A NUMERICAL SCHEME FOR TIME-DOMAIN FE ANALYSIS OF VISCOELASTIC STRUCTURES WITH FRACTIONAL DERIVATIVE CONSTITUTIVE EQUATIONS

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Abstract. *A numerical scheme is presented for time-domain simulations of structural dynamic problems with viscoelastic materials described by fractional derivative constitutive equations. The proposed approach combines a classical Newmark time-integration method used to solve second-order mechanical systems (obtained for example after finite element discretization), with a diffusive representation based on the transformation of the fractional operator into a system of linear differential equations. The focus is given on the general formulation of the problem, the algorithm implementation into a finite element framework, and the developpement of a closed-form solution for a fractionnally damped single degree-of-freedom equation.*

1 INTRODUCTION

The importance of fractional calculus for modeling viscoelastic material behavior has been recognized by the mechanical scientific community since the pioneering work of Bagley and Torvik [1]. The merits of using fractional differential operator lie in the fact that few parameters are needed to accurately describe the constitutive law of damping materials and the resulting model can be easily fitted to experimental data over a broad range of frequencies. The numerical approximation of such damped mechanical systems is today intensively studied with a special interest concerning the implementation of fractional constitutive equations within a time-domain finite element framework.

The resolution methods are classically either based on time discretization of the fractional dynamics (see e.g. [2, 3, 4]), or on diffusive representations (cf. [5, 6, 7]). For large scale systems, the first method proves memory consuming because it is necessary to store the whole displacement history of the system due to the non-local character of the fractional derivatives. The second method, based on diffusive realizations of fractional derivatives, is numerically more efficient because it has no hereditary behavior, thus avoiding the storage of the solution from all past time steps. In the second group of methods, a coupled Newmark-diffusive scheme has recently been proposed by the authors and analyzed through a single degree-of-freedom example [8].

In this contribution, we propose to extend our approach to more complex mechanical systems. The focus is given on the general formultation of the problem and the algorithm implementation compatible with the finite element method. The applications to damping prediction of complex structures containing viscoelastic materials will be presented at the conference.

2 FINITE ELEMENT VISCOELASTIC PROBLEM

We consider a structure composed of elastic and viscoelastic materials. The finite element discretization of such a problem leads to the following stiffness and mass matrices $\mathbf{K}_i$ and $\mathbf{M}_i$, associated to the volume of elastic ($i = 1$) and viscoelastic ($i = 2$) material:

$$\begin{aligned} \mathbf{K}_1 &= \mathop{\mathsf{A}}_{e=1}^{n_1^{\mathrm{el}}} \int_{\Omega^e} \mathbf{B}^{\mathrm{T}}\mathbf{C}_1\mathbf{B}\mathrm{d}\Omega \qquad & \mathbf{M}_1 &= \mathop{\mathsf{A}}_{e=1}^{n_1^{\mathrm{el}}} \int_{\Omega^e} \mathbf{N}^{\mathrm{T}}\rho_1\mathbf{N}\mathrm{d}\Omega \\ \mathbf{K}_2 &= \mathop{\mathsf{A}}_{e=1}^{n_2^{\mathrm{el}}} \int_{\Omega^e} \mathbf{B}^{\mathrm{T}}\mathbf{C}_2\mathbf{B}\mathrm{d}\Omega \qquad & \mathbf{M}_2 &= \mathop{\mathsf{A}}_{e=1}^{n_2^{\mathrm{el}}} \int_{\Omega^e} \mathbf{N}^{\mathrm{T}}\rho_2\mathbf{N}\mathrm{d}\Omega \end{aligned} \tag{1}$$

where A is the finite element assembly operator, n_i^{el} is the total number of finite elements meshing the volume i, $\mathbf{B}$ is the strain-displacement matrix, $\mathbf{N}$ is the shape function matrix, ρ_i and $\mathbf{C}_i$ are respectively the density and the material moduli matrix related to the volume i.

Assuming that the Poisson ratio of the viscoelastic material is constant, the stiffness matrix $\mathbf{K}_2$ in the frequency domain can be written as a constant stiffness matrix, calculated with unitary modulus $\mathbf{K}_2^0$, factor of the complex Young modulus $\hat{E}(\omega)$ of the viscoelastic material:

$$\mathbf{K}_2 = \hat{E}(\omega)\mathbf{K}_2^0 \tag{2}$$

In this work, the causal fractional Zener model is chosen to describe the frequency-dependent complex modulus:

$$\hat{E}(\omega) = \frac{E_0 + E_\infty(\mathrm{i}\omega\tau)^\alpha}{1 + (\mathrm{i}\omega\tau)^\alpha} = E_0 + \frac{(E_\infty - E_0)(\mathrm{i}\omega\tau)^\alpha}{1 + (\mathrm{i}\omega\tau)^\alpha} \tag{3}$$

where E_0 and E_∞ are respectively the relaxed ($E_0 = \hat{E}(\omega \to 0)$) and unrelaxed modulus ($E_\infty = \hat{E}(\omega \to \infty)$) satisfying $E_\infty > E_0$. The two other parameters are the fractional power α satisfying $0 < \alpha < 1$ and the relaxation time $\tau > 0$. This four-parameter fractional derivative model has been shown to be an effective tool to describe the weak frequency dependence of most viscoelastic materials [9, 10].

