Some remarks on the damage unilateral effect modelling for microcracked materials

Hélène WELEMANE and Fabrice CORMERY*

Laboratoire de Mécanique de Lille, URA CNRS 1441
Cité Scientifique, 59655 Villeneuve d'Ascq cedex, France

ABSTRACT : This study deals with the macroscopic modelling of the mechanical behaviour of microcracked materials and particularly with the unilateral aspect of such damage which leads, at the closure of microcracks, to a partial damage deactivation. By means of a micromechanical analysis, the aim of this article is first to point out the influence of the opening-closure of microdefects on the effective elastic properties of a microcracked medium. According to these considerations, a new elastic moduli recovery condition at damage deactivation is proposed. The introduction of this condition within the anisotropic damage model proposed by Halm and Dragon, 1996 allows to extend its micromechanical background while preserving its main advantages, in particular the continuity of the stress-strain response and the symmetry of the stiffness tensor.

KEY WORDS : damage, brittle materials, microcracks, unilateral effect, microcracks opening-closure, damage activation-deactivation, elastic moduli recovery, anisotropy, stress-strain response continuity.

* author to whom correspondence should be addressed (e-mail : Fabrice.Cormery@eudil.fr)

1. INTRODUCTION

The specific mechanical response of quasi-brittle materials (some rocks, concrete, ceramics...) has been widely explained by the existence, nucleation and growth of microcracks at a microscopic level, see Kranz, 1983. The oriented nature of these microdefects, associated with the unilateral aspect of the contact (with or without friction) of their lips (microcracks can be either open or closed according to the loading), lead to a complex irreversible behaviour characterized in particular by a recovery (partial or total) of the effective properties at the closure of microcracks. This recovery phenomenon, which typically reveals the transition between so-called activated and deactivated state of damage when microcracks are respectively open and closed, has been experimentally shown by Reinhardt, 1984, Mazars et al., 1990 and Ikogou, 1990.

The description of the damage activation-deactivation process, currently called unilateral effect, as part of macroscopic modelling requires to answer the both following questions : *when* does the transition between these two states of damage occur ($\mapsto$ opening-closure criterion), and *how* does damage deactivation affect the elastic properties of the material ($\mapsto$ recovery condition) ? This problem still remains a difficult (Chaboche, 1992) and open subject of interest even if two attractive theories have been recently proposed to this end :

- Chaboche, 1993 postulates a modification of the stiffness tensor which allows the recovery of the initial (i.e. of the undamaged material) normal stiffness in each of the “principal” damage (or strain) directions when the related normal strain becomes negative. Note that a dual formulation in stress has also been developed by this author,

- Halm and Dragon, 1996 propose a formulation based on the one hand on the introduction of a fourth-order tensor parameter controlling damage activation-deactivation, and on the other hand on the assumption that a deactivated damage no longer contributes to the stiffness degradation in the related normal direction.

These formulations, which ensure the continuity of the stress-strain response and the symmetry of the stiffness tensor during the aforementioned transition, yet allow a partial account for the elastic moduli recovery phenomenon occurring at damage deactivation (Chaboche, 1999).

The aim of this paper is first to clarify, through some micromechanical considerations, the consequences of the microcracks closure on the elastic properties of a microcracked medium. On account of this analysis, a new elastic moduli recovery condition is proposed, which is eventually introduced within the model of Halm and Dragon, 1996. This model provides indeed an appropriate framework for the new condition since some choices in its formulation are already motivated by a micromechanical analysis (notably for the internal damage variable and the fourth-order control parameter), and it remains efficient and convenient for structural analysis. Friction effects, namely blocking and dissipative sliding of closed microcracks lips, are not taken into account in this paper.

Usual intrinsic notations are employed throughout. In particular, the tensor products of two second order tensors **a** and **b** are defined by :

$$[\mathbf{a}\otimes\mathbf{b}]:\mathbf{x}=(\mathbf{b}:\mathbf{x})\,\mathbf{a}$$

$$[\mathbf{a}\underline{\otimes}\mathbf{b}]:\mathbf{x}=\mathbf{a}\cdot\mathbf{x}\cdot\mathbf{b}^{T}$$

$$[\mathbf{a}\overline{\otimes}\mathbf{b}]:\mathbf{x}=\mathbf{a}\cdot\mathbf{x}^{T}\cdot\mathbf{b}^{T}$$

$$\mathbf{a}\,\overline{\underline{\otimes}}\,\mathbf{b}=[\mathbf{a}\overline{\otimes}\mathbf{b}+\mathbf{a}\underline{\otimes}\mathbf{b}]/2\,,$$

for any second-order tensor **x**. We denote by $\mathbf{n}^{\otimes p}=\mathbf{n}\otimes\mathbf{n}\cdots\otimes\mathbf{n}$ the p^{th} tensor product power of any vector **n** and by **I** the unit second-order tensor.

2. SOME MICROMECHANICAL CONSIDERATIONS

Consider a representative volume element (RVE) V of an homogeneous isotropic elastic linear matrix (Young modulus E_0, Poisson ratio ν_0) weakened by an array of N randomly distributed flat penny-shaped microcracks (unit normal $\mathbf{n}_k$, radius a_k), whose radii are very small in comparison with the size of the RVE. Assuming non-interaction between microcracks and sliding without friction of their lips, the free enthalpy of the microcracked medium is given by, see for example Krajcinovic, 1987, Kachanov, 1993 :

$$u = \frac{(1+\nu_0)}{2E_0}\mathrm{tr}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) - \frac{\nu_0}{2E_0}(\mathrm{tr}\,\boldsymbol{\sigma})^2 + \frac{8}{3V}\frac{(1-\nu_0^2)}{(2-\nu_0)E_0}\,\boldsymbol{\sigma} : \sum_{k=1}^{N} a_k^3\left[\mathbf{n}_k^{\otimes 2}\,\underline{\bar{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\bar{\otimes}}\,\mathbf{n}_k^{\otimes 2} - [2-(2-\nu_0)\,H(\sigma_n^k)]\,\mathbf{n}_k^{\otimes 4}\right] : \boldsymbol{\sigma} \tag{1}$$

The Heaviside step function H depending on the normal stress $\sigma_n^k = \mathbf{n}_k \cdot \boldsymbol{\sigma} \cdot \mathbf{n}_k$ to each microcrack is introduced to enable simultaneous consideration of its contribution whether it is open ($\sigma_n^k \geq 0$) or closed ($\sigma_n^k < 0$).

From the free enthalpy u of the effective medium, one can derive expressions of effective generalized elastic moduli : $E(\mathbf{m})$ the Young modulus related to the direction of unit vector **m**, $\nu(\mathbf{m},\mathbf{p})$ the Poisson ratio and $\mu(\mathbf{m},\mathbf{p})$ the shear modulus related to orthogonal directions of respective unit vectors **m** and **p**, defined through (Hayes, 1972) :

$$\begin{aligned} E(\mathbf{m}) &= [\mathbf{m}^{\otimes 2} : \mathbf{S} : \mathbf{m}^{\otimes 2}]^{-1} \\ \nu(\mathbf{m},\mathbf{p}) &= -E(\mathbf{m})\,[\mathbf{m}^{\otimes 2} : \mathbf{S} : \mathbf{p}^{\otimes 2}] \\ \mu(\mathbf{m},\mathbf{p}) &= [4\mathbf{m}\otimes\mathbf{p} : \mathbf{S} : \mathbf{m}\otimes\mathbf{p}]^{-1} \end{aligned} \tag{2}$$

where $\mathbf{S} = \partial^2 u / \partial \boldsymbol{\sigma}^2$ denotes the effective compliance tensor. According to equation (1) and noting $A = 16\,(1-\nu_0^2)/(6-3\nu_0)$, relevant calculations give rise to the following results :

$$\begin{aligned} E(\mathbf{m}) &= E_0\left[1 + \frac{A}{V}\sum_{k=1}^{N} a_k^3 (\mathbf{m}\cdot\mathbf{n}_k)^2\left(2 - [2-(2-\nu_0)H(\sigma_n^k)](\mathbf{m}\cdot\mathbf{n}_k)^2\right)\right]^{-1} \\ \nu(\mathbf{m},\mathbf{p}) &= \nu_0 \frac{E(\mathbf{m})}{E_0}\left[1 + \frac{A}{\nu_0 V}\sum_{k=1}^{N} a_k^3 [2-(2-\nu_0)H(\sigma_n^k)](\mathbf{m}\cdot\mathbf{n}_k)^2(\mathbf{p}\cdot\mathbf{n}_k)^2\right] \\ \mu(\mathbf{m},\mathbf{p}) &= \mu_0\left[1 + \frac{A}{(1+\nu_0)V}\sum_{k=1}^{N} a_k^3 \begin{pmatrix} (\mathbf{m}\cdot\mathbf{n}_k)^2 + (\mathbf{p}\cdot\mathbf{n}_k)^2 \\ -2[2-(2-\nu_0)H(\sigma_n^k)](\mathbf{m}\cdot\mathbf{n}_k)^2(\mathbf{p}\cdot\mathbf{n}_k)^2 \end{pmatrix}\right]^{-1} \end{aligned} \tag{3}$$

