To link to this article: DOI: 10.1016/j.cma.2006.09.015
URL: http://dx.doi.org/10.1016/j.cma.2006.09.015

To cite this version: Salaün, Michel and Salmon, Stéphanie *Numerical stabilization of the Stokes problem in vorticity–velocity–pressure formulation.* (2007) Computer Methods in Applied Mechanics and Engineering, vol. 196 (n° 9-12). pp. 1767-1786. ISSN 0045-7825

Numerical stabilization of the Stokes problem in vorticity–velocity–pressure formulation

Michel Salaün [a], Stéphanie Salmon [b,*]

[a] Département de Génie Mécanique, ENSICA, 1 place Emile Blouin, F-31056 Toulouse Cedex 05, France
[b] IRMA – UMR 7501 CNRS/ULP, 7 rue René Descartes, F-67084 Strasbourg Cedex, France

Abstract

We work on a vorticity, velocity and pressure formulation of the bidimensional Stokes problem for incompressible fluids. In previous papers, the authors have developed a natural implementation of this scheme. We have then observed that, in case of unstructured meshes with Dirichlet boundary conditions on the velocity, the convergence is not optimal. In this paper, we propose to add "bubble" velocity functions with compact support along the boundary to improve convergence. We then prove a convergence theorem and illustrate by numerical results better behaviour of the scheme in general cases.

Keywords: Stokes problem; Vorticity–velocity–pressure formulation; Stream function-vorticity formulation; Mixed finite elements method; Bubble functions; Inf–sup conditions

1. Introduction

1.1. Motivation

Let Ω be a bounded connected domain of $\mathbb{R}^2$ with a regular boundary $\partial\Omega \equiv \Gamma$. We recall the Stokes problem which models the stationary equilibrium of an incompressible viscous fluid when the nonlinear terms are neglected (see e.g. [1])

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where u is the velocity, p the pressure, ν the kinematic viscosity, which is a strictly positive constant, and f the datum of external forces. For the sake of simplicity, we shall take $\nu = 1$ in all the following.

The HAWAY method (Harlow and Welch MAC scheme [2], Arakawa C-grid [3], Yee translated grids for Maxwell equations [4]) is a very popular way to solve the Navier–Stokes or Maxwell equations on quadrangular and regular meshes. It is now well extended in the Computer Graphics community [5] to simulate realistic movements of fluids. In 1992, Dubois [6] introduced a three-fields (vorticity, velocity and pressure) formulation in order to extend this HAWAY method to arbitrary triangular meshes. The idea of this formulation is to use exactly the same degrees of freedom as in the HAWAY one (see Figs. 1 and 2).

The boundary Γ of the domain Ω is decomposed with the help of *two* independent partitions and the problem we want to solve reads as

$$\Gamma = \overline{\Gamma_m} \cup \overline{\Gamma_p} \quad \text{with } \Gamma_m \cap \Gamma_p = \emptyset,$$
$$\Gamma = \overline{\Gamma_\theta} \cup \overline{\Gamma_t} \quad \text{with } \Gamma_\theta \cap \Gamma_t = \emptyset,$$

* Corresponding author. Tel.: +33 3 90240202; fax: +33 3 90240328.
E-mail addresses: michel.salaun@ensica.fr (M. Salaün), stephanie.salmon@math.u-strasbg.fr (S. Salmon).

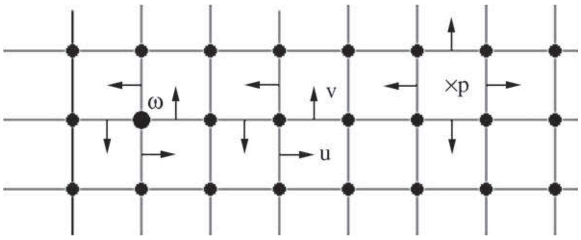

Fig. 1. HAWAY discretization on a cartesian mesh.

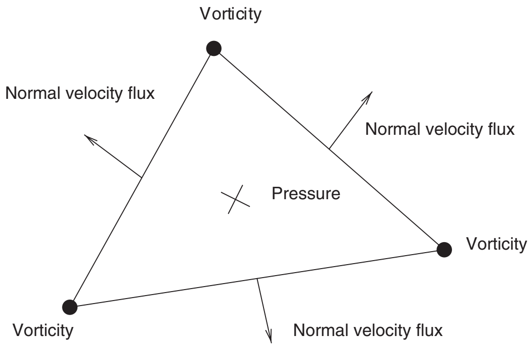

Fig. 2. Degrees of freedom on a triangular mesh.

$$\begin{cases} \omega - \operatorname{curl} u = 0 & \text{in } \Omega, \\ \operatorname{curl} \omega + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma_m, \\ p = \Pi_0 & \text{on } \Gamma_p, \\ \omega = 0 & \text{on } \Gamma_\theta \\ u \cdot t = \sigma_0 & \text{on } \Gamma_t, \end{cases}$$

where $u \cdot n$ and $u \cdot t$ stand respectively for the normal and the tangential components of the velocity.

We studied in [7] this three-fields mixed formulation in vorticity–velocity–pressure. This formulation asks two inf–sup conditions to be verified, the first one is the classical one in pressure and velocity and the second one links vorticity and velocity. We discretized the problem using conforming spaces compatible with the inf–sup condition in velocity–pressure (Raviart–Thomas of lower degree for velocity and constant functions for pressure [8]). For the second inf–sup condition, there is no problem with spaces as we chose piecewise linear continuous functions for the vorticity and Raviart–Thomas of lower degree for velocity. The only condition, which is needed, is a compatibility between boundary conditions on vorticity and velocity: the velocity should be known at least where the vorticity is known ($\Gamma_\theta \subset \Gamma_m$). Let us just observe that this compatibility condition is not really difficult to achieve as generally, there is no boundary condition on the vorticity. So, as soon as the inf–sup conditions are verified, the discrete problem is always well-posed.

Numerical experiments, using this scheme, were performed in [7]. On structured meshes with regular functions, we have optimal convergence for the three fields in L^2-norm: $\mathcal{O}(h^2)$ for the vorticity, $\mathcal{O}(h)$ for velocity and pressure. But on unstructured meshes, results were really not satisfying: vorticity and pressure fields are not well approached. In particular, on a test for which an analytical solution is known, we observe that values of vorticity and pressure are far from the expected ones along the boundary, even if the mesh is refined and that the order of convergence for all these fields, except the velocity, is more or less $\mathcal{O}(\sqrt{h})$. The theoretical study of convergence shows that the problem appears when we try to prove the stability as we then need a kind of "opposite" of the compatibility condition between boundary conditions on vorticity and velocity. Actually, the condition becomes the velocity and the vorticity should be known on the same part of the boundary ($\Gamma_\theta = \Gamma_m$). By the way, in this very particular case, an optimal rate of convergence is achieved, even on unstructured meshes (see [7]). Nevertheless, this condition is really too restrictive and we need to build a numerical velocity field which allows to release it. The idea of adding field called bubbles is well-known for the Stokes problem for verifying the discrete velocity–pressure inf–sup condition with piecewise linear continuous functions spaces, see Arnold et al. [9] and Franca and Oliveira [10]. But, where these bubble functions are introduced on the whole domain, here we only add them along the part of the boundary where the velocity is known but not the vorticity. The aim of this paper is to construct this bubble velocity field in order to get rid of the second compatibility condition and then allow to improve numerical results on the three fields.

Then the scope of this work is the following. We recall the variational formulation which was originally proposed and studied by the authors and its classical discretization, as it is mentioned above. As this formulation show numerical problems in the most general case of boundary conditions (see the first part of this work [7]), we though introduce "bubble functions" and the associated stabilized formulation in Section 2, which is numerically analyzed in Section 3. Section 4 is dedicated to some numerical results. Finally, the last section presents some extensions of results discussed in Section 3 and some particular cases.

1.2. Functional spaces and notation

Let Ω be a given bounded connected domain of $\mathbb{R}^2$ with a regular boundary Γ. We refer to Adams [11] for more details on the Sobolev spaces. We note $L^2(\Omega)$ the space of all (classes of) functions which are square integrable on Ω, equipped with its natural inner product, denoted by $(\cdot,\cdot)$, and the associated norm $\|\cdot\|_{0,\Omega}$. The subspace of $L^2(\Omega)$ containing square integrable functions whose mean value is zero, is denoted by $L^2_0(\Omega)$.

Space $H^1(\Omega)$ will be the space of functions $\varphi \in L^2(\Omega)$ for which the first partial derivatives (in the distribution sense) belong to $L^2(\Omega)$

$$H^1(\Omega) = \left\{ \varphi \in L^2(\Omega) \Big/ \frac{\partial \varphi}{\partial x_i} \in L^2(\Omega) \text{ for } i \in \{1,2\} \right\}.$$

The usual norm in space $H^1(\Omega)$ is denoted by $\|\cdot\|_{1,\Omega}$ while the semi-norm is written $|\cdot|_{1,\Omega}$. In a similar way, we define space $H^2(\Omega)$ as the space of functions of $H^1(\Omega)$ for which the first partial derivatives belong to $H^1(\Omega)$. The associated norms and semi-norms are respectively noted $\|\cdot\|_{2,\Omega}$ and $|\cdot|_{2,\Omega}$. We also introduce space $H^1_0(\Omega)$ which is the closure of the space of all indefinitely differentiable functions on Ω for the norm $\|\cdot\|_{1,\Omega}$.

For all vector field v in $\mathbb{R}^2$, the divergence of v is defined by

$$\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}.$$

Then, space $H(\operatorname{div}, \Omega)$ is the space of vector fields that belong to $(L^2(\Omega))^2$ with divergence (in the distribution sense) in $L^2(\Omega)$. We have classically

$$H(\operatorname{div}, \Omega) = \{v \in (L^2(\Omega))^2 / \operatorname{div} v \in L^2(\Omega)\}, \tag{2}$$

which is a Hilbert space for the norm

$$\|v\|_{\operatorname{div},\Omega} = \left(\sum_{j=1}^{2} \|v_j\|^2_{0,\Omega} + \|\operatorname{div} v\|^2_{0,\Omega} \right)^{1/2}. \tag{3}$$

We recall that functions of $H(\operatorname{div}, \Omega)$ have a normal trace, that we will shortly note $v \cdot n$.

Finally, let us recall that if v is a vector field in a bidimensional domain, then $\operatorname{curl} v$ is the scalar field defined by

$$\operatorname{curl} v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}. \tag{4}$$

In the following, we shall also use the curl of a scalar field, say φ, which is the bidimensional field defined by

$$\operatorname{curl} \varphi = \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1} \right)^t. \tag{5}$$

1.3. First vorticity–velocity–pressure numerical scheme for the Stokes problem

1.3.1. Continuous problem

Following [6,12], we write the Stokes problem with help of a vorticity–velocity–pressure formulation. So, we introduce the *vorticity* ω as

$$\omega = \operatorname{curl} u \tag{6}$$

and the equations of the Stokes problem become

$$\omega - \operatorname{curl} u = 0 \quad \text{in } \Omega, \tag{7}$$

$$\operatorname{curl} \omega + \nabla p = f \quad \text{in } \Omega, \tag{8}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega. \tag{9}$$

Then, we suppose that the boundary Γ of the domain Ω is decomposed with the help of *two* independent partitions

$$\Gamma = \overline{\Gamma_m} \cup \overline{\Gamma_p} \quad \text{with } \Gamma_m \cap \Gamma_p = \emptyset; \tag{10}$$

$$\Gamma = \overline{\Gamma_\theta} \cup \overline{\Gamma_t} \quad \text{with } \Gamma_\theta \cap \Gamma_t = \emptyset. \tag{11}$$

The general boundary conditions for the Stokes problem are

$$u \cdot n = 0 \quad \text{on } \Gamma_m, \tag{12}$$

$$p = \Pi_0 \quad \text{on } \Gamma_p, \tag{13}$$

$$\omega = 0 \quad \text{on } \Gamma_\theta, \tag{14}$$

$$u \cdot t = \sigma_0 \quad \text{on } \Gamma_t, \tag{15}$$

where $u \cdot n$ and $u \cdot t$ stand respectively for the normal and the tangential components of the velocity, n being the outer normal vector to the boundary Γ and t the tangent vector, chosen such that (n, t) is direct.

We finally introduce the following spaces. For velocity, we define space X by

$$X = \{v \in H(\operatorname{div}, \Omega) / v \cdot n = 0 \text{ on } \Gamma_m\}, \tag{16}$$

where Γ_m is the part of the boundary where the trace of the vector field v is given.

For the vorticity, we set

$$W = \{\varphi \in H^1(\Omega) / \gamma\varphi = 0 \text{ on } \Gamma_\theta\}. \tag{17}$$

Let us remark that we have noted $\gamma\varphi$ the trace of the function φ.

Finally, the space for pressure is parameterized by the fact that $\operatorname{meas}(\Gamma_p)$ is zero or not. We set

$$Y = \begin{cases} L^2(\Omega) & \text{if } \operatorname{meas}(\Gamma_p) \neq 0, \\ L^2_0(\Omega) & \text{if } \operatorname{meas}(\Gamma_p) = 0. \end{cases} \tag{18}$$

The variational formulation is easily obtained from Eqs. (7)–(9) and the associated boundary conditions. We obtain

$$\begin{cases} \text{Find } (\omega, u, p) \text{ in } W \times X \times Y \text{ such that:} \\ (\omega, \varphi) - (\operatorname{curl} \varphi, u) = \langle \sigma_0, \gamma\varphi \rangle_\Gamma \quad \forall \varphi \in W, \\ (\operatorname{curl} \omega, v) - (p, \operatorname{div} v) = (f, v) - \langle \Pi_0, v \cdot n \rangle_\Gamma \quad \forall v \in X, \\ (\operatorname{div} u, q) = 0 \quad \forall q \in Y. \end{cases} \tag{19}$$

In these expressions, $\langle \cdot, \cdot \rangle_\Gamma$ stands for a boundary integral. For more details about well-posedness of this continuous problem, the reader is referred to [12,13].

1.3.2. A first numerical discretization

Let $\mathscr{T}$ be a triangulation of the domain Ω. For the sake of simplicity, we shall assume that Ω is polygonal, in such a way that it is entirely covered by the mesh $\mathscr{T}$. Moreover, we will suppose that the trace of the triangulation on the boundary is such that the boundary edge of any triangle does not overlap different parts of the boundary, Γ_m and Γ_p on the one hand, Γ_θ and Γ_t on the other hand. Then, we denote by $\mathscr{E}_\mathscr{T}$ the set of triangles in $\mathscr{T}$. Moreover,

$\mathscr{A}_{\mathscr{T}}$ will be the set of all edges of triangles of $\mathscr{T}$. Finally, $h_{\mathscr{T}}$ is the maximum of the diameters of the triangles of $\mathscr{T}$.

Definition 1 (*Family $\mathscr{U}_\sigma$ of uniformly regular meshes*). We suppose that $\mathscr{T}$ belongs to the set $\mathscr{U}_\sigma$ of triangulations such that there exists two strictly positive constants τ and σ independent of $h_{\mathscr{T}}$ and K such that

$$\tau h_{\mathscr{T}} \leqslant h_K \leqslant \sigma \rho_K \quad \text{for all } K \in \mathscr{E}_{\mathscr{T}}, \tag{20}$$

where h_K is the diameter of the triangle K and ρ_K is the diameter of the circle inscribed in K.

Now, we shall introduce finite-dimensional spaces, say $W_{\mathscr{T}}$, $X_{\mathscr{T}}$ and $Y_{\mathscr{T}}$ which are respectively contained in W, X and Y.

For the vorticity, we choose piecewise linear continuous functions

$$P^1_{\mathscr{T}} = \left\{\varphi \in H^1(\Omega)/\varphi_{|_K} \in \mathbb{P}^1(K)\ \forall K \in \mathscr{E}_{\mathscr{T}}\right\}. \tag{21}$$

Then, including the boundary conditions, we set the following subspace of W:

$$W_{\mathscr{T}} = \left\{\varphi \in P^1_{\mathscr{T}}/\gamma\varphi = 0 \text{ on } \Gamma_\theta\right\}. \tag{22}$$

If we introduce the classical Lagrange interpolation operator, denoted by $\Pi^1_{\mathscr{T}}$, we have the following well-known result (see e.g. [14]):

Theorem 2 (Interpolation error for vorticity). *Let us assume that the mesh $\mathscr{T}$ belongs to a regular family of triangulations* (*see* Definition 1). *Then, there exists a strictly positive constant C, independent of $h_{\mathscr{T}}$, such that, for all $\omega \in H^2(\Omega)$, we have*

$$\|\omega - \Pi^1_{\mathscr{T}}\omega\|_{1,\Omega} \leqslant Ch_{\mathscr{T}}|\omega|_{2,\Omega}.$$

Then, velocity is given by its fluxes through edges of the triangles, by the use of the Raviart–Thomas finite element of degree one [8]

$$RT^0_{\mathscr{T}} = \left\{v \in H(\mathrm{div},\Omega)/v_{|_K} = \begin{pmatrix} a_K \\ b_K \end{pmatrix} + c_K\begin{pmatrix} x \\ y \end{pmatrix}\ \forall K \in \mathscr{E}_{\mathscr{T}}\right\}. \tag{23}$$

Now, we can state the discrete space for velocity

$$X_{\mathscr{T}} = \left\{v \in RT^0_{\mathscr{T}}/v\cdot n = 0 \text{ on } \Gamma_m\right\}. \tag{24}$$

Following [8], let us recall how the interpolation operator is defined.

