Fractional equations and diffusive systems: an overview

Denis Matignon *

* *University of Toulouse, ISAE; 10, av. E. Belin; BP 54032. 31055 Toulouse Cedex 4, France.* denis.matignon@isae.fr

Abstract: The aim of this discussion is to give a broad view of the links between fractional differential equations (FDEs) or fractional partial differential equations (FPDEs) and so-called diffusive representations (DR). Many aspects will be investigated: theory and numerics, continuous time and discrete time, linear and nonlinear equations, causal and anti-causal operators, optimal diffusive representations, fractional Laplacian. Many applications will be given, in acoustics, continuum mechanics, electromagnetism, identification, ...

Keywords: fractional equations, diffusive systems, pseudo-differential operators, hereditary mechanics, stability, numerical methods.

1. INTRODUCTION

Fractional differential systems have become quite popular in the recent decades, giving rise to a wide literature, both on the theoretical and on the applied sides; monographs, and special issues of international journals are now devoted to this active research field.

The aim of this discussion paper is two fold: first explain to what extent diffusive representations can be helful for fractional equations, second give many fields of application where this technique has proved useful.

1.1 Fractional integral and derivatives

These causal linear operators can be defined in many ways, see e.g. Matignon [2009a].

Fractional integral Let $\beta \in (0,1)$, and set $h_\beta(t) := \frac{1}{\Gamma(\beta)}\, t^{\beta-1}$ for $t > 0$ only; then, $h_\beta \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$. For any $T > 0$, let $u \in L^2(0,T)$, and define $I^\beta u := h_\beta \star u$ or, more explicitely:

$$I^\beta u(t) = \int_0^t \frac{1}{\Gamma(\beta)}\, \tau^{\beta-1}\, u(t-\tau)\,\mathrm{d}\tau\,.$$

This is the Riemann-Liouville fractional integral of order $\beta \in (0,1)$ of u: it is causal, and belongs to $L^2(0,T)$ also. In terms of causal Laplace transform, $H_\beta(s) = s^{-\beta}$ in $\Re e(s) > 0$; hence, the interpretation of the fractional integral is a causal low-pass filter, with a gain of $-6\,\beta$ dB per octave.

Fractional derivative The fractional derivative is the inverse of the fractional integral, but some technicalities are to be found in this case. Let $\alpha \in (0,1)$, and for any $T > 0$, let $u \in H^1(0,T)$, (that is $u \in L^2(0,T)$, u has a *weak* derivative say $\dot{u}$ which does belong to $L^2(0,T)$), and define $D^\alpha u = I^{1-\alpha} D u := h_{1-\alpha} \star \dot{u}$ or, more explicitely:

$$D^\alpha u(t) = \int_0^t \frac{1}{\Gamma(1-\alpha)}\, \tau^{-\alpha}\, \dot{u}(t-\tau)\,\mathrm{d}\tau\,.$$

This is the fractional derivative of order $\alpha \in (0,1)$ of u: it is causal, and belongs to $L^2(0,T)$. In terms of causal Laplace transform, $\widetilde{H}_\alpha(s) = s^{+\alpha}$ in $\Re e(s) > 0$; hence, the interpretation of the fractional derivative is a causal high-pass filter, with a gain of $+6\,\alpha$ dB per octave.

Open question Both these operators present the drawback of not being differential operators, not easily giving rise to a semigroup; moreover, a hereditary behaviour can be foreseen, with a long-memory decay. Therefore, we try to make a link between these fractional operators, and ordinary differential equations (ODEs), or just convolution by families of decaying exponentials: we derive functional identities for the kernel h_β and its Laplace transform H_β, in the time-domain and the frequency-domain; these are being used to define input-output representations, and even state-space realizations.

1.2 Diffusive representations

We refer to Staffans [1994] and Montseny [1998] for an introduction.

First diffusive representations With specific weight $\mu_\beta(\xi) := \frac{\sin(\beta\,\pi)}{\pi}\,\xi^{-\beta}$, one has $h_\beta(t) = \int_0^\infty \mu_\beta(\xi)\, e^{-\xi\, t}\,\mathrm{d}\xi$, which helps reformulate the fractional integral as the following input-output representation :

$$y(t) = \int_0^\infty \mu_\beta(\xi)\,[e_\xi \star u](t)\,\mathrm{d}\xi\,,$$

with $e_\xi(t) := e^{-\xi\, t}$, and $[e_\xi \star u](t) = \int_0^t e^{-\xi\,(t-\tau)}\, u(\tau)\,\mathrm{d}\tau$. Then, the following infinite-dimensional dynamical system can be seen as a state-space realization of the fractional intergral of order β:

$$\partial_t \varphi(\xi,t) = -\xi\,\varphi(\xi,t) + u(t),\;\; \varphi(\xi,0) = 0\,, \tag{1}$$

$$y(t) = \int_0^\infty \mu_\beta(\xi)\,\varphi(\xi,t)\,\mathrm{d}\xi\,. \tag{2}$$

The well-posedness condition $\int_0^\infty \frac{\mu_\beta(\xi)}{1+\xi}\,\mathrm{d}\xi < +\infty$ holds.

