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# Reconstructing initial data using iterative observers for wave type systems.

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**Abstract**

An iterative algorithm for solving initial data inverse problems from partial observations has been proposed in 2010 by Ramdani, Tucsnak and Weiss [1]. In this work, we are concerned with the convergence of this algorithm when the inverse problem is ill-posed, *i.e.* when the observations are not sufficient to reconstruct any initial data. We prove that the state space can be decomposed as a direct sum, stable by the algorithm, corresponding to the observable and unobservable part of the initial data. We show that this result holds for both locally distributed and boundary observation [2], [3].

**Introduction**

Let us start by briefly recalling the principle of the reconstruction method proposed in [1] in the simplified context of skew-adjoint generators and bounded observation operator. Given two Hilbert spaces $X$ and $Y$ (called *state* and *output* spaces respectively), let $A : \mathcal{D}(A) \to X$ be skew-adjoint operator generating a $C_0$-group $\mathbb{T}$ of isometries on $X$ and let $C \in \mathcal{L}(X, Y)$ be a bounded observation operator. Consider the infinite dimensional linear system given by

$$\left\{ \begin{array}{ll} \dot{z}(t) = Az(t), & \forall t \geq 0, \\ y(t) = Cz(t), & \forall t \in [0, \tau]. \end{array} \right. \tag{1}$$

where $z$ is the state and $y$ the output function (where the dot symbol is used to denote the time derivative). Such systems are often used as models of vibrating systems.

The inverse problem considered here is to reconstruct the initial state $z(0) = z_0 \in X$ of system (1) knowing *the observation* $y(t)$ on the time interval $[0, \tau]$.

Then, let $z_0^+ \in X$ be a first arbitrary guess of $z_0$ and let us denote $A^+ = A - C^*C$ and $A^- = -A - C^*C$ and introduce the following initial and final Cauchy problems, for all $n \geq 1$, called respectively *forward* and *backward observers* of (1)

$$\left\{ \begin{array}{l} \dot{z}_n^+(t) = A^+ z_n^+(t) + C^* y(t), \quad \forall t \in [0, \tau], \\ z_1^+(0) = z_0^+, \\ z_n^+(0) = z_{n-1}^-(0), \quad \forall n \geq 2, \end{array} \right. \tag{2}$$

$$\left\{ \begin{array}{l} \dot{z}_n^-(t) = -A^- z_n^-(t) - C^* y(t), \quad \forall t \in [0, \tau], \\ z_n^-(\tau) = z_n^+(\tau), \quad \forall n \geq 2. \end{array} \right. \tag{3}$$

If we assume that $(A, C)$ is exactly observable in time $\tau > 0$, i.e. that there exists $k_\tau > 0$ such that

$$\int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \forall z_0 \in \mathcal{D}(A), \tag{4}$$

then, it is well-known that $A^+$ (respectively $A^-$) generate an exponentially stable $C_0$-semigroup $\mathbb{T}^+$ (respectively $\mathbb{T}^-$) on $X$. If we set $\mathbb{L} = \mathbb{T}_\tau^- \mathbb{T}_\tau^+$, then by [1, Proposition 3.7], we have $\delta := \|\mathbb{L}\|_{\mathcal{L}(X)} < 1$ and we obtain

$$\|z_n^-(0) - z_0\| \leq \delta^n \|z_0^+ - z_0\|, \quad \forall z_0 \in X, n \geq 1.$$

Note that since the choice of $z_0^+$ is arbitrary, we often choose zero in the applications.