Definition 3 (*Interpolation operator in $H(\mathrm{div},\Omega)$*). For all vector field v in $(H^1(\Omega))^2$, the interpolation operator $\Pi^{\mathrm{div}}_{\mathscr{T}}$ is such that

$$\forall a \in \mathscr{A}_{\mathscr{T}}, \quad \int_a \Pi^{\mathrm{div}}_{\mathscr{T}} v\cdot n\,\mathrm{d}\gamma = \int_a v\cdot n\,\mathrm{d}\gamma,$$

where n is the unit normal vector to edge a.

Then, we recall the associated interpolation error (see [15]).

Theorem 4 (Interpolation error for velocity). *Let us assume that the mesh $\mathscr{T}$ belongs to a regular family of triangulation. Then, there exists a strictly positive constant C, independent of $h_{\mathscr{T}}$, such that, for all v in $(H^1(\Omega))^2$, we have*

$$\|v - \Pi^{\mathrm{div}}_{\mathscr{T}} v\|_{0,\Omega} \leqslant Ch_{\mathscr{T}}\|v\|_{1,\Omega}.$$

Remark 5. It is possible to define the interpolation operator for less regular functions i.e. for functions v belonging to $(H^\epsilon(\Omega))^2 \cap H(\mathrm{div},\Omega)$. Moreover, an associated interpolation theorem can also be given. But, as we shall not explicitly use it in this paper, we only refer to Mathew [16] for complements on this topic.

Finally, pressure is chosen piecewise constant. Setting

$$P^0_{\mathscr{T}} = \left\{q \in L^2(\Omega)/q_{|_K} \in \mathbb{P}^0(K)\ \forall K \in \mathscr{E}_{\mathscr{T}}\right\}, \tag{25}$$

we define

$$Y_{\mathscr{T}} = \left\{q \in P^0_{\mathscr{T}} \Big/ \int_\Omega q\,\mathrm{d}x = 0 \text{ if } \Gamma_p = \emptyset\right\}. \tag{26}$$

If we introduce the L^2 projection operator on space $Y_{\mathscr{T}}$, denoted by $\Pi^0_{\mathscr{T}}$, which is defined for all q in $L^2(\Omega)$ by

$$\int_K (\Pi^0_{\mathscr{T}} q - q)\,\mathrm{d}x = 0 \quad \text{for all } K \in \mathscr{E}_{\mathscr{T}},$$

we recall the following result (see e.g. [17]):

Theorem 6 (Interpolation error for pressure). *There exists a strictly positive constant C, independent of $h_{\mathscr{T}}$, such that, for all $q \in H^1(\Omega)$, we have*

$$\|q - \Pi^0_{\mathscr{T}} q\|_{0,\Omega} \leqslant Ch_{\mathscr{T}}|q|_{1,\Omega}.$$

Let us also recall the following basic property, which is a direct consequence of the previous definitions and of the Stokes formula (cf. [7]):

Proposition 7. *For all v in $(H^1(\Omega))^2$ and for all q in $Y_{\mathscr{T}}$, we have*

$$\int_\Omega q\,\mathrm{div}\,(\Pi^{\mathrm{div}}_{\mathscr{T}} v - v)\,\mathrm{d}x = 0.$$

The discrete problem is then to find $(\omega_{\mathscr{T}}, u_{\mathscr{T}}, p_{\mathscr{T}})$ in $W_{\mathscr{T}} \times X_{\mathscr{T}} \times Y_{\mathscr{T}}$ such that:

$$\begin{cases} (\omega_{\mathscr{T}}, \varphi) - (\mathrm{curl}\,\varphi, u_{\mathscr{T}}) = \langle \sigma_0, \gamma\varphi\rangle_\Gamma & \forall \varphi \in W_{\mathscr{T}}, \\ (\mathrm{curl}\,\omega_{\mathscr{T}}, v) - (p_{\mathscr{T}}, \mathrm{div}\,v) = (f, v) - \langle \Pi_0, v\cdot n\rangle_\Gamma & \forall v \in X_{\mathscr{T}}, \\ (\mathrm{div}\,u_{\mathscr{T}}, q) = 0 & \forall q \in Y_{\mathscr{T}}. \end{cases} \tag{27}$$

1.3.3. A partial convergence result

In [7], we prove that the discrete problem (27) is well-posed when Γ_θ is contained in Γ_m. Naturally, there are also other technical hypotheses but they are more classical (regularity of the mesh family, regularity of the Laplace operator on the domain $\Omega \ldots$). In particular, the proof needs two inf–sup conditions which are recalled here and are proved in the first part of this work (see [7]). The condition Γ_θ contained in Γ_m does not seem to be difficult to achieve. In fact, most of the time, Γ_θ is empty.

Proposition 8 (Inf–sup condition on velocity and pressure). *Let us assume that Ω is polygonal and bounded, and that the mesh $\mathcal{T}$ belongs to a regular family of triangulations. Then, there exists a strictly positive constant a, independent of $h_{\mathcal{T}}$, such that*

$$\inf_{q_{\mathcal{T}} \in Y_{\mathcal{T}}} \sup_{v_{\mathcal{T}} \in X_{\mathcal{T}}} \frac{(q_{\mathcal{T}}, \operatorname{div} v_{\mathcal{T}})}{\|v_{\mathcal{T}}\|_{\operatorname{div},\Omega} \|q_{\mathcal{T}}\|_{0,\Omega}} \geqslant a. \tag{28}$$

Proposition 9 (Inf–sup condition on vorticity and velocity). *Let Ω be a simply connected domain. Let us assume that Γ_m has a strictly positive measure and that Γ_θ is contained in Γ_m. We denote by $V_{\mathcal{T}}$ the discrete kernel of the divergence operator (see (38)). Then, there exists a strictly positive constant b, independent of $h_{\mathcal{T}}$, such that*

$$\inf_{v_{RT} \in V_{\mathcal{T}}} \sup_{\varphi \in W_{\mathcal{T}}} \frac{(v_{RT}, \operatorname{curl} \varphi)}{\|v_{RT}\|_X \|\varphi\|_W} \geqslant b. \tag{29}$$

Remark 10. For the second inf–sup condition, we use the fact that for any vector field v_{RT} of $RT^0_{\mathcal{T}}$, divergence free, such that $v_{RT} \cdot n = 0$ on Γ_m and then on Γ_θ contained in Γ_m, there exists a scalar field φ in $W_{\mathcal{T}}$ such that $v_{RT} = \operatorname{curl} \varphi$ on Ω.

The problem of this formulation appears when we try to prove the convergence result obtained in [7] and that we recall here.

Theorem 11 (Convergence of the discrete variational formulation). *Let us recall the two partitions of the boundary*

$$\Gamma = \Gamma_m \cup \Gamma_p = \Gamma_\theta \cup \Gamma_t.$$

Then, we assume that Γ_m has a strictly positive measure and that Γ_θ is equal to Γ_m

$$\Gamma_\theta = \Gamma_m.$$

Moreover, we also assume that Ω is polygonal, bounded and simply connected and that the mesh $\mathcal{T}$ belongs to a regular family of triangulations.

Let (ω, u, p) be the solution in $W \times X \times Y$ of the continuous problem (19) and $(\omega_{\mathcal{T}}, u_{\mathcal{T}}, p_{\mathcal{T}})$ in space $W_{\mathcal{T}} \times X_{\mathcal{T}} \times Y_{\mathcal{T}}$, the solution of the discrete problem (27). We suppose that the solution is such that: $\omega \in H^2(\Omega)$, $u \in (H^1(\Omega))^2$, with $\operatorname{div} u \in H^1(\Omega)$, and $p \in H^1(\Omega)$. Then, there exists a strictly positive constant C, independent of the mesh, such that

$$\|\omega - \omega_{\mathcal{T}}\|_{1,\Omega} + \|u - u_{\mathcal{T}}\|_{div,\Omega} + \|p - p_{\mathcal{T}}\|_{0,\Omega}$$
$$\leqslant C h_{\mathcal{T}} \Big(|\omega|_{2,\Omega} + \|u\|_{1,\Omega} + \|\operatorname{div} u\|_{1,\Omega} + |p|_{1,\Omega} \Big).$$

To obtain this result, let us remark that we had to enforce the link between Γ_θ and Γ_m. More precisely, they must be equal: $\Gamma_\theta = \Gamma_m$. But this equality is clearly too restrictive. Let us observe that the reason of this hypothesis is that, to conclude on the convergence of the scheme, we need to set $v_{\mathcal{T}} = \operatorname{curl} \theta_{\mathcal{T}}$, with $v_{\mathcal{T}}$ in $X_{\mathcal{T}}$ and $\theta_{\mathcal{T}}$ in $W_{\mathcal{T}}$.

Moreover, numerical experiments, using this scheme, were performed but the results were not satisfying, when the equality $\Gamma_\theta = \Gamma_m$ is not true. On unstructured meshes, vorticity and pressure fields are not well approached. In particular, on a test proposed by Bercovier and Engelman [18], for which an analytical solution is known, we observe that values of vorticity and pressure are far from the expected ones along the boundary, even if the mesh is refined (see [7]).

Then, the aim of the following section is to build a velocity field which permits us to set, in a weaker sense, $v_{\mathcal{T}} = \operatorname{curl} \theta_{\mathcal{T}}$, with $v_{\mathcal{T}}$ in $X_{\mathcal{T}}$ and $\theta_{\mathcal{T}}$ in $W_{\mathcal{T}}$. Finally, let us add that numerical experiments have guided the choice of these extra velocity fields.

2. Numerical stabilization

2.1. Description of the bubble velocity functions

The problem is to build a velocity field which belongs to $H(\operatorname{div}, \Omega)$ and satisfies the boundary conditions. For any vertex S which is on the boundary of the domain Ω and for any triangle K for which S is a vertex, we define the following vector field:

$$w_S = B \operatorname{curl} \lambda_S, \tag{30}$$

where λ_S is the function associated with the barycentric coordinates relatively to S (see Fig. 3). Moreover, we set: $B = 60\lambda_1\lambda_2\lambda_3$, which is the "bubble" function on triangle K (in this case, to avoid heavy notation, the three vertices of K are denoted by 1, 2 and 3). We recall the classical formula

$$\int_K \lambda_1^n \lambda_2^m \lambda_3^p \,\mathrm{d}x = 2|K| \frac{n!m!p!}{(n+m+p+2)!} \tag{31}$$

for all integers n, m and p, where $|K|$ stands for the area of K. Then, it is easy to check that the multiplicative coefficient 60 is such that

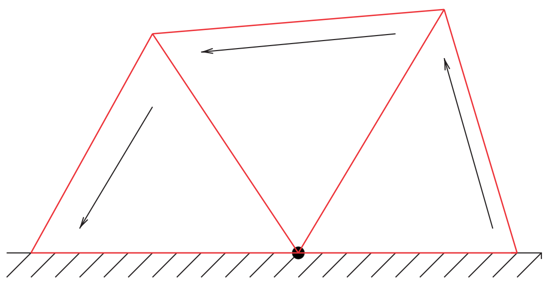

Fig. 3. Support of an added function.

$$\int_K B\,\mathrm{d}x = |K|. \tag{32}$$

This "bubble" function ensures, first, that functions w_S are zero on the boundary of Ω, and then satisfy the boundary conditions, and, second, that they are also zero on the edges of each triangle of their support. So their normal fluxes are zero and by the way are continuous across the edges. Consequently, the vector fields w_S belong to the continuous velocity space $X = \{v \in H(\mathrm{div}, \Omega)/v \cdot n = 0 \text{ on } \Gamma_m\}$ defined in (16).

Let us recall that, if $\theta_{\mathscr{T}}$ is a vorticity field, which is zero along Γ_θ, then the velocity field $v_{\mathscr{T}} = \mathrm{curl}\,\theta_{\mathscr{T}}$ belongs to $RT^0_{\mathscr{T}}$ and is such that $v_{\mathscr{T}} \cdot n = 0$ on Γ_θ. So, in practice, extra functions are only added on the part $\Gamma_m \backslash \Gamma_\theta$ of the boundary where normal velocity is prescribed but not the vorticity. Then, we set:

Definition 12 (*Space of extra velocity functions*). The space $X^S_{\mathscr{T}}$ of extra velocity functions v_S is spanned by the functions w_S associated to the vertices of the triangulation that are on $\Gamma_{m\backslash\theta} = \Gamma_m \backslash \Gamma_\theta$. Let us denote these vertices by S_i. Then, if $N(\mathscr{T}, \Gamma_{m\backslash\theta})$ is their number, it is equal to the dimension of space $X^S_{\mathscr{T}}$. Finally, each function v_S in $X^S_{\mathscr{T}}$ can be written

$$v_S = \sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})} \alpha_i w_{Si} = \sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})} \alpha_i B\,\mathrm{curl}\,\lambda_{Si}.$$

Remark 13. Let us observe that these extra fields v_S are not divergence free. However, due to Stokes formula, on each triangle K of their support, they obviously satisfy

$$\int_K \mathrm{div}\,v_S\,\mathrm{d}x = 0. \tag{33}$$

Now, we introduce the first degree polynomial function associated with v_S.

Definition 14. To any extra velocity functions v_S of $X^S_{\mathscr{T}}$, which can be written

$$v_S = \sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})} \alpha_i B\,\mathrm{curl}\,\lambda_{Si},$$

we associate the first degree polynomial function λ_S defined as

$$\lambda_S = \sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})} \alpha_i \lambda_{Si},$$

where λ_{Si} is the barycentric coordinate function relatively to node S_i.

We have the following relation between the norms of these vector fields

Lemma 15. *For any extra velocity functions v_S of $X^S_{\mathscr{T}}$, associated with the first degree polynomial function λ_S, we have*:

$$\|v_S\|_{0,\Omega} = \sqrt{\frac{10}{7}}\|\mathrm{curl}\,\lambda_S\|_{0,\Omega}.$$

Proof. Using the previous definition, we have

$$\begin{aligned}\|v_S\|^2_{0,\Omega} &= \left(\sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})} \alpha_i B\,\mathrm{curl}\,\lambda_{Si}, \sum_{j=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})} \alpha_j B\,\mathrm{curl}\,\lambda_{Sj}\right) \\ &= \sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})} \sum_{j=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})} \sum_{K \ni S_i, S_j} \alpha_i \alpha_j\,\mathrm{curl}\,\lambda_{Si}\,\mathrm{curl}\,\lambda_{Sj} \int_K B^2\,\mathrm{d}x,\end{aligned}$$

which leads to the result as $\int_K B^2\,\mathrm{d}x = \frac{10}{7}|K|$ (see (31)). $\square$

Then, using the equality (32), it is easy to check that for all triangle K

$$\int_K \mathrm{curl}\,\lambda_S\,\mathrm{d}x = |K|\,\mathrm{curl}\,\lambda_S = \int_K v_S\,\mathrm{d}x. \tag{34}$$

We have not exactly $v_{\mathscr{T}} = \mathrm{curl}\,\theta_{\mathscr{T}}$ with $v_{\mathscr{T}}$ in $X_{\mathscr{T}}$ and $\theta_{\mathscr{T}}$ in $W_{\mathscr{T}}$, but the equality is true in mean value on each triangle.

2.2. Stabilized variational formulation

Let us recall the expression of the continuous variational formulation

Find (ω, u, p) in $W \times X \times Y$ such that:

$$\begin{cases} (\omega, \varphi) - (\mathrm{curl}\,\varphi, u) = \langle \sigma_0, \gamma\varphi \rangle_\Gamma & \forall \varphi \in W, \\ (\mathrm{curl}\,\omega, v) - (p, \mathrm{div}\,v) = (f, v) - \langle \Pi_0, v \cdot n \rangle_\Gamma & \forall v \in X, \\ (\mathrm{div}\,u, q) = 0 & \forall q \in Y. \end{cases}$$

As we have the following imbeddings: $W_{\mathscr{T}} \subset W$, $X_{\mathscr{T}} \subset X$, $X^S_{\mathscr{T}} \subset X$ and $Y_{\mathscr{T}} \subset Y$, the discrete problem is directly deduced from the previous one and consists in finding $\omega_{\mathscr{T}} \in W_{\mathscr{T}}$, $u_{\mathscr{T}} = u_{RT} + u_S \in X_{\mathscr{T}} \oplus X^S_{\mathscr{T}}$ and $p_{\mathscr{T}} \in Y_{\mathscr{T}}$ such that

$$\begin{cases} (\omega_{\mathscr{T}}, \varphi) - (\mathrm{curl}\,\varphi, u_{\mathscr{T}}) = \langle \sigma_0, \gamma\varphi \rangle_\Gamma & \forall \varphi \in W_{\mathscr{T}}, \\ (\mathrm{curl}\,\omega_{\mathscr{T}}, v) - (p_{\mathscr{T}}, \mathrm{div}\,v) = (f, v) - \langle \Pi_0, v \cdot n \rangle_\Gamma & \forall v \in X_{\mathscr{T}} \oplus X^S_{\mathscr{T}}, \\ (\mathrm{div}\,u_{\mathscr{T}}, q) = 0 & \forall q \in Y_{\mathscr{T}}. \end{cases} \tag{35}$$

However, due to some basic properties of the extra velocity fields, this formulation will be slightly modified.

