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Reconstructing initial data using observers: error analysis of the semi-discrete and fully discrete approximations

Ghislain Haine · **Karim Ramdani**

Abstract A new iterative algorithm for solving initial data inverse problems from partial observations has been recently proposed in Ramdani et al. (Automatica 46(10), 1616–1625, 2010). Based on the concept of observers (also called Luenberger observers), this algorithm covers a large class of abstract evolution PDE's. In this paper, we are concerned with the convergence analysis of this algorithm. More precisely, we provide a complete numerical analysis for semi-discrete (in space) and fully discrete approximations derived using finite elements in space and an implicit Euler method in time. The analysis is carried out for abstract Schrödinger and wave conservative systems with bounded observation (locally distributed).

Mathematics Subject Classification (2000) Primary 35Q93; Secondary 35L05 · 35J10 · 65M22

1 Introduction

The goal of this paper is to present a convergence analysis for the iterative algorithm recently proposed in Ramdani et al. [24] for solving initial state inverse problems from measurements over a time interval. This algorithm is based on the use back and forth in time of observers (sometimes called Luenberger observers or Kalman observers; see for instance Curtain and Zwart [6]). Inspired by the works of Mathias Fink on time

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reversal [9,10], Phung and Zhang [22] used this algorithm in the particular case of the Kirchhoff plate equation with distributed observation, while Ito et al. [15] considered more general evolution PDE's with locally distributed observation. Let us mention also Auroux and Blum [1] who implemented a similar algorithm in the context of data assimilation. More generally, during the last decade, observers have been designed for linear and nonlinear infinite-dimensional systems in many works, among which we can mention for instance Deguenon et al. [8], Guo and Guo [13], Guo and Shao [14] in the context of wave-type systems, Lasiecka and Triggiani [19], Smyshlyaev and Krstic [26] for parabolic systems and Krstic et al. [17] for the non linear viscous Burgers equation.

Let us first briefly describe the principle of the reconstruction method proposed in [24] in the simplified context of skew-adjoint generators and bounded observation operator. We will always work under these assumptions throughout the paper. Given two Hilbert spaces X and Y (called *state* and *output* spaces respectively), let $A : \mathcal{D}(A) \to X$ be skew-adjoint operator generating a C_0-group $\mathbb{T}$ of isometries on X and let $C \in \mathcal{L}(X, Y)$ be a bounded observation operator. Consider the infinite dimensional linear system given by

$$\begin{cases} \dot{z}(t) = Az(t), & \forall t \geqslant 0, \\ y(t) = Cz(t), & \forall t \in [0, \tau]. \end{cases} \tag{1.1}$$

where z is the state and y the output function (throughout the paper, the dot symbol is used to denote the time derivative). Such systems are often used as models of vibrating systems (e.g., the wave equation, the beam equation,...), electromagnetic phenomena (Maxwell's equations) or in quantum mechanics (Schrödinger's equation).

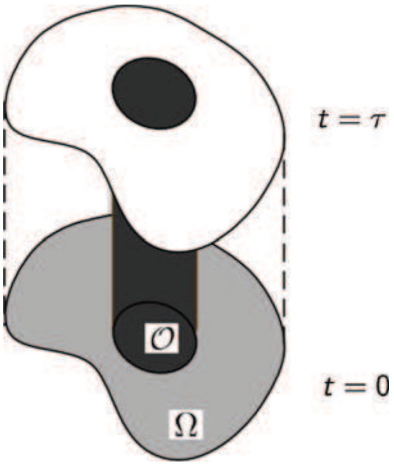
The inverse problem considered here is to reconstruct the initial state $z_0 = z(0)$ of system (1.1) knowing (*the observation*) $y(t)$ on the time interval $[0, \tau]$ (see Fig. 1). Such inverse problems arise in many applications, like thermoacoustic tomography Kuchment and Kunyansky [18] or data assimilation Puel [23]. To solve this inverse problem, we assume here that it is well-posed, i.e. that (A, C) is exactly observable in time $\tau > 0$. In other words, we assume that there exists $k_\tau > 0$ such that

$$\int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \qquad \forall\, z_0 \in \mathcal{D}(A).$$

For instance, in the case of the wave equation on a bounded domain Ω, this inequality holds provided we observe the state on $\mathcal{O} \times (0, \tau)$ where $\mathcal{O} \subset \Omega$ and τ are chosen such that the geometric optics condition of Bardos et al. [2] holds. For similar results related to other equations, see for instance Burq [3], Burq and Lebeau [4] and Jaffard [16] and the monograph of Lions [20].

Following Liu [21, Theorem 2.3.], we know that $A^+ = A - C^*C$ (respectively $A^- = -A - C^*C$) generate an exponentially stable C_0-semigroup $\mathbb{T}^+$ (respectively $\mathbb{T}^-$) on X. Then, we introduce the following initial and final Cauchy problems, called respectively *forward* and *backward observers* of (1.1)

Fig. 1 An initial data inverse problem for evolution PDE's: How to reconstruct the initial state (*light grey*) for a PDE set on a domain Ω from partial observation on $\mathcal{O} \times [0, \tau]$ (*dark grey*)?



$$\begin{cases} \dot{z}^+(t) = A^+ z^+(t) + C^* y(t), & \forall t \in [0, \tau], \\ z^+(0) = 0, \end{cases} \tag{1.2}$$

$$\begin{cases} \dot{z}^-(t) = -A^- z^-(t) - C^* y(t), & \forall t \in [0, \tau], \\ z^-(\tau) = z^+(\tau). \end{cases} \tag{1.3}$$

Note that the states z^+ and z^- of the forward and backward observers are completely determined by the knowledge of the output y. If we set $\mathbb{L}_\tau = \mathbb{T}_\tau^- \mathbb{T}_\tau^+$, then by [24, Proposition 3.7], we have $\eta := \|\mathbb{L}_\tau\|_{\mathcal{L}(X)} < 1$ and by [24, Proposition 3.3], the following remarkable relation holds true

$$z_0 = (I - \mathbb{L}_\tau)^{-1} z^-(0). \tag{1.4}$$

In particular, one can invert the operator $(I - \mathbb{L}_\tau)$ using a Neumann series and get the following expression for the initial state

$$z_0 = \sum_{n=0}^{\infty} \mathbb{L}_\tau^n z^-(0). \tag{1.5}$$

Thus, at least theoretically, the reconstruction of the initial state is given by the above formula. Note that the computation of the first term in the above sum requires to solve the two non-homogeneous systems (1.2) and (1.3), while the terms for $n \geq 1$ involve the resolution of the two homogeneous systems associated with (1.2) and (1.3) (i.e. for $y \equiv 0$). In practice, the reconstruction procedure requires the discretization of these two systems and the truncation of the infinite sum in (1.5) to keep only a finite number of back and forth iterations. For instance, if we consider a space semi-discretization corresponding to a mesh size h (typically a finite element approximation), one can only compute

$$z_{0,h} = \sum_{n=0}^{N_h} \mathbb{L}_{h,\tau}^n z_h^-(0), \tag{1.6}$$

where

- $\mathbb{L}_{h,\tau} = \mathbb{T}_{h,\tau}^- \mathbb{T}_{h,\tau}^+$, where $\mathbb{T}_{h,\tau}^\pm \in \mathcal{L}(X)$ are suitable space discretizations of $\mathbb{T}_\tau^\pm$,
- $z_h^-(0) \in X_h$ is an approximation of $z^-(0)$ in a suitable finite dimensional subspace X_h of X,
- N_h is a suitable truncation parameter.

Similarly, if a full discretization described by a mesh size h and a time step Δt is considered, one can compute

$$z_{0,h,\Delta t} = \sum_{n=0}^{N_{h,\Delta t}} \mathbb{L}_{h,\Delta t,K}^n \left(z_h^-\right)^0. \tag{1.7}$$

where

- $\mathbb{L}_{h,\Delta t,K} = \mathbb{T}_{h,\Delta t,K}^- \mathbb{T}_{h,\Delta t,K}^+$, where $\mathbb{T}_{h,\Delta t,K}^\pm$ are suitable space and time discretizations of $\mathbb{T}_\tau^\pm$,
- $\left(z_h^-\right)^0 \in X_h$ is an approximation of $z^-(0)$,
- $N_{h,\Delta t}$ is a suitable truncation parameter.

For the sake of clarity, the precise definition of the spaces and discretizations used will be given later in the paper.

Our objective in this work is to present a convergence analysis of $z_{0,h}$ and $z_{0,h,\Delta t}$ towards z_0. A particular attention will be devoted to the optimal choice of the truncation parameters N_h and $N_{h,\Delta t}$ for given discretization parameters (mesh size h and time step Δt). Let us emphasize that our error estimates (see (2.8), (2.27), (3.15) and (3.25)) provide in particular an upper bound for the maximum admissible noise under which convergence of the algorithm is guaranteed. As usually in approximation error theory of PDE's, some regularity assumptions are needed to obtain our error estimates. Namely, our result allows us to reconstruct only initial data contained in some subspace of X (namely $\mathcal{D}\left(A^2\right)$).

Let us emphasize that similar error estimates have been recently obtained by Cîndea et al. [5] in the context of control problems. Using Russel's "stabilizability implies controllability" principle, the authors derived a new approximation method of exact controls for second order wave type systems with bounded input operator. The convergence analysis is carried out in the case of a Galerkin type semi-discretization.

Let us now make some comments on the type of observation for which we have been able to prove convergence results. First of all, we assume throughout the paper that $C \in \mathcal{L}(X, Y)$ is a bounded observation operator (locally distributed observation). This assumption is crucially used many times in the proofs and it seems difficult to extend our result to the case of unbounded observation. However, the reconstruction algorithm seems to be still efficient in this case, as it can be seen from the numerical results given in [24].

In addition to the boundedness of C, we assume that $C^*C \in \mathcal{L}\left(\mathcal{D}\left(A^2\right)\right) \cap \mathcal{L}\left(\mathcal{D}\left(A\right)\right)$. The fact that $C^*C \in \mathcal{L}\left(\mathcal{D}\left(A\right)\right)$ ensures that the contraction property for $\mathbb{T}^+$ and $\mathbb{T}^-$ is still satisfied when restricted to $\mathcal{D}\left(A\right)$ and $\mathcal{D}\left(A^2\right)$ (see Lemma 1 of the Appendix). Let us point out that this is proved for the damped wave equation in Cîndea et al. [5, Proposition 2.5]. Moreover, we also have $\|\mathbb{L}_\tau\|_{\mathcal{D}(A)} < 1$ and $\|\mathbb{L}_\tau\|_{\mathcal{D}(A^2)} < 1$ (by application of [27, Proposition 2.10.4]). The second technical assumption $C^*C \in \mathcal{L}\left(\mathcal{D}\left(A^2\right)\right)$ appears naturally in our analysis, but not in the one carried out in Cîndea et al. [5]. Indeed, this assumption is used to bound a term which does not appear in the context of control problems they considered. Finally, let us point out that these assumptions are in particular satisfied when the locally distributed observation is obtained via a smooth cut-off function.

Remark 1 Using an implicit Euler method preserves the dissipative properties of the high frequency part of the solution (see (2.30) and (3.30)). This is the main reason for which we did not use an explicit or midpoint Euler scheme, but we do not know if this restriction is only technical or not.

The paper is organized as follows: in Sect. 2 we provide a convergence analysis of the algorithm for an abstract Schrödinger type system, by considering successively the semi-discretization (Sect. 2.1) and the full discretization (Sect. 2.2). In Sect. 3, similar results are given for an abstract wave system. Once again, we tackle successively the semi-discretization (Sect. 3.1) and the full discretization (Sect. 3.2). However, since the proofs are very similar to those of the Schrödinger case, they will not be given with full details. Finally, the Appendix is devoted to the proof of two technical lemmas which are used several times throughout the paper.

Throughout the paper, we denote by M a constant independent of τ, of the initial state z_0 and of the discretization parameters h and Δt, but which may differ from line to line in the computations.

2 Schrödinger equation

Let X be a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$. Let $A_0 : \mathcal{D}\left(A_0\right) \to X$ be a strictly positive self-adjoint operator and $C \in \mathcal{L}(X, Y)$ a bounded observation operator, where Y is an other Hilbert space. The norm in $\mathcal{D}(A_0^\alpha)$ will be denoted by $\|\cdot\|_\alpha$. We assume that there exists some $\tau > 0$ such that (iA_0, C) is exactly observable in time τ. Thus by Liu [21, Theorem 2.3.], $A^+ = iA_0 - C^*C$ (resp. $A^- = -iA_0 - C^*C$) is the generator of an exponentially stable C_0-semigroup $\mathbb{T}^+$ (resp. $\mathbb{T}^-$). We want to reconstruct the initial value z_0 of the following system

$$\begin{cases} \dot{z}(t) = iA_0 z(t), & \forall t \geqslant 0, \\ y(t) = Cz(t), & \forall t \in [0, \tau]. \end{cases} \tag{2.1}$$

Throughout this section we always assume that $z_0 \in \mathcal{D}\left(A_0^2\right)$. Thus by applying Theorem 4.1.6 of Tucsnak and Weiss [27], we have

$$z \in C\left([0, \tau], \mathcal{D}\left(A_0^2\right)\right) \cap C^1\left([0, \tau], \mathcal{D}\left(A_0\right)\right).$$

The forward and backward observers (1.2) and (1.3) read then as follows

$$\begin{cases} \dot{z}^+(t) = iA_0 z^+(t) - C^*Cz^+(t) + C^*y(t), & \forall t \in [0,\tau], \\ z^+(0) = 0, \end{cases} \tag{2.2}$$

$$\begin{cases} \dot{z}^-(t) = iA_0 z^-(t) + C^*Cz^-(t) - C^*y(t), & \forall t \in [0,\tau], \\ z^-(\tau) = z^+(\tau). \end{cases} \tag{2.3}$$

Clearly, the above systems can be rewritten in the general form of an initial value Cauchy problem (simply by using a time reversal for the second system)

$$\begin{cases} \dot{q}(t) = \pm iA_0 q(t) - C^*Cq(t) + F(t), & \forall t \in [0,\tau], \\ q(0) = q_0, \end{cases} \tag{2.4}$$

where we have set

- for the forward observer (2.2) : $F(t) = C^*y(t) = C^*Cz(t)$ and $q_0 = 0$,
- for the backward observer (2.3) : $F(t) = C^*y(\tau - t) = C^*Cz(\tau - t)$ and $q_0 = z^+(\tau) \in \mathcal{D}\left(A_0^2\right)$.

