrm{tr}(G)])\tilde{G}_{pk}] \\ & +\frac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p,k)(\mathrm{tr}(G)-\mathbb{E}[\mathrm{tr}(G)])] \\ & +\frac{\kappa_4}{2N^2}\frac{1}{N}\sum_q \mathbb{E}[\tilde{B}(p,k,q)] + O(\frac{1}{N^2}) \end{aligned}$$

and

$$(z-\gamma_k-\sigma^2 g_N(z))\mathbb{E}[\tilde{G}_{pk}(\mathrm{tr}(G)-\mathbb{E}[\mathrm{tr}(G)])] =$$

$$\sigma^2\mathbb{E}[(\mathrm{tr}(G)-\mathbb{E}[\mathrm{tr}(G)])^2\tilde{G}_{pk}]+\frac{\sigma^2}{N^2}\mathbb{E}[(UG^3U^*)_{pk}]$$
$$+\frac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p,k)(\mathrm{tr}(G)-\mathbb{E}[\mathrm{tr}(G)])]$$
$$+\frac{\kappa_4}{2N^2}\frac{1}{N}\sum_q \mathbb{E}[\tilde{B}(p,k,q)]+O(\frac{1}{N^2}).$$

We now prove that the right-hand side of the above equation is of order $\frac{1}{N^2}$. This is obvious for the second and first term (since $\mathbf{V}(\mathrm{tr}(G_N(z)))=O(\frac{1}{N^2})$). Now, we have seen that

$$\frac{1}{N^2}\tilde{A}(p,k)\le\frac{C|\Im z|^{-4}}{N}.$$

By Cauchy-Schwarz inequality,

$$\frac{1}{N^2}\mathbb{E}[\tilde{A}(p,k)(\mathrm{tr}(G)-\mathbb{E}[\mathrm{tr}(G)])]=O(\frac{1}{N^2}).$$

It remains to study the last term

$$\frac{1}{N^3}\sum_q \mathbb{E}[\tilde{B}(p,k,q)]=\frac{1}{N^3}\sum_{i,j,q}U^*_{ik}U_{pj}\mathbb{E}[B(i,j,q)].$$

This term contains derivatives of G_{qq} of order a with *a strictly positive* ($a=1,2,3$) applied to a 3-tuple (v_1,v_2,v_3) where $v_u=E_{il}$ or E_{li} (with a product of the derivative of order $3-a$ of G_{jl}). Thus, the index q appears in $\tilde{B}(p,k,q)$ under the form of a product $G_{qm}G_{nq}$ with $m,n\in\{i,l\}$. Thus, the sum in q will give G^2_{nm}. Moreover, the term in j in the derivative appears as G_{jm} with $m\in\{i,l\}$ and we can do the sum in j to obtain $(UG)_{pm}$. Thus, $\frac{1}{N^3}\sum_q\tilde{B}(p,k,q)$ can be written as $\frac{1}{N^3}\sum_{i,l}$ of terms of the form

$$U^*_{ik}(G^2)_{i_1j_1}(UG)_{pj_2}G_{i_3j_3}G_{i_4j_4},$$

where $i_r,j_r\in\{i,l\}$ and $j_2=l$ for $a=3$ (no derivative in G_{jl}), $j_4=l$ for $a<3$. As in the previous computations, either the product G^2_{il} (or $G_{il}G_{li}$) appears and we can apply Lemma 6.0.4 (the others terms are bounded). In the other cases, we can always perform one sum in i (or l) and obtain $\frac{1}{N^3}\sum_{l(\text{ or } i)}$ of bounded terms. Let us just give an example of terms which can be obtained (for $a=1$):

$$U^*_{ik}(G^2)_{li}(UG)_{pl}G_{ii}G_{ll}.$$

Then,

$$\frac{1}{N^3}\sum_{i,l}U^*_{ik}(G^2)_{li}(UG)_{pl}G_{ii}G_{ll}=\frac{1}{N^3}\sum_i U^*_{ik}(UGG^{(d)}G^2)_{pi}G_{ii}.$$

Therefore, $\frac{1}{N^3}\sum_q\mathbb{E}[\tilde{B}(p,k,q)]$ is of order $\frac{1}{N^2}$. This proves Lemma 6.8.2 since $|\frac{1}{z-\gamma_k-\sigma^2 g_N(z)}|\le|\Im z|^{-1}$. □

Chapter 7

Largest eigenvalues of deformations of Wishart matrices

This chapter is a joint work in progress with M. Capitaine.

We consider the following model :

$$M_N(p(N)) = \frac{1}{N} X_N + A_N$$

- X_N is a white complex Wishart matrix of size N with $p(N)$ degrees of freedom.
- $\lim_{N\to+\infty} \frac{p(N)}{N} = \alpha > 0$.
- A_N is a deterministic diagonal Hermitian matrix. To simplify, we denote its eigenvalues $\gamma_i^{(N)}$ by γ_i. We assume that the spectral measure of A_N defined by $\mu_{A_N} := \frac{1}{N}\sum_{i=1}^N \delta_{\gamma_i}$ converges to some probability measure ν with compact support and $\sup_N \sup_{i=1,\dots N} |\gamma_i^{(N)}| \leq M$ for some constant M. We assume that A_N has a number J of fixed eigenvalues $\theta_1 > \dots > \theta_J$ which are independent of N, each θ_j having a fixed multiplicity k_j, $\sum_j k_j = r$ and A_N has $N-r$ eigenvalues $\beta_i(N)$ such that
$$\max_{i=1}^{N-r} \operatorname{dist}(\beta_i(N), \operatorname{supp}\nu) \underset{N\to+\infty}{\longrightarrow} 0$$
when N goes to infinity.

Throughout this chapter, we will use the following notations:

- G_N denotes the resolvent of $M_N(p(N))$

- g_N the mean of the Stieltjes transform of the spectral measure of $M_N(p(N))$, that is

$$g_N(z) = \mathbb{E}(\mathrm{tr} G_N(z)),\ z \in \mathbb{C} \setminus \mathbb{R}.$$

- $\pi_{\gamma,\sigma}$ denotes the Marchenko-Pastur distribution of parameters (γ, σ) defined by (1.3).
- $\tilde{g}_N$ denotes the Stieltjes transform of the probability measure $\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N}$.
- When we state that some quantity $\Delta_N(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, is $O(\frac{1}{N^p})$, this means precisely that:

$$|\Delta_N(z)| \leq (|z| + K)^a \frac{P(|\Im z|^{-1})}{N^p}$$

for some $K \geq 0$, $a > 0$ and some polynomial P with nonnegative coefficients.

- For any set S in $\mathbb{R}$, we denote the set $\{x \in \mathbb{R}, \mathrm{dist}(x, S) \leq \varepsilon\}$ (resp. $\{x \in \mathbb{R}, \mathrm{dist}(x, S) < \varepsilon\}$) by $S + [-\varepsilon, +\varepsilon]$ (resp. $S + (-\varepsilon, +\varepsilon)$).

7.1 Basic tools

We recall some useful properties of the resolvent (see [KKP96], [CDM07]).

Lemma 7.1.1. *For a $N \times N$ Hermitian or symmetric matrix M, for any $z \in \mathbb{C} \setminus \mathrm{Spect}(M)$, we denote by $G(z) := (zI_N - M)^{-1}$ the resolvent of M. Let $z \in \mathbb{C} \setminus \mathbb{R}$,*

(i) $\|G(z)\| \leq |\Im z|^{-1}$ *where* $\|.\|$ *denotes the operator norm.*

(ii) $|G(z)_{ij}| \leq |\Im z|^{-1}$ *for all* $i, j = 1, \dots N$.

(iii) For $p \geq 2$,

$$\frac{1}{N} \sum_{i,j=1}^{N} |G(z)_{ij}|^p \leq (|\Im z|^{-1})^p. \tag{7.1}$$

(iv) The derivative with respect to M of the resolvent $G(z)$ satisfies:

$$G'_M(z).B = G(z)BG(z) \quad \text{for any matrix } B.$$

(v) Let $z \in \mathbb{C}$ *such that* $|z| > \|M\|$; *we have*

$$\|G(z)\| \leq \frac{1}{|z| - \|M\|}.$$

7.2 Subordination

The subordination phenomenon for the free additive convolution of probability measures may be precised when one of the measures is freely infinitely divisible. Indeed, under this additional assumption, the subordination map is one-to-one :

Theorem 7.2.1. *[Bia97a] Let μ and ν be two probability measures on $\mathbb{R}$, μ being freely infinitely divisible, the subordination map of $\mu \boxplus \nu$ with respect to ν is*

$$\omega_{\nu,\mu}(z) = z - R_\mu(G_{\mu\boxplus\nu}(z)), \forall z \in \mathbb{C}^+,$$

and is a conformal bijection from $\mathbb{C}^+$ onto a simply connected domain $\Omega_{\nu,\mu} \subseteq \mathbb{C}^+$.

Moreover, the inverse map of $\omega_{\nu,\mu}$ is the restriction to $\Omega_{\nu,\mu}$ of the analytic map $H_{\nu,\mu} : \mathbb{C}^+ \longrightarrow \mathbb{C}$ defined by :

$$\forall z \in \mathbb{C}^+, H_{\nu,\mu}(z) := z + R_\mu(G_\nu(z)).$$

Since $\Im H_{\nu,\mu}(z) \leq \Im z$ for $z \in \mathbb{C}^+$ and $\frac{H_{\nu,\mu}(iy)}{iy} \underset{y\to+\infty}{\longrightarrow} 1$, it is proved in [BB05] that

$$\Omega_{\nu,\mu} = H_{\nu,\mu}^{-1}(\mathbb{C}^+).$$

The paper [BB05] also provides the study of the behavior of the subordination map on the boundary of the upper half-plane :

Theorem 7.2.2. *[BB05] Let μ and ν be two probability measures on $\mathbb{R}$, μ being freely infinitely divisible, the subordination map $\omega_{\nu,\mu}$ of $\mu \boxplus \nu$ with respect to ν has a continuous extension to $\mathbb{R}$ which yields a bijection from $\mathbb{C}^+ \cup \mathbb{R}$ onto $\overline{\Omega_{\nu,\mu}} \subseteq \mathbb{C}^+ \cup \mathbb{R}$. Moreover,*

- *A point $z \in \mathbb{C}^+$ belongs to $\partial\Omega_{\nu,\mu}$ if and only if $H_{\nu,\mu}(z) \in \mathbb{R}$.*
- *A point $x \in \mathbb{R}$ belongs to $\partial\Omega_{\nu,\mu}$ if and only if the limit*

$$H_{\nu,\mu}(x) := \lim_{y\downarrow 0} H_{\nu,\mu}(x+iy)$$

exists in $\mathbb{R}$ and

$$H'_{\nu,\mu}(x) := \lim_{y\downarrow 0} \frac{H_{\nu,\mu}(x+iy) - H_{\nu,\mu}(x)}{iy} \in [0;1[.$$

When applied to the free additive convolution of the Marchenko-Pastur distribution $\pi_{\gamma,1}$ with the probability measure τ, the subordination map of $\pi_{\gamma,1} \boxplus \tau$ with respect to τ writes :

$$\omega_{\tau,\gamma}(z) = z - \frac{\gamma}{1 - G_{\pi_{\gamma,1}\boxplus\tau}(z)}.$$

Hence, we get the following subordination equation :

$$\forall z \in \mathbb{C}^+, \ G_{\pi_{\gamma,1} \boxplus \tau}(z) = G_\tau \left(z - \frac{\gamma}{1 - G_{\pi_{\gamma,1} \boxplus \tau}(z)} \right). \tag{7.2}$$

In particular,

$$\forall z \in \mathbb{C}^+, \ \tilde{g}_N(z) = G_{\mu_{A_N}} \left(z - \frac{p(N)}{N} \frac{1}{1 - \tilde{g}_N(z)} \right). \tag{7.3}$$

Moreover, we will denote by $\Omega_{\tau,\gamma}$ the range of $\omega_{\tau,\gamma}$ and by

$$H_{\tau,\gamma}(z) := z + \frac{\gamma}{1 - G_\tau(z)}$$

the inverse of $\omega_{\tau,\gamma}$.

