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ESTIMATION OF TRANSLATION, ROTATION AND SCALING BETWEEN NOISY IMAGES USING THE FOURIER MELLIN TRANSFORM

Jérémie Bigot, Fabrice Gamboa & Myriam Vimond

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Abstract

In this paper we focus on extended Euclidean registration of a set of noisy images. We provide an appropriate statistical model for this kind of registration problems, and a new criterion based on Fourier-type transforms is proposed to estimate the translation, rotation and scaling parameters to align a set of images. This criterion is a two step procedure which does not require the use of a reference template onto which aligning all the images. Our approach is based on M-estimation and we prove the consistency of the resulting estimators. A small scale simulation study and real examples are used to illustrate the numerical performances of our procedure.

Key words and phrases: Image registration, Extended Euclidean transformation, Semi-parametric estimation, White noise model, M-estimation, Fourier transform, Fourier-Mellin transform.

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1 Introduction

A fundamental task in image analysis is the comparison of two images or multiple sets of pictures. Generally, to compare similar objects, it is necessary to find a common referential to represent them. In many applications, it is therefore required to perform image registration i.e. to compute a function that warps an observed image to a given template (a reference image) or that aligns multiple sets of pictures. There are many applications of image registration:

among others we can cite face and pattern recognition, characterization of population variation when comparing images arising from different subjects, construction of brain atlas. The transformation may be parametric or nonparametric and constrained to be one-to-one. There is a wide literature on the subject and many methods have been proposed (see e.g. Brown [5], Glasbey and Mardia [16] for detailed reviews on image warping).

Choosing a proper definition of an image is a difficult task. There are several properties, each one highlighting different properties of images. The images may be viewed as continuous functions, sets of points on a picture (pixels or landmarks) or shapes. In practice, we always observe noisy images. The observation noise may be due either to observations measures or directly to the way the images are generated. This makes difficult the comparisons between different images. Over the last decade, progress has been made in defining appropriate distances between shapes or images to compare them. These distances are based on the use of deformation costs and transformation groups to model the variability of natural images. Originating in Grenander's pattern theory the study of the properties and intrinsic geometries of such deformation groups is now an active field of research (Beg *et al.* [3], Klassen *et al.* [24], Marsland & Twining [27], Michor & Mumford [28], Trouvé & Younes [35]). Most of the existing results are stated in a deterministic setting while results in a random framework that are concerned with the estimation of deformations or a template image are scarce.

Techniques in the statistical framework for noisy images include the penalized likelihood approach of Glasbey and Mardia [17], and the small deformations shape analysis using Bayesian procedures in parametric models recently proposed by Allasonnière *et al.* [1]. Note that in Allasonnière *et al.* [1], the template is not treated as an image, but is parameterized in some fashion using some functional approximation method. Then a prior is set on the parameters of the model, and a Bayesian estimation procedure using the EM algorithm is performed. Goodall [18] has proposed a statistical parametric model and maximum likelihood estimation for the Procrustes analysis of a set of shapes (i.e. landmarks) that differ by at most a rotation, a translation and isotropic scaling, while a large overview of existing methods for the statistical analysis of shapes can be found in Dryden & Mardia [12]. However, to the best of our knowledge, none work present fulfill responses to the specific problem of estimating rigid transformations between noisy images.

In this paper we focus on extended Euclidean registration of a set of images. We define an extended Euclidean transform as a rigid transform composed of three basic transforms: translation, scaling and rotation. When two images differ only by a translation, the phase correlation technique is a well known technique based on the shift property of the Fourier transform to estimate the translation parameter (Kuglin & Hines [20]). To account for rotations and scaling, the images can be transformed into a polar or log-polar Fourier-type domain. In these representations, rotation and scaling are reduced to translations and can then be estimated using phase correlation. Various techniques have thus been proposed to match two images that are translated, rotated and scaled with respect to one another. All these methods are usually based on the Fourier transform (see e.g. Reddy & Chatterji [31], De Castro & Morandi [10]), the pseudo-polar Fourier transform (Keller *et al.* [22], [23]) or the Fourier-Mellin transform which is a Fourier-type transform defined on the similarity group that is

invariant to scaling and rotation (Derrode & Ghorbel [9]). Generally, the estimation of the extended Euclidean transform between two images is performed in two steps. First, one uses the modulus of the Fourier transform expressed in polar coordinates to estimate the rotation and the scaling differences between the two images. Then, in a second step, one of the image is registered onto the other one using the estimated parameters for the rotation and the scaling, and the translation displacement is found by using the classical phase correlation technique.

We propose to analyze one of these two steps procedures from a statistical point of view to register a set of noisy images which are translated, rotated and scaled version of the same but unknown template. Our contribution is twofold. First, we give a new criterion to simultaneously register a set of images that does not require the use of a reference template onto which the images are aligned. More precisely, the estimators of the warping parameters are defined as the minimum of appropriate contrast functions. These contrast functions are built from the "shift property" of the Fourier Transform and the Fourier-Mellin Transform. Consequently, we show that a gradient descent algorithm can be easily implemented to compute estimators of extended-Euclidean transforms. Secondly, we prove the consistency (Theorem 4.1 and Theorem 4.3) of this two step procedure from an asymptotic point of view when the images are observed with a Gaussian noise. The technique follows the classical guide lines for the proof of convergence of M-estimators for parametric models as described e.g. in van der Vaart [36].

In the next section, we present a semi-parametric model for the problem of extended Euclidean registration for a set of images. We introduce some notations and we give some mathematical details about this model. In particular we justify the use of a white noise model which can be interpreted as a kind of continuous model for gray-level images. Then, we discuss identifiability conditions for our model. In section 3, we recall the standard invariance property of the Fourier transform for translation and we present the analytical Fourier Mellin transform which is a Fourier type transform proposed by Ghorbel [15] that is invariant to scaling and rotation. Then, in section 4, we propose a new criterion to simultaneously register a set of images. In a first step, we estimate the scaling and rotation parameters, and in a second step we find the shift differences between the images. We analyze this procedure from an asymptotic point of view, and we prove the consistency of the resulting estimators. Section 5 addresses the practical issues of the proposed estimation procedure. We conduct a short Monte Carlo study on the efficiency and rate of convergence of the estimator, and we illustrate our methodology on a real example. Furthermore, we discuss some possible extensions of the method in more general contexts. We conclude the paper by a technical appendix that provides the proofs of the results.

2 A parametric model for extended Euclidean registration

To motivate our parametric model for extended Euclidean registration, consider the following real example of image registration. In Figure 2.1, we display images (possibly noisy) of two butterflies with different shapes that have been observed with various size, orientation and

location. The purpose of image registration is to recover the scaling, rotation and translation parameters to align the images corresponding to the same butterfly (first or second row in Figure 2.1).

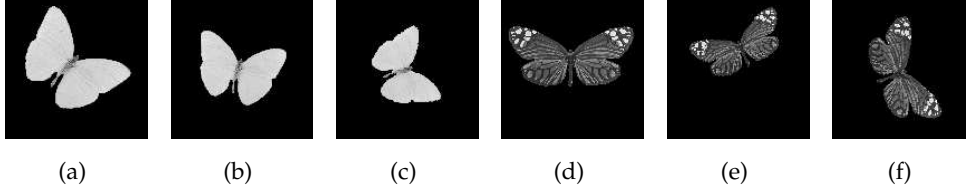


Figure 2.1: Images (a),(b) and (c) represent the same butterfly observed with various size, orientation and location. Images (d),(e) and (f) represent another butterfly with a different shape observed with various size, orientation and location.

2.1 Model assumptions

An extended Euclidean transformation is a parametric mapping $\phi_{a,b,\theta} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ composed of three basic operations: translation (or shift) by a factor $b \in \mathbb{R}^2$, scaling by a factor $a \in \mathbb{R}^+$, and a rotation of angle $\theta \in [0, 2\pi[$. For any vector $x = (x_1, x_2) \in \mathbb{R}^2$, $\phi_{a,b,\theta}(x)$ can thus be written as

$$\phi_{a,b,\theta}(x) = \frac{1}{a}A_\theta(x - b),$$

where A_θ is the 2×2 matrix given by

$$A_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

A grey level image can be modelled by a real valued function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ with a compact support $\mathcal{D} \subset \mathbb{R}^2$. For simplicity, we shall consider that \mathcal{D} is a square of side length $d > 0$ centered at the origin. We denote by $L^p(\mathcal{D})$ ($p = 1, 2$) the usual class of p -integrable functions on \mathbb{R}^2 which are supported on \mathcal{D} . This space is endowed with the norm $\|g\|_{L^p(\mathcal{D})} = (\int_{\mathcal{D}} |g(x)|^p dx / (2\pi))^{1/p}$. Then, for any function $g \in L^p(\mathcal{D})$, its deformation by an extended Euclidean transform $\phi_{a,b,\theta} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the function $g \circ \phi_{a,b,\theta}$.

For a set of J noisy images, we consider the following white noise model : for $j = 1, \dots, J$ and $x = (x_1, x_2) \in \mathcal{D}$

$$dY_j(x_1, x_2) = f(\phi_{a_j^*, b_j^*, \theta_j^*}(x_1, x_2))dx_1dx_2 + \epsilon dW_j(x_1, x_2), \quad (2.1)$$

where $W_j, j = 1, \dots, J$ are standard Brownian sheets on \mathcal{D} , and ϵ is an unknown noise level parameter. The function $f : \mathcal{D} \rightarrow \mathbb{R}$ is an unknown image and $a_j^*, b_j^*, \theta_j^*, j = 1, \dots, J$ are respectively unknown scaling, translation and rotation parameters that we wish to estimate. The image f can thus be viewed as an unknown template onto which the images $f_j = f \circ \phi_{a_j^*, b_j^*, \theta_j^*}, j = 1, \dots, J$ should be registered. To properly define the transformed images f_j , we make the following assumptions:

A1 the image f is zero outside a sub-domain Ω of \mathcal{D} and satisfy certain regularity conditions to be defined later.

A2 the scaling, translation and rotation parameters belong to a set \mathcal{C} of the form

$$(a_j^*, b_j^*, \theta_j^*) \in \mathcal{C} = [a_{min}, a_{max}] \times [-b_{max}, b_{max}]^2 \times [0, 2\pi[, \text{ for all } j = 1, \dots, J, \quad (2.2)$$

where the values a_{min} , a_{max} and b_{max} are user-defined (strictly positive) parameters which reflect our prior knowledge on the amount of mis-alignment between the images.

A3 the scaling, translation and rotation parameters are such that for any $1 \leq j \leq J$ the image of the sub-domain Ω by the transformation $\phi_{a_j^*, b_j^*, \theta_j^*}$ is contained in \mathcal{D} i.e. for all $(x_1, x_2) \in \Omega$ and all $j = 1, \dots, J$

$$\phi_{a_j^*, b_j^*, \theta_j^*}(x_1, x_2) \in \mathcal{D}. \quad (2.3)$$

With these assumptions (**A1-A3**), all the images f_j belong to $L^2(\mathcal{D})$, and correspond to translated, rotated and scaled versions of the same image observed on a black background (if we choose to code a zero-valued pixel by the black color). This model corresponds therefore to the real example that we have presented at the beginning of this section (see the first and second rows of Figure 2.1).

We shall analyze the properties of M -estimators for the parameters a_j^* , b_j^* and θ_j^* from an asymptotic point of view i.e. when the noise level parameter ϵ tends to 0 in the model (2.1). The white noise model (2.1) is a continuous model which has been proven to be a very useful theoretical tool for the study of nonparametric regression problems for 2D images. Since real images are typically discretely sampled on a regular grid, the model (2.1) may seem inappropriate at a first glance. However, it has been shown by several authors that asymptotic results obtained in the white noise model lead to comparable asymptotic theory in a sampled data model such as the standard nonparametric regression problem with an equi-spaced design. We shall not elaborate more on this point and we refer to Brown & Low [6], Donoho & Johnstone [11] for a clean and extensive study of the relationship between optimal procedures in continuous and sampled data models and further references. The white noise model should be interpreted in the following sense: for any function $g \in L^2(\mathcal{D})$ each integral $\int_{\mathcal{D}} g(x_1, x_2) dY(x_1, dx_2)$ of the "data" $dY(x_1, x_2) = f(x_1, x_2) dx_1 dx_2 + \epsilon dW(x_1, x_2)$ is a random variable normally distributed with mean $\int_{\mathcal{D}} f(x_1, x_2) g(x_1, x_2) dx_1 dx_2$ and variance $\epsilon^2 \int_{\mathcal{D}} (g(x_1, x_2))^2 dx_1 dx_2$.

Therefore, if $\int_{\mathcal{D}} f(x_1, x_2) g(x_1, x_2) dx_1 dx_2$ represents a coefficient of f into some basis of functions, then $\int_{\mathcal{D}} g(x_1, x_2) dY(x_1, dx_2)$ is modeled as the sum of the true coefficient of f plus some Gaussian noise. However, in our proofs of consistency for the estimators that we shall propose, the Gaussian assumption can be substituted by an assumption on the moment of the noise. Therefore, our procedure is somewhat robust to non-Gaussian noise.

In the context of inverse problems for 2D images, Candès & Donoho [7] have obtained nice theoretical results by modelling noisy tomographic data within the white noise model. The analysis of Candès & Donoho [7] yields estimators which are computationally tractable for sampled data and which lead to very satisfactory results for real images as expected by their

theoretical approach. Results such as those obtained by Candès & Donoho [7] are thus our main motivation for using white noise models such as (2.1) to analyze registration problems for 2D images from a statistical point of view. Moreover, when dealing with registration problems for discrete images defined on a Cartesian grid, one usually faces the problem of interpolating an image to calculate its deformation by a rigid transformation. Such interpolation problems complicate significantly the asymptotic analysis of registration problems for sampled images. Hence, we prefer to avoid this approach in order to focus on the statistical properties of our estimators rather than on the bias introduced by the discretization of the problem.