## 1 Main results

In this work, we investigate the case without exact observability (for the wave equation for instance, this corresponds to the case where $\tau$ is too small for the geometric optic condition of Bardos, Lebeau and Rauch [4] to hold true). Remarking that systems (2) and (3) are still well defined in this case (at least when $C$ is bounded), and that we still have

$$z_n^-(0) - z_0 = \mathbb{L}^n \left( z_0^+ - z_0 \right),$$

**the following questions naturally arise : does the sequence $z_n^-(0)$ converge and if so, to what limit ?**

Assume that $C \in \mathcal{L}(X, Y)$ is a bounded observation operator. Let us denote $\mathbb{S}$ the unitary $C_0$-group generated by $A$. Let $\Psi_\tau \in \mathcal{L}(X, L^2([0, \infty), Y)$ be the state-to-output operator defined by

$$(\Psi_\tau z_0)(t) = \left\{ \begin{array}{ll} C\mathbb{S}_t z_0, & \forall t \in [0, \tau], \\ 0, & \forall t > \tau. \end{array} \right.$$

**Proposition 1.** *We have the following decomposition of the state space $X$*

$$X = \operatorname{Ker} \Psi_\tau \oplus (\operatorname{Ker} \Psi_\tau)^\perp := \mathrm{V}_{\mathrm{Unobs}} \oplus \mathrm{V}_{\mathrm{Obs}},$$

*and this decomposition is $\mathbb{L}$-stable.*

*Furthermore,* $(\operatorname{Ker} \Psi_\tau)^\perp = \overline{\operatorname{Ran} \Phi_\tau}$*, where*

$$\Phi_\tau u = \int_0^\tau \mathbb{S}_{\tau - t}^* C^* u(t) dt,$$

*is the input-to-state operator.*

**Theorem 2.** *Denote by* $\Pi$ *the orthogonal projection from* $X$ *onto* $\mathrm{V}_{\mathrm{Obs}}$. *Then the following statements hold true:*

1. *We have for all* $z_0 \in X$, $z_0^+ \in \mathrm{V}_{\mathrm{Obs}}$, *and* $n \geq 1$,
$$\left\|(I-\Pi)\left(z_n^-(0)-z_0\right)\right\| = \|(I-\Pi)\, z_0\| .$$

2. *The sequence* $\left(\|\Pi\left(z_n^-(0)-z_0\right)\|\right)_{n\geq 1}$ *is strictly decreasing and verifies*
$$\left\|\Pi\left(z_n^-(0)-z_0\right)\right\| = \left\|z_n^-(0)-\Pi z_0\right\| \underset{n\to\infty}{\longrightarrow} 0.$$

3. *There exists a constant* $\alpha \in (0,1)$, *independent of* $z_0$ *and* $z_0^+$, *such that for all* $n \geq 1$,
$$\left\|\Pi\left(z_n^-(0)-z_0\right)\right\| \leq \alpha^n \left\|z_0^+ - \Pi z_0\right\|,$$
*if and only if* $\operatorname{Ran}\Phi_\tau$ *is closed in* $X$.

Using the framework of well-posed linear systems, we can use a result of Curtain and Weiss [5] to handle the case of (some) unbounded observation operators and derive a result similar to Theorem 2 (formally, we take $A^{\pm} = \pm A - \gamma C^* C$, with a suitably chosen $\gamma > 0$).

## 2 Application

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\Gamma_0$ and $\Gamma_1$ being relatively open in $\partial\Omega$. Denote by $\nu$ the unit normal vector of $\Gamma_1$ pointing towards the exterior of $\Omega$. Consider the following wave system

$$\begin{cases} \ddot{w}(x,t) - \Delta w(x,t) = 0, & \forall x \in \Omega, t > 0, \\ w(x,t) = 0, & \forall x \in \Gamma_0, t > 0, \\ w(x,t) = u(x,t), & \forall x \in \Gamma_1, t > 0, \\ w(x,0) = w_0(x),\ \dot{w}(x,0) = w_1(x), & \forall x \in \Omega, \end{cases} \tag{5}$$

with $u$ the input function (the control), and $(w_0, w_1)$ the initial state. We observe this system on $\Gamma_1$, leading to

$$y(x,t) = -\frac{\partial(-\Delta)^{-1}\dot{w}(x,t)}{\partial\nu}, \quad \forall x \in \Gamma_1, t > 0. \tag{6}$$

Using a result of Guo and Zhang [6, Theorem 1.1], we can show that the system (5)–(6) fits into the framework described above and we can thus apply Theorem 2 (in its generalized version to unbounded observation operators) to recover the observable part of the initial data $(w_0, w_1)$.