Relations (3) put forward the intricate effect due to open and closed microcracks in the elastic moduli degradation of a microcracked medium. Let examine first for clearness the simple case of a material weakened by a single array of parallel microcracks with unit normal **n**. Since the matrix is assumed to be isotropic, the effective medium exhibits the symmetry associated with the geometric shape of microcracks, namely transverse isotropy with the privileged direction **n**. The entire set of elastic moduli (3) is then fully determined by five independent coefficients (see Appendix 1) : $E(\mathbf{n})$, $E(\mathbf{t})$, $\nu(\mathbf{n},\mathbf{t})$, $\nu(\mathbf{t},\mathbf{k})$ and $\mu(\mathbf{n},\mathbf{t})$, for any vectors **t** and **k** forming with **n** an orthonormal basis of P^3. Their expressions are obtained from (3) :

$$
\begin{aligned}
E(\mathbf{n}) \quad &= E_0 \left[1 + \frac{A}{V} \sum_{k=1}^{N} a_k^3 \, (2-\nu_0) \, H(\sigma_n^k) \right]^{-1} \\
\nu(\mathbf{n},\mathbf{t}) &= \nu_0 \left[1 + \frac{A}{V} \sum_{k=1}^{N} a_k^3 \, (2-\nu_0) \, H(\sigma_n^k) \right]^{-1} \\
E(\mathbf{t}) \quad &= E_0 \\
\nu(\mathbf{t},\mathbf{k}) &= \nu_0 \\
\mu(\mathbf{n},\mathbf{t}) &= \mu_0 \left[1 + \frac{A}{(1+\nu_0)V} \sum_{k=1}^{N} a_k^3 \right]^{-1}
\end{aligned}
\tag{4}
$$

According to relations (4), some important aspects of the unilateral effect of damage can be emphasized :

- Among the aforementioned moduli, the Young modulus $E(\mathbf{n})$, but also the Poisson ratio $\nu(\mathbf{n},\mathbf{t})$, related to the direction normal to parallel microcracks, are affected by the microdefects change of state. In particular, they recover their initial values, i.e. those of the undamaged material (that is respectively E_0 and ν_0), at the closure of microcracks. It can notably be noted that the recovery of the Young modulus $E(\mathbf{n})$ to its initial value E_0 has been experimentally shown by Reinhardt, 1984 and Ramtani, 1990 during an uniaxial compression test in the direction of **n** on a concrete specimen previously microcracked orthogonally to **n** by an uniaxial tension test in this same direction.
- On the other hand, the shear modulus $\mu(\mathbf{n},\mathbf{t})$ remains unchanged by the closure of microdefects (partial damage deactivation). Elastic moduli $E(\mathbf{m})$, $\nu(\mathbf{m},\mathbf{p})$ and $\mu(\mathbf{m},\mathbf{p})$

related to directions different from the principal axes of the microdefects (**n**, **t**, **k**) are then only partially restored when microcracks get closed (except for the shear moduli $\mu(\mathbf{m},\mathbf{p})$ when $(\mathbf{m}\cdot\mathbf{n})^2 = (\mathbf{p}\cdot\mathbf{n})^2 = 1/2$).

Figure 1 illustrates these remarks on the example of a concrete weakened by a single array of parallel microcracks of normal **n**, when unit vectors **n**, **m** and **p** are supposed to be coplanar (vector **m** is defined by spherical angles θ and φ in the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{n})$ of P^3) ; owing to the problem symmetry, we have then $E(\mathbf{m}) = E(\phi)$, $\nu(\mathbf{m},\mathbf{p}) = \nu(\phi)$ and $\mu(\mathbf{m},\mathbf{p}) = \mu(\phi)$.

The particular nature of the microdefects contribution allows easily to extend these considerations for an array of N microcracks with different normal vectors : when microcracks of normal $\mathbf{n}_i$ are closed, they no longer contribute to the degradation of Young modulus $E(\mathbf{n}_i)$ and Poisson ratio $\nu(\mathbf{n}_i, \mathbf{t}_i)$, for any unit vector $\mathbf{t}_i$ orthogonal to $\mathbf{n}_i$. On the other hand, other moduli $E(\mathbf{t}_i)$, $\nu(\mathbf{t}_i, \mathbf{k}_i)$ and $\mu(\mathbf{n}_i, \mathbf{t}_i)$, for any unit vector $\mathbf{k}_i$ orthogonal to $\mathbf{n}_i$ and $\mathbf{t}_i$, remain unchanged by the change of state of microcracks of normal $\mathbf{n}_i$.

According to the previous remarks, taking into account the unilateral effect of damage as part of macroscopic modelling can no longer be limited to the single restoration of the Young modulus in the direction normal to closed microcracks. Indeed, it should be noted that the recovery affects also the Poisson ratio related to the same direction. In view of the lack of experimental results concerning this last point[1], the micromechanical analysis seems to us a judicious guide to macroscopic modelling of unilateral effect. The purpose of the next section is now to exploit it in the framework of the model proposed by Halm and Dragon, 1996.

[1] The recovery of Poisson ratios to their initial value can yet be observed on initially anisotropic composites during an uniaxial tension-compression test (cf. for example Allix et al., 1993).

3. A NEW RECOVERY CONDITION : APPLICATION TO AN ANISOTROPIC DAMAGE MODEL

The three-dimensional formulation proposed by Halm and Dragon, 1996 postulates the existence of a thermodynamic potential *w*, namely the Helmholtz free energy, built upon the following assumptions :

(i) Evolution of damage by microcrack growth is supposed to be the only dissipative mechanism, occurring in small strain, rate-independent and isothermal conditions. A symmetric second-order tensor **d** is chosen as the single damage internal variable to account for orientation and extent of microcracks :

$$\mathbf{d} = \sum_{k} \mathrm{d}_k(\mathrm{S})\, \mathbf{n}_k^{\otimes 2} \tag{5}$$

$\mathbf{n}_k$ represents the unit normal to the set k of parallel microcracks and $\mathrm{d}_k(\mathrm{S})$ a dimensionless scalar function characterizing the related density of microcracks. According to spectral decomposition, tensor **d** can be written in its principal axes :

$$\mathbf{d} = \sum_{i=1}^{3} \mathrm{d}_i\, \mathbf{v}_i^{\otimes 2} \tag{6}$$

where $\mathbf{v}_i$ and d_i are respectively the eigenvectors and eigenvalues of tensor **d**. Given the previous decomposition, any damage configuration is thus equivalent to three mutually orthogonal sets of parallel microcracks.

(ii) In the undamaged state, materials are supposed to be isotropic and linear elastic (with λ_0 and μ_0 the Lamé constants). The potential is taken linear in **d**, which corresponds to the hypothesis of a dilute density of non-interacting microcracks, and at most quadratic in $\boldsymbol{\varepsilon}$, as multilinear elasticity is assumed at constant damage. Residual effects due to damage (Acker, 1987) are eventually represented by a linear term in $\boldsymbol{\varepsilon}$, indifferent to damage activation-deactivation.

(iii) The thermodynamic potential (1) of micromechanics introduces a fourth-order tensorial quantity whose influence, that can be neglected when microcracks are open, is crucial when microcracks are closed (Kachanov, 1993). In order to account for this aspect within a simple and efficient formulation, Halm and Dragon, 1996 use a tensorial parameter built upon the eigenvalues and eigenvectors of **d** :

$$\mathbf{D} = \sum_{i=1}^{3} \mathrm{d}_i \, \mathbf{v}_i^{\otimes 4} \qquad (7)$$

According to these assumptions, Halm and Dragon, 1996 propose a thermodynamic potential holding two forms depending on whether microcracks are open or closed (for clearness, only a single set of parallel microcracks of unit normal **v** is first considered, thus $\mathbf{d} = \mathrm{d}\,\mathbf{v}^{\otimes 2}$) :

$$\begin{aligned} w^{\text{open}}(\boldsymbol{\varepsilon},\mathbf{d}) &= w_0(\boldsymbol{\varepsilon}) + \mathrm{g}\ \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + \alpha\ \mathrm{tr}\,\boldsymbol{\varepsilon}\ \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + 2\beta\ \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{d}) \\ w^{\text{clos}}(\boldsymbol{\varepsilon},\mathbf{d}) &= w_0(\boldsymbol{\varepsilon}) + \mathrm{g}\ \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + \alpha^{\text{clos}}\ \mathrm{tr}\,\boldsymbol{\varepsilon}\ \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + 2\beta^{\text{clos}}\ \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{d}) + \gamma\,\boldsymbol{\varepsilon}:\mathbf{D}:\boldsymbol{\varepsilon} \end{aligned} \qquad (8)$$

where $w_0(\boldsymbol{\varepsilon}) = (1/2)\lambda_0(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 + \mu_0\,\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon})$ is the free energy of the virgin material, g a constant characterizing residual effects, and α, β, and γ constant coefficients related to elastic properties degradation due to damage. The coefficients α^{clos}, β^{clos} and γ depend on the recovery condition selected in the modelling and on the mathematical restrictions imposed to the thermodynamic potential in order to ensure the continuity of the stress-strain response at the transition between open and closed microdefects states.