2.2 Identifiability of the model

Let \mathcal{A} denote the space $[a_{min}, a_{max}]^J \times [-b_{max}, b_{max}]^{2J} \times [0, 2\pi]^J$. Note that if we replace $\alpha^* = (a_1^*, \dots, a_J^*, b_1^*, \dots, b_J^*, \theta_1^*, \dots, \theta_J^*) \in \mathcal{A}$ by

$$\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_J \\ b_1 \\ \vdots \\ b_J \\ \theta_1 \\ \vdots \\ \theta_J \end{pmatrix} = \begin{pmatrix} a_1^* a_0 \\ \vdots \\ a_J^* a_0 \\ b_1^* + a_1 A_{-\theta_1} b_0 \\ \vdots \\ b_J^* + a_J A_{-\theta_J} b_0 \\ \theta_1^* + \theta_0 \\ \vdots \\ \theta_J^* + \theta_0 \end{pmatrix}$$

with $a_0 \in]0, +\infty[$, $b_0 \in \mathbb{R}^2$, $\theta_0 \in [0, 2\pi[$, then it is easy to see that the model (2.1) is left unchanged with f replaced by $f \circ \phi_{a_0, b_0, \theta_0}$. This model is therefore not identifiable. To ensure identification, we further assume that the set of parameters \mathcal{A} is reduced to the subset $\mathcal{A}_1 \times \mathcal{A}_2 \subset \mathcal{A}$ such that

$$\mathcal{A}_1 = \{(a_1, \dots, a_J, \theta_1, \dots, \theta_J) \in [a_{min}, a_{max}]^J \times [0, 2\pi]^J, \text{ such that } a_1 = 1, \theta_1 = 0\},$$

and

$$\begin{aligned} \mathcal{A}_2 = \{(b_1, \dots, b_J) \in [-b_{max}, b_{max}]^{2J}, \text{ such that } b_1 = (0, 0)\}. \\ (a_1^* = 1, b_1^* = (0, 0), \theta_1^* = 0) \end{aligned} \quad (2.4)$$

Note that with the above identifiability condition, we implicitly assume that the sub-domain Ω and the constants a_{min}, a_{max} defined previously are such that $a_{min} \leq 1 \leq a_{max}$ (see Assumptions **A1** and **A2**). All the images will thus have the same orientation as f_1 since the scaling, rotation and translation parameters for $j = 2, \dots, J$ are defined with respect to the first image. However, as we shall see in the next section, the condition (2.4) does not mean that the image f_1 is considered as a template on to which we align all the other images.

3 Fourier transforms invariant to translation, scaling and rotation

3.1 The standard Fourier transform and its shift property for translation

For $\omega \in \mathbb{R}^2$, we recall that the Fourier transform of a function $g \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ is defined as

$$\hat{g}(\omega) = \int_{\mathbb{R}^2} g(x) e^{-i\omega \cdot x} \frac{dx}{2\pi},$$

where $\omega \cdot x$ denotes the usual scalar product between ω and x . Note that the Fourier transform of $\hat{g}_{a,b,\theta} = g \circ \phi_{a,b,\theta}$ is

$$\hat{g}_{a,b,\theta}(\omega) = a^2 e^{i\omega \cdot b} \hat{g}(aA_\theta \omega). \quad (3.1)$$

In particular, for $a = 1$ and $\theta = 0$, we obtain the classical shift property of the Fourier transform for translation,

$$\hat{g}_{1,b,0}(\omega) = e^{i\omega \cdot b} \hat{g}(\omega). \quad (3.2)$$

These two last properties are used thereafter.

3.2 The analytical Fourier-Mellin transform and its shift property for scaling and rotation

The Fourier-Mellin transform is a Fourier-type transform defined on the similarity group. Several authors have developed Fourier-type transforms for studying various specific groups of transformations in the context of 2D or 3D image analysis. Some references on this topic include Derrode & Ghorbel [9], Ghorbel [15], Lenz [25], Gauthier, Bornard & Silbermann [14], Segman, Rubinstein & Zeevi [33].

Throughout this paper, we denote by \mathbb{Z} the additive group of integers, \mathbb{R} the additive group on the real line, \mathbb{R}_+^* the multiplicative group of strictly positive reals and \mathbb{S}^1 the unit sphere in \mathbb{R}^2 . All these sets are locally compact groups. In what follows, it will be convenient to represent any element of \mathbb{S}^1 by an element of the interval $[0, 2\pi[$. The direct product $\mathcal{G} = \mathbb{R}_+^* \times \mathbb{S}^1$ forms a locally compact group of planar similarities whose Haar measure is given by $d\mu(r, \theta) = \frac{dr}{r} \frac{d\theta}{2\pi}$ where dr and $d\theta$ denote the standard Lebesgue measures on \mathbb{R}_+^* and $[0, 2\pi[$ respectively. The measure $d\mu(r, \theta)$ is the positive and invariant measure on \mathcal{G} and the dual group of \mathcal{G} is $\hat{\mathcal{G}} = \mathbb{Z} \times \mathbb{R}$. It is therefore possible to define a Fourier transform for functions defined on \mathcal{G} (see Rudin [32]). To be more precise, we define $L^p(\mathcal{G})$ as the space of integrable ($p = 1$) and square integrable ($p = 2$) real valued functions defined on \mathcal{G} such that:

$$\|f\|_{L^p(\mathcal{G})}^p = \int_0^{+\infty} \int_0^{2\pi} |f(r, \theta)|^p \frac{d\theta}{2\pi} \frac{dr}{r} < +\infty$$

Let $g \in L^2(\mathbb{R}^2)$ be a 2D image such that $g \in L^1(\mathcal{G})$ when expressed in polar coordinates (i.e. when we make the change of variables $x_1 = r \cos(\theta)$, $x_2 = r \sin(\theta)$ for $(x_1, x_2) \in \mathbb{R}^2$). The standard Fourier-Mellin transform (FMT) of g is then given by

$$\forall (k, v) \in \mathbb{Z} \times \mathbb{R}, \tilde{\mathcal{M}}_g(k, v) = \int_0^{+\infty} \int_0^{2\pi} g(r, \theta) r^{-iv} e^{-ik\theta} \frac{d\theta}{2\pi} \frac{dr}{r}. \quad (3.3)$$

As explained e.g. in Derrode & Ghorbel [9], the FMT may be difficult to compute in practice since real images are generally not in $L^1(\mathcal{G})$. To see this, note that the FMT transform exists for functions $g(r, \theta)$ which are equivalent to r^σ in the neighborhood of the origin ($r = 0$) for some constant $\sigma > 0$. However, a real image usually does not meet this condition since in the vicinity of the origin the gray level of an image is generally different from zero. To overcome these difficulties, Derrode & Ghorbel [9] and Ghorbel [15] have proposed to replace the function $g(r, \theta)$ in equation (3.3) by the function $g_\sigma(r, \theta) = r^\sigma g(r, \theta)$ for some positive and fixed constant σ whose choice will be discussed later. This re-weighting by the factor r^σ leads to the definition of the so-called Analytical Fourier-Mellin Transform (AFMT) for a function g such that $g_\sigma \in L^1(\mathcal{G})$:

$$\forall (k, v) \in \mathbb{Z} \times \mathbb{R}, \mathcal{M}_g(k, v) = \int_0^{+\infty} \int_0^{2\pi} g(r, \theta) r^{\sigma-iv} e^{-ik\theta} \frac{d\theta}{2\pi} \frac{dr}{r}. \quad (3.4)$$

If $\int_{-\infty}^{+\infty} \sum_{k \in \mathbb{Z}} |\mathcal{M}_g(k, v)| dv < +\infty$, then the inverse AFMT can be used to retrieve g as:

$$\forall (r, \theta) \in [0, +\infty[\times [0, 2\pi[, r^\sigma g(r, \theta) = \int_{-\infty}^{+\infty} \sum_{k \in \mathbb{Z}} \mathcal{M}_g(k, v) r^{iv} e^{ik\theta} dv. \quad (3.5)$$

Moreover, if $g_\sigma \in L^1(\mathcal{G}) \cap L^2(\mathcal{G})$, the AFMT preserves the energy of the function. More precisely, the following Parseval relationship holds (see Rudin [32] for further details)

$$\|g_\sigma\|_{L^2(\mathcal{G})}^2 = \int_{-\infty}^{+\infty} \sum_{k \in \mathbb{Z}} |\mathcal{M}_g(k, v)|^2 dv. \quad (3.6)$$

Now, let $a > 0$ and $\theta \in [0, 2\pi[$, and consider the function $g_{a,\theta}$ defined as $g_{a,\theta}(x) = g(\frac{1}{a}A_\theta x)$ for $x \in \mathbb{R}^2$. When we express the function $g_{a,\theta}$ in polar coordinates, we easily see that its AFMT is given by

$$\forall (k, v) \in \mathbb{Z} \times \mathbb{R}, \mathcal{M}_{g_{a,\theta}}(k, v) = a^{\sigma-iv} e^{ik\theta} \mathcal{M}_g(k, v) \quad (3.7)$$

Hence, while the standard Fourier transform converts translation into a pure phase change into the Fourier domain, equation (3.7) shows that the AFMT converts scaling and rotation into a complex multiplication in the Fourier-Mellin domain. This relation can be seen as a shift property of the AFMT for the similarity transforms.

4 A two-step procedure for the registration of a set of images

For

$$\alpha = (a_1, \dots, a_J, b_1, \dots, b_J, \theta_1, \dots, \theta_J) \in \mathcal{A},$$

we could propose to minimize the following criterion inspired by recent results of Gamboa *et al.* [13] and Vimond [37] for the estimation of shifts between curves:

$$Q(\alpha) = \frac{1}{J} \sum_{j=1}^J \|f_j \circ \phi_{\frac{1}{a_j}, \bar{b}_j, -\theta_j} - \frac{1}{J} \sum_{j'=1}^J f_{j'} \circ \phi_{\frac{1}{a_{j'}}, \bar{b}_{j'}, -\theta_{j'}}\|_{L^2(\mathbb{R}^2)}^2. \quad (4.1)$$

where $\tilde{b}_j = -\frac{1}{a_j} A_{\theta_j} b_j$. This criterion is closely related to Procrustes analysis which is classically used for the statistical analysis of shapes (see e.g. Mardia & Dryden [26]) and the criterion which has been proposed by Ramsay and Li [30] for the registration of a set of curves onto a common target function. Here, the common template function is defined as the average of the synchronized images by the transformations $\phi_{a_j, b_j, \theta_j}$. Obviously, this criterion $Q(\alpha)$ has a minimum at

$$\alpha^* = (a_1^*, \dots, a_J^*, b_1^*, \dots, b_J^*, \theta_1^*, \dots, \theta_J^*)$$

such that $Q(\alpha^*) = 0$. In practice, we observe noisy images, and we could therefore choose to minimize the equation (4.1) by replacing the f_j 's by the Y_j 's to obtain a simultaneous estimation of the scaling, translation and rotation parameters. By using the principle of M-estimation [36] with an appropriate denoising of the images, one could prove that the minimum of the cost function (4.1) converges to the parameters α^* when the level of noise ϵ tends to 0. However, from a numerical point of view, to evaluate $Q(\alpha)$ for any set of parameters α we need to calculate the deformation of the f_j 's by the transformations $\phi_{a_j, b_j, \theta_j}$ which requires an interpolation of the images. A simple minimization of $Q(\alpha)$ is certainly not obvious as it is a highly nonlinear function of α and for instance a gradient-based algorithm would require the evaluation of the derivatives of the images which can be very difficult in the presence of noise. Therefore, to minimize such a criterion for real data, one needs to find an efficient way to smooth the observed images to reduce the noise, and also to find some procedure to align the images.

Our method provides an efficient solution to both the problem of aligning the images, and to the problem of noise removal. It relies mainly on the Fourier transform of the data, the shift properties (3.1) and (3.4) and Parseval equalities. Indeed in the Fourier domain, aligning images only amounts to a modification of the amplitude and the phase of the observed Fourier coefficients. Moreover, we carefully study the problem of noise removal by selecting only low-frequency coefficients to perform both image alignment and image denoising, which is necessary to obtain consistent estimators.

To be more precise, suppose first that all the b_j 's are zero, i.e. the images differ only by a scaling and a rotation. In this case, thanks to the shift property (3.7) of the AFMT for scaling and rotation and from the Parseval relation (3.6) we derive that the criterion $Q(\alpha)$ can be written as (if we take the $L^2(\mathcal{G})$ -norm instead of the $L^2(\mathbb{R}^2)$ -norm in the definition of $Q(\alpha)$)

$$Q_1(\alpha_1) = \frac{1}{J} \sum_{j=1}^J \int_{-\infty}^{+\infty} \sum_{k \in \mathbb{Z}} \left| a_j^{-\sigma+iv} e^{-ik\theta_j} \mathcal{M}_{f_j}(k, v) - \frac{1}{J} \sum_{j'=1}^J a_{j'}^{-\sigma+iv} e^{-ik\theta_{j'}} \mathcal{M}_{f_{j'}}(k, v) \right|^2 dv, \quad (4.2)$$

with $\alpha_1 = (a_1, \dots, a_J, \theta_1, \dots, \theta_J)$. Similarly, if we suppose that all the a_j 's are equal to one and all the θ_j 's are zero (i.e. the images differ only by a translation), then due to the shift property (3.2) of the Fourier transform for translation and by Parseval, we obtain that the criterion $Q(\alpha)$ can be written as

$$Q_2(\alpha_2) = \frac{1}{J} \sum_{j=1}^J \int_{\mathbb{R}^2} \left| e^{i\omega \cdot b_j} \hat{f}_j(\omega) - \frac{1}{J} \sum_{j'=1}^J e^{i\omega \cdot b_{j'}} \hat{f}_{j'}(\omega) \right|^2 d\omega, \quad (4.3)$$

with $\alpha_2 = (b_1, \dots, b_J)$. Equation (4.2) shows that we only have to properly modify the phase and the amplitude of the AMFT of the f_j 's to calculate the value of the criterion $Q_1(\alpha_1)$, while equation (4.3) shows that the calculation of $Q_2(\alpha_2)$ only requires to modify the phase of the Fourier transform of the f_j 's. Note that computing Fourier-type transforms and modifying the Fourier coefficients to align the images is thus a simple way of interpolating and smoothing the images. Moreover, we will show that the minimum of criterions similar to $Q_1(\alpha_1)$ and $Q_2(\alpha_2)$ can be easily obtained by a gradient descent algorithm.

However, one cannot find a transformation which has a shift property simultaneously for the scaling, rotation and translation parameters of an image. Hence, in the next section we use a two-step procedure which relies on two M-type criterions inspired by the formulation (4.1) using first the AMFT for the estimation of scaling and rotation, and then the Fourier transform for the estimation of translations.