Let us begin by the following properties of the extra velocity fields.

Lemma 16. *For all q in $Y_{\mathscr{T}}$ and for all v_S in $X^S_{\mathscr{T}}$, we have* $(q, \mathrm{div}\,v_S) = 0$.

Proof. As q is constant on each triangle, we have

$$(q, \mathrm{div}\,v_S) = \sum_{K \in \mathscr{T}} q_{|K} \int_K \mathrm{div}\,v_S\,\mathrm{d}x = 0$$

as v_S is divergence free in mean value on each triangle (see (33)). $\square$

An immediate consequence of this lemma is that the third equation of (35) becomes

$$(\operatorname{div} u_{\mathcal{T}}, q) = (\operatorname{div} u_{RT}, q) + (\operatorname{div} u_S, q) = (\operatorname{div} u_{RT}, q) = 0 \tag{36}$$

for all q in $Y_{\mathcal{T}}$. In a similar way, the second equation of (35), written for the extra velocity fields, is

$$(\operatorname{curl} \omega_{\mathcal{T}}, v_S) - (p_{\mathcal{T}}, \operatorname{div} v_S) = (f, v_S) - \langle \Pi_0, v_S \cdot n \rangle_\Gamma$$

and, due to the above property and the fact that v_S are zero on the whole boundary Γ, it leads to:

$$(\operatorname{curl} \omega_{\mathcal{T}}, v_S) = (f, v_S) \tag{37}$$

for all v_S in $X_{\mathcal{T}}^S$.

This last equation will be modified in the following way. Let us recall that, in the original Stokes problem (1), the velocity appears through its Laplacian. Well, we have

$$-\Delta u = \operatorname{curl}(\operatorname{curl} u) - \nabla(\operatorname{div} u) = \operatorname{curl} \omega - \nabla(\operatorname{div} u).$$

As the velocity is divergence free, the term $\nabla(\operatorname{div} u)$ is dropped out. In the variational formulation, it would have led to the additional term $(\operatorname{div} u, \operatorname{div} v)$ (and also to the associated boundary term). The Raviart–Thomas part of the discrete velocity will be exactly divergence free because of (36) and of the following lemma, whose proof can be found in [7].

Lemma 17. *Let us define the discrete kernel of the divergence operator by*

$$V_{\mathcal{T}} = \{ v \in X_{\mathcal{T}} / (\operatorname{div} v, q) = 0 \text{ for all } q \in Y_{\mathcal{T}} \}. \tag{38}$$

Then, we have the following characterization of $V_{\mathcal{T}}$:

$$V_{\mathcal{T}} = \{ v \in X_{\mathcal{T}} / \operatorname{div} v = 0 \text{ on } \Omega \}. \tag{39}$$

But the extra velocity fields are not exactly divergence free and it seems natural to include an additional term $(\operatorname{div} u_S, \operatorname{div} v_S)$ in the discrete variational formulation (the associated boundary term is zero as v_S is zero on Γ), and more precisely in Eq. (37). For reasons which will clearly appear in the last Section, we prefer to add the following term:

$$[\operatorname{div} u_S, \operatorname{div} v_S] \equiv D \sum_{K \in \mathcal{T}} |K| \int_K \operatorname{div} u_S \operatorname{div} v_S \, dx, \tag{40}$$

where D is an arbitrary strictly positive scalar. This term looks like a penalization term as it appears in Galerkin-least-square method [19,10].

Remark 18. As the divergence of a Raviart–Thomas polynomial function is constant on each triangle, it is easy to check that, for all $u_{\mathcal{T}} = u_{RT} + u_S$ and for all $v_{\mathcal{T}} = v_{RT} + v_S$, we have

$$(\operatorname{div} u_{\mathcal{T}}, \operatorname{div} v_{\mathcal{T}}) = (\operatorname{div} u_{RT}, \operatorname{div} v_{RT}) + (\operatorname{div} u_S, \operatorname{div} v_S). \tag{41}$$

It is why, when we noticed above that the discrete formulation should have included an additional term $(\operatorname{div} u, \operatorname{div} v)$, only the part $(\operatorname{div} u_S, \operatorname{div} v_S)$ really occurs.

The practical value of coefficient D will be discussed further.

Now, we can state the *stabilized discrete variational formulation*: Find $\omega_{\mathcal{T}} \in W_{\mathcal{T}}$, $u_{\mathcal{T}} = u_{RT} + u_S \in X_{\mathcal{T}} \oplus X_{\mathcal{T}}^S$ and $p_{\mathcal{T}} \in Y_{\mathcal{T}}$ such that

$$\begin{cases} \forall \varphi \in W_{\mathcal{T}}, & (\omega_{\mathcal{T}}, \varphi) - (\operatorname{curl} \varphi, u_{RT}) - (\operatorname{curl} \varphi, u_S) \\ & \quad = \langle \sigma_0, \gamma\varphi \rangle_\Gamma, \\ \forall v_{RT} \in X_{\mathcal{T}}, & (\operatorname{curl} \omega_{\mathcal{T}}, v_{RT}) - (p_{\mathcal{T}}, \operatorname{div} v_{RT}) \\ & \quad = (f, v_{RT}) - \langle \Pi_0, v_{RT} \cdot n \rangle_\Gamma, \\ \forall v_S \in X_{\mathcal{T}}^S, & (\operatorname{curl} \omega_{\mathcal{T}}, v_S) + [\operatorname{div} u_S, \operatorname{div} v_S] = (f, v_S), \\ \forall q \in Y_{\mathcal{T}}, & (\operatorname{div} u_{RT}, q) = 0. \end{cases} \tag{42}$$

3. Convergence results

3.1. Preliminary results

In all this section, we have to suppose that the mesh $\mathcal{T}$ is uniformly regular (see Definition 1).

To study the convergence of the discrete solution towards the continuous one, we need some technical results. This point deals with the extra velocity fields and the term $[\operatorname{div} u_S, \operatorname{div} v_S]$. Let us recall that $h_{\mathcal{T}}$ is the maximal diameter of the triangles of $\mathcal{T}$. Then, we have the following fundamental property.

Proposition 19 (Property of the extra velocity fields). *Let us assume that the triangulation $\mathcal{T}$ is uniformly regular. Then, there exists two strictly positive constants C_1 and C_2 independent of $h_{\mathcal{T}}$ such that, for all v_S in $X_{\mathcal{T}}^S$*

$$C_1 \|v_S\|_{0,\Omega} \leqslant [[\operatorname{div} v_S]] \equiv [\operatorname{div} v_S, \operatorname{div} v_S]^{1/2} \leqslant C_2 \|v_S\|_{0,\Omega}. \tag{43}$$

Proof. As it plays no role in the following, the constant D, which appears in the definition of $[\operatorname{div} v_S, \operatorname{div} v_S]$ (see (40)), is dropped out in this proof.

The right inequality is a direct consequence of the classical inverse inequalities (see [14]) which needs the triangulation to be uniformly regular. So, we have

$$[[\operatorname{div} v_S]]^2 = \sum_{K \in \mathcal{E}_{\mathcal{T}}} |K| \|\operatorname{div} v_S\|_{0,K}^2 \leqslant h_{\mathcal{T}}^2 \|\operatorname{div} v_S\|_{0,\Omega}^2 \leqslant C_2 \|v_S\|_{0,\Omega}^2$$

as $|K|$ is smaller than $h_{\mathcal{T}}^2$. The constant C_2 is associated with the one which appears in the inverse inequality.

Following Definition 14, as any function v_S can be written

$$v_S = \sum_{i=1}^{N(\mathcal{T}, \Gamma_{m \setminus \theta})} \alpha_i B \operatorname{curl} \lambda_{Si},$$

we introduce the associated first degree polynomial function λ_S

$$\lambda_S = \sum_{i=1}^{N(\mathcal{T},\Gamma_{m\backslash 0})} \alpha_i \lambda_{Si}.$$

Then, on each triangle K, we have: $v_S = B\,\mathrm{curl}\,\lambda_S$, and consequently: $\mathrm{div}\, v_S = \nabla B \cdot \mathrm{curl}\,\lambda_S$. This leads to:

$$[[\mathrm{div}\, v_S]]^2 = \sum_{K\in\mathcal{E}_{\mathcal{T}}} |K| \int_K (\nabla B \cdot \mathrm{curl}\,\lambda_S)^2\, \mathrm{d}x. \tag{44}$$

Using (31), it is fairly easy to check the following equality on each K:

$$\int_K (\nabla B \cdot \mathrm{curl}\,\lambda_S)^2\, \mathrm{d}x = 10|K|\{(\nabla\lambda_1\, \mathrm{curl}\,\lambda_S)^2 + (\nabla\lambda_2\, \mathrm{curl}\,\lambda_S)^2 + (\nabla\lambda_3\, \mathrm{curl}\,\lambda_S)^2\}. \tag{45}$$

Here again, to simplify notation, λ_i, for $i \in \{1,2,3\}$, stands for the three usual barycentric coordinate functions on triangle K. Let us recall that we have also: $\|\nabla\lambda_i\| = \frac{l_i}{2|K|}$, where l_i is the length of the opposite side to vertex i. To simplify the demonstration, let us assume that l_1 is the longest side of K and let us work in a system of coordinates such that we have

$$\nabla\lambda_1 = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}, \quad \nabla\lambda_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}, \quad \nabla\lambda_3 = \begin{pmatrix} -\alpha_1 - \alpha_2 \\ -\beta_2 \end{pmatrix}.$$

As l_1 is the longest side, we obtain

$$\|\nabla\lambda_2\| = \sqrt{\alpha_2^2 + \beta_2^2} = \frac{l_2}{2|K|} \leqslant \frac{l_1}{2|K|} = |\alpha_1| = \|\nabla\lambda_1\|,$$

from which we deduce that

$$|\alpha_2| \leqslant |\alpha_1|. \tag{46}$$

Now, let us introduce the two vectors of $\mathbb{R}^3$ which are built on the two opposite sides to the vertices 1 and 2, say μ_1 and μ_2. The third component of these vectors is taken to 0. Then, the norm of the vector product $\mu_1 \times \mu_2$ is equal to twice the area of the triangle K. As we have: $\mu_i = 2|K|\nabla\lambda_i$ (if we set to 0 the third component of $\nabla\lambda_i$), we obtain

$$\|\nabla\lambda_1 \times \nabla\lambda_2\| = |\alpha_1\beta_2| = \frac{1}{2|K|}. \tag{47}$$

Let us denote by r_1 and r_2 the two components of $\mathrm{curl}\,\lambda_S$. Then, we deduce from (45)

$$\begin{aligned}\int_K (\nabla B \cdot \mathrm{curl}\,\lambda_S)^2\, \mathrm{d}x &\geqslant 10|K|\left\{(\nabla\lambda_1\, \mathrm{curl}\,\lambda_S)^2 + (\nabla\lambda_2\, \mathrm{curl}\,\lambda_S)^2\right\} \\ &\geqslant 10|K|\left\{(\alpha_1 r_1)^2 + (\alpha_2 r_1 + \beta_2 r_2)^2\right\}.\end{aligned}$$

Let us set: $Q = (\alpha_1 r_1)^2 + (\alpha_2 r_1 + \beta_2 r_2)^2$. Then, we have also

$$\begin{aligned} Q &= (\alpha_1^2 - \alpha_2^2) r_1^2 + 2\alpha_2^2 r_1^2 + 2\alpha_2\beta_2 r_1 r_2 + \beta_2^2 r_2^2 \\ &\geqslant (\alpha_1^2 - \alpha_2^2) r_1^2 + 2\alpha_2^2 r_1^2 - 2|\alpha_2\beta_2 r_1 r_2| + \beta_2^2 r_2^2 \\ &\geqslant (\alpha_1^2 - \alpha_2^2) r_1^2 + \frac{1}{2}\alpha_2^2 r_1^2 + \frac{1}{3}\beta_2^2 r_2^2, \end{aligned}$$

by using the following inequality: $2|ab| \leqslant \frac{3}{2}a^2 + \frac{2}{3}b^2$. So we obtain

$$Q \geqslant \frac{1}{2}\alpha_1^2 r_1^2 + \frac{1}{2}(\alpha_1^2 - \alpha_2^2) r_1^2 + \frac{1}{3}\beta_2^2 r_2^2 \geqslant \frac{1}{3}(\alpha_1^2 r_1^2 + \beta_2^2 r_2^2),$$

with (46). Finally, as $|\alpha_1| = \frac{l_1}{2|K|}$ and, with (47), $|\beta_2| = \frac{1}{l_1}$, it leads to:

$$\int_K (\nabla B \cdot \mathrm{curl}\,\lambda_S)^2\, \mathrm{d}x \geqslant \frac{10}{3}|K|\left(\left(\frac{l_1}{2|K|}\right)^2 r_1^2 + \left(\frac{1}{l_1}\right)^2 r_2^2\right).$$

Introducing this relation in (44) gives

$$\begin{aligned}[[\mathrm{div}\, v_S]]^2 &\geqslant \frac{10}{3}\sum_{K\in\mathcal{E}_{\mathcal{T}}} |K|^2 \left(\frac{l_1^2}{4|K|^2} r_1^2 + \frac{1}{l_1^2} r_2^2\right) \\ &\geqslant \frac{5}{6}\sum_{K\in\mathcal{E}_{\mathcal{T}}} |K| \left(\frac{l_1^2}{|K|} r_1^2 + \frac{|K|}{l_1^2} r_2^2\right).\end{aligned}$$

As the triangulation is uniformly regular, and $l_1 = h_K$, we have

$$\frac{l_1^2}{|K|} \geqslant \frac{h_K^2}{h_{\mathcal{T}}^2} \geqslant \tau^2 \quad \text{and} \quad \frac{|K|}{l_1^2} \geqslant \frac{\rho_K^2}{2h_K^2} \geqslant \frac{1}{2\sigma^2}.$$

Thus, there exists a strictly positive constant C independent of $h_{\mathcal{T}}$ such that

$$\begin{aligned}[[\mathrm{div}\, v_S]]^2 &\geqslant C\sum_{K\in\mathcal{E}_{\mathcal{T}}} |K|(r_1^2 + r_2^2) = C\sum_{K\in\mathcal{E}_{\mathcal{T}}} \|\mathrm{curl}\,\lambda_S\|_{0,K}^2 \\ &= C\|\mathrm{curl}\,\lambda_S\|_{0,\Omega}^2 = \frac{7}{10}C\|v_S\|_{0,\Omega}^2.\end{aligned}$$

with Lemma 15, which achieves the proof. □

Now, let us give two properties of the first degree polynomial functions λ_S introduced in Definition 14

$$\lambda_S = \sum_{i=1}^{N(\mathcal{T},\Gamma_{m\backslash 0})} \alpha_i \lambda_{Si}.$$

Before all, let us remark that, by definition of the barycentric coordinates functions, we have: $\alpha_i = \lambda_S(S_i)$.

Lemma 20. *Let us assume that the triangulation $\mathcal{T}$ is uniformly regular. Then, there exists two strictly positive constants C_1 and C_2 independent of $h_{\mathcal{T}}$ such that, for all function $\lambda_S = \sum_{i=1}^{N(\mathcal{T},\Gamma_{m\backslash 0})} \alpha_i\lambda_{Si}$, we have*

$$C_1\sqrt{h_{\mathcal{T}}}\left[\sum_{i=1}^{N(\mathcal{T},\Gamma_{m\backslash 0})} \lambda_S^2(S_i)\right]^{\frac{1}{2}} \leqslant \|\lambda_S\|_{0,\Gamma} \leqslant C_2\sqrt{h_{\mathcal{T}}}\left[\sum_{i=1}^{N(\mathcal{T},\Gamma_{m\backslash 0})} \lambda_S^2(S_i)\right]^{\frac{1}{2}}.$$

Proof. Let us denote by $\mathcal{A}_{\mathcal{T}}(\Gamma_{m\backslash 0})$ the set of all triangles edges which are contained in $\Gamma_m\backslash\Gamma_0$. Then, by definition, we have

$$\|\lambda_S\|_{0,\Gamma}^2 = \sum_{A\in\mathscr{A}_{\mathscr{T}}(\Gamma_{m\backslash\theta})} \int_A \lambda_S^2 \,\mathrm{d}\gamma.$$

As λ_S is a first degree polynomial function, λ_S^2 is a polynomial function of degree 2. Then, using Simpson's formula, which is exact for third degree polynomial function, we obtain

$$\|\lambda_S\|_{0,\Gamma}^2 = \sum_{A\in\mathscr{A}_{\mathscr{T}}(\Gamma_{m\backslash\theta})} \frac{|A|}{6}\left[\lambda_S^2(a) + 4\lambda_S^2\left(\frac{a+b}{2}\right) + \lambda_S^2(b)\right], \tag{48}$$

where we have set $A=[a,b]$, $|A|$ its length and $\frac{a+b}{2}$ its middle.