The semi-discrete equation of motion for the damped system can be expressed in the frequency domain as:

$$\left[\mathbf{K}_e + \mathrm{i}\omega\hat{h}(\omega)\mathbf{K}_v - \omega^2\mathbf{M}\right]\hat{\mathbf{u}} = \hat{\mathbf{f}} \tag{4}$$

where $\hat{\mathbf{u}}$ is the displacement response to an external harmonic excitation of amplitude $\hat{\mathbf{f}}$, $\mathbf{K}_v = (E_\infty - E_0)\mathbf{K}_2^0$ is a positive matrix, $\mathbf{K}_e = \mathbf{K}_1 + E_0\mathbf{K}_2^0$ and $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$ are definite positive matrices. In addition, according to the expression of the complex modulus in Eq. (3) and to the previuous notations, the function $\hat{h}(\omega)$ is given by:

$$\hat{h}(\omega) = \frac{\tau^\alpha}{(\mathrm{i}\omega)^{1-\alpha}[1 + (\mathrm{i}\omega\tau)^\alpha]}\,. \tag{5}$$

3 DIFFUSIVE FORMULATION

From the previous Eqs. (4) and (5), we obtain the following system of equations in time-domain:

$$\mathbf{M}\,\ddot{\mathbf{u}} + h(t) \star \mathbf{K}_v\,\dot{\mathbf{u}} + \mathbf{K}_e\,\mathbf{u} = \mathbf{f}(t) \tag{6}$$

where an over-dot indicates a time-derivative, the symbol $\star$ represents a convolution product, and the function $h(t)$, for $t > 0$, is given by:

$$h(t) = \int_0^\infty \mu(\xi)\, e^{-\xi\, t}\,\mathrm{d}\xi \tag{7}$$

with

$$\mu(\xi) := \frac{\sin(\alpha\pi)}{\pi}\frac{\tau^\alpha}{\xi^{1-\alpha}\left[1 + 2\,\cos(\alpha\pi)\,(\tau\xi)^\alpha + (\tau\xi)^{2\alpha}\right]}\,. \tag{8}$$

Following e.g. [5, 6, 11, 12], the function h is applied on the velocity field $\mathbf{v} := \dot{\mathbf{u}}$ as input, and will be realized equivalently by a standard diffusive representation of the form:

$$\partial_t\boldsymbol{\varphi}(\xi, t) = -\xi\,\boldsymbol{\varphi}(\xi, t) + \mathbf{v}(t),\ \boldsymbol{\varphi}(\xi, 0) = \mathbf{0}\,, \tag{9}$$

observed through the continuous superposition:

$$(h \star \mathbf{v})(t) = \int_0^\infty \mu(\xi)\,\boldsymbol{\varphi}(\xi, t)\,\mathrm{d}\xi\,. \tag{10}$$

It can be noted that Eq. (9) corresponds to a familly of first order differential equations indexed by ξ. The previous diffusive representation, which is *exact*, can be approximated by stable numerical schemes using standard *interpolation*, i.e.

$$\int_0^\infty \mu(\xi)\,\boldsymbol{\varphi}(\xi, t)\,\mathrm{d}\xi \approx \sum_{k=1}^{K} \mu_k\,\boldsymbol{\varphi}(\xi_k), \tag{11}$$

where K is the number of approximation nodes, ξ_k a sequence of angular frequencies in the frequency range of interest, and μ_k the corresponding *interpolated* or *optimized* weights. It is important to note that this finite-dimensional representation is only approximate and the quality of the approximation depends on the choice of these three parameters. More details on the optimization procedure of diffusive models have been presented first in [13], and fully detailed on a series of fractional systems in e.g. [14].

4 TIME-INTEGRATION SCHEME

Using the previous diffusive representation, a predictor-corrector algorithm based on the Newmark integration scheme is proposed for the computation of the dynamical system given by Eq. (6). This algorithm is detailed below:

1. Initialization
   $\mathbf{S} = \mathbf{M} + \beta\,\Delta t^2\,\mathbf{K}_e$
   $\mathbf{u}(0) = \mathbf{u}^0$, $\mathbf{v}(0) = \mathbf{v}^0$
   $\boldsymbol{\varphi}_k(0) = \mathbf{0}$ for $1 \leq k \leq K$
   $\mathbf{a}^0 = \mathbf{M}^{-1}(\mathbf{f}^0 - \mathbf{K}_e\,\mathbf{u}^0)$