For instance, let us consider the configuration of Figure 1. We can easily obtain two subdomains of $\Omega$ (the striped ones on Figure 1), such that all initial data with support in the left (resp. right) one are in $\mathrm{V}_{\mathrm{Obs}}$ (resp. in $\mathrm{V}_{\mathrm{Unobs}}$).

We choose a suitable initial data to bring out these inclusions (in particular $w_1 \equiv 0$). We perform some simulations (using GMSH and GetDP) and obtain Figure 2, with 6% of relative error (in $L^2(\Omega)$) on the reconstruction of the observable part of the data after three iterations.

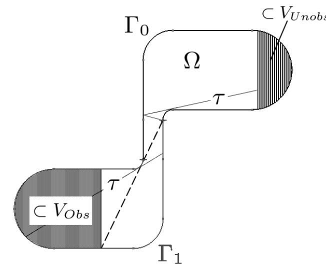


Figure 1: An example of configuration in 2D

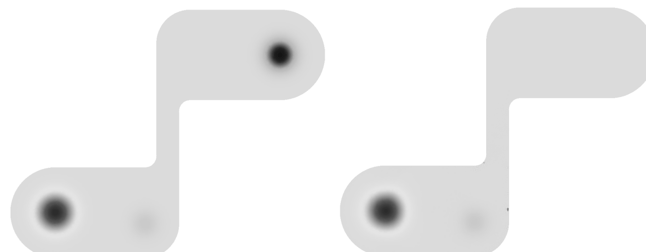

Figure 2: The initial position and its reconstruction after 3 iterations


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[1] K. Ramdani, M. Tucsnak, and G. Weiss, *Recovering the initial state of an infinite-dimensional system using observers*, Automatica, 46 (2010), pp. 1616–1625.

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[4] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992), pp. 1024–1065.

[5] R. F. Curtain and G. Weiss, *Exponential stabilization of well-posed systems by colocated feedback*, SIAM J. Control Optim., 45 (2006), pp. 273–297 (electronic).

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$$\begin{cases} \dot{z}(t) = Az(t), & \forall\, t \in [0, \infty), \\ z(0) = z_0 \in \mathcal{D}(A). \end{cases}$$

For instance:

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \text{ (+ Dirichlet boundary conditions) on } \Omega \subset \mathbb{R}^n$$

and $X = H_0^1(\Omega) \times L^2(\Omega)$

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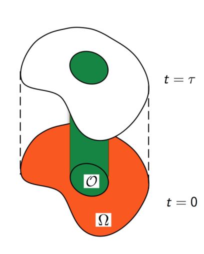
- $Y$ be another Hilbert space
- $C \in \mathcal{L}(X, Y)$
- $\tau > 0$

We observe $z$ *via* $y(t) = Cz(t)$ for all $t \in [0, \tau]$.

For instance, for the classical wave equation, let $\mathcal{O} \subset \Omega$:

$$\begin{aligned} y(t) &= \begin{bmatrix} 0 & \chi_{\mathcal{O}} \end{bmatrix} \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}, & \forall t \in [0, \tau], \\ &= \chi_{\mathcal{O}} \dot{w}(t), & \forall t \in [0, \tau]. \end{aligned}$$

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**Reconstruct the unknown $z_0$ in $X$ from the measurement $y(t)$.**

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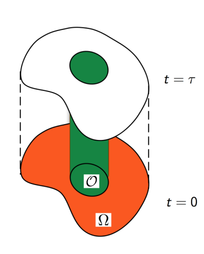


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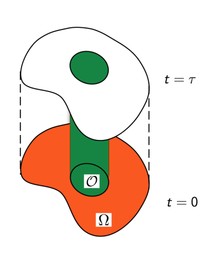


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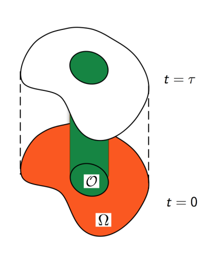