A direct consequence of (48) is

$$\begin{aligned}\|\lambda_S\|_{0,\Gamma}^2 &\geqslant \sum_{A\in\mathscr{A}_{\mathscr{T}}(\Gamma_{m\backslash\theta})} \frac{|A|}{6}\left[\lambda_S^2(a)+\lambda_S^2(b)\right]\\ &\geqslant \frac{1}{6}\min_{A\in\mathscr{A}_{\mathscr{T}}(\Gamma_{m\backslash\theta})}|A| \sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})}\lambda_S^2(S_i).\end{aligned}$$

A being one edge of a triangle K of $\mathscr{T}$, $|A|$ is greater than the diameter ρ_K of the circle inscribed in K. Then, as the mesh is uniformly regular, with (20), we obtain

$$|A| \geqslant \rho_K \geqslant \frac{\tau}{\sigma} h_{\mathscr{T}}.$$

So, there exists a strictly positive constant C_1 independent of $h_{\mathscr{T}}$ such that

$$\|\lambda_S\|_{0,\Gamma}^2 \geqslant C_1 h_{\mathscr{T}} \sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})}\lambda_S^2(S_i).$$

To obtain the other inequality, we proceed as follows. As λ_S is a first degree polynomial function, we have

$$\lambda_S\left(\frac{a+b}{2}\right) = \frac{1}{2}(\lambda_S(a)+\lambda_S(b)).$$

Then, using $2ab\leqslant a^2+b^2$, we easily obtain the overestimation

$$\lambda_S^2\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\left(\lambda_S^2(a)+\lambda_S^2(b)\right).$$

Now, let us introduce the above inequality in (48). It leads to:

$$\begin{aligned}\|\lambda_S\|_{0,\Gamma}^2 &\leqslant \sum_{A\in\mathscr{A}_{\mathscr{T}}(\Gamma_{m\backslash\theta})} \frac{|A|}{2}\left(\lambda_S^2(a)+\lambda_S^2(b)\right)\\ &\leqslant \max_{A\in\mathscr{A}_{\mathscr{T}}(\Gamma_{m\backslash\theta})}|A| \sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})}\lambda_S^2(S_i).\end{aligned}$$

As $|A|$ is smaller than the triangle diameter h_K, which is smaller than $h_{\mathscr{T}}$, we obtain the second inequality. □

Proposition 21. *Let us assume that the triangulation $\mathscr{T}$ is uniformly regular. Then, there exists a strictly positive constant C independent of $h_{\mathscr{T}}$ such that, for all function $\lambda_S = \sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})}\alpha_i\lambda_{S_i}$, we have*

$$\|\mathrm{curl}\,\lambda_S\|_{0,\Omega} \leqslant \frac{C}{\sqrt{h_{\mathscr{T}}}}\|\lambda_S\|_{0,\Gamma}.$$

Proof. First, let us observe that, because λ_S is a first degree polynomial function, its curl is constant on each triangle. So we have

$$\|\mathrm{curl}\,\lambda_S\|_{0,\Omega}^2 = \sum_K |K||\mathrm{curl}\,\lambda_{S|K}|^2.$$

Let us remark that the integrals on K are all zero except if K contains a vertex of $\Gamma_m\backslash\Gamma_\theta$. By the way, we have also

$$\mathrm{curl}\,\lambda_{S|K} = \sum_{S_i\in K}\lambda_S(S_i)\,\mathrm{curl}\,\lambda_{S_i} = \sum_{S_i\in K}\lambda_S(S_i)\frac{v_{S_i}}{2|K|},$$

where λ_{S_i} is the barycentric coordinate function associated with the vertex S_i and v_{S_i} is the vector associated to the opposite edge of K, relatively to S_i. Let us set l_{S_i} the length of v_{S_i}. Then, by definition of $h_{\mathscr{T}}$, we obtain

$$|\mathrm{curl}\,\lambda_{S|K}|^2 \leqslant C\sum_{S_i\in K}\lambda_S^2(S_i)\frac{l_{S_i}^2}{|K|^2} \leqslant C\frac{h_{\mathscr{T}}^2}{|K|^2}\sum_{S_i\in K}\lambda_S^2(S_i),$$

where C is a constant only dependent on the number of vertices of K. This estimate and the previous inequality lead to:

$$\|\mathrm{curl}\,\lambda_S\|_{0,\Omega}^2 \leqslant C\sum_K\frac{h_{\mathscr{T}}^2}{|K|}\sum_{S_i\in K}\lambda_S^2(S_i) \leqslant C\sum_K\sum_{S_i\in K}\lambda_S^2(S_i)$$

as the triangulation is uniformly regular. Let N be the maximum number of elements containing a vertex. When a triangulation is uniformly regular, this number is bounded independently of $h_{\mathscr{T}}$ and we have

$$\|\mathrm{curl}\,\lambda_S\|_{0,\Omega}^2 \leqslant CN\sum_{S_i}\lambda_S^2(S_i),$$

where the summation is done on all the vertices of the mesh. In our particular case, $\lambda_S^2(S_i)$ is zero except when S_i is on $\Gamma_m\backslash\Gamma_\theta$. So we have

$$\|\mathrm{curl}\,\lambda_S\|_{0,\Omega}^2 \leqslant CN\sum_{i=1}^{N(\mathscr{T},\Gamma_{m\backslash\theta})}\lambda_S^2(S_i)$$

and the result is a consequence of Lemma 20. □

Let us now analyse a term which will appear as a consistency error in the sequel (for more details on consistency see [20]).

Proposition 22 (General estimate of the consistency error). *Let us assume that the triangulation $\mathscr{T}$ is uniformly regular and that the pressure p solution of the Stokes problem belongs to $W^{1,\infty}(\Omega)$. Then, for all v_S in $X_{\mathscr{T}}^S$, there exists a strictly positive constant C independent of $h_{\mathscr{T}}$ such that*

$$|(p,\mathrm{div}\,v_S)| \leqslant C\sqrt{h_{\mathscr{T}}}|p|_{1,\infty,\Omega}\|v_S\|_{0,\Omega}. \tag{49}$$

Proof. The argument of the proof is the same as the one Scholz [21] used for the biLaplacian operator. First, we observe that any function v_S has a support reduced to the set of triangles which are connected with $\Gamma_m \backslash \Gamma_\theta$, say $\Sigma_{\mathscr{T}}$. So, we have

$$(p, \operatorname{div} v_S) = \sum_{K \in \Sigma_{\mathscr{T}}} \int_K p \operatorname{div} v_S \, \mathrm{d}x$$
$$= \sum_{K \in \Sigma_{\mathscr{T}}} \int_K (p - \Pi^0_{\mathscr{T}} p) \operatorname{div} v_S \, \mathrm{d}x$$

with Lemma 16. We recall that $\Pi^0_{\mathscr{T}}$ is the L^2 projection operator on space $Y_{\mathscr{T}}$. Moreover, if p belongs to $W^{1,\infty}(\Omega)$, there exists a strictly positive constant C, independent of h_K, such that for all triangle K (see e.g. [17])

$$\|p - \Pi^0_{\mathscr{T}} p\|_{0,K} \leqslant C|K|^{1/2} h_K |p|_{1,\infty,K}.$$

Then, we obtain

$$\begin{aligned}
|(p, \operatorname{div} v_S)| &\leqslant \sum_{K \in \Sigma_{\mathscr{T}}} \|p - \Pi^0_{\mathscr{T}} p\|_{0,K} \|\operatorname{div} v_S\|_{0,K} \\
&\leqslant \sum_{K \in \Sigma_{\mathscr{T}}} C|K|^{1/2} h_K |p|_{1,\infty,K} \|\operatorname{div} v_S\|_{0,K} \\
&\leqslant C|p|_{1,\infty,\Omega} \sum_{K \in \Sigma_{\mathscr{T}}} |K|^{1/2} h_K \|\operatorname{div} v_S\|_{0,K} \\
&\leqslant C|p|_{1,\infty,\Omega} \sum_{K \in \Sigma_{\mathscr{T}}} |K|^{1/2} \|v_S\|_{0,K},
\end{aligned}$$

using the inverse inequality (which is possible as the triangulation is uniformly regular). Finally, using the Cauchy–Schwarz inequality, we deduce

$$|(p, \operatorname{div} v_S)| \leqslant C|p|_{1,\infty,\Omega} \left(\sum_{K \in \Sigma_{\mathscr{T}}} |K| \right)^{1/2} \|v_S\|_{0,\Sigma_{\mathscr{T}}},$$

which leads to the announced result as, first, $\|v_S\|_{0,\Sigma_{\mathscr{T}}}$ and $\|v_S\|_{0,\Omega}$ are equal and, second: $\sum_{K \in \Sigma_{\mathscr{T}}} |K| = |\Sigma_{\mathscr{T}}| = \mathcal{O}(h_{\mathscr{T}})$. $\square$

3.2. A new convergence result

First of all, we can prove that the stabilized discrete problem (42) is well-posed.

Proposition 23 (Well-posedness of the stabilized variational formulation). *Let Ω be polygonal, bounded, and simply connected domain. Let us recall the two partitions of the boundary*

$$\Gamma = \Gamma_m \cup \Gamma_p = \Gamma_\theta \cup \Gamma_t.$$

Then, we assume that Γ_m has a strictly positive measure and that Γ_θ is contained in Γ_m. Finally, we suppose that the mesh $\mathscr{T}$ belongs to a uniformly regular family of triangulation.

Then, the discrete problem which consists in finding $(\omega_{\mathscr{T}}, u_{RT} + u_S, p_{\mathscr{T}})$ in $W_{\mathscr{T}} \times X_{\mathscr{T}} \oplus X^S_{\mathscr{T}} \times Y_{\mathscr{T}}$ such that

$$\begin{cases}
\forall \varphi \in W_{\mathscr{T}}, & (\omega_{\mathscr{T}}, \varphi) - (\operatorname{curl} \varphi, u_{RT}) - (\operatorname{curl} \varphi, u_S) \\
& \quad = \langle \sigma_0, \gamma\varphi \rangle_\Gamma \\
\forall v_{RT} \in X_{\mathscr{T}}, & (\operatorname{curl} \omega_{\mathscr{T}}, v_{RT}) - (p_{\mathscr{T}}, \operatorname{div} v_{RT}) \\
& \quad = (f, v_{RT}) - \langle \Pi_0, v_{RT} \cdot n \rangle_\Gamma \\
\forall v_S \in X^S_{\mathscr{T}}, & (\operatorname{curl} \omega_{\mathscr{T}}, v_S) + [\operatorname{div} u_S, \operatorname{div} v_S] = (f, v_S) \\
\forall q \in Y_{\mathscr{T}}, & (\operatorname{div} u_{RT}, q) = 0,
\end{cases}$$

has a unique solution.

Proof. The proof is very close to the one we did in [7] for the non-stabilized problem (27). First, let us observe that the hypotheses are such that the two inf–sup conditions (28) and (29) are true. Second, as we consider a finite-dimensional square linear system, the only point is to prove that the solution associated with σ_0, f and Π_0 equal to zero, is zero. For this, in the above system, we choose $\varphi = \omega_{\mathscr{T}}$, $v_{RT} = u_{RT}$, $v_S = u_S$ and $q = p_{\mathscr{T}}$, and we add the four equations. We obtain

$$(\omega_{\mathscr{T}}, \omega_{\mathscr{T}}) + [\operatorname{div} u_S, \operatorname{div} u_S] = 0,$$

which implies $\omega_{\mathscr{T}} = 0$ and $\operatorname{div} u_S = 0$. So u_S is also equal to zero because of (43). Then, the second equation becomes

$$(p_{\mathscr{T}}, \operatorname{div} v_{RT}) = 0 \quad \forall v_{RT} \in X_{\mathscr{T}}.$$

Then, using the inf–sup condition (28), we deduce that $p_{\mathscr{T}} = 0$. Finally, the last equation shows that u_{RT} belongs to the kernel $V_{\mathscr{T}}$, and the first one becomes

$$(\operatorname{curl} \varphi, u_{RT}) = 0 \quad \forall \varphi \in W_{\mathscr{T}}$$

as $\omega_{\mathscr{T}} = 0$. So u_{RT} is zero thanks to the inf–sup condition (29). $\square$

We can now study the stability of the stabilized discrete problem. So, let (ω, u, p) be the solution in $W \times X \times Y$ of the continuous problem

$$\begin{cases}
(\omega, \varphi) - (\operatorname{curl} \varphi, u) = \langle \sigma_0, \gamma\varphi \rangle_\Gamma & \forall \varphi \in W, \\
(\operatorname{curl} \omega, v) - (p, \operatorname{div} v) = (f, v) - \langle \Pi_0, v \cdot n \rangle_\Gamma & \forall v \in X, \\
(\operatorname{div} u, q) = 0 & \forall q \in Y,
\end{cases}$$

and $(\omega_{\mathscr{T}}, u_{RT} + u_S, p_{\mathscr{T}})$ in $W_{\mathscr{T}} \times X_{\mathscr{T}} \oplus X^S_{\mathscr{T}} \times Y_{\mathscr{T}}$ the solution of the stabilized discrete problem

$$\begin{cases}
\forall \varphi_{\mathscr{T}} \in W_{\mathscr{T}}, & (\omega_{\mathscr{T}}, \varphi_{\mathscr{T}}) - (\operatorname{curl} \varphi_{\mathscr{T}}, u_{RT}) - (\operatorname{curl} \varphi_{\mathscr{T}}, u_S) = \langle \sigma_0, \gamma\varphi_{\mathscr{T}} \rangle_\Gamma, \\
\forall v_{RT} \in X_{\mathscr{T}}, & (\operatorname{curl} \omega_{\mathscr{T}}, v_{RT}) - (p_{\mathscr{T}}, \operatorname{div} v_{RT}) = (f, v_{RT}) - \langle \Pi_0, v_{RT} \cdot n \rangle_\Gamma, \\
\forall v_S \in X^S_{\mathscr{T}}, & (\operatorname{curl} \omega_{\mathscr{T}}, v_S) + [\operatorname{div} u_S, \operatorname{div} v_S] = (f, v_S), \\
\forall q_{\mathscr{T}} \in Y_{\mathscr{T}}, & (\operatorname{div} u_{RT}, q_{\mathscr{T}}) = 0.
\end{cases}$$

As the discrete spaces $W_{\mathscr{T}}, X_{\mathscr{T}}, X^S_{\mathscr{T}}$ and $Y_{\mathscr{T}}$ are respectively contained in the continuous ones W, X (twice) and Y, we can take $\varphi = \varphi_{\mathscr{T}}$, $v = v_{RT}$, $v = v_S$ and $q = q_{\mathscr{T}}$ in the continuous problem. It means that the second equation of the continuous problem is written for each type of velocity vector field. Then, subtracting each corresponding equations in the two systems, we obtain

$$\begin{cases} (\omega-\omega_{\mathcal{T}},\varphi_{\mathcal{T}})-(u-u_{RT},\operatorname{curl}\varphi_{\mathcal{T}})+(u_S,\operatorname{curl}\varphi_{\mathcal{T}})=0 & \forall\varphi_{\mathcal{T}}\in W_{\mathcal{T}},\\ (\operatorname{curl}(\omega-\omega_{\mathcal{T}}),v_{RT})-(p-p_{\mathcal{T}},\operatorname{div}v_{RT})=0 & \forall v_{RT}\in X_{\mathcal{T}},\\ (\operatorname{curl}(\omega-\omega_{\mathcal{T}}),v_S)-[\operatorname{div}u_S,\operatorname{div}v_S]-(p,\operatorname{div}v_S)=0 & \forall v_S\in X^S_{\mathcal{T}},\\ (\operatorname{div}(u-u_{RT}),q_{\mathcal{T}})=0 & \forall q_{\mathcal{T}}\in Y_{\mathcal{T}} \end{cases}$$

The term $(p,\operatorname{div}v_S)$ which appears in the third equation is the consistency error term. Let us now introduce the interpolants on the mesh $\mathcal{T}$ of each field. Let us remark that we assume that the solution is smooth enough in order that these interpolants are well-defined. For the vorticity field, we denote by $\Pi^1_{\mathcal{T}}$ the classical Lagrange interpolation operator. For the velocity field, the interpolation operator in $H(\operatorname{div},\Omega)$ is $\Pi^{\operatorname{div}}_{\mathcal{T}}$ (see Definition 3). Finally, the pressure field is interpolated by the use of the L^2-projection operator on space $Y_{\mathcal{T}}$, say $\Pi^0_{\mathcal{T}}$. Then, we have for each equation:

- *First equation*. For all $\varphi_{\mathcal{T}}$ in $W_{\mathcal{T}}$
$$(\omega_{\mathcal{T}}-\Pi^1_{\mathcal{T}}\omega,\varphi_{\mathcal{T}})-(u_{RT}-\Pi^{\operatorname{div}}_{\mathcal{T}}u,\operatorname{curl}\varphi_{\mathcal{T}})-(u_S,\operatorname{curl}\varphi_{\mathcal{T}}) = (\omega-\Pi^1_{\mathcal{T}}\omega,\varphi_{\mathcal{T}})-(u-\Pi^{\operatorname{div}}_{\mathcal{T}}u,\operatorname{curl}\varphi_{\mathcal{T}}).$$
- *Second equation*. For all v_{RT} in $X_{\mathcal{T}}$:
$$(\operatorname{curl}(\omega_{\mathcal{T}}-\Pi^1_{\mathcal{T}}\omega),v_{RT})-(p_{\mathcal{T}}-\Pi^0_{\mathcal{T}}p,\operatorname{div}v_{RT}) = (\operatorname{curl}(\omega-\Pi^1_{\mathcal{T}}\omega),v_{RT})-(p-\Pi^0_{\mathcal{T}}p,\operatorname{div}v_{RT}).$$
- *Third equation*. For all v_S in $X^S_{\mathcal{T}}$:
$$(\operatorname{curl}(\omega_{\mathcal{T}}-\Pi^1_{\mathcal{T}}\omega),v_S)+[\operatorname{div}u_S,\operatorname{div}v_S] = (\operatorname{curl}(\omega-\Pi^1_{\mathcal{T}}\omega),v_S)-(p,\operatorname{div}v_S).$$
- *Fourth equation*. For all $q_{\mathcal{T}}$ in $Y_{\mathcal{T}}$:
$$(\operatorname{div}(u_{RT}-\Pi^{\operatorname{div}}_{\mathcal{T}}u),q_{\mathcal{T}})=(\operatorname{div}(u-\Pi^{\operatorname{div}}_{\mathcal{T}}u),q_{\mathcal{T}}).$$

Let us remark that this last equation becomes

$$(\operatorname{div}(u_{RT}-\Pi^{\operatorname{div}}_{\mathcal{T}}u),q_{\mathcal{T}})=0$$

for all $q_{\mathcal{T}}$ in $Y_{\mathcal{T}}$ because $(\operatorname{div}(u-\Pi^{\operatorname{div}}_{\mathcal{T}}u),q_{\mathcal{T}})=0$ (see Proposition 7). Let us observe that it supposes that u belongs to $(H^1(\Omega))^2$, which is also needed for the existence of the interpolant. This regularity condition on u can be relaxed as long as the result of Proposition 7 remains (see [16,7]). Nevertheless, in the following, for the error estimates, we shall assume that u belongs to $(H^1(\Omega))^2$. Finally, the following auxiliary problem appears:

Find $(\theta_{\mathcal{T}},w_{RT},w_S,r_{\mathcal{T}})$ in $W_{\mathcal{T}}\times X_{\mathcal{T}}\times X^S_{\mathcal{T}}\times Y_{\mathcal{T}}$ such that:

$$\begin{cases} \forall\varphi_{\mathcal{T}}\in W_{\mathcal{T}}, & (\theta_{\mathcal{T}},\varphi_{\mathcal{T}})-(w_{RT},\operatorname{curl}\varphi_{\mathcal{T}})-(w_S,\operatorname{curl}\varphi_{\mathcal{T}})\\ & \quad =(f,\varphi_{\mathcal{T}})+(g,\operatorname{curl}\varphi_{\mathcal{T}}),\\ \forall v_{RT}\in X_{\mathcal{T}}, & (\operatorname{curl}\theta_{\mathcal{T}},v_{RT})-(r_{\mathcal{T}},\operatorname{div}v_{RT})\\ & \quad =(k,v_{RT})+(l,\operatorname{div}v_{RT}),\\ \forall v_S\in X^S_{\mathcal{T}}, & (\operatorname{curl}\theta_{\mathcal{T}},v_S)+[\operatorname{div}w_S,\operatorname{div}v_S]\\ & \quad =(k,v_S)-(p,\operatorname{div}v_S),\\ \forall q_{\mathcal{T}}\in Y_{\mathcal{T}}, & (\operatorname{div}w_{RT},q_{\mathcal{T}})=0, \end{cases} \tag{50}$$

where we have set

- $f=\omega-\Pi^1_{\mathcal{T}}\omega$, which belongs to $L^2(\Omega)$;
- $g=-u+\Pi^{\operatorname{div}}_{\mathcal{T}}u$, which belongs to $(L^2(\Omega))^2$;
- $k=\operatorname{curl}(\omega-\Pi^1_{\mathcal{T}}\omega)$, which is in $(L^2(\Omega))^2$;
- $l=-p+\Pi^0_{\mathcal{T}}p$, which is in $L^2(\Omega)$.

Now, we can prove a partial stability result, in the general case.

Proposition 24 (Partial stability of the discrete variational formulation). *Let Ω be a polygonal, bounded, and simply connected domain of $\mathbb{R}^2$. Let us recall the two partitions of the boundary: $\Gamma=\Gamma_m\cup\Gamma_p=\Gamma_\theta\cup\Gamma_t$. Then, we assume that Γ_m has a strictly positive measure and that Γ_θ is contained in Γ_m:*

$$\Gamma_\theta\subset\Gamma_m.$$

Finally, we suppose that the mesh $\mathcal{T}$ belongs to a uniformly regular family of triangulation and that the pressure p, solution of the Stokes problem, belongs to $W^{1,\infty}(\Omega)$.

Then, the problem (50) is well-posed and there exists a strictly positive constant C, independent of the mesh, such that

$$\|\theta_{\mathcal{T}}\|^2_{0,\Omega}+\|w_{RT}\|^2_X+\|w_S\|^2_{0,\Omega} \leqslant C\Bigg(\|f\|^2_{0,\Omega}+\|k\|^2_{0,\Omega}+\|g\|^2_{0,\Omega}+h_{\mathcal{T}}|p|^2_{1,\infty,\Omega} + \frac{\|g\|_{0,\Omega}\|k\|_{0,\Omega}}{\sqrt{h_{\mathcal{T}}}}+\frac{\|g\|^2_{0,\Omega}}{h_{\mathcal{T}}}+\|g\|_{0,\Omega}|p|_{1,\infty,\Omega}\Bigg).$$

Proof. We observe that the hypotheses are such that the two inf–sup conditions (28) and (29) are true. Then, exactly as in Proposition 23, the problem (50) is well-posed. Moreover, the fourth equation of (50) shows that w_{RT} is divergence free (see Proposition 17). Then, we have: $\|w_{RT}\|_X=\|w_{RT}\|_{0,\Omega}$. Finally, we recall that, in two dimension, we have

$$\|\theta_{\mathcal{T}}\|^2_W=\|\theta_{\mathcal{T}}\|^2_{0,\Omega}+\|\operatorname{curl}\theta_{\mathcal{T}}\|^2_{0,\Omega}.$$

So, the proof of the inequality is given in four steps, in which, as usual, C will denote various constants, independent of the mesh.

First step. We take $\varphi_{\mathcal{T}}=\theta_{\mathcal{T}}$, $v_{RT}=w_{RT}$, $v_S=w_S$ and $q_{\mathcal{T}}=r_{\mathcal{T}}$ in (50). As w_{RT} is divergence free, after adding the four equations, we obtain

$$\begin{aligned}\|\theta_{\mathcal{T}}\|^2_{0,\Omega}+[[\operatorname{div}w_S]]^2 &= (f,\theta_{\mathcal{T}})+(g,\operatorname{curl}\theta_{\mathcal{T}})+(k,w_{RT})\\ &\quad +(k,w_S)-(p,\operatorname{div}w_S)\\ &\leqslant \|f\|_{0,\Omega}\|\theta_{\mathcal{T}}\|_{0,\Omega}+\|k\|_{0,\Omega}\|w_{RT}\|_{0,\Omega}\\ &\quad +\|k\|_{0,\Omega}\|w_S\|_{0,\Omega}+|(g,\operatorname{curl}\theta_{\mathcal{T}})|\\ &\quad +|(p,\operatorname{div}w_S)|.\end{aligned}$$

Then, using the classical inequality: $ab \leqslant \frac{1}{2}(a^2+b^2)$, and the equivalence between the two norms $[[\operatorname{div} w_S]]$ and $\|w_S\|_{0,\Omega}$ (see (43)), we deduce

$$\|\theta_{\mathcal{T}}\|_{0,\Omega}^2 + \|w_S\|_{0,\Omega}^2 \leqslant C\Big(\|f\|_{0,\Omega}^2 + \|k\|_{0,\Omega}^2 + \|k\|_{0,\Omega}\|w_{RT}\|_{0,\Omega} + |(g, \operatorname{curl}\theta_{\mathcal{T}})| + |(p, \operatorname{div} w_S)|\Big). \tag{51}$$

Second step. We apply the inf–sup condition (29) to w_{RT}, which is divergence free, in the first equation of (50). We deduce

$$b\|w_{RT}\|_X \leqslant \sup_{\varphi\in W_{\mathcal{T}}} \frac{(w_{RT}, \operatorname{curl}\varphi)}{\|\varphi\|_W} \leqslant \sup_{\varphi\in W_{\mathcal{T}}} \frac{(\theta_{\mathcal{T}}, \varphi) - (f, \varphi) - (g, \operatorname{curl}\varphi) - (w_S, \operatorname{curl}\varphi)}{\|\varphi\|_W}.$$

As the norm in W is the norm in $H^1(\Omega)$, we obtain

$$\|w_{RT}\|_X \leqslant C(\|\theta_{\mathcal{T}}\|_{0,\Omega} + \|f\|_{0,\Omega} + \|g\|_{0,\Omega} + \|w_S\|_{0,\Omega}), \tag{52}$$

where the constant C is equal to $1/b$, in this case.

Third step. The previous inequality and (51) lead to:

$$\|\theta_{\mathcal{T}}\|_{0,\Omega}^2 + \|w_S\|_{0,\Omega}^2 \leqslant C(\|f\|_{0,\Omega}^2 + \|k\|_{0,\Omega}^2 + \|k\|_{0,\Omega}(\|\theta_{\mathcal{T}}\|_{0,\Omega} + \|f\|_{0,\Omega} + \|g\|_{0,\Omega} + \|w_S\|_{0,\Omega}) + |(g, \operatorname{curl}\theta_{\mathcal{T}})| + |(p, \operatorname{div} w_S)|)$$

or else, using the classical inequality: $2ab \leqslant \frac{a^2}{\varepsilon} + \varepsilon b^2$, true for all strictly positive number ε, we obtain

$$\|\theta_{\mathcal{T}}\|_{0,\Omega}^2 + \|w_S\|_{0,\Omega}^2 \leqslant C\Big(\|f\|_{0,\Omega}^2 + \|k\|_{0,\Omega}^2 + \|g\|_{0,\Omega}^2 + \frac{1}{2\varepsilon}\Big(\|\theta_{\mathcal{T}}\|_{0,\Omega}^2 + \|w_S\|_{0,\Omega}^2\Big) + |(g, \operatorname{curl}\theta_{\mathcal{T}})| + |(p, \operatorname{div} w_S)|\Big).$$

Finally, choosing ε equal to C in the above inequality, we have

$$\|\theta_{\mathcal{T}}\|_{0,\Omega}^2 + \|w_S\|_{0,\Omega}^2 \leqslant C\Big(\|f\|_{0,\Omega}^2 + \|k\|_{0,\Omega}^2 + \|g\|_{0,\Omega}^2 + |(g, \operatorname{curl}\theta_{\mathcal{T}})| + |(p, \operatorname{div} w_S)|\Big). \tag{53}$$

So, the two inequalities (52) and (53) lead to:

$$\|\theta_{\mathcal{T}}\|_{0,\Omega}^2 + \|w_S\|_{0,\Omega}^2 + \|w_{RT}\|_X^2 \leqslant C\Big(\|f\|_{0,\Omega}^2 + \|k\|_{0,\Omega}^2 + \|g\|_{0,\Omega}^2 + |(g, \operatorname{curl}\theta_{\mathcal{T}})| + |(p, \operatorname{div} w_S)|\Big). \tag{54}$$

Now, we use the result, obtained in the analysis of the consistency error term (see (49)), which is

$$|(p, \operatorname{div} w_S)| \leqslant C\sqrt{h_{\mathcal{T}}}|p|_{1,\infty,\Omega}\|w_S\|_{0,\Omega}.$$

Then, inequality (54) becomes

$$\|\theta_{\mathcal{T}}\|_{0,\Omega}^2 + \|w_S\|_{0,\Omega}^2 + \|w_{RT}\|_X^2 \leqslant C\Big(\|f\|_{0,\Omega}^2 + \|k\|_{0,\Omega}^2 + \|g\|_{0,\Omega}^2 + |(g, \operatorname{curl}\theta_{\mathcal{T}})| + \sqrt{h_{\mathcal{T}}}|p|_{1,\infty,\Omega}\|w_S\|_{0,\Omega}\Big)$$

and, using classical arguments, we obtain

$$\|\theta_{\mathcal{T}}\|_{0,\Omega}^2 + \|w_S\|_{0,\Omega}^2 + \|w_{RT}\|_X^2 \leqslant C\Big(\|f\|_{0,\Omega}^2 + \|k\|_{0,\Omega}^2 + \|g\|_{0,\Omega}^2 + h_{\mathcal{T}}|p|_{1,\infty,\Omega}^2 + |(g, \operatorname{curl}\theta_{\mathcal{T}})|\Big). \tag{55}$$

Fourth step. This step is the longest. We have to study and overestimate the term: $|(g, \operatorname{curl}\theta_{\mathcal{T}})|$. First, we split $\theta_{\mathcal{T}}$ as $\theta_{\mathcal{T}} = \theta_{\mathcal{T}}^0 + \theta_{\mathcal{T}}^b$, where $\theta_{\mathcal{T}}^0$ is the "interior" part of $\theta_{\mathcal{T}}$; more precisely, $\theta_{\mathcal{T}}^0(S)$ is equal to 0 if the node S is on the part of the boundary $\Gamma_m\backslash\Gamma_\theta$ and, in the other case, $\theta_{\mathcal{T}}^0(S) = \theta_{\mathcal{T}}(S)$. So $\theta_{\mathcal{T}}^b$ can be seen as the "boundary" part of $\theta_{\mathcal{T}}$. Now, we can take $v_{RT} = \operatorname{curl}\theta_{\mathcal{T}}^0$ in the second equation of (50), as $\operatorname{curl}\theta_{\mathcal{T}}^0$ belongs to $X_{\mathcal{T}}$ (see [7]). And, $\operatorname{curl}\theta_{\mathcal{T}}^0$ being divergence free, we obtain

$$(\operatorname{curl}\theta_{\mathcal{T}}, \operatorname{curl}\theta_{\mathcal{T}}^0) = (k, \operatorname{curl}\theta_{\mathcal{T}}^0). \tag{56}$$

Then, in the third equation of (50), we choose for v_S the extra velocity field associated with $\theta_{\mathcal{T}}^b$. We shall denote it by: $v_S = B\operatorname{curl}\theta_{\mathcal{T}}^b$, and we obtain

$$(\operatorname{curl}\theta_{\mathcal{T}}, B\operatorname{curl}\theta_{\mathcal{T}}^b) + [\operatorname{div} w_S, \operatorname{div}(B\operatorname{curl}\theta_{\mathcal{T}}^b)] = (k, B\operatorname{curl}\theta_{\mathcal{T}}^b) - (p, \operatorname{div}(B\operatorname{curl}\theta_{\mathcal{T}}^b)). \tag{57}$$

Let us observe that we have

$$(\operatorname{curl}\theta_{\mathcal{T}}, B\operatorname{curl}\theta_{\mathcal{T}}^b) = \sum_{K\in\Sigma_{\mathcal{T}}}\int_K \operatorname{curl}\theta_{\mathcal{T}} B\operatorname{curl}\theta_{\mathcal{T}}^b\,\mathrm{d}x = \sum_{K\in\Sigma_{\mathcal{T}}}\int_K \operatorname{curl}\theta_{\mathcal{T}}\operatorname{curl}\theta_{\mathcal{T}}^b\,\mathrm{d}x = (\operatorname{curl}\theta_{\mathcal{T}}, \operatorname{curl}\theta_{\mathcal{T}}^b)$$

as $\theta_{\mathcal{T}}$ and $\theta_{\mathcal{T}}^b$ are first order polynomial functions and the "bubble" function B has been chosen as its integral on any triangle is equal to 1. We deduce that

$$(\operatorname{curl}\theta_{\mathcal{T}}, B\operatorname{curl}\theta_{\mathcal{T}}^b) + (\operatorname{curl}\theta_{\mathcal{T}}, \operatorname{curl}\theta_{\mathcal{T}}^0) = (\operatorname{curl}\theta_{\mathcal{T}}, \operatorname{curl}\theta_{\mathcal{T}}^b + \operatorname{curl}\theta_{\mathcal{T}}^0) = \|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega}^2.$$