2. Enter time step loop
   a) Prediction
   $\mathbf{u}_{\text{pr}}^{n+1} = \mathbf{u}^n + \Delta t\,\mathbf{v}^n + (0.5 - \beta)\,\Delta t^2\,\mathbf{a}^n$
   $\mathbf{v}_{\text{pr}}^{n+1} = \mathbf{v}^n + (1 - \gamma)\,\Delta t\,\mathbf{a}^n$

   b) Evaluation of $\boldsymbol{\varphi}_k^{n+1}$ for $1 \leq k \leq K$
   $$\boldsymbol{\varphi}_k^{n+1} = \exp(-\xi_k\,\Delta t)\,\boldsymbol{\varphi}_k^n + \frac{1 - \exp(-\xi_k\,\Delta t)}{\xi_k}\,\mathbf{v}_{\text{pr}}^{n+1}$$
   c) Evaluation of $\mathbf{a}^{n+1}$
   $$\mathbf{a}^{n+1} = \mathbf{S}^{-1}(\mathbf{f}^{n+1} - \mathbf{K}_e\mathbf{u}_{\text{pr}}^{n+1} - \mathbf{K}_v\sum_{k=1}^{K}\mu_k\,\boldsymbol{\varphi}_k^{n+1})$$
   d) Correction
   $\mathbf{u}^{n+1} = \mathbf{u}_{\text{pr}}^{n+1} + \beta\,\Delta t^2\,\mathbf{a}^{n+1}$
   $\mathbf{v}^{n+1} = \mathbf{v}_{\text{pr}}^{n+1} + \gamma\,\Delta t\,\mathbf{a}^{n+1}$

3. Update time step and return to step 2

Some remarks can be made on this algorithm:

- In the Newmark algorithm we use $\beta = 1/4$ and $\gamma = 1/2$ corresponding to the *average acceleration method* which is unconditionally stable and second order accurate for *non-dissipative* linear systems;
- The prediction velocity vector $\mathbf{v}_{\text{pr}}$ is frozen as input of the diffusive sub-scheme b).
- Only the diffusive components $\boldsymbol{\varphi}_k^n$ for $1 \leq k \leq K$ at time step $n\Delta t$ are stored.

5 SINGLE DEGREE-OF-FREEDOM MODEL

We consider the single-degree-of-freedom dynamical model obtained from Eq. (4):

$$\left[K_e + \frac{(\mathrm{i}\omega\tau)^\alpha}{1 + (\mathrm{i}\omega\tau)^\alpha}K_v - \omega^2 M\right]\,\hat{u} = \hat{f}\,, \tag{12}$$

In the Laplace domain, this equation writes:

$$[M\tau^\alpha s^{2+\alpha} + Ms^2 + (K_e + K_v)\tau^\alpha s^\alpha + K_e]\,U(s) = [1 + \tau^\alpha s^\alpha]\,F(s)\,, \tag{13}$$

and a time-domain version is given by:

$$\left[M\tau^{\alpha}\,(D_t)^{2+\alpha} + M\,(D_t)^2 + (K_e + K_v)\tau^{\alpha}\,(D_t)^{\alpha} + K_e\right]\,u(t) = \left[1 + \tau^{\alpha}\,(D_t)^{\alpha}\right]\,f(t)\,. \quad (14)$$

where $(D_t)^{\beta}$ represents the time derivative of order β (integer or fractional).

In order to solve this equation analytically, let us introduce the following notations:

- $0 < \alpha = \frac{p}{q} < 1$, so $p < q$
- $\gamma = \frac{1}{q}$, so $\alpha = p\,\gamma$ and $1 = q\,\gamma$

With this notation at hand, we can rewrite the problem as:

$$[M\tau^{\alpha}\,(D_t^{\gamma})^{2q+p} + M\,(D_t^{\gamma})^{2q} + (K_e + K_v)\tau^{\alpha}\,(D_t^{\gamma})^{p} + K_e]\,u(t) = [1 + \tau^{\alpha}\,(D_t^{\gamma})^{p}]\,f(t)\,. \quad (15)$$

The trick consists in looking for a solution expressed by means of Mittag-Leffler functions (see [5]), which can also be expanded as fractional power series (by definition). This theoretical result legitimate the following Ansatz:

$$u(t) := \sum_{n=0}^{\infty} u_n\,\frac{t^{n\gamma}}{\Gamma(1+n\gamma)} \quad (16)$$

with initial conditions displacement $u(0) = u_0$ and velocity $v(0) = \dot{u}(0) = v_0$.

Reintroducing this Ansatz into equation (14) gives the following recursion on the coefficients:

$$M\tau^{\alpha}\,u_{n+2q+p} + M\,u_{n+2q} + (K_e + K_v)\tau^{\alpha}\,u_{n+p} + K_e\,u_n = 0\,, \quad \forall n \in \mathbb{N}\,, \quad (17)$$

which is easy to solve by induction, starting from known u_0 and $u_q = v_0$ and other low order coefficients equal to 0. This analytical solution will be helpful to test our numerical solutions.

6 CONLUSION AND PERSPECTIVES

We have proposed in this contribution a general approach for time-domain simulations of structural dynamic problems with viscoelastic materials described by fractional derivative constitutive equations. This approach combines a classical Newmark time-integration method with a diffusive representation used in fractional calculus. The proposed methodology is applicable to complex mechanical systems thanks to an appropriate finite element space discretization. We have also proposed in this paper a closed-form solution for a single degree-of-freedom system with fractional Zener model. This solution will be used to validate the numerical implementation of our algorithm. Fully 3D simulations will be presented at the conference and used to analyse the efficiency of our approach.

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