4.1 A criteria for simultaneously estimating scaling and rotation

Note that given our assumptions on the parameters a_j^* , b_j^* and θ_j^* , we have that

$$\hat{Y}_j(\omega) = \int_{\mathcal{D}} e^{-i\omega \cdot x} \frac{dY_j(x)}{2\pi} = \hat{f}_j(\omega) + \epsilon \hat{W}_j(\omega), \quad \omega \in \mathbb{R}^2, \quad (4.4)$$

with $\hat{W}_j(\omega) = \int_{\mathcal{D}} e^{-i\omega \cdot x} \frac{dW_j(x)}{2\pi}$. If we consider the squared modulus of the Fourier Transform of the data, then equation (4.4) becomes

$$|\hat{Y}_j|^2(\omega) = |\hat{f}_j|^2(\omega) + \epsilon w_j(\omega), \quad \omega \in \mathbb{R}^2, \quad (4.5)$$

with for all $\omega \in \mathbb{R}^2$,

$$\begin{aligned} |\hat{f}_j|^2(\omega) &= |a_j^*|^4 |\hat{f}|^2(a_j^* A_{\theta_j^*} \omega), \\ w_j(\omega) &= 2\Re \left[\hat{f}_j(\omega) \overline{\hat{W}_j(\omega)} \right] + |\hat{W}_j|^2(\omega), \end{aligned}$$

where $\Re(c)$ denotes the real part of a complex number c . In order to simplify the notations, all points (r, θ) of \mathcal{G} is associated to its Cartesian coordinates $\omega = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2$, and vice-versa. As we have taken the modulus of the Fourier transform of the images, the functions $|\hat{f}_j|^2$ differ from $|\hat{f}|^2$ only by a scaling and a rotation. If we apply the AFMT to the functions $|\hat{f}_j|^2$, where σ is chosen such that $r^\sigma |\hat{f}|^2 \in \mathbb{L}^1(\mathcal{G})$ (for example $\sigma = 2$), then equation (3.1) implies that for all $v \in \mathbb{R}$, $k \in \mathbb{Z}$

$$\mathcal{M}_{|\hat{f}_j|^2}(k, v) = a_j^{*4-\sigma+iv} \mathcal{M}_{|\hat{f}|^2}(k, v) e^{ik\theta_j^*}. \quad (4.6)$$

Thus, for $\alpha_1 = (a_1, \dots, a_J, b_1, \dots, b_J) \in \mathcal{A}_1$, we propose to minimize the following criterion:

$$M(\alpha_1) = \frac{1}{J} \sum_{j=1}^J \int_0^{2\pi} \int_0^{+\infty} \left| a_j^{-4} r^\sigma |\hat{f}_j|^2\left(\frac{r}{a_j}, \theta - \theta_j\right) - \frac{1}{J} \sum_{j'=1}^J a_{j'}^{-4} r^\sigma |\hat{f}_{j'}|^2\left(\frac{r}{a_{j'}}, \theta - \theta_{j'}\right) \right|^2 \frac{dr d\theta}{r 2\pi}.$$

Obviously, the criterion $M(\alpha_1)$ has a minimum at $\alpha_1^* = (a_1^*, \dots, a_J^*, \theta_1^*, \dots, \theta_J^*)$, such that $M(\alpha_1^*) = 0$. Given our assumptions on the f_j 's, the Parseval relation (3.6) for the AFMT and equation (4.6) we have that the criterion $M(\alpha_1)$ can also be expressed as:

$$M(\alpha_1) = \frac{1}{J} \sum_{j=1}^J \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left| c_j(k, v) - \frac{1}{J} \sum_{j'=1}^J c_{j'}(k, v) \right|^2 dv, \quad (4.7)$$

where

$$c_j(k, v) = a_j^{-4+\sigma-iv} \mathcal{M}_{|\hat{f}_j|^2}(k, v) e^{-ik\theta_j}. \quad (4.8)$$

However, due to the noise in the original images, we do not observe directly the $c_j(k, v)$'s but noisy coefficients. Moreover, the stochastic functions $\omega \mapsto |\hat{Y}_j|^2(\omega)$ are certainly not integrable on \mathbb{R}^2 . Thus, in order to satisfy integrability properties, the Fourier-Mellin coefficients are estimated by:

$$\mathcal{M}_{|\hat{Y}_j|^2}^\epsilon(k, v) = \int_0^{\delta_\epsilon} \int_0^{2\pi} |\hat{Y}_j|^2(r, \theta) r^{\sigma-iv} e^{-ik\theta} \frac{d\theta}{2\pi} \frac{dr}{r} \quad (4.9)$$

$$= \mathcal{M}_{|\hat{f}_j|^2}^\epsilon(k, v) + \epsilon \mathcal{M}_{w_j}^\epsilon(k, v), \quad (4.10)$$

where

$$\mathcal{M}_{|\hat{f}_j|^2}^\epsilon(k, v) = \int_0^{\delta_\epsilon} \int_0^{2\pi} |\hat{f}_j|^2(r, \theta) r^{\sigma-iv} e^{-ik\theta} \frac{d\theta}{2\pi} \frac{dr}{r}$$

$$\mathcal{M}_{w_j}^\epsilon(k, v) = \int_0^{\delta_\epsilon} \int_0^{2\pi} w_j(r, \theta) r^{\sigma-iv} e^{-ik\theta} \frac{d\theta}{2\pi} \frac{dr}{r},$$

and δ_ϵ is appropriate smoothing parameter to be defined below. Then we define,

$$d_j^\epsilon(k, v) = a_j^{-4+\sigma-iv} \mathcal{M}_{|\hat{Y}_j|^2}^\epsilon(k, v) e^{-ik\theta_j},$$

and we propose the following M -type criterion

$$M_\epsilon(\alpha_1) = \frac{1}{J} \sum_{j=1}^J \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \left| d_j^\epsilon(k, v) - \frac{1}{J} \sum_{j'=1}^J d_{j'}^\epsilon(k, v) \right|^2 dv \quad (4.11)$$

where k_ϵ and v_ϵ are also smoothing parameters to be defined below.

Obviously, the template image f should not be invariant to some rotation else one cannot guarantee a unique minimum for the criterion $M(\alpha_1)$. For any real function f , we have that $|\hat{f}|^2(r, \theta) = |\hat{f}|^2(r, \theta + \pi)$, and thus the criterion $M(\alpha_1)$ is left unchanged if one replaces the rotation parameter θ_j by $\theta_j + \pi$. Hence, to guarantee the existence of a unique minimizer for $M(\alpha_1)$, we shall minimize this criterion over the set $\tilde{\mathcal{A}}_1$ defined by

$$\tilde{\mathcal{A}}_1 = \{(a_1, \dots, a_J, \theta_1, \dots, \theta_J) \in [a_{min}, a_{max}]^J \times [0, \pi]^J, \text{ such that } a_1 = 1, \theta_1 = 0\}.$$

The invariance to rotation of an image can be characterized via the coefficients of its AFMT. Since $\mathcal{M}_{|\hat{f}|^2}(2k+1, v) = 0$ for any $k \in \mathbb{Z}$, we introduce the following definition:

Definition 4.1 A function $f \in L^2(\mathcal{D})$ such that $r^\sigma |\hat{f}|^2 \in L^1(\mathcal{G})$ is said to be not rotation invariant if there exists two relatively prime integer $(k, k') \in \mathbb{Z}^2$ such that the functions $v \mapsto \mathcal{M}_{|\hat{f}|^2}(2k, v)$ and $v \mapsto \mathcal{M}_{|\hat{f}|^2}(2k', v)$ are not identically equal to 0.

Then, if the image f is not rotation invariant and satisfies some smoothness properties, the following theorem provides the consistency of the M -estimator defined by

$$\hat{\alpha}_1^\epsilon = \arg \min_{\alpha_1 \in \bar{\mathcal{A}}_1} M_\epsilon(\alpha_1).$$

Theorem 4.1 Assume that f is not rotation invariant. Let $s > 0$ and assume that

$$\int_{v \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\mathcal{M}_{|\hat{f}|^2}(k, v)|^2 dv < +\infty \quad (4.12)$$

$$\int_{\omega \in \mathbb{R}^2} |\hat{f}(\omega)|^2 |\omega|^{2s} d\omega < +\infty \quad (4.13)$$

$$\sigma < 2s + 2 \quad (4.14)$$

Suppose that k_ϵ , v_ϵ and δ_ϵ are such that as $\epsilon \rightarrow 0$

$$\begin{aligned} k_\epsilon, v_\epsilon, \delta_\epsilon &\rightarrow +\infty \\ \delta_\epsilon^{-4-4s+2\sigma} k_\epsilon v_\epsilon &= \mathcal{O}(1) \end{aligned} \quad (4.15)$$

$$\epsilon^2 \delta_\epsilon^\sigma k_\epsilon v_\epsilon = o(1), \quad (4.16)$$

then $\hat{\alpha}_1^\epsilon$ converges in probability to α_1^* for the standard Euclidean norm in $\mathbb{R}^J \times \mathbb{R}^J$.

A few remarks about the conditions of Theorem 4.1 can be made. First, since the function f is supported on a compact set, f lies in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $\hat{f} \in L^2(\mathbb{R}^2)$. Thus there exist $\sigma \leq 2$ such that $r^\sigma |\hat{f}|^2$ lies in $L^1(\mathcal{G})$: the AFMT of $|\hat{f}|^2$ is well defined.

The condition (4.12) means that $r^\sigma |\hat{f}|^2$ lies in $L^2(\mathcal{G})$. By Parseval equality, we may write

$$\int_{v \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\mathcal{M}_{|\hat{f}|^2}(k, v)|^2 dv = \int_{\omega \in \mathbb{R}^2} |\hat{f}(\omega)|^4 |\omega|^{2\sigma-2} \frac{d\omega}{2\pi}.$$

Observe when it is well defined, $|\hat{f}(\omega)|^2$ is the Fourier transform of the auto-convolution,

$$x \in \mathbb{R}^2 \rightarrow \int_{\omega \in \mathbb{R}^2} f(y) f(y - x) dy. \quad (4.17)$$

So that, when $\sigma \geq 1$, Assumption (4.12) implies that the function (4.17) lies in the Sobolev space of order $\sigma - 1$. When $\sigma < 1$, Assumption (4.12) is a consequence of Assumption (4.13). The condition (4.13) is a classical assumption on f namely that it belongs to a Sobolev space of order $s > 0$. The condition (4.14) implies that σ should not be too large with respect to the smoothness of f measured by the parameter s . Condition (4.14) is used in the proof of Theorem 4.1 to guarantee the existence of some integrals. Finally the conditions (4.15) and (4.16) gives empirical conditions on the choice of the smoothing parameters k_ϵ , v_ϵ and δ_ϵ to perform both image alignment and image denoising. For example, if $k_\epsilon = \epsilon^{-1+\sigma/(2+2s)}$, $v_\epsilon = \epsilon^{-1+\sigma/(2+2s)}$

and $\delta_\epsilon = \epsilon^{-1/(2+2s)}$, Condition (4.15) and Condition (4.16) hold. Additionally, we point out that the criterion (4.11) provides consistent estimators even if the parameters k_ϵ and v_ϵ are held fixed (they do not converges to infinity) provided that the corresponding limit criterion has a unique minimum (see the proof in the appendix). Moreover, the above theorem also shows that one can select only a finite set of Fourier-Mellin coefficients to perform the estimation.

Moreover, with stronger assumptions, the following theorem provides the ϵ^{-1} -consistency of the estimator.

Theorem 4.2 *Let us denote by $g(r, \theta)$ the function $r^\sigma |\hat{f}(r, \theta)|^2$. Assume that g is differentiable such that the partial derivatives $\partial_\theta g$ and $\partial_r g$ with respect the polar coordinates θ and r respectively are in $L^2(\mathcal{G})$, and that,*

$$\int_{v \in \mathbb{R}} \sum_{k \in \mathbb{Z}} (|k| + |v|) |\mathcal{M}_{|\hat{f}|^2}(k, v)| dv < \infty \quad (4.18)$$

$$\int_{\omega \in \mathbb{R}^2} |\hat{f}(\omega)|^2 |\omega|^{2s} d\omega < \infty \quad (4.19)$$

$$s > 0 \text{ and } 2\sigma < 2s + 2, \quad (4.20)$$

and that $f, \partial_\theta g$ are not identically equal to zero. Moreover assume that:

$$(\sigma - 5)(\sigma - 4) \|g\|_{L^2(\mathcal{G})}^2 + \|\partial_r g\|_{L^2(\mathcal{G})}^2 \neq 0, \quad \text{and} \quad (\sigma - 4)^2 \frac{1}{J} - (\sigma - 4) > 0. \quad (4.21)$$

Suppose that $k_\epsilon, v_\epsilon, \delta_\epsilon$ and ω_ϵ are such that as $\epsilon \rightarrow 0$

$$k_\epsilon, v_\epsilon, \delta_\epsilon \rightarrow +\infty \quad (4.22)$$

$$\delta_\epsilon^{-4-4s+2\sigma} k_\epsilon v_\epsilon (k_\epsilon + v_\epsilon) \epsilon^{-1} = \mathcal{O}(1) \quad \text{and} \quad \delta_\epsilon^{-2-2s+\sigma} \epsilon^{-1} = \mathcal{O}(1) \quad (4.23)$$

$$\epsilon k_\epsilon v_\epsilon (k_\epsilon + v_\epsilon) \delta_\epsilon^\sigma = o(1), \quad (4.24)$$

$$k_\epsilon v_\epsilon (k_\epsilon + v_\epsilon) \delta_\epsilon^{-2s-2+3\sigma/2} = \mathcal{O}(1) \quad (4.25)$$

then $\epsilon^{-1}(\hat{\alpha}_1^\epsilon - \alpha_1^*)$ converges in distribution to a centered Gaussian variable.

