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Coupling Harmonic Functions-Finite Elements for Solving the Stream Function-Vorticity Stokes Problem

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We consider the bidimensional Stokes problem for incompressible fluids in stream function-vorticity form. The classical finite element method of degree one usually used does not allow the vorticity on the boundary of the domain to be computed satisfactorily when the meshes are unstructured and does not converge optimally. To better approach the vorticity along the boundary, we propose that harmonic functions obtained by integral representation be used. Numerical results are very satisfactory, and we prove that this new numerical scheme leads to an optimal convergence rate of order 1 for the natural norm of the vorticity and, under higher regularity assumptions, from 3/2 to 2 for the quadratic norm of the vorticity. © 2004 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 20: 765–788, 2004

I. INTRODUCTION
A. Motivations

Let Ω be a bounded connected open set in $\mathbb{R}^2$. In order that error estimates may be obtained for our numerical schemes, we assume that Ω is polygonal. We denote by $\Gamma = \partial\Omega$ the boundary of Ω. Modeling of the equilibrium of an incompressible and viscous fluid leads to the Navier-Stokes problem [1]. If we neglect convection terms, when the viscosity is sufficiently dominant or the velocity of the fluid sufficiently small, we obtain the stationary Stokes problem which is (in primitive variables i.e., velocity **u** and pressure p):

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$$
\begin{aligned}
-\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\
\operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega, \\
\mathbf{u} &= 0 && \text{on } \Gamma,
\end{aligned}
$$

where ν is the kinematic viscosity and $\mathbf{f}$ a field of given external forces.

If Ω is supposed to be simply connected and the velocity is divergence free, this two-dimensional problem is often rewritten in terms of stream function and vorticity variables. Velocity is the curl of some stream function and vorticity is the curl of the velocity. A usual way of discretizing this new problem is to choose a finite element method and to use polynomial approximations of degree one for each variable. The stream function and the vorticity are though assumed to be in $H^1(\Omega)$, but it is well known that first this problem is not mathematically well-posed (see for example, Girault-Raviart [2]), and that second the vorticity is not satisfactorily approximated on the boundary of the domain when the meshes are unstructured (see e.g., Salmon [3]). Nevertheless, it can be shown that, if Ω is convex, the scheme is convergent (Scholz [2, 4]). The convergence for the quadratic norm of the vorticity is of order $\sqrt{h_\tau}$, where h_τ is the maximum diameter of the triangles in the mesh, and of order $h_\tau^{1-\varepsilon}$ for the H^1-norm of the stream function (ε is an arbitrary strictly positive real). Notice that in the case of structured meshes, the usual scheme gives optimal numerical results (see e.g., [3, 5]) and moreover, superconvergence can be observed [2].

However, as has been said, a major problem of the stream function-vorticity formulation arises in trying to obtain correct boundary values for the vorticity. Many articles deal with this aspect and propose new formulas for inclusion in the numerical scheme (see e.g., Napolitano et al. [6] and references therein). Another idea due to Amara and Bernardi [7] is to stabilize the usual formulation by adding jumps at interfaces of the triangles and thus improve the convergence.

However, from our point of view, the mathematically well-posed formulation of the problem should lead to a good numerical scheme. So, we work with the well-posed stream function-vorticity variational formulation which was introduced by Ruas [8] and Bernardi-Girault-Maday [9]. This formulation consists of looking for the vorticity in the space $M(\Omega) = \{\varphi \in L^2(\Omega), \Delta\varphi \in H^{-1}(\Omega)\}$, containing less regular functions than $H^1(\Omega)$. We propose in the sequel to study a natural discretization of this space, which leads to a numerical scheme using harmonic functions to compute the vorticity on the boundary. Let us observe that the idea of using harmonic functions was first introduced by Glowinski-Pironneau [10] and also used by Quartapelle and Valz-Gris [11]. But these authors do not present any theoretical convergence results.

In the sequel, we prove that, in a polygonal domain, our numerical scheme is convergent of order at least $\mathbb{O}(h_\tau)$ for the natural norm of the vorticity. Some of the authors have previously proposed a method based on discrete harmonic functions computed on refined meshes (Dubois et al. [12]), which is quite time-consuming. Here, the novelty is to use integral representation for computing the real harmonic functions. Then, we prove both theoretically and numerically that, when Ω is moreover assumed to be convex, if the vorticity is assumed to belong to $H^2(\Omega)$ [respectively to $H^{5/2}(\Omega)$] and if the stream function belongs to $H^2(\Omega)$, their quadratic norms converge in fact as $\mathbb{O}(h_\tau^{3/2})$ [respectively $\mathbb{O}(h_\tau^2)$]. The last part of the article is devoted to numerical experiments on different geometries and unstructured meshes, which are in agreement with theoretical results. In the whole article, Ω will be assumed to be at least connected and simply connected.

B. Notation

We shall consider the following spaces (see for example, Adams [13]). $\mathcal{D}(\Omega)$ denotes the space of all indefinitely differentiable functions from Ω to $\mathbb{R}$ with compact support and $L^2(\Omega)$ the space of all classes of square integrable functions. For any integer $m \geq 0$ and any real p such that $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ denotes the space of all functions $v \in L^p(\Omega)$, whose partial derivatives in the sense of distributions, $\partial^\alpha v/\partial x_i$, $|\alpha| \leq m$, belong to $L^p(\Omega)$. We define as usually $H^1(\Omega) = W^{1,2}(\Omega)$ and $H^2(\Omega) = W^{2,2}(\Omega)$. We denote by $\|\cdot\|_{m,p,\Omega}$ (respectively, $|\cdot|_{m,p,\Omega}$) the norms (respectively, the semi-norms) in the Sobolev spaces $W^{m,p}(\Omega)$. We make the usual modification for $p = \infty$, and we drop the index 2 when $p = 2$. The space $H_0^1(\Omega)$ [respectively, $H_0^2(\Omega)$] is the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{1,\Omega}$ (respectively, $\|\cdot\|_{2,\Omega}$). Then, $(\cdot, \cdot)$ denotes the standard inner product in $L^2(\Omega)$ and $\langle \cdot, \cdot \rangle_{-1,1}$ the duality product between $H_0^1(\Omega)$ and its topological dual space $H^{-1}(\Omega)$. Finally, γ denotes the trace operator from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$, or from $H^2(\Omega)$ onto $H^{3/2}(\Gamma)$ (see Lions-Magenes [14]).

II. THE STREAM FUNCTION-VORTICITY FORM OF THE STOKES PROBLEM

Let $\mathbf{f}$ in $(L^2(\Omega))^2$ be a field of given forces, we define curl $\mathbf{f}$ as $(\partial f_1/\partial x_2) - (\partial f_2/\partial x_1)$. The steady-state Stokes problem consists of finding a stream function ψ and a vorticity field ω solutions of

$$\omega + \Delta\psi = 0 \qquad \text{in } \Omega \tag{2.1}$$

$$-\Delta\omega = \text{curl } \mathbf{f} \qquad \text{in } \Omega \tag{2.2}$$

$$\psi = 0 \qquad \text{on } \Gamma \tag{2.3}$$

$$\frac{\partial\psi}{\partial\mathbf{n}} = 0 \qquad \text{on } \Gamma. \tag{2.4}$$

Indeed, it consists of finding a velocity field $\mathbf{u}$ that is divergence free and can be written with the help of a stream function ψ. We have

$$\mathbf{u} = \mathbf{curl}\ \psi = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1} \right)^T \qquad \text{in } \Omega.$$

Equation (2.1) means that the vorticity is the curl of the velocity. Equation (2.2) is the equilibrium equation for a viscous fluid, with kinematic viscosity equal to 1, and where convection terms are neglected. Boundary conditions (2.3) and (2.4) are consequences of $\mathbf{u} = 0$ on Γ. For more details about this problem, we refer to [2].

It is natural to discretize the problem (2.1)–(2.4) with a piecewise linear and continuous finite element method. As Ω is assumed to be polygonal, we can exactly cover it with a mesh $\mathcal{T}$ composed of triangular finite elements. The mesh $\mathcal{T}$ is assumed to be regular (in the sense defined in Ciarlet [15]). The set H_τ^1 denotes the space of continuous functions defined on $\bar{\Omega}$, which are polynomials of degree 1 in each triangle of $\mathcal{T}$ and $H_{0,\tau}^1 = H_\tau^1 \cap H_0^1(\Omega)$:

$$H^1_\tau = \{\varphi \in \mathscr{C}^0(\bar{\Omega}),\ \forall K \in \mathscr{E}_\tau,\ \varphi_{|K} \in \mathbb{P}^1(K)\}, \tag{2.5}$$

where $\mathscr{E}_\tau$ is the set of triangles in $\mathscr{T}$ and $\mathbb{P}^1$ the space of all polynomials of total degree less or equal to 1. The discretization of the problem (2.1)–(2.4) consists of finding ω_τ and ψ_τ such that:

$$\psi_\tau \in H^1_{0,\tau} \text{ (boundary condition (2.3) is then satisfied)} \tag{2.6}$$

$$\omega_\tau \in H^1_\tau \tag{2.7}$$

in the following way. We first multiply (2.1) by a scalar function $\varphi \in H^1_\tau$ and integrate by parts $(\Delta\psi_\tau, \varphi)$, we obtain, taking the boundary condition (2.4) into account:

$$(\omega_\tau, \varphi) - (\nabla\psi_\tau, \nabla\varphi) = 0 \qquad \forall\varphi \in H^1_\tau. \tag{2.8}$$

Then we multiply (2.2) by a test function ξ in $H^1_{0,\tau}$, integrate by parts $-(\Delta\omega_\tau, \xi)$ and (curl $\mathbf{f}$, ξ) and, as ξ vanishes on the boundary, we obtain:

$$(\nabla\omega_\tau, \nabla\xi) = (\mathbf{f}, \mathbf{curl}\ \xi) \qquad \forall\xi \in H^1_{0,\tau}. \tag{2.9}$$

This formulation (2.6)–(2.9) has been studied extensively (see Ciarlet-Raviart [16], Glowinski-Pironneau [10] and [2] among others) and presents some difficulties.