2.1 Space semi-discretization

2.1.1 Statement of the main result

We use a Galerkin method to approximate system (2.4). More precisely, consider a family $(X_h)_{h>0}$ of finite-dimensional subspaces of $\mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ endowed with the norm in X. We denote π_h the orthogonal projection from $\mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ onto X_h. We assume that there exist $M > 0$, $\theta > 0$ and $h^* > 0$ such that we have for all $h \in (0, h^*)$

$$\|\pi_h\varphi - \varphi\| \le Mh^\theta \|\varphi\|_{\frac{1}{2}}, \qquad \forall \varphi \in \mathcal{D}\left(A_0^{\frac{1}{2}}\right). \tag{2.5}$$

Given $q_0 \in \mathcal{D}\left(A_0^2\right)$, the variational formulation of (2.4) reads for all $t \in [0,\tau]$ and all $\varphi \in \mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ as follows

$$\begin{cases} \langle \dot{q}(t), \varphi \rangle = \pm i \langle q(t), \varphi \rangle_{\frac{1}{2}} - \langle C^*Cq(t), \varphi \rangle + \langle F(t), \varphi \rangle, \\ q(0) = q_0. \end{cases} \tag{2.6}$$

Suppose that $q_{0,h} \in X_h$ and F_h are given approximations of q_0 and F respectively in the spaces X and $L^1([0,\tau], X)$. For all $t \in [0,\tau]$, we define $q_h(t) \in X_h$ as the unique solution of the variational problem

$$\begin{cases} \langle \dot{q}_h(t), \varphi_h \rangle = \pm i \langle q_h(t), \varphi_h \rangle_{\frac{1}{2}} - \langle C^*Cq_h(t), \varphi_h \rangle + \langle F_h(t), \varphi_h \rangle, \\ q_h(0) = q_{0,h}. \end{cases} \tag{2.7}$$

for all $\varphi_h \in X_h$.

The above approximation procedure leads in particular to the definition of the semi-discretized versions $\mathbb{T}_h^{\pm}$ of the semigroups $\mathbb{T}^{\pm}$ that we will use. Indeed, we simply set

$$\mathbb{T}_t^{+} q_0 \simeq \mathbb{T}_{h,t}^{+} q_0 = q_h(t) \qquad \mathbb{T}_t^{-} q_0 \simeq \mathbb{T}_{h,t}^{-} q_0 = q_h(\tau - t)$$

where q_h is the solution of Eq. (2.7) with the corresponding sign and for $F_h = 0$ and $q_{0,h} = \pi_h q_0$. The approximation of $\mathbb{L}_\tau = \mathbb{T}_\tau^{-}\mathbb{T}_\tau^{+}$ follows immediately by setting

$$\mathbb{L}_{h,\tau} = \mathbb{T}_{h,\tau}^{-}\mathbb{T}_{h,\tau}^{+}.$$

Assume that y_h is an approximation of the output y in $L^1([0,\tau], Y)$ and let z_h^+ and z_h^- denote the Galerkin approximations of the solutions of systems (2.2) and (2.3), satisfying for all $t \in [0,\tau]$ and all $\varphi_h \in X_h$

$$\begin{cases} \left\langle \dot{z}_h^+(t), \varphi_h \right\rangle = i \left\langle z_h^+(t), \varphi_h \right\rangle_{\frac{1}{2}} - \left\langle C^*Cz_h^+(t), \varphi_h \right\rangle + \left\langle C^* y_h(t), \varphi_h \right\rangle, \\ z_h^+(0) = 0. \end{cases}$$

$$\begin{cases} \left\langle \dot{z}_h^-(t), \varphi_h \right\rangle = i \left\langle z_h^-(t), \varphi_h \right\rangle_{\frac{1}{2}} + \left\langle C^*Cz_h^-(t), \varphi_h \right\rangle - \left\langle C^* y_h(t), \varphi_h \right\rangle, \\ z_h^-(\tau) = z_h^+(\tau). \end{cases}$$

Thus, our main result in this subsection reads as follows.

Theorem 1 *Let* $A_0 : \mathcal{D}(A_0) \to X$ *be a strictly positive self-adjoint operator and* $C \in \mathcal{L}(X,Y)$ *such that* $C^*C \in \mathcal{L}\left(\mathcal{D}\left(A_0^2\right)\right) \cap \mathcal{L}\left(\mathcal{D}\left(A_0\right)\right)$*. Assume that the pair* (iA_0, C) *is exactly observable in time* $\tau > 0$ *and set* $\eta := \|\mathbb{L}_\tau\|_{\mathcal{L}(X)} < 1$*. Let* $z_0 \in \mathcal{D}\left(A_0^2\right)$ *be the initial value of* (2.1) *and* $z_{0,h}$ *be defined by* (1.6)*.*

Then there exist $M > 0$ *and* $h^* > 0$ *such that for all* $h \in (0, h^*)$

$$\|z_0 - z_{0,h}\| \le M \left[\left(\frac{\eta^{N_h+1}}{1-\eta} + h^\theta \tau N_h^2 \right) \|z_0\|_2 + N_h \int_0^\tau \|C^*\left(y(s) - y_h(s)\right)\| ds \right].$$

A particular choice of N_h leads to an explicit error estimate (with respect to h) as shown in the next Corollary (the proof is left to the reader because of its simplicity)

Corollary 1 *Under the assumptions of Theorem 1, we set*

$$N_h = \theta \frac{\ln h}{\ln \eta}.$$

Then, there exist $M_\tau > 0$ *and* $h^* > 0$ *such that for all* $h \in (0, h^*)$

$$\|z_0 - z_{0,h}\| \le M_\tau \left(h^\theta \ln^2 h \, \|z_0\|_2 + |\ln h| \int_0^\tau \|C^*\left(y(s) - y_h(s)\right)\| ds \right). \qquad (2.8)$$

Remark 2 In fact, Theorem 1 still holds true for $z_0 \in \mathcal{D}\left(A_0^{\frac{3}{2}}\right)$ (with the same proofs and slightly adapting the spaces). Nevertheless, we have not been able to carry out this analysis for the fully discrete approximation in this case. This is why we restricted our analysis to the case of an initial data $z_0 \in \mathcal{D}\left(A_0^2\right)$.

2.1.2 Proof of Theorem 1

Before proving Theorem 1, we first need to prove some auxiliary results. The next Proposition, which constitutes one of the main ingredients of the proof, provides the error estimate for the approximation in space of the initial value problem (2.6) by using the Galerkin scheme (2.7).

Proposition 1 *Given* $q_0 \in \mathcal{D}\left(A_0^2\right)$ *and* $q_{0,h} \in X_h$*, let* q *and* q_h *be the solutions of* (2.6) *and* (2.7) *respectively. Assume that* $C^*C \in \mathcal{L}\left(\mathcal{D}\left(A_0\right)\right)$*. Then, there exist* $M > 0$ *and* $h^* > 0$ *such that for all* $t \in [0, \tau]$ *and all* $h \in (0, h^*)$

$$\|\pi_h q(t) - q_h(t)\| \leq \|\pi_h q_0 - q_{0,h}\| + M h^\theta \Big[t \left(\|q_0\|_2 + \|F\|_{1,\infty} \right) + t^2 \|F\|_{2,\infty} \Big] + \int_0^t \|F(s) - F_h(s)\| ds,$$

where $\|F\|_{\alpha,\infty} = \sup_{t\in[0,\tau]} \|F(t)\|_\alpha$.

Proof First, we substract (2.7) from (2.6) and obtain (we omit the time dependence for the sake of clarity) for all $\varphi_h \in X_h$

$$\langle \dot{q} - \dot{q}_h, \varphi_h \rangle = \pm i \, \langle q - q_h, \varphi_h \rangle_{\frac{1}{2}} - \left\langle C^*C(q - q_h), \varphi_h \right\rangle + \langle F - F_h, \varphi_h \rangle .$$

Noting that $\langle \pi_h q - q, \varphi_h \rangle_{\frac{1}{2}} = 0$ for all $\varphi_h \in X_h$ and that $\pi_h \dot{q}$ makes sense by the regularity of q (see (4.1)), we obtain from the above equality that for all $\varphi_h \in X_h$

$$\begin{aligned} \langle \pi_h \dot{q} - \dot{q}_h, \varphi_h \rangle = {} & \langle \pi_h \dot{q} - \dot{q}, \varphi_h \rangle \pm i \, \langle \pi_h q - q_h, \varphi_h \rangle_{\frac{1}{2}} \\ & - \left\langle C^*C\left(q - q_h\right), \varphi_h \right\rangle + \langle F - F_h, \varphi_h \rangle . \end{aligned} \tag{2.9}$$

On the other hand, setting

$$\mathcal{E}_h = \frac{1}{2} \|\pi_h q - q_h\|^2,$$

we have

$$\dot{\mathcal{E}}_h = \mathrm{Re}\, \langle \pi_h \dot{q} - \dot{q}_h, \pi_h q - q_h \rangle .$$

Applying (2.9) with $\varphi_h = \pi_h q - q_h$ and substituting the result in the above relation, we obtain by using Cauchy-Schwarz inequality and the boundedness of C that there exists $M > 0$ such that

$$\dot{\mathcal{E}}_h \leq (\|\pi_h \dot{q} - \dot{q}\| + M\|\pi_h q - q\| + \|F - F_h\|) \underbrace{\|\pi_h q - q_h\|}_{=\sqrt{2\mathcal{E}_h}}.$$

Since $\frac{\dot{\mathcal{E}}_h}{\sqrt{2\mathcal{E}_h}} = \frac{d}{dt}\sqrt{2\mathcal{E}_h}$, the integration of the above inequality from 0 to t yields

$$\|\pi_h q(t) - q_h(t)\| \leq \|\pi_h q_0 - q_{0,h}\| + \int_0^t (\|\pi_h \dot{q}(s) - \dot{q}(s)\| + M\|\pi_h q(s) - q(s)\|)\, ds + \int_0^t \|F(s) - F_h(s)\| ds. \tag{2.10}$$

Thus, it remains to bound $\|\pi_h \dot{q}(t) - \dot{q}(t)\|$ and $\|\pi_h q(t) - q(t)\|$ for all $t \in [0, \tau]$. Using (2.5) and the classical continuous embedding from $\mathcal{D}(A^\alpha)$ to $\mathcal{D}(A^\beta)$ for $\alpha > \beta$, we get that

$$\begin{cases} \|\pi_h \dot{q}(t) - \dot{q}(t)\| \leq Mh^\theta \|\dot{q}(t)\|_{\frac{1}{2}} \leq Mh^\theta \|\dot{q}(t)\|_1, \\ \|\pi_h q(t) - q(t)\| \leq Mh^\theta \|q(t)\|_{\frac{1}{2}} \leq Mh^\theta \|q(t)\|_2, \end{cases} \qquad \forall t \in [0, \tau],\ h \in (0, h^*).$$

Using relations (4.2) and (4.3) proved in Lemma 2 of the Appendix, we get for all $t \in [0, \tau]$ and all $h \in (0, h^*)$

$$\|\pi_h \dot{q}(t) - \dot{q}(t)\| + \|\pi_h q(t) - q(t)\| \leq Mh^\theta \left(\|q_0\|_2 + t\|F\|_{2,\infty} + \|F\|_{1,\infty}\right).$$

Substituting the above inequality in (2.10), we get the result.

Using the last result, we derive an error approximation for the semigroups $\mathbb{T}^\pm$ and for the operator $\mathbb{L}_t = \mathbb{T}_t^- \mathbb{T}_t^+$.

Proposition 2 *Under the assumptions of Proposition 1, the following assertions hold true*

1. *There exist $M > 0$ and $h^* > 0$ such that for all $t \in (0, \tau)$ and all $h \in (0, h^*)$*

$$\left\|\pi_h \mathbb{T}_t^+ q_0 - \mathbb{T}_{h,t}^+ q_0\right\| \leq Mth^\theta \|q_0\|_2. \tag{2.11}$$

$$\left\|\pi_h \mathbb{T}_t^- q_0 - \mathbb{T}_{h,t}^- q_0\right\| \leq M(\tau - t)h^\theta \|q_0\|_2. \tag{2.12}$$

2. *There exist $M > 0$ and $h^* > 0$ such that for all $n \in \mathbb{N}$, all $t \in [0, \tau]$ and all $h \in (0, h^*)$, we have*

$$\|\mathbb{L}_t^n q_0 - \mathbb{L}_{h,t}^n q_0\| \leq M(1 + n\tau)h^\theta \|q_0\|_2. \tag{2.13}$$

Proof

1. It suffices to take $F = F_h = 0$ and $q_{0,h} = \pi_h q_0$ in Proposition 1.
2. We first note that

$$\|\mathbb{L}_t^n q_0 - \mathbb{L}_{h,t}^n q_0\| \leq \|\mathbb{L}_t^n q_0 - \pi_h \mathbb{L}_t^n q_0\| + \|\pi_h \mathbb{L}_t^n q_0 - \mathbb{L}_{h,t}^n q_0\|. \tag{2.14}$$

Using (2.5) and the fact that $\|\mathbb{L}_t\|_{\mathcal{L}(\mathcal{D}(A))} \leq 1$ proved in Lemma 1 of the Appendix, the first term in the above relation can be estimated as follows

$$\|\mathbb{L}_t^n q_0 - \pi_h \mathbb{L}_t^n q_0\| \leq M h^\theta \|q_0\|_2, \quad \forall h \in (0, h^*). \tag{2.15}$$

For the second term in (2.14), we prove by induction that for all $n \in \mathbb{N}$

$$\|\pi_h \mathbb{L}_t^n q_0 - \mathbb{L}_{h,t}^n q_0\| \leq M n \tau h^\theta \|q_0\|_2, \quad \forall h \in (0, h^*). \tag{2.16}$$

By definition, we have

$$\begin{aligned}\|\pi_h \mathbb{L}_t q_0 - \mathbb{L}_{h,t} q_0\| &= \|\pi_h \mathbb{T}_t^- \mathbb{T}_t^+ q_0 - \mathbb{T}_{h,t}^- \mathbb{T}_{h,t}^+ q_0\|, \\ &\leq \|\pi_h \mathbb{T}_t^- \mathbb{T}_t^+ q_0 - \mathbb{T}_{h,t}^- \mathbb{T}_t^+ q_0\| + \|\mathbb{T}_{h,t}^- (\mathbb{T}_t^+ q_0 - \mathbb{T}_{h,t}^+ q_0)\|.\end{aligned}$$