Finally, by adding equalities (56) and (57), the previous equation gives

$$\begin{aligned}\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega}^2 &= (k, \operatorname{curl}\theta_{\mathcal{T}}^0 + B\operatorname{curl}\theta_{\mathcal{T}}^b) - (p, \operatorname{div}(B\operatorname{curl}\theta_{\mathcal{T}}^b)) - [\operatorname{div} w_S, \operatorname{div}(B\operatorname{curl}\theta_{\mathcal{T}}^b)]\\ &= (k, \operatorname{curl}\theta_{\mathcal{T}}) + (k, (B-1)\operatorname{curl}\theta_{\mathcal{T}}^b) - (p, \operatorname{div}(B\operatorname{curl}\theta_{\mathcal{T}}^b)) - [\operatorname{div} w_S, \operatorname{div}(B\operatorname{curl}\theta_{\mathcal{T}}^b)].\end{aligned}$$

Then, using the Cauchy–Schwarz inequality, the fact that $(B-1)$ is bounded independently of $h_{\mathcal{T}}$, the analysis of the consistency error term again (see (49)) and the equivalence between the two norms $[[\operatorname{div} w_S]]$ and $\|w_S\|_{0,\Omega}$, we obtain the following inequality:

$$\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega}^2 \leqslant C\Big(\|k\|_{0,\Omega}\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega}+\|k\|_{0,\Omega}\|\operatorname{curl}\theta_{\mathcal{T}}^b\|_{0,\Omega} + \sqrt{h_{\mathcal{T}}}|p|_{1,\infty,\Omega}\|\operatorname{curl}\theta_{\mathcal{T}}^b\|_{0,\Omega}+\|w_S\|_{0,\Omega}\|\operatorname{curl}\theta_{\mathcal{T}}^b\|_{0,\Omega}\Big). \tag{58}$$

Moreover, applying Proposition 21 to $\theta_{\mathcal{T}}^b$, we deduce that

$$\|\operatorname{curl}\theta_{\mathcal{T}}^b\|_{0,\Omega} \leqslant \frac{C}{\sqrt{h_{\mathcal{T}}}}\|\theta_{\mathcal{T}}^b\|_{0,\Gamma} \leqslant \frac{C}{\sqrt{h_{\mathcal{T}}}}\|\theta_{\mathcal{T}}\|_{0,\Gamma}$$

because, on the part of the boundary $\Gamma_m\backslash\Gamma_\theta$, $\theta_{\mathcal{T}}^b$ and $\theta_{\mathcal{T}}$ are equal, while the first one is zero on the other part of the boundary. Next, using the continuity of the trace application from $H^1(\Omega)$ to $L^2(\Gamma)$, there exists a constant C independent of the mesh size such that

$$\|\operatorname{curl}\theta_{\mathcal{T}}^b\|_{0,\Omega} \leqslant \frac{C}{\sqrt{h_{\mathcal{T}}}}\|\theta_{\mathcal{T}}\|_{1,\Omega}.$$

Introducing this result in (58), we obtain

$$\begin{aligned}\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega}^2 &\leqslant C\left(\|k\|_{0,\Omega}\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega}+\frac{\|\theta_{\mathcal{T}}\|_{1,\Omega}}{\sqrt{h_{\mathcal{T}}}}\left(\|k\|_{0,\Omega}+\sqrt{h_{\mathcal{T}}}|p|_{1,\infty,\Omega}+\|w_S\|_{0,\Omega}\right)\right)\\ &\leqslant C\left(\frac{\varepsilon}{2}\|k\|_{0,\Omega}^2+\frac{1}{2\varepsilon}\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega}^2+\frac{1}{2\varepsilon}\|\theta_{\mathcal{T}}\|_{1,\Omega}^2 +\frac{\varepsilon}{2h_{\mathcal{T}}}\left(\|k\|_{0,\Omega}+\sqrt{h_{\mathcal{T}}}|p|_{1,\infty,\Omega}+\|w_S\|_{0,\Omega}\right)^2\right).\end{aligned}$$

With an appropriate choice of ε, and using the definition of the norm in $H^1(\Omega)$, we finally deduce

$$\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega}^2 \leqslant C\left(\|k\|_{0,\Omega}^2+\|\theta_{\mathcal{T}}\|_{0,\Omega}^2+\frac{1}{h_{\mathcal{T}}}\left(\|k\|_{0,\Omega}^2+h_{\mathcal{T}}|p|_{1,\infty,\Omega}^2+\|w_S\|_{0,\Omega}^2\right)\right),$$

which leads to:

$$\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega} \leqslant C\left(\|k\|_{0,\Omega}+\|\theta_{\mathcal{T}}\|_{0,\Omega}+\frac{\|k\|_{0,\Omega}}{\sqrt{h_{\mathcal{T}}}}+|p|_{1,\infty,\Omega}+\frac{\|w_S\|_{0,\Omega}}{\sqrt{h_{\mathcal{T}}}}\right). \tag{59}$$

Now, it is easy to overestimate the term $|(g,\operatorname{curl}\theta_{\mathcal{T}})|$

$$\begin{aligned}|(g,\operatorname{curl}\theta_{\mathcal{T}})| &\leqslant \|g\|_{0,\Omega}\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega}\\ &\leqslant C\left(\|g\|_{0,\Omega}\|k\|_{0,\Omega}\left(1+\frac{1}{\sqrt{h_{\mathcal{T}}}}\right)+\|g\|_{0,\Omega}|p|_{1,\infty,\Omega} +\left\|g\|_{0,\Omega}\|\theta_{\mathcal{T}}\|_{0,\Omega}+\|g\|_{0,\Omega}\frac{\|w_S\|_{0,\Omega}}{\sqrt{h_{\mathcal{T}}}}\right)\\ &\leqslant C\left(\|g\|_{0,\Omega}\|k\|_{0,\Omega}\left(1+\frac{1}{\sqrt{h_{\mathcal{T}}}}\right)+\|g\|_{0,\Omega}|p|_{1,\infty,\Omega} +\frac{1}{2\varepsilon}(\|\theta_{\mathcal{T}}\|_{0,\Omega}^2+\|w_S\|_{0,\Omega}^2)+\frac{\varepsilon}{2}\|g\|_{0,\Omega}^2\left(1+\frac{1}{h_{\mathcal{T}}}\right)\right).\end{aligned}$$

This inequality given in the proposition is an obvious consequence of (55) and this result. □

We can now state a partial convergence result, related to the previous proposition.

Theorem 25 (Convergence of the discrete variational formulation). *Let Ω be a polygonal, bounded, and simply connected domain of $\mathbb{R}^2$. Let us recall the two partitions of the boundary: $\Gamma=\Gamma_m\cup\Gamma_p=\Gamma_\theta\cup\Gamma_t$. Then, we assume that Γ_m has a strictly positive measure and that Γ_θ is contained in Γ_m*

$$\Gamma_\theta\subset\Gamma_m.$$

Finally, we suppose that the mesh $\mathcal{T}$ belongs to a uniformly regular family of triangulations and that the mesh size $h_{\mathcal{T}}$ is small enough: $h_{\mathcal{T}}\leqslant 1$.

Let (ω,u,p) be the solution in $W\times X\times Y$ of the continuous problem (19) and $(\omega_{\mathcal{T}},u_{RT}+u_S,p_{\mathcal{T}})$ in $W_{\mathcal{T}}\times X_{\mathcal{T}}\oplus X_{\mathcal{T}}^S\times Y_{\mathcal{T}}$ the solution of the stabilized discrete problem (42). We suppose that the solution is such that: $\omega\in H^2(\Omega)$, $u\in(H^1(\Omega))^2$ and $p\in W^{1,\infty}(\Omega)$. Then, there exists a strictly positive constant C, independent of the mesh, such that

$$\|\omega-\omega_{\mathcal{T}}\|_{0,\Omega}+\|u-u_{RT}\|_{\operatorname{div},\Omega}+\|u_S\|_{0,\Omega}\leqslant C\sqrt{h_{\mathcal{T}}}. \tag{60}$$

Moreover, we have also

$$\|\operatorname{curl}\omega-\operatorname{curl}\omega_{\mathcal{T}}\|_{0,\Omega}\leqslant C, \tag{61}$$

$$\|p-p_{\mathcal{T}}\|_{0,\Omega}\leqslant C, \tag{62}$$

where C are various strictly positive constants independent of the mesh.

Proof. First, let us recall the basic inequalities

$$\begin{aligned}\|\omega-\omega_{\mathcal{T}}\|_{1,\Omega} &\leqslant \|\omega-\Pi_{\mathcal{T}}^1\omega\|_{1,\Omega}+\|\Pi_{\mathcal{T}}^1\omega-\omega_{\mathcal{T}}\|_{1,\Omega},\\ \|u-u_{RT}\|_{\operatorname{div},\Omega} &\leqslant \|u-\Pi_{\mathcal{T}}^{\operatorname{div}}u\|_{\operatorname{div},\Omega}+\|\Pi_{\mathcal{T}}^{\operatorname{div}}u-u_{RT}\|_{\operatorname{div},\Omega},\\ \|p-p_{\mathcal{T}}\|_{0,\Omega} &\leqslant \|p-\Pi_{\mathcal{T}}^0 p\|_{0,\Omega}+\|\Pi_{\mathcal{T}}^0 p-p_{\mathcal{T}}\|_{0,\Omega}.\end{aligned} \tag{63}$$

In these relations, the first terms are well-known: they are the classical interpolation errors. And the second terms are precisely the solutions of the auxiliary problem (50) where we have

$$\theta_{\mathcal{T}}=\omega_{\mathcal{T}}-\Pi_{\mathcal{T}}^1\omega,\quad w_{RT}=u_{RT}-\Pi_{\mathcal{T}}^{\operatorname{div}}u,\quad r_{\mathcal{T}}=p_{\mathcal{T}}-\Pi_{\mathcal{T}}^0 p$$

and $w_S=u_S$. Moreover, in (50), we had set: $f=\omega-\Pi_{\mathcal{T}}^1\omega$, $g=-u+\Pi_{\mathcal{T}}^{\operatorname{div}}u$, $k=\operatorname{curl}(\omega-\Pi_{\mathcal{T}}^1\omega)$ and $l=-p+\Pi_{\mathcal{T}}^0 p$. Then, using the interpolation errors recalled in Theorems 2, 4 and 6, we obtain the existence of a constant C, independent of the mesh size, such that

$$\|f\|_{0,\Omega}+\|g\|_{0,\Omega}+\|k\|_{0,\Omega}+\|l\|_{0,\Omega}\leqslant Ch_{\mathcal{T}}. \tag{64}$$

We can notice that the pressure p belongs to $H^1(\Omega)$ since it belongs to $W^{1,\infty}(\Omega)$ and Ω is bounded. So Theorem 6 is true. Using this inequality in Proposition 24 ensures that there exists a strictly positive constant C, independent of the mesh, such that

$$\|\theta_{\mathcal{T}}\|_{0,\Omega}^2+\|w_{RT}\|_X^2+\|w_S\|_{0,\Omega}^2\leqslant C\left(h_{\mathcal{T}}^2+h_{\mathcal{T}}^{3/2}+h_{\mathcal{T}}\right)\leqslant Ch_{\mathcal{T}} \tag{65}$$

as $h_{\mathcal{T}}$ is assumed smaller than 1. This inequality can also be written as

$$\|\omega_{\mathcal{T}} - \Pi^1_{\mathcal{T}}\omega\|^2_{0,\Omega} + \|u_{RT} - \Pi^{\mathrm{div}}_{\mathcal{T}}u\|^2_X + \|u_S\|^2_{0,\Omega} \leqslant Ch_{\mathcal{T}}.$$

Finally, using (63) and Theorems 2 and 4, we obtain

$$\|\omega - \omega_{\mathcal{T}}\|^2_{0,\Omega} + \|u - u_{RT}\|^2_X + \|u_S\|^2_{0,\Omega} \leqslant Ch_{\mathcal{T}},$$

which obviously leads to the inequality (60) given in the theorem, as u, u_{RT} and $\Pi^{\mathrm{div}}_{\mathcal{T}}u$ are divergence free.

Second, let us recall the following inequality, obtained in the proof of Proposition 24

$$\|\mathrm{curl}\,\theta_{\mathcal{T}}\|_{0,\Omega} \leqslant C\left(\|k\|_{0,\Omega} + \|\theta_{\mathcal{T}}\|_{0,\Omega} + \frac{\|k\|_{0,\Omega}}{\sqrt{h_{\mathcal{T}}}} + |p|_{1,\infty,\Omega} + \frac{\|w_S\|_{0,\Omega}}{\sqrt{h_{\mathcal{T}}}}\right).$$

Then, we deduce from (60), (64) and (65) that

$$\|\mathrm{curl}\,\theta_{\mathcal{T}}\|_{0,\Omega} \leqslant C\left(h_{\mathcal{T}} + \sqrt{h_{\mathcal{T}}} + |p|_{1,\infty,\Omega} + 1\right).$$

The inequality (61) is a direct consequence of the definition of $\theta_{\mathcal{T}}$, the first inequality of (63) and this result, as $h_{\mathcal{T}}$ is smaller than 1.

Finally, let us use the inf–sup condition (28) in the second equation of (50). We obtain

$$a\|r_{\mathcal{T}}\|_{0,\Omega} \leqslant \sup_{v\in X_{\mathcal{T}}} \frac{(\mathrm{div}\,v, r_{\mathcal{T}})}{\|v\|_X} \leqslant \sup_{v\in X_{\mathcal{T}}} \frac{(\mathrm{curl}\,\theta_{\mathcal{T}}, v) - (l, \mathrm{div}\,v) - (k, v)}{\|v\|_X}.$$

Using the fact that the norm in X is the norm in $H(\mathrm{div},\Omega)$, we deduce that

$$\|r_{\mathcal{T}}\|_{0,\Omega} \leqslant C\left(\|\mathrm{curl}\,\theta_{\mathcal{T}}\|_{0,\Omega} + \|l\|_{0,\Omega} + \|k\|_{0,\Omega}\right). \tag{66}$$

Let us recall that: $r_{\mathcal{T}} = p_{\mathcal{T}} - \Pi^0_{\mathcal{T}}p$. Then, using the third inequality (63), (60) and (64), we obtain

$$\|p - p_{\mathcal{T}}\|_{0,\Omega} \leqslant C\left(h_{\mathcal{T}} + \|\mathrm{curl}\,\theta_{\mathcal{T}}\|_{0,\Omega}\right), \tag{67}$$

which lead to the inequality (62). □

To conclude this subsection, let us observe that this result is far to be optimal. By the way, a convergence of $\mathcal{O}(\sqrt{h_{\mathcal{T}}})$ for the vorticity in quadratic norm is very classical on a convex domain (see [17,21]). The only point we have improved is the fact that we do not need the convexity and that $\mathrm{curl}\,(\omega - \omega_{\mathcal{T}})$ is bounded. It seems to be very poor but the numerical results are much better, as it will appear in the next section. Finally, some complements to this theorem will be given further.

4. Numerical experiments

4.1. Bercovier–Engelman test case

The first numerical experiments have been performed on a unit square with an analytical solution proposed by Bercovier and Engelman [18]. The velocity is zero on the whole boundary Γ and there is no boundary condition on the pressure and the vorticity. So, $\Gamma_m = \Gamma$ has a strictly positive measure and Γ_θ, which is empty, is contained in Γ_m. Then, the hypothesis on the boundary, needed in the previous theorem, is true. Finally, the exact pressure p is a polynomial function equal to

$$p(x,y) = \left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right),$$

on the domain, so obviously belongs to $W^{1,\infty}(\Omega)$.

Fig. 4 gives the numerical results we obtained on triangular unstructured meshes, with the classical numerical scheme, while Fig. 5 gives the results we obtained on the same meshes with our stabilization. If the convergence rate on the velocity remains the same, it varies from 0.41 to 1.36 for the quadratic norm of the vorticity, and from 0.40 to 0.65 for the pressure. Very surprisingly, the *curl* of the vorticity, which is not bounded in the classical case, becomes convergent with a slope close to 1, with the stabilized scheme! Moreover, as far as numerical values of the fields are concerned, we had noticed in [7] that both vorticity and pressure explode along the boundary. More precisely, for the vorticity, the numerical maximum is 27.8 instead of 16, which is the analytical solution. And, pressure varies

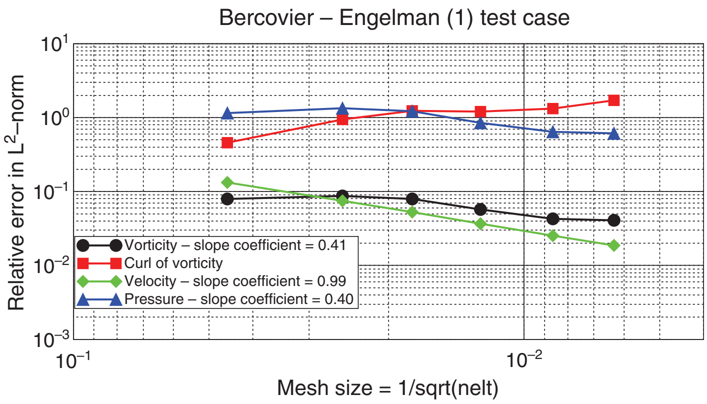

Fig. 4. Convergence order without stabilization – Bercovier–Engelman's test.