First, the continuous formulation associated to it is not well-posed for any $\mathbf{f}$ in $(L^2(\Omega))^2$. Indeed, the Stokes problem can be seen as a biharmonic problem for the stream function: $\psi \in H^2_0(\Omega)$ and $\Delta^2\psi = $ curl $\mathbf{f}$. In a variational form, this problem can be rewritten as follows:

$$(\Delta\psi, \Delta\mu) = (\mathbf{f}, \mathbf{curl}\ \mu) \qquad \forall\mu \in H^2_0(\Omega). \tag{2.10}$$

Problem (2.10) is well posed in $H^2_0(\Omega)$ (see [15]) and as $\omega = -\Delta\psi$, the vorticity cannot be more regular than square integrable. Second, error estimates derived for the scheme (2.6–2.9) are not optimal. When Ω is assumed to be convex, the bound is in $h_\tau^{1/2}$ for the quadratic norm of the vorticity, where h_τ is the maximum diameter of the elements of the triangulation [2, 4] (it is numerically illustrated in Figure 3 at the end of this article).

So, a different weak formulation was proposed by Ruas [8] and Bernardi et al. [9] who introduced the space $M(\Omega) = \{\varphi \in L^2(\Omega),\ \Delta\varphi \in H^{-1}(\Omega)\}$, which is a Hilbert space for the norm $\|\varphi\|_M = \sqrt{\|\varphi\|^2_{0,\Omega} + \|\Delta\varphi\|^2_{-1,\Omega}}$. It consists of finding (ω, ψ) in $M(\Omega) \times H^1_0(\Omega)$ with the following method. We test the first equation (2.1) with a function $\varphi \in M(\Omega)$ and the second one (2.2) with a function $\xi \in H^1_0(\Omega)$. It is then necessary to integrate twice by parts the term $\langle\Delta\psi, \varphi\rangle_{-1,1}$ in order to include boundary conditions (2.3) and (2.4). We get

$$\begin{cases} (\omega, \varphi) + \langle\Delta\varphi, \psi\rangle_{-1,1} = 0 & \forall\varphi \in M(\Omega) \\ \langle-\Delta\omega, \xi\rangle_{-1,1} = (\mathbf{f}, \mathbf{curl}\ \xi) & \forall\xi \in H^1_0(\Omega). \end{cases} \tag{2.11}$$

This formulation (2.11) leads to a well-posed problem [9].

Proposition 2.1. *The Sobolev space $H^1(\Omega)$ is contained in $M(\Omega)$ with continuous imbedding. We have, $\|\varphi\|_M \leq \|\varphi\|_{1,\Omega}$ for all function φ in $H^1(\Omega)$. Moreover if $\varphi \in M(\Omega) \cap H_0^1(\Omega)$, its M-norm is equal to its H_0^1-norm.*

Proof. First we notice that the laplacian of a function in $H^1(\Omega)$ belongs to $H^{-1}(\Omega)$ and $H^1(\Omega)$ is a subspace of $M(\Omega)$. Then to prove the continuity of the imbedding, we need the definition of the H^{-1}-norm. For all $\varphi \in H^1(\Omega)$, we have

$$\|\Delta\varphi\|_{-1,\Omega} = \sup_{\chi \in H_0^1(\Omega)} \frac{\langle \Delta\varphi, \chi \rangle_{-1,1}}{|\chi|_{1,\Omega}} = \sup_{\chi \in H_0^1(\Omega)} \frac{-(\nabla\varphi, \nabla\chi)}{\|\nabla\chi\|_{0,\Omega}} \quad \text{because } \varphi \text{ is in } H^1(\Omega) \text{ and } \chi_{|\Gamma} = 0,$$

$$\leq \|\nabla\varphi\|_{0,\Omega} \text{ (using the Cauchy-Schwarz inequality).} \quad (2.12)$$

An immediate consequence of this inequality is $\forall \varphi \in H^1(\Omega)$, $\|\varphi\|_M \leq \|\varphi\|_{1,\Omega}$.

Second, if $\varphi \in M(\Omega) \cap H_0^1(\Omega)$, it suffices to take $\chi = -\varphi$ in the inequality (2.12) to obtain $\|\Delta\varphi\|_{-1,\Omega} = \|\nabla\varphi\|_{0,\Omega}$ and then $\|\varphi\|_M = \|\varphi\|_{1,\Omega}$. ∎

Let us now introduce the kernel $\mathscr{H}(\Omega)$ of the bilinear form $\langle \Delta\cdot, \cdot \rangle_{-1,1}$:

$$\mathscr{H}(\Omega) = \{\varphi \in M(\Omega), \langle \Delta\varphi, \xi \rangle_{-1,1} = 0, \quad \forall \xi \in H_0^1(\Omega)\}.$$

Proposition 2.2. *Characterization of the space $\mathscr{H}(\Omega)$: $\mathscr{H}(\Omega) = \{\varphi \in L^2(\Omega), \Delta\varphi = 0 \text{ in } \mathscr{D}'(\Omega)\}$. There also exists a trace operator (still denoted by γ) from $\mathscr{H}(\Omega)$ on $H^{-1/2}(\Gamma)$.*

The proof is quite classical and completely developed in [12].

So, when we restrict the first equation of (2.11) to functions in $\mathscr{H}(\Omega)$, this new problem is well-posed according to Lax-Milgram's lemma [17]. Indeed

Proposition 2.3. *The L^2-scalar product: $\mathscr{H}(\Omega) \times \mathscr{H}(\Omega) \ni (\omega, \varphi) \mapsto \int_\Omega \omega \cdot \varphi \, dx \in \mathbb{R}$ is elliptic on $\mathscr{H}(\Omega)$.*

Proof. It is obvious that for ω in $\mathscr{H}(\Omega)$, $\|\omega\|_M^2 = \|\omega\|_{0,\Omega}^2$. ∎

Proposition 2.4. *The space $M(\Omega)$ can be decomposed as follows: $M(\Omega) = H_0^1(\Omega) \oplus \mathscr{H}(\Omega)$.*

Proof. The proof can also be found in [18]. We split $\varphi \in M(\Omega)$ into two parts: $\varphi = \varphi^0 + \varphi^\Delta$. On the one hand, since $\Delta\varphi \in H^{-1}(\Omega)$, the component φ^0 is uniquely defined in $H_0^1(\Omega)$ by the Dirichlet problem:

$$\begin{cases} \Delta\varphi^0 = \Delta\varphi & \text{in } \Omega \\ \gamma\varphi^0 = 0 & \text{on } \Gamma. \end{cases}$$

On the other hand, we define φ^Δ by $\varphi^\Delta = \varphi - \varphi^0$. Then, it verifies $\Delta\varphi^\Delta = 0$ in Ω, so φ^Δ belongs to $\mathscr{H}(\Omega)$. ∎

Moreover, we can observe that $\gamma\varphi^\Delta = \gamma\varphi$ on Γ, with $\gamma\varphi$ well defined in $H^{-1/2}(\Gamma)$ (see Proposition 2.2).

We now rewrite the well-posed formulation of the Stokes problem (2.11) taking into account the decomposition given in Proposition 2.4.

Find $\omega = \omega^0 + \omega^\Delta \in H_0^1(\Omega) \oplus \mathscr{H}(\Omega)$ and $\psi \in H_0^1(\Omega)$ such that

$$
\begin{cases}
(a) & (\omega^0 + \omega^\Delta, \chi) + \langle \Delta\chi, \psi \rangle_{-1,1} = 0 \qquad\qquad \forall \chi \in H_0^1(\Omega). \\[2ex]
(b) & (\omega^0 + \omega^\Delta, \varphi) + \underbrace{\langle \Delta\varphi, \psi \rangle_{-1,1}}_{=0} = 0 \qquad\qquad \forall \varphi \in \mathcal{H}(\Omega). \\[2ex]
(c) & \langle -\Delta\omega^0, \xi \rangle_{-1,1} - \underbrace{\langle \Delta\omega^\Delta, \xi \rangle_{-1,1}}_{=0} = (\mathbf{f}, \mathbf{curl}\ \xi) \qquad \forall \xi \in H_0^1(\Omega).
\end{cases}
$$

Then it is obvious that the previous problem can be solved in the following way:

$$
\begin{cases}
(c) & \text{find } \omega^0 \in H_0^1(\Omega) \quad \text{such that} \\
& \langle -\Delta\omega^0, \xi \rangle_{-1,1} = (\nabla\omega^0, \nabla\xi) = (\mathbf{f}, \mathbf{curl}\ \xi) \qquad \forall \xi \in H_0^1(\Omega). \\
(b) & \text{find } \omega^\Delta \in \mathcal{H}(\Omega) \quad \text{such that} \\
& (\omega^\Delta, \varphi) = -(\omega^0, \varphi) \qquad\qquad\qquad \forall \varphi \in \mathcal{H}(\Omega). \\
(a) & \text{find } \psi \in H_0^1(\Omega) \quad \text{such that} \\
& \langle -\Delta\chi, \psi \rangle_{-1,1} = (\nabla\psi, \nabla\chi) = (\omega^0 + \omega^\Delta, \chi) \qquad \forall \chi \in H_0^1(\Omega).
\end{cases}
\tag{2.13}
$$

A previous work of the authors used with success homothetic mesh refinements to solve the second step of the above problem (Dubois et al. [3, 12]). The following section gives an alternative way to solve this step by a direct use of harmonic functions, based on an integral representation.

III. DISCRETIZATION OF THE STOKES PROBLEM USING HARMONIC FUNCTIONS
A. Discretization of the Space $\mathcal{H}(\Omega)$

We recall that Ω being polygonal allows to entirely cover it with a mesh $\mathcal{T}$. We denote by $\mathcal{E}_\tau$ the set of triangles in $\mathcal{T}$.

Definition. *Family $\mathcal{U}_\sigma$ of regular meshes* [15]. *We assume that $\mathcal{T}$ belongs to the set $\mathcal{U}_\sigma$ of triangulations satisfying*:

$$
\exists \sigma > 0, \qquad \forall K \in \mathcal{E}_\tau, \qquad \frac{h_K}{\rho_K} \le \sigma,
$$

where $h_K = \operatorname{diam} K$ and ρ_K is the diameter of the circle inscribed in K.