After a preliminary discussion on the recovery condition initially postulated by the authors, we propose subsequently to introduce in this model a new elastic moduli recovery condition at the closure of microcracks, which refers to the micromechanical considerations developed in Section 2. To this end, we will follow the process introduced by Halm and Dragon, 1996, with the determination of conditions imposed first by the recovery and then by the continuity. The question of damage evolution will however not be treated here, as this part of the formulation proposed by Halm and Dragon, 1996 is not concerned by the recovery condition introduced in this section.

(a) Recovery condition

The recovery condition proposed by Halm and Dragon, 1996 is based on the assumption that the stiffness in the direction normal to closed microcracks is equal to its initial value, i.e. :

$$\mathbf{v}^{\otimes 2} : \mathbf{C}^{\text{clos}} : \mathbf{v}^{\otimes 2} = \mathbf{v}^{\otimes 2} : \mathbf{C}_0 : \mathbf{v}^{\otimes 2} \qquad (9)$$

where $\mathbf{C}_0 = \partial^2 w_0 / \partial \boldsymbol{\varepsilon}^2$ denotes the initial stiffness tensor and $\mathbf{C}^{\text{clos}} = \partial^2 w^{\text{clos}} / \partial \boldsymbol{\varepsilon}^2$ its form when the material is weakened by closed microcracks. This condition imposes the form of the coefficient $\gamma = -\alpha^{\text{clos}} - 2\beta^{\text{clos}}$ related to the control parameter $\mathbf{D}$ and leads in particular to the following expressions for elastic moduli $\mathrm{E}(\mathbf{v})$ and $\nu(\mathbf{v},\mathbf{t})$ when microcracks are closed (for any unit vector **t** orthogonal to **v**) :

$$\begin{aligned} \mathrm{E}(\mathbf{v}) &= \lambda_0 + 2\mu_0 - \frac{(\lambda_0 + \alpha^{\text{clos}}\,\mathrm{d})^2}{\lambda_0 + \mu_0} \\ \nu(\mathbf{v},\mathbf{t}) &= \frac{\lambda_0 + \alpha^{\text{clos}}\,\mathrm{d}}{2(\lambda_0 + \mu_0)} \end{aligned} \qquad (10)$$

The clear dependence of these elastic moduli on α^{clos} emphasizes the incomplete character of their recovery whose intensity depends on this parameter value. Besides it requires to restrict the choice of this coefficient to a validity domain ($\alpha^{\text{clos}} \geq 0$) in order to ensure the physical admissibility of Young modulus $\mathrm{E}(\mathbf{v})$, which cannot indeed take a higher value than initial one. One may also note that, according to continuity conditions that will be detailed later on, Poisson ratio value $\nu(\mathbf{v},\mathbf{t})$ remains in fact unchanged, namely degraded, at the closure of microcracks.

The recovery condition postulated by Halm and Dragon, 1996 hence does not provide in the general case a satisfactory description of the unilateral effect of damage if we refer to the micromechanical analysis. Accordingly, we suggest to introduce a new recovery condition which assumes, at the closure of microcracks of unit normal **v**, the total recovery of the Young modulus in the direction given by **v** and of the Poisson ratio related to orthogonal directions of vectors **v** and **t**, for any unit vector **t**, that is :

$$\begin{cases} E(\mathbf{v}) \quad = E_0 \\ \\ \nu(\mathbf{v},\mathbf{t}) = \nu_0 \end{cases} \tag{11}$$

where E_0 and ν_0 are respectively the Young modulus and the Poisson ratio of the virgin material. From the thermodynamic potential (8), one derives the following expressions for effective elastic moduli $E(\mathbf{v})$ and $\nu(\mathbf{v},\mathbf{t})$ when microcracks of unit normal **v** are closed :

$$\begin{aligned} E(\mathbf{v}) \quad &= \lambda_0 + 2\mu_0 + 2(\alpha^{clos} + 2\beta^{clos} + \gamma)\,d - \frac{(\lambda_0 + \alpha^{clos}\,d)^2}{\lambda_0 + \mu_0} \\ \\ \nu(\mathbf{v},\mathbf{t}) &= \frac{\lambda_0 + \alpha^{clos}\,d}{2(\lambda_0 + \mu_0)} \end{aligned} \tag{12}$$

for any unit vector **t** orthogonal to **v**. The recovery assumption (11) implies consequently the necessary and sufficient conditions on the model parameters that follow :

$$\begin{cases} \alpha^{clos} \quad = 0 \\ \\ \gamma \qquad = -2\beta^{clos} \end{cases} \tag{13}$$

One can remark that the new recovery condition (11) on Young modulus $E(\mathbf{v})$ and Poisson ratio $\nu(\mathbf{v},\mathbf{t})$ implies the full restoration postulated by Halm and Dragon, 1996 of the normal stiffness in the direction given by **v**.

(b) Continuity and opening-closure criterion

The continuity of the stress-strain response at the transition between the two states of damage (division of the strain space into two distinct subdomains separated by boundary *H*) imposes to the thermodynamic potential to be of class C^1. This property is satisfied provided the stiffness tensor discontinuity $[\mathbf{C}] = \mathbf{C}^{open} - \mathbf{C}^{clos}$ (with $\mathbf{C}^{open} = \partial^2 w^{open} / \partial \boldsymbol{\varepsilon}^2$) takes such a form :

$$[\mathbf{C}] = s\,\mathbf{N}^{\otimes 2} \tag{14}$$

where s is a continuous scalar function and **N** is the unit normal to the hyperplane *H* (Curnier et al., 1995). Then $[\mathbf{C}]$ must be singular and in particular of rank one. According to relations (13), this is verified if and only if :

$$\begin{cases} \beta^{clos} = \beta \\ \alpha = 0 \end{cases} \tag{15}$$

At this time, it only remains to establish the opening-closure criterion for microcracks, i.e. the equation of the hyperplane *H* defined by $\mathbf{N}:\boldsymbol{\varepsilon}=0$. As conditions (13) and (15) lead to $[\mathbf{C}]=4\beta\,\mathrm{d}\,\mathbf{v}^{\otimes 4}$, we obtain the following criterion :

$$\mathbf{v}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{v}=0 \tag{16}$$

The set of microcracks of unit normal **v** is considered as active (respectively partially inactive) if the related normal strain $\varepsilon_v = \mathbf{v}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{v}$ is positive (respectively strictly negative). Although the recovery conditions employed are different, we can remark herein that we obtain the same opening-closure criterion as Halm and Dragon, 1996.

The spectral decomposition of **d** eventually provides the form of the thermodynamic potential for any damage configuration, that is :

$$w(\boldsymbol{\varepsilon},\mathbf{d}) = w_0(\boldsymbol{\varepsilon}) + g\,\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + 2\beta\,[\,\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{d}) - \boldsymbol{\varepsilon}:\sum_{i=1}^{3} \mathrm{H}(-\varepsilon_v^i)\,\mathrm{d}_i\,\mathbf{v}_i^{\otimes 4}:\boldsymbol{\varepsilon}\,] \tag{17}$$

with $\varepsilon_v^i = \mathbf{v}_i\cdot\boldsymbol{\varepsilon}\cdot\mathbf{v}_i$ the normal strain to the i^{th} equivalent set of microcracks. Figure 2 presents a simulation of elastic moduli $\mathrm{E}(\mathbf{m})$, $\nu(\mathbf{m},\mathbf{p})$ and $\mu(\mathbf{m},\mathbf{p})$ of a sandstone weakened by a single set of parallel microcracks of normal **v**, when unit vectors **v**, **m** and **p** are supposed to be coplanar (vector **m** is defined by spherical angles θ and ϕ in the orthonormal basis $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3=\mathbf{v})$ of P^3) ; owing to the problem symmetry, we have then $\mathrm{E}(\mathbf{m})=\mathrm{E}(\phi)$, $\nu(\mathbf{m},\mathbf{p})=\nu(\phi)$ and $\mu(\mathbf{m},\mathbf{p})=\mu(\phi)$.

4. DISCUSSION

(a) The formulation (17) allows a characterization of the damage unilateral effect in better accordance with the micromechanical analysis, nevertheless it leads to a particular form for the stiffness tensor : in the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ built upon the eigenvectors of **d**, damage affects only the diagonal components of the representative matrix of this tensor (see Appendix 2). This leads in particular to a constant value of the Poisson ratio $\nu(\mathbf{v},\mathbf{t})$, namely its initial value ν_0, during an uniaxial tension test of axis **v** (cf. figure 2). The introduction of the recovery condition (11) within the approach proposed by Halm and Dragon, 1996 consequently restricts the range of the modelling applications to a particular kind of quasi-brittle materials.