7.3 Approximate subordination equation for g_N

We look for an approximate equation for $g_N(z)$ of the form (7.3). We shall rely on the following integration by parts formula established in [Kon09].

Theorem 7.3.1. *Let B be a $p \times N$ complex Gaussian random matrix*

$$B = (b_{ij})_{i=1 \ldots p, j=1 \ldots N}$$

with independent entries of variance 1, that is such that the real random variables $\Re(b_{ij}), \Im(b_{ij}), 1 \leq i \leq p, 1 \leq j \leq N$ form a family of $2pN$ independent $\mathcal{N}(0, \frac{1}{2})$ random variables.
Assume that $\Phi := \Phi(B^*B) = (\Phi_{i,j})_{i,j=1,\ldots,N}$ is a $N \times N$ complex random matrix such that each Φ_{ij} is a differentiable function of B through $X_N = B^*B$ and satisfies the following conditions for $i_1, i_2, i_3, i_4 = 1, \ldots, N$, $j_1, j_2 = 1, \ldots, p$:

$$\mathbb{E}\left\{ |b_{j_1 i_1}^2 \Phi_{i_2 i_3}| \right\} < \infty,$$

$$\mathbb{E}\left\{ |b_{j_1 i_1} \frac{\partial \Phi_{i_2 i_3}}{\partial b_{j_2 i_4}}| \right\} < \infty.$$

Then,

$$\mathbb{E}\left[\mathrm{Tr}(X_N \Phi) \right] = p \mathbb{E}\left[\mathrm{Tr}\Phi \right] + \mathbb{E}\left[\mathrm{Tr}(B^T \nabla_B \Phi^T) \right], \tag{7.4}$$

where

$$\nabla_B = \left(\frac{\partial}{\partial b_{ij}} \right) = \left(\frac{1}{2} \frac{\partial}{\partial (\Re b_{ij})} - \frac{\sqrt{-1}}{2} \frac{\partial}{\partial (\Im b_{ij})} \right)_{i=1,\ldots,p, j=1,\ldots,N}.$$

To simplify the notations here, we will denote by G (instead of G_N) the resolvent of $M_N(p(N))$:

$$\begin{aligned} G(z) &:= (zI_N - M_N(p(N)))^{-1} \\ &= (zI_N - X_N - A_N)^{-1} \\ &= (zI_N - \frac{1}{N}B^*B - A_N)^{-1}. \end{aligned}$$

Denote by e_{ij} (resp. E_{ij}) the $p \times N$ matrix (resp. $N \times N$ matrix) such that $(e_{ij})_{lq}$ (resp. $(E_{ij})_{lq}$)$= \delta_{il}\delta_{jq}$. Note that

$$\begin{aligned} \frac{\partial G_{ql}}{\partial b_{ik}} &= \frac{1}{2N}\left\{G\left(B^*\frac{\partial B}{\partial(\Re b_{ik})} + \frac{\partial B^*}{\partial(\Re b_{ik})}B\right)G\right\}_{ql} \\ &\quad -\frac{\sqrt{-1}}{2N}\left\{G\left(B^*\frac{\partial B}{\partial(\Im b_{ik})} + \frac{\partial B^*}{\partial(\Im b_{ik})}B\right)G\right\}_{ql} \\ &= \frac{1}{2N}\left\{G\left(B^*e_{ik} + e_{ik}^*B\right)G\right\}_{ql} \\ &\quad -\frac{\sqrt{-1}}{2N}\left\{G\left(B^*\sqrt{-1}e_{ik} - \sqrt{-1}e_{ik}^*B\right)G\right\}_{ql} \\ &= \frac{1}{N}\left\{GB^*e_{ik}G\right\}_{ql}. \end{aligned}$$

Hence applying Theorem 7.3.1 with $\Phi = GE_{kk}$ one can readily get the following formula

$$\mathbb{E}((X_NG)_{kk}) = \mathbb{E}(G_{kk}\mathrm{tr}(GX_N)) + \frac{p(N)}{N}\mathbb{E}(G_{kk}). \tag{7.5}$$

Now since G satisfies $X_NG = -I - A_NG + zG$, we deduce from (7.5) that

$$\mathbb{E}(G_{kk})\left\{\frac{p(N)}{N} - z + \gamma_k + \mathbb{E}(\mathrm{tr}(GX_N))\right\} = -1 - \Delta_k, \tag{7.6}$$

where

$$\Delta_k = \mathbb{E}\left[\mathrm{tr}(GX_N)(G_{kk} - \mathbb{E}(G_{kk}))\right].$$

Now, taking the sum in k in (7.5) we get

$$\mathbb{E}(\mathrm{tr}(GX_N)\mathrm{tr}G) - \mathbb{E}(\mathrm{tr}(GX_N)) + \frac{p(N)}{N}g_N(z) = 0. \tag{7.7}$$

Since, according to Lemma 7.8.4,

$$\mathbb{E}(\mathrm{tr}(GX_N)\mathrm{tr}G) = \mathbb{E}(\mathrm{tr}(GX_N))g_N(z) + O(\frac{1}{N^2}),$$

it follows from (7.7) that

$$\mathbb{E}(\mathrm{tr}(GX_N))(1 - g_N(z)) = \frac{p(N)}{N}g_N(z) + O(\frac{1}{N^2})$$

and then, using Lemma 7.8.3, that

$$\mathbb{E}(\operatorname{tr}(GX_N)) = \frac{p(N)}{N}\frac{g_N(z)}{1-g_N(z)} + O(\frac{1}{N^2}). \tag{7.8}$$

Plugging (7.8) into (7.6), we get

$$\mathbb{E}(G_{kk})\left\{\frac{p(N)}{N} - z + \gamma_k + \frac{p(N)}{N}\frac{g_N(z)}{1-g_N(z)}\right\} = (-1-\Delta_k) + O(\frac{1}{N^2}) \tag{7.9}$$

and then

$$\mathbb{E}(G_{kk})\left\{z - \gamma_k - \frac{p(N)}{N}\frac{1}{1-g_N(z)}\right\} = (1+\Delta_k) + O(\frac{1}{N^2}).$$

Now, for any $z \in \mathbb{C}^+$, $\Im\frac{1}{1-g_N(z)} < 0$ so that

$$\Im\left\{z - \gamma_k - \frac{p(N)}{N}\frac{1}{1-g_N(z)}\right\} > \Im z > 0.$$

Hence

$$\mathbb{E}(G_{kk}) = (1+\Delta_k)\frac{1}{z-\gamma_k-\frac{p(N)}{N}\frac{1}{1-g_N(z)}} + O(\frac{1}{N^2}).$$

Summing on k and dividing by N, we deduce that

$$g_N(z) = G_{\mu_{A_N}}\left(z - \frac{p(N)}{N}\frac{1}{1-g_N(z)}\right) + \Gamma_N + O(\frac{1}{N^2}),$$

where

$$\Gamma_N = \frac{1}{N}\sum_{k=1}^{N}\frac{\Delta_k}{z-\gamma_k-\frac{p(N)}{N}\frac{1}{1-g_N(z)}}.$$

Lemma 7.3.2.

$$\Gamma_N = O(\frac{1}{N^2}).$$

Proof of Lemma 7.3.2 Applying Theorem 7.3.1 with $\Phi = GG_{kk}$ we readily get the following formula

$$\begin{aligned}\mathbb{E}\left(\operatorname{tr}(X_N G)G_{kk}\right) &= \mathbb{E}\left(\operatorname{tr}(X_N G)\operatorname{tr}(G)G_{kk}\right)\\ &+ \frac{p(N)}{N}\mathbb{E}\left(\operatorname{tr}(G)G_{kk}\right)\\ &+ \frac{1}{N^2}\mathbb{E}\left((GX_N G^2)_{kk}\right).\end{aligned} \tag{7.10}$$

Now (7.10) - $\mathbb{E}(G_{kk})\times$ (7.7) gives

$$\begin{aligned}\mathbb{E}\left(\operatorname{tr}(X_N G)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right) &= \mathbb{E}\left(\operatorname{tr}(X_N G)\operatorname{tr}(G)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right)\\ &+\frac{p(N)}{N}\mathbb{E}\left(\operatorname{tr}(G)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right)\\ &+\frac{1}{N^2}\mathbb{E}\left((GX_NG^2)_{kk}\right).\end{aligned}$$

Thus,

$$\begin{gathered}\mathbb{E}\left(\operatorname{tr}(X_N G)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right) =\\ \mathbb{E}\left(\operatorname{tr}(X_N G)\left(\operatorname{tr}(G)-\mathbb{E}(\operatorname{tr}(G))\right)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right)\\ +\frac{p(N)}{N}\mathbb{E}\left(\left(\operatorname{tr}(G)-\mathbb{E}(\operatorname{tr}(G))\right)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right)\\ +g_N(z)\mathbb{E}\left(\operatorname{tr}(X_N G)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right)\\ +\frac{1}{N^2}\mathbb{E}\left((GX_NG^2)_{kk}\right)\end{gathered}$$

and

$$\begin{gathered}(1-g_N(z))\mathbb{E}\left[\operatorname{tr}(X_N G)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right] =\\ \mathbb{E}\left[\left(\operatorname{tr}(X_N G)-\mathbb{E}(\operatorname{tr}(X_N G))\right)\left(\operatorname{tr}(G)-\mathbb{E}(\operatorname{tr}(G))\right)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right]\\ \frac{p(N)}{N}\mathbb{E}\left[\left(\operatorname{tr}(G)-\mathbb{E}(\operatorname{tr}(G))\right)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right]\\ \mathbb{E}[\operatorname{tr}(X_N G)]\mathbb{E}\left(\left[\left(\operatorname{tr}(G)-\mathbb{E}(\operatorname{tr}(G))\right)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right]\right.\\ \frac{1}{N^2}\mathbb{E}\left[(GX_NG^2)_{kk})\right].\end{gathered}$$

Now using Lemma 7.1.1 (ii) and then Cauchy Schwarz inequality and Lemma 7.8.4 one can easily see that

$$\mathbb{E}\left[\left(\operatorname{tr}(X_N G)-\mathbb{E}(\operatorname{tr}(X_N G))\right)\left(\operatorname{tr}(G)-\mathbb{E}(\operatorname{tr}(G))\right)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right] =$$

$$O(\frac{1}{N^2}).$$

Moreover, using (7.8) and Lemma 7.1.1, one can see that

$$\mathbb{E}[\operatorname{tr}(X_N G)]\mathbb{E}\left[\left(\operatorname{tr}(G)-\mathbb{E}(\operatorname{tr}(G))\right)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right]$$

$$=\frac{p(N)}{N}\frac{1}{1-g_N(z)}\mathbb{E}\left[\left(\operatorname{tr}(G)-\mathbb{E}(\operatorname{tr}(G))\right)\left(G_{kk}-\mathbb{E}(G_{kk})\right)\right]+O(\frac{1}{N^2}).$$

Finally, using Lemma 7.1.1,

$$\begin{aligned}\frac{1}{N^2}\mathbb{E}\left[(GXG^2)_{kk})\right] &\leq \frac{1}{N^2}\frac{1}{|\Im z|^3}\mathbb{E}\left[\lambda_1(M_N))\right]\\ &\leq \frac{1}{N^2}\frac{1}{|\Im z|^3}\left\{\mathbb{E}\left[\lambda_1(X_N))\right]+\|A_N\|\right\}\end{aligned}$$

According to Lemma 7.8.1, $\mathbb{E}[\lambda_1(X_N)]$ is bounded independently of N. $\|A_N\|$ is also obviously bounded independently of N according to the assumptions on A_N. Thus

$$\frac{1}{N^2}\mathbb{E}\left[(GX_NG^2)_{kk})\right] = O(\frac{1}{N^2}).$$

It follows that

$$\Delta_k = \frac{p(N)}{N}\frac{1}{(1-g_N(z))^2}\mathbb{E}\left[(\mathrm{tr}(G) - \mathbb{E}(\mathrm{tr}(G)))\,(G_{kk} - \mathbb{E}(G_{kk}))\right] + O(\frac{1}{N^2}). \tag{7.11}$$