We introduce the trace of mesh $\mathcal{T}$ on the boundary Γ. It is a set $\mathcal{A}(\mathcal{T}, \Gamma)$ of edges of triangles of the mesh which are contained in Γ. If $N_a(\mathcal{T}, \Gamma)$ is the number of these edges, we denote them by Γ_i, $1 \le i \le N_a(\mathcal{T}, \Gamma)$. As Γ is closed, $N_a(\mathcal{T}, \Gamma)$ is also equal to the number of vertices of the mesh $\mathcal{T}$ on the boundary Γ. Then we define the vector space $\mathcal{E}_\tau$ generated by the characteristic functions of the edges $\Gamma_i \in \mathcal{A}(\mathcal{T}, \Gamma)$ of Γ:

$$
\mathcal{E}_\tau = \operatorname{Span}\{ q_i = \mathbb{1}_{\Gamma_i}, \Gamma_i \in \mathcal{A}(\mathcal{T}, \Gamma) \},
\tag{3.1}
$$

where $\mathbb{1}_{\Gamma_i}$ is the characteristic function defined from Γ to $\mathbb{R}$ by

$$\mathbb{1}_{\Gamma_i}(x) = \begin{cases} 1 & \text{if } x \in \Gamma_i \\ 0 & \text{if } x \notin \Gamma_i. \end{cases}$$

The dimension of $\mathscr{C}_\tau$ is exactly equal to the number $N_a(\mathscr{T}, \Gamma)$.

Remark 3.1. We can define the vector space $\mathscr{L}_\tau$ of continuous polynomial functions of degree one on the edges $\Gamma_i \in \mathscr{A}(\mathscr{T}, \Gamma)$ of Γ.

$$\mathscr{L}_\tau = \{q \in \mathscr{C}^0(\Gamma), \qquad q_{|\Gamma_i} \in \mathbb{P}^1(\Gamma_i), \qquad \Gamma_i \in \mathscr{A}(\mathscr{T}, \Gamma)\}. \tag{3.2}$$

Then we denote by $\mathscr{S}$ the simple layer operator applied to functions of $\mathscr{C}_\tau$.

Definition. *Simple layer potential.*

$$\mathscr{S} : \mathscr{C}_\tau \ni q_i \mapsto \varphi_i \in \mathscr{H}_{\tau,I}$$

where $\varphi_i(x) = \mathscr{S}q_i(x) = \int_\Gamma G(x, y)q_i(y)\, d\gamma_y \; \forall x \in \bar{\Omega}$, and $G(x, y) = (1/2\pi)\log|x - y|$ is the Green kernel.

We denote by $\mathscr{H}_{\tau,I}$ the discrete space spanned by functions $\varphi_i = \mathscr{S}q_i$, for all $q_i \in \mathscr{C}_\tau$. The space $\mathscr{H}_{\tau,I}$ is finite dimensional and, clearly, its dimension is equal to the dimension of $\mathscr{C}_\tau$. By construction, functions of $\mathscr{H}_{\tau,I}$ are harmonic. We shall denote by $S = \gamma\mathscr{S}$ the operator $\mathscr{S}$ on the boundary. We introduce $\gamma\varphi_i(x) = Sq_i(x) = \int_{\Gamma_i} G(x, y)\, d\gamma_y$ for all x on the boundary Γ and $q_i = \mathbb{1}_{\Gamma_i}$ in $\mathscr{C}_\tau$.

Remark 3.2. Our discretization will be conforming as $\mathscr{H}_{\tau,I}$ is contained in $\mathscr{H}(\Omega) \subset M(\Omega)$ and that functions in $\mathscr{H}_{\tau,I}$ have a trace on the boundary in $H^{-1/2}(\Gamma)$ (see Proposition 2.2).

Theorem 3.3. *The operator S is an isomorphism from the Sobolev space $H^s(\Gamma)$ onto $H^{s+1}(\Gamma)$ for all real numbers (see e.g., Nédélec [19], Dautray-Lions [20]).*

Definition. *We define the following subspace of $H^{-3/2}(\Gamma)$:*

$$_\circ H^{-3/2}(\Gamma) = \{\mu \in H^{-3/2}(\Gamma), \langle \mu, 1 \rangle_{-3/2,3/2} = 0\}.$$

Proposition 3.4. *For all $q \in {_\circ}H^{-3/2}(\Gamma)$, we can define the harmonic function $\mathscr{S}q$ of $L^2(\Omega)$:*

$$\mathscr{S}q(x) = \int_\Gamma G(x, y)q(y)\, d\gamma_y \qquad \forall x \in \bar{\Omega}.$$

Then, there exists a constant $C > 0$ such that, for all $q \in {_\circ}H^{-3/2}(\Gamma)$:

$$\|\mathscr{S}q\|_{0,\Omega} \le C\|q\|_{-3/2,\Gamma}. \tag{3.3}$$

Proof. Let q be in $_\circ H^{-3/2}(\Gamma)$ and Ω' be the interior of the complementary of Ω. Notice that $\mathcal{S}q$ is, by definition, the solution of the following problem [19–21]:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \cup \Omega' \\ [u] = 0 & \text{on } \Gamma \\ \left[\dfrac{\partial u}{\partial n}\right] = q & \text{on } \Gamma, \end{cases}$$

where $[\chi] = \chi_{|\Omega} - \chi_{|\Omega'}$ is the jump of χ across Γ. Vector n is the outer normal on Γ. Notice that the condition $\langle q, 1 \rangle_{-3/2,3/2} = 0$ is necessary to obtain the existence of a solution to the previous problem.

Following Nédélec [19], we know that for all $q \in H^{-1/2}(\Gamma)$, whose mean value is zero, $\mathcal{S}q$ belongs to $H^1(\Omega)$ and then to $L^2(\Omega)$. So, we can consider the L^2-norm of $\mathcal{S}q$ when q is sufficiently regular. Then, if (3.3) is proven for regular functions, a classical density argument leads to the expected result.

Let us now prove this proposition for regular functions. We shall use methods similar to those used in the Aubin-Nitsche argument (see Aubin [22] and Nitsche [23]). Indeed,

$$\|\mathcal{S}q\|_{0,\Omega} = \sup_{g \in L^2(\Omega)} \frac{(\mathcal{S}q, g)_{0,\Omega}}{\|g\|_{0,\Omega}}.$$

We extend a function $g \in L^2(\Omega)$ by zero outside Ω and we define $\tilde{g}$:

$$\tilde{g} = \begin{cases} g & \text{in } \Omega \\ 0 & \text{in } \Omega'. \end{cases}$$

Notice that, as g belongs to $L^2(\Omega)$, $\tilde{g}$ belongs to $L^2(\mathbb{R}^2)$. We consider a function φ such that

$$-\Delta\varphi = \tilde{g} \qquad \text{in } \mathbb{R}^2.$$

$$\varphi(x) = \mathbb{O}(\log|x|) \qquad \text{when } |x| \text{ becomes large.}$$

By local regularity of the Laplacian operator, the solution $\varphi_{|\Omega}$ belongs to $H^2(\Omega)$ if Ω is bounded (Nédélec [24]) and verifies

$$\|\varphi_{|\Omega}\|_{2,\Omega} \leq C\|g\|_{0,\Omega}. \tag{3.4}$$

Because Ω is bounded, let B_R be a ball, of radius R sufficiently large to contain Ω (we denote by S_R the boundary of B_R). We have

$$(\mathcal{S}q, g)_{0,\Omega} = (\mathcal{S}q, \tilde{g})_{0,\mathbb{R}^2} = \lim_{R \to \infty} \int_{B_R} -\Delta\varphi \cdot \mathcal{S}q \, dx.$$

Then, as $\Delta \mathscr{S}q = 0$:

$$\int_{B_R} -\Delta\varphi \cdot \mathscr{S}q \, dx = \int_{B_R} (\Delta\mathscr{S}q \cdot \varphi - \Delta\varphi \cdot \mathscr{S}q) \, dx$$

$$= \int_{\Omega} (\Delta\mathscr{S}q \cdot \varphi - \Delta\varphi \cdot \mathscr{S}q) \, dx + \int_{B_R\backslash\Omega} (\Delta\mathscr{S}q \cdot \varphi - \Delta\varphi \cdot \mathscr{S}q) \, dx.$$

Notice that, on the one hand, $\varphi_{|B_R}$ belongs to $H^2(B_R)$ [respectively, $\varphi_{|\Omega}$ to $H^2(\Omega)$] as B_R is bounded and so, $\gamma\varphi \in H^{3/2}(\mathscr{S}_R)$ [respectively $H^{3/2}(\Gamma)$] and $\partial\varphi/\partial n \in H^{1/2}(S_R)$ [respectively $H^{1/2}(\Gamma)$]. On the other hand, functions in $L^2(\Omega)$ whose Laplacian is also in $L^2(\Omega)$ have their normal trace in $H^{-3/2}(\Gamma)$ (see [20]). So, we obtain by definition:

$$\int_{\Omega} (\Delta\mathscr{S}q \cdot \varphi - \Delta\varphi \cdot \mathscr{S}q) \, dx + \int_{B_R\backslash\Omega} (\Delta\mathscr{S}q \cdot \varphi - \Delta\varphi \cdot \mathscr{S}q) \, dx = \left\langle \left[\frac{\partial Sq}{\partial n}\right], [\gamma\varphi] \right\rangle_{-3/2,3/2}$$

$$+ \left\langle -\left[\frac{\partial\varphi}{\partial n}\right], [Sq] \right\rangle_{-1/2,1/2} + \int_{S_R} \left(\frac{\partial Sq}{\partial n} \cdot \varphi - \frac{\partial\varphi}{\partial n} \cdot Sq\right) d\gamma.$$

First, $[\gamma\varphi]$ and $[\partial\varphi/\partial n]$ are continuous across Γ and S_R since φ belongs to $H^2(B_R)$. Second, by construction: $[\partial Sq/\partial n] = q$, $[Sq] = 0$. Then, we obtain

$$\int_{B_R} -\Delta\varphi \cdot \mathscr{S}q \, dx = \langle q, \gamma\varphi\rangle_{-3/2,3/2} + \int_{S_R} \left(\frac{\partial Sq}{\partial n} \cdot \varphi - \frac{\partial\varphi}{\partial n} \cdot Sq\right) d\gamma.$$