The predictive ability of the model may yet be enlarged to a larger class of quasi-brittle materials by modifying the thermodynamic potential (8) initially proposed by Halm and Dragon, 1996. In agreement with the basic assumptions, we suggest to introduce the additional invariant $\mathrm{tr}\,\mathbf{d}\,(\mathrm{tr}\,\boldsymbol{\varepsilon})^2$ in the two forms of *w*, that is for a single set of parallel microcracks of unit normal **v** :

$$w^{\mathrm{open}}(\boldsymbol{\varepsilon},\mathbf{d}) = w_0(\boldsymbol{\varepsilon}) + g\,\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + \alpha\,\mathrm{tr}\boldsymbol{\varepsilon}\,\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + 2\beta\,\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{d}) + \delta\,\mathrm{tr}\mathbf{d}\,(\mathrm{tr}\boldsymbol{\varepsilon})^2$$

$$w^{\mathrm{clos}}(\boldsymbol{\varepsilon},\mathbf{d}) = w_0(\boldsymbol{\varepsilon}) + g\,\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + \alpha^{\mathrm{clos}}\,\mathrm{tr}\boldsymbol{\varepsilon}\,\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + 2\beta^{\mathrm{clos}}\,\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{d}) + \delta^{\mathrm{clos}}\,\mathrm{tr}\mathbf{d}\,(\mathrm{tr}\boldsymbol{\varepsilon})^2 + \gamma\,\boldsymbol{\varepsilon}:\mathbf{D}:\boldsymbol{\varepsilon} \tag{18}$$

Note that such an invariant appears notably in the dual expression obtained by means of a Legendre transform of the thermodynamic potential (1) of micromechanics (Krajcinovic, 1989).

A similar reasoning as carried out before allows to establish the conditions imposed to the parameters α^{clos}, β^{clos}, γ and δ^{clos} by the recovery postulate (11) and by the stress-strain response continuity. From the thermodynamic potential (18), one obtains the following expressions for the elastic moduli $E(\mathbf{v})$ and $\nu(\mathbf{v},\mathbf{t})$ when microcracks are closed :

$$E(\mathbf{v}) = \lambda_0 + 2\mu_0 + 2(\alpha^{\mathrm{clos}} + 2\beta^{\mathrm{clos}} + \delta^{\mathrm{clos}} + \gamma)\,d - \frac{[\lambda_0 + (\alpha^{\mathrm{clos}} + 2\delta^{\mathrm{clos}})\,d]^2}{\lambda_0 + \mu_0 + 2\delta^{\mathrm{clos}}\,d}$$

$$\nu(\mathbf{v},\mathbf{t}) = \frac{\lambda_0 + (\alpha^{\mathrm{clos}} + 2\delta^{\mathrm{clos}})\,d}{2(\lambda_0 + \mu_0 + 2\delta^{\mathrm{clos}}\,d)} \tag{19}$$

for any unit vector **t** orthogonal to **v**. The full recovery of these moduli is thus ensured by the following necessary and sufficient conditions :

$$\begin{cases} \alpha^{clos} + 2(1-2\nu_0)\,\delta^{clos} = 0 \\ (1-2\nu_0)\,\alpha^{clos} + 4\beta^{clos} + 2\gamma = 0 \end{cases} \tag{20}$$

On the other hand, the continuity of the stress-strain response is systematically satisfied if and only if :

$$\begin{cases} \beta^{clos} = \beta \\ (\alpha - \alpha^{clos})^2 + 4\gamma(\delta - \delta^{clos}) = 0 \end{cases} \tag{21}$$

According to relations (20) and (21), parameters α^{clos}, β^{clos}, γ and δ^{clos} must then be solution of the following linear system :

$$\begin{cases} \alpha^{clos} + 2(1-2\nu_0)\,\delta^{clos} = 0 \\ (1-2\nu_0)\,\alpha^{clos} + 4\beta^{clos} + 2\gamma = 0 \\ \beta^{clos} = \beta \\ 2\,[2\beta + (1-2\nu_0)^2\,\delta + (1-2\nu_0)\,\alpha]\,\alpha^{clos} = (1-2\nu_0)\,(\alpha^2 - 8\,\delta\,\beta) \end{cases} \tag{22}$$

for any coefficients α, β and δ. One distinguishes two cases for which this system has a solution :

(i) Parameters α, β and δ are such that $2\beta + (1-2\nu_0)^2\,\delta + (1-2\nu_0)\,\alpha \neq 0$; the system is then fully determined, i.e. of rank four. The unique solution is :

$$
\begin{cases}
\alpha^{clos} = \dfrac{(1-2\nu_0)(\alpha^2 - 8\beta\delta)}{2[2\beta + (1-2\nu_0)^2\delta + (1-2\nu_0)\alpha]} \\[2ex]
\beta^{clos} = \beta \\[2ex]
\gamma \quad = -\dfrac{[4\beta + (1-2\nu_0)\alpha]^2}{4[2\beta + (1-2\nu_0)^2\delta + (1-2\nu_0)\alpha]} \\[2ex]
\delta^{clos} = -\dfrac{(\alpha^2 - 8\beta\delta)}{4[2\beta + (1-2\nu_0)^2\delta + (1-2\nu_0)\alpha]}
\end{cases}
\tag{23}
$$

(ii) Parameters α, β and δ are related by $2\beta+(1-2\nu_0)^2\delta+(1-2\nu_0)\alpha=0$ and $\alpha^2-8\beta\delta=0$ (or equivalently $\alpha=-4\beta/(1-2\nu_0)$ and $\delta=2\beta/(1-2\nu_0)^2$) ; the system (22) is then undetermined with one parameter, i.e. of rank three. With α^{clos}, β^{clos} and γ as principal unknowns, the system has consequently a unique solution :

$$
\begin{cases}
\alpha^{clos} = -2(1-2\nu_0)\,\delta^{clos} \\[2ex]
\beta^{clos} = \beta \\[2ex]
\gamma = -2\beta + (1-2\nu_0)^2\,\delta^{clos}
\end{cases}
\tag{24}
$$

for any value of δ^{clos}.

One may note that when coefficients α, β and δ verify $2\beta+(1-2\nu_0)^2\delta+(1-2\nu_0)\alpha=0$ and are such that $\alpha^2-8\beta\delta\neq 0$, the system (22) has no solution. In the subsequent part of this paper, it is assumed that the validity domain of these parameters excludes such an eventuality.

According to relations (23) and (24), the stiffness tensor discontinuity takes the form :

$$
[\mathbf{C}] = \Lambda\, d\,(K_1\,\mathbf{v}^{\otimes 2} + K_2\,\mathbf{I})^{\otimes 2} \tag{25}
$$

where coefficients Λ, K_1 and K_2 are defined in table 1. This leads consequently to a new expression for the microcracks opening-closure criterion :

$$K_1\ \mathbf{v}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{v}+K_2\ \mathrm{tr}\,\boldsymbol{\varepsilon}=0 \tag{26}$$

The set of microcracks of normal **v** is considered as active (respectively partially inactive) if the value of $\in_{v}=K_1\ \varepsilon_v+K_2\ \mathrm{tr}\,\boldsymbol{\varepsilon}$ is positive (respectively strictly negative).

According to the spectral decomposition and noting $\in_{v}^{i}=K_1\ \varepsilon_v^{i}+K_2\ \mathrm{tr}\,\boldsymbol{\varepsilon}$, the form of the thermodynamic potential for any damage configuration is eventually given by :

$$\begin{aligned} w(\boldsymbol{\varepsilon},\mathbf{d}) &= w_0(\boldsymbol{\varepsilon})+g\ \mathrm{tr}\,(\boldsymbol{\varepsilon}\cdot\mathbf{d})+\alpha\ \mathrm{tr}\,\boldsymbol{\varepsilon}\ \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d})+2\beta\ \mathrm{tr}\,(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{d})+\delta\ \mathrm{tr}\,\mathbf{d}\ (\mathrm{tr}\,\boldsymbol{\varepsilon})^2 \\ &-\frac{\Lambda}{2}\,\boldsymbol{\varepsilon}:\sum_{i=1}^{3}H(-\in_{v}^{i})\ d_i\ [K_1\ \mathbf{v}_i^{\otimes 2}+K_2\,\mathbf{I}]^{\otimes 2}:\boldsymbol{\varepsilon} \end{aligned} \tag{27}$$

under the conditions recapitulated in table 1.