Let us set $D_N = \mathrm{diag}(d_1, \ldots, d_N)$, where $d_k = \frac{1}{z-\gamma_k-\frac{p(N)}{N}\frac{1}{1-g_N(z)}}$.
We can deduce from (7.11) that Γ_N is equal to

$$\frac{p(N)}{N(1-g_N(z))^2}\mathbb{E}\left[(\mathrm{tr}(G) - \mathbb{E}(\mathrm{tr}(G)))\,(\mathrm{tr}(D_NG) - \mathbb{E}(\mathrm{tr}(D_NG)))\right]$$

$$+O(\frac{1}{N^2}).$$

It follows that $\Gamma_N = O(\frac{1}{N^2})$ using Lemma 7.8.4 and Lemma 7.8.3. □
Thus, we can deduce the following

Proposition 7.3.3. *For $z \in \mathbb{C}^+$, $g_N(z)$ satisfies:*

$$g_N(z) = G_{\mu_{A_N}}\left(z - \frac{p(N)}{N}\frac{1}{1-g_N(z)}\right) + O(\frac{1}{N^2}). \tag{7.12}$$

7.4 Estimation of $g_N - \tilde{g}_N$

Proposition 7.4.1. $\forall z \in \mathbb{C}^+$,

$$g_N(z) - \tilde{g}_N(z) = O(\frac{1}{N^2}). \tag{7.13}$$

Proof of proposition 7.4.1: For a fixed $z \in \mathbb{C}^+$, one may write the subordination equation (7.3) :

$$\tilde{g}_N(z) = G_{\mu_{A_N}}\left(z - \frac{p(N)}{N}\frac{1}{1-\tilde{g}_N(z)}\right),$$

and the approximative matricial subordination equation (7.12) :

$$g_N(z) = G_{\mu_{A_N}}\left(z - \frac{p(N)}{N}\frac{1}{1-g_N(z)}\right) + O(\frac{1}{N^2}).$$

The main idea is to simplify the difference $g_N(z) - \tilde{g}_N(z)$ by introducing a complex number z' likely to satisfy

$$\omega_{\mu_{A_N},\frac{p(N)}{N}}(z') = z' - \frac{p(N)}{N}\frac{1}{1-\tilde{g}_N(z')} = z - \frac{p(N)}{N}\frac{1}{1-g_N(z)}. \tag{7.14}$$

We know by Theorem 7.2.1 that $\omega_{\mu_{A_N},\frac{p(N)}{N}}$ is a homeomorphism from $\mathbb{C}^+$ onto $\Omega_{\mu_{A_N},\frac{p(N)}{N}}$ whose inverse $H_{\mu_{A_N},\frac{p(N)}{N}}$ has an analytic continuation to the whole upper half-plane $\mathbb{C}^+$.
Since $z - \frac{p(N)}{N}\frac{1}{1-g_N(z)} \in \mathbb{C}^+$, $z' \in \mathbb{C}$ is well-defined by the formula :

$$z' := H_{\mu_{A_N},\frac{p(N)}{N}}(z - \frac{p(N)}{N}\frac{1}{1-g_N(z)}).$$

One has

$$\begin{aligned} z' - z &= \frac{p(N)}{N}\frac{1}{1 - G_{\mu_{A_N}}(z - \frac{p(N)}{N}\frac{1}{1-g_N(z)})} - \frac{p(N)}{N}\frac{1}{1-g_N(z)} \\ &= -\frac{p(N)}{N}\frac{g_N(z) - G_{\mu_{A_N}}\left(z - \frac{p(N)}{N}\frac{1}{1-g_N(z)}\right)}{(1 - G_{\mu_{A_N}}(z - \frac{p(N)}{N}\frac{1}{1-g_N(z)}))(1-g_N(z))} \end{aligned}$$

By (7.12), Lemma 7.8.2 and Lemma 7.8.3, there exists a polynomial P with non negative coefficients such that

$$|z' - z| \leq (|z| + K_1)^{a_1}\frac{P(|\Im z|^{-1})}{N^2}.$$

On the one hand, if

$$|z' - z| \geq \frac{|\Im z|}{2},$$

one obtains :

$$1 \leq \frac{2|\Im z|^{-1}(|z| + K_1)^{a_1}P(|\Im z|^{-1})}{N^2}. \tag{7.15}$$

It is then enough to prove that

$$g_N(z) - \tilde{g}_N(z) = O(1). \tag{7.16}$$

Indeed, if we assume that (7.15) and (7.16) hold, then there exists a polynomial Q with non negative coefficients such that

$$\begin{aligned} |g_N(z) - \tilde{g}_N(z)| &\leq (|z| + K_2)^{a_2}(|z| + K)^a Q(|\Im z|^{-1}) \\ &\leq (|z| + K_2)^{a_2}Q(|\Im z|^{-1})\frac{2|\Im z|^{-1}(|z| + K_1)^{a_1}P(|\Im z|^{-1})}{N^2} \end{aligned}$$

Hence,

$$g_N(z) - \tilde{g}_N(z) = O(\frac{1}{N^2}).$$

To prove (7.16), one may claim that both $g_N(z)$ and $\tilde{g}_N(z)$ are bounded by $\frac{1}{|\Im z|}$.
On the other hand, if

$$|z' - z| \leq \frac{|\Im z|}{2},$$

one has :

$$|\Im z' - \Im z| \leq |z' - z| \leq \frac{|\Im z|}{2}$$

which implies $\Im z' \geq \frac{\Im z}{2}$ and therefore $z' \in \mathbb{C}^+$. We have proved that

$$z - \frac{p(N)}{N}\frac{1}{1 - g_N(z)} \in H^{-1}_{\mu_{A_N}, \frac{p(N)}{N}}(\mathbb{C}^+) = \Omega_{\mu_{A_N}, \frac{p(N)}{N}}$$

and therefore (7.14) is satisfied. Thus,

$$g_N(z) - \tilde{g}_N(z') = O(\frac{1}{N^2}). \tag{7.17}$$

On the other hand,

$$\tilde{g}_N(z') - \tilde{g}_N(z) = (z - z') \int_{\mathbb{R}} \frac{d(\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N})(x)}{(z' - x)(z - x)} \tag{7.18}$$

Taking into account the estimation of $z' - z$ above, one has :

$$(z - z') \int_{\mathbb{R}} \frac{d(\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N})(x)}{(z' - x)(z - x)} = O(\frac{1}{N^2}).$$

Hence

$$\tilde{g}_N(z') - \tilde{g}_N(z) = O(\frac{1}{N^2}). \tag{7.19}$$

Conclusion follows from (7.17) and (7.19) since

$$|g_N(z) - \tilde{g}_N(z)| \leq |g_N(z) - \tilde{g}_N(z')| + |\tilde{g}_N(z') - \tilde{g}_N(z)|. \quad \square \tag{7.20}$$

7.5 Study of $\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N}$

In the notations of Section 7.2, we define

$$F_{\tau,\gamma} := \overline{\mathbb{R} \setminus (\mathbb{R} \cap \partial\Omega_{\tau,\gamma})} \cup \{x \in \mathbb{R} | \tau(\{x\}) > \gamma\}.$$

Remark 7.5.1. The set $F_{\tau,\gamma}$ is clearly closed. Actually, it is a compact subset of $\mathbb{R}$ because it is included in $\{x \in \mathbb{R} | \mathrm{dist}(x, \mathrm{supp}(\tau)) \leq \sqrt{\gamma} + 2\}$.
Indeed, consider $x \in \mathbb{R}$ such that $\mathrm{dist}(x, \mathrm{supp}(\tau)) > \sqrt{\gamma} + 2$.
Then, G_τ is analytic at $x \notin \mathrm{supp}(\tau)$ and $G_\tau(x) \in [-\frac{1}{2}; \frac{1}{2}]$ since

$$\mathrm{dist}(x, \mathrm{supp}(\tau)) > 2.$$

It follows that $H_{\tau,\gamma}$ is analytic and real-valued at x. In particular, the limits

$$\lim_{y \downarrow 0} H_{\tau,\gamma}(x + iy) = H_{\tau,\gamma}(x),$$

$$\lim_{y\downarrow 0} \frac{H_{\tau,\gamma}(x+iy) - H_{\tau,\gamma}(x)}{iy} = H'_{\tau,\gamma}(x)$$

exist respectively in $\mathbb{R}$ and $\mathbb{C}$. Moreover,

$$H'_{\tau,\gamma}(x) = 1 + \frac{\gamma G'_\tau(x)}{(1-G_\tau(x))^2} \in \mathbb{R}.$$

Since $\mathrm{dist}(x, \mathrm{supp}(\tau)) > \sqrt{\gamma}+1$, we have

$$-\frac{1}{(\sqrt{\gamma}+1)^2} \leq G'_\tau(x) < 0$$

and

$$0 < \frac{1}{(1-G_\tau(x))^2} \leq \frac{1}{(1-\frac{1}{\sqrt{\gamma}+1})^2},$$

which implies $H'_{\tau,\gamma}(x) \in [0;1[$ and shows that $x \in \mathbb{R} \cap \partial\Omega_{\tau,\gamma}$.

We will need the following

Lemma 7.5.2.

$$\mathrm{supp}(\tau) \subseteq F_{\tau,\gamma}.$$

Proof of Lemma 7.5.2: We show successively that $\mathrm{supp}(\tau^{sc}), \mathrm{supp}(\tau^{ac})$ and $\{\text{atoms of } \tau\}$ are included in $F_{\tau,\gamma}$, where τ^{sc} (respectively τ^{ac}) denotes the singular continuous (resp. absolutely continuous) part of τ.
Assume that $\tau^{sc}(\mathbb{R} \setminus F_{\tau,\gamma}) > 0$, it means that there is an uncountable set of $x \in \mathbb{R} \setminus F_{\tau,\gamma}$ satisfying $\tau(\{x\}) = 0$ and $\lim_{y\to 0^+} |G_\tau(x+iy)| = +\infty$. For such x,

$$H_{\tau,\gamma}(x) = x$$

and

$$\lim_{y\to 0^+} |\frac{H_{\tau,\gamma}(x+iy) - H_{\tau,\gamma}(x)}{iy}| = +\infty,$$

which is in contradiction with the assumption that $x \in \mathbb{R} \setminus F_{\tau,\gamma}$.
Then, we show that there exists a real Borel set O such that $\mathbb{R}\setminus O$ is negligible and τ^{ac} satisfies :

$$\forall x \in O \cap (\mathbb{R} \setminus F_{\tau,\gamma}), \frac{d\tau^{ac}}{dx}(x) = 0.$$

Define

$$O_1 := \{x \in \mathbb{R} \mid G_\tau \text{ has a finite nontangential limit at } x\},$$

$$O_2 := \{x \in \mathbb{R} \mid \lim_{y\to 0^+} \Im G_\tau(x+iy) = -\pi \frac{d\tau^{ac}}{dx}(x)\}$$

and

$$O = O_1 \cap O_2,$$

whose complementary is indeed negligible.
Take $x \in O \cap (\mathbb{R} \setminus F_{\tau,\gamma})$. Then

$$\lim_{y \to 0^+} H_{\tau,\gamma}(x+iy) \in \mathbb{R} \setminus \{x\},$$

and consequently $\lim_{y\to 0^+} \Im G_\tau(x+iy) = 0$.
Let x be an atom of τ such that $\tau(\{x\}) < \gamma$. Then

$$H_{\tau,\gamma}(x+iy) \underset{y\to 0^+}{\longrightarrow} x,$$

and

$$\frac{H_{\tau,\gamma}(x+iy) - x}{iy} \underset{y\to 0^+}{\longrightarrow} 1 - \frac{\gamma}{\tau(\{x\})} < 0.$$

