Puzzles in pipes with negative curvature: from the Webster PDE to stable numerical simulation in real time*

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Abstract

Minimal realizations of a class of delay-differential systems are derived for the digital simulation of waveguides, modelled by the Webster horn equation. Studying their stability is an interesting issue, since *negative* curvatures could lead to unstable systems. Spectral properties of Toeplitz matrix play a key role in this work.

Keywords

Webster equation, delay-differential systems, BIBO-stability, numerical simulation.

1 Introduction

The wave equation with space-varying coefficients that models propagation in horns is the Webster PDE, which is known to be *conservative*, whatever the shape of the horn; hence, stable numerical schemes can be derived for it, e.g. forward Euler on the vector (pressure, flow) with a CFL condition. But from an input-ouptput point of view, in the case of *negative curvature*, it is well-known that unstable subsystems are to be found, a paradox that has recently been fully understood in [MED08] thanks to minimal realization of a delay system modelling a convergent cone. In the present work, the same methodology is applied to a pipe with negative (constant) curvature, the discretization of which now gives rise to a delay-differential system.

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2 1D propagation in a convex acoustic pipe

Consider the conservative acoustic 1D propagation into a pipe with varying radius $R(\ell) = R_0 \cos(\ell)$ where $\ell \in [-L/2, L/2]$ (with $L < \pi$) denotes the curvilinear abscissa measuring the length of the shape from $\ell = 0$ (see Fig. 1a). Inside this pipe symmetrical w.r.t $\ell = 0$, the acoustic pressure $p(\ell, t)$ is governed by the following Webster equation [?], for the adimentional celerity $c = 1$, $\partial_\ell^2 p(\ell,t) + 2\zeta(\ell)\partial_\ell p(\ell,t) - \partial_t^2 p(\ell,t) = 0$, where $\zeta = r'/r$. Denoting $X(\ell,t) = [p(\ell,t),\, r(\ell)^2 v(\ell,t)]^T$ where v is the particle velocity for the adimensional massic density $\rho = 1$, the acoustic problem can be described by (see [?]) $\partial_\ell X(\ell,t) + \begin{pmatrix} 0 & 1/r(\ell)^2 \\ r(\ell)^2 & 0 \end{pmatrix} \partial_t X(\ell,t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or equivalently, and introducing the state variable $\phi^\pm = r(p \pm v)$,

$$\partial_\ell \phi^\pm(\ell,t) \pm \partial_t \phi^\pm(\ell,t) = \zeta(\ell)\, \phi^\mp(\ell,t). \tag{1}$$

Consider the scattering matrix $M(s)$ defined by $[\hat{\phi}^+(L/2,s), \hat{\phi}^-(-L/2,s)]^T = M(s)[\hat{\phi}^+(-L/2,s), \hat{p}^-(L/2,s$ in the Laplace domain for $s \in \mathbb{C}_0^+ = \{s \in \mathbb{C} | \Re e(s) > 0\}$. For $r(\ell) = r_0 \cos(\ell)$, its computation yields $M(s) = \begin{pmatrix} T(s) & R(s) \\ R(s) & T(s) \end{pmatrix}$ where a square root of $s^2 - 1$ is to be found...

3 Approximation with pieces of conical pipes

For all $N \in \mathbb{N}^*$ and $n \in [0, N]_{\mathbb{N}}$, define $\ell_n^N = \left(-\frac{1}{2} + \frac{n}{N}\right)L$ and approximate the shape r with the continuous piecewise affine model $\widetilde{r}$ as follows: if $1 \le n \le N$ and $\ell \in [\ell_{n-1}, \ell_n]$, then $\widetilde{r}_N(\ell) = r(\ell_{n-1}) + \xi_n^N(\ell - \ell_{n-1})$ with slope $\xi_n^N = \big(r(\ell_n) - r(\ell_{n-1})\big)/\epsilon_N$ where $\epsilon_N = L/N$. In each conical piece of pipe, the propagation of ideal spherical

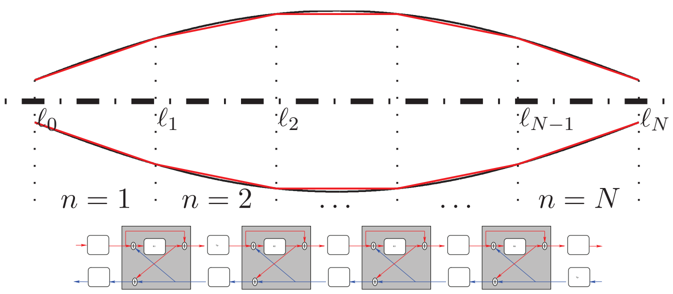

Figure 1: Shape of the pipe with radius r and its piecewise afinne approximation $\widetilde{r}_N$... A AFFINER !

waves is assumed. Solving (1) and assuming the continuity of $\phi^\pm$ (that is, of acoustic pressure and flow), we find the structure in Fig. 1b in which reflections functions are all given by $R_{\epsilon_N}(s)$ where $R_\epsilon(s) = \alpha_\epsilon/(s - \alpha_\epsilon)$ and $\alpha_\epsilon = (1 - \cos\epsilon)/\epsilon \underset{0}{\sim} \epsilon/2$ (see e.g. [?]). This structure corresponds to the following global state-space-like representation, with input $U = []^T$, output $Y_N = []^T$, and state $X_N = [?]^T$,

$$s\, X_N(s) = A_N(e^{-\tau_n s}) X_N(s) + B_N(e^{-\tau_n s}) U(s), \tag{2}$$

$$Y_N = C_N(e^{-\tau_n s}) X_N(s), \tag{3}$$

where $A_N(w) = \alpha_{\epsilon_N} W_N(w)$, $W_N(w)$ is the $N \times N$-symmetrical Toeplitz matrix such that $[W_N(w)]_{ij} = w^{|i-j|}$, $B_N(w)$ is composed of the first and last columns of $W_N(w)$ and $C_N(w) = w\, B_N(w)^T$.

4 Simulations

References

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