Compared with the thermodynamic potential (17), the formulation (27) provides many advantages :

- A more general expression for the stiffness tensor where damage affects also the non-diagonal terms of its representative matrix (see Appendix 3). It allows then a more realistic description of the evolution of some elastic moduli, like for example the decrease of the Poisson ratio $\nu(\mathbf{v},\mathbf{t})$ during an uniaxial tension test with axis **v**, which is typical of some quasi-brittle materials like concrete (cf. for example Ramtani, 1990). Figure 3 illustrates the improvements provided by this new approach on the example of a concrete weakened by a single set of parallel microcracks of unit normal **v** (as in section 3, elastic moduli $E(\mathbf{m})$, $\nu(\mathbf{m},\mathbf{p})$ and $\mu(\mathbf{m},\mathbf{p})$ are studied in directions for which unit vectors **v**, **m** and **p** are coplanar).
- The particular form of the opening-closure criterion whose modular character allows an enriched description of the damage unilateral effect. For example during an uniaxial compression-tension test, the damage activation-deactivation process may occur for a non-zero lateral strain. One can note that we obtain the criterion $\mathbf{v}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{v}=0$ of the previous

section when parameters α, β and δ are such that $2\beta + (1-2\nu_0)^2\delta + (1-2\nu_0)\,\alpha \neq 0$ and $\alpha + 2(1-2\nu_0)\,\delta = 0$.

An other key aspect of the formulation (27) concerns the stiffness tensor whose non-diagonal terms of its representative matrix in the basis ($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$) are affected by the damage activation-deactivation process (see Appendix 3), while the symmetry of this tensor and the continuity of the stress-strain response remain preserved. One may note that this feature is directly owed to the new recovery condition (11).

(b) It can also be interesting to examine the modelling proposed by Chaboche, 1993, which assumes, in the case of a strain formulation, the following form for the effective stiffness tensor **C** of the material :

$$\mathbf{C} = \tilde{\mathbf{C}} + \eta \sum_{i=1}^{3} \mathrm{H}(-\varepsilon_v^i)\, \mathbf{v}_i^{\otimes 4} : [\mathbf{C}_0 - \tilde{\mathbf{C}}] : \mathbf{v}_i^{\otimes 4} \tag{28}$$

where $\tilde{\mathbf{C}}$ denotes the stiffness tensor for fully active conditions (all microcracks are open), $\mathbf{v}_i$ represents a “principal” direction of damage (or strain) and η is a material parameter aimed at characterizing the recovery intensity ($0 \leq \eta \leq 1$). This formulation, based on the exclusive modification of diagonal terms of the representative matrix of **C** in the “principal” directions of damage, is introduced in order to avoid any discontinuity of the stress-strain response and dissymetry of the stiffness tensor.

As the formulation proposed by Halm and Dragon, 1996, this approach accounts only for a partial elastic moduli recovery at damage deactivation. For example, if we apply the condition (28) to the thermodynamic potential proposed in (18) ($\tilde{\mathbf{C}} = \partial^2 w^{\mathrm{open}} / \partial \boldsymbol{\varepsilon}^2$), with, for $\mathbf{v}_i$, the eigenvectors of tensor **d** and $\eta = 1$ (maximum recovery), we obtain the following expression for the thermodynamic potential :

$$\begin{aligned} w(\boldsymbol{\varepsilon},\mathbf{d}) = {} & w_0(\boldsymbol{\varepsilon}) + \mathrm{g}\ \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + \alpha\ \mathrm{tr}\,\boldsymbol{\varepsilon}\ \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{d}) + 2\beta\ \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{d}) + \delta\ \mathrm{tr}\,\mathbf{d}\ (\mathrm{tr}\,\boldsymbol{\varepsilon})^2 \\ & - \boldsymbol{\varepsilon} : \sum_{i=1}^{3} \mathrm{H}(-\varepsilon_v^i)[(\alpha + 2\beta)\,\mathrm{d}_i + \delta\ \mathrm{tr}\,\mathbf{d}]\ \mathbf{v}_i^{\otimes 4} : \boldsymbol{\varepsilon} \end{aligned} \tag{29}$$

which leads, in the case of a single set of closed parallel microcracks of unit normal $\mathbf{v}$ ($\mathbf{d} = \mathrm{d}\,\mathbf{v}^{\otimes 2}$), to the elastic moduli $E(\mathbf{v})$ and $\nu(\mathbf{v},\mathbf{t})$ that follow :

$$E(\mathbf{v}) = \lambda_0 + 2\mu_0 - \frac{[\lambda_0 + (\alpha + 2\delta)\,\mathrm{d}]^2}{\lambda_0 + \mu_0 + 2\delta\,\mathrm{d}}$$

$$\nu(\mathbf{v},\mathbf{t}) = \frac{\lambda_0 + (\alpha + 2\delta)\,\mathrm{d}}{2(\lambda_0 + \mu_0 + 2\delta\,\mathrm{d})} \tag{30}$$

for any unit vector $\mathbf{t}$ orthogonal to $\mathbf{v}$. The single condition (28) is then not sufficient to corroborate the results derived from micromechanics. Indeed, it must be completed by some additional conditions ($\alpha = \delta = 0$) which lead in such a case to the formulation (17) also obtained from the approach proposed by Halm and Dragon, 1996.

(c) In the recovery condition (11), perfect microcracks closure is implicitly assumed. An alternative form of this condition may be proposed in order to account for an uncompleted microdefects closure.

Let consider a medium weakened by a single set of parallel microcracks of unit normal $\mathbf{v}$. At the closure of microcracks, the following elastic moduli recovery condition is postulated (for any unit vector $\mathbf{t}$ orthogonal to $\mathbf{v}$) :

$$\begin{cases} E(\mathbf{v}) = E_0 + (1-\phi)\,\Delta E_d(\mathbf{v}) \\ \nu(\mathbf{v},\mathbf{t}) = \nu_0 + (1-\varphi)\,\Delta\nu_d(\mathbf{v},\mathbf{t}) \end{cases} \qquad (\phi,\varphi) \in [0,1]^2 \tag{31}$$

where $\Delta E_d(\mathbf{v})$ and $\Delta\nu_d(\mathbf{v},\mathbf{t})$ represent the damage contribution in the expressions respectively of the Young modulus $E(\mathbf{v})$ and of the Poisson ratio $\nu(\mathbf{v},\mathbf{t})$ when microcracks are open, and coefficients ϕ and φ characterize the intensity of their recovery. In particular, when parameters ϕ and φ are such that $\phi = \varphi = 1$, the recovery condition (31) is the one postulated in (11).

(d) In the approach for the unilateral effect proposed in (17) and (27), the shear modulus $\mu(\mathbf{v},\mathbf{t})$ of a medium weakened by a single set of parallel microcracks of unit normal $\mathbf{v}$ remains unchanged by the closure of microdefects (cf. for example figures 2 and 3). This result is directly owed to the basic assumption of non-dissipative sliding of closed microcracks lips ; microdefects lips are indeed implicitly assumed to be perfectly lubricated. This modulus recovery, which has been clearly experimentally shown by Pecqueur, 1995 for a rock and Maire and Pacou, 1996 for a composite, would imply to take into account the friction on the microdefects lips. The theory proposed in this aim by Halm, 1997, Halm and Dragon, 1998 would then constitute an appropriate framework to include the recovery condition (11) or (31), allowing consequently the recovery of the whole elastic moduli $E(\mathbf{v})$, $\nu(\mathbf{v},\mathbf{t})$ and $\mu(\mathbf{v},\mathbf{t})$ related to the direction normal to microcracks. Note that a formulation accounting for the shear modulus recovery has also been proposed by Boursin et al., 1996 and Pottier, 1998.

5. CONCLUSION

In spite of recent advances in the macroscopic modelling of unilateral effect - which solve in particular the continuity problems arising when induced anisotropy is simultaneously described (Chaboche, 1993, Halm and Dragon, 1996) -, this question still remains an open research field. In view of the lack of exhaustive experimental results on this subject, the aim of this paper was to establish, by means of a micromechanical analysis, the influence of the microdefects opening-closure on the effective elastic properties of a microcracked medium. The evidence of some inconsistencies between derived results and existing macroscopic theories has induced us to propose a new elastic moduli recovery condition at the closure of microcracks. This original postulate, which has been introduced in the anisotropic damage model of Halm and Dragon, 1996, allows to extend the physical background of this model, while preserving its initial advantages. An improved version of this model has also been studied, which enlarges the scope of its applications and allows a better agreement with some experimental results. Further investigations need now to be conducted in order to evaluate the both aspects presented in this paper, namely the recovery condition (*How?*) and the microcracks opening-closure criterion (*When?*), by means of an extensive experimental study on various materials and for different loading paths.

REFERENCES

Acker P. 1987. "Comportement mécanique du béton : apports de l'approche physico-chimique," *Ph. D.-thesis*, Ecole Nationale des Ponts et Chaussées.

Allix O., C. Cluzel, A. Gasser and P. Ladevèze 1993. "Modélisation des Composites Céramique-Céramique à Différentes Echelles," *Revue des Composites et des Matériaux Avancés*, 3 (HS) : 277-297.

Boursin A., J. -L. Chaboche and F. Roudolff 1996. "Mécanique de l'endommagement avec conditions unilatérales et stockage d'énergie élastique," *C. R. Acad. Sci.*, Paris, 323 (Série IIb) : 369-376.