As φ acts as $\log R$ and Sq as $1/R$ (because the mean value of q is null) when R becomes large (see [19]) we obtain

$$\left|\int_{S_R} \frac{\partial Sq}{\partial n} \cdot \varphi \, d\gamma\right| \leq \int_0^{2\pi} \frac{1}{R^2} \cdot \log R \cdot R \, d\theta \sim \frac{\log R}{R} \xrightarrow{R \to +\infty} 0,$$

and

$$\left|\int_{S_R} \frac{\partial\varphi}{\partial n} \cdot Sq \, d\gamma\right| \leq \int_0^{2\pi} \frac{1}{R} \cdot \frac{1}{R} \cdot R \, d\theta \sim \frac{1}{R} \xrightarrow{R \to +\infty} 0.$$

Finally, we deduce

$$(\mathscr{S}q, g)_{0,\Omega} = (\mathscr{S}q, \tilde{g})_{0,\mathbb{R}^2} = \langle q, \gamma\varphi\rangle_{-3/2,3/2} \leq \|q\|_{-3/2,\Gamma}\|\gamma\varphi\|_{3/2,\Gamma}$$

$$\leq C\|q\|_{-3/2,\Gamma}\|\varphi\|_{2,\Omega} \qquad \text{by continuity of the trace.}$$

Using relation (3.4), it leads to

$$\|\mathscr{S}q\|_{0,\Omega} = \sup_{g \in L^2(\Omega)} \frac{(\mathscr{S}q, g)}{\|g\|_{0,\Omega}} \le C \sup_{g \in L^2(\Omega)} \frac{\|q\|_{-3/2,\Gamma}\|g\|_{0,\Omega}}{\|g\|_{0,\Omega}} \le C\|q\|_{-3/2,\Gamma}.$$

∎

Definition. *For the discretization of $M(\Omega)$, we set*

$$H_\tau^{1,I} = H_{0,\tau}^1 \; \oplus \; \mathscr{H}_{\tau,I}. \tag{3.5}$$

The dimension of $H_\tau^{1,I}$ is the same as H_τ^1 [see (2.5)] but near the boundary, we use harmonic functions instead of piecewise linear continuous functions.

B. Discrete Formulation

We propose the following discrete variational formulation of the Stokes problem based on problem (2.13):

$$\psi_\tau \in H_{0,\tau}^1, \qquad \omega_\tau = \omega_\tau^0 + \omega_\tau^\Delta \in H_\tau^{1,I} = H_{0,\tau}^1 \; \oplus \; \mathscr{H}_{\tau,I} \tag{3.6}$$

$$(\nabla\omega_\tau^0, \nabla\xi) = (\mathbf{f}, \mathbf{curl}\,\xi) \qquad \forall \xi \in H_{0,\tau}^1 \tag{3.7}$$

$$(\omega_\tau^\Delta, \varphi) = -(\omega_\tau^0, \varphi) \qquad \forall \varphi \in \mathscr{H}_{\tau,I} \tag{3.8}$$

$$(\nabla\psi_\tau, \nabla\chi) = (\omega_\tau^0 + \omega_\tau^\Delta, \chi) \qquad \forall \chi \in H_{0,\tau}^1. \tag{3.9}$$

The above method is a conforming discretization of problem (2.13) since, as it was said in remark 3.2, $H_\tau^{1,I}$ is a subset of $M(\Omega)$. Our approach for studying problem (3.6)–(3.9) follows ideas of [10] and Ruas [25].

Proposition 3.5. *Existence and uniqueness of a solution to problem (3.6)–(3.9). If $\mathbf{f} \in (L^2(\Omega))^2$, problem (3.6)–(3.9) has a unique solution $(\psi_\tau, \omega_\tau) \in H_{0,\tau}^1 \times H_\tau^{1,I}$, which depends continuously on the datum $\mathbf{f}$. There exists a strictly positive constant C independent of the mesh such that*

$$\|\omega_\tau\|_M + \|\nabla\psi_\tau\|_{0,\Omega} \le C\|\mathbf{f}\|_{0,\Omega}. \tag{3.10}$$

Proof. As Ω is bounded, if C_p denotes the Poincaré constant, we have

$$\forall \mu \in H_0^1(\Omega), \qquad \|\mu\|_{0,\Omega} \le C_p|\mu|_{1,\Omega}.$$

Problem (3.7) is well posed according to Lax-Milgram's lemma, and Poincaré's inequality. There exists a unique $\omega_\tau^0 \in H_{0,\tau}^1$ satisfying [take $\xi = \omega_\tau^0$ in (3.7)]

$$\|\nabla\omega_\tau^0\|_{0,\Omega}^2 = \|\mathbf{curl}\,\omega_\tau^0\|_{0,\Omega}^2 \le \|\mathbf{f}\|_{0,\Omega}\|\mathbf{curl}\,\omega_\tau^0\|_{0,\Omega}.$$

As $\omega_\tau^0 \in H_0^1(\Omega)$, its M-norm is equal to its H^1-norm (see Proposition 2.1) and

$$\|\omega_\tau^0\|_M \leq \sqrt{1 + C_p^2}\,\|\mathbf{f}\|_{0,\Omega}. \tag{3.11}$$

We now study Equation (3.8). Function ω_τ^0 is given and problem (3.8) has a unique solution ω_τ^Δ, as the L^2-scalar product is M-elliptic on $\mathcal{H}_{\tau,I}$ (Proposition 2.3):

$$(\omega_\tau^\Delta, \varphi) = -(\omega_\tau^0, \varphi) \qquad \forall \varphi \in \mathcal{H}_{\tau,I}.$$

Taking $\varphi = \omega_\tau^\Delta$ and thanks to (3.11), function ω_τ^Δ verifies

$$\|\omega_\tau^\Delta\|_M = \|\omega_\tau^\Delta\|_{0,\Omega} \leq \|\omega_\tau^0\|_{0,\Omega} \leq \sqrt{1 + C_p^2}\,\|\mathbf{f}\|_{0,\Omega}. \tag{3.12}$$

Finally, Equation (3.9) is formally identical to (3.7), so there exists (Lax-Milgram's lemma and Poincaré's inequality) a unique $\psi_\tau \in H_{0,\tau}^1$ such that

$$(\nabla \psi_\tau, \nabla \chi) = (\omega_\tau^0, \chi) + (\omega_\tau^\Delta, \chi) \qquad \forall \chi \in H_{0,\tau}^1,$$

and then

$$\|\nabla \psi_\tau\|_{0,\Omega} \leq C_p(\|\omega_\tau^0\|_{0,\Omega} + \|\omega_\tau^\Delta\|_{0,\Omega}).$$

Using (3.11) and (3.12), we obtain

$$\|\nabla \psi_\tau\|_{0,\Omega} \leq C'\|\mathbf{f}\|_{0,\Omega}. \tag{3.13}$$

Combining (3.11), (3.12), and (3.13), we obtain the announced result (3.10). ∎

To obtain error estimates and thus convergence, we need a stability result (Proposition 3.14) and an interpolation error (Proposition 3.12).

C. Interpolation Error

We recall the following result [14].

Theorem 3.6. *Interpolation between Sobolev spaces. Let s_i and t_i be two couples of positive reals for $i = 0$ or $i = 1$ and $p \in \mathbb{R}$ such that $1 \leq p \leq \infty$. Let Π be an operator of $\mathcal{L}(W^{s_0,p}(\Omega); W^{t_0,p}(\Omega)) \cap \mathcal{L}(W^{s_1,p}(\Omega); W^{t_1,p}(\Omega))$ ($\mathcal{L}$ is the space of all linear and continuous functions). Then, for all $\theta \in \mathbb{R} \cap [0; 1]$, operator Π belongs to the interpolate space $\mathcal{L}_\theta = \mathcal{L}(W^{(1-\theta)s_0 + \theta s_1, p}(\Omega); W^{(1-\theta)t_0 + \theta t_1, p}(\Omega))$ and we have*

$$\|\Pi\|_{\mathcal{L}_\theta} \leq \|\Pi\|_{\mathcal{L}_0}^{1-\theta}\|\Pi\|_{\mathcal{L}_1}^{\theta}.$$

Proposition 3.7. *Regularity of components of functions in $M(\Omega)$. Let φ be in $M(\Omega)$, we know (see Proposition 2.4) that φ can be split: $\varphi = \varphi^0 + \varphi^\Delta$ with $\varphi^0 \in H_0^1(\Omega)$ and $\varphi^\Delta \in \mathcal{H}(\Omega)$ (i.e., harmonic). Then, if Ω is convex and if φ belongs to $H^2(\Omega) \cap M(\Omega) = H^2(\Omega)$, φ^0 and φ^Δ belong also to $H^2(\Omega)$. Moreover, there exists a constant $C > 0$ such that*

$$\|\varphi^0\|_{2,\Omega} \leq C\|\varphi\|_{2,\Omega}. \tag{3.14}$$

$$\|\varphi^\Delta\|_{2,\Omega} \leq C\|\varphi\|_{2,\Omega}. \tag{3.15}$$

Proof.   Let us recall that $\varphi^0 \in H_0^1(\Omega)$ is solution of

$$\begin{cases} \Delta\varphi^0 = \Delta\varphi & \text{in } \Omega \\ \gamma\varphi^0 = 0 & \text{on } \Gamma. \end{cases}$$

If φ belongs to $H^2(\Omega)$, its Laplacian belongs to $L^2(\Omega)$ and by the regularity of the Laplacian operator on a convex domain, φ^0 belongs to $H^2(\Omega)$ (Agmon et al. [26]). Then, $\varphi^\Delta = \varphi - \varphi^0$ belongs also to $H^2(\Omega)$. Moreover, we have

$$\|\varphi^0\|_{2,\Omega} \leq C\|\Delta\varphi\|_{0,\Omega} \leq C\|\varphi\|_{2,\Omega}.$$

The inequality for $\varphi^\Delta = \varphi - \varphi^0$ is deduced from the previous one by the triangular inequality. ∎

Definition.   *Projection operator: Let p_c be the L^2-projection on the space of piecewise constants $\mathscr{C}_\tau$ defined in (3.1).*

$$p_c : L^2(\Gamma) \to \mathscr{C}_\tau$$

$$\rho \mapsto p_c\rho$$

$$\text{such that } \int_\Gamma (p_c\rho - \rho) \cdot q \, d\gamma = 0 \quad \forall q \in \mathscr{C}_\tau.$$

Remark 3.8.   It is also possible to define the L^2-projection on the space of piecewise linear functions $\mathscr{L}_\tau$ defined in (3.2), $p_\mathscr{L} : L^2(\Gamma) \to \mathscr{L}_\tau$, which is such that, for all $\rho \in L^2(\Gamma)$, $\int_\Gamma (p_\mathscr{L}\rho - \rho) \cdot q \, d\gamma = 0 \ \forall q \in \mathscr{L}_\tau$.