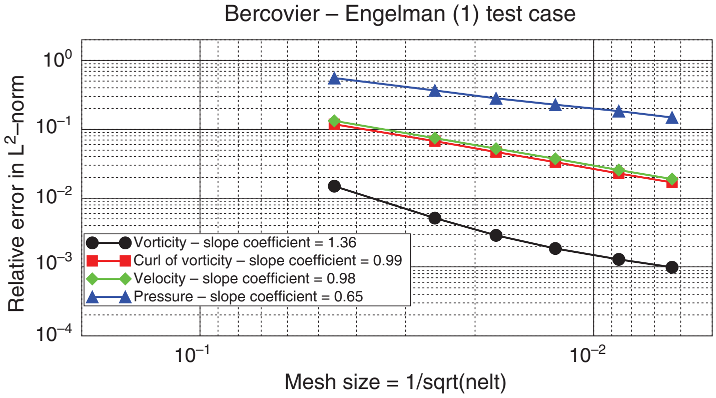

Fig. 5. Convergence order with stabilization – Bercovier–Engelman's test.

between −7.67 and 6.44 instead of −0.25 and 0.25, while the quadratic error on the pressure remains at a very important level: more than 200%. With the stabilized scheme, these numerical explosions disappear: the maximum of the vorticity becomes 15.97, the pressure varies from −0.18 and 0.19 and the error on the finest mesh is close to 10%.

4.2. *Ruas test case*

Then, we have worked on a circle with an analytical solution proposed by Ruas [22]. The boundary conditions are exactly the same as in the previous case and the exact pressure p is constant (equal to 1) on the whole domain, so is as regular as needed. For sake of symmetry, we work on a quarter of the domain. Then, the "bubble" velocities are added only on the circular part of the boundary. The numerical results, we obtained with the classical scheme, are given in Fig. 6.

Then, Fig. 7 gives the results of the stabilized scheme. Here again, we observe that the *curl* of the vorticity, which is not bounded in the classical case (Fig. 6), is convergent with a slope close to 1, with the stabilized scheme! Moreover, there is also a kind of super-convergence on the velocity and on the pressure (slope close to 2). We shall try to explain these results in the next sections.

5. Extensions and particular cases

5.1. *An improved convergence result*

As we have seen previously on the convergence curves, the *curl* of the vorticity is numerically convergent, with a slope close to 1, with the stabilized scheme, even if we have not obtained any convergence result. To try to understand this surprising behaviour, we were induced to make a new hypothesis. To do that, let us recall two results we obtained during the proof of Proposition 24

$$\|\theta_{\mathcal{T}}\|^2_{0,\Omega} + \|w_S\|^2_{0,\Omega} + \|w_{RT}\|^2_X \leqslant C\left(\|f\|^2_{0,\Omega} + \|k\|^2_{0,\Omega} + \|g\|^2_{0,\Omega} + h_{\mathcal{T}}|p|^2_{1,\infty,\Omega} + |(g, \operatorname{curl}\theta_{\mathcal{T}})|\right) \tag{68}$$

and

$$\|\operatorname{curl}\theta_{\mathcal{T}}\|^2_{0,\Omega} \leqslant C\Big(\|k\|_{0,\Omega}\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega} + \|k\|_{0,\Omega}\|\operatorname{curl}\theta^b_{\mathcal{T}}\|_{0,\Omega} + \sqrt{h_{\mathcal{T}}}|p|_{1,\infty,\Omega}\|\operatorname{curl}\theta^b_{\mathcal{T}}\|_{0,\Omega} + \|w_S\|_{0,\Omega}\|\operatorname{curl}\theta^b_{\mathcal{T}}\|_{0,\Omega}\Big). \tag{69}$$

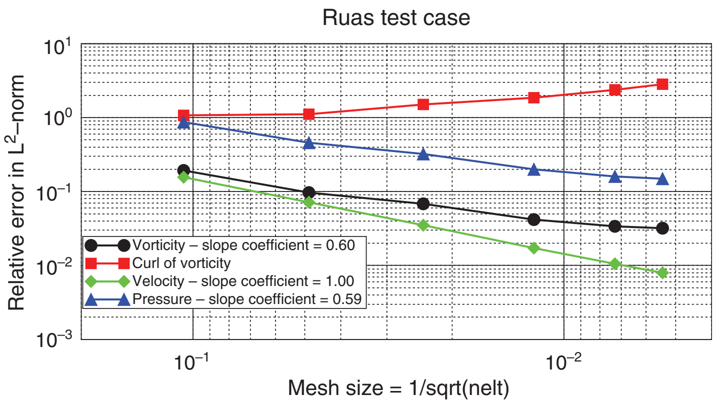

Fig. 6. Convergence without stabilization – test proposed by Ruas.

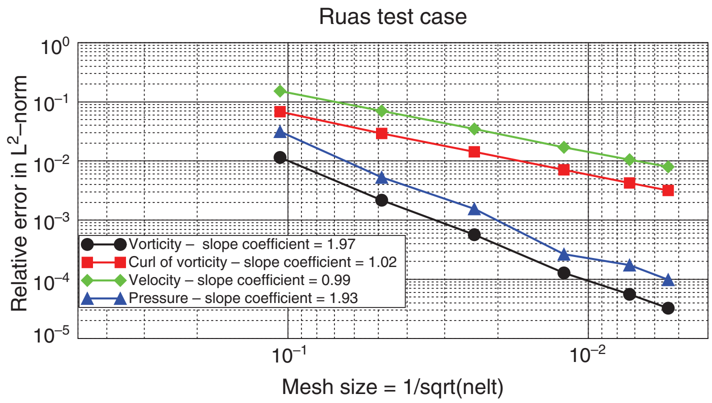

Fig. 7. Convergence with stabilization – test proposed by Ruas.

Let us also recall that we have split $\theta_{\mathscr{T}}$ as $\theta_{\mathscr{T}} = \theta^0_{\mathscr{T}} + \theta^b_{\mathscr{T}}$, where $\theta^0_{\mathscr{T}}$ is the "interior" part of $\theta_{\mathscr{T}}$ while $\theta^b_{\mathscr{T}}$ can be seen as the "boundary" part of $\theta_{\mathscr{T}}$.

Now, *we will assume that there exists a strictly positive constant C, independent of the mesh, such that*

$$\|\operatorname{curl}\theta^b_{\mathscr{T}}\|_{0,\Omega} \leqslant C\|\operatorname{curl}\theta_{\mathscr{T}}\|_{0,\Omega}. \tag{70}$$

Thanks to this inequality, the result (69) leads to:

$$\|\operatorname{curl}\theta_{\mathscr{T}}\|_{0,\Omega} \leqslant C\Big(\|k\|_{0,\Omega} + \sqrt{h_{\mathscr{T}}}|p|_{1,\infty,\Omega} + \|w_S\|_{0,\Omega}\Big), \tag{71}$$

which allows to obtain a new overestimate of $|(g, \operatorname{curl}\theta_{\mathscr{T}})|$. Using classical arguments, with (68), we finally have

$$\|\theta_{\mathscr{T}}\|^2_{0,\Omega} + \|w_S\|^2_{0,\Omega} + \|w_{RT}\|^2_X \leqslant C\Big(\|f\|^2_{0,\Omega} + \|k\|^2_{0,\Omega} + \|g\|^2_{0,\Omega} + h_{\mathscr{T}}|p|^2_{1,\infty,\Omega}\Big).$$

Then, following the proof of Theorem 25, and under the same assumptions, we obtain the existence of a strictly positive constant C, independent of the mesh, such that

$$\|\omega - \omega_{\mathscr{T}}\|_{1,\Omega} + \|u - u_{RT}\|_{\operatorname{div},\Omega} + \|u_S\|_{0,\Omega} + \|p - p_{\mathscr{T}}\|_{0,\Omega} \leqslant C\sqrt{h_{\mathscr{T}}}. \tag{72}$$

Compared to Theorem 25, the gain does not seem very obvious. Nevertheless, we have obtained the convergence on the *curl* of the vorticity and on the pressure, even if it is not optimal. Moreover, this convergence result can appear as more relevant when we examine Fig. 5: the numerical convergence rate on the pressure is close to 0.65, and the rate, which is obtained for the vorticity, is close to 1.36. This last result could be interpreted as a consequence of the Aubin–Nitsche lemma as the domain Ω is convex (see e.g. Ciarlet [14]).

Remark 26. Let us discuss briefly on hypothesis (70). First, even if the numerical convergence may lead us to this assumption, we cannot suppose that: $\|\operatorname{curl}\theta^b_{\mathscr{T}}\|_{0,\Omega} \leqslant Ch_{\mathscr{T}}$, with C independent of the mesh. If this inequality was true, because of (67), the convergence rate on the pressure should also be 1 and it is far to be the case. Second, an inequality as (70) is certainly not true in the general case: it is enough to take $\theta_{\mathscr{T}}$ equal to a constant. Nevertheless, this case, which is the worst for (70), is the best for the convergence theory (we have then: $\|\operatorname{curl}\theta_{\mathscr{T}}\|_{0,\Omega} = 0$!). Moreover, the study of the numerical convergence on the vorticity shows that the numerical problems occur near the boundary (see [7]). Then assuming that the "interior" part of $\theta_{\mathscr{T}}$ is negligible leads to something like (70).

5.2. *Choice of the numerical coefficient of the term [$\operatorname{div} u_S, \operatorname{div} v_S$]*

An interesting consequence of the previous theory deals with the choice of the numerical coefficient D, which was introduced in the definition (40) as

$$[\operatorname{div} u_S, \operatorname{div} v_S] = D\sum_{K\in\mathscr{T}}|K|\int_K \operatorname{div} u_S \operatorname{div} v_S \,\mathrm{d}x,$$

and had no influence until now. To understand the way to chose it, let us recall some estimates, which were proved above and in which we shall make the scalar D appear. First, we had (see (54))

$$\|\theta_{\mathscr{T}}\|^2_{0,\Omega} + D\|w_S\|^2_{0,\Omega} + \|w_{RT}\|^2_X \leqslant C\Big(\|f\|^2_{0,\Omega} + \|k\|^2_{0,\Omega} + \|g\|^2_{0,\Omega} + |(g, \operatorname{curl}\theta_{\mathscr{T}})| + |(p, \operatorname{div} w_S)|\Big). \tag{73}$$

In a similar manner, under the hypothesis (70), the inequality (71) becomes

$$\|\operatorname{curl}\theta_{\mathscr{T}}\|_{0,\Omega} \leqslant C\Big(\|k\|_{0,\Omega} + \sqrt{h_{\mathscr{T}}}|p|_{1,\infty,\Omega} + D\|w_S\|_{0,\Omega}\Big).$$

Finally, we recall the result, obtained in the analysis of the consistency error term (see (49))

$$|(p, \operatorname{div} w_S)| \leqslant C\sqrt{h_{\mathscr{T}}}|p|_{1,\infty,\Omega}\|w_S\|_{0,\Omega}.$$

Introducing this two inequalities in (73), we obtain

$$\|\theta_{\mathscr{T}}\|^2_{0,\Omega} + D\|w_S\|^2_{0,\Omega} + \|w_{RT}\|^2_X \leqslant C\Big(\|f\|^2_{0,\Omega} + \|k\|^2_{0,\Omega} + \|g\|^2_{0,\Omega} + \sqrt{h_{\mathscr{T}}}|p|_{1,\infty,\Omega}\|w_S\|_{0,\Omega} + \sqrt{h_{\mathscr{T}}}|p|_{1,\infty,\Omega}\|g\|_{0,\Omega} + D\|w_S\|_{0,\Omega}\|g\|_{0,\Omega}\Big).$$

Now, let us remark that f, g and k are connected to the interpolation error (see (64)). Then, the previous inequality becomes

$$\|\theta_{\mathcal{T}}\|_{0,\Omega}^2 + D\|w_S\|_{0,\Omega}^2 + \|w_{RT}\|_X^2 \leqslant C\Big(h_{\mathcal{T}}^2 + h_{\mathcal{T}}^{3/2}|p|_{1,\infty,\Omega} + Dh_{\mathcal{T}}\|w_S\|_{0,\Omega} + \sqrt{h_{\mathcal{T}}}|p|_{1,\infty,\Omega}\|w_S\|_{0,\Omega}\Big).$$

We can use again the classical overestimate: $2ab \leqslant \frac{a^2}{\varepsilon} + \varepsilon b^2$, true for all strictly positive number ε, and we obtain

$$\|\theta_{\mathcal{T}}\|_{0,\Omega}^2 + D\|w_S\|_{0,\Omega}^2 + \|w_{RT}\|_X^2 \leqslant C\left(h_{\mathcal{T}}^2 + h_{\mathcal{T}}^{3/2}|p|_{1,\infty,\Omega} + Dh_{\mathcal{T}}^2 + \frac{h_{\mathcal{T}}|p|_{1,\infty,\Omega}^2}{D}\right).$$

Then, if we chose D equal to $h_{\mathcal{T}}^{\alpha}$, it is easy to see that the optimal value of α, to obtain the best convergence rate, is $-1/2$. So, following again the proof of Theorem 25, and under the same assumptions, we obtain the existence of a strictly positive constant C, independent of the mesh, such that

$$\|\omega - \omega_{\mathcal{T}}\|_{0,\Omega} + \|u - u_{\mathcal{T}}\|_{\mathrm{div},\Omega} + \frac{\|u_S\|_{0,\Omega}}{h_{\mathcal{T}}^{1/4}} \leqslant Ch_{\mathcal{T}}^{3/4} \tag{74}$$

and

$$\|\mathrm{curl}\,\omega - \mathrm{curl}\,\omega_{\mathcal{T}}\|_{0,\Omega} + \|p - p_{\mathcal{T}}\|_{0,\Omega} \leqslant C\sqrt{h_{\mathcal{T}}}. \tag{75}$$

Numerical experiments have been performed again on the unit square with the analytical solution proposed by Bercovier and Engelman [18]. The numerical results we obtained on triangular unstructured meshes, with the stabilized numerical scheme and the previous choice of D are given on Fig. 8.

Compared with Fig. 4 (classical scheme) and Fig. 5 (stabilized scheme with $D = 1$), we observe a real improvement of the convergence orders on the vorticity and the pressure (from 1.36 to 1.81 and from 0.65 to 1.26).

5.3. Particular case of a constant pressure along the boundary

In the case of the test proposed by Ruas, the stabilized scheme exhibits an optimal convergence behaviour, with super-convergence on the pressure (see Fig. 7). To understand this phenomenon, we were led to examine again the consistency error term. Before, we have proved that the term $(p, \mathrm{div}\, v_S)$ is in $\mathcal{O}(\sqrt{h_{\mathcal{T}}})$ if the triangulation is uniformly regular and the exact pressure p belongs to $W^{1,\infty}(\Omega)$. Let us study again this term when the pressure p is constant along the boundary.

Proposition 27 (Estimate of the consistency error in a particular case). *Let us assume that the triangulation $\mathcal{T}$ is uniformly regular. Moreover, we suppose that the pressure p solution of the Stokes problem belongs to $H^2(\Omega)$ and that it is constant on $\Gamma_m \backslash \Gamma_0$. Then, for all v_S in $X_{\mathcal{T}}^S$, there exists a strictly positive constant C independent of $h_{\mathcal{T}}$ such that*

$$|(p, \mathrm{div}\, v_S)| \leqslant Ch_{\mathcal{T}}|p|_{2,\Omega}\|v_S\|_{0,\Omega}. \tag{76}$$

Proof. Let us introduce the $\Pi_{\mathcal{T}}^1$ interpolate of the pressure, which is well defined as p belongs to $H^2(\Omega)$. So, we have

$$(p, \mathrm{div}\, v_S) = (p - \Pi_{\mathcal{T}}^1 p, \mathrm{div}\, v_S) + (\Pi_{\mathcal{T}}^1 p, \mathrm{div}\, v_S).$$

We shall study successively each term of the right-hand side of this equality.

First, introducing the support $\Sigma_{\mathcal{T}}$ of any function v_S, we obtain with the Cauchy–Schwarz inequality

$$|(p - \Pi_{\mathcal{T}}^1 p, \mathrm{div}\, v_S)| \leqslant \sum_{K \in \Sigma_{\mathcal{T}}} \|p - \Pi_{\mathcal{T}}^1 p\|_{0,K}\|\mathrm{div}\, v_S\|_{0,K}.$$

Moreover, as p belongs to $H^2(\Omega)$, using the classical interpolation result (see [14]), there exists a strictly positive constant C, independent of h_K, such that for any triangle K

$$\|p - \Pi_{\mathcal{T}}^1 p\|_{0,K} \leqslant Ch_K^2|p|_{2,K}.$$

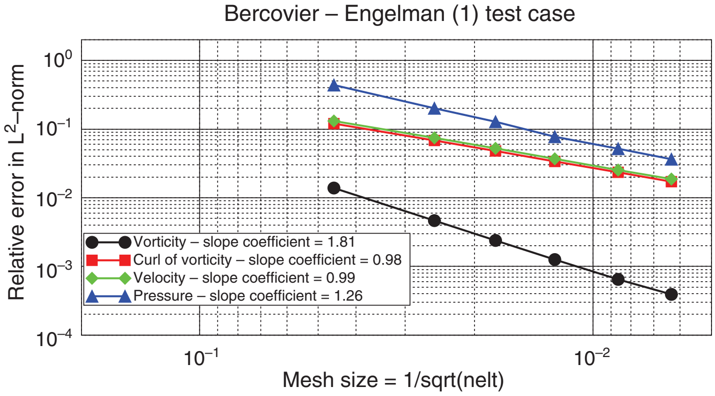

Fig. 8. Convergence order with stabilization – $D = h_{\mathcal{T}}^{-1/2}$ – Bercovier–Engelman's test.