This implies that $x \notin \mathbb{R} \cap \partial\Omega_{\tau,\gamma}$. In particular, $x \in F_{\tau,\gamma}$.
So far, we have proved that

$$\operatorname{supp}(\tilde{\tau}) \subseteq F_{\tau,\gamma},$$

where $\tilde{\tau} = \tau - \sum_{x\in\mathbb{R};\tau(\{x\})=\gamma} \gamma\delta_x$. Let $x \notin \operatorname{supp}(\tilde{\tau})$ satisfying $\tau(\{x\}) = \gamma$, one can find a sequence $(x_n)_{n\in\mathbb{N}} \in (\mathbb{R} \setminus \operatorname{supp}(\tau))^{\mathbb{N}}$ such that $x_n \underset{n\to+\infty}{\longrightarrow} x$. Since $G_\tau(x_n) \in \mathbb{R}$ and $|G_\tau(x_n)| \underset{n\to+\infty}{\longrightarrow} +\infty$, for n sufficiently large, we have $H_{\tau,\gamma}(x_n) \in \mathbb{R}$. More precisely, we have

$$G_\tau(x_n) = \frac{\gamma}{x_n - x} + G_{\tau-\gamma\delta_x}(x) + (x_n - x)G'_{\tau-\gamma\delta_x}(x) + o(x_n - x)$$

and

$$G'_\tau(x_n) = -\frac{\gamma}{(x_n-x)^2} + G'_{\tau-\gamma\delta_x}(x) + o(1).$$

This implies that

$$(1 - G_\tau(x_n))^2 + \gamma G'_\tau(x_n) = 2\gamma\frac{G_{\tau-\gamma\delta_x}(x) - 1}{x_n - x}$$

$$+(1 - G_{\tau-\gamma\delta_x}(x))^2 + 3\gamma G'_{\tau-\gamma\delta_x}(x) + o(1).$$

If $G_{\tau-\gamma\delta_x}(x) - 1 > 0$ (resp. < 0), we choose $(x_n)_{n\in\mathbb{N}} \in (\mathbb{R} \setminus \operatorname{supp}(\tau))^{\mathbb{N}}$ so that $x_n \underset{n\to+\infty}{\longrightarrow} x^-$ (resp. x^+) and

$$\begin{aligned} H'_{\tau,\gamma}(x_n) &\sim \frac{2\gamma\frac{G_{\tau-\gamma\delta_x}(x)-1}{x_n-x}}{\frac{\gamma^2}{(x_n-x)^2}} \\ &\sim \frac{2(G_{\tau-\gamma\delta_x}(x) - 1)}{\gamma}(x_n - x). \end{aligned}$$

If $G_{\tau-\gamma\delta_x}(x) - 1 = 0$,

$$\begin{aligned} H'_{\tau,\gamma}(x_n) &\sim \frac{3\gamma G'_{\tau-\gamma\delta_x}(x)}{\frac{\gamma^2}{(x_n-x)^2}} \\ &\sim \frac{3G'_{\tau-\gamma\delta_x}(x)}{\gamma}(x_n-x)^2. \end{aligned}$$

In each case, for n sufficiently large, $x_n \in \mathbb{R} \setminus \overline{\mathbb{R} \cap \partial\Omega_{\tau,\gamma}}$, and finally $x \in \overline{\mathbb{R} \setminus \mathbb{R} \cap \partial\Omega_{\tau,\gamma}} \subseteq F_{\tau,\gamma}$, which concludes the proof. □

Proposition 7.5.3. *$H_{\tau,\gamma}$ is an increasing $\mathcal{C}^1$-diffeomorphism from $\mathbb{R} \setminus F_{\tau,\gamma}$ onto $\mathbb{R} \setminus \operatorname{supp}(\pi_{\gamma,1} \boxplus \tau)$.*

Proof of Proposition 7.5.3: Since $\pi_{\gamma,1}$ is freely infinitely divisible, by Theorem 7.2.2, $H_{\tau,\gamma}$ is well defined and real-valued on $\mathbb{R} \setminus F_{\tau,\gamma} \subseteq \mathbb{R} \cap \partial\Omega_{\tau,\gamma}$. Notice also that $1 - G_\tau$ does not vanish on $\mathbb{R} \setminus F_{\tau,\gamma}$, otherwise $H_{\tau,\gamma}$ would explode.
Since $\mathbb{R} \setminus F_{\tau,\gamma} \subseteq \mathbb{R} \setminus \operatorname{supp}(\tau)$, $1 - G_\tau$ is continuously differentiable on $\mathbb{R} \setminus F_{\tau,\gamma}$, and so is $H_{\tau,\gamma}$. Moreover,

$$\forall x \in \mathbb{R} \setminus F_{\tau,\gamma}, H'_{\tau,\gamma}(x) = 1 - \frac{\gamma}{(1 - G_\tau(x))^2} \int_{\mathbb{R}} \frac{d\tau(t)}{(x-t)^2} \geq 0.$$

Take $x, y \in \mathbb{R} \setminus F_{\tau,\gamma}$, and assume $x < y$. Then

$$\frac{H_{\tau,\gamma}(y) - H_{\tau,\gamma}(x)}{y - x} = 1 - \frac{\gamma}{(1 - G_\tau(x))(1 - G_\tau(y))} \int_{\mathbb{R}} \frac{d\tau(t)}{(x-t)(y-t)}.$$

Apply Cauchy-Schwarz inequality to get

$$\left| \frac{\gamma}{(1 - G_\tau(x))(1 - G_\tau(y))} \int_{\mathbb{R}} \frac{d\tau(t)}{(x-t)(y-t)} \right| \leq$$

$$\Big(\frac{\gamma}{(1 - G_\tau(x))^2} \int_{\mathbb{R}} \frac{d\tau(t)}{(x-t)^2}\Big)^{\frac{1}{2}} \Big(\frac{\gamma}{(1 - G_\tau(y))^2} \int_{\mathbb{R}} \frac{d\tau(t)}{(y-t)^2}\Big)^{\frac{1}{2}},$$

and obtain that $H_{\tau,\gamma}$ is increasing on $\mathbb{R} \setminus F_{\tau,\gamma}$.
For $x \in \mathbb{R} \setminus F_{\tau,\gamma} \subseteq \mathbb{R} \cap \partial\Omega_{\tau,\gamma}$, $x + iy \in \Omega_{\tau,\gamma}$ for any $y > 0$. This implies that

$$\forall y > 0, \omega_{\tau,\gamma}(H_{\tau,\gamma}(x + iy)) = x + iy.$$

Let y tend to 0^+ in the expression above, and get, by continuity of $\omega_{\tau,\gamma}$,

$$\omega_{\tau,\gamma}(H_{\tau,\gamma}(x)) = x.$$

This has for consequence that $H_{\tau,\gamma}$ is a bijection from $\mathbb{R} \setminus F_{\tau,\gamma}$ onto its image $H_{\tau,\gamma}(\mathbb{R} \setminus F_{\tau,\gamma})$, whose inverse is precisely the restriction of $\omega_{\tau,\gamma}$ to $H_{\tau,\gamma}(\mathbb{R} \setminus F_{\tau,\gamma})$. We have to prove that

$$H_{\tau,\gamma}(\mathbb{R} \setminus F_{\tau,\gamma}) = \mathbb{R} \setminus \operatorname{supp}(\pi_{\gamma,1} \boxplus \tau).$$

We first show the inclusion

$$\mathbb{R} \setminus \operatorname{supp}(\pi_{\gamma,1} \boxplus \tau) \subseteq H_{\tau,\gamma}(\mathbb{R} \setminus F_{\tau,\gamma}).$$

We know that $\omega_{\tau,\gamma}$ is a homeomorphism from $\mathbb{R}\setminus\operatorname{supp}(\pi_{\gamma,1}\boxplus\tau)$ onto its image, which is an open subset of $\mathbb{R} \cap \partial\Omega_{\tau,\gamma}$. In particular, $\omega_{\tau,\gamma}$ is continuous from $\mathbb{R} \setminus \operatorname{supp}(\pi_{\gamma,1} \boxplus \tau)$ into $\mathbb{R} \setminus \overline{\mathbb{R} \setminus (\mathbb{R} \cap \partial\Omega_{\tau,\gamma})}$.
The atoms x of τ satisfying $\tau(\{x\}) > \gamma$, because they are atoms of $\pi_{\gamma,1} \boxplus \tau$ satisfying $\omega_{\tau,\gamma}(x) = x$, are not in $\omega_{\tau,\gamma}(\mathbb{R} \setminus \operatorname{supp}(\pi_{\gamma,1} \boxplus \tau)$, due to univalence of $\omega_{\tau,\gamma}$ on $\mathbb{C}^+ \cup \mathbb{R}$. This has for consequence that $\omega_{\tau,\gamma}$ is continuous from $\mathbb{R} \setminus \operatorname{supp}(\pi_{\gamma,1} \boxplus \tau)$ into $\mathbb{R} \setminus F_{\tau,\gamma}$.
Since in addition $H_{\tau,\gamma}$ is continuous on $\mathbb{R} \setminus F_{\tau,\gamma}$, we have, for any $x \in \mathbb{R} \setminus \operatorname{supp}(\pi_{\gamma,1} \boxplus \tau)$,

$$x = \lim_{y\to 0^+} x + iy = \lim_{y\to 0^+} H_{\tau,\gamma}(\omega_{\tau,\gamma}(x+iy)) = H_{\tau,\gamma}(\omega_{\tau,\gamma}(x)) \in H_{\tau,\gamma}(\mathbb{R} \setminus F_{\tau,\gamma}).$$

To prove the other inclusion

$$H_{\tau,\gamma}(\mathbb{R} \setminus F_{\tau,\gamma}) \subseteq \mathbb{R} \setminus \operatorname{supp}(\pi_{\gamma,1} \boxplus \tau),$$

we recall that it is proved in [Bel08] that the singular part of $\pi_{\gamma,1} \boxplus \tau$ is purely atomic and has support equal to $\{x \in \mathbb{R} | \tau(\{x\}) > \gamma\}$. It is therefore sufficient to show the existence of a real Borel set O such that $\mathbb{R} \setminus O$ is negligible and

$$\forall x \in O \cap H_{\tau,\gamma}(\mathbb{R} \setminus F_{\tau,\gamma}), \frac{d(\pi_{\gamma,1} \boxplus \tau)^{ac}}{dx}(x) = 0.$$

Consider

$$O := \{x \in \mathbb{R} \mid \lim_{y\to 0^+} \Im G_{\pi_{\gamma,1}\boxplus\tau}(x+iy) = -\pi \frac{d(\pi_{\gamma,1} \boxplus \tau)^{ac}}{dx}(x)\},$$

whose complementary is indeed negligible.
For $x \in \mathbb{R} \setminus F_{\tau,\gamma}$ such that $H_{\tau,\gamma}(x) \in O$, and for $y > 0$, we define

$$z := \omega_{\tau,\gamma}(H_{\tau,\gamma}(x) + iy) \in \Omega_{\tau,\gamma}$$

Then

$$G_{\pi_{\gamma,1}\boxplus\tau}(H_{\tau,\gamma}(x) + iy) = G_{\pi_{\gamma,1}\boxplus\tau}(H_{\tau,\gamma}(z)) = G_\tau(z).$$

When y tends to 0^+, z goes to $\omega_{\tau,\gamma}(H_{\tau,\gamma}(x)) = x$. We know that G_τ is continuous and real-valued at $x \in \mathbb{R} \setminus F_{\tau,\gamma} \subseteq \mathbb{R} \setminus \operatorname{supp}(\tau)$. We may therefore conclude that $\lim_{y\to 0^+} \Im G_{\pi_{\gamma,1}\boxplus\tau}(H_{\tau,\gamma}(x) + iy) = 0$ and we are done. □

We also define

$$\Theta = \{\theta_i,\ 1 \le i \le J\} \quad \text{and} \quad \Theta_{\nu,\pi_{\alpha,1}} = \Theta \cap (\mathbb{R} \setminus F_{\nu,\alpha}). \tag{7.21}$$

Furthermore for all $\theta_i \in \Theta_{\nu,\pi_{\alpha,1}}$, we let

$$\rho_{\theta_i}^{(\alpha)} = H_{\nu,\alpha}(\theta_i) = \theta_i + \frac{\alpha}{1 - G_\nu(\theta_i)} \tag{7.22}$$

which is outside the support of $\pi_{\alpha,1} \boxplus \nu$ according to Proposition 7.5.3 and we let

$$K_{\nu,\alpha}(\theta_1, \ldots, \theta_J) := \operatorname{supp}(\pi_{\alpha,1} \boxplus \nu) \bigcup \left\{\rho_{\theta_i}^{(\alpha)},\, \theta_i \in \Theta_{\nu,\pi_{\alpha,1}}\right\}. \tag{7.23}$$

Theorem 7.5.4. *For any $\varepsilon > 0$,*

$$\operatorname{supp}(\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N}) \subset K_{\nu,\alpha}(\theta_1, \ldots, \theta_J) + (-\varepsilon, \varepsilon),$$

when N is large enough.