Let h_τ be the maximum diameter of triangles in $\mathscr{T}$. The standard interpolation error for one-dimensional problems [15] gives, for all ρ in $H^1(\Gamma)$:

$$\|\rho - p_c\rho\|_{0,\Gamma} \leq Ch_\tau|\rho|_{1,\Gamma}. \tag{3.16}$$

Lemma 3.9.   *For all $\rho \in H^{1/2}(\Gamma)$, we have*

$$\|\rho - p_c\rho\|_{0,\Gamma} \leq Ch_\tau^{1/2}\|\rho\|_{1/2,\Gamma}. \tag{3.17}$$

Proof.   Let I be the identity operator. As $I - p_c$ is a continuous operator from $L^2(\Gamma)$ into $L^2(\Gamma)$ with its norm bounded by a constant C_1 and is also continuous from $L^2(\Gamma)$ into $H^1(\Gamma)$ with its norm bounded by $C_2 h_\tau$ [see (3.16)], we deduce that $I - p_c$ is continuous from $L^2(\Gamma)$ into $H^{1/2}(\Gamma)$ ($\theta = 1/2$ in Theorem 3.6) with its norm bounded by a constant times $h_\tau^{1/2}$ which is inequality (3.17). ∎

Proposition 3.10. *For all $\rho \in H^{1/2}(\Gamma)$, we have*

$$\|\rho - p_c\rho\|_{-3/2,\Gamma} \leq Ch_\tau^{3/2}\|\rho\|_{1/2,\Gamma}. \tag{3.18}$$

For all $\rho \in H^1(\Gamma)$, we have

$$\|\rho - p_c\rho\|_{-3/2,\Gamma} \leq Ch_\tau^2\|\rho\|_{1,\Gamma}. \tag{3.19}$$

Proof. By definition of the norm in $H^{-3/2}(\Gamma)$, we have

$$\|\rho - p_c\rho\|_{-3/2,\Gamma} = \sup_{\eta \in H^{3/2}(\Gamma)} \frac{\langle \rho - p_c\rho, \eta \rangle_{-3/2,3/2}}{\|\eta\|_{3/2,\Gamma}}.$$

As $H^{1/2}(\Gamma)$ and $H^{3/2}(\Gamma)$ are contained in $L^2(\Gamma)$, the duality product $\langle \rho - p_c\rho, \eta \rangle_{-3/2,3/2}$ can be rewritten as $\int_\Gamma (\rho - p_c\rho) \cdot \eta \, d\gamma$.

By definition $\int_\Gamma (\rho - p_c\rho) \cdot \chi \, d\gamma = 0 \ \forall \chi \in \mathscr{C}_\tau$, so

$$\langle \rho - p_c\rho, \eta \rangle_{-3/2,3/2} = \int_\Gamma (\rho - p_c\rho) \cdot (\eta - \chi) \, d\gamma \quad \forall \chi \in \mathscr{C}_\tau$$

$$\leq \|\rho - p_c\rho\|_{0,\Gamma}\|\eta - \chi\|_{0,\Gamma} \quad \forall \chi \in \mathscr{C}_\tau \text{ (Cauchy-Schwarz inequality)}$$

$$\leq \|\rho - p_c\rho\|_{0,\Gamma}\|\eta - p_c\eta\|_{0,\Gamma} \quad \text{with } \chi = p_c\eta \in \mathscr{C}_\tau$$

$$\leq C_1 h_\tau^{1/2}\|\rho\|_{1/2,\Gamma} C_2 h_\tau\|\eta\|_{1,\Gamma} \quad \text{using (3.17) and (3.16)}$$

$$\leq Ch_\tau^{3/2}\|\rho\|_{1/2,\Gamma}\|\eta\|_{3/2,\Gamma},$$

which ends the proof of (3.18).

Notice that if ρ belongs to $H^1(\Gamma)$, using a classical interpolation estimate [see (3.16)], we have $\|\rho - p_c\rho\|_{0,\Gamma} \leq Ch_\tau\|\rho\|_{1,\Gamma}$, so we obtain

$$\langle \rho - p_c\rho, \eta \rangle_{-3/2,3/2} \leq \|\rho - p_c\rho\|_{0,\Gamma}\|\eta - p_c\eta\|_{0,\Gamma} \quad \text{with } \chi = p_c\eta \in \mathscr{C}_\tau$$

$$\leq C_1 h_\tau\|\rho\|_{1,\Gamma} C_2 h_\tau\|\eta\|_{1,\Gamma}$$

$$\leq Ch_\tau^2\|\rho\|_{1,\Gamma}\|\eta\|_{3/2,\Gamma},$$

which leads to formula (3.19). ∎

Let $\Pi_\tau : H^2(\Omega) \to H_\tau^1$ be the classical Lagrange interpolation operator associated with mesh $\mathscr{T}$.

Definition. *The interpolation operator $\phi_\tau : \mathscr{H}(\Omega) \cap H^2(\Omega) \to \mathscr{H}_{\tau,I}$ is defined by $\phi_\tau\varphi^\Delta = \zeta$ where ζ is such that*

$$\zeta(x) = \mathscr{S}p_c(S^{-1}\varphi^\Delta)(x) = \int_\Gamma G(x, y) \cdot p_c(S^{-1}\varphi^\Delta)(y) \, d\gamma_y \qquad \forall x \in \bar{\Omega}.$$

We define the interpolation operator $\mathscr{P}_\tau$ from $M(\Omega) \cap H^2(\Omega)$ to $H^{1,I}_{\mathcal{T}} = H^1_{0,\tau} \oplus \mathscr{H}_{\tau,I}$ by the relations:

$$\mathscr{P}_\tau : M(\Omega) \cap H^2(\Omega) \ni \varphi = \varphi^0 + \varphi^\Delta \mapsto \mathscr{P}_\tau\varphi = \Pi_\tau\varphi^0 + \phi_\tau\varphi^\Delta \in H^{1,I}_\tau.$$

Remark 3.11. We can also define the interpolation operator $\phi'_\tau : \mathscr{H}(\Omega) \cap H^2(\Omega) \to \mathscr{H}_{\tau,I}$ by $\phi'_\tau\varphi^\Delta = \zeta$, where ζ is such that

$$\zeta(x) = \mathscr{S}p_{\mathscr{L}}(S^{-1}\varphi^\Delta)(x) = \int_\Gamma G(x, y) \cdot p_{\mathscr{L}}(S^{-1}\varphi^\Delta)(y) \, d\gamma_y \qquad \forall x \in \bar{\Omega},$$

and $\mathscr{P}'_\tau$ from $M(\Omega) \cap H^2(\Omega)$ to $H^{1,I}_{\mathcal{T}} = H^1_{0,\tau} \oplus \mathscr{H}_{\tau,I}$ by the relations:

$$\mathscr{P}'_\tau : M(\Omega) \cap H^2(\Omega) \ni \varphi = \varphi^0 + \varphi^\Delta \mapsto \mathscr{P}'_\tau\varphi = \Pi_\tau\varphi^0 + \phi'_\tau\varphi^\Delta \in H^{1,I}_\tau.$$

Proposition 3.12. *Error estimates*: *For $\mathcal{T}$ in a regular family of triangulations $\mathcal{U}_\sigma(\sigma > 0$ fixed), for h_τ small enough, and for φ given in $M(\Omega)$ decomposed into φ^0 and φ^Δ, if we assume $\varphi^0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\varphi \in H^2(\Omega)$, then there exists some strictly positive constants, say C, only dependent on σ, such that*

$$\|\varphi^0 - \Pi_\tau\varphi^0\|_M \leq Ch_\tau|\varphi^0|_{2,\Omega}. \tag{3.20}$$

$$\|\varphi^\Delta - \phi_\tau\varphi^\Delta\|_M \leq Ch_\tau^{3/2}\|\varphi\|_{2,\Omega}. \tag{3.21}$$

Moreover, if φ belongs to $H^{5/2}(\Omega)$, we have

$$\|\varphi^\Delta - \phi_\tau\varphi^\Delta\|_M \leq Ch_\tau^2\|\varphi\|_{5/2,\Omega}. \tag{3.22}$$

Proof. As $H^2(\Omega) \subset \mathscr{C}^0(\Omega)$ when Ω is two-dimensional, we can use the classical interpolation operator, and we have the following interpolation error estimate (Ciarlet-Raviart [27]):

$$\|\varphi^0 - \Pi_\tau\varphi^0\|_{1,\Omega} \leq Ch_\tau|\varphi^0|_{2,\Omega}.$$

But $\|\varphi^0 - \Pi_\tau\varphi^0\|_M = \|\varphi^0 - \Pi_\tau\varphi^0\|_{1,\Omega}$ because $\varphi^0 - \Pi_\tau\varphi^0 \in H^1_0(\Omega)$ (see Proposition 2.1), so relation (3.20) is established, if h_τ is small enough.

