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Fluid boundary of a viscoplastic Bingham flow for finite solid deformations

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The modelling of viscoplastic Bingham fluids often relies on a rheological constitutive law based on a "plastic rule function" often identical to the yield criterion of the solid state. It is also often assumed that this plastic rule function vanishes at the boundary between the solid and fluid states, based on the fact that it is true in the limit of small deformations of the solid state or for simple yield criteria. We show that this is not the case for finite deformations by considering the example of a two state flow on a tilted plane where the solid state is described by a Neo-Hookean model with a Von Mises yield criterion. This opens new approaches for the modelling and the computation of the fluid state boundaries.

Keywords: yield stress; plastic rule function; Bingham fluid; Neo-Hookean rheology; Von Mises criterion

1. INTRODUCTION

We consider viscoplastic Bingham materials for which coexist "solid state" domains, in which their rheology is elastic, and "fluid state" domains, in which it is viscous. Most of the models assume that the boundary between the elastic solid state and the viscous fluid state is based on a yield criterion derived from the level of stress in the material. This yield criterion for the solid state is then used as the plastic rule function to build the constitutive rheological law of the fluid state ([1] and references therein).

For small deformations or simple yield criteria, one can show ([11]) that the plastic rule function vanishes on the yield surface, due to the continuity of the contact forces. In this case, it is thus possible to compute the fluid flow by ignoring the deformation of the solid state since the location of the interface between the solid and fluid states is simultaneously the solid yield surface and the surface for which the fluid plastic rule function vanishes.

For finite deformations and general yield criteria, the same procedure is used to compute flows for various geometries and realistic applications ([2,1,5,8,7,14]). Numerical simulations of such models then compute the surface for which the plastic function vanishes, which involves the removal of singularities of the mathematical problem ([15], [14]).

To our knowledge, these computations are not followed by a posteriori analyses checking that the resulting solid deformation really leads to a yield surface which matches its boundary with the Bingham fluid. On the numerical side, such analyses would require the computation of the coupled deformations of the solid and fluid states. On the laboratory experiments side, there are indications that the answer to the interface determination problem is complex. For instance, [10] have suggested that the yield stress depends on the final strain reached by a granular suspensions prior to yielding. In the field of soil mechanics, [9] have shown how the elastic and plastic behaviour of a three-phase porous media is influenced by the presence of water and air. In the field of waxy crude oil, [4] have proposed a conceptual model with two yield stresses to describe the material close to the interface.

Here, we investigate, on a simple case, how the coupling between the solid and fluid state can adress the issue of their interface determination. In Section 2, we recall the formulation of a wide range of rheological constitutive laws for the solid state in finite deformation and for the Bingham fluid state. The example of the Neo-Hookean model for the solid behavior and of the Von-Mises based potential for the fluid is given. In Section 3, we consider the case of a flow driven by gravity on a tilted slope and show, with our choice of constitutive laws, that the plastic rule function does not vanish at the solid and fluid interface where the yield criterion vanishes. We conclude by opening new approaches for the modelling of the fluid boundaries.

2. CONSTITUTIVE LAWS OF BINGHAM FLUIDS

We express the general form for the constitutive laws of elastic solids and Bingham fluids, only focusing on incompressible media for simplicity. The boundary between these two states is defined with a yield criterion as explained below. We make choices among these laws and criteria in order to build an example for the following section.

2.1. Elastic laws in finite deformation

We denote by $\underline{\underline{\sigma}}(\underline{x}, t)$ the Cauchy stress tensor in Eulerian representation and by $\underline{\underline{e}}(\underline{X}, t)$ the strain tensor in Lagrangian representation. In the case of finite deformations, we can use the second Piola-Kirchhoff stress tensor $\underline{\underline{\Pi}}(\underline{X}, t)$ (see for instance [13] or [6]) to measure the stress, which is linked to $\underline{\underline{\sigma}}$ by the relation $\underline{\underline{\sigma}} = {}^t\underline{\underline{F}} \cdot \underline{\underline{\Pi}} \cdot \underline{\underline{F}}/J$, where $\underline{\underline{F}}(\underline{X}, t)$ is the Jacobian deformation tensor and $J = \det \underline{\underline{F}}$ the Jacobian. A general constitutive law for an incompressible solid encountering finite deformation reads