Proof of Theorem 7.5.4: For a fixed $\varepsilon > 0$, we rather prove the existence of a $N_0 \in \mathbb{N}$ such that, for $N \geq N_0$, we have :

$$\mathbb{R} \setminus (K_{\nu,\alpha}(\theta_1, \ldots, \theta_J) + (-\varepsilon, \varepsilon)) \subseteq \mathbb{R} \setminus \operatorname{supp}(\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N}).$$

We break the argument in two lemmas.

Lemma 7.5.5. $\forall \eta > 0, \exists N_1 \in \mathbb{N}, \forall N \geq N_1,$

$$\{u \in \mathbb{R} \mid \operatorname{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) > \eta\} \subseteq \mathbb{R} \setminus F_{\mu_{A_N}, \frac{p(N)}{N}}.$$

Proof of Lemma 7.5.5 : Fix $\eta > 0$, and choose $N_1' \in \mathbb{N}$ such that :

$$\forall N \geq N_1', \max_{1 \leq i \leq N-r} \operatorname{dist}(\beta_i(N), \operatorname{supp}(\nu)) < \frac{\eta}{2}.$$

This implies that, for $N \geq N_1'$,

$$\begin{aligned} \{u \in \mathbb{R} \mid \operatorname{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) > \eta\} &\subseteq \mathbb{R} \setminus \operatorname{supp}(\mu_{A_N}) \\ &\subseteq \mathbb{R} \setminus \{x \in \mathbb{R} | \mu_{A_N}(\{x\}) > \frac{p(N)}{N}\}. \end{aligned}$$

It remains to prove that, for N sufficiently large,

$$\{u \in \mathbb{R} \mid \operatorname{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) > \eta\} \subseteq \partial\Omega_{\mu_{A_N}, \frac{p(N)}{N}},$$

or, in other words, that there exists $N_1 \geq N_1'$ such that for all $N \geq N_1$, and each $u \in \mathbb{R}$ satisfying $\operatorname{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) > \eta\}$, the limit

$$H_{\mu_{A_N}, \frac{p(N)}{N}}(u) := \lim_{v \downarrow 0} H_{\mu_{A_N}, \frac{p(N)}{N}}(u + iv)$$

exists in $\mathbb{R}$ and

$$H'_{\mu_{A_N},\frac{p(N)}{N}}(u) := \lim_{v\downarrow 0} \frac{H_{\mu_{A_N},\frac{p(N)}{N}}(u+iv) - H_{\mu_{A_N},\frac{p(N)}{N}}(u)}{iv} \in [0;1[.$$

When $N \geq N_1'$,

$$\{u \in \mathbb{R} \mid \mathrm{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) > \eta\} \subseteq \mathbb{R} \setminus \mathrm{supp}(\mu_{A_N}),$$

hence $G_{\mu_{A_N}}$ takes real values on $\{u \in \mathbb{R} \mid \mathrm{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) > \eta\}$. To conclude that $H_{\mu_{A_N},\frac{p(N)}{N}}(u) \in \mathbb{R}$, it remains to prove that, for N sufficiently large, $G_{\mu_{A_N}}$ stays away from 1 on this set. We know that $|1 - G_\nu|$ is continuous positive on $\mathbb{R} \setminus F_{\nu,\alpha}$, so in particular on the compact set

$$K := \big\{u \in \mathbb{R} \mid \quad \mathrm{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) \geq \eta, \\ \mathrm{dist}(u, (\mathrm{supp}(\nu) \cup \Theta) + (-\tfrac{\eta}{2}; \tfrac{\eta}{2})) \leq 2\big\}.$$

Denote then by m the positive quantity $m := \inf_K |1 - G_\nu| > 0$. Then, by uniform convergence of $(G_{\mu_{A_N}})_{N \geq N_1'}$ toward G_ν on K,

$$\exists N_1'' \geq N_1', \forall N \geq N_1'', \forall u \in K, |1 - G_{\mu_{A_N}}(u)| \geq \frac{1}{2} m.$$

If $u \in \{u \in \mathbb{R} \mid \mathrm{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) > \eta\} \setminus K$, we notice that $|1 - G_{\mu_{A_N}}(u)| \geq \frac{1}{2}$. This concludes the first part of the proof.
On $\mathbb{R} \setminus F_{\nu,\alpha}$, we know that $H'_{\nu,\pi_{\alpha,1}}$ is nonnegative ; it is actually positive, since, by taking the derivative (possible by Proposition 7.5.3) in the relation

$$\forall x \in \mathbb{R} \setminus F_{\nu,\alpha}, \omega_{\nu,\alpha}(H_{\nu,\alpha}(x)) = x,$$

one gets

$$\forall x \in \mathbb{R} \setminus F_{\nu,\alpha}, H'_{\nu,\alpha}(x)\omega'_{\nu,\alpha}(H_{\nu,\alpha}(x)) = 1$$

which shows that $H'_{\nu,\alpha}$ cannot vanish on $\mathbb{R} \setminus F_{\nu,\alpha}$. By uniform convergence of $(H'_{\mu_{A_N},\frac{p(N)}{N}})_{N \geq N_1'}$ toward $H'_{\nu,\alpha}$ on the compact set K' consisting of the numbers $u \in \mathbb{R}$ satisfying

$$\mathrm{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) \geq \eta$$

and

$$\mathrm{dist}(u, (\mathrm{supp}(\nu) \cup \Theta) + (-\frac{\eta}{2}; \frac{\eta}{2})) \leq \sqrt{\sup_{N \in \mathbb{N}} \frac{p(N)}{N}} + 1,$$

we get :

$$\exists N_1 \geq N_1'', \forall N \geq N_1, \forall u \in K', H'_{\mu_{A_N},\frac{p(N)}{N}}(u) \in [0;1[.$$

Notice that, if $u \in \{u \in \mathbb{R} \mid \mathrm{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) > \eta\} \setminus K'$, one has $H'_{\mu_{A_N},\frac{p(N)}{N}}(u) \in [0;1[$. □

Lemma 7.5.6. $\exists \eta > 0, \exists N_2 \in \mathbb{N}, \forall N \geq N_2,$

$$\mathbb{R} \setminus (K_{\nu,\alpha}(\theta_1, \ldots, \theta_J) + (-\varepsilon, \varepsilon)) \subseteq$$

$$H_{\mu_{A_N}, \frac{p(N)}{N}}(\{u \in \mathbb{R} \mid \mathrm{dist}(u, F_{\nu,\alpha} \cup \Theta_{\nu,\pi_{\alpha,1}}) > \eta\}).$$

Proof of Lemma 7.5.6 : We write the compact set $F_{\nu,\alpha}$ as the union of its connected components :

$$F_{\nu,\alpha} = \bigcup_{1 \leq l \leq L} [s_l; t_l].$$

Using Proposition 6.4, we obtain that the set $\mathbb{R} \setminus \mathrm{supp}(\pi_{\alpha,1} \boxplus \nu)$ is equal to

$$]-\infty; H_{\nu,\alpha}(s_1^-)[\cup \bigcup_{1 \leq l \leq L-1}]H_{\nu,\alpha}(t_l^+); H_{\nu,\alpha}(s_{l+1}^-)[\cup]H_{\nu,\alpha}(t_L^+); +\infty[.$$

Then the set

$$\mathbb{R} \setminus (K_{\nu,\alpha}(\theta_1, \ldots, \theta_J) + (-\varepsilon, \varepsilon))$$

is of the form

$$]-\infty; h_1[\cup \bigcup_{1 \leq m \leq \tilde{L}-1}]k_m; h_{m+1}[\cup]k_{\tilde{L}}; +\infty[,$$

where either

$$(h_m, k_m) = (H_{\nu,\alpha}(s_{l_m}^-) - \varepsilon, H_{\nu,\alpha}(t_{l_m}^+) + \varepsilon),$$

or

$$(h_m, k_m) = (H_{\nu,\alpha}(\theta_{i_m}) - \varepsilon, H_{\nu,\alpha}(\theta_{i_m}) + \varepsilon).$$

By continuity (or definition of left/right limit) of $H_{\nu,\alpha}$, there exists an $\eta > 0$ such that, for each $1 \leq m \leq \tilde{L}$,

$$H_{\nu,\alpha}(s_{l_m} - \eta) \geq H_{\nu,\alpha}(s_{l_m}^-) - \frac{\varepsilon}{2},$$

$$H_{\nu,\alpha}(t_{l_m} + \eta) \leq H_{\nu,\alpha}(t_{l_m}^+) + \frac{\varepsilon}{2},$$

$$H_{\nu,\alpha}(\theta_{i_m} - \eta) \geq H_{\nu,\alpha}(\theta_{i_m}) - \frac{\varepsilon}{2},$$

$$H_{\nu,\alpha}(\theta_{i_m} + \eta) \leq H_{\nu,\alpha}(\theta_{i_m}) + \frac{\varepsilon}{2}.$$

Finally, there exists an $N_2 \in \mathbb{N}$ such that, for all $N \geq N_2$,

$$H_{\mu_{A_N}, \frac{p(N)}{N}}(s_{l_m} - \eta) \geq H_{\nu,\alpha}(s_{l_m} - \eta) - \frac{\varepsilon}{2},$$

$$H_{\mu_{A_N}, \frac{p(N)}{N}}(t_{l_m} + \eta) \leq H_{\nu,\alpha}(t_{l_m} + \eta) + \frac{\varepsilon}{2},$$

$$H_{\mu_{A_N}, \frac{p(N)}{N}}(\theta_{i_m} - \eta) \geq H_{\nu,\alpha}(\theta_{i_m} - \eta) - \frac{\varepsilon}{2},$$

$$H_{\mu_{A_N},\frac{p(N)}{N}}(\theta_{i_m}+\eta) \leq H_{\nu,\alpha}(\theta_{i_m}+\eta)+\frac{\varepsilon}{2}.$$

We have thus proved, for $N \geq N_2$, the required inclusion. $\square$

End of proof of Theorem 7.5.4: Let $\varepsilon > 0$ be fixed, and let $\eta > 0, N_2 \in \mathbb{N}$ be given by Lemma 7.5.6. Apply then Lemma 7.5.5 to this η and let $N_1 \in \mathbb{N}$ be given by this lemma. Since, by Proposition 7.5.3,

$$H_{\mu_{A_N},\frac{p(N)}{N}}(\mathbb{R}\setminus F_{\mu_{A_N},\frac{p(N)}{N}}) = \mathbb{R}\setminus \operatorname{supp}(\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N}),$$

the conclusion of Theorem 7.5.4 holds for $N_0 = \max(N_1, N_2)$. $\square$

7.6 Inclusion of the spectrum in an neighborhood of $K_{\nu,\alpha}$

We are now in position to prove the following theorem :

Theorem 7.6.1. $\forall \varepsilon > 0,$

$$\mathbb{P}(\textit{For large } N, \operatorname{Spect}(M_N(p(N))) \subset \{x, \operatorname{dist}(x, K_{\nu,\alpha}(\theta_1,\ldots,\theta_J)) \leq \varepsilon\}) = 1.$$