We now interpolate the harmonic part φ^Δ of $\varphi \in M(\Omega)$. By definition φ^Δ verifies

$$\begin{cases} \Delta\varphi^\Delta = 0 & \text{in } \Omega \\ \gamma\varphi^\Delta = \gamma\varphi & \text{on } \Gamma. \end{cases}$$

As φ is assumed to be in $H^2(\Omega)$, its trace on Γ, $\gamma\varphi$, belongs to $H^{3/2}(\Gamma)$. As S is an isomorphism from $H^s(\Gamma)$ onto $H^{s+1}(\Gamma)$ (see Theorem 3.3), there exists a unique $q \in H^{1/2}(\Gamma)$

such that $Sq = \gamma \mathcal{S} q = \gamma \varphi$ on Γ. And because of uniqueness, $\mathcal{S} q = \varphi^\Delta$ on Ω. Let now $q_\tau = p_c q \in \mathcal{C}_\tau$, we recall that $\phi_\tau \varphi^\Delta$ is

$$\phi_\tau \varphi^\Delta(x) = \mathcal{S} q_\tau(x) = \int_\Gamma G(x, y) \cdot q_\tau(y) \, d\gamma_y \qquad \forall x \in \bar{\Omega}.$$

So,

$$\|\varphi^\Delta - \phi_\tau \varphi^\Delta\|_M = \|\varphi^\Delta - \phi_\tau \varphi^\Delta\|_{0,\Omega} = \|\mathcal{S} q - \mathcal{S} q_\tau\|_{0,\Omega}.$$

As constants belong to $\mathcal{C}_\tau$, $\int_\Gamma (q - q_\tau) \, d\gamma = 0$, i.e. $q - q_\tau \in {}_\circ H^{-3/2}(\Gamma)$. Then proposition 3.4 gives

$$\|\mathcal{S} q - \mathcal{S} q_\tau\|_{0,\Omega} \le C\|q - q_\tau\|_{-3/2,\Gamma}.$$

Then, from inequality (3.18), we obtain

$$\|q - q_\tau\|_{-3/2,\Gamma} = \|q - p_c q\|_{-3/2,\Gamma} \le Ch_\tau^{3/2}\|q\|_{1/2,\Gamma}.$$

And finally

$$\|\varphi^\Delta - \phi_\tau \varphi^\Delta\|_M \le Ch_\tau^{3/2}\|q\|_{1/2,\Gamma} = Ch_\tau^{3/2}\|S^{-1}(\varphi)\|_{1/2,\Gamma}$$

$$\le Ch_\tau^{3/2}\|\gamma \varphi\|_{3/2,\Gamma} \qquad \text{because } S \text{ is an isomorphism}$$

$$\le Ch_\tau^{3/2}\|\varphi\|_{2,\Omega} \qquad \text{by continuity of the trace operator.}$$

Notice that if φ is assumed to be in $H^{5/2}(\Omega)$, its trace on Γ, $\gamma \varphi$, belongs to $H^2(\Gamma)$. So, there exists a unique $q \in H^1(\Gamma)$ such that $Sq = \gamma \mathcal{S} q = \gamma \varphi$ on Γ. The same arguments as above lead to the inequality:

$$\|\mathcal{S} q - \mathcal{S} q_\tau\|_{0,\Omega} \le C\|q - q_\tau\|_{-3/2,\Gamma}.$$

Then, from formula (3.19), we have

$$\|q - q_\tau\|_{-3/2,\Gamma} = \|q - p_c q\|_{-3/2,\Gamma} \le Ch_\tau^2\|q\|_{1,\Gamma}.$$

And finally

$$\|\varphi^\Delta - \phi_\tau \varphi^\Delta\|_M \le Ch_\tau^2\|q\|_{1,\Gamma} = Ch_\tau^2\|S^{-1}(\varphi)\|_{1,\Gamma}$$

$$\le Ch_\tau^2\|\gamma \varphi\|_{2,\Gamma} \qquad \text{because } S \text{ is an isomorphism}$$

$$\le Ch_\tau^2\|\varphi\|_{5/2,\Omega} \qquad \text{by continuity of the trace operator.} \qquad \blacksquare$$

Remark 3.13. Using linear interpolation on the boundary, we obtain

$$\|\varphi^\Delta - \phi_\tau' \varphi^\Delta\|_M \le Ch_\tau^2\|\varphi\|_{2,\Omega}. \tag{3.23}$$

To prove the inequality with ϕ'_τ, we use the same arguments as above. Projection is done on $\mathscr{L}_\tau : q_\tau = p_{\mathscr{L}} q \in \mathscr{L}_\tau$, we define $\phi'_\tau \varphi^\Delta$ as the simple layer potential associated to q_τ, i.e.,

$$\phi'_\tau \varphi^\Delta(x) = \mathscr{S} q_\tau(x) = \int_\Gamma G(x,\,y)\cdot q_\tau(y)\,d\gamma_y \qquad \forall x \in \bar{\Omega}.$$

As we have proved for projection p_c, using classical interpolation and interpolation between Sobolev spaces results, we have

$$\|q - q_\tau\|_{-3/2,\Gamma} = \|q - p_{\mathscr{L}} q\|_{-3/2,\Gamma} \leq C h_\tau^2 \|q\|_{1/2,\Gamma}.$$

The end of the proof, which leads to inequality (3.23), is the same as above.

D. Error Estimates

We define an auxiliary problem: Find $\theta_\tau \in H^1_{0,\tau}$, $\eta_\tau = \eta_\tau^0 + \eta_\tau^\Delta \in H_\tau^{1,I}$ such that

$$\begin{cases} (\nabla \eta_\tau^0, \nabla \xi) = (\nabla g, \nabla \xi) & \forall \xi \in H^1_{0,\tau}. \\ (\eta_\tau^\Delta, \varphi) = (m, \varphi) & \forall \varphi \in \mathscr{H}_{\tau,I}. \\ (\nabla \theta_\tau, \nabla \chi) = (\nabla l, \nabla \chi) + (n, \chi) & \forall \chi \in H^1_{0,\tau}. \end{cases} \qquad (3.24)$$

Proposition 3.14. *Stability of discrete formulation* (3.24). *Let* $g \in H^1_0(\Omega)$, $m \in L^2(\Omega)$, $l \in H^1_0(\Omega)$, $n \in L^2(\Omega)$. *Then problem* (3.24) *has a unique solution* $(\theta_\tau, \eta_\tau) \in H^1_{0,\tau} \times H_\tau^{1,I}$, *which is stable in the following sense*: *there exists a constant C only dependent on the mesh family such that the following stability inequality holds*:

$$\|\eta_\tau\|_M + \|\nabla \theta_\tau\|_{0,\Omega} \leq C\{\|\nabla g\|_{0,\Omega} + \|m\|_{0,\Omega} + \|\nabla l\|_{0,\Omega} + \|n\|_{0,\Omega}\}.$$

Proof. In the following, *C* will denote various constants.

In the first equation of problem (3.24) we take $\xi = \eta_\tau^0$, and using Poincaré's inequality, we obtain

$$\|\eta_\tau^0\|_M \leq C\|\nabla \eta_\tau^0\|_{0,\Omega} \leq C\|\nabla g\|_{0,\Omega},$$

as η_τ^0 belongs to $H^1_0(\Omega)$, its H^1-norm is equal to its *M*-norm.

In the second equation of problem (3.24), we take $\varphi = \eta_\tau^\Delta$, as $\Delta \eta_\tau^\Delta = 0$, we obtain

$$\|\eta_\tau^\Delta\|_M = \|\eta_\tau^\Delta\|_{0,\Omega} \leq \|m\|_{0,\Omega}.$$

Finally, in the last equation of problem (3.24), we take $\chi = \theta_\tau$. As θ_τ belongs to $H^1_0(\Omega)$, using Poincaré's inequality, we have

$$\|\nabla \theta_\tau\|_{0,\Omega} \leq C(\|\nabla l\|_{0,\Omega} + \|n\|_{0,\Omega}).$$

By combining these three inequalities, Proposition 3.14 is proven. ∎

Proposition 3.15. *The error is bounded by the interpolation error. Let (ω, ψ) be the solution of the continuous stream function-vorticity formulation (2.13) and (ω_τ, ψ_τ) solution of the associated discrete problem (3.6)–(3.9). There exists a constant $C > 0$ independent of $\mathcal{T}$ such that*

$$\|\omega - \omega_\tau\|_M + \|\psi - \psi_\tau\|_{1,\Omega} \leq C(\|\omega^0 - \Pi_\tau \omega^0\|_M + \|\omega^\Delta - \phi_\tau \omega^\Delta\|_M + \|\psi - \Pi_\tau \psi\|_{1,\Omega}).$$

Proof. The continuous problem (2.13) is written with a discrete test function $\varphi = \varphi^0 + \varphi^\Delta \in H_\tau^{1,I}$ since $H_\tau^{1,I} \subset M(\Omega)$ and with $\xi \in H_{0,\tau}^1 \subset H_0^1(\Omega)$:

$$\begin{cases} (\nabla \omega^0, \nabla \xi) = (\mathbf{f}, \mathbf{curl}\, \xi) & \forall \xi \in H_{0,\tau}^1. \\ (\omega^\Delta, \varphi) = -(\omega^0, \varphi) & \forall \varphi \in \mathcal{H}_{\tau,I}. \\ (\nabla \psi, \nabla \chi) = (\omega^0 + \omega^\Delta, \chi) & \forall \chi \in H_{0,\tau}^1. \end{cases} \tag{3.25}$$

The discrete problem is

$$\begin{cases} (\nabla \omega_\tau^0, \nabla \xi) = (\mathbf{f}, \mathbf{curl}\, \xi) & \forall \xi \in H_{0,\tau}^1 \\ (\omega_\tau^\Delta, \varphi) = -(\omega_\tau^0, \varphi) & \forall \varphi \in \mathcal{H}_{\tau,I}. \\ (\nabla \psi_\tau, \nabla \chi) = (\omega_\tau^0 + \omega_\tau^\Delta, \chi) & \forall \chi \in H_{0,\tau}^1. \end{cases} \tag{3.26}$$

Subtracting (3.25) from (3.26), we obtain

$$\begin{cases} (\nabla(\omega^0 - \omega_\tau^0), \nabla \xi) = 0 & \forall \xi \in H_{0,\tau}^1. \\ (\omega^\Delta - \omega_\tau^\Delta, \varphi) = -(\omega^0 - \omega_\tau^0, \varphi) & \forall \varphi \in \mathcal{H}_{\tau,I}. \\ (\nabla(\psi - \psi_\tau), \nabla \chi) = ((\omega^0 - \omega_\tau^0) + (\omega^\Delta - \omega_\tau^\Delta), \chi) & \forall \chi \in H_{0,\tau}^1. \end{cases} \tag{3.27}$$