4.2 A criteria for simultaneously estimating shifts

In the previous section, we built an ϵ^{-1} -estimator $\hat{\alpha}_1^\epsilon$ of α_1^* , with $\hat{\alpha}_1^\epsilon = (\hat{a}_1, \dots, \hat{a}_J, \hat{\theta}_1, \dots, \hat{\theta}_J)$. For $\alpha_2 = (b_1, \dots, b_J) \in \mathcal{A}_2$, these estimators are used to align the images in the following way:

$$\begin{aligned} \hat{Z}_j(\omega) &= \hat{a}_j^{-2} \hat{Y}_j(\hat{a}_j^{-1} A_{-\hat{\theta}_j} \omega) e^{i\omega \tilde{\beta}_j} \\ &= \left(\frac{a_j^*}{\hat{a}_j} \right)^2 \hat{f} \left(\frac{a_j^*}{\hat{a}_j} A_{\theta_j^* - \hat{\theta}_j} \omega \right) e^{i\omega(\tilde{\beta}_j - \tilde{\beta}_j^*)} + \epsilon \hat{a}_j^{-2} \hat{W}_j(\hat{a}_j^{-1} A_{-\hat{\theta}_j} \omega) e^{i\omega \tilde{\beta}_j}, \end{aligned} \quad (4.26)$$

where the following notation are used:

$$\begin{aligned} \beta_j &= a_j^{*-1} A_{\theta_j^*} b_j, & \beta_j^* &= a_j^{*-1} A_{\theta_j^*} b_j^* \\ \tilde{\beta}_j &= \hat{a}_j^{-1} A_{\hat{\theta}_j} b_j, & \tilde{\beta}_j^* &= \hat{a}_j^{-1} A_{\hat{\theta}_j} b_j^*. \end{aligned}$$

Then, we define the criteria N_ϵ as

$$N_\epsilon(\alpha_2) = \frac{1}{J} \sum_{j=1}^J \int_{|\omega| \leq \omega_\epsilon} \left| \hat{Z}_j(\omega) - \frac{1}{J} \sum_{j'=1}^J \hat{Z}_{j'}(\omega) \right|^2, \quad \alpha_2 \in \mathcal{A}_2, \quad (4.27)$$

where ω_ϵ is appropriate smoothing parameter to be defined below. The following theorem provides the consistency of the M -estimator defined by

$$\hat{\alpha}_2^\epsilon = \arg \min_{\alpha_2 \in \mathcal{A}_2} N_\epsilon(\alpha_2).$$

Theorem 4.3 *Assume that the assumptions of Theorem 4.2 hold and that \hat{f} is 1-Lipschitz i.e. that there exists a constant $C > 0$ such that for all $\omega, \omega' \in \mathbb{R}^2$*

$$|\hat{f}(\omega) - \hat{f}(\omega')| \leq C \|\omega - \omega'\|.$$

Suppose that ω_ϵ is such that as $\epsilon \rightarrow 0$, then $\omega_\epsilon \rightarrow +\infty$ and $\epsilon \omega_\epsilon^2 = o(1)$. Then $\hat{\alpha}_2^\epsilon$ converges in probability to $\alpha_2^ = (b_1^*, \dots, b_J^*)$ for the standard Euclidean norm in \mathbb{R}^{2J} .*

The interpretation of the above Theorem, in particular the meaning of the condition $\epsilon \omega_\epsilon^2 = o(1)$ is as before. This gives a way to empirically choose the frequency cut-off parameter ω_ϵ to both align and denoise the observed images. Again, with stronger assumptions, one can prove the ϵ^{-1} -consistency of the estimator $\hat{\alpha}_2^\epsilon$. The proof of this result is similar to the proof of Theorem 4.2 but for reasons of space, we omit it.

5 Numerical performances and Perspectives

5.1 Simulations

In this section we report the results of some simulations to study the numerical performances of our two-step procedure. All the simulations have been carried out with Matlab. We took a simulated squared image f of size $N \times N$ as a reference (but unknown) template with $N = 100$. The image f is displayed in Figure 5.2(a), and $J = 6$ deformed images are created with various values for the scaling, translation and rotation parameters (see the bold numbers in Table 1). For simplicity, we denote the value of the image f_j at pixel (p_1, p_2) by $f_j(p_1, p_2)$ for $p_1 = -(N-1)/2, \dots, (N-1)/2$ and $p_2 = -(N-1)/2, \dots, (N-1)/2$. Noisy images Y_{p_1, p_2}^j are generated from the additive model

$$Y_{p_1, p_2}^j = f_j(p_1, p_2) + \tau z_{p_1, p_2}^j,$$

where τ is an unknown level of noise, and z_{p_1, p_2}^j are independent realizations of normal variables with zero mean and variance 1. An important quantity in the simulations is the root of the signal-to-noise ratio defined by $s2n = \text{std}(f) / \tau$, where $\text{std}(f)$ denotes the empirical standard deviation of the pixel-values of the image f . We present results for $s2n = 5$ and $s2n = 1$ (respectively a low and a high level of noise). A typical realization of a set of noisy images for $s2n = 1$ is displayed in Figure 5.3.

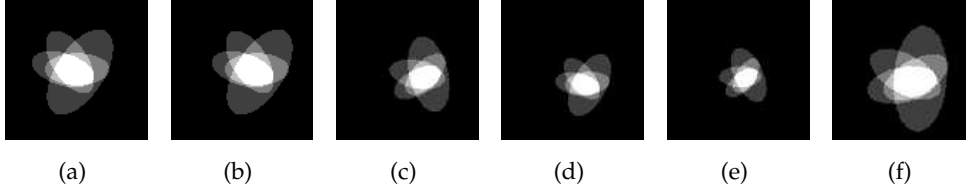


Figure 5.2: **Simulated images** f_j : (a) $j = 1$, (b) $j = 2$, (c) $j = 3$, (d) $j = 4$, (e) $j = 5$, (f) $j = 6$.

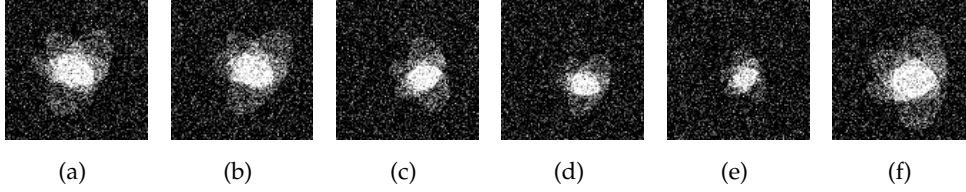


Figure 5.3: **Noisy images with $s2n = 1$** : (a) $j = 1$, (b) $j = 2$, (c) $j = 3$, (d) $j = 4$, (e) $j = 5$, (f) $j = 6$.

We have compared our two steps procedure with a direct minimization of the cost function (4.1). To evaluate the criterion (4.1) for a given set of parameters, we use cubic spline smoothing to reduce the noise in the observed images and bilinear interpolation to compute the deformation of the smoothed images by a set of scaling, rotation and translation parameters.

In the next paragraphs, we describe the implementation of our two steps procedure. To compute an approximation of the Fourier Mellin transform of a discrete image g defined on a squared grid of size $N \times N$, we have chosen to approximate first its Fourier transform \hat{g} on a polar grid (r_{q_1}, θ_{q_2}) with $r_{q_1} = \frac{q_1}{N}$, $\theta_{q_2} = 2\pi \frac{q_2-1}{N}$ and

$$\hat{g}(r_{q_1}, \theta_{q_2}) = \sum_{p_1=-(N-1)/2}^{(N-1)/2} \sum_{p_2=-(N-1)/2}^{(N-1)/2} e^{-i(p_1 r_{q_1} \cos(\theta_{q_2}) + p_2 r_{q_1} \sin(\theta_{q_2}))} g(p_1, p_2)$$

for $q_1 = 1, \dots, N$ and $q_2 = 1, \dots, N$. Faster and more accurate implementations of the discrete polar Fourier transform have been recently proposed (Averbuch *et al.* [2]), but we have not investigated this possibility as the goal of our simulations is to illustrate the good finite sample properties of our method in terms of estimation rather than providing a fast algorithm. For any $(k, v) \in \mathbb{Z} \times \mathbb{R}$, a numerical approximation of the AFMT of $|\hat{g}|^2$ is then obtained by

$$\mathcal{M}_{|\hat{g}|^2}(k, v) = \sum_{q_1=1}^N \sum_{q_2=1}^N |\hat{g}|^2(r_{q_1}, \theta_{q_2}) r_{q_1}^{\sigma-1-iv} e^{-ik\theta_{q_2}}.$$

For simplicity, we have chosen $\sigma = 1$ since this guarantees the conditions (4.14) and (4.20) of Theorems 4.1 and 4.2 for any $s > 0$. Finally, to compute the criteria $M_\epsilon(\alpha)$ defined by equation (4.11), it remains to choose the values of the smoothing parameters k_ϵ, v_ϵ and δ_ϵ . Our sampling of the polar Fourier domain corresponds to the choice $\delta_\epsilon = 1$ and we have found that the choices $k_\epsilon = 5$ and $v_\epsilon = 5$ yield satisfactory results for various values of the signal-to-noise ratio. Of course it would nice to have data-dependent smoothing parameters but it seems that

taking such small values for these parameters is a reasonable choice. Note that to approximate the integral over v in (4.11), we simply use $\{-v_\epsilon, -v_\epsilon + 1, -v_\epsilon + 2, \dots, v_\epsilon\}$ as a discretization of the integration interval $[-v_\epsilon, v_\epsilon]$.

For the set of noisy discrete images $Y^j, j = 1, \dots, J$ and for any vector of scaling and rotation parameters α , the criteria $M_\epsilon(\alpha)$ can then be numerically evaluated using the above approximation of the AFMT for $|\hat{Y}^j|^2$. To find the minimum of the criterion $M_\epsilon(\alpha)$ we use a gradient descent algorithm with an adaptive step, i.e. the following iterative procedure

$$\begin{aligned} a_j^{n+1} &= a_j^n - \gamma_n \frac{d}{da_j} M_\epsilon(\alpha^n) \\ \theta_j^{n+1} &= \theta_j^n - \gamma_n \frac{d}{d\theta_j} M_\epsilon(\alpha^n) \end{aligned}$$

where γ_n in the step (small enough number) at iteration n , with a standard stopping rule on the value of the objective function $M_\epsilon(\alpha^n)$. The above partial derivatives are given in Lemma 5.1. Note that for a fair comparison with our method we also use a gradient descent algorithm for a direct minimization of the criterion (4.1).

Once estimators \hat{a}_j and $\hat{\theta}_j$ for the scaling and rotation parameters (with the identifiability constraints $\hat{a}_1 = 1$ and $\hat{\theta}_1 = 0$) have been found, we compute the registered Fourier transform $\hat{a}_j^{-2} \hat{Y}^j(\hat{a}_j^{-1} A_{-\hat{\theta}_j} \omega_{q_1, q_2})$ with $\omega_{q_1, q_2} = (r_{q_1} \cos(\theta_{q_2}), r_{q_1} \sin(\theta_{q_2}))$ for $q_1, q_2 = 1, \dots, N$ using cubic spline interpolation with vanishing conditions at the boundaries of the polar Fourier domain. For any set of translation parameters α_2 the criteria $N_\epsilon(\alpha_2)$ is then numerically evaluated by replacing the integration over the set of frequencies such that $|\omega| \leq \omega_\epsilon$ in equation (4.27) by the sum over the discrete frequency ω_{q_1, q_2} for $q_1 = 1, \dots, q_\epsilon$ and $q_2 = 1, \dots, N$. In our simulations, we have found that the choice $q_\epsilon = 3$ gives good results which corresponds to the choice $\omega_\epsilon = 3/N$. Again we use the a gradient descent algorithm as described above to find the minimum of $N_\epsilon(\alpha_2)$.

For each value of the signal-to-noise ratio $s2n$ ($s2n = 1, 5$) we have simulated $M = 100$ sets of J noisy images. For each sample, an estimation of the scaling, translation and rotation parameters has been computed for each image with a direct minimization of the criterion (4.1) and with our two steps procedure. In Table 1 we give the empirical mean and standard deviation of these estimators over the $M = 100$ simulations. Note that we only give the results for $j = 2, \dots, 6$ as the estimators for the first image ($j = 1$) are fixed (due to the identifiability condition).

The results given in Table 1 show that our procedure performs very well for the estimation of the parameters for all images. As expected the standard deviation is inversely proportional to $s2n$, but it is always very small and the empirical means are very close to the true values. The results obtained with a direct minimization of (4.1) are satisfactory for the estimation of the scaling and translation parameters, but the estimation of the rotation is not very good for some images. Moreover, these simulations show that the standard deviation of our estimators are smaller. One of the drawbacks of a direct minimization of (4.1) by smoothing and interpolation is its computational time which is quite large compared to a search in the Fourier domain. We have tried to implement this direct minimization using downsampling of the images and

Table 1: Empirical mean and standard deviation (in brackets) of the estimators $\hat{a}_j, \hat{b}_j = (\hat{b}_j^1, \hat{b}_j^2), \hat{\theta}_j$ over $M = 100$ simulations for various values of the signal-to-noise ratio $s2n$. Bold numbers represent the true values of the parameters. The estimators, which are computed with the original cost function (4.1), are marked with #

	$s2n$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
a_j^*		1	0.8	0.7	0.6	1.1
\hat{a}_j	1	1.0183 (0.0096)	0.8181 (0.0080)	0.7169 (0.0091)	0.6150 (0.0087)	1.1182 (0.0126)
	1 [#]	1.0983 (0.0657)	0.8873 (0.0801)	0.7758 (0.0630)	0.6546 (0.0436)	1.1926 (0.0754)
	5	1.0144 (0.0020)	0.8144 (0.0015)	0.7136 (0.0018)	0.6117 (0.0018)	1.1155 (0.0026)
	5 [#]	1.0407 (0.0182)	0.8241 (0.0126)	0.7216 (0.0107)	0.6183 (0.0090)	1.1329 (0.0172)
$b_j^{1,*}$		10	12	10	5	8
\hat{b}_j^1	1	9.8494 (0.9937)	12.1135 (0.8859)	9.9208 (0.8268)	5.0669 (0.8694)	8.0786 (1.3407)
	1 [#]	5.7089 (2.4988)	7.2506 (2.1544)	7.2647 (1.7269)	4.2156 (2.0557)	4.5981 (2.7619)
	5	9.9342 (0.5550)	12.2087 (0.4991)	9.9960 (0.4208)	5.1820 (0.3394)	8.0946 (1.2855)
	5 [#]	6.9275 (1.2935)	10.9098 (0.8012)	9.0888 (1.0992)	4.6799 (0.5116)	6.7378 (1.2669)
b_j^2		0	4	10	5	6
$\hat{b}_j^{2,*}$	1	0.0618 (0.7259)	5.7963 (0.7836)	10.4611 (0.8580)	6.5932 (0.7247)	7.7587 (0.6854)
	1 [#]	3.9313 (2.5415)	4.7819 (1.8635)	7.2898 (1.5668)	4.6641 (1.6660)	5.0971 (2.6990)
	5	0.0519 (0.3594)	5.8635 (0.1954)	10.2942 (0.2702)	6.6405 (0.2022)	7.7411 (0.1553)
	5 [#]	1.9357 (1.2829)	4.3582 (1.1497)	9.6215 (0.5310)	5.3261 (0.9161)	5.7736 (1.1782)
θ_j^*		0	0.8	0.1	1	0.5
$\hat{\theta}_j$	1	0.0768 (0.3056)	0.8049 (0.2013)	0.1751 (0.3048)	0.8561 (0.3768)	0.5314 (0.1613)
	1 [#]	2.0337 (1.3737)	1.3412 (0.6931)	0.4580 (0.4936)	0.6534 (0.9766)	0.7636 (0.8225)
	5	0.0141 (0.0644)	0.8058 (0.0633)	0.1167 (0.0692)	0.9954 (0.0685)	0.5099 (0.0610)
	5 [#]	2.2372 (1.6292)	1.2330 (0.9403)	0.5284 (0.9350)	0.6904 (1.1277)	0.6889 (0.8373)

their gradient to reduce the computational cost but unfortunately subsampling yields worse estimates. Finally note that the regularizing parameter for smoothing the images with cubic spline has been chosen to obtain results as good as possible but we do not have any automatic rule to choose it. To the contrary, our analysis provides some data-based choices for the various regularizing parameters $k_\epsilon, v_\epsilon, \delta_\epsilon$ and ω_ϵ .