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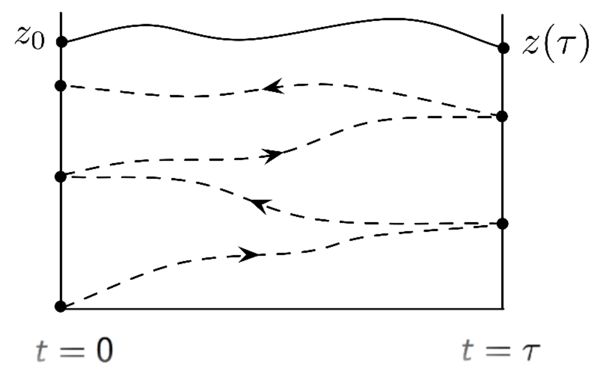
K. Ramdani, M. Tucsnak, and G. Weiss
*Recovering the initial state of an infinite-dimensional system using observers* (Automatica, 2010)

## Intuitive representation




*2 iterations, observation on* $[0, \tau]$.

# Some remarks

- **2005:** Auroux and Blum (*C. R. Math. Acad. Sci. Paris*) introduced the Back and Forth Nuding (BFN), based on the generalization of Kalmann's filters
- **2008:** Phung and Zhang (*SIAM J. Appl. Math.*) introduced the Time Reversal Focusing (TRF), for the Kirchhoff plate equation
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$$\begin{cases} \dot{z}^+(t) = Az^+(t) - C^*Cz^+(t) + C^*y(t), & \forall\, t \in [0,\tau], \\ z^+(0) = z_0^+ \in \mathcal{D}(A). \end{cases}$$

If we subtract the observed system

$$\begin{cases} \dot{z}(t) = Az(t), & \forall\, t \in [0,\tau], \\ z(0) = z_0, \end{cases}$$

to obtain *(remember that $y(t) = Cz(t)$)*, denoting $e = z^+ - z$,

$$\begin{cases} \dot{e}(t) = (A - C^*C)\, e(t), & \forall\, t \in [0,\tau], \\ e(0) = z_0^+ - z_0, \end{cases}$$

which is known to be exponentially stable if and only if $(A, C)$ is exactly observable, *i.e.*

$$\exists T > 0, \exists k_T > 0, \int_0^T \|y(t)\|^2 dt \geq k_T^2 \|z_0\|^2, \qquad \forall\, z_0 \in \mathcal{D}(A).$$

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Exponential stability $\Rightarrow \exists M > 0, \beta > 0$ such that

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We construct a similar system: the **backward observer**,

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$$\|z^-(0) - z_0\| \le M e^{-\beta\tau} \|z^+(\tau) - z(\tau)\| \le M^2 e^{-2\beta\tau} \|z_0^+ - z_0\|.$$

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However, the algorithm doesn't need this assumption to be well-posed.

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In this work, the exact observability assumption in time $\tau$

$$\exists k_\tau > 0, \int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \qquad \forall\, z_0 \in \mathcal{D}(A),$$

**is not supposed to be satisfied !**
However, the algorithm doesn't need this assumption to be well-posed.

## Questions

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**Decomposition of $X$:**

- Let us denote $\Psi_\tau$ the following continuous linear operator

$$\begin{array}{rcccc} \Psi_\tau & : & X & \longrightarrow & L^2([0,\tau],Y), \\ & & z_0 & \mapsto & y(t). \end{array}$$

*Intuitively, if $z_0$ is in Ker $\Psi_\tau$, then $y(t) \equiv 0$, and we have no information on $z_0$ !*

- We decompose $X = \text{Ker } \Psi_\tau \oplus (\text{Ker } \Psi_\tau)^\perp$ and define

$$\mathrm{V}_{\mathrm{Unobs}} = \text{Ker } \Psi_\tau, \quad \mathrm{V}_{\mathrm{Obs}} = (\text{Ker } \Psi_\tau)^\perp = \overline{\text{Ran } \Psi_\tau^*}.$$

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**Stability under the algorithm:**

Let us denote $\mathbb{T}^+$ (resp. $\mathbb{T}^-$) the semigroup generated by $A^+ := A - C^*C$ (resp. $A^- := -A - C^*C$) on $X$.