$$\underline{\underline{\Pi}}(\underline{X}, t) = \rho_0 \frac{\partial \psi_e(\underline{\underline{e}})}{\partial \underline{\underline{e}}} + \eta_e \frac{\partial \varphi_e(\underline{\underline{e}})}{\partial \underline{\underline{e}}} \qquad \text{and} \qquad \varphi_e(\underline{\underline{e}}) = 0 \ , \tag{1}$$

where ρ_0 is the mass density in the reference configuration, $\psi_e(\underline{\underline{e}})$ is a thermodynamic potential and $\varphi_e(\underline{\underline{e}}) = \det\left(\underline{\underline{\mathbb{1}}} + 2\,\underline{\underline{e}}\right) - 1$ traduces the isochore constraint. The $\eta_e(\underline{x}, t)$ function is the Lagrange multiplier associated to this incompressibility constraint. For an homogeneous and isotropic medium, the thermodynamic potential is an arbitrary function $\psi_e(I'_1, I'_2, I'_3)$ of the three invariants $I'_1 = \operatorname{tr} \underline{\underline{e}}$, $I'_2 = \frac{1}{2}\operatorname{tr}(\underline{\underline{e}}^2)$ and $I'_3 = \frac{1}{3}\operatorname{tr}(\underline{\underline{e}}^3)$.

As an example of elastic constitutive law for incompressible medium encountering finite deformations, we consider the Neo-Hookean model for which $\rho_0\, \psi_e(\underline{\underline{e}}) = \mu_e \operatorname{tr} \underline{\underline{e}}$, where μ_e

is the elasticity modulus of the medium. One then checks that

$$\underline{\underline{\sigma}}(\underline{x}) = \mu_e \, \underline{\underline{F}} \cdot {}^t\underline{\underline{F}} + \eta_e(\underline{x}) \, \underline{\underline{\mathbb{1}}} \qquad \text{and} \quad \det \underline{\underline{F}} = 1 \; . \tag{2}$$

In the case of the small-deformation approximation $\underline{\underline{e}} \sim \underline{\underline{\epsilon}}$ where $\underline{\underline{\epsilon}} = \frac{1}{2}\left(\underline{\underline{F}} + {}^t\underline{\underline{F}}\right)$ is the tensor of small deformations, the Cauchy stress tensor reads

$$\underline{\underline{\sigma}}(\underline{x}) = 2\,\mu_e \, \underline{\underline{\epsilon}} + \eta_e \, \underline{\underline{\mathbb{1}}} \; . \tag{3}$$

Back to the general deformations, an elastic medium subjected to an increasing constraint yields at the points for which a yield criterion $f_e(\underline{\underline{\sigma}})$ becomes positive. For an homogeneous and isotropic medium, this yield criterion $f_e(I_1, I_2, I_3)$ is a function of the three invariants $I_1 = \text{tr}\ \underline{\underline{\sigma}}$, $I_2 = \frac{1}{2}\text{tr}(\underline{\underline{\sigma}}^2)$ and $I_3 = \frac{1}{3}\text{tr}(\underline{\underline{\sigma}}^3)$. This model leads to the definition of the yield surface $f_e(\underline{\underline{\sigma}}) = 0$ and the solid state is the domain for which $f_e(\underline{\underline{\sigma}}) \leq 0$.

As an example of yield criterion, we choose the Von Mises criterion based on the sign of the function

$$f_e(\underline{\underline{s}}) = \sqrt{\frac{1}{2}\text{tr}\left(\underline{\underline{s}}^2\right)} - k_e \; , \tag{4}$$

where $\underline{\underline{s}} = \underline{\underline{\sigma}} - \frac{1}{3}\text{tr}\underline{\underline{\sigma}} \; \underline{\underline{\mathbb{1}}}$ is the deviatoric part of $\underline{\underline{\sigma}}$ and k_e a critical stress value.