Moreover, the energy balance can be checked, $\forall T > 0$:
$$\int_0^T u(t)\, y(t)\, \mathrm{d}t = E_\phi(T) + \int_0^T \int_0^{+\infty} \xi\, \mu_\beta(\xi)\, \phi(\xi,t)^2\, \mathrm{d}\xi\, \mathrm{d}t\,,$$
with storage function $E_\phi(T) := \frac{1}{2} \int_0^\infty \phi(\xi,T)^2\, \mu_\beta(\xi)\, \mathrm{d}\xi$.

Extended diffusive representations A careful computation shows that the following input-output representation holds:
$$\widetilde{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\, [u - \xi\, e_\xi \star u](t)\, \mathrm{d}\xi\,.$$
The following infinite-dimensional dynamical system can be seen as a state-space realization of the fractional derivative of order α:
$$\partial_t \widetilde{\varphi}(\xi,t) = -\xi\, \widetilde{\varphi}(\xi,t) + u(t),\ \ \widetilde{\varphi}(\xi,0) = 0\,, \tag{3}$$
$$\widetilde{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\, [u(t) - \xi\, \widetilde{\varphi}(\xi,t)]\, \mathrm{d}\xi\,. \tag{4}$$
The well-posedness condition $\int_0^\infty \frac{\mu_{1-\alpha}(\xi)}{1+\xi}\, \mathrm{d}\xi < +\infty$ holds.

Moreover, the energy balance can be checked, $\forall T > 0$:
$$\int_0^T u(t)\, \widetilde{y}(t)\, \mathrm{d}t = \widetilde{E}_{\widetilde{\phi}}(T) + \int_0^T \int_0^{+\infty} \mu_{1-\alpha}(\xi)\, (u - \xi\, \widetilde{\phi})^2\, \mathrm{d}\xi\, \mathrm{d}t\,,$$
with storage function $\widetilde{E}_{\widetilde{\phi}}(T) := \frac{1}{2} \int_0^\infty \widetilde{\phi}(\xi,T)^2\, \xi\, \mu_{1-\alpha}(\xi)\, \mathrm{d}\xi$.

Conclusion on DR The main interests or advantages of the diffusive realizations are:

- the existence of an associated semigroup,
- the dissipativity of the realization, whenever the operator is positive,
- the possibility of deriving numerical schemes without heredity, hence memory-saving algorithms.

2. GENERAL ASPECTS

2.1 Equivalent formulation of fractionally damped equations

First an infinite-dimensional state-space realization of the fractional derivative or integral input-ouput relation is being used. Thus, another way of considering the problem consists in interpreting the fractionally damped equation as a *coupled* problem between a conservative classical system (either linear or non-linear, either ODE or PDE) and a diffusion equation: this naturally introduces a global energy (either quadratic or non-quadratic) in the augmented state-space, and leads, at least formally, to the decay of the global energy, see e.g. Staffans [1994] first, Montseny et al. [2000], Haddar et al. [2004] for a nonlinear FDE, Haddar et al. [2008] for a linear FPDE.

2.2 Stability issues

On FDEs, there is an if and only if condition Matignon [1998], and some related results in Bonnet et al. [2000]. With DR at hand, we only have an if condition, which does not impose constant coefficients, so the range of application is larger; but still if the formal computation of the global energy balance is quite easy, the mathematical proof remains quite involved.

The main difficulty or drawback of the diffusive realizations is the lack of compactness proprety,

- which makes the canonical embeddings of dense subspaces *not* compact, and forbids the use of many existence theorems where it is required,
- which does not allow the use of LaSalle's invariance principle for asymptotic stability analysis, which would require a precompactness prorerty of the trajectories.

Hence, we have to resort to Arendt-Batty stability theorem to conclude on asymptotic stability of the coupled system, after a careful study of the spectrum of the generator of the associated semigroup, see Matignon et al. [2005] for an FDE and Matignon [2006] for an FPDE.

2.3 Numerical methods

For sure, discretizing the continuous diffusive representations in the ξ-domain is possible, but it must be done in a careful way, so as to ensure an equivalent energy balance at the discrete level, see e.g. Haddar et al. [2008]; also the weights can be optimized w.r.t a criterion, see e.g. Deü et al. [2010]. More involved numerical methods can be found in Diethelm [2008], Haddar et al. [2010].