Proof of Theorem 7.6.1 The proof still uses the ideas of [HT05]. Using the inverse Stieltjes tranform, we get respectively that, for any φ in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ with compact support,

$$\mathbb{E}[\operatorname{tr}(\varphi(M_N(p(N))))] - \int_{\mathbb{R}} \varphi(x) d(\mu_{A_N} \boxplus \pi_{\frac{p}{N},1})(x) =$$

$$-\frac{1}{\pi}\lim_{y\to 0^+} \Im \int_{\mathbb{R}} \varphi(x) r_N(x+iy)dx,$$

where $r_N = \tilde{g}_N - g_N$ satisfies, according to Proposition 7.4.1, for any $z \in \mathbb{C}\setminus\mathbb{R}$,

$$|r_N(z)| \leq \frac{1}{N^2}(|z|+K)^\alpha P(|\Im(z)^{-1}|),$$

for some nonnegative numbers α and K and some polynomial P. We refer the reader to the Appendix of [CDM07] where it is proved using the ideas of [HT05] that

$$\limsup_{y\to 0^+} |\int_{\mathbb{R}} \varphi(x) h(x+iy)dx| \leq C < +\infty$$

when h is an analytic function on $\mathbb{C}\setminus\mathbb{R}$ which satisfies

$$|h(z)| \leq (|z|+K)^\alpha P_k(|\Im(z)^{-1}|).$$

Hence, dealing with $h(z) = N^2 r_N(z)$, we deduce that for all large N,

$$\limsup_{y\to 0^+} |\int_{\mathbb{R}} \varphi(x) r_N(x+iy)dx| \leq \frac{C}{N^2}$$

and then

$$\mathbb{E}[\mathrm{tr}(\varphi(M_N(p(N))))] - \int \varphi(x) d(\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N})(x) = O(\frac{1}{N^2}). \quad (7.24)$$

The function ρ defined by

$$\begin{aligned} \rho(x) &= \exp\{\frac{1}{x^2-1}\} \text{ if } |x|<1 \\ &= 0 \text{ if } |x| \geq 1; \end{aligned}$$

is in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, its support is included in $\{|x| \leq 1\}$ and $\int_{\mathbb{R}} \rho(x)dx = 1$. Let $0<\varepsilon<1$ be such that 4ε is strictly smaller than the minimal distance between two connected components of $K_{\nu,\alpha}(\theta_1,\ldots,\theta_J)$. Define

$$\rho_{\frac{\varepsilon}{2}}(x) = \frac{2}{\varepsilon}\rho(\frac{2x}{\varepsilon}),$$

$$K(\varepsilon) = \{x, \mathrm{dist}(x, K_{\nu,\alpha}(\theta_1,\ldots,\theta_J)) \leq \varepsilon\}$$

and

$$f(\varepsilon)(x) = \int \mathbf{1}_{K(\varepsilon)}(y)\rho_{\frac{\varepsilon}{2}}(x-y)dy.$$

The function $f(\varepsilon)$ is in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, $f(\varepsilon) \equiv 1$ on $K(\frac{\varepsilon}{2})$; its support is included in the compact set $K(2\varepsilon)$. Thus, according to (7.24),

$$\mathbb{E}[\mathrm{tr}(f(\varepsilon)(M_N(p(N))))] - \int_{\mathbb{R}} f(\varepsilon)(x) d(\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N})(x) = O_\varepsilon(\frac{1}{N^2}). \quad (7.25)$$

Then, the function $\psi(\varepsilon) \equiv 1 - f(\varepsilon)$ also satisfies

$$\mathbb{E}[\mathrm{tr}(\psi(\varepsilon)(M_N(p(N))))] - \int_{\mathbb{R}} \psi(\varepsilon)(x) d(\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N})(x) = O_\varepsilon(\frac{1}{N^2}). \quad (7.26)$$

Now, since according to Theorem 7.5.4, for all large N, the support of $\pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N}$ is included in $K(\frac{\varepsilon}{2})$ and $\psi(\varepsilon) \equiv 0$ on $K(\frac{\varepsilon}{2})$, we deduce that

$$\mathbb{E}[\mathrm{tr}(\psi(\varepsilon)(M_N(p(N))))] = O_\varepsilon(\frac{1}{N^2}). \quad (7.27)$$

Using Gaussian Poincaré inequality (7.42) and Lemma 4.6 in [HT05], we have

$$\mathbf{V}[\mathrm{tr}(\psi(\varepsilon)(M_N(p(N))))] \leq$$

$$\frac{C}{N^2}\mathbb{E}\left[\sup_{\substack{V \in \mathcal{M}_{p\times N}(\mathbb{C}) \\ TrVV^*=1}} |\mathrm{tr}\{\psi'(\varepsilon)(M_N(p(N)))(B^*V+V^*B)\}|^2\right] \leq$$

$$\frac{2C}{N^2}\mathbb{E}\left[\lambda_1(\frac{BB^*}{N})\mathrm{tr}\left(\psi^{'}(\varepsilon)^2(M_N(p))\right)\right],$$

the last inequality coming from the following inequalities

$$|\mathrm{tr}\{\psi^{'}(\varepsilon)(M_N(p(N)))(B^*V)\}| \leq$$

$$\left[\mathrm{tr}\left(\psi^{'}(\varepsilon)^2(M_N(p(N)))\right)\right]^{\frac{1}{2}}[\mathrm{tr}\{(B^*VV^*B)]^{\frac{1}{2}} \leq$$

$$\left[\mathrm{tr}\left(\psi^{'}(\varepsilon)^2(M_N(p(N)))\right)\right]^{\frac{1}{2}}\|\frac{BB^*}{N}\|^{\frac{1}{2}}[\mathrm{Tr}\{(VV^*)]^{\frac{1}{2}} \leq$$

$$\left[\mathrm{tr}\left(\psi^{'}(\varepsilon)^2(M_N(p(N)))\right)\right]^{\frac{1}{2}}\left[\lambda_1(\frac{BB^*}{N})\right]^{\frac{1}{2}}.$$

Now, by Hölder inequality,

$$\mathbf{V}[\mathrm{tr}(\psi(\varepsilon)(M_N(p(N))))] \leq$$

$$\frac{2C}{N^2}\mathbb{E}\left[\lambda_1(\frac{BB^*}{N})^3\right]^{\frac{1}{3}}\mathbb{E}\left[\left(\mathrm{tr}\{\psi^{'}(\varepsilon)^2(M_N(p(N)))\}\right)^{\frac{3}{2}}\right]^{\frac{2}{3}} \leq$$

$$\frac{2C}{N^2}\mathbb{E}\left[\lambda_1(\frac{BB^*}{N})^3\right]^{\frac{1}{3}}\mathbb{E}\left[\mathrm{tr}\{|\psi^{'}(\varepsilon)|^3(M_N(p(N)))\}\right]^{\frac{2}{3}}.$$

Since when $\psi^{'}(\varepsilon)$ vanishes then $\psi^{(k+1)}(\varepsilon)$ vanishes too for any $k \geq 0$, $|\psi^{'}(\varepsilon)|$ is still a $\mathcal{C}^\infty$ function with compact support such that $\psi^{'}(\varepsilon) \equiv 0$ on the support of $\mu_{A_N} \boxplus \pi_{\frac{p(N)}{N},1}$. Hence, we deduce from (7.24) that

$$\mathbb{E}\left[\mathrm{tr}\{|\psi^{'}(\varepsilon)|^3(M_N(p(N)))\}\right] = O_\varepsilon(\frac{1}{N^2}).$$

Since moreover, according to Lemma 7.8.1, $\mathbb{E}\left[\lambda_1(\frac{BB^*}{N})^3\right]$ is bounded independently of N, we can deduce that

$$\mathbf{V}[\mathrm{tr}(\psi(\varepsilon)(M_N(p(N))))] = O_\varepsilon(\frac{1}{N^{\frac{10}{3}}}). \tag{7.28}$$

Now, set

$$Z_{N,\varepsilon} := \mathrm{tr}(\psi(\varepsilon)(M_N(p(N))))$$

and

$$\Omega_{N,\varepsilon} = \{Z_{N,\varepsilon} > N^{-13/12}\}.$$

From (7.27) and (7.28), we deduce that

$$\mathbb{E}\{|Z_{N,\varepsilon}|^2\} = O_\varepsilon(\frac{1}{N^{10/3}}).$$

Hence

$$\mathbb{P}(\Omega_{N,\varepsilon}) \le N^{\frac{26}{12}} \mathbb{E}\{|Z_{N,\varepsilon}|^2\} = O_\varepsilon(\frac{1}{N^{\frac{14}{12}}}).$$

By Borel-Cantelli lemma, we deduce that almost surely for all large N, $Z_{N,\varepsilon} \le N^{-13/12}$. Since $Z_{N,\varepsilon} \ge \mathbf{1}_{\mathbb{R}\setminus K(2\varepsilon)}$, it follows that almost surely for all large N, the number of eigenvalues of $M_N(p(N))$ which are in $\mathbb{R} \setminus K(2\varepsilon)$ is lower than $N^{-1/12}$ and thus obviously has to be equal to zero. The proof of Theorem 7.6.1 is complete. □

7.7 Convergence of the largest eigenvalues

Remark 7.7.1. Let α and $\hat{\alpha}$ be such that $\hat{\alpha} < \alpha$. Let z be in $\mathbb{C}^+$.

$$\Im\left(H_{\nu,\hat{\alpha}}(z) - H_{\nu,\alpha}(z)\right) = \frac{\hat{\alpha} - \alpha}{|1 - G_\nu(z)|^2} \Im G_\nu(z) > 0.$$

Hence, if $H_{\nu,\alpha}(z)$ belongs to $\mathbb{C}^+$, so does $H_{\nu,\hat{\alpha}}(z)$ and therefore $\Omega_{\nu,\alpha} \subset \Omega_{\nu,\hat{\alpha}}$. Thus $\mathbb{R} \setminus F_{\nu,\alpha} \subset \mathbb{R} \setminus F_{\nu,\hat{\alpha}}$ so that it makes sense to consider the following composition of homeomorphisms :

$$H_{\nu,\hat{\alpha}} \circ \omega_{\nu,\hat{\alpha}} : \mathbb{R} \setminus K_{\nu,\alpha}(\theta_1, \ldots, \theta_J) \to H_{\nu,\hat{\alpha}}(\mathbb{R} \setminus (F_{\nu,\alpha} \cup \Theta))$$

which is increasing on each connected component of $\mathbb{R} \setminus K_{\nu,\alpha}(\theta_1, \ldots, \theta_J)$ and with values in $\mathbb{R} \setminus K_{\nu,\alpha}(\theta_1, \ldots, \theta_J)$.

Lemma 7.7.2. *Let $[a,b]$ be a compact set contained in $\mathbb{R} \setminus \mathbb{K}_{\nu,\alpha}(\theta_1, \ldots, \theta_J)$. Then,*

(i) For all large N, $[\omega_{\nu,\alpha}(a), \omega_{\nu,\alpha}(b)] \subset \mathbb{R} \setminus \mathrm{Spect}(A_N)$.

(ii) For all $0 < \hat{\alpha} < \alpha$, the interval $[H_{\nu,\hat{\alpha}}(\omega_{\nu,\alpha}(a)), H_{\nu,\hat{\alpha}}(\omega_{\nu,\alpha}(b))]$ is contained in $\mathbb{R} \setminus K_{\nu,\hat{\alpha}}(\theta_1, \ldots, \theta_J)$ and

$$H_{\nu,\hat{\alpha}}(\omega_{\nu,\alpha}(b)) - H_{\nu,\hat{\alpha}}(\omega_{\nu,\alpha}(a)) \ge b - a.$$

Proof of Lemma 7.7.2: For simplicity, we define $K^\varepsilon_{\alpha,J} = K_{\nu,\alpha}(\theta_1, \ldots, \theta_J) + [-\varepsilon, \varepsilon]$, $\omega_N = \omega_{\mu_{A_N}, \frac{p(N)}{N}}$, $F_N = F_{\mu_{A_N}, \frac{p(N)}{N}}$ and $\tilde{\mu}_N = \pi_{\frac{p(N)}{N},1} \boxplus \mu_{A_N}$. As $[a,b]$ is a compact set, there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$[a-\delta, b+\delta] \subset \mathbb{R} \setminus K^\varepsilon_{\alpha,J} \quad \text{and} \quad d([a-\delta, b+\delta]; K^\varepsilon_{\alpha,J}) \ge \delta.$$

According to Theorem 7.5.4, there exists some $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$, $\mathrm{supp}(\tilde{\mu}_N)$ is contained in $K^\varepsilon_{\alpha,J}$. Thus, using (6.4) and since ω_N is strictly increasing on $[a-\delta, b+\delta]$, we have

$$\forall N \ge N_0, \quad [\omega_N(a-\delta), \omega_N(b+\delta)] \subset \mathbb{R} \setminus F_N \subset \mathbb{R} \setminus \mathrm{Spect}(A_N). \quad (7.29)$$