Chaboche J. -L. 1992. "Damage induced anisotropy : on the difficulties associated with the active/passive unilateral condition," *Int. J. Damage Mech.*, 1 : 148-171.

Chaboche J. -L. 1993. "Development of continuum damage mechanics for elastic solids sustaining anisotropic and unilateral damage," *Int. J. Damage Mech.*, 2 : 311-329.

Chaboche J. -L. 1999. "Continuum damage mechanics : initial and induced anisotropy in composite materials," in *Mechanics of heterogeneous materials*, *Conf. to the memory of Prof. J. -P. Boehler*, *Grenoble*, *France*, Ed. Darve F. and B. Loret .

Curnier A., Q. -C. He and P. Zysset 1995. "Conewise linear elastic materials," *J. Elasticity*, 37 : 1-38.

Halm D. 1997. "Contribution à la modélisation du comportement unilateral et du frottement dans les matériaux mésofissurés," *Ph. D.-thesis*, Université de Poitiers-ENSMA.

Halm D. and A. Dragon 1996. "A model of anisotropic damage by mesocrack growth ; unilateral effect," *Int. J. Damage Mech.*, 5 : 384-402.

Halm D. and A. Dragon 1998. "An anisotropic model of damage and frictional sliding for brittle materials," *Eur. J. Mech., A/Solids*, 17 : 439-460.

Hayes M. 1972. "Connections between the moduli for anisotropic elastic materials," *J. Elasticity*, 2 : 135-141.

Ikogou S. 1990. "Etude expérimentale et modélisation du comportement d'un grès," *Ph. D.-thesis*, Université des Sciences et Techniques de Lille 1.

Kachanov M. 1993. "Elastic solids with many cracks and related problems," in *Advances in applied mechanics*, Ed. Hutchinson J. and Wu T., New York, 30 : 259-445.

Krajcinovic D. 1987. “Micromechanical basis of phenomenological models,” in *Continuum Damage Mechanics, Theory and Applications*, Ed. Krajcinovic D. and J. Lemaitre, Wien, 195-206.

Krajcinovic D. 1989. “Damage mechanics,” *Mech. Mater.*, 8 : 117-197.

Kranz R. L. 1983. “Microcracks in rocks : a review,” *Tectonophysics*, 100 : 449-480.

Maire J.F. and D. Pacou 1996. “Essais de traction-compression-torsion sur tubes composites céramique-céramique,” *10èmes Journées Nationales sur les Composites, Paris, France*.

Mazars J., Y. Berthaud and S. Ramtani 1990. “The unilateral behaviour of damaged concrete,” *Eng. Fracture Mech.*, 35 : 629-635.

Pecqueur G. 1995. “Etude expérimentale et modélisation du comportement d’une craie et d’un grès en torsion,” *Ph. D.-thesis*, Université de Lille I.

Pottier T. 1998. “Modélisation multiéchelle du comportement et de l’endommagement de composites à matrice métallique,” *Ph. D.-thesis,* Ecole Nationale des Ponts et Chaussées.

Ramtani S. 1990. “Contribution à la modélisation du comportement multiaxial du béton endommagé avec description du caractère unilatéral,” *Ph. D.-thesis*, Université de Paris VI.

Reinhardt H. W. 1984. “Fracture mechanics of elastic softening materials like concrete,” *Heron* 29 (2).

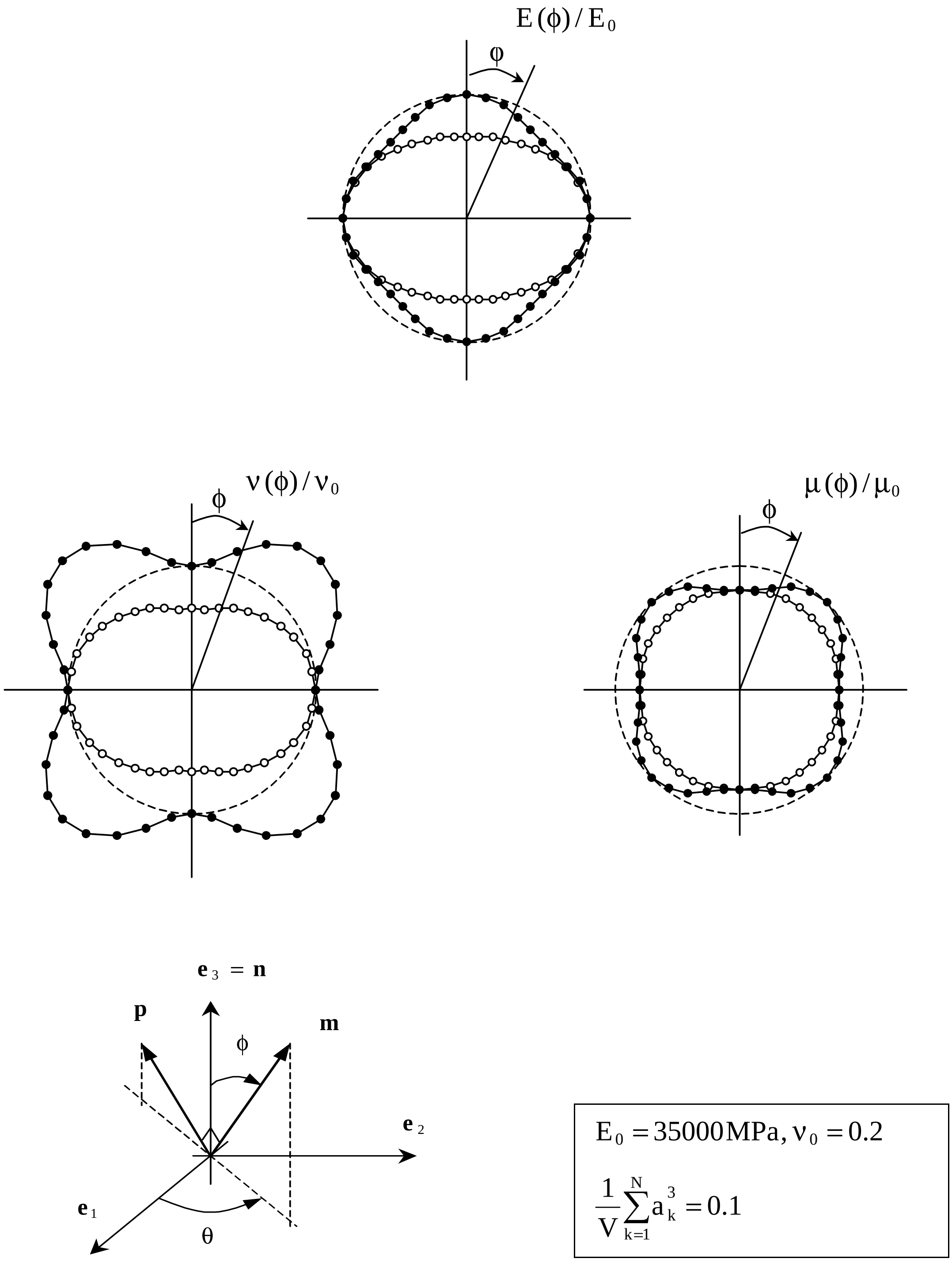

Figure 1. Generalized elastic moduli normalized by their initial values of a concrete weakened by a single array of parallel microcracks of unit normal **n** (–○– open microcracks, –●– closed microcracks, --- unit circle).