2.2. Bingham fluid rheological law

We denote by $\underline{\underline{d}}$ the strain rate tensor of the medium under study. A general constitutive rheological law for an incompressible fluid reads

$$\underline{\underline{\sigma}} = \rho_0 \frac{\partial \psi_b(\underline{\underline{d}})}{\partial \underline{\underline{d}}} + \eta_b \frac{\partial \varphi_b(\underline{\underline{d}})}{\partial \underline{\underline{d}}} \qquad \text{with} \qquad \varphi_b(\underline{\underline{d}}) = 0 \; , \tag{5}$$

where ρ_0 is the mass density, $\psi_b(\underline{\underline{d}})$ is a thermodynamic potential and $\varphi_b(\underline{\underline{d}}) = \text{tr}\,\underline{\underline{d}}$ traduces the isochore constraint. Since $\partial\varphi_b/\partial\underline{\underline{d}} = \underline{\underline{\mathbb{1}}}$, the Lagrange multiplier $\eta_b(\underline{x}, t)$ can be viewed as a pressure term and ψ_b is then chosen such that $\text{tr}\left(\partial\psi_b/\partial\underline{\underline{d}}\right) = 0$. The rheological constitutive law is thus characterized by the relation $\underline{\underline{s}} = \rho_0\, \partial\psi_b/\partial\underline{\underline{d}}$, where $\underline{\underline{s}}$ is the deviatoric part of the stress tensor. For a homogenous and isotropic medium, the potential $\psi_b(J_1, J_2, J_3)$ is a function of the three invariants $J_1 = \text{tr}\ \underline{\underline{d}}$, $J_2 = \frac{1}{2}\,\text{tr}(\underline{\underline{d}}^2)$ and $J_3 = \frac{1}{3}\text{tr}(\underline{\underline{d}}^3)$.

The rheology of an incompressible Bingham fluid is often based on the choice of a "plastic rule function" $f_b(\underline{\underline{s}})$ such that the rheological constitutive law reads

$$\underline{\underline{d}} = \frac{1}{2\mu_b} \frac{\partial}{\partial \underline{\underline{s}}} \left[f_b^2(\underline{\underline{s}}) \right] \; , \tag{6}$$

where μ_b is normalisation factor which has the dimension of a Lamé coefficient. We first note that $\underline{\underline{d}} = \underline{\underline{0}}$ for $f_b(\underline{\underline{s}}) = 0$. It is supposed that the fluid state only exists for $f_b(\underline{\underline{s}}) > 0$. The choice of the plastic rule function f_b is often the same than the yield criterion f_e used for the solid.

As an example of Bingham constitutive law, we choose the very common case where $f_b(\underline{\underline{s}}) = \sqrt{\frac{1}{2}\mathrm{tr}\left(\underline{\underline{s}}^2\right)} - k_b$ is based on a Von Mises criterion. We can show that the associated thermodynamic potential $\psi_b(\underline{\underline{d}})$ is such that

$$\rho_0\,\psi_b(\underline{\underline{d}}) = \left(\sqrt{\mu_b\,\mathrm{tr}\left(\underline{\underline{d}}^2\right)} + k_b/\sqrt{2\,\mu_b}\right)^2 . \tag{7}$$

We can also show that the relation between $\underline{\underline{s}}$ and $\underline{\underline{d}}$ reads

$$\underline{\underline{s}} = \left[2\mu_b + k_b/\sqrt{\frac{1}{2}\mathrm{tr}\left(\underline{\underline{d}}^2\right)}\right]\underline{\underline{d}} \quad \Longleftrightarrow \quad \underline{\underline{d}} = \frac{1}{2\mu_b}\left[1 - k_b/\sqrt{\frac{1}{2}\mathrm{tr}\left(\underline{\underline{s}}^2\right)}\right]\underline{\underline{s}} . \tag{8}$$

2.3. Boundaries between the solid and fluid state

We now consider a two state incompressible flow in which the rheological constitutive law is characterized by the potential $\psi_e(\underline{\underline{e}})$ for the solid state and by the potential $\psi_b(\underline{\underline{d}})$ associated to the plastic rule $f_b(\underline{\underline{s}})$ in the fluid state. We assume that suitable boundary conditions, in displacements or in constraints, are imposed at the boundary of the domain. At the interface between the two states, the contact force $\underline{T} = \underline{\underline{\sigma}} \cdot \underline{n}$, where $\underline{n}$ is normal to the surface, is continuous. This is a consequence of the momentum conservation. We also suppose that the displacement, and thus the velocity, are continuous. This is due to the viscous nature of the fluid which must stick to the solid state.