- Forward–backward observers cycle $\Rightarrow$ operator $\mathbb{T}^-_\tau \mathbb{T}^+_\tau$, *i.e.*

$$z^-(0) - z_0 = \mathbb{T}^-_\tau \mathbb{T}^+_\tau \left( z_0^+ - z_0 \right).$$

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$$\mathbb{S}_\tau z_0 = \mathbb{T}^+_\tau z_0 + \int_0^\tau \mathbb{T}^+_{\tau - t} C^* \underbrace{C \mathbb{S}_t z_0}_{\Psi_\tau z_0} dt, \qquad \forall\, z_0 \in X.$$

- Using this (type of) Duhamel formula(s), we obtain

$$\mathbb{T}^-_\tau \mathbb{T}^+_\tau \mathrm{V}_{\mathrm{Unobs}} \subset \mathrm{V}_{\mathrm{Unobs}}, \quad \mathbb{T}^-_\tau \mathbb{T}^+_\tau \mathrm{V}_{\mathrm{Obs}} \subset \mathrm{V}_{\mathrm{Obs}}.$$

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## Convergence of the algorithm:

- It is obvious that the algorithm has no influence on $\mathrm{V}_{\mathrm{Unobs}}$.
- Let us denote $L = \mathbb{T}_\tau^- \mathbb{T}_\tau^+|_{\mathrm{V}_{\mathrm{Obs}}}$, we have:
  1. $$\lim_{n\to\infty} L^n z = 0, \qquad \forall\, z \in X$$
  2. $$\|L\|_{\mathcal{L}(\mathrm{V}_{\mathrm{Obs}})} < 1 \iff \text{Ran } \Psi_\tau^* \text{ is closed in } X$$

### Sketch of proof

1. 
   - $L$ is positive self-adjoint.
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## Theorem

Denote by $\Pi$ the orthogonal projection from $X$ onto $\mathrm{V}_{\mathrm{Obs}}$. Then the following statements hold true for all $z_0 \in X$ and $z_0^+ \in \mathrm{V}_{\mathrm{Obs}}$:

1. For all $n \geq 1$,

$$\left\|(I-\Pi)\left(z_n^-(0)-z_0\right)\right\| = \|(I-\Pi)\, z_0\| .$$

2. The sequence $\left(\|\Pi\left(z_n^-(0)-z_0\right)\|\right)_{n\geq 1}$ is strictly decreasing and

$$\left\|\Pi\left(z_n^-(0)-z_0\right)\right\| = \left\|z_n^-(0)-\Pi z_0\right\| \underset{n\to\infty}{\longrightarrow} 0.$$

3. There exists a constant $\alpha \in (0,1)$, independent of $z_0$ and $z_0^+$, such that for all $n \geq 1$,

$$\left\|\Pi\left(z_n^-(0)-z_0\right)\right\| \leq \alpha^n \left\|z_0^+ - \Pi z_0\right\| ,$$

if and only if Ran $\Psi_\tau^*$ is closed in $X$.

## Remark

Using the framework of well-posed linear systems, we obtain the same result for some unbounded observation operator $C \in \mathcal{L}(\mathcal{D}(A), Y)$.

## Example

Let

- $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$
- $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$

Consider the following wave system

$$\begin{cases} \ddot{w}(x,t) - \Delta w(x,t) = 0, & \forall x \in \Omega, t > 0, \\ w(x,t) = 0, & \forall x \in \Gamma_0, t > 0, \\ w(x,t) = u(x,t), & \forall x \in \Gamma_1, t > 0, \\ w(x,0) = w_0(x),\ \dot{w}(x,0) = w_1(x), & \forall x \in \Omega, \end{cases}$$

with $u$ the control, and $(w_0, w_1)$ the initial state.

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## Observation

Let $\nu$ be the unit normal vector of $\Gamma_1$ pointing towards the exterior of $\Omega$, we observe the system *via*

$$y(x,t) = -\frac{\partial(-\Delta)^{-1}\dot{w}(x,t)}{\partial\nu}, \quad \forall x \in \Gamma_1, t > 0.$$

- Guo and Zhang (SIAM J. Control Optim., 2005) $\Rightarrow$ well-posed linear system.
- Curtain and Weiss (SIAM J. Control Optim., 2006) $\Rightarrow$ construction of forward and backward observers (formally $A^{\pm} = \pm A - C^*C$).
- So we can use the algorithm.