5.2 A real example

First, we return to the real example of the alignment of the butterfly images. We have used the same methodology as described in the previous section with the same choices for the various smoothing parameters. We align the three images of the first row of Figure 2.1 by choosing the image in Figure 2.1(a) as the “reference image” in the sense that the alignment will be performed with respect to the scale, orientation and location of this butterfly. We perform the same kind of alignment for the three images of the second row of Figure 2.1 with Figure 2.1(d) as the “reference image”. The result displayed in Figure 5.4 is very satisfactory as they are no

difference between the images after the alignment. Note that to register the images we have simply used a linear interpolation of the images.

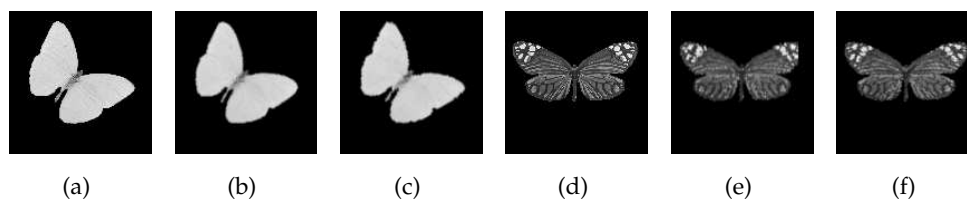


Figure 5.4: Alignment of the butterfly images.

Now, the two steps procedure is applied to a set of images which are more complex. We align the six images of the first row of Figure 5.5 by choosing the image in Figure 5.5(a) as "the reference images". It should be noted that these images do not fit exactly our mode since they are not embedded in a black background. The result is displayed in the second row of Figure 5.5: each image in the second row is the aligned image which is above. We observe that the images are approximately aligned in scale and in rotation, but they are misaligned in translation. This could be explained by the fact that the images are aligned on the average of the synchronized images (this effect is less visible when $J = 2$), but clearly the results are not very satisfactory.

Note that a qualitative measure of alignment could be derived from the value of the criterion (4.1) after alignment of the images, but we have not tried to use this. A perspective is to develop statistical inference, such as testing procedures, in order to decide if the images are aligned or not. The test statistic may be defined from our criteria. We refer to [29], [19] [38], which treat this question for nearby models.

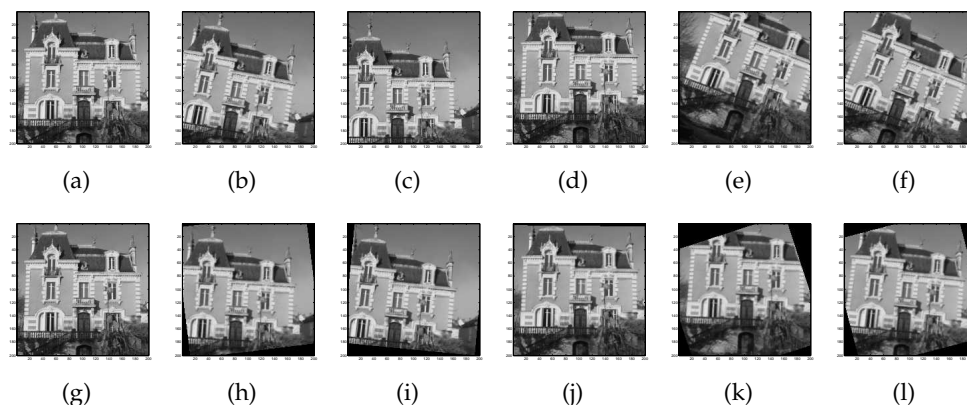


Figure 5.5: Alignment of the house images.

5.3 A synthetic example with clutter noise

The example shown in Figure 5.5 suggests that our approach does not perform well if the noise is not gaussian but rather a clutter type noise. To study this, another synthetic example has

been created with the simulated images of Figure 5.2 to which a white circle of small radius has been added outside the support of the deformed template to represent a kind of clutter noise. Gaussian noise is then added to these perturbed images, see Figure 5.6. For $s2n = 1, 5$, we have simulated $M = 100$ sets of J noisy images. Again we have compared our approach with a direct minimization of (4.1) using cubic spline smoothing and bilinear interpolation. Results are given in Table 2. Both approaches generally fail in recovering the true parameters. Hence, this tends to show that the proposed matching criterion (4.1) is certainly not very robust from deviations of the ideal model (2.1).

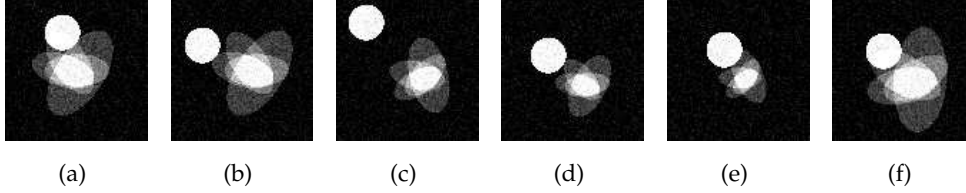


Figure 5.6: **Simulated images with clutter noise and $s2n = 5$:** (a) $j = 1$, (b) $j = 2$, (c) $j = 3$, (d) $j = 4$, (e) $j = 5$, (f) $j = 6$.

Table 2: Simulations with clutter noise: empirical mean and standard deviation (in brackets) of the estimators $\hat{a}_j, \hat{b}_j = (\hat{b}_j^1, \hat{b}_j^2), \hat{\theta}_j$ over $M = 100$ simulations for $s2n = 5$. Bold numbers represent the true values of the parameters. The estimators, which are computed with the original cost function (4.1), are marked with #

	$s2n$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
a_j^*		1	0.8	0.7	0.6	1.1
\hat{a}_j	5	1.0663 (0.0021)	0.9528 (0.0019)	0.8718 (0.0016)	0.8277 (0.0018)	1.1720 (0.0029)
	5 [#]	1.3134 (0.4041)	1.4004 (0.5314)	1.0881 (0.6529)	0.9606 (0.3887)	1.1768 (0.4759)
$b_j^{1,*}$		10	12	10	5	8
\hat{b}_j^1	5	-6.1134 (0.1416)	-4.8594 (0.1489)	-4.6419 (0.1563)	-2.5379 (0.2103)	1.0387 (0.1662)
	5 [#]	4.8926 (1.4939)	6.9050 (5.4946)	6.1604 (1.2638)	4.1619 (1.0381)	8.2347 (1.2291)
b_j^2		0	4	10	5	6
$\hat{b}_j^{2,*}$	5	3.8804 (0.6389)	-1.4111 (0.3152)	8.3201 (0.3139)	2.1259 (0.2667)	12.0945 (0.3421)
	5 [#]	0.7085 (1.9110)	2.0118 (2.1931)	8.2228 (0.8983)	5.6468 (1.0016)	8.7203 (0.9404)
θ_j^*		0	0.8	0.1	1	0.5
$\hat{\theta}_j$	5	0.8653 (0.0064)	0.5269 (0.0066)	0.6351 (0.0059)	0.3515 (0.0058)	0.4840 (0.0073)
	5 [#]	0.6917 (0.0527)	0.6302 (0.5071)	0.5230 (0.0398)	0.2375 (0.0399)	0.4284 (0.0359)

5.4 Perspectives

Developing a fully satisfactory theory for an asymptotic analysis as J grows to infinity remains a challenge. Some works in this direction have been recently proposed in one-dimensional and

two-dimensional settings. In such models, the scaling, rotation and translation are not fixed parameters but are realization of random variables which makes the problem different. The goal is then to estimate the template of the signals, and also to estimate of the law governing such random deformation parameters. For more details, we refer to the papers by Chafaï and Loubes [8], Bigot *et al.* [4], Ke and Wang [21], and Allasonnière *et al.* [1].

An interesting extension of this work would be to compare from a theoretical point of view our two steps procedure with the direct minimization (4.1) by spline smoothing and bilinear interpolation as investigated in our simulated experiments.

Finally, alignment methods for 3D images is also an active field of research in the domain of medical images. These methods can be based for example on entropy measures (see Studholme *et al.* [34] for a detailed review). Therefore, it would be interesting to extend the present methodology based on Fourier transform to a 3D setting.

Appendix

Proof of Theorem 4.1

To simplify the notation we write $\alpha = \alpha_1$ for $\alpha_1 \in \mathcal{A}_1$ and $\alpha^* = \alpha_1^*$. The proof of this theorem follows the classical guidelines of the convergence of M -estimators. We shall prove that $M(\alpha)$ has an unique minimum at $\alpha = \alpha^*$, and that M_ϵ converges uniformly (in α) in probability to M i.e.

$$\sup_{\alpha \in \mathcal{A}_1} |M_\epsilon(\alpha) - M(\alpha)| \rightarrow 0 \text{ in probability as } \epsilon \rightarrow 0.$$

Then, these two conditions ensure that $\hat{\alpha}_\epsilon$ converges in probability to α^* as $\epsilon \rightarrow 0$ (see e.g. Theorem 5.7 in van der Vaart [36]).

Unicity of the minimum of $M(\alpha)$: recall that

$$M(\alpha) = \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left| \mathcal{M}_{|\hat{f}|^2}(k, v) \right|^2 \frac{1}{J} \sum_{j=1}^J \left| \left(\frac{a_j}{a_j^*} \right)^{-4+\sigma-iv} e^{ik(\theta_j^* - \theta_j)} - \frac{1}{J} \sum_{l=1}^J \left(\frac{a_l}{a_l^*} \right)^{-4+\sigma-iv} e^{ik(\theta_l^* - \theta_l)} \right|^2.$$

has a minimum at α^* such that $M(\alpha^*) = 0$. Assume that there exists $\alpha \in \mathcal{A}_1$ such that $M(\alpha) = 0$. Since \hat{f} is not rotation invariant, $(k, v) \mapsto \mathcal{M}_{|\hat{f}|^2}(k, v)$ is a non zero function. Let (k, v) be in $\mathbb{Z} \times \mathbb{R}$ such that $\mathcal{M}_{|\hat{f}|^2}(k, v) \neq 0$. Then, the condition $M(\alpha) = 0$ implies that:

$$\frac{1}{J} \sum_{j=1}^J \left| \left(\frac{a_j}{a_j^*} \right)^{-4+\sigma-iv} e^{ik(\theta_j^* - \theta_j)} - \frac{1}{J} \sum_{l=1}^J \left(\frac{a_l}{a_l^*} \right)^{-4+\sigma-iv} e^{ik(\theta_l^* - \theta_l)} \right|^2 = 0.$$

This means that there exists $\lambda_{k,v} \in \mathbb{C}$ such that

$$\left(\frac{a_j}{a_j^*} \right)^{-4+\sigma-iv} e^{ik(\theta_j^* - \theta_j)} = \lambda_{k,v}, \quad \forall j = 1 \dots J. \quad (5.1)$$

Using the identifiability constraints ($a_1 = a_1^* = 1$ and $\theta_1 = \theta_1^* = 0$) we deduce that $\lambda_{k,v} = 1$. Considering the complex modulus, we obtain that:

$$\left(\frac{a_j}{a_j^*}\right)^{-4+\sigma} = 1, \quad \forall j = 1 \dots J.$$

If $\sigma \neq 4$, we deduce that for all $j = 1, \dots, J$, $a_j = a_j^*$. If $\sigma = 4$, since the function $v' \rightarrow \mathcal{M}_{|\hat{f}|^2}(k, v')$ is continuous and non null, there exist v_0 and v_1 such that $v - v_0 \in \mathbb{Q}$, $v - v_1 \in \mathbb{R} \setminus \mathbb{Q}$ (because \mathbb{Q} and $\mathbb{R} \setminus \mathbb{Q}$ are dense in \mathbb{R}) and $\mathcal{M}_{|\hat{f}|^2}(k, v_0) \neq 0$, $\mathcal{M}_{|\hat{f}|^2}(k, v_1) \neq 0$. Then, from equation (5.1), we have,

$$e^{-i(v-v_0)\log(a_j/a_j^*)} = 1 \quad \text{and} \quad e^{-i(v-v_1)\log(a_j/a_j^*)} = 1, \quad j = 1 \dots J.$$

Necessarily, for all $j = 1 \dots J$, we have that $\log(a_j/a_j^*) = 0$. We deduce that for all $j = 1, \dots, J$, $a_j = a_j^*$.