Another condition is necessary to determine the location of the interface. An obvious candidate is the solid yield criterion $f_e(\underline{\underline{s}}) = 0$, where $\underline{\underline{s}}$ is the stress deviator in the solid state, meaning that the interface between the two states coincides with the yield surface of the solid state. This set of boundary conditions leads to a well posed problem for the coupled solid-fluid equations provided that $f_b(\underline{\underline{s}}) \geq 0$ in the fluid state. It must be noted that we do not assume that f_b and f_e coincide.

As an example, we assume that both f_e and f_b are built with Von Mises criteria with respective critical constraints k_e and k_b. Assuming that the rheology is Neo-Hookean for the solid state (finite deformations), the stress tensor is

$$\underline{\underline{\sigma}} = \mu_e\,\underline{\underline{F}}\cdot{}^t\underline{\underline{F}} + \eta_e\,\underline{\underline{\mathbb{1}}} \quad \text{with} \quad \det\underline{\underline{F}} = 1 \quad \text{for} \quad f_e(\underline{\underline{s}}) \leq 0, \tag{9}$$

$$\underline{\underline{\sigma}} = \left\{2\mu_b + k_b\left[\mathrm{tr}\left(\tfrac{1}{2}\underline{\underline{d}}^2\right)\right]^{-\frac{1}{2}}\right\}\underline{\underline{d}} + \eta_b\,\underline{\underline{\mathbb{1}}} \quad \text{with} \quad \mathrm{tr}\,\underline{\underline{d}} = 0 \quad \text{otherwise.} \tag{10}$$

3. FLOW ON A TILTED PLANE

Since it is often assumed in the literature that the plastic rule function f_b vanishes at the interface of the solid and the fluid, we consider a simple example for which this is not the case, even when the plastic rule function f_b and the yield criterion f_e are identical. By considering first the case of small solid deformations, we show that finite deformations are required to exhibit this behaviour.

3.1. Solid equilibrium for small elastic deformations

We consider a layer of material of thickness h on a tilted plane making an angle α with the horizontal. The deformation of this layer is due to the gravity $\underline{g} = -g\sin\alpha\;\underline{e}_1 +$

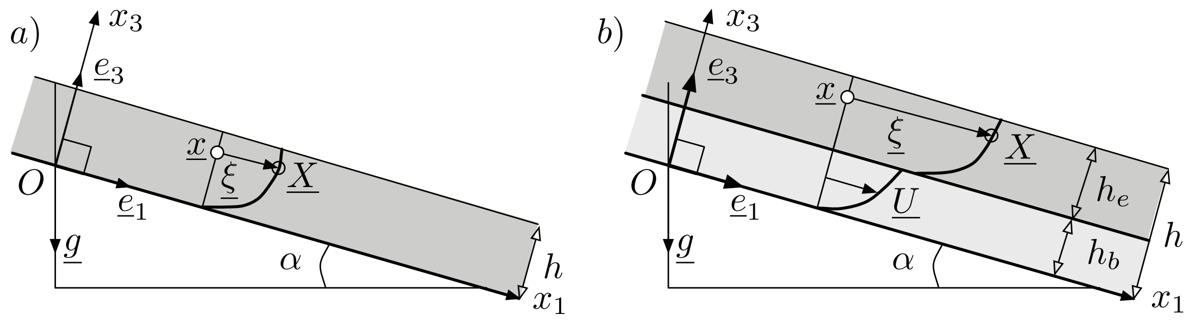

Figure 1. a) Layer of elastic solid on a tilted plane. b) Layer of Bingham fluid for $0 \le x_3 \le h_b$ under a layer of solid for $h_b \le x_3 \le h$.

$g \cos\alpha\, \underline{e}_3$, where the axes Ox_1 and Ox_3 are respectively parallel and perpendicular to the tilted plane (Figure 1a).