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$$y(x,t) = -\frac{\partial(-\Delta)^{-1}\dot{w}(x,t)}{\partial\nu}, \quad \forall x \in \Gamma_1, t > 0.$$

- Guo and Zhang (SIAM J. Control Optim., 2005) $\Rightarrow$ well-posed linear system.
- Curtain and Weiss (SIAM J. Control Optim., 2006) $\Rightarrow$ construction of forward and backward observers (formally $A^{\pm} = \pm A - C^*C$).
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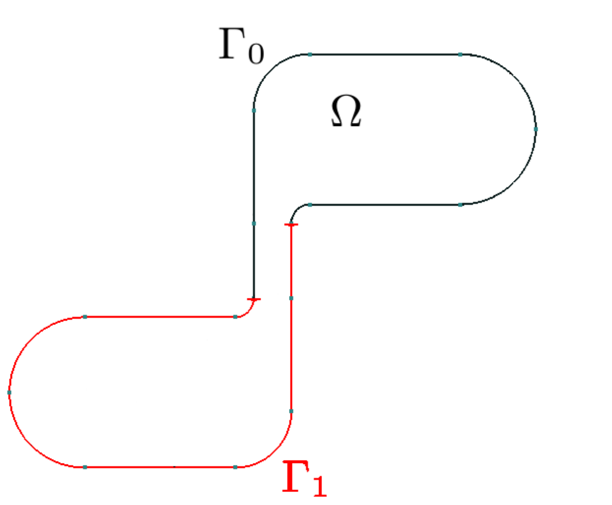
For instance, let us consider the following configuration



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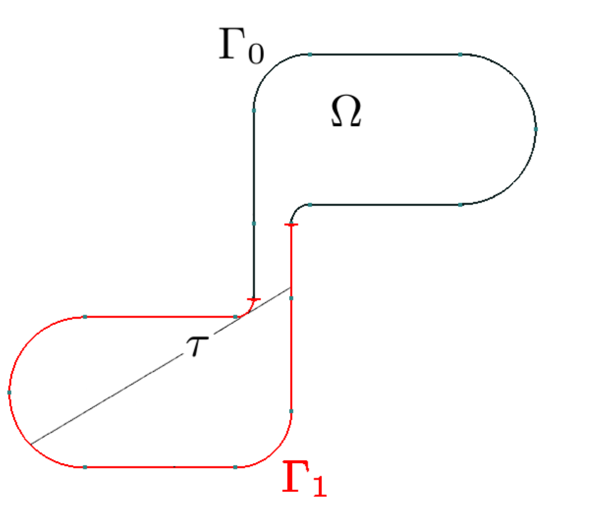

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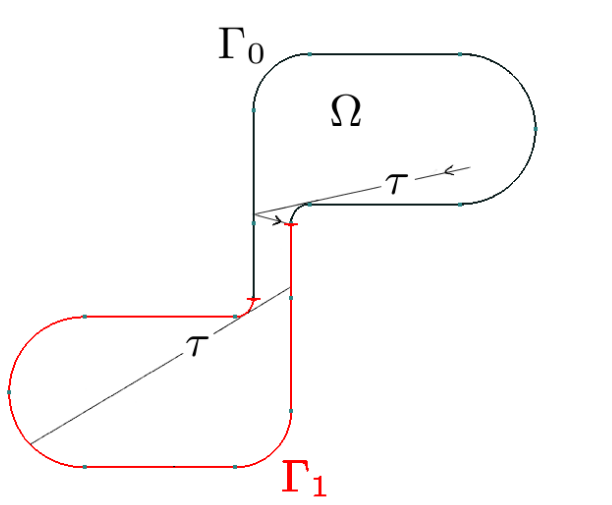