By Lemma 1 of the Appendix and Eq. (2.12), we get

$$\|\pi_h \mathbb{T}_t^- \mathbb{T}_t^+ q_0 - \mathbb{T}_{h,t}^- \mathbb{T}_t^+ q_0\| \leq M(\tau - t) h^\theta \|q_0\|_2, \quad \forall h \in (0, h^*).$$

Obviously $\|\mathbb{T}_h^-\|_{\mathcal{L}(X)}$ is uniformly bounded with respect to h (this follows for example from (2.12)), and thus by (2.5) and Eq. (2.11), we have

$$\begin{aligned}\|\mathbb{T}_{h,t}^- (\mathbb{T}_t^+ q_0 - \mathbb{T}_{h,t}^+ q_0)\| &\leq \|\mathbb{T}_t^+ q_0 - \pi_h \mathbb{T}_t^+ q_0\| + \|\pi_h \mathbb{T}_t^+ q_0 - \mathbb{T}_{h,t}^+ q_0\| \\ &\leq M t h^\theta \|q_0\|_2, \quad \forall h \in (0, h^*).\end{aligned}$$

Consequently

$$\|\pi_h \mathbb{L}_t q_0 - \mathbb{L}_{h,t} q_0\| \leq M \tau h^\theta \|q_0\|_2, \quad \forall h \in (0, h^*), \tag{2.17}$$

which shows that (2.16) holds for $n = 1$. Suppose now that for a given $n \geq 2$, there holds

$$\|\pi_h \mathbb{L}_t^{n-1} q_0 - \mathbb{L}_{h,t}^{n-1} q_0\| \leq M(n-1)\tau h^\theta \|q_0\|_2. \tag{2.18}$$

We write

$$\|\pi_h \mathbb{L}_t^n q_0 - \mathbb{L}_{h,t}^n q_0\| \leq \|\pi_h \mathbb{L}_t \mathbb{L}_t^{n-1} q_0 - \mathbb{L}_{h,t} \mathbb{L}_t^{n-1} q_0\| + \|\mathbb{L}_{h,t}(\mathbb{L}_t^{n-1} q_0 - \mathbb{L}_{h,t}^{n-1} q_0)\|.$$

Thanks to Lemma 1 of the Appendix and to the uniform boundedness of $\|\mathbb{L}_{h,t}\|_{\mathcal{L}(X)}$ with respect to h (which follows from the uniform boundedness of $\|\mathbb{T}_{h,t}^{\pm}\|$) and using (2.17) and (2.18), we obtain

$$\|\pi_h \mathbb{L}_t^n q_0 - \mathbb{L}_{h,t}^n q_0\| \leq M(\tau + (n-1)\tau)h^\theta \|q_0\|_2,$$

which is exactly (2.16). Substituting (2.15) and (2.16) in (2.14), we obtain the result.

We are now able to prove Theorem 1.

Proof of Theorem 1 Introducing the term $\sum_{n=0}^{N_h} \mathbb{L}_{h,\tau}^n z^-(0)$, we rewrite $z_0 - z_{0,h}$ in the following form

$$\begin{aligned} z_0 - z_{0,h} &= \sum_{n=0}^{\infty} \mathbb{L}_\tau^n z^-(0) - \sum_{n=0}^{N_h} \mathbb{L}_{h,\tau}^n z_h^-(0), \\ &= \sum_{n>N_h} \mathbb{L}_\tau^n z^-(0) + \sum_{n=0}^{N_h} \left(\mathbb{L}_\tau^n - \mathbb{L}_{h,\tau}^n\right) z^-(0) + \sum_{n=0}^{N_h} \mathbb{L}_{h,\tau}^n \left(z^-(0) - z_h^-(0)\right). \end{aligned}$$

Therefore, we have

$$\|z_0 - z_{0,h}\| \leq S_1 + S_2 + S_3, \tag{2.19}$$

where we have set

$$\begin{cases} S_1 = \sum_{n>N_h} \left\|\mathbb{L}_\tau^n z^-(0)\right\|, \\ S_2 = \sum_{n=0}^{N_h} \left\|\left(\mathbb{L}_\tau^n - \mathbb{L}_{h,\tau}^n\right) z^-(0)\right\|, \\ S_3 = \left(\sum_{n=0}^{N_h} \left\|\mathbb{L}_{h,\tau}^n\right\|_{\mathcal{L}(X)}\right) \left\|z^-(0) - z_h^-(0)\right\|. \end{cases}$$

Note that the term S_1 is the truncation error of the tail of the infinite sum (1.5), the term S_2 represents the cumulated error due to the approximation of the semigroups $\mathbb{T}^{\pm}$ while the term S_3 comes from the approximation of the first iterate $z^-(0)$ of the algorithm.

Since $\eta = \|\mathbb{L}_\tau\|_{\mathcal{L}(X)} < 1$, using relation (1.4), the first term can be estimated very easily

$$S_1 \leq M \frac{\eta^{N_h+1}}{1-\eta} \|z_0\|_2. \tag{2.20}$$

The term S_2 can be estimated using the estimate (2.13) from Proposition 2

$$S_2 \leq M \left(\sum_{n=0}^{N_h} (1 + n\tau)\right) h^\theta \|z^-(0)\|_2, \quad \forall h \in (0, h^*).$$

Therefore, using (1.4) and the fact that $\|\mathbb{L}_\tau\|_{\mathcal{D}(A^2)} < 1$ in the above relation, we finally get that

$$S_2 \leq M\Big[1 + (1+\tau)N_h + N_h^2\tau\Big]h^\theta \|z_0\|_2, \quad \forall h \in (0, h^*). \tag{2.21}$$

It remains to estimate the term S_3. As $\eta = \|\mathbb{L}_\tau\|_{\mathcal{L}(X)} < 1$, (2.13) implies that $\|\mathbb{L}_{h,\tau}\|_{\mathcal{L}(X)}$ is also uniformly with respect to h bounded by 1, provided h is small enough. Hence, we have

$$\begin{aligned} S_3 &\leq MN_h \left\|z^-(0) - z_h^-(0)\right\| \\ &\leq MN_h \left(\left\|z^-(0) - \pi_h z^-(0)\right\| + \left\|\pi_h z^-(0) - z_h^-(0)\right\|\right). \end{aligned} \tag{2.22}$$

By using (2.5) and (1.4), we immediately obtain that

$$\left\|z^-(0) - \pi_h z^-(0)\right\| \leq M h^\theta \|z_0\|_2. \tag{2.23}$$

To estimate the second term $\pi_h z^-(0) - z_h^-(0)$, we apply twice Proposition 1 first for the time reversed backward observer $z^-(\tau - \cdot)$ and then for the forward observer z^+ (the time reversal step is introduced as in the formulation of Proposition 1, only initial value Cauchy problems can be considered). After straightforward calculation we obtain that for all $h \in (0, h^*)$

$$\begin{aligned} \left\|\pi_h z^-(0) - z_h^-(0)\right\| \leq M h^\theta &\Big[\tau(\|z^+(\tau)\|_2 + \|C^* y\|_{1,\infty}) + \tau^2 \|C^* y\|_{2,\infty}\Big] \\ &+ \int_0^\tau \|C^*\left(y(\tau - s) - y_h(\tau - s)\right)\| ds \\ &+ \int_0^\tau \|C^*\left(y(s) - y_h(s)\right)\| ds. \end{aligned} \tag{2.24}$$

Applying (4.2) of Lemma 2 of the Appendix with zero initial data, we obtain that

$$\|z^+(\tau)\|_2 \leq \tau \|C^* y\|_{2,\infty}.$$

Therefore (2.24) also reads

$$\left\|\pi_h z^-(0) - z_h^-(0)\right\| \leq M h^\theta (\tau + \tau^2)\|C^* y\|_{2,\infty} + 2\int_0^\tau \|C^*\left(y(s) - y_h(s)\right)\| ds.$$

As $C^*C \in \mathcal{L}\left(\mathcal{D}\left(A_0^2\right)\right) \cap \mathcal{L}\left(\mathcal{D}\left(A_0\right)\right)$ and $\|z\|_{2,\infty} = \|z_0\|_2$ (since iA_0 is skew-adjoint), the last relation becomes

$$\left\|\pi_h z^-(0) - z_h^-(0)\right\| \leq M h^\theta (\tau + \tau^2)\|z_0\|_2 + 2\int_0^\tau \|C^*\left(y(s) - y_h(s)\right)\| ds.$$

Substituting the above relation and (2.23) in (2.22), we get

$$S_3 \leq M N_h \left(h^\theta (1 + \tau + \tau^2)\|z_0\|_2 + \int_0^\tau \|C^*\left(y(s) - y_h(s)\right)\| ds \right). \tag{2.25}$$

Substituting (2.20), (2.21) and (2.25) in (2.19), we get for all $h \in (0, h^*)$

$$\|z_0 - z_{0,h}\| \leq M \left[\left(\frac{\eta^{N_h+1}}{1-\eta} + h^\theta \left[1 + (1 + \tau + \tau^2)N_h + \tau N_h^2\right] \right) \|z_0\|_2 \right.$$
$$\left. + N_h \int_0^\tau \|C^*\left(y(s) - y_h(s)\right)\| ds \right],$$

which leads to the result (with possibly reducing the value of h^*).

2.2 Full discretization

2.2.1 Statement of the main result

In order to approximate (2.6), we use an implicit Euler scheme in time combined with the previous Galerkin approximation in space. In others words, we discretize the time interval $[0, \tau]$ using a time step $\Delta t > 0$. We obtain a discretization $t_k = k\Delta t$, where $0 \leq k \leq K$ and where we assumed, without loss of generality, that $\tau = K\Delta t$. Given a continuously differentiable function of time f, we approximate its derivative at time t_k by the formula

$$f'(t_k) \simeq D_t f(t_k) := \frac{f(t_k) - f(t_{k-1})}{\Delta t}.$$

We suppose that $q_{0,h} \in X_h$ and F_h^k, for $0 \leq k \leq K$, are given approximations of q_0 and $F(t_k)$ in the space X. We define (q_h^k), for $0 \leq k \leq K$, as the solution of the following problem: for all $\varphi_h \in X_h$:

$$\begin{cases} \left\langle D_t q_h^k, \varphi_h \right\rangle = \pm i \left\langle q_h^k, \varphi_h \right\rangle_{\frac{1}{2}} - \left\langle C^*C q_h^k, \varphi_h \right\rangle + \left\langle F_h^k, \varphi_h \right\rangle, \\ q_h^0 = q_{0,h}. \end{cases} \tag{2.26}$$

Note that the above procedure leads to a natural approximation $\mathbb{T}^{\pm}_{h,\Delta t,k}$ of the continuous semigroup $\mathbb{T}^{\pm}_{t_k}$ by setting

$$\mathbb{T}^{+}_{t_k} q_0 \simeq \mathbb{T}^{+}_{h,\Delta t,k} q_0 := q_h^k, \qquad \mathbb{T}^{-}_{t_k} q_0 \simeq \mathbb{T}^{-}_{h,\Delta t,k} q_0 := q_h^{K-k},$$

where q_h^k solves (2.26) with $F_h^k = 0$ for all $0 \leq k \leq K$ and for $q_{0,h} = \pi_h q_0$. Obviously, this also leads to an approximation of $\mathbb{L}_\tau = \mathbb{T}^{-}_{\tau}\mathbb{T}^{+}_{\tau}$ by setting

$$\mathbb{L}_{h,\Delta t,K} = \mathbb{T}^{-}_{h,\Delta t,K}\mathbb{T}^{+}_{h,\Delta t,K}.$$

Assume that for all $0 \leq k \leq K$, y_h^k is a given approximation of $y(t_k)$ in Y and let $\left(z_h^+\right)^k$ and $\left(z_h^-\right)^k$ be respectively the approximations of (2.2) and (2.3) obtained via (2.26) as follows:

- For all $0 \leq k \leq K$, $\left(z_h^+\right)^k = q_h^k$ where q_h^k solves (2.26) with $F_h^k = C^* y_h^k$ and $q_h^0 = 0$,
- For all $0 \leq k \leq K$, $\left(z_h^-\right)^k = q_h^{K-k}$ where q_h^k solves (2.26) with $F_h^k = C^* y_h^{K-k}$ and $q_h^0 = (z_h^+)^K$.

Then, our main result (which is the fully discrete counterpart of Theorem 1) reads as follows

Theorem 2 *Let $A_0 : \mathcal{D}(A_0) \to X$ be a strictly positive self-adjoint operator and $C \in \mathcal{L}(X, Y)$ such that $C^*C \in \mathcal{L}\left(\mathcal{D}\left(A_0^2\right)\right) \cap \mathcal{L}\left(\mathcal{D}\left(A_0\right)\right)$. We assume that the pair (iA_0, C) is exactly observable in time $\tau > 0$. Let $z_0 \in \mathcal{D}\left(A_0^2\right)$ be the initial value of (2.1). With the above notation, let $z_{0,h,\Delta t}$ be defined by (1.7) and denote $\eta := \|\mathbb{L}_\tau\|_{\mathcal{L}(X)} < 1$. Then there exist $M > 0$, $h^* > 0$ and $\Delta t^* > 0$ such that for all $h \in (0, h^*)$ and all $\Delta t \in (0, \Delta t^*)$ we have*

$$\|z_0 - z_{0,h,\Delta t}\| \leq M \left[\left(\frac{\eta^{N_{h,\Delta t}+1}}{1-\eta} + (h^\theta + \Delta t)(1+\tau) N_{h,\Delta t}^2 \right) \|z_0\|_2 \right.$$
$$\left. + N_{h,\Delta t} \Delta t \sum_{\ell=0}^{K} \left\| C^*(y(t_\ell) - y_h^\ell) \right\| \right].$$

Corollary 2 *Under the assumptions of Theorem 2, we set*

$$N_{h,\Delta t} = \frac{\ln(h^\theta + \Delta t)}{\ln \eta}$$

Then, there exist $M_\tau > 0, h^* > 0$ *and* $\Delta t^* > 0$ *such that for all* $h \in (0, h^*)$ *and* $\Delta t \in (0, \Delta t^*)$

$$\|z_0 - z_{0,h,\Delta t}\| \leq M_\tau \Bigg[(h^\theta + \Delta t) \ln^2(h^\theta + \Delta t) \|z_0\|_2 + \left|\ln(h^\theta + \Delta t)\right| \Delta t \sum_{\ell=0}^{K} \left\| C^*(y(t_\ell) - y_h^\ell) \right\| \Bigg]. \tag{2.27}$$

Remark 3 Contrarily to the semi-discrete case, we have not been able to extend our results for z_0 in a larger space than $\mathcal{D}\left(A_0^2\right)$.