In order to compute the deformation of this continuous medium, we assume that the displacement $\underline{\xi}(\underline{X}, t)$ reads $x_1 = X_1 + \zeta(X_3)$, $x_2 = X_2$ and $x_3 = X_3 + \xi_3(X_3)$, where X_1, X_2 and X_3 are the Lagrangian coordinates. We assume that the displacement vanishes at the bottom, of equation $x_3 = 0$, and that the contact force $\underline{\underline{\sigma}} \cdot \underline{e}_3 = -p_a\, \underline{e}_3$ at the free surface of equation $x_3 = h$ is due to a constant pressure p_a. Since we assume that the incompressibility constraint holds, we have $\xi_3'(X_3) = 0$ and thus $\xi_3(X_3) = 0$ using the boundary condition $\xi_3(0) = 0$.

For small deformations, the Lagrangian and Eulerian representations are approximatively the same and the stress tensor reads $\underline{\underline{\sigma}}(\underline{x}) = 2\,\mu_e\, \underline{\underline{\epsilon}} + \eta_e\, \underline{\underline{\mathbb{I}}}$, where the tensor of small deformations is $\underline{\underline{\epsilon}} = \zeta'(x_3)\ (\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1)$. The equilibrium equation $\rho_0\, \underline{g} + \underline{\mathrm{div}}\, \underline{\underline{\sigma}} = \underline{0}$ and the boundary conditions lead to $\eta_e = -p_a$ and $\mu\, \zeta(X_3) = -\frac{1}{2}\, \rho_0\, g\, \sin\alpha\, X_3\, (X_3 - 2\,h)$. Denoting $\Lambda(x_3) = \zeta'(x_3) = \rho_0\, g\, \sin\alpha\, (h - x_3)$, the stress tensor finally reads $\underline{\underline{\sigma}} = -p_a\, \underline{\underline{\mathbb{I}}} + \mu_e\, \Lambda(\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1)$.

The Von Mises criterion $f_e(\underline{\underline{s}}) = \sqrt{\frac{1}{2} \underline{\underline{s}} : \underline{\underline{s}}}\ - k_e \le 0$ reads $\Lambda^2 - k_e^2/\mu_e^2 \le 0$ and thus $\Lambda(x_3) \le k_e/\mu_e$. At points where $\Lambda(x_3) = k_e/\mu_e$, the shear stress $\tau = \underline{e}_1 \cdot \underline{\underline{\sigma}} \cdot \underline{e}_3$ is $\tau_*(k_e) = k_e$. Since the trace of $\underline{\underline{s}}^2$ is maximum for $x_3 = 0$, one deduces that the maximum layer thickness beyond which the material yields is $h_e = \tau_*/(\rho_0\, g\, \sin\alpha)$ with $\tau_* = k_e$

However, this small deformations analysis is ofen inadequate since finite deformations (not small) are likely to be encountered before yield.

3.2. Solid equilibrium for finite deformations

In the case of finite deformation, we consider the example of a Neo-Hookean constitutive law for which the stress tensor reads

$$\underline{\underline{\sigma}}(\underline{x}) = \mu_e\, \underline{\underline{F}} \cdot\ {}^t\underline{\underline{F}} + \eta_e\, \underline{\underline{\mathbb{I}}} \qquad \text{with} \qquad \det \underline{\underline{F}} = 1\ \ . \tag{11}$$

The Jacobian matrix reads $\underline{\underline{F}}(\underline{X}) = \underline{\underline{\mathbb{I}}} + \zeta'(X_3)\, \underline{e}_1 \otimes \underline{e}_3 + \xi_3'(X_3)\, \underline{e}_3 \otimes \underline{e}_3$. The stress tensor reads

$$\underline{\underline{\sigma}}(\underline{x}) = [\mu_e + \eta_e(\underline{x})]\ \underline{\underline{\mathbb{I}}} + \mu_e\, \left[\zeta'^2(x_3)\, \underline{e}_1 \otimes \underline{e}_1 + \zeta'(x_3)\, (\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1)\right] \tag{12}$$

and the equilibrium equation div $\underline{\underline{\sigma}}(\underline{x}) + \rho_0\, \underline{g} = \underline{0}$ leads to

$$\left(\frac{\partial \eta_e}{\partial x_1} + \mu_e\, \zeta'' + \rho_0\, g\, \sin\alpha\right) \underline{e}_1 + \frac{\partial \eta_e}{\partial x_2}\, \underline{e}_2 + \left(\frac{\partial \eta_e}{\partial x_3} - \rho_0\, g\, \cos\alpha\right) \underline{e}_3 = \underline{0}\ . \tag{13}$$