Moreover since f is not rotation invariant, we have that there exists two relative prime integers k and k' such that $\mathcal{M}_{|\hat{f}|^2}(2k, v) \neq 0$ and $\mathcal{M}_{|\hat{f}|^2}(2k', v) \neq 0$, which implies from (5.1) (since $a = a^*$) that for all $j = 1, \dots, J$

$$e^{i2k(\theta_j^* - \theta_j)} = 1 \quad \text{and} \quad e^{i2k'(\theta_j^* - \theta_j)} = 1.$$

Hence if there exists some $j \geq 2$ such that $\theta_j^* - \theta_j \neq 0$ then the above equations implies that there exists two integers p and p' such that

$$\frac{k}{p} = \frac{\pi}{\theta_j^* - \theta_j} = \frac{k'}{p'}.$$

Hence, unless $\theta_j^* - \theta_j = \pi$, the above equation is a contradiction since k and k' are two relative prime integers. Hence, this implies that for all $j = 1, \dots, J$, $\theta_j = \theta_j^*$ (modulo $\pi\mathbb{Z}$), which proves the unicity of the minimum of $M(\alpha)$ (modulo $\pi\mathbb{Z}$ for the translation parameters). \square

Uniform convergence of M_ϵ : in the proof C will denote a constant whose value may change from line to line. Recall that:

$$\begin{aligned} c_j^\epsilon(k, v) &= a_j^{-4+\sigma-iv} e^{-ik\theta_j} \mathcal{M}_{|\hat{f}_j|^2}^\epsilon(k, v), \\ s_j^\epsilon(k, v) &= a_j^{-4+\sigma-iv} e^{-ik\theta_j} \mathcal{M}_{w_j}^\epsilon(k, v), \\ \tilde{c}^\epsilon &= \frac{1}{J} \sum_{j=1}^J c_j^\epsilon \quad \text{and} \quad \tilde{s}^\epsilon = \frac{1}{J} \sum_{j=1}^J s_j^\epsilon. \end{aligned}$$

and remark that $M_\epsilon(\alpha)$ is the sum of three terms using (4.9):

$$M_\epsilon(\alpha) = D_\epsilon(\alpha) + \epsilon L_\epsilon(\alpha) + \epsilon^2 Q_\epsilon(\alpha), \tag{5.2}$$

where

$$\begin{aligned}
D_\epsilon(\alpha) &= \frac{1}{J} \sum_{j=1}^J \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |c_j^\epsilon(k, v)|^2 dv - \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |\tilde{c}^\epsilon(k, v)|^2 dv \\
L_\epsilon(\alpha) &= \frac{2\epsilon}{J} \sum_{j=1}^J \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \Re \left\{ \left(c_j^\epsilon(k, v) - \tilde{c}^\epsilon(k, v) \right) \overline{\left(s_j^\epsilon(k, v) - \tilde{s}^\epsilon(k, v) \right)} \right\} dv \\
Q_\epsilon(\alpha) &= \frac{\epsilon^2}{J} \sum_{j=1}^J \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |s_j^\epsilon(k, v)|^2 dv - \epsilon^2 \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |\tilde{s}^\epsilon(k, v)|^2 dv,
\end{aligned}$$

where $\Re(x)$ denotes the real part of a complex number x . Let us notice that by applying the Cauchy Schwarz inequality we have that:

$$|\epsilon L_\epsilon(\alpha)| \leq 2 \left\{ \sup_{(\alpha) \in \tilde{\mathcal{A}}_1} |D_\epsilon(\alpha) - M(\alpha)| + \sup_{(\alpha) \in \tilde{\mathcal{A}}_1} M(\alpha) \right\}^{1/2} \left\{ \sup_{(\alpha) \in \tilde{\mathcal{A}}_1} \epsilon^2 Q_\epsilon(\alpha) \right\}^{1/2}$$

Since the function M is continuous on the compact set $\tilde{\mathcal{A}}_1$, we have just to consider the uniform convergence of D_ϵ to M and the uniform convergence in probability of $\epsilon^2 Q_\epsilon$ to zero.

First, we study the uniform convergence of $D_\epsilon(\alpha)$ to $M(\alpha)$. Since α belongs to a compact set, and using the Cauchy Schwarz inequality, it suffices to prove that for a fixed $1 \leq j \leq J$ the following term converges to zero:

$$(I) = \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |\mathcal{M}_{|\hat{f}_j|^2}^\epsilon(k, v)|^2 dv - \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |\mathcal{M}_{|\hat{f}_j|^2}(k, v)|^2 dv.$$

Using the inequality $|a|^2 - |b|^2 \leq |a - b|^2 + 2|(a - b)b| \forall (a, b) \in \mathbb{C}^2$, we have

$$|(I)| \leq (I - 1) + (I - 2) + (I - 3)$$

where

$$\begin{aligned}
(I - 1) &= \left\{ \int_{|v| > v_\epsilon} \sum_{|k| \leq k_\epsilon} + \int_{|v| > v_\epsilon} \sum_{|k| > k_\epsilon} + \int_{|v| \leq v_\epsilon} \sum_{|k| > k_\epsilon} \right\} |\mathcal{M}_{|\hat{f}_j|^2}(k, v)|^2 dv, \\
(I - 2) &= \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \left| \mathcal{M}_{|\hat{f}_j|^2}^\epsilon(k, v) - \mathcal{M}_{|\hat{f}_j|^2}(k, v) \right|^2 dv, \\
(I - 3) &= 2 \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |\mathcal{M}_{|\hat{f}_j|^2}(k, v)| \left| \mathcal{M}_{|\hat{f}_j|^2}^\epsilon(k, v) - \mathcal{M}_{|\hat{f}_j|^2}(k, v) \right| dv.
\end{aligned}$$

Since Assumption (4.12) holds, the term $(I - 1)$ converges to zero. Furthermore, due to Assumption (4.13) and to Assumption (4.14), we have:

$$\left| \mathcal{M}_{|\hat{f}_j|^2}^\epsilon(k, v) - \mathcal{M}_{|\hat{f}_j|^2}(k, v) \right| \leq \int_{|\omega| > \delta_\epsilon} |\hat{f}(\omega)|^2 |\omega|^{2s} \frac{d\omega}{\delta_\epsilon^{2s+2-\sigma}}. \quad (5.3)$$

Then, using the Cauchy Schwarz inequality and since $\delta_\epsilon \rightarrow \infty$, we have

$$(I - 2) + (I - 3) = o(k_\epsilon v_\epsilon \delta_\epsilon^{-4s-4+2\sigma} + k_\epsilon^{1/2} v_\epsilon^{1/2} \delta_\epsilon^{-2s-2+\sigma}).$$

From Assumption (4.15), D_ϵ converges uniformly to M .

We show that $\epsilon^2 Q_\epsilon$ converges uniformly in probability to 0. Let $j \in \{1 \dots J\}$ be fixed. Using the Cauchy Schwarz inequality and that the parameter (θ, a) lies in a compact set, it suffices to consider the variable (II):

$$(II) = \epsilon^2 \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |\mathcal{M}_{w_j}^\epsilon|^2(k, v) dv.$$

Using the inequality $|a + b|^2 \leq 2|a|^2 + 2|b|^2 \quad \forall (a, b) \in \mathbb{C}^2$, we have that:

$$(II) \leq 8(II - 1) + 2(II - 2),$$

where

$$(II - 1) = \epsilon^2 \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |\mathcal{M}_{\Re \hat{f}_j \overline{W}_j}^\epsilon(k, v)|^2 dv,$$

$$(II - 2) = \epsilon^4 \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |\mathcal{M}_{|\hat{W}_j|^2}^\epsilon(k, v)|^2 dv.$$

We compute an upper-bound for the expectation of the term (II - 1). First we have:

$$\begin{aligned} \mathbb{E} |\mathcal{M}_{\Re \hat{f}_j \overline{W}_j}^\epsilon(k, v)|^2 &= \int_{\mathcal{D}} \left| \int_0^{2\pi} \int_0^{\delta_\epsilon} \left(\cos(w \cdot x) \Re \hat{f}_j(\omega) - \sin(w \cdot x) \Im \hat{f}_j(\omega) \right) e^{-ik\theta} r^{\sigma-iv} \frac{dr d\theta}{2\pi r} \right|^2 dx \\ &\leq d^2 \left| \int_0^{2\pi} \int_0^{\delta_\epsilon} |\hat{f}_j(\omega)| r^{\sigma-1} \frac{dr d\theta}{2\pi} \right|^2, \end{aligned}$$

for $\omega = (r \cos(\theta), r \sin(\theta))$, and where $\Im(c)$ denotes the imaginary part of a complex number c . Since $\sigma - 2 < 2s$, and using the Cauchy Schwarz inequality, we obtain that:

$$\mathbb{E} |\mathcal{M}_{\Re \hat{f}_j \overline{W}_j}^\epsilon(k, v)|^2 \leq d^2 \int_{|w| \leq \delta_\epsilon} |\hat{f}_j(\omega)|^2 |\omega|^{\sigma-2} \frac{d\omega}{2\pi} \int_{|w| \leq \delta_\epsilon} |\omega|^{\sigma-2} \frac{d\omega}{2\pi}. \quad (5.4)$$

Then $\mathbb{E} |\mathcal{M}_{\Re \hat{f}_j \overline{W}_j}^\epsilon(k, v)|^2 = \mathcal{O}(\delta_\epsilon^\sigma)$. Similarly, we get:

$$\mathbb{E} |\mathcal{M}_{|\hat{W}_j|^2}^\epsilon(k, v)|^2 = \mathcal{O}(\delta_\epsilon^{2\sigma}). \quad (5.5)$$

Then, Assumption (4.16) ensures the uniform convergence in probability to zero of (II) because:

$$\mathbb{E}(II) = \mathcal{O}(\epsilon^2 k_\epsilon v_\epsilon \delta_\epsilon^\sigma (1 + \epsilon^2 \delta_\epsilon^\sigma)).$$

□

Proof of Theorem 4.2

Again, the proof of this theorem follows the classical guidelines of the convergence of M -estimators. Recall that the M -estimator is defined as the minimum of the criterion function $M_\epsilon(\cdot)$. Hence, we get :

$$\nabla M_\epsilon(\hat{\alpha}_\epsilon) = 0,$$

where ∇ is the gradient operator. Thanks to a second order expansion, there exists $\bar{\alpha}_\epsilon$ in a neighborhood of (a^*, θ^*) such that

$$\nabla^2 M_\epsilon(\bar{\alpha}_\epsilon) \epsilon^{-1}(\hat{\alpha}_\epsilon - a^*) = -\epsilon^{-1} \nabla M_\epsilon(a^*),$$

where ∇^2 is the Hessian operator. In the sequel we will prove the following two results: the gradient converges in distribution to a Gaussian variable (see Proposition 5.1), and the hessian matrix converges uniformly in probability to an invertible matrix H (see Proposition 5.2). Then, using Slutsky's Lemma, Theorem 4.2 follows.

Proposition 5.1 and Proposition 5.2 require the calculation of the first and second derivative of M_ϵ . After some calculus, we have the following lemma:

Lemma 5.1 *Let $2 \leq j, l \leq J$ be two integers such that $j \neq l$. For $\alpha \in \tilde{\mathcal{A}}_1$, the first and second order partial derivatives of M_ϵ are:*

$$\begin{aligned} \frac{d}{da_j} M_\epsilon(\alpha) &= \frac{2\Re}{J} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \frac{-4 + \sigma - iv}{a_j} \left\{ |d_j^\epsilon(k, v)|^2 - d_j^\epsilon(k, v) \overline{\tilde{d}^\epsilon(k, v)} \right\} dv, \\ \frac{d}{d\theta_j} M_\epsilon(\alpha) &= \frac{2\Re}{J} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \left\{ ik d_j^\epsilon(k, v) \overline{\tilde{d}^\epsilon(k, v)} \right\} dv, \\ \frac{d^2}{da_j^2} M_\epsilon(\alpha) &= \frac{2\Re}{J} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \frac{(-4 + \sigma - iv)(-5 + \sigma - iv)}{a_j^2} \left\{ 2|d_j^\epsilon(k, v)|^2 - d_j^\epsilon(k, v) \overline{\tilde{d}^\epsilon(k, v)} \right\} dv \\ &\quad - \frac{2\Re}{J} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \frac{|-4 + \sigma - iv|^2}{a_j^2} 2|d_j^\epsilon(k, v)|^2 dv, \\ \frac{d^2}{da_j da_l} M_\epsilon(\alpha) &= \frac{-2\Re}{J^2} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \frac{|-4 + \sigma - iv|^2}{a_j a_l} d_j^\epsilon(k, v) \overline{\tilde{d}_l^\epsilon(k, v)} dv, \\ \frac{d^2}{d\theta_j^2} M_\epsilon(\alpha) &= \frac{2\Re}{J} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} k^2 \left\{ d_j^\epsilon(k, v) \overline{\tilde{d}^\epsilon(k, v)} - |d_j^\epsilon(k, v)|^2 / J \right\} dv, \\ \frac{d^2}{d\theta_j d\theta_l} M_\epsilon(\alpha) &= \frac{-2\Re}{J^2} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} k^2 d_j^\epsilon(k, v) \overline{\tilde{d}_l^\epsilon(k, v)} dv, \\ \frac{d^2}{d\theta_j a_j} M_\epsilon(\alpha) &= \frac{2\Re}{J} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \frac{k}{a_j} \left\{ i \frac{-4 + \sigma + iv}{J} |d_j^\epsilon(k, v)|^2 + i(4 - \sigma + iv) d_j^\epsilon(k, v) \overline{\tilde{d}^\epsilon(k, v)} \right\} dv, \\ \frac{d^2}{d\theta_j a_l} M_\epsilon(\alpha) &= \frac{2\Re}{J^2} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} ik \frac{-4 + \sigma + iv}{a_l} d_j^\epsilon(k, v) \overline{\tilde{d}^\epsilon(k, v)} dv. \end{aligned}$$

Proposition 5.1 *Under assumptions and notations of Theorem 4.2, the random variable $\epsilon^{-1} \nabla M_\epsilon(\alpha^*)$ is tight.*

Proof of Proposition 5.1: First, using the notations of the proof of Theorem 4.1, we show that

$$\epsilon^{-1} \nabla M_\epsilon(\alpha^*) = \nabla L_\epsilon(\alpha^*) + o_{\mathbb{P}}(1).$$

At the end, we prove that the vector $\epsilon \nabla L_\epsilon(\alpha^*)$ is tight.