- Thermodynamic potential (1) -

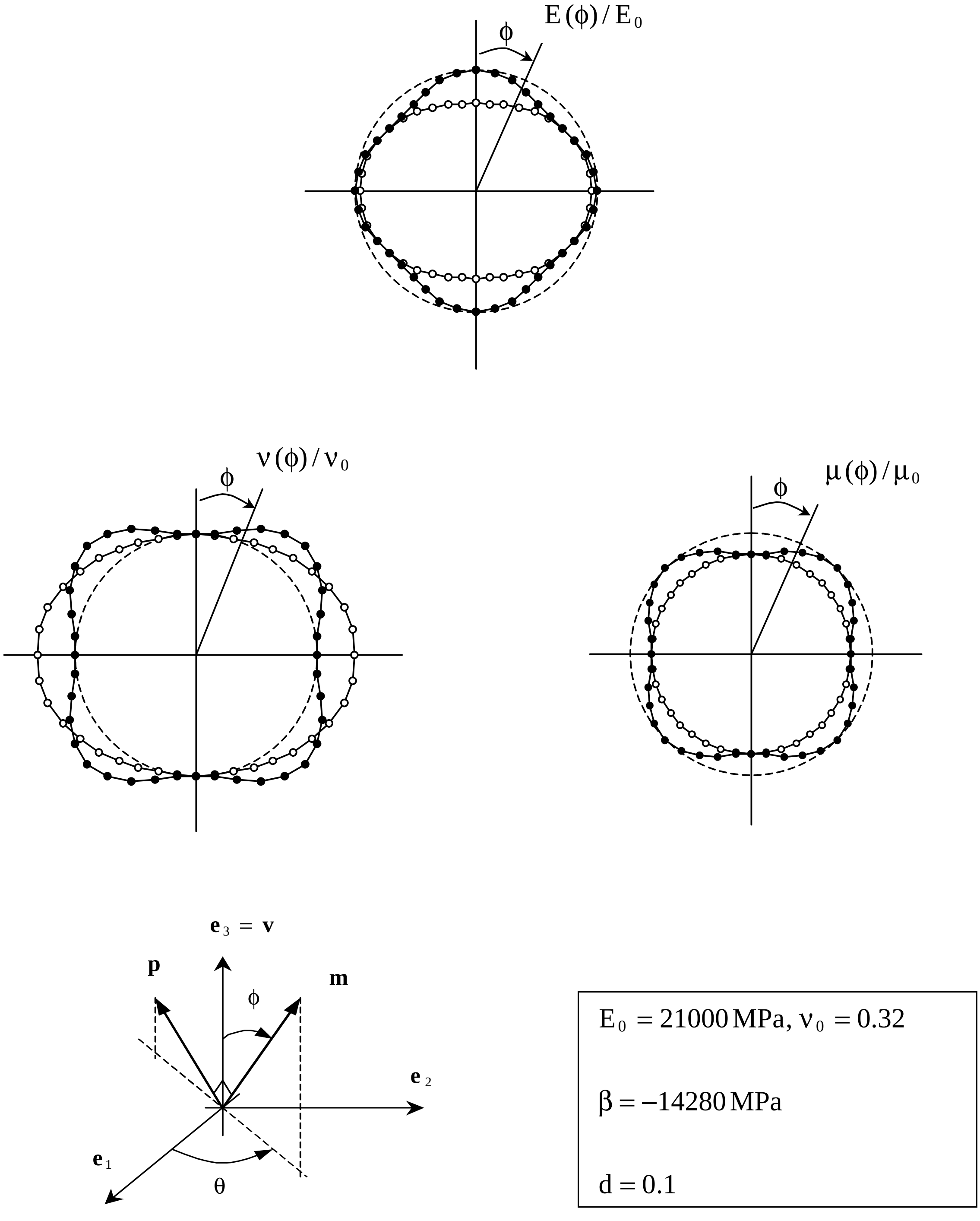

Figure 2. Generalized elastic moduli normalized by their initial values of a sandstone weakened by a single set of parallel microcracks of unit normal **v** (–o– open microcracks, –●– closed microcracks, - - - unit circle).

\- Thermodynamic potential (17) -

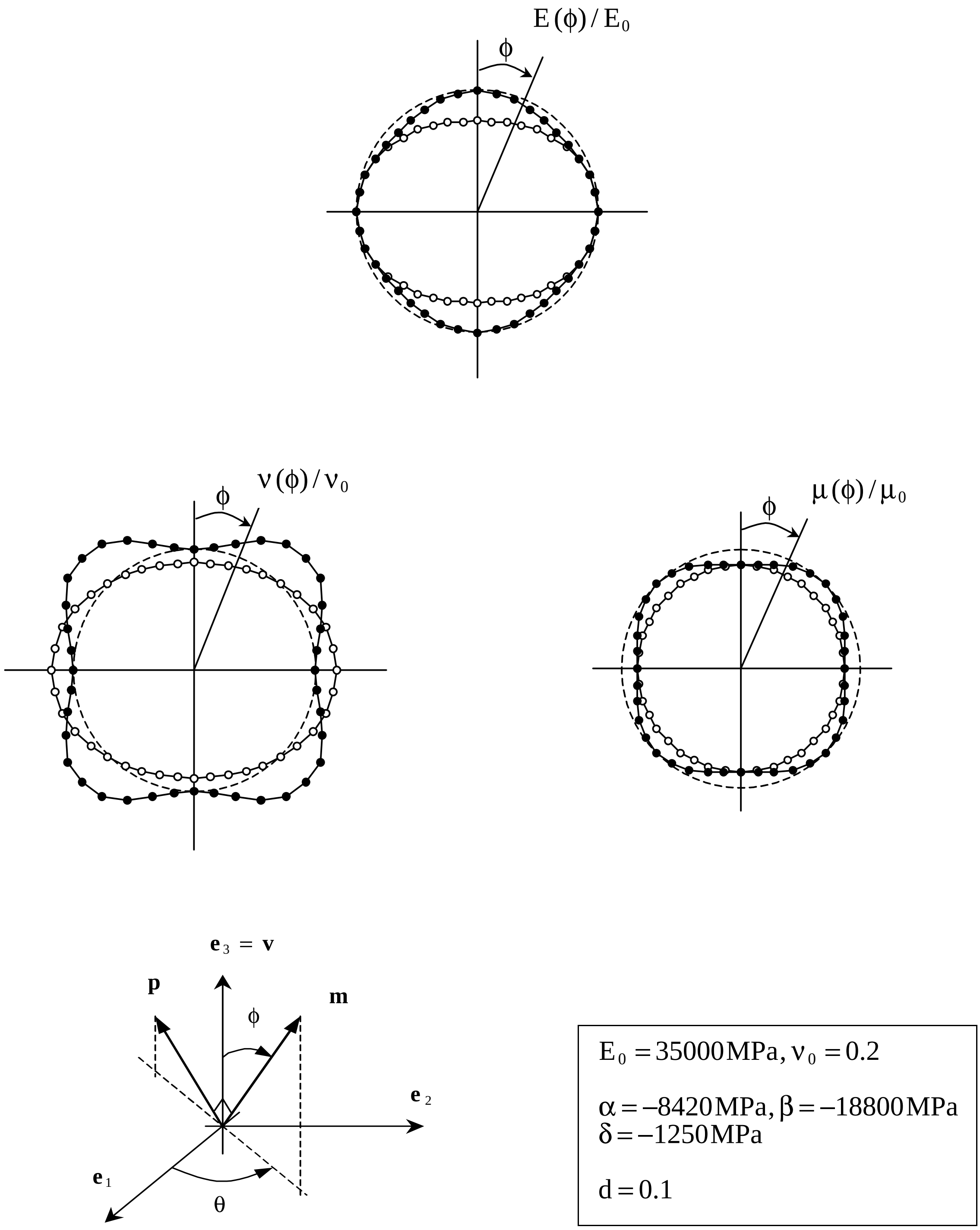

Figure 3. Generalized elastic moduli normalized by their initial values of a concrete weakened by a single set of parallel microcracks of unit normal **v** (–○– open microcracks, –●– closed microcracks, --- unit circle).

- Thermodynamic potential (27) -

	(i) α, β, δ such that $2\beta + (1-2\nu_0)^2 \delta + (1-2\nu_0)\,\alpha \neq 0$	**(ii)** α, β, δ such that $2\beta + (1-2\nu_0)^2 \delta + (1-2\nu_0)\,\alpha = 0$ and $\alpha^2 - 8\beta\delta = 0$
Λ	$\dfrac{1}{2\,[2\beta + (1-2\nu_0)^2\,\delta + (1-2\nu_0)\,\alpha]}$	$2[\dfrac{2\beta}{(1-2\nu_0)^2} - \delta^{clos}]$
K_1	$4\beta + (1-2\nu_0)\,\alpha$	$1-2\nu_0$
K_2	$\alpha + 2(1-2\nu_0)\,\delta$	-1

Table 1. Coefficients Λ, K_1 and K_2.

APPENDIX 1

Generalized moduli expression for a transversely isotropic material with axe **n**. $(\mathbf{n},\mathbf{t},\mathbf{k})$ is an orthonormal basis of P^3 and **m** and **p** two unit orthogonal vectors :

$$\mathrm{E}(\mathbf{m}) = \Big[(\mathbf{m}\cdot\mathbf{n})^4 \Big(\frac{1+2\,\nu(\mathbf{n},\mathbf{t})}{\mathrm{E}(\mathbf{n})}+\frac{1}{\mathrm{E}(\mathbf{t})}-\frac{1}{\mu(\mathbf{n},\mathbf{t})}\Big) - (\mathbf{m}\cdot\mathbf{n})^2 \Big(\frac{2\,\nu(\mathbf{n},\mathbf{t})}{\mathrm{E}(\mathbf{n})}+\frac{2}{\mathrm{E}(\mathbf{t})}-\frac{1}{\mu(\mathbf{n},\mathbf{t})}\Big)+\frac{1}{\mathrm{E}(\mathbf{t})} \Big]^{-1}$$