Remark 4 Let us emphasize that our results hold without assuming a CFL type condition.

2.2.2 Proof of Theorem 2

The proof of Theorem 2 goes along the same lines as the one of Theorem 1 in the semi-discrete case and uses energy estimates similar to those developed in Fujita and Suzuki [11, p. 865]. The main ingredient for the convergence analysis is the following result (the counterpart of Proposition 1) which gives the error estimate for the approximation (in space and time) of system (2.6) by (2.26).

Proposition 3 *Given initial states* $q_0 \in \mathcal{D}\left(A_0^2\right)$ *and* $q_{0,h} \in X_h$*, let* q *and* q_h^k*, for* $0 \leq k \leq K$*, be respectively the solutions of* (2.6) *and* (2.26)*. Assume that* $C^*C \in \mathcal{L}(\mathcal{D}(A_0))$*. Then, there exist* $M > 0$, $h^* > 0$ *and* $\Delta t^* > 0$ *such that for all* $h \in (0, h^*)$*, all* $\Delta t \in (0, \Delta t^*)$ *and all* $0 \leq k \leq K$*:*

$$\|\pi_h q(t_k) - q_h^k\| \leq \|\pi_h q_0 - q_{0,h}\| + M \Bigg\{ \Delta t \sum_{\ell=1}^{k} \|F(t_\ell) - F_h^\ell\| + \left(h^\theta + \Delta t\right) \left[t_k \left(\|q_0\|_2 + \|F\|_{1,\infty} + \|\dot{F}\|_\infty \right) + t_k^2 \|F\|_{2,\infty} \right] \Bigg\}.$$

Proof Let $r_1(t_k)$ denote the residual term in the first order Taylor expansion of q around t_{k-1}, so that

$$\dot{q}(t_k) = \frac{q(t_k) - q(t_{k-1})}{\Delta t} - \frac{1}{\Delta t} r_1(t_k) = D_t q(t_k) - \frac{1}{\Delta t} r_1(t_k), \tag{2.28}$$

Subtracting (2.26) from the continuous weak formulation (2.6) applied for $t = t_k$ and for an arbitrary test function $\varphi = \varphi_h \in X_h$, we immediately get by using (2.28) that for all $1 \leq k \leq K$

$$\left\langle D_t\left(q(t_k)-q_h^k\right),\varphi_h\right\rangle = \pm i\,\langle \pi_h q(t_k),\varphi_h\rangle_{\frac{1}{2}} - \left\langle C^*C\left(q(t_k)-q_h^k\right),\varphi_h\right\rangle$$
$$+\frac{1}{\Delta t}\langle r_1(t_k),\varphi_h\rangle + \left\langle F(t_k)-F_h^k,\varphi_h\right\rangle.$$

The above relation implies that

$$\begin{aligned}\left\langle D_t\left(\pi_h q(t_k)-q_h^k\right),\varphi_h\right\rangle &= \langle D_t\left(\pi_h q(t_k)-q(t_k)\right),\varphi_h\rangle\\ &\quad \pm i\left\langle \pi_h q(t_k)-q_h^k,\varphi_h\right\rangle_{\frac{1}{2}} - \left\langle C^*C\left(q(t_k)-q_h^k\right),\varphi_h\right\rangle\\ &\quad +\frac{1}{\Delta t}\langle r_1(t_k),\varphi_h\rangle + \left\langle F(t_k)-F_h^k,\varphi_h\right\rangle. \end{aligned} \tag{2.29}$$

Now, for all $1\le k\le K$, let

$$\mathcal{E}_h^k = \frac{1}{2}\|\pi_h q(t_k)-q_h^k\|^2.$$

Using the identity

$$\frac{1}{2}\left(\|u\|^2-\|v\|^2+\|u-v\|^2\right) = \mathrm{Re}\,\langle u-v,u\rangle,\quad \forall u,v\in X,$$

one easily obtains that for all $1\le k\le K$

$$D_t\mathcal{E}_h^k \le \mathrm{Re}\left\langle D_t\left(\pi_h q(t_k)-q_h^k\right),\pi_h q(t_k)-q_h^k\right\rangle. \tag{2.30}$$

Substituting (2.29) with $\varphi_h = \pi_h q(t_k)-q_h^k$ in the above inequality and using the boundedness of C, we obtain the existence of $M>0$ such that for all $1\le k\le K$

$$\begin{aligned} D_t\mathcal{E}_h^k \le \big[&\|D_t\left(\pi_h q(t_k)-q(t_k)\right)\| + M\|\pi_h q(t_k)-q(t_k)\|\\ &+\frac{1}{\Delta t}\|r_1(t_k)\| + \|F(t_k)-F_h^k\|\big]\|\pi_h q(t_k)-q_h^k\|. \end{aligned} \tag{2.31}$$

Using the straightforward relations

$$D_t\mathcal{E}_h^k = \left(D_t\sqrt{\mathcal{E}_h^k}\right)\left(\sqrt{\mathcal{E}_h^k}+\sqrt{\mathcal{E}_h^{k-1}}\right), \tag{2.32}$$

and

$$\|\pi_h q(t_k)-q_h^k\| \le \sqrt{2}\left(\sqrt{\mathcal{E}_h^k}+\sqrt{\mathcal{E}_h^{k-1}}\right), \tag{2.33}$$

we obtain from (2.5) and (2.31) that for all $h\in(0,h^*)$

$$D_t\sqrt{\mathcal{E}_h^k} \le M\left\{h^\theta\left(\|D_t q(t_k)\|_{\frac{1}{2}} + \|q(t_k)\|_{\frac{1}{2}}\right) + \frac{1}{\Delta t}\|r_1(t_k)\| + \|F(t_k)-F_h^k\|\right\}.$$

By (2.28) and relations (4.2) and (4.3) in Lemma 2 of the Appendix, the last estimate yields

$$D_t\sqrt{\mathcal{E}_h^k} \le M\Bigg\{h^\theta\left(\|q_0\|_2 + t_k\|F\|_{2,\infty} + \|F\|_{1,\infty}\right) + \|F(t_k) - F_h^k\| + \frac{h^\theta}{\Delta t}\|r_1(t_k)\|_{\frac{1}{2}} + \frac{1}{\Delta t}\|r_1(t_k)\|\Bigg\}. \tag{2.34}$$

To conclude, it remains to bound the two last terms in the above estimate. By definition of r_1, we have

$$r_1(t_k) = q(t_{k-1}) - q(t_k) + \Delta t\,\dot{q}(t_k),$$

in $\mathcal{D}\left(A_0^{\frac{1}{2}}\right)$, and thus by the mean value theorem, we get

$$\|r_1(t_k)\|_{\frac{1}{2}} \le \Delta t \sup_{s\in[t_{k-1},t_k]}\|\dot{q}(s)\|_{\frac{1}{2}} + \Delta t\|\dot{q}(t_k)\|_{\frac{1}{2}}.$$

Using once again (4.3), we obtain that there exists $M > 0$ such that

$$\|r_1(t_k)\|_{\frac{1}{2}} \le M\Delta t\left(\|q_0\|_2 + t_k\|F\|_{2,\infty} + \|F\|_{1,\infty}\right). \tag{2.35}$$

Now by the regularity of q (see Lemma 2), the residual r_1 can be expressed via the integral

$$r_1(t_k) = \int_{t_{k-1}}^{t_k} \ddot{q}(s)\,(t_{k-1} - s)\,ds,$$

in X, and thus

$$\|r_1(t_k)\| \le \Delta t^2 \sup_{s\in[t_{k-1},t_k]}\|\ddot{q}(s)\|.$$

Using Eq. (2.4) verified by q and the boundedness of C, we have

$$\begin{aligned}\|\ddot{q}(t)\| = \left\|\frac{d\dot{q}}{dt}(t)\right\| &= \left\|\frac{d}{dt}\Big\{\pm iA_0q(t) - C^*Cq(t) + F(t)\Big\}\right\|,\\ &\le \|\dot{q}(t)\|_1 + M\|\dot{q}(t)\| + \|\dot{F}(t)\|.\end{aligned}$$

Hence, once again by (4.3), we get

$$\|r_1(t_k)\| \le \Delta t^2\left(\|q_0\|_2 + t_k\|F\|_{2,\infty} + \|F\|_{1,\infty} + \|\dot{F}\|_\infty\right). \tag{2.36}$$

Substituting inequalities (2.35) and (2.36) in relation (2.34) provides estimates for $D_t\sqrt{\mathcal{E}_h^k} = \frac{\sqrt{\mathcal{E}_h^k}-\sqrt{\mathcal{E}_h^{k-1}}}{\Delta t}$, for $k = 1, \dots, K$, that can be added together to get the desired inequality (since $\|\pi_h q(t_k) - q_h^k\| = \sqrt{2\mathcal{E}_h^k}$).

Using this Proposition, we can derive an error estimate for the semigroup $\mathbb{T}_{t_k}^{\pm}$ (for all $1 \le k \le K$) and for the operator $\mathbb{L}_\tau = \mathbb{T}_\tau^- \mathbb{T}_\tau^+$ (the counterpart of Proposition 2).

Proposition 4 *Under the assumptions of Proposition 3, the following assertions hold true*

1. *There exist* $M > 0, h^* > 0$ *and* $\Delta t^* > 0$ *such that for all* $h \in (0, h^*)$*, all* $\Delta t \in (0, \Delta t^*)$ *and all* $0 \le k \le K$

$$\left\|\pi_h \mathbb{T}_{t_k}^+ q_0 - \mathbb{T}_{h,\Delta t,k}^+ q_0\right\| \le M t_k (h^\theta + \Delta t)\|q_0\|_2. \tag{2.37}$$

$$\left\|\pi_h \mathbb{T}_{t_k}^- q_0 - \mathbb{T}_{h,\Delta t,k}^- q_0\right\| \le M(\tau - t_k)(h^\theta + \Delta t)\|q_0\|_2. \tag{2.38}$$

2. *There exist* $M > 0, h^* > 0$ *and* $\Delta t^* > 0$ *such that for all* $n \in \mathbb{N}$*, all* $h \in (0, h^*)$*, all* $\Delta t \in (0, \Delta t^*)$ *and all* $0 \le k \le K$

$$\|(\mathbb{L}_{t_k}^n - \mathbb{L}_{h,\Delta t,k}^n) q_0\| \le M\left[h^\theta + n\tau\left(h^\theta + \Delta t\right)\right]\|q_0\|_2. \tag{2.39}$$

Proof

1. It suffices to apply Proposition 3 with $F(t_k) = F_h^k = 0$ for all $0 \le k \le K$ and $q_{0,h,\Delta t} = \pi_h q_0$.
2. First, we note that

$$\|\mathbb{L}_{t_k}^n q_0 - \mathbb{L}_{h,\Delta t,k}^n q_0\| \le \|\mathbb{L}_{t_k}^n q_0 - \pi_h \mathbb{L}_{t_k}^n q_0\| + \|\pi_h \mathbb{L}_{t_k}^n q_0 - \mathbb{L}_{h,\Delta t,k}^n q_0\|. \tag{2.40}$$

Using (2.5), the fact that $\|\mathbb{L}_t^n\|_{\mathcal{L}(\mathcal{D}(A))} \le 1$ (proved in Lemma 1 of the Appendix), the first term in the above relation can be estimated as follows

$$\|\mathbb{L}_{t_k}^n q_0 - \pi_h \mathbb{L}_{t_k}^n q_0\| \le M h^\theta \|q_0\|_2, \quad \forall h \in (0, h^*). \tag{2.41}$$

For the second term in (2.40), we prove by induction that for all $n \in \mathbb{N}$, all $h \in (0, h^*)$ and all $\Delta t \in (0, \Delta t^*)$ (for some $\Delta t^* > 0$)

$$\|\pi_h \mathbb{L}_{t_k}^n q_0 - \mathbb{L}_{h,\Delta t,k}^n q_0\| \le M n\tau\left(h^\theta + \Delta t\right)\|q_0\|_2. \tag{2.42}$$

By definition, we have

$$\begin{aligned}\left\|\pi_h \mathbb{L}_{t_k} q_0 - \mathbb{L}_{h,\Delta t,k} q_0\right\| &= \left\|\pi_h \mathbb{T}_{t_k}^- \mathbb{T}_{t_k}^+ q_0 - \mathbb{T}_{h,\Delta t,k}^- \mathbb{T}_{h,\Delta t,k}^+ q_0\right\|, \\ &\le \left\|\left(\pi_h \mathbb{T}_{t_k}^- - \mathbb{T}_{h,\Delta t,k}^-\right)\pi_h \mathbb{T}_{t_k}^+ q_0\right\| \\ &\quad + \left\|\mathbb{T}_{h,\Delta t,k}^-\left(\pi_h \mathbb{T}_{t_k}^+ - \mathbb{T}_{h,\Delta t,k}^+\right) q_0\right\|.\end{aligned}$$

Using (2.38) and Lemma 1, we get

$$\left\| \left(\pi_h \mathbb{T}^{-}_{t_k} - \mathbb{T}^{-}_{h,\Delta t,k}\right) \pi_h \mathbb{T}^{+}_{t_k} q_0 \right\| \leq M(\tau - t_k)\left(h^\theta + \Delta t\right) \|q_0\|_2.$$

Obviously $\|\mathbb{T}^{-}_{h,\Delta t,k}\|_{\mathcal{L}(X)}$ is uniformly bounded (with respect to h and Δt), and thus again by (2.37) we have

$$\left\| \mathbb{T}^{-}_{h,\Delta t,k} \left(\pi_h \mathbb{T}^{+}_{t_k} - \mathbb{T}^{+}_{h,\Delta t,k}\right) q_0 \right\| \leq M t_k \left(h^\theta + \Delta t\right) \|q_0\|_2.$$

So, by adding the two last inequalities, we obtain that

$$\left\| \pi_h \mathbb{L}_{t_k} q_0 - \mathbb{L}_{h,\Delta t,k} q_0 \right\| \leq M\tau \left(h^\theta + \Delta t\right) \|q_0\|_2, \tag{2.43}$$

showing that (2.42) holds for $n = 1$. Suppose now that for some $n \geq 2$

$$\|\pi_h \mathbb{L}^{n-1}_{t_k} q_0 - \mathbb{L}^{n-1}_{h,\Delta t,k} q_0\| \leq M(n-1)\tau \left(h^\theta + \Delta t\right) \|q_0\|_2. \tag{2.44}$$

Writing

$$\begin{aligned}\|\pi_h \mathbb{L}^{n}_{t_k} q_0 - \mathbb{L}^{n}_{h,\Delta t,k} q_0\| \leq \|&\pi_h \mathbb{L}_{t_k} \mathbb{L}^{n-1}_{t_k} q_0 - \mathbb{L}_{h,\Delta t,k} \pi_h \mathbb{L}^{n-1}_{t_k} q_0\| \\ &+ \|\mathbb{L}_{h,\Delta t,k}(\pi_h \mathbb{L}^{n-1}_{t_k} q_0 - \mathbb{L}^{n-1}_{h,\Delta t,k} q_0)\|,\end{aligned}$$

we get by using Lemma 1, the uniform boundedness of $\|\mathbb{L}_{h,\Delta t,k}\|_{\mathcal{L}(X)}$ with respect to h and Δt, (2.43) and (2.44) that

$$\|\pi_h \mathbb{L}^{n}_{t_k} q_0 - \mathbb{L}^{n}_{h,\Delta t,k} q_0\| \leq M\left[(1 + (n-1))\tau \left(h^\theta + \Delta t\right)\right] \|q_0\|_2,$$

which is exactly (2.42). Substituting (2.41) and (2.42) in (2.40), we obtain the result.