The boundary conditions lead to $\mu_e + \eta_e(x_3) = -p(x_3)$ with $p(x_3) = p_a + \rho_0\, g\, \cos\alpha\, (h - x_3)$ and $\mu_e\, \zeta(X_3) = -\frac{1}{2}\, \rho_0\, g\, \sin\alpha\, X_3\, (X_3 - 2\,h)$. Denoting $\Lambda(x_3) = \zeta'(x_3) = \rho_0\, g\, \sin\alpha\, (h - x_3)$, the stress tensor finally reads

$$\underline{\underline{\sigma}} = -p\, \underline{\underline{\mathbb{I}}} + \mu_e \left[\Lambda^2\, \underline{e}_1 \otimes \underline{e}_1 + \Lambda\, (\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1)\right]\ . \tag{14}$$

In that configuration, the Von Mises criterion $f_e(\underline{\underline{s}}) = \sqrt{\frac{1}{2}\underline{\underline{s}} : \underline{\underline{s}}} - k_e \leq 0$ gives $\Lambda^2 - k_e^2/\mu_e^2 + \Lambda^4/3 \leq 0$ and thus $\Lambda(x_3) \leq L(k_e/\mu_e)$ with $L(\chi) = \sqrt{\frac{3}{2}} \left(\sqrt{1 + 4\,\chi^2/3} - 1\right)^{1/2}$. At points where $\Lambda(x_3) = L(k_e/\mu_e)$, the shear stress $\tau = \underline{e}_1 \cdot \underline{\underline{\sigma}} \cdot \underline{e}_3$ is $\tau_*(k_e) = \mu_e\, L(k_e/\mu_e)$. We note that $\tau_*(k_e) \sim k_e$ for $k_e \ll \mu_e$, which can be shown to recover the small deformation limit up to the the rigid limit $\mu_e = \infty$.

Since the trace of $\underline{\underline{s}}^2$ is maximum for $x_3 = 0$, one deduces that the maximum layer thickness beyond which the material yields is $h_e = \tau_*/(\rho_0\, g\, \sin\alpha)$. For $h > h_e$, a two state flow including a Bingham fluid has to be considered.

3.3. Stationnary flow for a two layer material

One supposes that the displacement $\zeta(\underline{X}, t)$, in Lagrangian representation, and the velocity $\underline{U}(\underline{x}, t)$, in Eulerian representation, reads

$$\begin{array}{llll} \underline{\xi}(\underline{X}, t) = & [A(t) + \zeta(X_3)]\ \underline{e}_1 & \text{for} & h_b \leq X_3 \leq h \quad \text{and} \\ \underline{U}(\underline{x}, t) = & U(x_3)\ \underline{e}_1 & \text{for} & 0 \leq x_3 \leq h_b\ , \end{array} \tag{15}$$

where the layer $h_b \leq x_3 \leq h$ is in the solid state and the layer $0 \leq x_3 \leq h_b$ is in the fluid state (Figure 1b). We suppose that $\zeta(h_b) = 0$ and $A(0) = 0$. We still assume that the displacement, and thus the velocity, vanish for $x_3 = 0$ and that the contact force at $x_3 = h$ is the one of constant pressure p_a.

As before, we have $\mu_e\, \zeta''(x_3) = -\rho_0\, g\, \sin\alpha$ and $\eta_e'(x_3) = \rho_0\, g\, \cos\alpha$ for the solid state. We notice that this problem is the same as the one dealing with a single solid layer on a flat bottom which would be located at $h = h_b$. Using the Von Mises criterion in the solid state, one sees that the yield interface is such that $h_b = h - h_e$, where $h_e = k_e/(\rho_0\, g\, \sin\alpha)$ in the case of small deformation or small k_e/μ_e and $h_e = \mu_e L(k_e/\mu_e)/(\rho_0\, g\, \sin\alpha)$ in the case of finite Neo-Hookean deformations.