As $\omega_{\nu,\alpha}$ is strictly increasing on the compact set $[a-\delta, b+\delta]$ (since $\mathrm{supp}(\pi_{\alpha,1} \boxplus \nu) \subset K^{\varepsilon}_{\sigma,J}$), one can consider $\eta > 0$ such that

$$\omega_{\nu,\alpha}(a-\delta) \leq \omega_{\nu,\alpha}(a) - \eta \quad \text{and} \quad \omega_{\nu,\alpha}(b+\delta) \geq \omega_{\nu,\alpha}(b) + \eta. \tag{7.30}$$

Now, one can readily notice that the probability measures $\tilde{\mu}_N$ weakly converge to $\pi_{\alpha,1} \boxplus \nu$. This will lead to the result recalling from the definition of the subordination functions that for all $x \in [a-\delta, b+\delta]$, $\omega_{\nu,\alpha}(x) = x - \frac{\alpha}{1-G_{\pi_{\alpha,1}\boxplus\nu}(x)}$ and $\omega_N(x) = x - \frac{p(N)}{N}\frac{1}{1-G_{\tilde{\mu}_N}(x)}$ (at least for all $N \geq N_0$). Indeed, observing that for any x in $[a-\delta, b+\delta]$, the map $h : t \mapsto 1/(x-t)$ is bounded on $K^{\varepsilon}_{\sigma,J}$ one readily get the simple convergence of $G_{\tilde{\mu}_N}$ to $G_{\pi_{\alpha,1}\boxplus\nu}$ as well as the one of the corresponding subordination functions, by considering a bounded continuous function which coincide with h on $K^{\varepsilon}_{\sigma,J}$. We then deduce that there exists $N_0' \geq N_0$ such that, for $N \geq N_0'$,

$$\omega_N(a-\delta) \leq \omega_{\nu,\alpha}(a-\delta) + \eta \quad \text{and} \tag{7.31}$$

$$\omega_N(b+\delta) \geq \omega_{\nu,\alpha}(b+\delta) - \eta. \tag{7.32}$$

Combining (7.29), (7.30) and (7.31) prove that the inclusion of point (i) holds true for all $N \geq N_0'$.

The first part of (ii) is obvious from Remark 7.7.1. Now set $a' = H_{\nu,\hat{\alpha}}(\omega_{\nu,\alpha}(a))$ and $b' = H_{\nu,\hat{\alpha}}(\omega_{\nu,\alpha}(b))$ then

$$\begin{aligned} b' - a' &= \omega_{\nu,\alpha}(b) - \omega_{\nu,\alpha}(a) + \hat{\alpha}\frac{G_{\pi_{\alpha,1}\boxplus\nu}(b) - G_{\pi_{\alpha,1}\boxplus\nu}(a)}{(1-G_{\pi_{\alpha,1}\boxplus\nu}(a))(1-G_{\pi_{\alpha,1}\boxplus\nu}(b))} \\ &\geq H_{\nu,\alpha}(\omega_{\nu,\alpha}(b)) - H_{\nu,\alpha}(\omega_{\nu,\alpha}(a)) = b - a, \end{aligned}$$

since $G_{\pi_{\alpha,1}\boxplus\nu}(b) - G_{\pi_{\alpha,1}\boxplus\nu}(a) < 0$ and $(1-G_{\pi_{\alpha,1}\boxplus\nu}(a))(1-G_{\pi_{\alpha,1}\boxplus\nu}(b)) > 0$.
□

Let $[a, b]$ be in $\mathbb{R} \setminus K_{\nu,\alpha}(\theta_1, \ldots, \theta_J)$. From Lemma 7.7.2 (i), it corresponds an interval $I = [a', b']$ outside the spectrum of A_N i.e. there is $i_N \in \{0, \ldots, N\}$ such that

$$\lambda_{i_N+1}(A_N) < \omega_{\nu,\alpha}(a) := a' \quad \text{and} \quad \lambda_{i_N}(A_N) > \omega_{\nu,\alpha}(b) := b'. \tag{7.33}$$

Theorem 7.7.3. *With i_N satisfying* (7.33)*, one has*

$$\mathbb{P}[\lambda_{i_N}(M_N) > b, \textit{ for all large } N] = 1. \tag{7.34}$$

The proof closely follows the proof of Theorem 4.5 in [CDFF10] by introducing a continuum of matrices interpolating from M_N to A_N. Let δ be such that

$$0 < \delta \leq \frac{b-a}{4\alpha}|1 - G_{\pi_{\alpha,1}\boxplus\nu}(b)|. \tag{7.35}$$

Set for any $k \geq 0$,

$$\alpha_k = \left(1 - \frac{1}{1+\delta k}\right)\alpha$$

and

$$p_k = [\alpha_k N].$$

Let us introduce the following matrices

$$M_N^{(0)} = A_N,$$

$$\forall k \geq 1,\ M_N^{(p_k)} := \frac{1}{N}\sum_{i=1}^{p_k} Y_i Y_i^* + A_N \ .$$

For all $k \geq 0$, define

$$a_k = H_{\nu,\alpha_k}(\omega_{\nu,\alpha}(a)) \ \text{ and } \ b_k = H_{\nu,\alpha_k}(\omega_{\nu,\alpha}(b))$$

and set

$$\mathrm{E}_k = \{\text{no eigenvalues of } M_N^{(p_k)} \text{ in } [a_k, b_k], \text{ for all large } N\}.$$

By Lemma 7.7.2 (ii) and Theorem 7.6.1, we know that for all k

$$\mathbb{P}(\mathrm{E}_k) = 1. \tag{7.36}$$

Now, we shall show by induction on k that with i_N satisfying (7.33), one has

$$\mathbb{P}[\lambda_{i_N}(M_N^{(p_k)}) > b_k, \text{ for all large } N] = 1. \tag{7.37}$$

The proof mainly relies on the following classical result (due to Weyl).

Lemma 7.7.4. *(cf. Theorem 4.3.7 of [HJ90]) Let B and C be two $N \times N$ Hermitian matrices. For any pair of integers j, k such that $1 \leq j, k \leq N$ and $j + k \leq N + 1$, we have*

$$\lambda_{j+k-1}(B + C) \leq \lambda_j(B) + \lambda_k(C).$$

For any pair of integers j, k such that $1 \leq j, k \leq N$ and $j + k \geq N + 1$, we have

$$\lambda_j(B) + \lambda_k(C) \leq \lambda_{j+k-N}(B + C).$$

(7.37) is obviously true for $k = 0$ since $M_N^{(p_0)} = M_N^{(0)} = A_N$ and $b_0 = F_{\nu,\alpha}$. Now, let us assume that (7.37) holds true for k. Since

$$M_N^{(p_{k+1})} = M_N^{(p_k)} + M_N^{(p_{k+1}-p_k)},$$

we can deduce from Lemma 7.7.4 that,

$$\begin{aligned} \lambda_{i_N}(M_N^{(p_{k+1})}) &\geq \lambda_{i_N}(M_N^{(p_k)}) + \lambda_N(M_N^{(p_{k+1}-p_k)}) \\ &> b_k \end{aligned}$$

since $\lambda_N(M_N^{(p_{k+1}-p_k)}) \geq 0$.
If $b_k \geq b_{k+1}$, we are done.
Now, if $b_k < b_{k+1}$, we have $a_{k+1} \leq b_k$ since using (7.35), we have $|b_k - b_{k+1}| \leq \frac{b-a}{4}$ and according to Lemma (7.7.2)(ii) we have $b_{k+1} - a_{k+1} \geq b - a$. Thus $\lambda_{i_N}(M_N^{(p_{k+1})}) > a_{k+1}$. Since according to (7.36), there is no eigenvalues of $M_N^{(p_{k+1})}$ in $[a_{k+1}, b_{k+1}]$ for N large, we can deduce that $\lambda_{i_N}(M_N^{(p_{k+1})}) > b_{k+1}$. This complete the proof of (7.37) by induction.

Now, let us complete the proof of Theorem 7.7.3. Since

$$M_N^{(p(N))} = M_N^{(p_k)} + M_N^{(p(N)-p_k)},$$

we can deduce from Lemma 7.7.4 and (7.37) that,

$$\begin{aligned} \lambda_{i_N}(M_N^{(p(N))}) &\geq \lambda_{i_N}(M_N^{(p_k)}) + \lambda_N(M_N^{(p-p_k)}) \\ &> b_k \end{aligned}$$

since $\lambda_N(M_N^{(p(N)-p_k)}) \geq 0$.
If $b_k \geq b$, we are done.
We have

$$\begin{aligned} |b_k - b| &\leq \frac{\alpha - \alpha_k}{|1 - G_{\pi_{\alpha,1} \boxplus \nu}(b)|} \\ &= \frac{\alpha}{(1+\delta k)|1 - G_{\pi_{\alpha,1} \boxplus \nu}(b)|} \\ &\leq \frac{b-a}{4} \end{aligned}$$

for $k \geq k_0$. Thus, if $b_k < b$, we have $a \leq b_k$. Thus $\lambda_{i_N}(M_N^{(p(N))}) > a$. Since according to Theorem 7.6.1, there is no eigenvalues of $M_N^{(p_{k+1})}$ in $[a, b]$ for N large, we can deduce that $\lambda_{i_N}(M_N^{(p(N))}) > b$.□
We readily deduce the following

Corollary 7.7.5. *Let $\varepsilon > 0$. Let us fix u in $\{\Theta_{\sigma,\nu}\}$. Let us choose $\delta > 0$ small enough such that for large N, $[u - 2\delta; u - \delta]$ is included in $\mathbb{R} \setminus F_{\nu,\alpha} \cap \mathbb{R} \setminus \mathrm{Spect}(A_N)$ and $|H_{\nu,\alpha}(u - 2\delta) - H_{\nu,\alpha}(u)| < \varepsilon$. Let $i_N = i_N(u)$ be such that*

$$\lambda_{i_N}(A_N) > u - \delta \text{ and } \lambda_{i_N+1}(A_N) < u - 2\delta.$$

Then

$$\mathbb{P}\big[\lambda_{i_N}(M_N) > H_{\nu,\alpha}(u) - \varepsilon \text{ for all large } N\big] = 1.$$

Theorem 7.7.6. *1) If $\theta_1 > t_L$, the k_1 eigenvalues $(\lambda_j(M_N), 1 \leq j \leq k_1)$ converge almost surely outside the support of $\pi_{\alpha,1} \boxplus \nu$ towards $\rho_{\theta_1} = H_{\nu,\alpha}(\theta_1)$.*

2) Else, the largest eigenvalues ($\lambda_j(M_N)$, $1 \le j \le k$) converge almost surely to $H_{\nu,\alpha}(t_L)$ which is the right endpoint of the support of $\pi_{\alpha,1} \boxplus \nu$.

Proof of Theorem 7.7.6: 1) Corollary 7.7.5 (choosing $u = \theta_1$) and Theorem 7.6.1 readily give for any $\varepsilon > 0$

$$\mathbb{P}[\rho_{\theta_1} - \varepsilon \le \lambda_{k_1}(M_N) \le \cdots \le \lambda_1(M_N) \le \rho_{\theta_1} + \varepsilon, \text{ for } N \text{ large}] = 1. \quad (7.38)$$

Hence

$$\forall 1 \le j \le k_1, \quad \lambda_j(M_N) \xrightarrow[N\to+\infty]{a.s.} \rho_{\theta_1}.$$

2) Let $\varepsilon > 0$. According to Theorem 7.6.1,

$$\mathbb{P}\big[\lambda_1(M_N) < H_{\nu,\alpha}(t_L) + \varepsilon \text{ for all large } N\big] = 1.$$

Now, almost surely, the number of eigenvalues of M_N being in $]H_{\nu,\alpha}(t_L) - \varepsilon, H_{\nu,\alpha}(t_L)]$ should tend to infinity when N goes to infinity. Hence we should have

$$H_{\nu,\alpha}(t_L) - \varepsilon \le \lambda_{k_1}(M_N).$$

Hence, we deduce that for all $1 \le j \le k_1$, $\quad \lambda_j(M_N) \xrightarrow[N\to+\infty]{a.s.} H_{\nu,\alpha}(t_L)$. □.