We are now able to prove Theorem 2.

Proof of Theorem 2 We first introduce the term $\sum_{n=0}^{N_{h,\Delta t}} \mathbb{L}^{n}_{h,\Delta t,K} z^{-}(0)$ to rewrite the approximation error $z_0 - z_{0,h,\Delta t}$ in the following form:

$$\begin{aligned} z_0 - z_{0,h,\Delta t} &= \sum_{n=0}^{\infty} \mathbb{L}^{n}_{\tau} z^{-}(0) - \sum_{n=0}^{N_{h,\Delta t}} \mathbb{L}^{n}_{h,\Delta t,K} \left(z^{-}_{h}\right)^{0} \\ &= \sum_{n>N_{h,\Delta t}} \mathbb{L}^{n}_{\tau} z^{-}(0) + \sum_{n=0}^{N_{h,\Delta t}} \left(\mathbb{L}^{n}_{\tau} - \mathbb{L}^{n}_{h,\Delta t,K}\right) z^{-}(0) \\ &\quad + \sum_{n=0}^{N_{h,\Delta t}} \mathbb{L}^{n}_{h,\Delta t,K} \left(z^{-}(0) - \left(z^{-}_{h}\right)^{0}\right). \end{aligned}$$

Therefore, we have

$$\|z_0 - z_{0,h,\Delta t}\| \leq S_1 + S_2 + S_3, \tag{2.45}$$

where we have set

$$\begin{cases} S_1 = \sum_{n>N_{h,\Delta t}} \left\| \mathbb{L}_\tau^n z^-(0) \right\|, \\ S_2 = \sum_{n=0}^{N_{h,\Delta t}} \left\| \left(\mathbb{L}_\tau^n - \mathbb{L}_{h,\Delta t,K}^n \right) z^-(0) \right\|, \\ S_3 = \left(\sum_{n=0}^{N_{h,\Delta t}} \left\| \mathbb{L}_{h,\Delta t,K}^n \right\|_{\mathcal{L}(X)} \right) \left\| z^-(0) - (z_h^-)^0 \right\|. \end{cases}$$

Since $\eta = \|\mathbb{L}_\tau\|_{\mathcal{L}(X)} < 1$, the first term can be estimated very easily

$$S_1 \leq M \frac{\eta^{N_{h,\Delta t}+1}}{1-\eta} \|z_0\|_2. \tag{2.46}$$

The second term S_2 can be estimated using the estimate (2.39) from Proposition 4

$$S_2 \leq M \left\{ \sum_{n=0}^{N_{h,\Delta t}} \left(h^\theta + n\tau(h^\theta + \Delta t) \right) \right\} \|z^-(0)\|_2, \quad \forall h \in (0, h^*), \Delta t \in (0, \Delta t^*).$$

Therefore, using (1.4), the fact that $\|\mathbb{L}_\tau\|_{\mathcal{D}(A^2)} < 1$ (see Lemma 1) in the above relation, we get that for all $h \in (0, h^*)$ and $\Delta t \in (0, \Delta t^*)$

$$S_2 \leq M \Big[1 + (1+\tau) N_{h,\Delta t} + (1+\tau) N_{h,\Delta t}^2 \Big] \left(h^\theta + \Delta t \right) \|z_0\|_2. \tag{2.47}$$

It remains to estimate the term S_3. As for the semi-discrete case, one can easily show that $\|\mathbb{L}_{h,\Delta t,K}\|_{\mathcal{L}(X)}$ is uniformly bounded by 1 (with respect to h and Δt), and thus we have

$$\begin{aligned} S_3 &\leq M N_{h,\Delta t} \left\| z^-(0) - (z_h^-)^0 \right\| \\ &\leq M N_{h,\Delta t} \left(\left\| z^-(0) - \pi_h z^-(0) \right\| + \left\| \pi_h z^-(0) - (z_h^-)^0 \right\| \right). \end{aligned} \tag{2.48}$$

By using (2.5) and (1.4), we immediately obtain that

$$\left\| z^-(0) - \pi_h z^-(0) \right\| \leq M h^\theta \|z_0\|_2. \tag{2.49}$$

To estimate the second term $\pi_h z^-(0) - (z_h^-)^0$, we apply twice Proposition 3 first for the time reversed backward observer $z^-(\tau - \cdot)$ and then for the forward observer z^+ (the time reversal step is introduced simply because Proposition 3 is written for initial (and not final) value Cauchy problems). After straightforward calculation we obtain that for all $h \in (0, h^*)$ and all $\Delta t \in (0, \Delta t^*)$

$$\begin{aligned}\left\|\pi_h z^-(0) - (z_h^-)^0\right\| \le{}& M(h^\theta + \Delta t)\Big[\tau(\|z^+(\tau)\|_2 + \|C^* y\|_{1,\infty} + \|C^*\dot{y}\|_\infty) \\ &+\tau^2\|C^* y\|_{2,\infty}\Big] + \Delta t\sum_{\ell=1}^{K}\left\|C^*\left(y(\tau - t_\ell) - y_h^{K-\ell}\right)\right\| \\ &+\Delta t\sum_{\ell=1}^{K}\left\|C^*\left(y(t_\ell) - y_h^\ell\right)\right\|.\end{aligned} \tag{2.50}$$

Applying (4.2) of Lemma 2 of the Appendix with zero initial data, we obtain that

$$\|z^+(\tau)\|_2 \le \tau\|C^* y\|_{2,\infty}.$$

As $C^*C \in \mathcal{L}\left(\mathcal{D}\left(A_0^2\right)\right) \cap \mathcal{L}\left(\mathcal{D}\left(A_0\right)\right)$ and $\|z\|_{2,\infty} = \|z_0\|_2$ (since iA_0 is skew-adjoint), (2.50) also reads

$$\left\|\pi_h z^-(0) - (z_h^-)^0\right\| \le M(h^\theta + \Delta t)(\tau + \tau^2)\|z_0\|_2 + 2\Delta t\sum_{\ell=0}^{K}\left\|C^*\left(y(t_\ell) - y_h^\ell\right)\right\|.$$

Substituting the above relation and (2.49) in (2.48), we get

$$S_3 \le M N_{h,\Delta t}\left\{(h^\theta + \Delta t)(1 + \tau + \tau^2)\|z_0\|_2 + \Delta t\sum_{\ell=0}^{K}\left\|C^*\left(y(t_\ell) - y_h^\ell\right)\right\|\right\}. \tag{2.51}$$

Substituting (2.46), (2.47) and (2.51) in (2.45), we get for all $h \in (0, h^*)$ and all $\Delta t \in (0, \Delta t^*)$

$$\begin{aligned}\|z_0 - z_{0,h,\Delta t}\| \le M\Bigg\{& N_{h,\Delta t}\Delta t\sum_{\ell=0}^{K}\left\|C^*\left(y(t_\ell) - y_h^\ell\right)\right\| + \frac{\eta^{N_{h,\Delta t}+1}}{1-\eta}\|z_0\|_2 \\ &+(h^\theta + \Delta t)\Big[1 + (1 + \tau + \tau^2)N_{h,\Delta t} + (1 + \tau)N_{h,\Delta t}^2\Big]\|z_0\|_2\Bigg\},\end{aligned}$$

which leads to the result (with possibly reducing the value of h^* and Δt^*).

3 The wave equation

Let H be a Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$. The corresponding norm of H is denoted by $\|\cdot\|$. Let $A_0 : \mathcal{D}(A_0) \to H$ be a strictly positive self-adjoint operator and $C_0 \in \mathcal{L}(H, Y)$ a bounded observation operator, where Y is an other Hilbert space. The norm in $\mathcal{D}(A_0^\alpha)$ will be denoted by $\|\cdot\|_\alpha$. Given $\tau > 0$, we deal with the general wave type system

$$\begin{cases}\ddot{w}(t) + A_0 w(t) = 0, & \forall t \geqslant 0,\\ y(t) = C_0\dot{w}(t), & \forall t \in [0,\tau],\end{cases} \tag{3.1}$$

and we want to reconstruct the initial value $(w_0, w_1) = (w(0), \dot{w}(0))$ of (3.1) knowing $y(t)$ for $t \in [0, \tau]$. In order to use the general iterative algorithm described in the introduction, we first rewrite (3.1) as a first order system of the form (1.1). To achieve this, it suffices to introduce the following notation:

$$z(t) = \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}, \qquad X = \mathcal{D}\left(A_0^{\frac{1}{2}}\right) \times H,$$

$$A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}\left(A_0^{\frac{1}{2}}\right), \tag{3.2}$$

$$C \in \mathcal{L}(X, Y), \qquad C = \begin{bmatrix} 0 & C_0 \end{bmatrix}. \tag{3.3}$$

The space X is endowed with the norm

$$\|z\| = \sqrt{\|z_1\|_{\frac{1}{2}}^2 + \|z_2\|^2}, \qquad \forall\, z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in X.$$

Note that the operator iA is selfadjoint but has no sign so that the problem studied here does not fit into the framework of Sect. 2. We assume that the pair (A, C) is exactly observable in time $\tau > 0$. Thus, according to Liu [21, Theorem 2.3.], $A^+ = A - C^*C$ (resp. $A^- = -A - C^*C$) is the generator of an exponentially stable C_0-semigroup $\mathbb{T}^+$ (resp. $\mathbb{T}^-$). We set as usually

$$\mathbb{L}_\tau = \mathbb{T}_\tau^- \mathbb{T}_\tau^+.$$

Throughout this section we always assume that $(w_0, w_1) \in \mathcal{D}\left(A^2\right) = \mathcal{D}\left(A_0^{\frac{3}{2}}\right) \times \mathcal{D}(A_0)$. Thus by applying Theorem 4.1.6 of Tucsnak and Weiss [27], we have

$$w \in C\left([0, \tau], \mathcal{D}\left(A_0^{\frac{3}{2}}\right)\right) \cap C^1\left([0, \tau], \mathcal{D}(A_0)\right) \cap C^2\left([0, \tau], \mathcal{D}\left(A_0^{\frac{1}{2}}\right)\right).$$

The forward and backward observers (1.2) and (1.3) read then as follows (as second-order systems)

$$\begin{cases} \ddot{w}^+(t) + A_0 w^+(t) + C_0^* C_0 \dot{w}^+(t) = C_0^* y(t), & \forall t \in [0, \tau], \\ w^+(0) = 0, \quad \dot{w}^+(0) = 0, \end{cases} \tag{3.4}$$

$$\begin{cases} \ddot{w}^-(t) + A_0 w^-(t) - C_0^* C_0 \dot{w}^-(t) = -C_0^* y(t), & \forall t \in [0, \tau], \\ w^-(\tau) = w^+(\tau), \quad \dot{w}^-(\tau) = \dot{w}^+(\tau). \end{cases} \tag{3.5}$$

Clearly, the above two systems can be written as a general initial value Cauchy problem of the same form (simply by using a time reversal for the second system)

$$\begin{cases} \ddot{p}(t) + A_0 p(t) + C_0^* C_0 \dot{p}(t) = f(t), & \forall t \in [0, \tau], \\ p(0) = p_0, \quad \dot{p}(0) = p_1 \end{cases} \tag{3.6}$$

where we have set

- for the forward observer (3.4): $f(t) = C_0^* y(t) = C_0^* C_0 \dot{w}(t)$ and $(p_0, p_1) = (0, 0)$,
- for the backward observer (3.5): $f(t) = -C_0^* y(\tau - t) = -C_0^* C_0 \dot{w}(\tau - t)$ and $(p_0, p_1) = (w^+(\tau), -\dot{w}^+(\tau)) \in \mathcal{D}\left(A^2\right) = \mathcal{D}\left(A_0^{\frac{3}{2}}\right) \times \mathcal{D}(A_0)$.

Let us emphasize that with these notation, the semigroups $\mathbb{T}^{\pm}$ are given by the relations

$$\mathbb{T}_t^+ \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} \qquad \mathbb{T}_t^- \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} p(\tau - t) \\ -\dot{p}(\tau - t) \end{bmatrix} \tag{3.7}$$

where p solves (3.6) with $f = 0$.

In the next two subsections, we present a convergence analysis of semi-discretized and fully discretized approximation schemes for the forward and backward observers (3.4) and (3.5). Our proof is based on the convergence analysis of the semi and fully discretizations of (3.6). For the sake of clarity, we dropped in the proofs some of the details which are very close to the ones given in the Schrödinger. As far as we know, the existing literature on the convergence analysis of full discretizations of wave-type systems concern only the particular cases of conservative systems (i.e. without damping), see e.g. Raviart and Thomas [25, p. 197] or Dautray and Lions [7, p. 921] and systems with constant damping coefficients Geveci and Kok [12]. For a recent review of numerical approximation issues related to the control and the observation of waves, we refer the reader to the review paper of Zuazua [28].