$$\nu(\mathbf{m},\mathbf{p}) = -\mathrm{E}(\mathbf{m})\Big[(\mathbf{m}\cdot\mathbf{n})^2 (\mathbf{p}\cdot\mathbf{n})^2 \Big(\frac{1+2\,\nu(\mathbf{n},\mathbf{t})}{\mathrm{E}(\mathbf{n})}+\frac{1}{\mathrm{E}(\mathbf{t})}-\frac{1}{\mu(\mathbf{n},\mathbf{t})}\Big) - ((\mathbf{m}\cdot\mathbf{n})^2+(\mathbf{p}\cdot\mathbf{n})^2)\Big(\frac{\nu(\mathbf{n},\mathbf{t})}{\mathrm{E}(\mathbf{n})}-\frac{\nu(\mathbf{t},\mathbf{k})}{\mathrm{E}(\mathbf{t})}\Big)-\frac{\nu(\mathbf{t},\mathbf{k})}{\mathrm{E}(\mathbf{t})} \Big]$$

$$\mu(\mathbf{m},\mathbf{p}) = \Big[4(\mathbf{m}\cdot\mathbf{n})^2 (\mathbf{p}\cdot\mathbf{n})^2 \Big(\frac{1+2\,\nu(\mathbf{n},\mathbf{t})}{\mathrm{E}(\mathbf{n})}+\frac{1}{\mathrm{E}(\mathbf{t})}-\frac{1}{\mu(\mathbf{n},\mathbf{t})}\Big) - ((\mathbf{m}\cdot\mathbf{n})^2+(\mathbf{p}\cdot\mathbf{n})^2)\Big(\frac{2(1+\nu(\mathbf{t},\mathbf{k}))}{\mathrm{E}(\mathbf{t})}-\frac{1}{\mu(\mathbf{n},\mathbf{t})}\Big)+\frac{2(1+\nu(\mathbf{t},\mathbf{k}))}{\mathrm{E}(\mathbf{t})} \Big]^{-1}$$

APPENDIX 2

Expression of the representative matrix of the stiffness tensor associated with formulation (17) in the orthonormal basis ($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$) built upon the eigenvectors of tensor **d** (Voigt notation) : case of a material weakened by a single set of parallel microcracks of unit normal $\mathbf{v}_1$ ($\mathbf{d} = d\,\mathbf{v}_1^{\otimes 2}$).

- *Open microcracks* $(\varepsilon_v^1 = \mathbf{v}_1 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v}_1 \geq 0)$:

$$\begin{bmatrix} \lambda_0 + 2\mu_0 + 4\beta d & \lambda_0 & \lambda_0 & 0 & 0 & 0 \\ \lambda_0 & \lambda_0 + 2\mu_0 & \lambda_0 & 0 & 0 & 0 \\ \lambda_0 & \lambda_0 & \lambda_0 + 2\mu_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_0 + \beta d & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_0 + \beta d \end{bmatrix}$$

- *Closed microcracks* $(\varepsilon_v^1 = \mathbf{v}_1 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v}_1 < 0)$:

$$\begin{bmatrix} \lambda_0 + 2\mu_0 & \lambda_0 & \lambda_0 & 0 & 0 & 0 \\ \lambda_0 & \lambda_0 + 2\mu_0 & \lambda_0 & 0 & 0 & 0 \\ \lambda_0 & \lambda_0 & \lambda_0 + 2\mu_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_0 + \beta d & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_0 + \beta d \end{bmatrix}$$

APPENDIX 3

Expression of the representative matrix of the stiffness tensor associated with formulation (27) in the orthonormal basis ($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$) built upon the eigenvectors of tensor **d** (Voigt notation) : case of a material weakened by a single set of parallel microcracks of unit normal $\mathbf{v}_1$ ($\mathbf{d} = d\,\mathbf{v}_1^{\otimes 2}$).

(i) α, β et δ are such that $2\beta + (1-2\nu_0)^2\,\delta + (1-2\nu_0)\,\alpha \neq 0$:

- *Open microcracks* $(\in_v^1 = K_1\,\mathbf{v}_1 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v}_1 + K_2\,\mathrm{tr}\boldsymbol{\varepsilon} \geq 0)$:

$$\begin{bmatrix} \lambda_0 + 2\mu_0 + 2(\alpha + 2\beta + \delta)\,d & \lambda_0 + (\alpha + 2\delta)\,d & \lambda_0 + (\alpha + 2\delta)\,d & 0 & 0 & 0 \\ \lambda_0 + (\alpha + 2\delta)\,d & \lambda_0 + 2\mu_0 + 2\delta\,d & \lambda_0 + 2\delta\,d & 0 & 0 & 0 \\ \lambda_0 + (\alpha + 2\delta)\,d & \lambda_0 + 2\delta\,d & \lambda_0 + 2\mu_0 + 2\delta\,d & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_0 + \beta\,d & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_0 + \beta\,d \end{bmatrix}$$

- *Closed microcracks* $(\in_v^1 = K_1\,\mathbf{v}_1 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{v}_1 + K_2\,\mathrm{tr}\boldsymbol{\varepsilon} < 0)$:

$$\begin{bmatrix} \lambda_0 + 2\mu_0 - \dfrac{\lambda_0^2}{(\lambda_0 + \mu_0)^2}\chi\,d & \lambda_0 - \dfrac{\lambda_0}{\lambda_0 + \mu_0}\chi\,d & \lambda_0 - \dfrac{\lambda_0}{\lambda_0 + \mu_0}\chi\,d & 0 & 0 & 0 \\ \lambda_0 - \dfrac{\lambda_0}{\lambda_0 + \mu_0}\chi\,d & \lambda_0 + 2\mu_0 - \chi\,d & \lambda_0 - \chi\,d & 0 & 0 & 0 \\ \lambda_0 - \dfrac{\lambda_0}{\lambda_0 + \mu_0}\chi\,d & \lambda_0 - \chi\,d & \lambda_0 + 2\mu_0 - \chi\,d & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_0 + \beta\,d & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_0 + \beta\,d \end{bmatrix}$$

with $\chi = \dfrac{(\lambda_0 + \mu_0)^2\;(\alpha^2 - 8\,\beta\,\delta)}{2[2(\lambda_0 + \mu_0)^2\beta + \mu_0^2\,\delta + \mu_0\,(\lambda_0 + \mu_0)\,\alpha]}$.

(ii) α, β et δ are such that $2\beta+(1-2\nu_0)^2\delta+(1-2\nu_0)\alpha=0$ and $\alpha^2-8\beta\delta=0$ (or equivalently $\alpha=-4\beta/(1-2\nu_0)$ and $\delta=2\beta/(1-2\nu_0)^2$) :

- *Open microcracks* $(\in_v^1 = K_1\, \mathbf{v}_1\cdot\boldsymbol{\varepsilon}\cdot\mathbf{v}_1 + K_2\, \mathrm{tr}\boldsymbol{\varepsilon} \geq 0)$:

$$\begin{bmatrix} \lambda_0+2\mu_0+\frac{4\lambda_0^2\beta}{\mu_0^2}d & \lambda_0+\frac{4\lambda_0(\lambda_0+\mu_0)\beta}{\mu_0^2}d & \lambda_0+\frac{4\lambda_0(\lambda_0+\mu_0)\beta}{\mu_0^2}d & 0 & 0 & 0 \\ \lambda_0+\frac{4\lambda_0(\lambda_0+\mu_0)\beta}{\mu_0^2}d & \lambda_0+2\mu_0+\frac{4(\lambda_0+\mu_0)^2\beta}{\mu_0^2}d & \lambda_0+\frac{4(\lambda_0+\mu_0)^2\beta}{\mu_0^2}d & 0 & 0 & 0 \\ \lambda_0+\frac{4\lambda_0(\lambda_0+\mu_0)\beta}{\mu_0^2}d & \lambda_0+\frac{4(\lambda_0+\mu_0)^2\beta}{\mu_0^2}d & \lambda_0+2\mu_0+\frac{4(\lambda_0+\mu_0)^2\beta}{\mu_0^2}d & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_0+\beta d & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_0+\beta d \end{bmatrix}$$

- *Closed microcracks* $(\in_v^1 = K_1\, \mathbf{v}_1\cdot\boldsymbol{\varepsilon}\cdot\mathbf{v}_1 + K_2\, \mathrm{tr}\boldsymbol{\varepsilon} < 0)$:

$$\begin{bmatrix} \lambda_0+2\mu_0+\frac{2\lambda_0^2\,\delta^{clos}}{(\lambda_0+\mu_0)^2}d & \lambda_0+\frac{2\lambda_0\,\delta^{clos}}{\lambda_0+\mu_0}d & \lambda_0+\frac{2\lambda_0\,\delta^{clos}}{\lambda_0+\mu_0}d & 0 & 0 & 0 \\ \lambda_0+\frac{2\lambda_0\,\delta^{clos}}{\lambda_0+\mu_0}d & \lambda_0+2\mu_0+2\delta^{clos}d & \lambda_0+2\delta^{clos}d & 0 & 0 & 0 \\ \lambda_0+\frac{2\lambda_0\,\delta^{clos}}{\lambda_0+\mu_0}d & \lambda_0+2\delta^{clos}d & \lambda_0+2\mu_0+2\delta^{clos}d & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_0+\beta d & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_0+\beta d \end{bmatrix}$$