We now introduce the interpolants of ψ and $\omega = \omega^0 + \omega^\Delta$ on the mesh $\mathcal{T}$. In the first equation of problem (3.27), we add and subtract $\Pi_\tau \omega^0$ and obtain

$$(\nabla(\omega_\tau^0 - \Pi_\tau \omega^0), \nabla \xi) = (\nabla(\omega^0 - \Pi_\tau \omega^0), \nabla \xi) \qquad \forall \xi \in H_{0,\tau}^1. \tag{3.28}$$

Adding and subtracting $\phi_\tau \omega^\Delta$ in the second equation, we obtain

$$(\omega_\tau^\Delta - \phi_\tau \omega^\Delta, \varphi) = (\omega^\Delta - \phi_\tau \omega^\Delta, \varphi) + (\omega^0 - \omega_\tau^0, \varphi) \qquad \forall \varphi \in \mathcal{H}_{\tau,I}. \tag{3.29}$$

And finally using same techniques for the last equation, we have

$$(\nabla(\psi_\tau - \Pi_\tau \psi), \nabla \chi) = (\nabla(\psi - \Pi_\tau \psi), \nabla \chi) - (\omega^0 - \omega_\tau^0, \chi) - (\omega^\Delta - \omega_\tau^\Delta, \chi) \qquad \forall \chi \in H_{0,\tau}^1. \tag{3.30}$$

Equations (3.28), (3.29), and (3.30) lead to the following problem:

$$\begin{cases} (\nabla(\omega_\tau^0 - \Pi_\tau \omega^0), \nabla \xi) = (\nabla g, \nabla \xi) & \forall \xi \in H_{0,\tau}^1. \\ (\omega_\tau^\Delta - \phi_\tau \omega^\Delta, \varphi) = (m, \varphi) & \forall \varphi \in \mathcal{H}_{\tau,I} \\ (\nabla(\psi_\tau - \Pi_\tau \psi), \nabla \chi) = (\nabla l, \nabla \chi) + (n, \chi) & \forall \chi \in H_{0,\tau}^1, \end{cases}$$

which is the auxiliary problem (3.24) previously defined with $g = \omega^0 - \Pi_\tau \omega^0$ in $H_0^1(\Omega)$, $m = (\omega^\Delta - \phi_\tau \omega^\Delta) + (\omega^0 - \omega_\tau^0)$ in $L^2(\Omega)$, $l = \psi - \Pi_\tau \psi$ in $H_0^1(\Omega)$, $n = -(\omega^0 - \omega_\tau^0) - (\omega^\Delta - \omega_\tau^\Delta)$ in $L^2(\Omega)$.

By applying the triangular inequality and Proposition 3.14, we have

$$\|\omega - \omega_\tau\|_M + \|\psi - \psi_\tau\|_{1,\Omega} \leq \underbrace{\|\omega - \mathscr{P}_\tau \omega\|_M + \|\psi - \Pi_\tau \psi\|_{1,\Omega}}_{\text{interpolation error}} + \|\mathscr{P}_\tau \omega - \omega_\tau\|_M + \|\Pi_\tau \psi - \psi_\tau\|_{1,\Omega}$$

$$\leq \|\omega - \mathscr{P}_\tau \omega\|_M + \|\psi - \Pi_\tau \psi\|_{1,\Omega} + C(\|\nabla(\omega^0 - \Pi_\tau \omega^0)\|_{0,\Omega} + \|\omega^\Delta - \phi_\tau \omega^\Delta + \omega^0 - \omega_\tau^0\|_{0,\Omega}$$

$$+ \|\nabla(\psi - \Pi_\tau \psi)\|_{0,\Omega} + \|\omega^0 - \omega_\tau^0 + \omega^\Delta - \omega_\tau^\Delta\|_{0,\Omega})$$

$$\leq \|\omega - \mathscr{P}_\tau \omega\|_M + \|\psi - \Pi_\tau \psi\|_{1,\Omega} + C(\|\omega^0 - \Pi_\tau \omega^0\|_M + \|\omega^\Delta - \phi_\tau \omega^\Delta\|_M + \|\omega^0 - \omega_\tau^0\|_{0,\Omega}$$

$$+ \|\nabla(\psi - \Pi_\tau \psi)\|_{0,\Omega} + \|\omega^0 - \omega_\tau^0\|_{0,\Omega} + \|\omega^\Delta - \omega_\tau^\Delta\|_M),$$

which leads to the announced result. ∎

The inequality obtained in Proposition 3.15 and the interpolation error lead to the following.

Theorem 3.16. *First convergence result. If $\mathscr{T}$ belongs to a regular family of triangulation $\mathscr{U}_\sigma$ ($\sigma > 0$) and h_τ is small enough, the discrete problem (3.6)–(3.9) has a unique solution $(\psi_\tau, \omega_\tau) \in H_{0,\tau}^1 \times H_\tau^{1,I}$, associated with a stable discretization of the Stokes problem (2.1)–(2.4). If $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\omega = \omega^0 + \omega^\Delta$ such that $\omega \in H^2(\Omega)$ and $\omega^0 \in H^2(\Omega)$, then*

$$\exists\, C(\sigma) > 0, \qquad \forall \mathscr{T} \in \mathscr{U}_\sigma, \qquad \|\omega - \omega_\tau\|_M + \|\psi - \psi_\tau\|_{1,\Omega} \leq C(\sigma) h_\tau(|\omega^0|_{2,\Omega} + \|\omega\|_{2,\Omega} + |\psi|_{2,\Omega}).$$

Proof. With the help of the inequality in Proposition 3.15 (C denotes various constants),

$$\|\omega - \omega_\tau\|_M + \|\psi - \psi_\tau\|_{1,\Omega} \leq C(\|\omega^0 - \Pi_\tau \omega_\tau^0\|_M + \|\omega^\Delta - \phi_\tau \omega^\Delta\|_M + \|\psi - \Pi_\tau \psi\|_{1,\Omega})$$

$$\leq C(h_\tau |\omega^0|_{2,\Omega} + h_\tau^{3/2} \|\omega\|_{2,\Omega} + h_\tau |\psi|_{2,\Omega}) \qquad \text{by Proposition 3.12}$$

$$\leq C h_\tau(|\omega^0|_{2,\Omega} + \|\omega\|_{2,\Omega} + |\psi|_{2,\Omega}) \qquad \text{as } h_\tau \text{ can be assumed to be less or equal to 1.} \blacksquare$$

Remark 3.17. If Ω is assumed to be convex, proposition 3.7 leads to

$$\exists\, C(\sigma) > 0, \qquad \forall \mathscr{T} \in \mathscr{U}_\sigma, \qquad \|\omega - \omega_\tau\|_M + \|\psi - \psi_\tau\|_{1,\Omega} \leq C(\sigma) h_\tau(\|\omega\|_{2,\Omega} + |\psi|_{2,\Omega}).$$

Theorem 3.16 is important because it shows that using a space of harmonic functions along the boundary gives an error of order $\mathbb{O}(h_\tau)$ when $\omega \in H^2(\Omega)$, which improves previous known results and is equivalent to those proved in [12].

Theorem 3.18. *Second convergence result. If Ω is assumed to be convex and under the same assumptions as in Theorem 3.16, we have*

$$\exists\, C(\sigma) > 0, \qquad \forall \mathscr{T} \in \mathscr{U}_\sigma, \qquad \|\omega - \omega_\tau\|_{0,\Omega} + \|\psi - \psi_\tau\|_{0,\Omega} \leq C(\sigma) h_\tau^{3/2}(\|\omega\|_{2,\Omega} + \|\psi\|_{2,\Omega}).$$

Under the same assumptions, if moreover $\omega \in H^{5/2}(\Omega)$, we have

$$\exists\, C(\sigma) > 0, \qquad \forall \mathcal{T} \in \mathcal{U}_\sigma, \qquad \|\omega - \omega_\tau\|_{0,\Omega} + \|\psi - \psi_\tau\|_{0,\Omega} \leq C(\sigma)h_\tau^2(\|\omega\|_{5/2,\Omega} + \|\psi\|_{2,\Omega}).$$

Proof. If ω belongs to $H^2(\Omega)$, from Proposition 3.12, formula (3.21), we know that

$$\|\omega^\Delta - \omega_\tau^\Delta\|_{0,\Omega} \leq C(\sigma)h_\tau^{3/2}\|\omega\|_{2,\Omega}.$$

If ω belongs to $H^{5/2}(\Omega)$, from Proposition 3.12, formula (3.22), we know that

$$\|\omega^\Delta - \omega_\tau^\Delta\|_{0,\Omega} \leq C(\sigma)h_\tau^2\|\omega\|_{5/2,\Omega}.$$

We just have to use

$$\|\omega^0 - \omega_\tau^0\|_{0,\Omega} \leq C(\sigma)h_\tau^2\|\omega\|_{2,\Omega} \leq C(\sigma)h_\tau^2\|\omega\|_{5/2,\Omega},$$

and

$$\|\psi - \psi_\tau\|_{0,\Omega} \leq C(\sigma)h_\tau^2(\|\psi\|_{2,\Omega} + \|\omega\|_{5/2,\Omega}).$$

These inequalities are proven using the classical Aubin-Nitsche argument when the domain Ω is convex so that regularity on the adjoint problem is obtained ([22, 23]). ∎

The last part of the previous theorem says that if the solution is more regular than usual, the convergence is of order 2. This is illustrated in the numerical examples in the next section where the solutions are very regular.

Remark 3.19. If we use the linear interpolation for the boundary values of the vorticity instead of piecewise constant one, we know from formula (3.23) that

$$\|\omega^\Delta - \omega_\tau^\Delta\|_{0,\Omega} \leq C(\sigma)h_\tau^2\|\omega\|_{2,\Omega},$$

from which we deduce the following theorem.