3.1 Space semi-discretization

3.1.1 Statement of the main result

We use a Galerkin method to approximate system (3.6). More precisely, consider a family $(H_h)_{h>0}$ of finite-dimensional subspaces of $\mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ endowed with the norm in H. We denote π_h the orthogonal projection from $\mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ onto H_h. We assume that there exist $M > 0, \theta > 0$ and $h^* > 0$ such that we have for all $h \in (0, h^*)$

$$\|\pi_h \varphi - \varphi\| \leq M h^{\theta} \|\varphi\|_{\frac{1}{2}}, \qquad \forall \varphi \in \mathcal{D}\left(A_0^{\frac{1}{2}}\right). \tag{3.8}$$

Given $(p_0, p_1) \in \mathcal{D}\left(A^2\right)$, the variational formulation of (3.6) reads for all $t \in [0, \tau]$ and all $\varphi \in \mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ as follows

$$\begin{cases} \langle \ddot{p}(t), \varphi \rangle + \langle p(t), \varphi \rangle_{\frac{1}{2}} + \left\langle C_0^* C_0 \dot{p}(t), \varphi \right\rangle = \langle f(t), \varphi \rangle, & \forall t \in [0, \tau], \\ p(0) = p_0, \quad \dot{p}(0) = p_1. \end{cases} \tag{3.9}$$

Suppose that $(p_{0,h}, p_{1,h}) \in H_h \times H_h$ and f_h are given approximations of (p_0, p_1) and f respectively in the spaces X and $L^1([0, \tau], H)$. We define $p_h(t)$ as the solution of the variational problem

$$\begin{cases} \langle \ddot{p}_h(t), \varphi_h\rangle + \langle p_h(t), \varphi_h\rangle_{\frac{1}{2}} + \left\langle C_0^* C_0 \dot{p}_h(t), \varphi_h\right\rangle = \langle f_h(t), \varphi_h\rangle, \quad \forall t \in [0, \tau], \\ p_h(0) = p_{0,h}, \quad \dot{p}_h(0) = p_{1,h}. \end{cases} \tag{3.10}$$

for all $t \in [0, \tau]$ and all $\varphi_h \in H_h$.

The above approximation procedure leads in particular to the definition of the semi-discretized versions $\mathbb{T}_h^{\pm}$ of the semigroups $\mathbb{T}^{\pm}$ that we will use. Indeed, we simply set

$$\mathbb{T}_{h,t}^{+} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} p_h(t) \\ \dot{p}_h(t) \end{bmatrix} \qquad \mathbb{T}_{h,t}^{-} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} p_h(\tau - t) \\ -\dot{p}_h(\tau - t) \end{bmatrix} \tag{3.11}$$

where p_h solves (3.10) for $f_h = 0$ and $(p_{0,h}, p_{1,h}) = (\pi_h p_0, \pi_h p_1)$. The semi-discretized counterpart of $\mathbb{L}_\tau = \mathbb{T}_\tau^- \mathbb{T}_\tau^+$ is then given by

$$\mathbb{L}_{h,\tau} = \mathbb{T}_{h,\tau}^{-} \mathbb{T}_{h,\tau}^{+}.$$

Assume that y_h is an approximation of the output y in $L^1([0, \tau], Y)$ and let w_h^+ and w_h^- denote the Galerkin approximations of the solutions of systems (3.4) and (3.5), satisfying for all $t \in [0, \tau]$ and all $\varphi_h \in H_h$

$$\begin{cases} \left\langle \ddot{w}_h^+(t), \varphi_h\right\rangle + \left\langle w_h^+(t), \varphi_h\right\rangle_{\frac{1}{2}} + \left\langle C_0^* C_0 \dot{w}_h^+(t), \varphi_h\right\rangle = \left\langle C_0^* y_h(t), \varphi_h\right\rangle, \\ w_h^+(0) = 0, \quad \dot{w}_h^+(0) = 0, \end{cases} \tag{3.12}$$

$$\begin{cases} \left\langle \ddot{w}_h^-(t), \varphi_h\right\rangle + \left\langle w_h^-(t), \varphi_h\right\rangle_{\frac{1}{2}} - \left\langle C_0^* C_0 \dot{w}_h^-(t), \varphi_h\right\rangle = -\left\langle C_0^* y_h(t), \varphi_h\right\rangle, \\ w_h^-(\tau) = w_h^+(\tau), \quad \dot{w}_h^-(\tau) = \dot{w}_h^+(\tau). \end{cases} \tag{3.13}$$

With the above notation, the main result of this section reads as follows.

Theorem 3 *Let* $A_0 : \mathcal{D}(A_0) \to H$ *be a strictly positive self-adjoint operator and* $C_0 \in \mathcal{L}(H, Y)$ *such that* $C_0^* C_0 \in \mathcal{L}\left(\mathcal{D}\left(A_0^{\frac{3}{2}}\right)\right) \cap \mathcal{L}\left(\mathcal{D}\left(A_0^{\frac{1}{2}}\right)\right)$. *Define* (A, C) *by* (3.2) *and* (3.3). *Assume that the pair* (A, C) *is exactly observable in time* $\tau > 0$ *and set* $\eta := \|\mathbb{L}_\tau\|_{\mathcal{L}(X)} < 1$. *Let* $(w_0, w_1) \in \mathcal{D}\left(A_0^{\frac{3}{2}}\right) \times \mathcal{D}(A_0)$ *be the initial value of* (3.1) *and let* $(w_{0,h}, w_{1,h})$ *be defined by*

$$\begin{bmatrix} w_{0,h} \\ w_{1,h} \end{bmatrix} = \sum_{n=0}^{N_h} \mathbb{L}_{h,\tau}^{n} \begin{bmatrix} w_h^-(0) \\ \dot{w}_h^-(0) \end{bmatrix}. \tag{3.14}$$

Then there exist $M > 0$ and $h^* > 0$ such that for all $h \in (0, h^*)$

$$\|w_0 - w_{0,h}\|_{\frac{1}{2}} + \|w_1 - w_{1,h}\| \leq M \left[\left(\frac{\eta^{N_h+1}}{1-\eta} + h^\theta \tau N_h^2 \right) \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1 \right) + N_h \int_0^\tau \|C_0^* \left(y(s) - y_h(s) \right) \| ds \right].$$

Corollary 3 *Under the assumptions of Theorem 3, we set*

$$N_h = \theta \frac{\ln h}{\ln \eta}.$$

Then, there exist $M_\tau > 0$ and $h^* > 0$ such that for all $h \in (0, h^*)$

$$\|w_0 - w_{0,h}\|_{\frac{1}{2}} + \|w_1 - w_{1,h}\| \leq M_\tau \left[h^\theta \ln^2 h \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1 \right) + |\ln h| \int_0^\tau \|C_0^* \left(y(s) - y_h(s) \right) \| ds \right]. \tag{3.15}$$

3.1.2 *Proof of Theorem 3*

The next Proposition provides the error estimate for the approximation of (3.9) by using the Galerkin scheme (3.10).

Proposition 5 *Given $(p_0, p_1) \in \mathcal{D}\left(A_0^{\frac{3}{2}}\right) \times \mathcal{D}(A_0)$ and $(p_{0,h}, p_{1,h}) \in H_h \times H_h$, let p and p_h be the solutions of (3.9) and (3.10) respectively. Assume that $C_0^* C_0 \in \mathcal{L}\left(\mathcal{D}\left(A_0^{\frac{1}{2}}\right)\right)$. Then, there exist $M > 0$ and $h^* > 0$ such that for all $t \in [0, \tau]$ and all $h \in (0, h^*)$*

$$\|\pi_h p(t) - p_h(t)\|_{\frac{1}{2}} + \|\pi_h \dot{p}(t) - \dot{p}_h(t)\| \leq M \Big\{ \|\pi_h p_0 - p_{0,h}\|_{\frac{1}{2}} + \|\pi_h p_1 - p_{1,h}\| + h^\theta \left[t \left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 + \|f\|_{\frac{1}{2},\infty} \right) + t^2 \|f\|_{1,\infty} \right] \Big\} + \int_0^t \|f(s) - f_h(s)\| ds.$$

Proof First, we substract (3.10) from (3.9) to obtain (we omit the time dependence for the sake of clarity) for all $\varphi_h \in H_h$

$$\langle \ddot{p} - \ddot{p}_h, \varphi_h \rangle + \langle p - p_h, \varphi_h \rangle_{\frac{1}{2}} + \left\langle C_0^* C_0 \left(\dot{p} - \dot{p}_h \right), \varphi_h \right\rangle = \langle f - f_h, \varphi_h \rangle .$$

Noting that $\langle \pi_h p - p, \varphi_h \rangle_{\frac{1}{2}} = 0$ for all $\varphi_h \in H_h$ and that $\pi_h \ddot{p}$ makes sense by the regularity of p (this is a direct consequence of relation (4.1) from Lemma 2 used with $q = \begin{bmatrix} p \\ \dot{p} \end{bmatrix}$), we obtain from the above equality that for all $\varphi_h \in H_h$

$$\begin{aligned} \langle \pi_h \ddot{p} - \ddot{p}_h, \varphi_h \rangle + \langle \pi_h p - p_h, \varphi_h \rangle_{\frac{1}{2}} = \langle \pi_h \ddot{p} - \ddot{p}, \varphi_h \rangle &+ \left\langle C_0^* C_0 \left(\dot{p}_h - \dot{p} \right), \varphi_h \right\rangle \\ &+ \langle f - f_h, \varphi_h \rangle . \end{aligned} \tag{3.16}$$

On the other hand, setting

$$\mathcal{E}_h = \frac{1}{2} \| \pi_h \dot{p} - \dot{p}_h \|^2 + \frac{1}{2} \| \pi_h p - p_h \|_{\frac{1}{2}}^2 ,$$

we have

$$\dot{\mathcal{E}}_h = \langle \pi_h \ddot{p} - \ddot{p}_h, \pi_h \dot{p} - \dot{p}_h \rangle + \langle \pi_h p - p_h, \pi_h \dot{p} - \dot{p}_h \rangle_{\frac{1}{2}} .$$

Applying (3.16) with $\varphi_h = \pi_h \dot{p} - \dot{p}_h$ and substituting the result in the above relation, we obtain by using Cauchy-Schwarz inequality and the boundedness of C_0 that there exists $M > 0$ such that

$$\dot{\mathcal{E}}_h \le \Big(\| \pi_h \ddot{p} - \ddot{p} \| + M \| \pi_h \dot{p} - \dot{p} \| + \| f - f_h \| \Big) \underbrace{\| \pi_h \dot{p} - \dot{p}_h \|}_{\le \sqrt{2 \mathcal{E}_h}} .$$

Since $\frac{\dot{\mathcal{E}}_h}{\sqrt{2\mathcal{E}_h}} = \frac{d}{dt} \sqrt{2 \mathcal{E}_h}$, the integration of the above inequality from 0 to t yields

$$\begin{aligned} \| \pi_h p(t) - p_h(t) \|_{\frac{1}{2}} + \| \pi_h \dot{p}(t) - \dot{p}_h(t) \| \le M \Bigg\{ & \| \pi_h p_0 - p_{0,h} \|_{\frac{1}{2}} + \| \pi_h p_1 - p_{1,h} \| \\ + \int_0^t \left(\| \pi_h \ddot{p}(s) - \ddot{p}(s) \| + \| \pi_h \dot{p}(s) - \dot{p}(s) \| \right) ds &+ \int_0^t \| f(s) - f_h(s) \| ds \Bigg\}. \end{aligned} \tag{3.17}$$

Thus, it remains to bound $\| \pi_h \ddot{p}(t) - \ddot{p}(t) \|$ and $\| \pi_h \dot{p}(t) - \dot{p}(t) \|$ for all $t \in [0, \tau]$. Using (3.8) and the classical continuous embedding from $\mathcal{D}(A^\alpha)$ to $\mathcal{D}(A^\beta)$ for $\alpha > \beta$, we get that

$$\begin{cases} \| \pi_h \ddot{p}(t) - \ddot{p}(t) \| \le M h^\theta \| \ddot{p}(t) \|_{\frac{1}{2}} , \\ \| \pi_h \dot{p}(t) - \dot{p}(t) \| \le M h^\theta \| \dot{p}(t) \|_{\frac{1}{2}} \le M h^\theta \| \dot{p}(t) \|_1 , \end{cases} \qquad \forall t \in [0, \tau], \;\; h \in (0, h^*).$$

Using relations (4.3) proved in Lemma 2 of the Appendix for the first order unknown $q = \begin{bmatrix} p \\ \dot{p} \end{bmatrix}$ and the right-hand side $F = \begin{bmatrix} 0 \\ f \end{bmatrix}$, we get for all $t \in [0, \tau]$ and all $h \in (0, h^*)$

$$\|\pi_h \ddot{p}(t) - \ddot{p}(t)\| + \|\pi_h \dot{p}(t) - \dot{p}(t)\| \leq M h^\theta \left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 + t\|f\|_{1,\infty} + \|f\|_{\frac{1}{2},\infty} \right).$$

Substituting the above inequality in (3.17), we get the result.

Thanks to the last result, we are now in position to derive an error approximation for the semigroups $\mathbb{T}^\pm$ and for the operator $\mathbb{L}_t = \mathbb{T}_t^- \mathbb{T}_t^+$. This result has been recently proved in [5] we refer the interested reader to the proof given there, which is similar to the one of Proposition 2.

Proposition 6 *Let* $\Pi_h = \begin{bmatrix} \pi_h & 0 \\ 0 & \pi_h \end{bmatrix}$. *Under the assumptions of Proposition 5, the following assertions hold true*

1. *There exist* $M > 0$ *and* $h^* > 0$ *such that for all* $t \in (0, \tau)$ *and all* $h \in (0, h^*)$

$$\left\| (\Pi_h \mathbb{T}_t^+ - \mathbb{T}_{h,t}^+) \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \right\| \leq M t h^\theta \left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 \right), \tag{3.18}$$

$$\left\| (\Pi_h \mathbb{T}_t^- - \mathbb{T}_{h,t}^-) \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \right\| \leq M(\tau - t) h^\theta \left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 \right). \tag{3.19}$$

2. *There exist* $M > 0$ *and* $h^* > 0$ *such that for all* $n \in \mathbb{N}$, *all* $t \in [0, \tau]$ *and all* $h \in (0, h^*)$, *we have*

$$\left\| (\mathbb{L}_t^n - \mathbb{L}_{h,t}^n) \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \right\| \leq M(1 + n\tau) h^\theta \left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 \right). \tag{3.20}$$

Now, we can turn to the proof of Theorem 3

Proof of Theorem 3 Introducing the term $\sum_{n=0}^{N_h} \mathbb{L}_{h,\tau}^n \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix}$, we first rewrite the error term $\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} - \begin{bmatrix} w_{0,h} \\ w_{1,h} \end{bmatrix} = \sum_{n=0}^{\infty} \mathbb{L}_\tau^n \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix} - \sum_{n=0}^{N_h} \mathbb{L}_{h,\tau}^n \begin{bmatrix} w_h^-(0) \\ \dot{w}_h^-(0) \end{bmatrix}$ in the following form

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} - \begin{bmatrix} w_{0,h} \\ w_{1,h} \end{bmatrix} = \sum_{n > N_h} \mathbb{L}_\tau^n \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix} + \sum_{n=0}^{N_h} (\mathbb{L}_\tau^n - \mathbb{L}_{h,\tau}^n) \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix}$$
$$+ \sum_{n=0}^{N_h} \mathbb{L}_{h,\tau}^n \begin{bmatrix} w^-(0) - w_h^-(0) \\ \dot{w}^-(0) - \dot{w}_h^-(0) \end{bmatrix}.$$

Therefore, we have

$$\left\| \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} - \begin{bmatrix} w_{0,h} \\ w_{1,h} \end{bmatrix} \right\| \leq S_1 + S_2 + S_3, \tag{3.21}$$

where we have set

$$\begin{cases} S_1 = \sum_{n>N_h} \left\| \mathbb{L}_\tau^n \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix} \right\|, \\ S_2 = \sum_{n=0}^{N_h} \left\| \left(\mathbb{L}_\tau^n - \mathbb{L}_{h,\tau}^n \right) \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix} \right\|, \\ S_3 = \left(\sum_{n=0}^{N_h} \left\| \mathbb{L}_{h,\tau}^n \right\|_{\mathcal{L}(X)} \right) \left\| \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix} \right\|. \end{cases}$$

Following exactly the same way than in the proof of the Schrödinger case, we get the claimed result.