2.4 Optimal control

Still very few papers are concerned with the question of *optimal* control of *fractional* differential systems, ad hoc finite-dimensional approximations of fractional derivatives are used in the first place, and classical optimal control methods are being applied in the second place; no proof of convergence of the process is provided.

A first reason for that could be that optimal control of infinite-dimensional systems is a quite involved and technical field, but a second one lies in the very nature of fractional operators. They are causal, but highly non-local in time (with a weakly integrable singularity at the origin); hence their adjoint becomes necessarily anti-causal and still non-local in time. Thus, one can easily imagine that the complexity of the theory for forward fractional dynamical systems becomes even more intricate when the coupled equations of the adjoint systems are derived; because we will be left with coupled forward and backward fractional dynamics in order to solve the optimal control problem: at first glance, it seems very unlikely that Riccati equations (if any) could be either analysed or even solved (not to speak of adequate numerical schemes for these) in such a complicated setting.

In order to overcome this intrinsic difficulty, in Matignon [2010] we propose to use the equivalent *diffusive* representations of fractional systems, and to work on it, as for infinite dimensional systems of integer order.

3. APPLICATIONS

3.1 Viscoelasticity in continuum mechanics

A large class of such systems, often found in mechanics, is based on causal pseudo-differential time-operators, sometimes with a long-memory behaviour: classical examples are fractional integrals and derivatives. Pseudo-differential operators are hereditary: the whole past of the physical state is involved in the dynamic expression of the system evolution. This generally introduces major technical difficulties. Moreover, from the thermodynamical point of view, consistency of the model is a difficult question in most cases. In e.g. Deü et al. [2010], the fractional Zener model is first translated into its equivqlent diffusive realization, then discretized using a coupled Newmark-diffusive scheme, with optimal choice of the weights. An FPDE in dimension 2 will be treated using the same approach, and the classical finite element method (FEM) in space.

3.2 Visco-thermal losses in musical acoustics

The famous Webster-Lokshin model which describes acoustic waves traveling in a duct with viscothermal losses at the lateral walls is a wave equation with spatially-varying coefficients, which involves fractional-order integrals and derivatives with respect to time: it has been presented in Lokshin and Rok [1978] and Polak [1991], and studied in e.g. Hélie et al. [2006a] and Haddar et al. [2008] thanks to DR.

3.3 Identification in signal processing

Once an equivalent diffusive model has been found, it is of interest to find a low order finite dimensional approximation: to this end, optimal formulation is preferred in practise: a first worked-out example can be found in Garcia et al. [1998], and a whole family of fractional and diffusive systems of increasing complexity has been explored in Hélie et al. [2006b].

3.4 Matched impedance of a beam in control of PDEs

For the formulation of second order systems in time, it can be necessary to deal with vector-valued X, hence positive *matrix-valued PDOs* of diffusive type; not only diagonal matrices of scalar positive PDOs are useful in practice, as an example, in order to solve the impedance matching problem for the Euler–Bernoulli beam, an impedance matrix of the form $\begin{bmatrix} a\,\partial_t^{+\alpha} & 1 \\ 1 & b\,\partial_t^{-\alpha} \end{bmatrix}$, was used with $\alpha = \frac{1}{2}$, see Montseny et al. [1997]: a necessary and sufficient condition for the positivity of this operator is $ab > \frac{1}{(\cos\alpha\pi/2)^2}$. Its equivalent DR should be useful to build an absorbing feedback, which realizes an impedance matching, but so far, the theoretical question is still open, even though numerical simulations prove efficient.

3.5 Polarized media in electomagnetism

In polarized media, a hierarchy of models can be found, see e.g. Petropoulos [2005]. First Debye ou Maxwell : $\frac{1}{1+\tau s}$, then Cole–Cole : $\frac{1}{1+(\tau s)^\alpha}$, then Davidson–Cole : $\frac{1}{(1+\tau s)^\gamma}$, and last Havriliak–Negami : $\frac{1}{(1+(\tau s)^\alpha)^\gamma}$. All of these are of diffusive type, meaning that there exists a specific positive measure μ, satisfying the well-posedness condition, which can be used to represent the system as diffusive.

4. CONCLUSION

Still many things are to be done in this very active field of research: study existence for nonlinear systems, both FDEs and FPDEs; study asymptotic stability in the nonlinear case; go to the MIMO case for Euler-Bernoulli matched impedance matrix; do some more progress on optimal numerical methods; develop the ideas, both theoretical and numerical, for the optimal control of such systems; address new family of examples, such as those in electromagnetism;together with computer science developpement, define some distributed systems strategy to simulate these types of distributed parameter systems.

REFERENCES

C. Bonnet and J. R. Partington, *Coprime factorizations and stability of fractional differential systems*, Systems Control Lett., 41 (2000), pp. 167174.