From Lemma 5.1, and after straightforward computations, we obtain that for $j = 2 \dots J$:

$$\begin{aligned}\frac{d}{da_j}D_\epsilon(\alpha^*) &= \frac{2\Re}{J} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \frac{-4 + \sigma - iv}{a_j^*} \left\{ |c_j^\epsilon(k, v)|^2 - c_j^\epsilon(k, v) \overline{\tilde{c}^\epsilon(k, v)} \right\} dv, \\ \frac{d}{d\theta_j}D_\epsilon(\alpha^*) &= \frac{2\Re}{J} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \left\{ ikc_j^\epsilon(k, v) \overline{\tilde{c}^\epsilon(k, v)} \right\} dv.\end{aligned}$$

Notice that:

$$\begin{aligned}0 &= \Re \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \frac{iv}{a_j^*} \left\{ |\mathcal{M}_{|\hat{f}|^2}(k, v)|^2 - \mathcal{M}_{|\hat{f}|^2}(k, v) \overline{\mathcal{M}_{|\hat{f}|^2}(k, v)} \right\} dv, \\ 0 &= \Re \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \left\{ ik\mathcal{M}_{|\hat{f}|^2}(k, v) \overline{\mathcal{M}_{|\hat{f}|^2}(k, v)} \right\} dv.\end{aligned}$$

Then, we get:

$$\begin{aligned}\frac{d}{da_j}D_\epsilon(\alpha^*) &= \mathcal{O} \left\{ \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |v| \left| \mathcal{M}_{|\hat{f}|^2}^\epsilon(k, v) - \mathcal{M}_{|\hat{f}|^2}(k, v) \right|^2 dv \right\} \\ &\quad + \mathcal{O} \left\{ \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |v| |\mathcal{M}_{|\hat{f}|^2}(k, v)| \left| \mathcal{M}_{|\hat{f}|^2}^\epsilon(k, v) - \mathcal{M}_{|\hat{f}|^2}(k, v) \right| dv \right\}, \\ \frac{d}{d\theta_j}D_\epsilon(\alpha^*) &= \mathcal{O} \left\{ \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |k| \left| \mathcal{M}_{|\hat{f}|^2}^\epsilon(k, v) - \mathcal{M}_{|\hat{f}|^2}(k, v) \right|^2 dv \right\} \\ &\quad + \mathcal{O} \left\{ \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} |k| |\mathcal{M}_{|\hat{f}|^2}(k, v)| \left| \mathcal{M}_{|\hat{f}|^2}^\epsilon(k, v) - \mathcal{M}_{|\hat{f}|^2}(k, v) \right| dv \right\}.\end{aligned}$$

Due to Assumption (4.19) and to Assumption (4.20), the inequality (5.3) holds. Consequently, we get:

$$\begin{aligned}\frac{d}{da_j}D_\epsilon(a^*, \theta^*) &= o \left(v_\epsilon^2 k_\epsilon \delta_\epsilon^{-4s-4+2\sigma} + \delta_\epsilon^{-2s-2+\sigma} \right), \\ \frac{d}{d\theta_j}D_\epsilon(a^*, \theta^*) &= o \left(v_\epsilon k_\epsilon^2 \delta_\epsilon^{-4s-4+2\sigma} + \delta_\epsilon^{-2s-2+\sigma} \right).\end{aligned}$$

Since Assumption (4.23) holds, we deduce that $\epsilon^{-1} \nabla D_\epsilon(\alpha^*)$ converges to 0.

From Lemma 5.1, we obtain for $j = 2 \dots J$:

$$\epsilon \frac{d}{da_j} Q_\epsilon(\alpha^*) = \frac{2\Re}{J^2} \sum_{\substack{l=1 \dots J \\ l \neq j}} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \frac{-4 + \sigma - iv}{a_j^*} s_j^\epsilon(k, v) \overline{\tilde{s}_l^\epsilon(k, v)} dv \quad (5.6)$$

$$+ \frac{2\Re}{J} \left(1 - \frac{1}{J}\right) \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \frac{-4 + \sigma - iv}{a_j^*} |s_j^\epsilon(k, v)|^2 dv, \quad (5.7)$$

$$\epsilon \frac{d}{d\theta_j} Q_\epsilon(\alpha^*) = \epsilon \frac{2\Re}{J^2} \sum_{\substack{l=1 \dots J \\ l \neq j}} \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \left\{ iks_j^\epsilon(k, v) \overline{\tilde{s}_l^\epsilon(k, v)} \right\} dv. \quad (5.8)$$

Using the computations (5.4) and (5.5) in the proof of Theorem 4.1, we get that:

$$(5.7) = \mathcal{O}_{\mathbb{P}}(\epsilon k_{\epsilon} v_{\epsilon} \delta_{\epsilon}^{\sigma} (1 + \epsilon^2 \delta_{\epsilon}^{\sigma})).$$

Since Assumption (4.24) holds, (5.7) converges in probability to 0. Similarly, we get:

$$\mathbb{E}|(5.6)| = \mathcal{O}(\epsilon k_{\epsilon} v_{\epsilon}^2 \delta_{\epsilon}^{\sigma} (1 + \epsilon^2 \delta_{\epsilon}^{\sigma})),$$

$$\mathbb{E}|(5.8)| = \mathcal{O}_{\mathbb{P}}(\epsilon k_{\epsilon}^2 v_{\epsilon} \delta_{\epsilon}^{\sigma} (1 + \epsilon^2 \delta_{\epsilon}^{\sigma})).$$

Then it results that $\epsilon \nabla Q_{\epsilon}$ converges in probability to 0 from Assumption (4.24). Consequently, using the equation (5.2), we have:

$$\epsilon^{-1} \nabla M_{\epsilon}(\alpha^*) = \epsilon \nabla L_{\epsilon}(\alpha^*) + o_{\mathbb{P}}(1).$$

From Lemma 5.1, we obtain that for $j = 2 \dots J$:

$$\frac{d}{da_j} L_{\epsilon}(\alpha^*) = \frac{2\Re}{J} \int_{|v| \leq v_{\epsilon}} \sum_{|k| \leq k_{\epsilon}} \frac{-4 + \sigma - iv}{a_j^*} \left\{ c_j^{\epsilon}(k, v) \overline{(s_j^{\epsilon} - \tilde{s}^{\epsilon})(k, v)} - (c_j^{\epsilon} - \tilde{c}^{\epsilon})(k, v) \overline{s_j^{\epsilon}(k, v)} \right\} dv,$$

$$\frac{d}{d\theta_j} L_{\epsilon}(\alpha^*) = \frac{2\Re}{J} \int_{|v| \leq v_{\epsilon}} \sum_{|k| \leq k_{\epsilon}} ik \left\{ c_j^{\epsilon}(k, v) \overline{\tilde{s}^{\epsilon}(k, v)} - \tilde{c}_j^{\epsilon}(k, v) \overline{s_j^{\epsilon}(k, v)} \right\} dv.$$

From Equation (5.3), we have:

$$c_j^{\epsilon}(k, v) = \mathcal{M}_{|\hat{f}|^2}(k, v) + o(\delta_{\epsilon}^{-2s-2+\sigma}) \quad \forall j = \dots J.$$

Moreover, using the result of the proof of Theorem 4.1, the following properties hold:

$$\mathbb{E}|s_j^{\epsilon}(k, v)| = \mathcal{O}(\delta_{\epsilon}^{\sigma/2} + \epsilon \delta_{\epsilon}^{\sigma}) \quad \forall j = \dots J.$$

Then, the partial derivatives of L_n may be rewritten as:

$$\begin{aligned} \frac{d}{da_j} L_{\epsilon}(\alpha^*) &= \frac{2\Re}{J} \int_{|v| \leq v_{\epsilon}} \sum_{|k| \leq k_{\epsilon}} \frac{-4 + \sigma - iv}{a_j^*} \left\{ \mathcal{M}_{|\hat{f}|^2}(k, v) \overline{(s_j^{\epsilon} - \tilde{s}^{\epsilon})(k, v)} \right\} dv \\ &\quad + o_{\mathbb{P}}(k_{\epsilon} v_{\epsilon}^2 \delta_{\epsilon}^{-2s-2+3\sigma/2} (1 + \epsilon \delta_{\epsilon}^{\sigma/2})), \\ \frac{d}{d\theta_j} L_{\epsilon}(\alpha^*) &= \frac{2\Re}{J} \int_{|v| \leq v_{\epsilon}} \sum_{|k| \leq k_{\epsilon}} ik \left\{ \mathcal{M}_{|\hat{f}|^2}(k, v) \overline{\tilde{s}^{\epsilon}(k, v)} - \mathcal{M}_{|\hat{f}|^2}(k, v) \overline{s_j^{\epsilon}(k, v)} \right\} dv, \\ &\quad + o_{\mathbb{P}}(k_{\epsilon}^2 v_{\epsilon} \delta_{\epsilon}^{-2s-2+3\sigma/2} (1 + \epsilon \delta_{\epsilon}^{\sigma/2})). \end{aligned}$$

Since $\mathbb{E}|\mathcal{M}_{|\hat{W}_j(\omega)|^2}^{\epsilon}(k, v)| = \mathcal{O}(\delta_{\epsilon}^{\sigma})$, we have that

$$s_j^{\epsilon}(k, v) = a_j^* - 4 + \sigma - iv e^{-ik\theta_j^*} \mathcal{M}_{2\Re\{\hat{f}_j \overline{\hat{W}_j}\}}^{\epsilon}(k, v) + \mathcal{O}_{\mathbb{P}}(\epsilon \delta_{\epsilon}^{\sigma}).$$

Consequently from Assumption (4.18) and Assumption (4.24-4.25),

$$\begin{aligned} \frac{d}{da_j} L_{\epsilon}(\alpha^*) &= \frac{2\Re}{J} (V_{a,j} - \frac{1}{J} \sum_{l=1}^J V_{a,l}) + o_{\mathbb{P}}(1), \\ \frac{d}{d\theta_j} L_{\epsilon}(\alpha^*) &= \frac{2\Re}{J} (\frac{1}{J} \sum_{l=1}^J V_{\theta,l} - V_{\theta,j}) + o_{\mathbb{P}}(1), \end{aligned}$$

where

$$V_{a,l} = \Re \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} \frac{-4 + \sigma - iv}{a_j^*} \left\{ \mathcal{M}_{|\hat{f}|^2}(k, v) a_l^{*-4 + \sigma + iv} e^{ik\theta_l^*} \overline{\mathcal{M}_{2\Re\{\hat{f}_l \overline{\hat{W}_l}\}}(k, v)} \right\} dv$$

$$V_{\theta,l} = \Re \int_{|v| \leq v_\epsilon} \sum_{|k| \leq k_\epsilon} ik \mathcal{M}_{|\hat{f}|^2}(k, v) a_l^{*-4 + \sigma + iv} e^{ik\theta_l^*} \overline{\mathcal{M}_{2\Re\{\hat{f}_l \overline{\hat{W}_l}\}}(k, v)} dv.$$

Then, it suffices to study the convergence of the two centered variables $V_{a,l}$ and $V_{\theta,l}$ (for a fixed $l = 1 \dots J$)

Let g denotes the function $r^\sigma |\hat{f}|^2(r, \theta)$, and h denotes the function $r^\sigma \hat{f}(r, \theta)$. From assumption (4.20), these functions are in $L^2(\mathcal{G})$. Let $\partial_r g$ and $\partial_\theta g$ be respectively the partial derivatives of g respect to the polar coordinates r and θ . After straightforward computations and using the inverse AFMT, we obtain:

$$V_{a,l} = \int_{\mathcal{D}} dW_l(x) \frac{2}{a_j^*} \left\langle (-4 + \sigma)g - \partial_r g, \Re \left(h e^{i\omega \cdot \frac{1}{a_l^*} A_{\theta_l^*} x} \right) \right\rangle_{L^2(\mathcal{G})} + o_{\mathbb{P}}(1),$$

$$V_{\theta,l} = \int_{\mathcal{D}} dW_l(x) 2 \left\langle \partial_\theta g, \Re \left(h e^{i\omega \cdot \frac{1}{a_l^*} A_{\theta_l^*} x} \right) \right\rangle_{L^2(\mathcal{G})} + o_{\mathbb{P}}(1).$$

Then we deduce that,

$$\begin{aligned} \text{var} \left(\frac{d}{da_j} L_\epsilon(\alpha^*) \right) &= \frac{16}{a_j^{*2} J^4} \sum_{l \neq j} \int_{\mathcal{D}} \left\langle (-4 + \sigma)g - \partial_r g, \Re \left(h e^{i\omega \cdot \frac{1}{a_l^*} A_{\theta_l^*} x} \right) \right\rangle_{L^2(\mathcal{G})}^2 dx \\ &\quad + \frac{16(J-1)^2}{a_j^{*2} J^4} \int_{\mathcal{D}} \left\langle (-4 + \sigma)g - \partial_r g, \Re \left(h e^{i\omega \cdot \frac{1}{a_j^*} A_{\theta_j^*} x} \right) \right\rangle_{L^2(\mathcal{G})}^2 dx, \\ \text{var} \left(\frac{d}{d\theta_j} L_\epsilon(\alpha^*) \right) &= \frac{16}{J^4} \sum_{l \neq j} \int_{\mathcal{D}} \left\langle \partial_\theta g, \Re \left(h e^{i\omega \cdot \frac{1}{a_l^*} A_{\theta_l^*} x} \right) \right\rangle_{L^2(\mathcal{G})}^2 dx \\ &\quad + \frac{16(J-1)^2}{J^4} \int_{\mathcal{D}} \left\langle \partial_\theta g, \Re \left(h e^{i\omega \cdot \frac{1}{a_j^*} A_{\theta_j^*} x} \right) \right\rangle_{L^2(\mathcal{G})}^2 dx. \end{aligned}$$

As Assumption (4.25) holds, this completes the proof of Proposition 5.1. \square

Proposition 5.2 Let $\mathcal{A}_{1,\epsilon}^{loc}$ denote the following space:

$$\mathcal{A}_{1,\epsilon}^{loc} = \left\{ (a, \theta) \in \tilde{\mathcal{A}}_1, \|(a - a^*, \theta - \theta^*)\| \leq \|(\hat{a}_\epsilon - a^*, \hat{\theta}_\epsilon - \theta^*)\| \right\}.$$

Under assumptions and notations of Theorem 4.2, there exists an invertible matrix H such that

$$\sup_{(\bar{\alpha}_\epsilon) \in \mathcal{A}_{1,\epsilon}^{loc}} \|\nabla^2 M_\epsilon(\bar{\alpha}_\epsilon) - H\| = o_{\mathbb{P}}(1)$$

Proof of Proposition 5.2: After computations, using Lemma 5.1, the matrix H is the value of the Hessian matrix of M in point α^* .

We study locally the Hessian matrix of M_ϵ . The sequences $\bar{\alpha}_\epsilon$ are in $\mathcal{A}_{1,\epsilon}^{loc}$. Notice that for $\eta > 0$:

$$\begin{aligned} \mathbb{P} \left(\sup_{\beta \in \mathcal{A}_{1,\epsilon}^{loc}} \left\| \nabla^2 M_\epsilon(\beta) - \nabla^2 M(\alpha^*) \right\| > 2\eta \right) &\leq \mathbb{P} \left(\sup_{\beta \in \mathcal{A}_{1,\epsilon}^{loc}} \left\| \nabla^2 M_\epsilon(\beta) - \nabla^2 M(\alpha^*) \right\| > \eta \right) \\ &\quad + \mathbb{P} \left(\sup_{\beta \in \mathcal{A}_{1,\epsilon}^{loc}} \left\| \nabla^2 M(\beta) - \nabla^2 M(\alpha^*) \right\| > \eta \right) \end{aligned}$$

As in proof of Theorem 4.1, the assumptions of Theorem 4.2 ensure the uniform convergence in probability of $\nabla^2 M_\epsilon$ to the Hessian matrix of M on $\mathcal{A}_{1,\epsilon}^{loc}$. Thus, the first term of inequality converges to 0 with n .