3.2 Full discretization

3.2.1 Statement of the main result

In order to approximate (3.9) in space and time, we use an implicit Euler scheme in time combined with the previous Galerkin approximation in space. We discretize the time interval $[0, \tau]$ using a time step $\Delta t > 0$. We obtain a discretization $t_k = k\Delta t$, where $0 \le k \le K$ and where we assumed, without loss of generality, that $\tau = K\Delta t$. Given a function of time f of class $\mathcal{C}^2$, we approximate its first and second derivative at time t_k by

$$f'(t_k) \simeq D_t f(t_k) := \frac{f(t_k) - f(t_{k-1})}{\Delta t}.$$
$$f''(t_k) \simeq D_{tt} f(t_k) := \frac{f(t_k) - 2f(t_{k-1}) + f(t_{k-2})}{\Delta t^2}.$$

We suppose that $(p_{0,h,\Delta t}, p_{1,h,\Delta t}) \in H_h \times H_h$ and f_h^k, for $0 \le k \le K$, are given approximations of (p_0, p_1) and $f(t_k)$ in the space X and H respectively. We define the approximate solution $(p_h^k)_{0\le k\le K}$ of (3.9) as the solution of the following problem: $p_h^k \in H_h$ such that for all $\varphi_h \in H_h$

$$\begin{cases} \left\langle D_{tt} p_h^k, \varphi_h \right\rangle + \left\langle p_h^k, \varphi_h \right\rangle_{\frac{1}{2}} + \left\langle C_0^* C_0 D_t p_h^k, \varphi_h \right\rangle = \left\langle f_h^k, \varphi_h \right\rangle, \qquad 2 \le k \le K \\ p_h^0 = p_{0,h,\Delta t}, \quad p_h^1 = p_h^0 + \Delta t\, p_{1,h,\Delta t}. \end{cases} \tag{3.22}$$

Note that the above procedure leads to a natural approximation $\mathbb{T}_{h,\Delta t,k}^{\pm}$ of the continuous operators $\mathbb{T}_{t_k}^{\pm}$ by setting

$$\begin{cases} \mathbb{T}_{t_k}^{+} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \simeq \mathbb{T}_{h,\Delta t,k}^{+} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} := \begin{bmatrix} p_h^k \\ D_t p_h^k \end{bmatrix} \\ \\ \mathbb{T}_{t_k}^{-} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \simeq \mathbb{T}_{h,\Delta t,k}^{-} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} := \begin{bmatrix} p_h^{K-k} \\ -D_t p_h^{K-k} \end{bmatrix} \end{cases} \tag{3.23}$$

where p_h^k solves (3.22) with $f_h^k = 0$ for all $0 \le k \le K$ and for $(p_{0,h,\Delta t}, p_{1,h,\Delta t}) = (\pi_h p_0, \pi_h p_1)$. Obviously, this also leads to a fully discretized approximation of the operator $\mathbb{L}_\tau = \mathbb{T}_\tau^- \mathbb{T}_\tau^+$ by setting

$$\mathbb{L}_{h,\Delta t,K} = \mathbb{T}_{h,\Delta t,K}^- \mathbb{T}_{h,\Delta t,K}^+.$$

Assume that for all $0 \le k \le K$, y_h^k is a given approximation of $y(t_k)$ in Y and let $\left(w_h^+\right)^k$ and $\left(w_h^-\right)^k$ be respectively the approximations of (3.4) and (3.5) obtained via (3.22) as follows:

- For all $0 \le k \le K$, $\left(w_h^+\right)^k = p_h^k$ where p_h^k solves (3.22) with $f_h^k = C_0^* y_h^k$ and $(p_{0,h,\Delta t}, p_{1,h,\Delta t}) = (0, 0)$,
- For all $0 \le k \le K$, $\left(w_h^-\right)^k = p_h^{K-k}$ where p_h^k solves (3.22) with $f_h^k = -C_0^* y_h^{K-k}$ and $(p_{0,h,\Delta t}, p_{1,h,\Delta t}) = ((w_h^+)^K, -D_t(w_h^+)^K)$.

Then, our main result (the fully discrete counterpart of Theorem 3) reads as follows

Theorem 4 *Let $A_0 : \mathcal{D}(A_0) \to H$ be a strictly positive self-adjoint operator and $C_0 \in \mathcal{L}(H, Y)$ such that $C_0^* C_0 \in \mathcal{L}\left(\mathcal{D}\left(A_0^{\frac{3}{2}}\right)\right) \cap \mathcal{L}\left(\mathcal{D}\left(A_0^{\frac{1}{2}}\right)\right)$. Define (A, C) by (3.2) and (3.3). Assume that the pair (A, C) is exactly observable in time $\tau > 0$ and set $\eta := \|\mathbb{L}_\tau\|_{\mathcal{L}(X)} < 1$. Let $(w_0, w_1) \in \mathcal{D}\left(A_0^{\frac{3}{2}}\right) \times \mathcal{D}(A_0)$ be the initial value of (3.1) and let $(w_{0,h,\Delta t}, w_{1,h,\Delta t})$ be defined by*

$$\begin{bmatrix} w_{0,h,\Delta t} \\ w_{1,h,\Delta t} \end{bmatrix} = \sum_{n=0}^{N_h} \mathbb{L}_{h,\Delta t,K}^n \begin{bmatrix} (w_h^-)^0 \\ D_t(w_h^-)^1 \end{bmatrix}, \tag{3.24}$$

where $D_t(w_h^-)^1 = \dfrac{(w_h^-)^1 - (w_h^-)^0}{\Delta t}$.

Then there exist $M > 0$, $h^* > 0$ and $\Delta t^* > 0$ such that for all $h \in (0, h^*)$ and $\Delta t \in (0, \Delta t^*)$

$$\begin{aligned} \|w_0 - w_{0,h,\Delta t}\|_{\frac{1}{2}} + \|w_1 - w_{1,h,\Delta t}\| \le M \Bigg[& \left(\frac{\eta^{N_{h,\Delta t}+1}}{1-\eta} + \left(h^\theta + \Delta t\right)(1+\tau) N_{h,\Delta t}^2 \right) \\ & \times \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1 \right) + N_{h,\Delta t} \Delta t \\ & \times \sum_{\ell=0}^{K} \left\| C_0^* (y(t_\ell) - y_h^\ell) \right\| \Bigg]. \end{aligned}$$

Corollary 4 *Under the assumptions of Theorem 4, we set*

$$N_{h,\Delta t} = \frac{\ln(h^\theta + \Delta t)}{\ln \eta}$$

Then, there exist $M_\tau > 0, h^* > 0$ and $\Delta t^* > 0$ such that for all $h \in (0, h^*)$ and $\Delta t \in (0, \Delta t^*)$

$$\|w_0 - w_{0,h,\Delta t}\|_{\frac{1}{2}} + \|w_1 - w_{1,h,\Delta t}\| \leq M_\tau \Bigg[(h^\theta + \Delta t) \ln^2(h^\theta + \Delta t) \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1 \right) \\ + \left| \ln(h^\theta + \Delta t) \right| \Delta t \sum_{\ell=0}^{K} \left\| C_0^* \left(y(t_\ell) - y_h^\ell \right) \right\| \Bigg]. \tag{3.25}$$

3.2.2 *Proof of Theorem 4*

As in the semi-discrete case, the main ingredient for the convergence analysis is the following result (the counterpart of Proposition 5) which gives the error estimate for the full approximation of the general system (3.9) by (3.22).

Proposition 7 *Given $(p_0, p_1) \in \mathcal{D}\left(A_0^{\frac{3}{2}}\right) \times \mathcal{D}(A_0)$ and $(p_{0,h,\Delta t}, p_{1,h,\Delta t}) \in H_h \times H_h$, let p and $(p_h^k)_k$ be the solutions of (3.9) and (3.22) respectively. Assume that $C_0^* C_0 \in \mathcal{L}\left(\mathcal{D}\left(A_0^{\frac{1}{2}}\right)\right)$. Then, there exist $M > 0, h^* > 0$ and $\Delta t^* > 0$ such that for all $1 \leq k \leq K$, all $h \in (0, h^*)$ and all $\Delta t \in (0, \Delta t^*)$*

$$\|\pi_h p(t_k) - p_h^k\|_{\frac{1}{2}} + \|\pi_h \dot{p}(t_k) - D_t p_h^k\| \leq M \Bigg\{ \|\pi_h p_0 - p_{0,h,\Delta t}\|_{\frac{1}{2}} + \|\pi_h p_1 - p_{1,h,\Delta t}\| \\ + \left(h^\theta + \Delta t\right) \left[t_k \left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 + \|f\|_{\frac{1}{2},\infty} + \|\dot{f}\|_\infty \right) + t_k^2 \|f\|_{1,\infty} \right] \\ + \Delta t \sum_{\ell=1}^{k} \|f(t_\ell) - f_h^\ell\| \Bigg\}.$$

Proof Denote by $r_1(t_k)$ the residual term in the first order Taylor expansion of p around t_{k-1}. Then

$$\dot{p}(t_k) = \frac{p(t_k) - p(t_{k-1})}{\Delta t} - \frac{1}{\Delta t} r_1(t_k) = D_t p(t_k) - \frac{1}{\Delta t} r_1(t_k), \tag{3.26}$$

We have

$$\begin{aligned} \|\pi_h \dot{p}(t_k) - D_t p_h^k\| &\leq \|\pi_h \dot{p}(t_k) - \pi_h D_t p(t_k)\| + \|D_t(\pi_h p(t_k) - p_h^k)\| \\ &\leq \frac{1}{\Delta t} \|r_1(t_k)\| + \|D_t(\pi_h p(t_k) - p_h^k)\| \end{aligned}$$

Therefore, the error we need to bound satisfies

$$\|\pi_h p(t_k) - p_h^k\|_{\frac{1}{2}} + \|\pi_h \dot{p}(t_k) - D_t p_h^k\| \leq 2\sqrt{\mathcal{E}_h^k} + \frac{1}{\Delta t} \|r_1(t_k)\| \tag{3.27}$$

where we have set for all $1 \leq k \leq K$

$$\mathcal{E}_h^k = \frac{1}{2}\left\{\left\| D_t\left(\pi_h p(t_k) - p_h^k\right)\right\|^2 + \left\|\pi_h p(t_k) - p_h^k\right\|_{\frac{1}{2}}^2\right\}.$$

On the other hand, if $r_2(t_k)$ denote the residual term first order the Taylor expansion of $\dot{p}$ around t_{k-1}, then

$$\ddot{p}(t_k) = D_{tt}p(t_k) - \gamma^k, \tag{3.28}$$

where

$$\gamma^k = \frac{1}{\Delta t^2}\left(r_1(t_k) - r_1(t_{k-1})\right) + \frac{1}{\Delta t} r_2(t_k).$$

Using (3.26) and (3.28), and subtracting (3.22) from the variational formulation (3.9) written for $t = t_k$ and for an arbitrary test function $\varphi = \varphi_h \in H_h$, one easily obtains

$$\begin{aligned}\left\langle D_{tt}\left(\pi_h p(t_k) - p_h^k\right), \varphi_h\right\rangle + \left\langle \pi_h p(t_k) - p_h^k, \varphi_h\right\rangle_{\frac{1}{2}} &= \left\langle D_{tt}\left(\pi_h p(t_k) - p(t_k)\right), \varphi_h\right\rangle \\ -\left\langle C_0^* C_0 D_t\left(p(t_k) - p_h^k\right), \varphi_h\right\rangle + \left\langle \gamma^k, \varphi_h\right\rangle &+ \frac{1}{\Delta t}\left\langle C_0^* C_0 r_1(t_k), \varphi_h\right\rangle \\ +\left\langle f(t_k) - f_h^k, \varphi_h\right\rangle.\end{aligned} \tag{3.29}$$

Using the identity

$$\frac{1}{2}\left(\|u\|^2 - \|v\|^2 + \|u - v\|^2\right) = \mathrm{Re}\,\langle u - v, u\rangle, \quad \forall u, v \in H,$$

one easily obtains that for all $2 \leq k \leq K$

$$\begin{aligned}D_t \mathcal{E}_h^k \leq &\left\langle D_{tt}\left(\pi_h p(t_k) - p_h^k\right), D_t\left(\pi_h p(t_k) - p_h^k\right)\right\rangle \\ &+\left\langle \pi_h p(t_k) - p_h^k, D_t\left(\pi_h p(t_k) - p_h^k\right)\right\rangle_{\frac{1}{2}}.\end{aligned} \tag{3.30}$$

Taking $\varphi_h = D_t\left(\pi_h p(t_k) - p_h^k\right)$ in (3.29) and substituting in the above inequality and using the boundedness of C_0, we obtain the existence of $M > 0$ such that for all $2 \leq k \leq K$

$$\begin{aligned}D_t \mathcal{E}_h^k \leq M\Big[&\|D_{tt}\left(\pi_h p(t_k) - p(t_k)\right)\| + \|D_t\left(\pi_h p(t_k) - p(t_k)\right)\| + \|\gamma^k\| \\ &+\frac{1}{\Delta t}\|r_1(t_k)\| + \|f(t_k) - f_h^k\|\Big]\left\| D_t(\pi_h p(t_k) - p_h^k)\right\|.\end{aligned} \tag{3.31}$$