J.-F. Deü and D. Matignon, *Simulation of fractionally damped mechanical systems by means of a Newmark-diffusive scheme*, Computers and Mathematics with Applications, vol. 59 (2010), pp. 1745–1753.

K. Diethelm, *An investigation of some nonclassical methods for the numerical approximation of Caputo-type fractional derivatives*, Numer. Algor., 47 (2008), 361-390.

G. Garcia and J. Bernussou (1998), *Identification of the dynamics of a lead acid battery by a diffusive model*, ESAIM Proc., 5 (1998), pp. 87–98.

H. Haddar and D. Matignon, *Well-posedness of non-linear conservative systems when coupled with diffusive systems*, in IFAC conf. on Nonlinear Control Systems (NOLCOS), vol. 1, Stuttgart, Germany, sept 2004

———, *Theoretical and numerical analysis of the Webster-Lokshin model*, Research Rep. RR-6558, Institut National de la Recherche en Informatique et Automatique (INRIA), 2008.

H. Haddar, J. R. Li, and D. Matignon, *Efficient solution of a wave equation with fractional-order dissipative terms*, Journal of Computational and Applied Mathematics, vol. 234 (2010), pp. 2003–2010.

T. Hélie and D. Matignon, *Diffusive representations for the analysis and simulation of flared acoustic pipes with visco-thermal losses*, Math. Models Meth. Appl. Sci., 16 (2006), pp. 503–536.

———, *Representations with poles and cuts for the time-domain simulation of fractional systems and irrational transfer functions*, Signal Processing (SP), 86 (2006), pp. 2516–2528.

Y. Le Gorrec, and D. Matignon, Diffusive systems coupled to an oscillator: a Hamiltonian formulation. In *IFAC conf. on Lagrangian and Hamiltonian Methods and Nonlinear Control (LHMNLC'12)*, August 29–31, 2012, Bertinoro, Italy. (invited session).

Lokshin, A. A. and V. E. Rok. Fundamental solutions of the wave equation with retarded time. *Dokl. Akad. Nauk SSSR* **239**, 1305–1308, 1978. (in Russian).

Matignon, D. (1998), *Stability properties for generalized fractional differential systems*, ESAIM Proc., 5 (1998), pp. 145–158.

———, (2006). *Asymptotic stability of the Webster-Lokshin model*, in Mathematical Theory of Networks and Systems (MTNS), Kyoto, Japan, jul 2006, pp. 11 p. CD–Rom. (invited session).

———, (2009). *An introduction to fractional calculus*, vol. 1 of Scaling, Fractals and Wavelets of Digital Signal and Image Processing Series, ISTE–Wiley, 2009, ch. 7, pp. 237–278.

———, (2009). Diffusive representations for fractional Laplacian: systems theory framework and numerical issues. *Physica Scripta*, vol. T136, 014009 (6 p.).

———, (2010). *Optimal control of fractional systems: a diffusive formulation*, in 19th International Symposium on Mathematical Theory of Networks and Systems - MTNS 2010, 2010, pp. 3 p. CD–Rom. (invited session).

D. Matignon and C. Prieur, *Asymptotic stability of linear conservative systems when coupled with diffusive systems*, ESAIM Control Optim. Calc. Var., 11 (2005), pp. 487–507.

D. Matignon and H. Zwart, *Standard diffusive systems as well-posed linear systems*. International Journal of Control, (2012). To appear.

Montseny, G. (1998), *Diffusive representation of pseudo-differential time-operators*, ESAIM Proc., 5 (1998), pp. 159–175.

Montseny, G., J. Audounet and D. Matignon (1997). Fractional integrodifferential boundary control of the Euler–Bernoulli beam. In: *Conference on Decision and Control*. IEEE-CSS, SIAM. San Diego, California. pp. 4973–4978.

Montseny, G., J. Audounet and D. Matignon (2000). Diffusive representation for pseudo-differentially damped non-linear systems. In: *Nonlinear Control in the Year 2000* (A. Isidori, F. Lamnabhi-Lagarrigue and W. Respondek, Eds.). Vol. 2. CNRS, NCN. Springer Verlag. pp. 163–182.

Polack, J.D., (1991). Time domain solution of Kirchhoff's equation for sound propagation in viscothermal gases: a diffusion process. *J. Acoustique*, 4, 47–67.

Petropoulos, P., (2005). On the time-domain response of Cole-Cole dielectrics, *IEEE Trans. on Antennas and Propagation*, 53, pp. 3741–3746.

Staffans, O. J. (1994). Well-posedness and stabilizability of a viscoelastic equation in energy space. *Trans. Amer. Math. Soc.* **345**(2), 527–575.