Theorem 3.20. *We assume that Ω is convex, that $\mathcal{T}$ belongs to a regular family of triangulations, that $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ and that $\omega \in H^2(\Omega)$. If we use space $\mathcal{L}_\tau$, we have*

$$\exists\, C(\sigma) > 0, \qquad \forall \mathcal{T} \in \mathcal{U}_\sigma, \qquad \|\omega - \omega_\tau\|_{0,\Omega} + \|\psi - \psi_\tau\|_{0,\Omega} \leq C(\sigma)h_\tau^2(\|\omega\|_{2,\Omega} + \|\psi\|_{2,\Omega}).$$

IV. NUMERICAL APPLICATIONS

The first numerical experiments have been performed on a unit square with an analytical solution (Bercovier-Engelman test [28]):

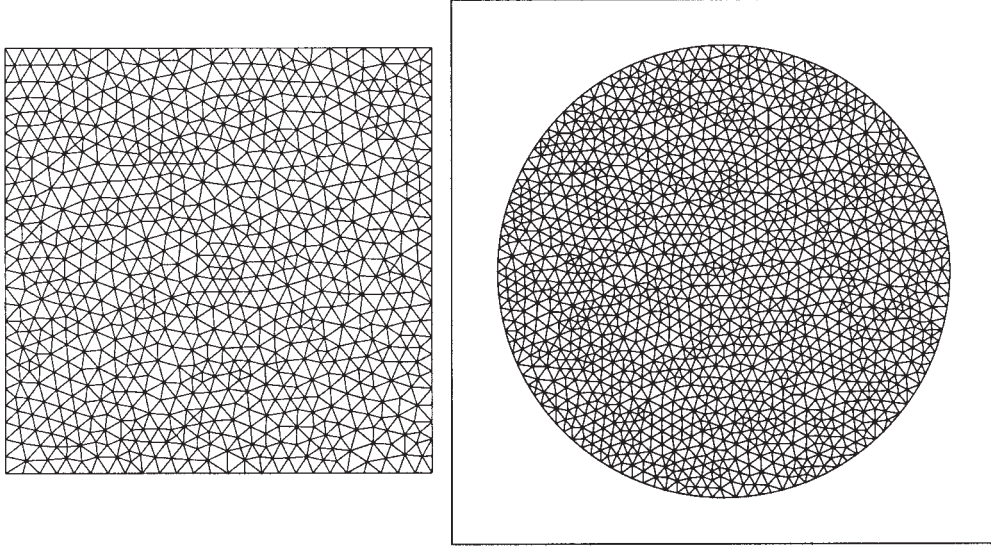$$f_1(x, y) = 256(x^2(x - 1)^2(12y - 6) + y(y - 1)(2y - 1)(12x^2 - 12x + 2))$$

FIG. 1. Two unstructured meshes obtained by EMC2.

$$f_2(x, y) = -f_1(y, x),$$

for which we obtain $\psi(x, y) = -128(y^2(y - 1)^2 x^2(x - 1)^2)$
and $\omega(x, y) = 256(y^2(y - 1)^2(6x^2 - 6x + 1) + x^2(x - 1)^2(6y^2 - 6y + 1))$. For the second numerical experiments, we have extended the tests to circles. They have been performed on a circle of radius 2 with an analytical solution (test suggested by Ruas [29]):

$$f_1(x, y) = -32y, \qquad f_2(x, y) = 32x,$$

which gives $\psi(x, y) = (4 - x^2 - y^2)^2$ and $\omega(x, y) = 32 - 16x^2 - 16y^2$.

Remark 4.1. Notice that for error estimates on the circle, we have to add a boundary approximation error. The boundary is approximated by polynomials of degree 1 and it seems that this error is of the same order than the approximation error with polynomials of degree zero.

We have worked with unstructured meshes obtained with EMC2, mesh generator of Modulef (Bernadou et al. [30]), see Fig. 1. Notice that structured meshes give good results without any stabilization, see [2] and for numerical results see [3] and [5]. For the sake of simplicity, we use the space $\mathscr{C}_\tau$ of functions which are constant on each edge for the integral representation. It would be interesting to test the linear integral representation but the results were so satisfying with the constant one that we did not even try.

Remark 4.2. All the integrals for assembling the mass matrix in Equation (3.8) were computed with the help of a Gauss formula using 7 quadrature points and we obtain results in accordance with the theory. The cost of such an integration could be reduced by using less points far from the boundary. Although errors due to numerical integration were not studied here, they seem to be dominated by other errors and do not pollute results.
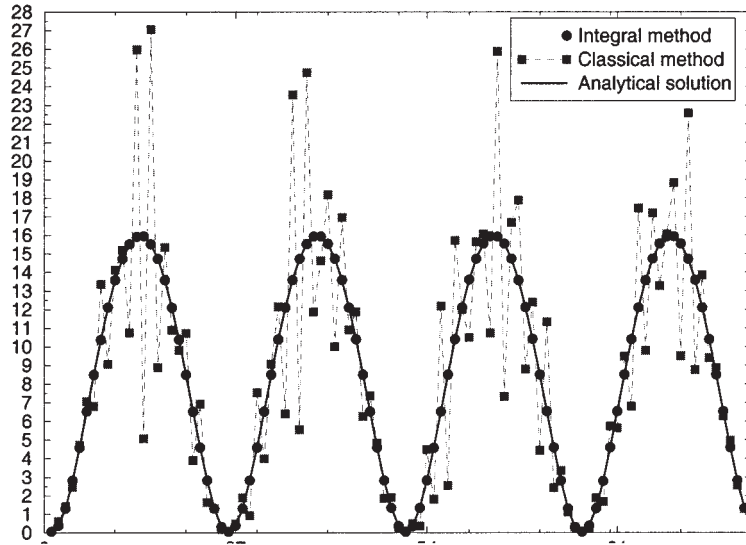
FIG. 2.   Comparison: vorticity on the boundary—Bercovier-Engelman test.

For the first test, the analytical vorticity attains its extremum on the middle of each edge of the square and its value is then $+16.00$. And for the second one the extremum (-32) is attained on the whole boundary. We recall that the number of discrete harmonic functions is equal to the number of vertices on the boundary and that the classical method uses piecewise linear functions for approximating the vorticity and the stream function. Figure 2 gives the values of the vorticity on the boundary obtained by the classical and the integral method on the same mesh. In fact with the classical method, extrema of the vorticity blow up on the boundary (see Fig. 2).

We recall that problem (2.6)–(2.9) is not stable and error bounds are in $h_\tau^{1/2}$ for the L^2-norm of the vorticity—as numerically illustrated on Figs. 3 and 4—and $h_\tau^{1-\varepsilon}$ for the H^1-norm of the stream function [2]. Using only constants on edges with regular solutions, we have obtained, as
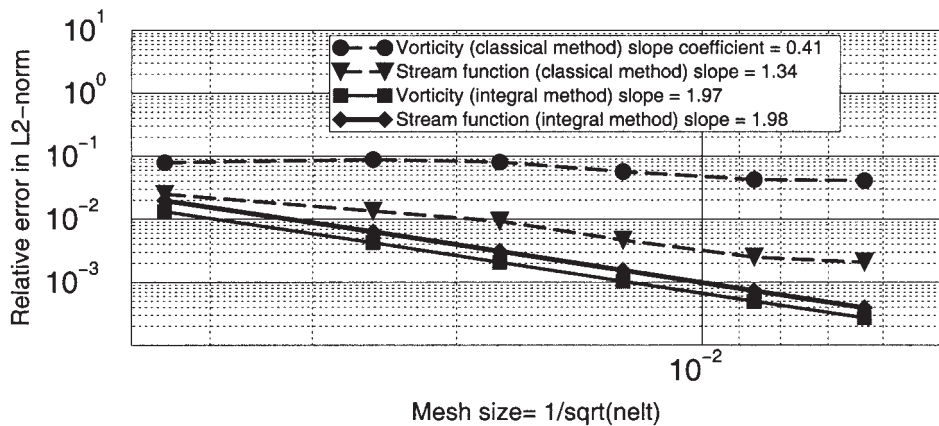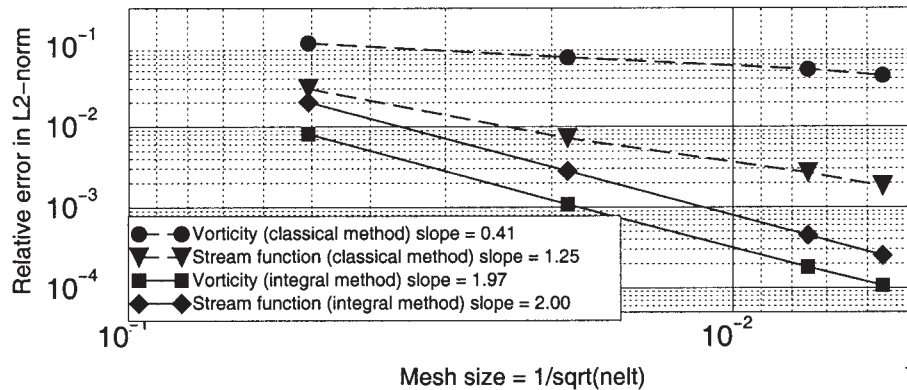

FIG. 3.   Convergence orders—Bercovier-Engelman test.

Relative error in L2-norm

- Vorticity (classical method) slope = 0.41
- Stream function (classical method) slope = 1.25
- Vorticity (integral method) slope = 1.97
- Stream function (integral method) slope = 2.00

Mesh size = 1/sqrt(nelt)

expected by Theorem 3.18, convergence of order 2 for the L^2-norm of the vorticity and for the L^2-norm of the stream function (see Figs. 3 and 4).

V. CONCLUSION

We have studied the well-posed Stokes problem in stream function and vorticity form. We have shown that using a space of *real* harmonic functions is sufficient to obtain on the one hand a better solution and on the other hand better estimations on the convergence than those obtained previously. We have proposed a way of approaching numerically the space of real harmonic functions with the help of an integral representation which yields a large gain of time compared to previous results obtained in [12]. We have shown theoretically and numerically that by this way we obtain convergence with optimal rate in some cases on the quadratic norm of the vorticity.

We use the same method for a vorticity-velocity-pressure formulation of the Stokes problem that allows more general boundary conditions [3, 5]. The results are not published yet but they are really satisfying as, once again, the solution and the convergence of the method are improved. We insist on the fact that the additional work of the method (the computation of the matrix) needs to be done only once. So, the additional cost of our scheme shall be less important for a more realistic problem like the time dependent Stokes or Navier-Stokes equations.

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