We thus have $\tau(h_b) = \mu_e\, \zeta'(h_b) = \tau_*(k_e)$ on this surface. In the fluid layer, the stress tensor reads

$$\underline{\underline{\sigma}} = \eta_b\, \underline{\underline{\mathbb{I}}} + \mu_b\, [U' + k_b]\, (\underline{e}_1 \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{e}_1) \tag{16}$$

and we have $\mu_b\, U''(x_3) = -\rho_0\, g\, \sin\alpha$ and $\eta_b'(x_3) = \rho_0\, g\, \cos\alpha$. The continuity boundary conditions at the interface leads to $A(t) = U(h_b)\, t$ and $\mu_b\, [U'(h_b) + k_b] = \tau_*$. One then deduces that $U(x_3) = -\frac{1}{2}\, \rho_0\, g\, \sin\alpha\, (x_3^2 - 2\,h\, x_3) - k\, x_3$. Thus, the value of the plastic function is $f_b(\underline{\underline{s}}) = \tau_*(k_e) - k_b$ at the interface of equation $x_3 = h_b$ with $\tau_*(k_e) - k_b = k_e - k_b$ in the case of small deformations of the solid state and

$$\tau_*(k_e) - k_b = L(k_e/\mu_e) - k_b = \mu_e \sqrt{\frac{3}{2}} \left(\sqrt{1 + \frac{4\,k_e^2}{3\,\mu_e^2}} - 1 \right)^{\frac{1}{2}} - k_b \tag{17}$$

in the case of finite Neo-Hookean deformations. Even if $k_b = k_e$, this expression does not vanish for Neo-Hookean deformations of arbitrary μ_e, excepted for the particular case $k_e \ll \mu_e$ which corresponds to the small deformation limit. If we define $\Lambda(x_3)$ by $\Lambda = \zeta'(x_3)$ in the solid domain and $\Lambda = \frac{\mu_b}{\mu_e} [U'(x_3) + k]$ in the fluid domain, we see that $\Lambda = (\rho_0\, g\, \sin\alpha/\mu_e)(h - x_3)$ is a continuous function on the whole domain. Since $f_e(\underline{\underline{s}}) + k_e = \mu_e^2\, \Lambda^2$ in the solid domain and $f_b(\underline{\underline{s}}) + k_b = \mu_e^2\, \Lambda^2 \left(1 + \frac{1}{3}\, \Lambda^2\right)$ in the fluid domain, the values of f_e and f_b do not coincide at the fluid and solid interface, for all μ_e, excepted when $k_b = k_e$ and $\Lambda \ll 1$.

4. CONCLUSION

We have shown, on a particular example of constitutive laws for the solid and the fluid state of a Bingham material, that the plastic function f_b did not vanish at the interface of the two states, where the yield criterion f_e vanishes, even when their expressions as functions of the stress tensor are identical. This results contradicts a widespread assumption in the modelling of realistic applications and relies on the fact that there exists finite deformations of the solid state. But one can think that the finite deformation approach is consistent with the fact that the solid reaches the yield criteria to become fluid.

Our choice of a parallel flow on a tilded slope could be easily substitued by other simple geometries where analytical solution can be obtained, replacing for instance the gravity forcing by a shear imposed at the top of the layer or a pressure gradient. Our choice of the incompressible constraint in the rheological law has been motivated by simplicity considerations for the presentation. Compressibility can been taken into account by specifying rheological laws for the spherical parts of the tensors in addition to laws, for their deviatoric parts, identical to those that has been presented for the incompressible case ([12], [3]). Our choice of the Von Mises criterion for the yield criterium and the plastic function is motivated by its wide use for Bingham fluid modeling. Other criterions would have also generically led to the same conclusion for the non vanishing value of f_b on the interface.

Since the interface between the solid and fluid states is not the surface for which the plastic rule function vanishes, new methods must be seeked for its determination. A complex approach is to consider the coupled problem in the domain gathering both the solid and fluid states. One advantage is the fact that the singularity problem encountered in usual numerical simulations does no longer exists. But cheaper and regular approaches would consist in new parametrizations of the interface between the solid and fluid states based on numerical or laboratory experiments. The existence of finite deformation near the yield surface does not facilitate this task.

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