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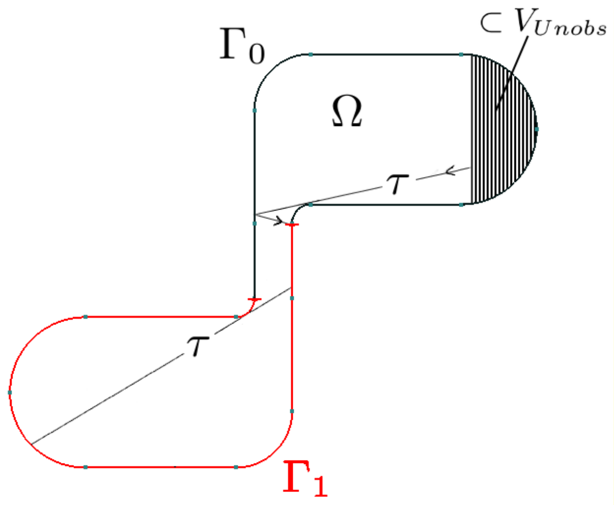

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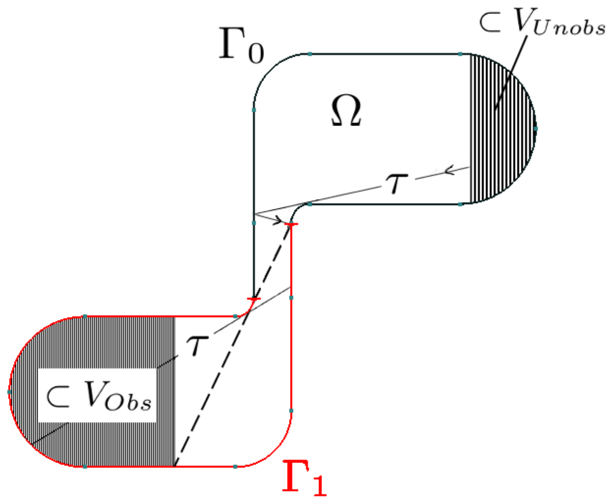

Choosing a suitable initial data

- Supp $w_0$ has three components $W_1, W_2$ and $W_3$, such that
  - $W_1 \subset \mathrm{V}_{\mathrm{Obs}}$
  - $W_2 \subset \mathrm{V}_{\mathrm{Unobs}}$
  - $W_3 \cap \mathrm{V}_{\mathrm{Obs}} \neq \emptyset$ and $W_3 \cap \mathrm{V}_{\mathrm{Unobs}} \neq \emptyset$
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To perform the test, we use

- Gmsh: a 3D finite element grid generator
- GetDP: a general finite element solver

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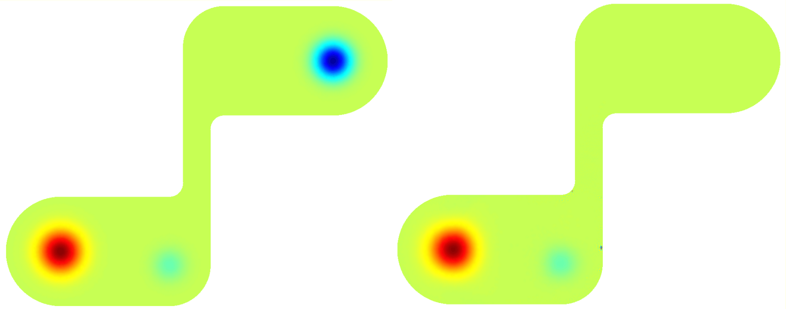

*The initial position and its reconstruction after 3 iterations*

$\Rightarrow$ 6% of relative error in $L^2(\Omega)$ on the "observable part".

**Work-in-progress:**

Application to thermo-acoustic tomography (simulations in progress)

**Still to be done:**

- Stability of $\mathrm{V}_{\mathrm{Obs}}$ and $\mathrm{V}_{\mathrm{Unobs}}$ with noisy observation $y$
- Generalization $(A^* \neq -A)$

# Thanks for your attention !

G. Haine
*Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint operator*
(Mathematics of Control, Signals, and Systems (MCSS), *In Revision*)