Using relations (2.32) and (2.33), we obtain from (3.8), (3.31), (3.26), (3.28) and relations (4.2) and (4.3) in Lemma 2 of the Appendix for the first order formulation of (3.6) that for all $h \in (0, h^*)$

$$\begin{aligned} D_t\sqrt{\mathcal{E}_h^k} \leq M\Big\{ & h^\theta \left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 + t_k\|f\|_{1,\infty} + \|f\|_{\frac{1}{2},\infty}\right) + \|f(t_k) - f_h^k\| \\ & + \frac{h^\theta}{\Delta t^2}\|r_1(t_k) - r_1(t_{k-1})\|_{\frac{1}{2}} + \frac{h^\theta}{\Delta t}\left(\|r_1(t_k)\|_{\frac{1}{2}} + \|r_2(t_k)\|_{\frac{1}{2}}\right) \\ & + \frac{1}{\Delta t^2}\|r_1(t_k) - r_1(t_{k-1})\| + \frac{1}{\Delta t}\Big(\|r_1(t_k)\| + \|r_2(t_k)\|\Big)\Big\}. \end{aligned} \tag{3.32}$$

To conclude, it remains to bound the terms including the residuals r_1 and r_2 in the above estimate. By definition of r_2, the mean value theorem and using once again (4.3), we obtain that there exists $M > 0$ such that

$$\|r_2(t_k)\|_{\frac{1}{2}} \leq M\Delta t\left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 + t_k\|f\|_{1,\infty} + \|f\|_{\frac{1}{2},\infty}\right). \tag{3.33}$$

Now by the regularity of p (see Lemma 2 applied to the first order formulation of (3.6)), the residual r_2 can be expressed via the integral

$$r_2(t_k) = \int_{t_{k-1}}^{t_k} \frac{d^3p}{ds^3}(s)\,(t_{k-1} - s)\,ds,$$

in H. Using Eq. (3.6) verified by p and the boundedness of C_0, we have

$$\begin{aligned} \left\|\frac{d^3p}{dt^3}(t)\right\| = \left\|\frac{d\ddot{p}}{dt}(t)\right\| = \left\|\frac{d}{dt}\Big\{-A_0p(t) - C_0^*C_0\dot{p}(t) + f(t)\Big\}\right\|, \\ \leq \|\dot{p}(t)\|_1 + M\|\ddot{p}(t)\| + \|\dot{f}(t)\|. \end{aligned} \tag{3.34}$$

Hence, once again by (4.3), we get

$$\|r_2(t_k)\| \leq M\Delta t^2\left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 + t_k\|f\|_{1,\infty} + \|f\|_{\frac{1}{2},\infty} + \|\dot{f}\|_\infty\right). \tag{3.35}$$

For the term implying r_1, we note that

$$r_1(t_k) = \int_{t_{k-1}}^{t_k} \ddot{p}(s)(t_{k-1} - s)ds,$$

in $\mathcal{D}\left(A_0^{\frac12}\right)$. Hence, by a similar argument and (4.3),

$$\|r_1(t_k)\| \le M\|r_1(t_k)\|_{\frac12} \le M\Delta t^2\left(\|p_0\|_{\frac32} + \|p_1\|_1 + t_k\|f\|_{1,\infty} + \|f\|_{\frac12,\infty}\right). \tag{3.36}$$

Then, we write in $\mathcal{D}\left(A_0^{\frac12}\right)$ the difference $r_1(t_k) - r_1(t_{k-1})$ on the integral form. Using the above relation, it comes by using once again (4.3)

$$\begin{aligned}\|r_1(t_k) - r_1(t_{k-1})\|_{\frac12} &\le M\Delta t^2 \sup_{s\in(t_{k-2},t_{k-1})} \|\ddot{p}(s)\|_{\frac12},\\ &\le M\Delta t^2\left(\|p_0\|_{\frac32} + \|p_1\|_1 + t_{k-1}\|f\|_{1,\infty} + \|f\|_{\frac12,\infty}\right).\end{aligned} \tag{3.37}$$

Finally

$$\begin{aligned}\|r_1(t_k) - r_1(t_{k-1})\| &\le \Delta t \int_{t_{k-2}}^{t_{k-1}} \int_{s-\Delta t}^{s} \left\|\frac{d^3p}{d\sigma^3}(\sigma)\right\| d\sigma\, ds,\\ &\le M\Delta t^3 \sup_{s\in(t_{k-3},t_{k-1})} \left\|\frac{d^3p}{ds^3}(s)\right\|.\end{aligned}$$

Using (3.34) and (4.3), we get

$$\|r_1(t_k) - r_1(t_{k-1})\| \le M\Delta t^3\left(\|p_0\|_{\frac32} + \|p_1\|_1 + t_{k-1}\|f\|_{1,\infty} + \|f\|_{\frac12,\infty} + \|\dot{f}\|_\infty\right). \tag{3.38}$$

Substituting (3.33), (3.35), (3.36), (3.37) and (3.38) in relation (3.32) provides estimates for $D_t\sqrt{\mathcal{E}_h^k} = \frac{\sqrt{\mathcal{E}_h^k} - \sqrt{\mathcal{E}_h^{k-1}}}{\Delta t}$, for $k = 1, \dots, K$. By adding all these inequalities, we immediately get an upper bound for $\sqrt{\mathcal{E}_h^k}$, and thus the desired inequality thanks to (3.27) and (3.36).

Using this Proposition, we can derive an error estimate for the semigroup $\mathbb{T}_{t_k}^{\pm}$ (for all $0 \le k \le K$) and for the operator $\mathbb{L}_\tau = \mathbb{T}_\tau^- \mathbb{T}_\tau^+$ (the counterpart of Proposition 6). We skip the proof, which is nearly the same as the one of Proposition 4.

Proposition 8 *Let* $\Pi_h = \begin{bmatrix}\pi_h & 0\\ 0 & \pi_h\end{bmatrix}$. *Under the assumptions of Proposition 7, the following assertions hold true*

1. *There exist* $M > 0, h^* > 0$ *and* $\Delta t^* > 0$ *such that for all* $h \in (0, h^*)$*, all* $\Delta t \in (0, \Delta t^*)$ *and all* $0 \le k \le K$

$$\left\| (\Pi_h \mathbb{T}^+_{t_k} - \mathbb{T}^+_{h,\Delta t,k}) \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \right\| \le M t_k (h^\theta + \Delta t) \left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 \right). \tag{3.39}$$

$$\left\| (\Pi_h \mathbb{T}^-_{t_k} - \mathbb{T}^-_{h,\Delta t,k}) \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \right\| \le M (\tau - t_k)(h^\theta + \Delta t) \left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 \right). \tag{3.40}$$

2. *There exist* $M > 0, h^* > 0$ *and* $\Delta t^* > 0$ *such that for all* $n \in \mathbb{N}$*, all* $h \in (0, h^*)$*, all* $\Delta t \in (0, \Delta t^*)$ *and all* $0 \le k \le K$

$$\left\| (\mathbb{L}^n_{t_k} - \mathbb{L}^n_{h,\Delta t,k}) \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \right\| \le M \left[h^\theta + n\tau (h^\theta + \Delta t) \right] \left(\|p_0\|_{\frac{3}{2}} + \|p_1\|_1 \right). \tag{3.41}$$

We are now able to prove Theorem 4.

Proof of Theorem 4 Introducing the term $\sum_{n=0}^{N_{h,\Delta t}} \mathbb{L}^n_{h,\Delta t,K} \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix}$, we can rewrite $\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} - \begin{bmatrix} w_{0,h,\Delta t} \\ w_{1,h,\Delta t} \end{bmatrix}$ in the following form

$$\begin{aligned} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} - \begin{bmatrix} w_{0,h,\Delta t} \\ w_{1,h,\Delta t} \end{bmatrix} &= \sum_{n=0}^{\infty} \mathbb{L}^n_\tau \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix} - \sum_{n=0}^{N_{h,\Delta t}} \mathbb{L}^n_{h,\Delta t,K} \begin{bmatrix} (w_h^-)^0 \\ D_t (w_h^-)^1 \end{bmatrix}, \\ &= \sum_{n > N_{h,\Delta t}} \mathbb{L}^n_\tau \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix} + \sum_{n=0}^{N_{h,\Delta t}} \left(\mathbb{L}^n_\tau - \mathbb{L}^n_{h,\Delta t,K} \right) \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix} \\ &\quad + \textstyle\sum_{n=0}^{N_{h,\Delta t}} \mathbb{L}^n_{h,\Delta t,K} \left(\begin{bmatrix} w^-(0) - (w_h^-)^0 \\ \dot{w}^-(0) - D_t (w_h^-)^1 \end{bmatrix} \right). \end{aligned}$$

Therefore, we have

$$\|w_0 - w_{0,h,\Delta t}\|_{\frac{1}{2}} + \|w_1 - w_{1,h,\Delta t}\| \le S_1 + S_2 + S_3, \tag{3.42}$$

where we have set

$$\begin{cases} S_1 = \sum_{n > N_{h,\Delta t}} \left\| \mathbb{L}^n_\tau \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix} \right\|, \\ S_2 = \sum_{n=0}^{N_{h,\Delta t}} \left\| \left(\mathbb{L}^n_\tau - \mathbb{L}^n_{h,\Delta t,K} \right) \begin{bmatrix} w^-(0) \\ \dot{w}^-(0) \end{bmatrix} \right\|, \\ S_3 = \left(\sum_{n=0}^{N_{h,\Delta t,K}} \left\| \mathbb{L}^n_{h,\Delta t,K} \right\|_{\mathcal{L}(X)} \right) \left\| \begin{bmatrix} w^-(0) - (w_h^-)^0 \\ \dot{w}^-(0) - D_t (w_h^-)^1 \end{bmatrix} \right\|. \end{cases}$$

Once again, using similar arguments as the ones detailed for the Schrödinger case, we get the claimed result.

Appendix

Let $A : \mathcal{D}(A) \to X$ a skew-adjoint operator and $C \in \mathcal{L}(X, Y)$ such that $C^*C \in \mathcal{L}(\mathcal{D}(A))$. Assume that $A - C^*C$ generates a C_0-semigroup $\mathbb{T}$ of contractions on X, *i.e.* that $\|\mathbb{T}_t\|_{\mathcal{L}(X)} \leq 1$ for all $t \geq 0$.

Lemma 1 *The operator $A - C^*C$ generates a C_0-semigroup of contractions on $\mathcal{D}(A)$ and $\mathcal{D}(A^2)$.*

Proof As $C \in \mathcal{L}(X, Y)$ is bounded, we clearly have $\mathcal{D}(A) = \mathcal{D}(A - C^*C)$. Moreover, $C^*C \in \mathcal{L}(\mathcal{D}(A))$ implies that $\mathcal{D}(A^2) = \mathcal{D}\left((A - C^*C)^2\right)$. The result follows then from [27, Proposition 2.10.4].

Lemma 2 *Given $q_0 \in \mathcal{D}(A^2)$ and $F \in C\left([0, \tau], \mathcal{D}(A^2)\right) \cap C^1([0, \tau], \mathcal{D}(A))$, let q denote the solution of the initial value problem*

$$\begin{cases} \dot{q}(t) = Aq(t) - C^*Cq(t) + F(t), & t \in (0, \tau), \\ q(0) = q_0. \end{cases}$$

Then, we have the following statements

1. *Regularity:*

$$q \in C\left([0, \tau], \mathcal{D}\left(A^2\right)\right) \cap C^1([0, \tau], \mathcal{D}(A)) \cap C^2([0, \tau], X), \tag{4.1}$$

2. *Bound for q:*

$$\|q(t)\|_\alpha \leq \|q_0\|_\alpha + t\|F\|_{\alpha,\infty}, \qquad \textit{for } \alpha = 0, 1, 2, \tag{4.2}$$

3. *Bound for $\dot{q}$: there exists $M > 0$ such that*

$$\|\dot{q}(t)\|_\alpha \leq M\left(\|q_0\|_{\alpha+1} + t\|F\|_{\alpha+1,\infty}\right) + \|F\|_{\alpha,\infty}, \quad \textit{for } \alpha = 0, 1, \tag{4.3}$$

where $\|F\|_{\alpha,\infty} = \sup_{t\in[0,\tau]} \|F(t)\|_\alpha$.

Proof

1. By [27, Theorem 4.1.6], we have $q \in C\left([0, \tau], \mathcal{D}\left(A^2\right)\right) \cap C^1([0, \tau], \mathcal{D}(A))$. But since $C^*C \in \mathcal{L}(\mathcal{D}(A))$ and $F \in C\left([0, \tau], \mathcal{D}\left(A^2\right)\right) \cap C^1([0, \tau], \mathcal{D}(A))$, we have

$$\left(A - C^*C\right) q(t) \in C([0, \tau], \mathcal{D}(A)) \cap C^1([0, \tau], X).$$

The last inclusion follows then from the fact that $\dot{q}(t) = (A - C^*C)\, q(t)$ in $\mathcal{D}(A)$.

2. By Duhamel's formula, we have

$$\begin{aligned}
\|q(t)\|_\alpha &= \left\| \mathbb{T}_t q_0 + \int_0^t \mathbb{T}_{t-s} F(s) ds \right\|_\alpha , \\
&\leq \|\mathbb{T}_t q_0\|_\alpha + \int_0^t \|\mathbb{T}_{t-s} F(s)\|_\alpha \, ds, \\
&\leq \|q_0\|_\alpha + t \|F\|_{\alpha,\infty},
\end{aligned}$$

   where we have used Lemma 1 of the Appendix for the last inequality.
3. Using the estimate (4.2) obtained for $q(t)$ and the continuity of the embeddings $\mathcal{D}\left(A^2\right) \hookrightarrow \mathcal{D}(A) \hookrightarrow X$, we easily get

$$\begin{aligned}
\|\dot{q}(t)\|_\alpha &= \| (A - C^* C)\, q(t) + F(t)\|_\alpha , \\
&\leq \|q(t)\|_{\alpha+1} + M\|q(t)\|_\alpha + \|F\|_{\alpha,\infty}, \\
&\leq M \left(\|q_0\|_{\alpha+1} + t\|F\|_{\alpha+1,\infty} \right) + \|F\|_{\alpha,\infty}.
\end{aligned}$$

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