Then, we deduce

$$
\begin{aligned}
|(p-\Pi^1_{\mathcal{T}}p,\operatorname{div} v_S)| &\leqslant \sum_{K\in\Sigma_{\mathcal{T}}} Ch_K^2|p|_{2,K}\|\operatorname{div} v_S\|_{0,K} \\
&\leqslant Ch_{\mathcal{T}}\left(\sum_{K\in\Sigma_{\mathcal{T}}}|p|^2_{2,K}\right)^{\frac{1}{2}}\left(\sum_{K\in\Sigma_{\mathcal{T}}} h_K^2\|\operatorname{div} v_S\|^2_{0,K}\right)^{\frac{1}{2}} \\
&\leqslant Ch_{\mathcal{T}}|p|_{2,\Omega}\left(\sum_{K\in\Sigma_{\mathcal{T}}}\|v_S\|^2_{0,K}\right)^{\frac{1}{2}} \\
&= Ch_{\mathcal{T}}|p|_{2,\Omega}\|v_S\|_{0,\Omega}
\end{aligned}
$$

using the inverse inequalities, which is possible as the triangulation is uniformly regular.

Let us observe that the above inequality leads to the announced result, if we prove that the second term is zero. This is what we do now. So, we have

$$
(\Pi^1_{\mathcal{T}}p,\operatorname{div} v_S) = \sum_{K\in\Sigma_{\mathcal{T}}}\int_K \Pi^1_{\mathcal{T}}p\operatorname{div} v_S\,\mathrm{d}x.
$$

Let us recall that on each triangle K, we have: $v_S = B\operatorname{curl}\lambda_S$. As λ_S is a first degree polynomial function, $\operatorname{curl}\lambda_S$ is constant on K. Moreover the "bubble" function B is null on the edges of K and satisfies $\int_K B\,\mathrm{d}x = |K|$. Then, integrating by parts, we obtain for any triangle K

$$
\begin{aligned}
\int_K \Pi^1_{\mathcal{T}}p\operatorname{div} v_S\,\mathrm{d}x &= -\int_K B\nabla\Pi^1_{\mathcal{T}}p\cdot\operatorname{curl}\lambda_S\,\mathrm{d}x \\
&= -|K|\nabla\Pi^1_{\mathcal{T}}p\cdot\operatorname{curl}\lambda_S \\
&= -\int_K \nabla\Pi^1_{\mathcal{T}}p\cdot\operatorname{curl}\lambda_S\,\mathrm{d}x \\
&= \int_{\partial K}\frac{\partial\Pi^1_{\mathcal{T}}p}{\partial t}\lambda_S\mathrm{d}\gamma,
\end{aligned}
$$

where $\frac{\partial\Pi^1_{\mathcal{T}}p}{\partial t}$ is the tangential derivative of $\Pi^1_{\mathcal{T}}p$ along ∂K. Let us now examine each edge of K. Three cases appear:

When the intersection between the edge and $\Gamma_m\backslash\Gamma_\theta$ is empty, λ_S is zero on this edge as it is associated with vertices of $\Gamma_m\backslash\Gamma_\theta$. So, the associated boundary integral vanishes.

When this intersection is reduced to one vertex, the edge belongs to two triangles of $\Sigma_{\mathcal{T}}$. As $\Pi^1_{\mathcal{T}}p$ is continuous on the mesh, the boundary integrals on such edges will appear twice and will cancel two by two.

When this intersection is equal to the edge, the associated boundary integral remains.

Finally, we deduce that

$$
(\Pi^1_{\mathcal{T}}p,\operatorname{div} v_S) = \int_{\Gamma_m\backslash\Gamma_\theta}\frac{\partial\Pi^1_{\mathcal{T}}p}{\partial t}\lambda_S\,\mathrm{d}\gamma.
$$

So, when p is constant on $\Gamma_m\backslash\Gamma_\theta$, we have $\Pi^1_{\mathcal{T}}p = p$ on this part of the boundary. Then, $\frac{\partial\Pi^1_{\mathcal{T}}p}{\partial t}$ vanishes there and the previous integral is equal to zero, which achieves the proof. $\square$

With the previous result, the inequality (68) becomes

$$
\begin{aligned}
&\|\theta_{\mathcal{T}}\|^2_{0,\Omega}+\|w_S\|^2_{0,\Omega}+\|w_{RT}\|^2_X \\
&\quad\leqslant C\Big(\|f\|^2_{0,\Omega}+\|k\|^2_{0,\Omega}+\|g\|^2_{0,\Omega}+h^2_{\mathcal{T}}|p|^2_{2,\Omega}+|(g,\operatorname{curl}\theta_{\mathcal{T}})|\Big),
\end{aligned}
$$

while the inequality (71) can be written

$$
\|\operatorname{curl}\theta_{\mathcal{T}}\|_{0,\Omega}\leqslant C\Big(\|k\|_{0,\Omega}+h_{\mathcal{T}}|p|_{2,\Omega}+\|w_S\|_{0,\Omega}\Big),
$$

if we make the hypothesis (70). Then, it is easy to obtain

$$
\begin{aligned}
&\|\theta_{\mathcal{T}}\|^2_{0,\Omega}+\|w_S\|^2_{0,\Omega}+\|w_{RT}\|^2_X \\
&\quad\leqslant C\Big(\|f\|^2_{0,\Omega}+\|k\|^2_{0,\Omega}+\|g\|^2_{0,\Omega}+h^2_{\mathcal{T}}|p|^2_{2,\Omega}\Big).
\end{aligned}
$$

Finally, following the proof of Theorem 25, under the assumptions of this Theorem, of Proposition 27 and (70), we obtain the existence of a strictly positive constant C, independent of the mesh, such that

$$
\|\omega-\omega_{\mathcal{T}}\|_{1,\Omega}+\|u-u_{RT}\|_{\operatorname{div},\Omega}+\|u_S\|_{0,\Omega}+\|p-p_{\mathcal{T}}\|_{0,\Omega}\leqslant Ch_{\mathcal{T}},
$$

which is optimal.

This convergence result explains the convergence curves we obtained for the test proposed by Ruas (see Fig. 7), except the superconvergence on pressure. But, in this case, the pressure is identically constant. To check that it is sufficient for the pressure to be constant along the boundary, we built a new test from Bercovier–Engelman's one. The

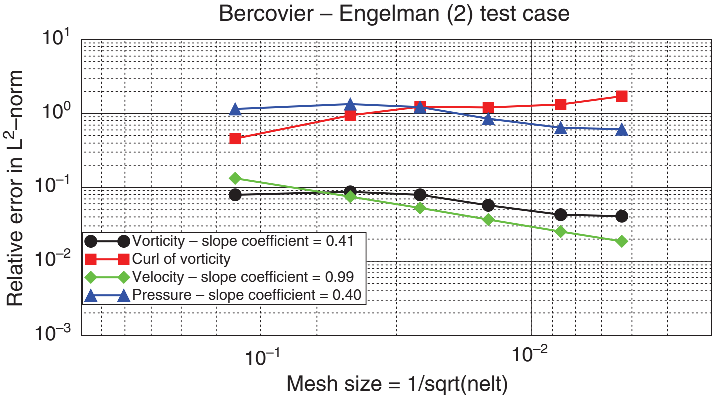

Fig. 9. Convergence without stabilization – $P = 0$ along the boundary – modified Bercovier–Engelman's test.

surfacic loading f is changed in such a way that the exact pressure p is given by

$$p(x,y) = \sin(\pi x)\sin(\pi y)$$

on the domain Ω, which is in this case: $\Omega =]0,1[\times]0,1[$. Let us recall that the boundary conditions are "velocity equal to zero along the whole boundary". First, we check that the convergence problems we have on triangular unstructured meshes for the classical scheme are the same as in the case of the classical Bercovier–Engelman's test (see Fig. 9).

Then, the stabilized scheme is used and exhibits results which are in complete accordance with the above theoretical result (see Fig. 10). Let us observe that, here, there is no superconvergence on the pressure field. This lead us to think that this one is linked with the fact that the pressure is constant on the whole domain in the case of the test suggested by Ruas.

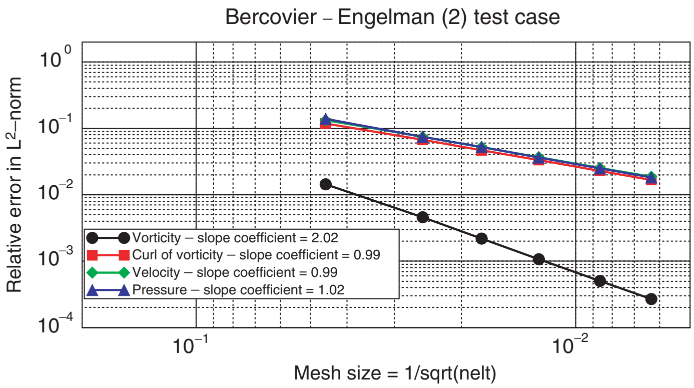

Fig. 10. Convergence order with stabilization – $P = 0$ along the boundary – modified Bercovier–Engelman's test.

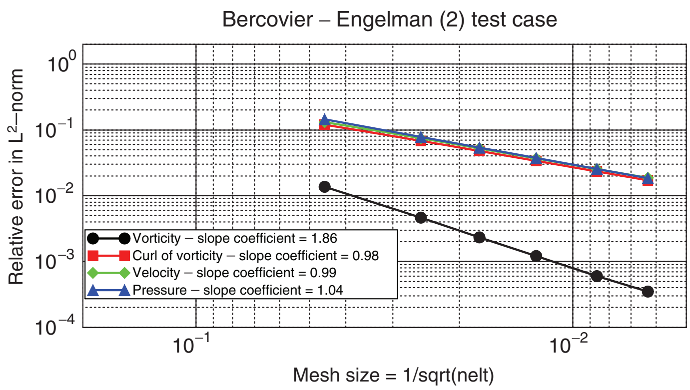

Fig. 11. Convergence order with stabilization – $D = h_{\mathcal{T}}^{-1/2}$ – $P = 0$ along the boundary – modified Bercovier–Engelman's test.

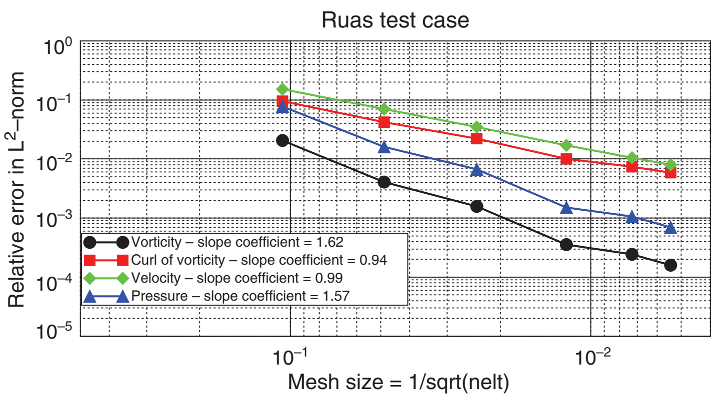

Fig. 12. Convergence order with stabilization – $D = h_{\mathcal{T}}^{-1/2}$ – test proposed by Ruas.

Remark 28. A last question we may ask is what happens when we use the stabilized scheme with the optimal choice of coefficient D *and* when the pressure is constant along the boundary. A careful examination of the estimates leading to (74) and (75) allows to see that, if the consistency error due to the pressure is in $h_{\mathscr{T}}$ instead of $\sqrt{h_{\mathscr{T}}}$, the "best" choice of D is 1 and not $h_{\mathscr{T}}^{-1/2}$. To illustrate this, we give below the convergence curves, obtained for the modified Bercovier–Engelman's test and for the Ruas test, with D equal to $h_{\mathscr{T}}^{-1/2}$. As expected, compared with Fig. 10, Fig. 11 exhibits a small lack of convergence. The conclusions are the same when we compare Fig. 7 with Fig. 12.

6. Conclusion

We have introduced in [7] a vorticity–velocity–pressure variational formulation of the bidimensional Stokes problem. For this formulation, we have defined a natural numerical scheme which can be viewed as an adaptation of the popular MAC scheme on triangular meshes. We have numerically studied this scheme and observed that it is not stable in the general case of boundary conditions. If it gives correct results on structured meshes, improvable ones are obtained on unstructured meshes.

In this paper, we have introduced a stabilization using "bubble" functions, which are added only along a part of the boundary: their numerical cost is then negligible. For this scheme, a general theoretical convergence result is given which is not optimal. But numerical experiments show a very good behaviour of this new scheme, in particular when the exact pressure is constant along the boundary. To try to understand this surprisingly good convergence, we had to make a new hypothesis which allows to improve the convergence and explain the optimal convergence we have obtained. Up to now, the complete comprehension of the convergence of this stabilized scheme is not achieved. It seems to be possible to get rid of the consistency error term by using arguments of Pierre [20]. Work is in progress and hopefully the scheme will be better understood soon.

Acknowledgements

The authors want to thank the referee for very helpful remarks.

References

[1] L. Landau, E. Lifchitz, Mécanique des fluides, vol. VI of Physique théorique, Editions Mir, 1971.

[2] F.H. Harlow, J.E. Welch, Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface, Phys. Fluids 8 (1965) 2182–2189.

[3] A. Arakawa, Computational design for long-term numerical integration of the equations of fluid motion, J. Comput. Phys. 1 (1966) 119–143.

[4] K.S. Yee, Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media, IEEE Trans. Antennas Propagat. AP-14 (1966) 302–307.

[5] O. Génevaux, A. Habibi, J.-M. Dischler, Simulating fluid-solid interaction, in: A.K. Peters (Ed.), Graphics Interface, CIPS, Canadian Human–Computer Communication Society, 2003, pp. 31–38, iSBN 1-56881-207-8, ISSN 0713-5424.

[6] F. Dubois, Une formulation tourbillon–vitesse–pression pour le problème de Stokes, CR Acad. Sci. Paris 314 (1992) 277–280.

[7] F. Dubois, M. Salaün, S. Salmon, First vorticity–velocity–pressure numerical scheme for the Stokes problem, Comput. Methods Appl. Mech. Engrg. 192 (2003) 4877–4907.

[8] P.-A. Raviart, J.-M. Thomas, A mixed finite element method for 2-nd order elliptic problems, Lect. Notes Math. 306 (1977) 292–315.

[9] D. Arnold, F. Brezzi, M. Fortin, A stable finite element for the Stokes equations, Calcolo 21 (1984) 337–344.

[10] L. Franca, S. Oliveira, Pressure bubbles stabilization features in the Stokes problem, Comput. Methods Appl. Mech. Engrg. 192 (2003) 1929–1937.

[11] R. Adams, Sobolev spaces, Pure and Applied Mathematics, vol. 65, Academic Press, New York, London, 1975.

[12] F. Dubois, Vorticity–velocity–pressure formulation for the Stokes problem, Math. Methods Appl. Sci. 25 (2002) 1091–1119.

[13] S. Salmon, Développement numérique de la formulation tourbillon–vitesse–pression pour le problème de Stokes, Ph.D. thesis, Université Paris VI, December 1999.

[14] P. Ciarlet, The finite element method for elliptic problems, Studies in Mathematics and Its Applications, vol. 4, North-Holland, 1978.

[15] J. Thomas, Méthode d'éléments finis mixtes et hybrides, Cours de DEA (1980).

[16] T.P. Mathew, Domain decomposition and iterative refinement methods for mixed finite element discretisations of elliptic problems, Technical Report (463), Courant Institute of Mathematical Sciences, New York.

[17] V. Girault, P.-A. Raviart, Finite element methods for Navier–Stokes equations, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986.

[18] M. Bercovier, M. Engelman, A finite element for the numerical solution of viscous incompressible flows, J. Comput. Phys. 30 (1979) 181–201.

[19] C. Baiocchi, F. Brezzi, L. Franca, Virtual bubbles and the Galerkin-least-squares method, Comput. Methods Appl. Mech. Engrg. 105 (1993) 125–141.

[20] R. Pierre, Simple C0 approximations for the computation of incompressible flows, Comput. Methods Appl. Mech. Engrg. 68 (1988) 205–227.

[21] R. Scholz, A mixed method for 4th order problems using linear finite elements, RAIRO Numer. Anal. 12 (1) (1978) 85–90.

[22] V. Ruas, Méthodes vitesse-tourbillon pour les équations de Navier–Stokes, in: Actes du 29ème congrès d'Analyse Numérique, Canum'97, Larnas, Ardèche, du 26 au 30 Mai 1997, pp. 329–330.