7.8 Appendix

Lemma 7.8.1. *$\exists K > 0$, $\forall N$, $\mathbb{E}\left(\left[\lambda_1(\frac{B^*B}{N})\right]^3\right) < K$.*

Proof It can be proved as in Lemma 5.1 of [HT05], using previous results in [HT03].

Lemma 7.8.2. *$\forall z \in \mathbb{C}^+$,*

$$\frac{1}{1 - _{\mu_{A_N}}(z - \frac{p}{N}\frac{1}{1-g_N(z)})} = O(1). \quad (7.39)$$

$$\frac{1}{1 - \tilde{g}_N(z)} = O(1). \quad (7.40)$$

Proof of Lemma 7.8.2: We will only give the proof of (7.39). The proof of (7.40) is similar.
Let $z \in \mathbb{C}^+$.
By assumption,

$$\underset{N\in\mathbb{N}^*}{\cup} \mathrm{sp(A_N)} \subseteq \mathrm{[-M; M]}.$$

If $\mathrm{dist}(z - \frac{p}{N}\frac{1}{1-g_N(z)}, [-M; M]) > 2$, then

$$|G_{\mu_{A_N}}(z - \frac{p}{N}\frac{1}{1 - g_N(z)})| \le \frac{1}{2}.$$

It follows that

$$|\frac{1}{1-G_{\mu_{A_N}}(z-\frac{p}{N}\frac{1}{1-g_N(z)})}| \leq 2.$$

Otherwise, if $\mathrm{dist}(z-\frac{p}{N}\frac{1}{1-g_N(z)},[-M;M]) \leq 2$,

$$\begin{aligned} -\Im G_{\mu_{A_N}}(z-\frac{p}{N}\frac{1}{1-g_N(z)}) &= \frac{1}{N}\sum_{i=1}^{N}\frac{\Im(z-\frac{p}{N}\frac{1}{1-g_N(z)})}{|z-\frac{p}{N}\frac{1}{1-g_N(z)}-\gamma_i|^2} \\ &\geq \frac{\Im z}{(2M+2)^2}. \end{aligned}$$

Since

$$|\frac{1}{1-G_{\mu_{A_N}}(z-\frac{p}{N}\frac{1}{1-g_N(z)})}| \leq \frac{1}{|\Im G_{\mu_{A_N}}(z-\frac{p}{N}\frac{1}{1-g_N(z)})|},$$

we are done. □

Lemma 7.8.3. $\forall z \in \mathbb{C}^+$,

$$\frac{1}{1-g_N(z)} = O(1). \tag{7.41}$$

Proof of Lemma 7.8.3: Let $z \in \mathbb{C}^+$, and notice that

$$|1-g_N(z)| \geq |\Im(g_N(z))| = |\Im z|\mathbb{E}\left[\int_{\mathbb{R}}\frac{d\mu_{M_N}(x)}{|z-x|^2}\right].$$

Observe that, for any $x \in \mathrm{Spect}(M_N)$, one has

$$|z-x| \leq |z| + ||M_N|| \leq |z| + ||\frac{S_N}{N}|| + ||A_N||.$$

It follows

$$|z-x|^2 \leq 3(|z|^2 + ||\frac{S_N}{N}||^2 + ||A_N||^2)$$

and then

$$\begin{aligned} \mathbb{E}\left[\int_{\mathbb{R}}\frac{d\mu_{M_N}(x)}{|z-x|^2}\right] &\geq \mathbb{E}\left[\frac{1}{3(|z|^2+||\frac{S_N}{N}||^2+||A_N||^2)}\right] \\ &\geq \frac{1}{3(|z|^2+\mathbb{E}\left[||\frac{S_N}{N}||^2\right]+||A_N||^2)}. \end{aligned}$$

We know that the sequences $\mathbb{E}\left[||\frac{S_N}{N}||^2\right]$ and $||A_N||^2$ are bounded. There exists thus a positive constant K such that

$$\mathbb{E}\left[\int_{\mathbb{R}}\frac{d\mu_{M_N}(x)}{|z-x|^2}\right] \geq \frac{1}{3(|z|+K)^2}.$$

Finally,

$$\frac{1}{|1-g_N(z)|} \leq (|z|+K)^2 \frac{3}{|\Im z|},$$

which concludes the proof. □

We present now the different estimates on the variance used throughout the paper. They rely on the Gaussian Poincaré inequality. Let $Z_1, \ldots, Z_q$ be q independent centered Gaussian variables with variance σ^2. For any $\mathcal{C}^1$ function $f : \mathbb{R}^q \to \mathbb{C}$ such that f and grad$f$ are in $L^2(\mathcal{N}(0, \sigma^2 I_q))$, we have

$$\mathbf{V}\{f(Z_1, \ldots, Z_q)\} \leq \sigma^2 \mathbb{E}\|(\mathrm{grad} f)(Z_1, \ldots, Z_q)\|^2, \tag{7.42}$$

denoting for any random variable a by $\mathbf{V}(a)$ its variance $\mathbb{E}(|a - \mathbb{E}(a)|^2)$.

Lemma 7.8.4.

$$\mathbb{E}\left(|\mathrm{tr} G - \mathbb{E}(\mathrm{tr} G)|^2\right) = O(\frac{1}{N^2}).$$

$$\mathbb{E}\left(|\mathrm{tr}(X_N G) - \mathbb{E}(\mathrm{tr}(X_N G))|^2\right) = O(\frac{1}{N^2}).$$

Let $D_N = \mathrm{diag}(d_1, \ldots, d_N)$ *where* $d_k = \frac{1}{z - \gamma_k - \frac{p(N)}{N}\frac{1}{1-g_N(z)}}$.

$$\mathbb{E}\left(|\mathrm{tr}(D_N G) - \mathbb{E}(\mathrm{tr}(D_N G))|^2\right) = O(\frac{1}{N^2}).$$

Proof Let us define $\Psi : \mathbb{R}^{2(p\times N)} \to \mathcal{M}_{p\times N}(\mathbb{C})$ by

$$\Psi : \{x_{ij}, y_{ij}, i = 1, \ldots, p, j = 1, \ldots, N\} \to \sum_{i=1,\ldots,p} \sum_{j=1,\ldots,N} \left(x_{ij} + \sqrt{-1} y_{ij}\right) e_{ij}.$$

Let F be a smooth complex function on $\mathcal{M}_{p\times N}(\mathbb{C})$ and define the complex function f on $\mathbb{R}^{2(p\times N)}$ by setting $f = F \circ \Psi$. Then,

$$\|\mathrm{grad} f(u)\| = \sup_{V \in \mathcal{M}_{p\times N}(\mathbb{C}), TrVV^*=1} |\frac{d}{dt} F(\Psi(u) + tV)|_{t=0}|.$$

Now, $B = \Psi(\Re(b_{ij}), \Im(b_{ij}), 1 \leq i \leq p, 1 \leq j \leq N)$ where the distribution of $\{\Re(b_{ij}), \Im(b_{ij}), 1 \leq i \leq p, 1 \leq j \leq N\}$ is $\mathcal{N}(0, I_{2pN})$.
Hence consider $F : \; H \to \mathrm{tr}(zI_N - \frac{H^*H}{N} - A_N)^{-1}$.
Let $V \in \mathcal{M}_{p\times N}(\mathbb{C})$ such that $TrVV^* = 1$.

$$\frac{d}{dt} F(B + tV)|_{t=0} = \frac{1}{N} \{\mathrm{tr}(GV^*BG) + \mathrm{tr}(GB^*VG)\}$$

Moreover

$$\begin{aligned} |\mathrm{tr}(GV^*BG)| &\leq \frac{1}{N^{\frac{1}{2}}} (\mathrm{Tr}VV^*)^{\frac{1}{2}} \left[\mathrm{tr}(B^*BG^2(G^*)^2)\right]^{\frac{1}{2}} \\ &\leq \frac{1}{|\Im z|^2} \left[\lambda_1(\frac{1}{N} B^*B)\right]^{\frac{1}{2}}. \end{aligned}$$

We get obviously the same bound for $|\mathrm{tr}(GB^*VG)|$. Thus,

$$\mathbb{E}\left(\|\mathrm{grad}f\|^2\right) \le \frac{4}{N^2}\mathbb{E}\left[\lambda_1(\frac{1}{N}B^*B)\right].$$

Since, according to Lemma 7.8.1, $\mathbb{E}\left[\lambda_1(\frac{1}{N}B^*B)\right]$ is bounded independently of N, we can conclude by (7.42) that

$$\mathbf{V}(\mathrm{tr}G) = O(1/N^2).$$

Since D_N is deterministic and its norm is obviously bounded by $\frac{1}{|\Im z|}$ it is clear that one can get in the same way

$$\mathbf{V}(\mathrm{tr}(D_N G)) = O(1/N^2).$$

Now, consider $F:\ H \to \mathrm{tr}\left[\frac{H^*H}{N}(zI_N - \frac{H^*H}{N} - A_N)^{-1}\right]$.
Let $V \in \mathcal{M}_{p\times N}(\mathbb{C})$ be such that $\mathrm{Tr}VV^* = 1$.

$$\begin{aligned}\frac{d}{dt}F(B+tV)|_{t=0} &= \frac{1}{N}\mathrm{tr}(V^*BG) + \frac{1}{N}\mathrm{tr}(B^*VG)\\ &+ \frac{1}{N}\mathrm{tr}\left[B^*BG(V^*B+B^*V)G\right]\end{aligned}$$

As in the previous analysis $|\mathrm{tr}(V^*BG)|$ and $|\mathrm{tr}(B^*VG)|$ are bounded by $\frac{1}{|\Im z|}\left[\lambda_1(\frac{1}{N}B^*B)\right]^{\frac{1}{2}}$. Moreover,

$$\begin{aligned}|\frac{1}{N}\mathrm{tr}B^*BGV^*BG| &\le \frac{1}{N}\frac{1}{\sqrt{N}}\left[\mathrm{tr}BGB^*BGG^*B^*BG^*B^*\right]^{\frac{1}{2}}\\ &\le \frac{1}{|\Im z|^2}\left[\lambda_1(\frac{1}{N}B^*B)\right]^{\frac{3}{2}}\end{aligned}$$

and the same bound obviously holds for $|\frac{1}{N}\mathrm{tr}B^*BGB^*VG|$.
According to Lemma 7.8.1, $\mathbb{E}[\left\{\lambda_1(\frac{1}{N}B^*B)\right\}^3]$ is bounded, independently of N. Hence the previous analysis allows to conclude that

$$\mathbf{V}(\mathrm{tr}(X_N G)) = O(1/N^2). \quad \Box$$

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Résumé : Le travail effectué dans cette thèse concerne les domaines de la théorie des matrices aléatoires et des probabilités libres, dont on connaît les riches connexions depuis le début des années 90. Les résultats s'organisent principalement en deux parties : la première porte sur la liberté infinitésimale, la seconde sur les matrices aléatoires déformées.

Plus précisément, on jette les bases d'une théorie combinatoire de la liberté infinitésimale, au premier ordre d'abord, telle que récemment introduite par Belinschi et Shlyakhtenko, puis aux ordres supérieurs. On en donne un cadre simple et général, et on introduit des fonctionnelles de cumulants non-croisés, caractérisant la liberté infinitésimale. L'accent est mis sur la combinatoire et les idées d'essence différentielle qui sous-tendent cette notion.

La seconde partie poursuit l'étude des déformations de modèles matriciels, qui a été ces dernières années un champ de recherche très actif. Les résultats présentés sont originaux en ce qu'ils concernent des perturbations déterministes Hermitiennes de rang non nécessairement fini de matrices de Wigner et de Wishart. En outre, un apport de ce travail est la mise en lumière du lien entre la convergence des valeurs propres de ces modèles et les probabilités libres, plus particulièrement le phénomène de subordination pour la convolution libre. Ce lien donne une illustration de la puissance des idées des probabilités libres dans les problèmes de matrices aléatoires.

Mots clés : Probabilités ; Matrices aléatoires ; Probabilités libres ; Probabilités libres de type B ; Liberté infinitésimale ; Cumulants non-croisés infinitésimaux ; Système dual de dérivation ; Subordination ; Modèle matriciel déformé ; Plus grande valeur propre ; Valeur propre extrêmale ; Matrice de Wigner ; Matrice de covariance empirique.

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