Since $\nabla^2 M$ is continuous in (a^*, θ^*) , there exists $\delta > 0$ such:

$$\nabla^2 M(B(\alpha^*, \delta)) \subseteq B(\nabla^2 M(\alpha^*), \eta).$$

Consequently, we have the following inclusion of event

$$\left(\sup_{\beta \in \mathcal{A}_{1,\epsilon}} \left\| \nabla^2 M(\beta) - \nabla^2 M(\alpha^*) \right\| > \eta \right) \subseteq (\|\hat{\alpha}_\epsilon - \alpha^*\| > \delta).$$

Thus from the Theorem 4.1, the second term of inequality converges to 0 too.

Furthermore, using the notation which are introduced in the proof of Proposition 5.1, the matrix H is equal to

$$H = \frac{2}{J} \begin{pmatrix} H_a & 0 \\ 0 & H_\theta \end{pmatrix},$$

where H_a and H_θ are defined as:

$$H_a = ((\sigma - 5)A_{\text{diag}}^{-2} - \frac{\sigma - 4}{J}a_{-1}a_{-1}^T)(\sigma - 4)\|g\|_{L^2(\mathcal{G})}^2 + (A_{\text{diag}}^{-2} - \frac{1}{J}a_{-1}a_{-1}^T)\|\partial_r g\|_{L^2(\mathcal{G})}^2,$$

$$H_\theta = (I_{J-1} - \frac{1}{J}\mathbb{I}_{J-1}I_{J-1} = \text{diag}(1 \dots 1)^T)\|\partial_\theta g\|_{L^2(\mathcal{G})}^2.$$

$A_{\text{diag}} = \text{diag}(a_2^* \dots a_J^*)$, $I_{J-1} = \text{diag}(1 \dots 1)$ are diagonal $J - 1$ squared matrix. $\mathbb{I}_{J-1} = (1, \dots, 1)^T$ and $a_1 = A_{\text{diag}}\mathbb{I}_{J-1}$ are two vectors of \mathbb{R}^{J-1} .

If $\partial_\theta g$ is not an identically null function, the matrix H_θ is invertible,

$$H_\theta^{-1} = (I_{J-1} + \mathbb{I}_{J-1})\|\partial_\theta g\|_{L^2(\mathcal{G})}^{-2}.$$

Furthermore, we may rewrite the matrix H_a as,

$$H_a = A_{\text{diag}}^{-2}\gamma_0 - A_{-1}\gamma_1,$$

where $\gamma_0 = (\sigma - 5)(\sigma - 4)\|g\|_{L^2(\mathcal{G})}^2 + \|\partial_r g\|_{L^2(\mathcal{G})}^2$ and $\gamma_1 = (\frac{(\sigma-4)^2}{J}\|g\|_{L^2(\mathcal{G})}^2 + \frac{1}{J}\|\partial_r g\|_{L^2(\mathcal{G})}^2)$. This matrix is invertible if, and only if:

$$\gamma_0 \neq 0 \quad \text{and} \quad \gamma_1(J - 1) - \gamma_0 \neq 0.$$

Then from Assumption (4.21), H_a is invertible, and

$$H_a^{-1} = \gamma_0^{-1} A_{\text{diag}}^2 - \frac{\gamma_1 \gamma_0^{-1}}{\gamma_0 - \gamma_1 (J-1)} a a^T,$$

where $a = (a_2^*, \dots, a_J^*) \in \mathbb{R}^{J-1}$. □

Proof of Theorem 4.3

The proof of this theorem is similar to the proof of Theorem 4.1. To simplify the notations, we write $b = \alpha_2$ for $\alpha_2 \in \mathcal{A}_2$ and $b^* = \alpha_2^*$. Let

$$N(b) = \int_{\mathbb{R}^2} |\hat{f}|^2(\omega) \left(1 - \left| \frac{1}{J} \sum_{j=1}^J e^{i\omega \cdot (\beta_j - \beta_j^*)} \right|^2 \right) d\omega.$$

We shall prove that $N(b)$ has an unique minimum at $b = b^*$, and that N_ϵ converges uniformly (in b) in probability to N . Then, these two conditions ensure that $\hat{\alpha}_2^\epsilon$ converges in probability to b^* as $\epsilon \rightarrow 0$ (see e.g. Theorem 5.7 in van der Vaart [36]).

Unicity of the minimum of $N(b)$: first one can remark that

$$N(b) = \int_{\mathbb{R}^2} |\hat{f}|^2(\omega) \left(1 - \left| \frac{1}{J} \sum_{j=1}^J e^{i\omega \cdot (\beta_j - \beta_j^*)} \right|^2 \right) d\omega.$$

has a minimum in b^* such that $N(b^*) = 0$. Assume that there exists $b \in \mathcal{A}_2$ such that $N(b) = 0$. Given our assumptions on f and \hat{f} , we have that \hat{f} is a continuous and non zero function. Hence, there exists an open set U of \mathbb{R}^2 such that $U \subset \text{supp}(\hat{f})$ and

$$1 = \left| \frac{1}{J} \sum_{j=1}^J \sum_{j=1}^J e^{i\omega \cdot (\beta_j - \beta_j^*)} \right|^2, \quad \forall \omega \in U,$$

which corresponds to the case of equality of the Cauchy Schwarz inequality. This means that for all $\omega \in U$, there exists $\lambda_\omega \in \mathbb{C}$ such that

$$\forall j = 1 \dots J, \quad e^{i\omega \cdot (\beta_j - \beta_j^*)} = \lambda_\omega.$$

Using the identifiability constraint ($b_1 = b_1^* = 0$) we deduce that $\lambda_\omega = 1$ for all $\omega \in U$. Then we deduce that:

$$\forall \omega \in U, \quad \forall j = 1 \dots J, \quad \omega \cdot (\beta_j - \beta_j^*) \equiv 0 \pmod{2\pi}.$$

Assume that there exists j such that $\beta_j - \beta_j^* \neq 0$. Since the rational numbers and the non rational numbers are dense in \mathbb{R} , there exist $\omega \in \mathbb{Q}^2 \cap U$ and $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ such that $\omega \neq 0$ and $\lambda \omega \in U$. Consequently, there exist $k \in \mathbb{Z}$ and $k' \in \mathbb{Z}$ such that

$$\omega \cdot (\beta_j - \beta_j^*) = (2\pi)k \quad \text{and} \quad \lambda \omega \cdot (\beta_j - \beta_j^*) = (2\pi)k',$$

which is a contraction. Hence, $\beta_j = \beta_j^*$ for all $j = 1, \dots, J$ which implies that $b = b^*$ and proves that $N(b)$ has a unique minimum. \square

Uniform convergence of N_ϵ : Let $N(b) = N_1 + N_2(b)$ with $N_1 = \int_{\mathbb{R}^2} |\hat{f}|^2(\omega) d\omega$ and

$$N_2(b) = \int_{\mathbb{R}^2} |\hat{f}|^2(\omega) \left| \frac{1}{J} \sum_{j=1}^J e^{i\omega \cdot (\beta_j - \beta_j^*)} \right|^2 d\omega.$$

We may rewrite the criterion $N_\epsilon(b)$ as

$$N_\epsilon(b) = \frac{1}{J} \sum_{j=1}^J \int_{|\omega| \leq \omega_\epsilon} |\hat{Z}_j(\omega)|^2 d\omega - \int_{|\omega| \leq \omega_\epsilon} \left| \frac{1}{J} \sum_{j=1}^J \hat{Z}_j(\omega) \right|^2 d\omega.$$

The first term in the above sum does not depend of the variable b , and we shall prove that it converges to N_1 . Hence, it suffices to study the uniform convergence of the second term in the above sum to prove the uniform convergence of N_ϵ to N . First, remark that

$$\int_{|\omega| \leq \omega_\epsilon} \left| \frac{1}{J} \sum_{j=1}^J \hat{Z}_j(\omega) \right|^2 d\omega = D_\epsilon(b) + \epsilon L_\epsilon(b) + \epsilon^2 Q_\epsilon(b),$$

where

$$\begin{aligned} D_\epsilon(b) &= \int_{|\omega| \leq \omega_\epsilon} \left| \frac{1}{J} \sum_{j=1}^J \left(\frac{a_j^*}{\hat{a}_{j,\epsilon}} \right)^2 \hat{f} \left(\frac{a_j^*}{\hat{a}_{j,\epsilon}} A_{\theta_j^* - \hat{\theta}_{j,\epsilon}} \omega \right) e^{i\omega \cdot (\tilde{\beta}_j - \tilde{\beta}_j^*)} \right|^2 d\omega, \\ L_\epsilon(b) &= 2\Re \int_{|\omega| \leq \omega_\epsilon} \left[\frac{1}{J} \sum_{j=1}^J \left(\frac{a_j^*}{\hat{a}_{j,\epsilon}} \right)^2 \hat{f} \left(\frac{a_j^*}{\hat{a}_{j,\epsilon}} A_{\theta_j^* - \hat{\theta}_{j,\epsilon}} \omega \right) e^{i\omega \cdot (\tilde{\beta}_j - \tilde{\beta}_j^*)} \right] \\ &\quad \times \left[\frac{1}{J} \sum_{j=1}^J \frac{e^{-i\omega \cdot \tilde{\beta}_j}}{\hat{a}_{j,\epsilon}^2} \overline{\hat{W} \left(\frac{1}{\hat{a}_{j,\epsilon}} \hat{A}_{-\hat{\theta}_{j,\epsilon}} \omega \right)} \right] d\omega, \\ Q_\epsilon(b) &= \int_{|\omega| \leq \omega_\epsilon} \left| \frac{1}{J} \sum_{j=1}^J \frac{e^{i\omega \cdot \tilde{\beta}_j}}{\hat{a}_{j,\epsilon}^2} \hat{W} \left(\frac{1}{\hat{a}_{j,\epsilon}} \hat{A}_{-\hat{\theta}_{j,\epsilon}} \omega \right) \right|^2 d\omega. \end{aligned}$$

for $\alpha_1 = (\hat{a}_{1,\epsilon}, \dots, \hat{a}_{J,\epsilon}, \hat{\theta}_{1,\epsilon}, \dots, \hat{\theta}_{J,\epsilon})$. As in the proof of Theorem 4.1, it suffices to prove the uniform convergence in probability of D_ϵ to N , and the uniform convergence in probability of $\epsilon^2 Q_\epsilon$ to 0. We may rewrite the difference between D_ϵ and $N_2(b)$ as

$$\begin{aligned} D_\epsilon(b) - N_2(b) &= D_\epsilon(b) - \int_{|\omega| \leq \omega_\epsilon} |\hat{f}(\omega)|^2 \left| \frac{1}{J} \sum_{j=1}^J e^{i\omega \cdot (\beta_j - \beta_j^*)} \right|^2 d\omega \\ &\quad - \int_{|\omega| > \omega_\epsilon} |\hat{f}(\omega)|^2 \left| \frac{1}{J} \sum_{j=1}^J e^{i\omega \cdot (\beta_j - \beta_j^*)} \right|^2 d\omega. \end{aligned}$$

Since Assumption (4.19) holds, the second term, which is deterministic, converges uniformly to 0. Using Cauchy-Schwarz inequality and the inequality $||a|^2 - |b|^2| \leq |a - b| + 2|b||a - b|$, $\forall (a, b) \in \mathbb{C}^2$, the first term converges uniformly in probability to 0 if, and only if, for all $j = 1, \dots, J$ the following terms converges uniformly in probability to 0,

$$(III)_j = \int_{|\omega| \leq \omega_\epsilon} \left| \left(\frac{a_j^*}{\hat{a}_{j,\epsilon}} \right)^2 \hat{f} \left(\frac{a_j^*}{\hat{a}_{j,\epsilon}} A_{\theta_j^* - \hat{\theta}_{j,\epsilon}} \omega \right) e^{i\omega \cdot (\tilde{\beta}_j - \tilde{\beta}_j^*)} - \hat{f}(\omega) e^{i\omega \cdot (\beta_j - \beta_j^*)} \right|^2 d\omega.$$

Since $\hat{a}_{j,\epsilon}$ and $\hat{\theta}_{j,\epsilon}$ are ϵ^{-1} -consistent estimators of a_j^* and θ_j^* , the delta method (see e.g. van der Vaart [36]) implies that

$$\left(\frac{a_j^*}{\hat{a}_{j,\epsilon}} \right)^2 - 1 = \mathcal{O}_{\mathbb{P}}(\epsilon) \quad \text{and} \quad \frac{a_j^*}{\hat{a}_{j,\epsilon}} A_{\theta_j^* - \hat{\theta}_{j,\epsilon}} - I_2 = \mathcal{O}_{\mathbb{P}}(\epsilon).$$

where I_2 is the 2×2 identity matrix. Moreover given that the function \hat{f} is 1-Lipschitz, we have:

$$(III)_j = \mathcal{O}_{\mathbb{P}} \left(\int_{|\omega| \leq \omega_\epsilon} \epsilon^2 |\omega|^2 + \epsilon |\hat{f}|^2(\omega) + \epsilon^2 |\omega|^2 |\hat{f}|^2(\omega) d\omega \right).$$

Consequently $(III)_j$ converges uniformly to zero because Assumption (??) holds. Hence D_ϵ converges uniformly in probability to 0.

Using the Cauchy-Schwarz inequality, $\epsilon^2 Q_\epsilon$ is uniformly bounded by the sums of the following variables

$$(IV)_j = \epsilon^2 \int_{|\omega| \leq \omega_\epsilon} \frac{1}{\hat{a}_{j,\epsilon}^4} \left| \hat{W}_j \left(\frac{1}{\hat{a}_{j,\epsilon}} A_{-\hat{\theta}_{j,\epsilon}} \omega \right) \right| d\omega \quad j = 1, \dots, J.$$

Since $\hat{a}_{j,\epsilon}$ is consistent and using a substitution rule, for a fixed $j = 1, \dots, J$ we have

$$(IV)_j = \mathcal{O}(\epsilon^2) \int_{|\omega| \leq \omega_\epsilon / a_{\min}} |\hat{W}_j(\omega)| d\omega.$$

Consequently, $\epsilon^2 Q_\epsilon$ converges uniformly to zero. Finally, the arguments used to prove that $(III)_j$ converges to zero, can also be used to derive that $\frac{1}{J} \sum_{j=1}^J \int_{|\omega| \leq \omega_\epsilon} |\hat{Z}_j(\omega)|^2 d\omega$ converges to N_1 , which finally proves that $N_\epsilon(b)$ converges uniformly to $N(b)$. \square

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